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A PROJECTION-BASED DERIVATION OF THE EQUATIONS OF MOTION FOR THE MOVING FRAME METHOD FOR MULTI-BODY DYNAMICS

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ABSTRACT

The moving frame method for multi-body dynamics, established by Murakami in [10] and [11], embodies a consistent notation and mathematical framework that simplifies the derivation of equations of motion of complex systems. The derivation of the equations of motion follows Hamilton's principle and requires the calculation of virtual angular velocities and the corresponding virtual rotational displacements. The goal of this paper is to present a projection-based approach, which only requires knowledge of Euler's first and second law, that results in the same equation of motion. The constraints need not satisfy d'Alembert's principle and the projection is based on a generalization of Gauss' principle of least constraint [14]. One advantage of the proposed method is that it avoids variational principles and therefore is more accessible to undergraduate students. In addition, the final form of the equation of motion is more easily understood. We motivate our approach using the example of the simple pendulum, derive the main result, and apply the methodology for derivation of the equations of motion for a modified Chaplygin sleigh and a rotary pendulum.

1 Introduction

The moving frame method for multi-body dynamics, established by Murakami in [10] and [11], details a systematic methodology for describing motion and computing multi-body dynamics. The motion is described in inertial and moving frames attached to each moving body. A frame consists of three orthonormal basis vectors and a position vector specifying the origin. These frames are related to one another by homogeneous transformation matrices that are referred to as frame connection matrices. Using these relations, a systematic method is developed for computing velocities and angular velocities of multi-body systems. The moving frame method also details the computation of constrained multi-body motion. It is tacitly assumed that the constraints are given in generalized velocity form. These constraints are incorporated in the derivation of the equation of motion which follows Hamilton's principle. The derivation is relatively complex and requires the calculation of virtual angular velocities and the corresponding virtual rotational displacements.

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The goal of this paper is to present a projection-based approach, which only requires knowledge of Euler's first and second law, that results in the same equation of motion. The idea is to consider the motion of connected rigid bodies as a motion of free rigid bodies with constraint forces, which are subsequently projected onto the fundamental subspaces of the constraint matrix that relates translational and angular velocities to generalized velocities. We note that the constraints need not satisfy d'Alembert's principle and the projection is based on a generalization of Gauss' principle of least constraint [14]. One advantage of the proposed method is that it avoids variational principles and we feel therefore is more accessible to undergraduate students. In addition, the final form of the equation of motion is more easily understood. We motivate our approach in Sec. 2 using the example of the simple pendulum. Section 3 details the methodology for rigid body motion. We illustrate our approach using the examples of a modified Chaplygin sleigh with an internal elastic rotor with nonholonomic constraints and a rotary (inverted) pendulum in Sec. 4.

2 Introductory example: simple pendulum

This section introduces the projection-based derivation of the equation of motion (EOM) using the familiar example of the simple pendulum: a particle of mass m in a gravitational field is constrained to move on a circle of radius ℓ (see Fig. 1). First, we consider the particle kinematics. Subsequently, we derive the equation of motion.

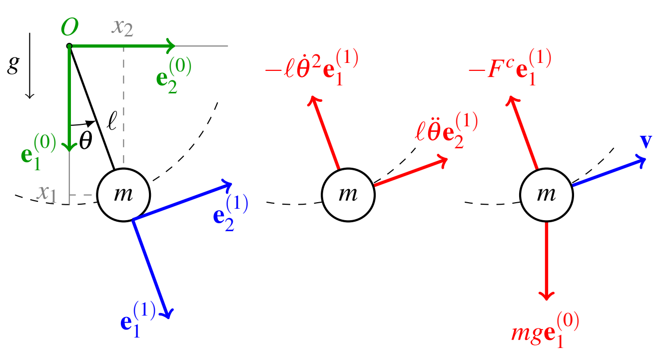

FIGURE 1. (a) Pendulum frames of reference, (b) Kinematic diagram, and (c) Free Body Diagram (FBD).

