Notes on the Grassmannian operations in 3D

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We specify the problem of using the Minkowski sum, intersection, orthogonal complement, and projection operations on Grassmannian manifolds with dimensions $d \leq 3$, under Exact Real Computation.

Lemma 1.1. Testing inequality under Exact Real Computation is equivalent to the Halting Problem. In particular, testing inequality of two Grassmannians with the same dimension is undecidable but semi-decidable.

Corollary 1.2. Testing whether a line is not contained in a plane is undecidable but semi-decidable.

Definition 1.3. A Grassmannian is represented as a linearly independent set of vectors in $\mathbb{R}^3$ that span the Grassmannian. The dimension of the Grassmannian is the cardinality of the basis set. By convention, the Grassmannian consisting of a single point has dimension zero, so it is represented by the empty set.

Proof. Given vectors $\mathbf{v}$ representing the line and $\mathbf{n}$ representing the normal vector to the plane, the problem is equivalent to testing whether $\mathbf{v}$ is orthogonal to $\mathbf{n}$. This holds if and only if $\mathbf{v} \cdot \mathbf{n} = 0$, which is undecidable but semi-decidable by Lemma 1.1. □

Definition 2.1. The Minkowski sum of two Grassmannians is defined to be the subspace spanned the union of the respective bases.

Proposition 2.2. Given Grassmannians G_1 and G_2 in $\mathbb{R}^3$, an algorithm for the Minkowski sum is given as follows. We specify that there are no cases where decidability is a problem (as in Lemma 1.1, Corollary 1.2):

Algorithm 1 Minkowski Addition

```
procedure ADD(G_1, G_2)
    if dim(G_1) > dim(G_2) then
        Swap G_1 and G_2
    if dim(G_2) = 3 then return R^3
    else if dim(G_1) = 0 then return G_2
    Assert G_1 ⊈ G_2
    if dim(G_1) = 2 then return R^3
    else if dim(G_2) = 1 then return G_1 ∪ G_2 (as bases)
    else return R^3                                  ▷ G_1 is a line, G_2 a plane
```

Proposition 2.3. The following is a modified variant of the algorithm that avoids dealing with casework, such as disallowing equal Grassmannians. It supplies an additional variable k denoting the desired dimension of the output, specifying that this is valid.

Algorithm 2 Modified Minkowski Addition

procedure ADD(G_1, G_2, k)
 if $dim(G_1) > dim(G_2)$ **then**
  Swap G_1 and G_2
 if $dim(G_2) = 3$ **then return** $\mathbb{R}^3$
 else if $dim(G_1) = 0$ **then return** G_2
 Assert $dim(G_2) \leq k \leq dim(G_1) + dim(G_2)$
 if $dim(G_1) = 2$ and $k = 3$ **then return** $\mathbb{R}^3$
 else if $dim(G_1) = 2$ and $k = 2$ **then return** G_1
 else if $dim(G_2) = 1$ and $k = 2$ **then return** $G_1 \cup G_2$ (as bases)
 else if $dim(G_2) = 1$ ad $k = 1$ **then return** G_1
 else if $k = 3$ **then return** $\mathbb{R}^3$
 else return G_2

Definition 3.1. The intersection of two Grassmannians is defined to be the intersections as sets.

Proposition 3.2. Given Grassmannians G_1 and G_2 in $\mathbb{R}^3$, an algorithm for the intersection is given as follows. Again, we specify that there are no cases where decidability is a problem:

Algorithm 3 Intersection

procedure INTERSECT(G_1, G_2)
 if $dim(G_1) > dim(G_2)$ **then**
  Swap G_1 and G_2
 if $dim(G_2) = 3$ **then return** G_1
 else if $dim(G_1) = 0$ **then return** $\emptyset$
 Assert $G_1 \not\subseteq G_2$
 if $dim(G_1) = 2$ **then return** $G_1 \cap G_2$ (as bases)
 else if $dim(G_2) = 1$ **then return** $\emptyset$
 else return $\emptyset$ $\triangleright$ G_1 is a line, G_2 a plane

Proposition 3.3. The following is a modified variant of the algorithm that avoids dealing with casework, such as disallowing equal Grassmannians. It supplies an additional variable k denoting the desired dimension of the output, specifying that this is valid.

Algorithm 4 Modified Intersection

```
procedure INTERSECT(G_1, G_2, k)
    if dim(G_1) > dim(G_2) then
        Swap G_1 and G_2
    if dim(G_2) = 3 then return G_1
    else if dim(G_1) = 0 then return ∅
    Assert dim(G_1) + dim(G_2) − 3 ≤ k ≤ dim(G_1)
    if dim(G_1) = 2 and k = 1 then return G_1 ∩ G_2 (as bases)
    else if dim(G_1) = 2 and k = 2 then return G_1
    else if dim(G_2) = 1 and k = 0 then return ∅
    else if dim(G_2) = 1 ad k = 1 then return G_1
    else if k = 0 then return ∅
    else return G_1
```

Definition 4.1. The orthogonal complement of a Grassmannian G is a Grassmannian H such that $G + H = \mathbb{R}^3$ and g and h are orthogonal for all $g \in G$ and $h \in H$.

Proposition 4.2. The algorithm for the orthogonal complement for a Grassmannian G is given as follows.

Algorithm 5 Orthogonal Complement

```
procedure ORTHOCOMP(G)
    d ← dim(G)
    Append vectors to basis of G to have dimension 3
    Apply Gram-Schmidt to make the basis orthogonal  return Last 3 − d vectors
```