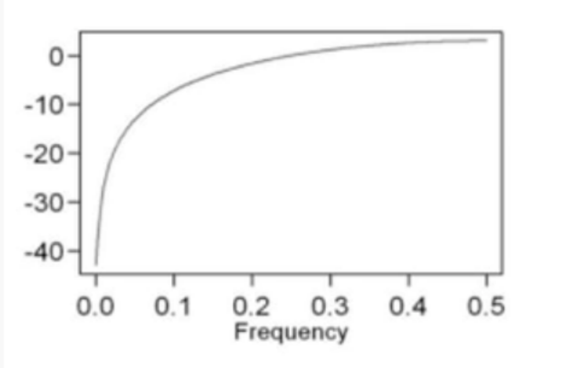
Moving Average(MA(q)) and ARMA(p,q) Models

More stationary model

- 1.MA(q)
  - Properties and Characteristics
    - 1. Model stationary data — not as useful as AR for actual stationary data
    - 2. Combine AR to create ARMA
    - 3.Model
      - Original form — $X_t = \mu + a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q}$
        - T时刻信号与其他时刻无关
        - Where the Theta's are real constants and a's are white
          - GLP: $X_t = \mu + \sum_{j=0}^{\infty} \psi_j a_{t-j}$ so for an MA(q) $\psi_0 = 1, \psi_1 = -\theta_1, \ldots, \psi_q = -\theta_q, \psi_k = 0, k > q$
          - An MA(q) is a finite GLP and is always stationary
            - $X_t - \mu = a_t - \theta_1 a_{t-1}$
              MA(1)
              - $E(X_t) = \mu$
                Mean
              - $\sigma_X^2 = \sigma_a^2(1+\theta_1^2)$
                Variance
              - $\rho_0 = 1$, $\rho_1 = \dfrac{-\theta_1}{1+\theta_1^2}$, $\rho_k = 0, k > 1$
                Autocorrelations
              - $S_X(f) = \dfrac{\sigma_a^2}{\sigma_X^2}\left|1-\theta_1 e^{-2\pi i f}\right|^2 = \dfrac{\sigma_a^2}{\sigma_X^2}\left|1-\theta_1(\cos 2\pi f - i\sin 2\pi f)\right|^2$
                Spectral density
              - MA(1) spectral densities do not have 'peaks' but have 'dips'



            - $X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$
              MA(2)
              - $\rho_0 = 1$, $\rho_1 = \dfrac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_2 = \dfrac{-\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_k = 0 \ when \ k > 2$
                Autocorrelations
      - Zero mean form — $X_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$
      - Opeartor zero mean form — $X_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t$
      - MA-charactersitic form — $1 - \theta_1 z - \cdots - \theta_q z^q = 0$
  - Invertibility — 2 models with same autocorrelations — MA invertible if and only if all roots of MA-characteristic equation are **outside the unit circle**
- 2.ARMA(p,q)
  - $X_t - \varphi_1 X_{t-1} - \cdots - \varphi_p X_{t-p} = \beta + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$
    Where
    - $\beta = (1 - \varphi_1 - \varphi_2 - \cdots - \varphi_p)\mu$
    - a_t is white noise
    - φ_i's and θ_j's are real constants
    - $\varphi_p \neq 0$ and $\theta_q \neq 0$
    - $\varphi(z)$ and $\theta(z)$ have no common factors
    - Properties and Characteristics
      - Zero mean form — $X_t - \varphi_1 X_{t-1} - \cdots - \varphi_p X_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$
      - Operator notation — $(1 - \varphi_1 B - \cdots - \varphi_p B^p) X_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t$ or $\varphi(B)X_t = \theta(B)a_t$
    - Examples — Cancellation
      - $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$
        What model is this? ARMA(2,1)?

k	ρ_k	
0	1	1
1	.5	$.5^1$
2	.25	$.5^2$
3	.125	$.5^3$
:	:	:

        That is, $\rho^k = .5^k$, which are the autocorrelations for the AR(1) model $(1 - .5B)X_t = a_t$.

        Note:
        $(1 - 1.3B + .4B^2)X_t = (1 - .8B)a_t$. Factoring, you get $(1 - .8B)(1 - .5B)X_t = (1 - .8B)a_t$. By cancellation, we get $(1 - .5B)X_t = a_t$ (i.e., the model above is an AR(1)).
    - Key result — $\varphi(B)X_t = \theta(B)a_t$ **is stationary and invertible if**
      - i. Roots of $\varphi(z) = 0$ are all outside the unit circle
      - ii. Roots of $\theta(z) = 0$ are all outside the unit circle
- 3.AIC: Akaike's information criterion
  - Given a set of data — Evaluate and compare the quality of models
  - Given a set of models — The model with the **lowest** AIC is thought to have the **most quality**
- 4.Psi weights — $X_t = \sum_{k=0}^{\infty} \psi_k a_{t-k} = \sum_{j=0}^{\infty} \phi_1^j B^j a_t$