2.1 Kinematics

In Fig. 1 (a) we define two frames: 1) an inertial frame with origin O and fixed basis vectors $\mathbf{e}_1^{(0)}$, $\mathbf{e}_2^{(0)}$, and 2) a body-fixed frame with same origin O but different basis vectors $\mathbf{e}_1^{(1)}$, $\mathbf{e}_2^{(1)}$, which depend on time t. The basis vectors of the two frames are related through a rotation matrix

$$\left[\mathbf{e}_1^{(1)}(t)\ \mathbf{e}_2^{(1)}(t)\right] = \left[\mathbf{e}_1^{(0)}\ \mathbf{e}_2^{(0)}\right]\begin{bmatrix}\cos(\theta(t)) & -\sin\theta(t))\\ \sin(\theta(t)) & \cos\theta(t))\end{bmatrix}, \quad (1)$$

or, in short-hand notation

$$\mathbf{E}^{(1)} = \mathbf{E}^{(0)}\mathbf{R}^{(0,1)}. \quad (2)$$

Similarly,

$$\mathbf{E}^{(0)} = \mathbf{E}^{(1)}\mathbf{R}^{(1,0)} \text{ where } \mathbf{R}^{(1,0)} = \left(\mathbf{R}^{(0,1)}\right)^{-1} = \left(\mathbf{R}^{(0,1)}\right)^{T}. \quad (3)$$

The time derivative of the body frame, expressed in the body frame, is given by

$$\dot{\mathbf{E}}^{(1)} = \mathbf{E}^{(0)}\dot{\mathbf{R}}^{(0,1)} = \mathbf{E}^{(1)}\underbrace{\mathbf{R}^{(1,0)}\dot{\mathbf{R}}^{(0,1)}}_{\Omega^{(1,1)}} \quad (4)$$

where the skew symmetric angular velocity matrix is given by

$$\Omega^{(1,1)} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\dot{\theta}. \quad (5)$$

Thus, the time derivatives of the body-fixed basis vectors are

$$\dot{\mathbf{e}}_1^{(1)} = \mathbf{e}_2^{(1)}\dot{\theta}, \quad \dot{\mathbf{e}}_2^{(1)} = -\mathbf{e}_1^{(1)}\dot{\theta}. \quad (6)$$

The position of the particle in the body-fixed frame is

$$\mathbf{r} = \ell\mathbf{e}_1^{(1)}. \quad (7)$$

The velocity is obtained by taking the time-derivative and using Eqn. (6), i.e.

$$\mathbf{v} = \dot{\mathbf{r}} = \ell\dot{\mathbf{e}}_1^{(1)} = \ell\dot{\theta}\mathbf{e}_2^{(1)} \quad (8)$$

and likewise for the acceleration:

$$\mathbf{a} = \dot{\mathbf{v}} = \ell\ddot{\theta}\mathbf{e}_2^{(1)} - \ell\dot{\theta}^2\mathbf{e}_1^{(1)}. \quad (9)$$

The accelerations are indicated in Fig. 1 (b).

2.2 Equation of motion

In this subsection, we illustrate the projection-based derivation of the equation of motion. In particular, we highlight the mechanics of the moving frame method (MFM) and how the constraint force is eliminated from the equation of motion.

Simple derivation As indicated in the Free Body Diagram (FBD) in Fig. 1 (c), assume constraint force, $-F^c\mathbf{e}_1^{(1)}$, is perpendicular to the circular path. Then dynamic equilibrium in the $\mathbf{e}_2^{(1)}$ direction, or moments with respect to O, yields

$$m\ell\ddot{\theta} = -mg\sin(\theta). \tag{10}$$

MFM - Geometry-based derivation (d'Alembert's principle) Employing Newton's 2nd law in Cartesian coordinates we obtain:

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{\mathbf{x}}} = \underbrace{\begin{bmatrix} mg \\ 0 \end{bmatrix}}_{\mathbf{F}} + \mathbf{F}^c, \tag{11}$$

where $\mathbf{F}$ is the external or impressed force (gravity in this case), and $\mathbf{F}^c$ the constraint force that enforces the circular path, i.e. $x_1^2 + x_2^2 = \ell^2$, or writing x in polar coordinates with fixed radius ℓ,

$$\mathbf{x} = \begin{bmatrix} \ell\cos(\theta) \\ \ell\sin(\theta) \end{bmatrix}. \tag{12}$$

