Filtering and Autoregressive models of lag1

- 1. Technical details
  - Complex numbers
    - Z= a+bi
    - i = sqrt(-1)
    - complex conjugate: z* = a-bi
    - |z|= sqrt(a^2+b^2)
      - If |z| is such that |z|>1, it's outside unit circle
- 2. Linear Filters
  - Filter out certain frequencies from a data set
    - Low-pass
      - Filter out higher-frequency
        - moving aervage smoother : X_t = (Z_1+Z_2+Z_3+Z_4+Z_5)/5
    - High-pass
      - Filter out lower-frequency
        - Difference: X_t = Z_t - Z_{t-1}
    - General linear process: A GLP is a linear filter with **white noise input**
      - $\sum_{j=0}^{\infty} \psi_j a_{t-j} = X_t - \mu$
        - AR,MA and ARMA processes are sepcial cases of GLP
- 3. Introduction and Stationarity
  - **AR(*p*) *p* = 1**

    Autogregressive models
    - useful for describing stationary data that move forward in time
      - $\boxed{X_t = \beta + \varphi_1 X_{t-1} + a_t}$ where $\beta = (1-\varphi_1)\mu$

        AR(1)
        - Value of the process at t depends on the value of the process at time t-1 plus a random noise component(and a constant)
      - $X_t = (1-\varphi_1)\mu + \varphi_1 X_{t-1} + a_t$

        AR(1)
        - An AR(1) process is stationary if and only if $|\varphi_1| < 1$.
          - Conditions:
            - $E[X_t] = \mu$ ← Mean does not depend on *t*.

              Expected value
            - **Condition 2: Variance**

              $\sigma_X^2 = \frac{\sigma_a^2}{1-\varphi_1^2}$ ← Note that the variance is finite as long as $|\phi_1| < 1$, and also note that it does not depend on *t*!

              Variance
            - **Condition 3: Autocorrelations**

              $\rho_k = \varphi_1^k, k \geq 0$ ← Note that the ρ_k decreases exponentially with *k* and only depends on *k* and not *t*.

              Autocorrelations
            - **Bonus: Spectral density**

              $S_X(f) = \frac{\sigma_a^2}{\sigma_X^2}\left(\frac{1}{\left|1-\varphi_1 e^{-2\pi i f}\right|^2}\right)$ ← Monotonically increasing or decreasing in *f* depending on φ_1

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              Spectral density
      - β is moving average constant
- 4. Phi Positive and Negative
  - Positive
    - Realizations seem to be 'wandering', aperiodic in nature
    - Autocorrelations are damped exponentials.
    - Spectral densities have peaks at f=0, which is consistent with the behavior of the realizations.
  - Negative
    - Realizations seem to be 'oscillating'. That is, if X_t is above the mean, then the strong tendency is for X_{t+1} to be below the mean and so on.
    - Autocorrelations are damped, oscillating exponentials.
    - Spectral density have peaks at f=.5. This is consistent with the up-and-down behavior in realizations.
- 5. Characteristic equation and stationarity
  - Characteristic equation
    - X_t = φ_1*X_{t+1} + a_t
      - 1-φ_1*Z=0
        - r=Z=1/φ_1
          - |r| >1 then stationary
          - |φ_1|<1 then stationary
  - Backshift operator
    - BX_t = X_{t-1}
    - Ba_t = a_{t-1}
      - (1-φ_1B)X_t = a_t