Frequency Domain/Spectral Density

Domain method of estimating frequency data in time series

- 1. Review:Trig Functions
  - Sin(t)
  - Cos(t)
    - Period = 2π
      - Horizontal Phase Shift: (t+Δ)
- 2. The Frequency domain
  - Discover the frequency content of a set of data(or lack), better understand process generating data
  - Periodicities
    - Periodic functions
    - Psuedo-periodic data — Data are pseudo-periodic with period p if p is the smallest value such that a cycle appears to repeat itself
    - Aperiodic functions/data
      - Wandering behavior
  - Frequency = 1/period — Consider: sin(Bt+C)
    - Period = 2π /B
    - Frequency = B/ 2π
  - Represent 'non-cylic type' functions as sines and cosines — Fourier expansion
    - $$f(x) = \lim_{n\to\infty}\left\{\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx\right\}$$
      where f is defined over [-π,π]
      - $$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt, \qquad k = 0,1,\ldots$$
      - $$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt, \qquad k = 0,1,\ldots$$
      - Example:
        - Fourier Expansion for f(x) = x — $$x = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx), \qquad x \in (-\pi,\pi)$$
        - Fourier Expansion for f(x) = e^x — $$e^x = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}[\cos kx - k\sin kx]$$
- 3. Spectrum/Spectral density
  - Tool to identify the frequency content of a time series
  - Estimate spectral density — Smoothing spectrum
    - Choose M(2sqrt(n)) and a window function(\lambda_k) — $$\hat{S}(f) = \lambda_0 1 + 2\sum_{k=1}^{M}\lambda_k\hat{\rho}_k cos2\pi fk \quad |f| \le .5$$
      Minimize impact of \hat{\rho_k} as k increases
      - Properties
        - $$\int_{-.5}^{.5} S(f)e^{2\pi ifk}df = \rho_k$$
        - $$\int_{-.5}^{.5} S(f)\,df = 1$$
        - White noise — $$S_x(f) = 1 + 2\sum_{k=1}^{\infty}\rho_k\cos 2\pi fk$$
          - $$\hat{S}_x(f) = \lambda_0\cdot 1 + 2\sum_{k=1}^{M}\lambda_k\hat{\rho}_k\cos 2\pi fk$$
            $$M = 2\sqrt{n}$$
            $|\lambda_k|$ decreases as k increases
            |f| <= 0.5
            - S(f) = 1
    - Window:
      - 1. Bartlett (triangular) window
        $$\lambda_k = 1 - \left(\frac{k}{M}\right), \quad k = 0,1,\ldots,M$$
        $$= 0, \quad k > M$$
      - 2. Tukey window
        $$\lambda_k = \frac{1}{2}\left[1 + \cos\left(\frac{k\pi}{M}\right)\right], \quad k = 0,1,\ldots,M$$
        $$= 0, \quad k > M.$$
      - 3. Parzen window
        $$\lambda_k = 1 - 6\left(\frac{k}{M}\right)^2 + 6\left(\frac{|k|}{M}\right)^3, \quad 0 < k \le \frac{M}{2}$$
        $$= 2\left[1 - \left(\frac{|k|}{M}\right)\right]^3, \quad M/2 < k \le M$$
        $$= 0, \quad k > M.$$
- 4. Nyquist frequency — f > 0.5 — Aliasing: Sampling at t= 0,+-1,+-2.. the two curves are indistinguishable — Lower frequency is identified
- 5. Estimation of spectrum — Log spectral density — accentuate secondary peaks