The velocity is

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} -\ell\sin(\theta) \\ \ell\cos(\theta) \end{bmatrix}}_{\mathbf{B}} \dot{\theta}. \tag{13}$$

Subsitution in Eqn. (11) yields

$$\mathbf{M}\left(\mathbf{B}\ddot{\theta} + \dot{\mathbf{B}}\dot{\theta}\right) = \mathbf{F} + \mathbf{F}^c \tag{14}$$

Again, assume that the constraint force $\mathbf{F}^c$ is perpendicular to the path, or, equivalently, is perpendicular to the velocity, see Fig. 1 (c). This implies that the constraint force does no work or power (d'Alembert's principle):

$$\dot{\mathbf{x}}^T\mathbf{F}^c = \dot{\theta}\mathbf{B}^T\mathbf{F}^c = 0 \;\forall\; \dot{\theta} \implies \mathbf{B}^T\mathbf{F}^c = 0 \tag{15}$$

Thus, taking the inner product of Eqn. (14) with $\mathbf{B}$, we obtain

$$\mathbf{B}^T\mathbf{M}\mathbf{B}\,\ddot{\theta} + \mathbf{B}^T\mathbf{M}\dot{\mathbf{B}}\,\dot{\theta} = \mathbf{B}^T\mathbf{F}, \tag{16}$$

which yields the EOM, see Eqn. (10).

Linear Algebra based derivation of constraint force We note that Eq. 11, $\mathbf{M}\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{F}^c$, is a matrix equation linear in $\ddot{\mathbf{x}}$. Using superposition we split up the solution in two parts:

$$\mathbf{M}\mathbf{a} = \mathbf{F}, \;\; \mathbf{M}\ddot{\mathbf{x}}^c = \mathbf{F}^c, \tag{17}$$

so that $\ddot{\mathbf{x}} = \mathbf{a} + \ddot{\mathbf{x}}^c$. Following [14] , the constraint $x_1^2 + x_2^2 = \ell^2$ is differentiated with respect to time to obtain the allowable velocity and again to obtain the allowable acceleration:

$$\underbrace{\begin{bmatrix} x_1 & x_2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = 0 \tag{18}$$

$$\mathbf{A}\ddot{\mathbf{x}} = \underbrace{-(\dot{x}_1^2 + \dot{x}_2^2)}_{\mathbf{b}} \tag{19}$$

We scale the equations as follows [7]:

$$\underbrace{\mathbf{M}^{1/2}\ddot{\mathbf{x}}^c}_{\ddot{\mathbf{x}}_s^c} = \underbrace{\mathbf{M}^{-1/2}\mathbf{F}^c}_{\mathbf{F}_s^c} \tag{20}$$

$$\underbrace{\mathbf{A}\mathbf{M}^{-1/2}}_{\mathbf{A}_s} \underbrace{\mathbf{M}^{1/2}\ddot{\mathbf{x}}^c}_{\ddot{\mathbf{x}}_s^c} = \mathbf{b} - \mathbf{A}\mathbf{a} \tag{21}$$

The minimum norm solution of Eqn. (21) is

$$\ddot{\mathbf{x}}_s^c = \mathbf{A}_s^T(\mathbf{A}_s\mathbf{A}_s^T)^{-1}(\mathbf{b} - \mathbf{A}\mathbf{a}), \tag{22}$$

resulting in[1]

$$\mathbf{F}^c = \mathbf{A}^T(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T)^{-1}(\mathbf{b} - \mathbf{A}\mathbf{a}). \tag{23}$$

In words, $\mathbf{F}^c$ is in the range of $\mathbf{A}^T$. Since the allowable velocity $\dot{\mathbf{x}}$ is in the nullspace of $\mathbf{A}$ (see Eqn. (18)), we have that the constraint force does no work:

$$\mathbf{A}\dot{\mathbf{x}} = 0 \iff \dot{\mathbf{x}}^T\mathbf{A}^T = 0 \implies \dot{\mathbf{x}}^T\mathbf{F}^c = 0. \tag{24}$$

Using Eqn. (13), the velocity constraint becomes

$$\mathbf{A}\underbrace{\mathbf{B}\dot{\theta}}_{\dot{\mathbf{x}}} = 0, \tag{25}$$

[1] In the following, the constraint force can also be written as $\mathbf{F}^c = \mathbf{A}^T(\mathbf{A}\mathbf{M}^{-1}\mathbf{A}^T)^{-1}\mathbf{A}\ddot{\mathbf{x}}^c = \mathbf{P}\ddot{\mathbf{x}}^c$, where the projector $\mathbf{P}$ can be used to eliminate the constraint force, see e.g. [9], Sec. 8.7.

and we see that (the colums of) $\mathbf{B}$ is (are) in the nullspace of $\mathbf{A}$. Thus,

$$\dot{\mathbf{x}}^T \mathbf{F}^c = 0 \iff \left(\mathbf{B}\dot{\theta}\right)^T \mathbf{F}^c = 0 \implies \mathbf{B}^T \mathbf{F}^c = 0, \tag{26}$$

which is the same as Eqn. (15) and, as above, leads to EOM (10). In summary, the constraint force given by Eqn. (22) satisfies d'Alembert's principle.

3 Rigid body motion

The equations of motion for a rigid body are often known as Euler's laws [12]. Let the center of mass position of a rigid body be given by $\mathbf{r}$ and its mass by m. Then, its linear momentum $\mathbf{P} = m\dot{\mathbf{r}}$ satisfies Euler's first law:

$$\dot{\mathbf{P}} = \mathbf{F}, \tag{27}$$

where $\mathbf{F}$ is the sum of all forces. Let the angular momentum of a rigid body with respect to the center of mass position be given by $\mathbf{L} = \mathbf{J}\omega$, where $\mathbf{J}$ is the mass moment of inertia matrix and ω is the angular velocity. Euler's second law is the balance of angular momentum

$$\dot{\mathbf{L}} = \tau, \tag{28}$$

where τ is the sum of all moments or torques with respect to the center of mass.

3.1 Euler's laws in inertial and body-fixed frame

Following the moving frame method [10], we explicitly write out Euler's laws in the inertial and body-fixed frame respectively [2].

[2]This choice of frames follows MFM papers [10, 11]. Depending on the appliation, another choice might be more convenient, however the derivations in the following are similar.

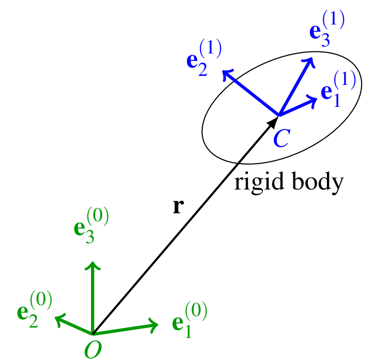

FIGURE 2. Frames of refrence: Inertial frame with fixed origin O (green), and body-fixed frame (blue) with origin C at center of mass.

The position of the center of mass $\mathbf{r}$ in the inertial frame, see Fig. 2, is

$$\mathbf{r} = x_1\mathbf{e}_1^{(0)} + x_2\mathbf{e}_2^{(0)} + x_3\mathbf{e}_3^{(0)} = \begin{bmatrix}\mathbf{e}_1^{(0)} & \mathbf{e}_2^{(0)} & \mathbf{e}_3^{(0)}\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{E}^{(0)}\mathbf{x}. \tag{29}$$

Likewise, the total force is also expressed in the inertial frame. Thus, Euler's first law, Eq. (30), in inertial frame $\mathbf{E}^{(0)}$ is given by

$$\mathbf{E}^{(0)} m\ddot{\mathbf{x}} = \mathbf{E}^{(0)}\mathbf{F}^{(0)},$$

or, in coordinate form

$$m\ddot{\mathbf{x}} = \mathbf{F}^{(0)}. \tag{30}$$

The angular momentum $\mathbf{L} = \mathbf{J}\omega$ is expressed in the body-fixed frame, see Fig. 2, as follows

$$\mathbf{L} = \mathbf{E}^{(1)}\mathbf{J}^{(1,1)}\omega^{(1)}, \tag{31}$$

where $\mathbf{J}^{(1,1)}$ is the mass moment of inertia matrix computed with respect to center of mass C and expressed in the body-fixed frame (See e.g. [8] for moments of inertia of some selected bodies), and $\omega^{(1)}$ are the components of the angular velocity vector, also expressed in the body-fixed frame. The time derivative of the angular momentum vector is given by

$$\dot{\mathbf{L}} = \dot{\mathbf{E}}^{(1)}\mathbf{J}^{(1,1)}\omega^{(1)} + \mathbf{E}^{(1)}\mathbf{J}^{(1,1)}\dot{\omega}^{(1)}. \tag{32}$$

We stipulate that the orientation of the rigid body with respect to the inertial frame changes over time and therefore we need to compute the time derivative of body-fixed basis vectors. This time derivative is expressed in the body-fixed frame $\mathbf{E}^{(1)}$,

$$\dot{\mathbf{E}}^{(1)} = \mathbf{E}^{(1)}\Omega^{(1,1)}, \tag{33}$$

where $\Omega^{(1,1)}$ is a skew-symmetric angular velocity matrix, which is uniquely related to the components of the angular velocity vector $\omega^{(1)}$ as follows

$$\Omega^{(1,1)} = \tilde{\omega}^{(1)} = \begin{bmatrix} 0 & -\omega_3^{(1)} & \omega_2^{(1)} \\ \omega_3^{(1)} & 0 & -\omega_1^{(1)} \\ -\omega_2^{(1)} & \omega_1^{(1)} & 0 \end{bmatrix} \leftrightarrow \omega^{(1)} = \begin{bmatrix} \omega_1^{(1)} \\ \omega_2^{(1)} \\ \omega_3^{(1)} \end{bmatrix}. \tag{34}$$

The derivation is analogous to the derivation of Eqn. (6) in Section 2, we refer to [10] for details. Thus, Euler's second law, Eqn. (35), in the body-fixed frame is

$$\mathbf{E}^{(1)}\left(\Omega^{(1,1)}\mathbf{J}^{(1,1)}\omega^{(1)} + \mathbf{J}^{(1,1)}\dot{\omega}^{(1)}\right) = \mathbf{E}^{(1)}\tau^{(1)},$$

or, in coordinate form

$$\Omega^{(1,1)}\mathbf{J}^{(1,1)}\omega^{(1)} + \mathbf{J}^{(1,1)}\dot{\omega}^{(1)} = \tau^{(1)}. \tag{35}$$

Equations (30), (35) are summarized using matrix vector notation as in [11]. Define the following vector and matrices

$$\mathbf{M} = \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{(1,1)} \end{bmatrix}, \dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{x}} \\ \omega^{(1)} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Omega^{(1,1)} \end{bmatrix}, \Gamma = \begin{bmatrix} \mathbf{F}^{(0)} \\ \tau^{(1)} \end{bmatrix},$$

where $\mathbf{I}$ is the 3×3 identity matrix. We note that, in general, $\Gamma = \Gamma(\mathbf{X},\dot{\mathbf{X}})$ and $\mathbf{D} = \mathbf{D}(\dot{\mathbf{X}})$. Euler's first and second laws in inertial and body-fixed coordinates respectively are now summarized by

$$\mathbf{M}\ddot{\mathbf{X}} = \mathbf{Q}(\mathbf{X},\dot{\mathbf{X}},t), \tag{36}$$

where,

$$\mathbf{Q}(\mathbf{X},\dot{\mathbf{X}},t) = \Gamma(\mathbf{X},\dot{\mathbf{X}}) - \mathbf{D}(\dot{\mathbf{X}})\mathbf{M}\dot{\mathbf{X}}. \tag{37}$$

3.2 Constrained multi-body motion

The equation of motion for unconstrained motion of one rigid body is given by Eqn. (36). For n bodies, we simply write down Eqn. (36) for each body. Numbering the bodies from $1,2,\ldots,n$ and assigning all associated variables with a corresponding subscript, the equations of motion are:

$$\mathbf{M}\ddot{\mathbf{X}} = \mathbf{Q}(\mathbf{X},\dot{\mathbf{X}},t), \tag{38}$$

where, $\mathbf{Q}(\mathbf{X},\dot{\mathbf{X}},t) = \Gamma - \mathbf{D}\mathbf{M}\dot{\mathbf{X}}$, and

$$\dot{\mathbf{X}} = \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \omega_1^{(1)} \\ \vdots \\ \dot{\mathbf{x}}_n \\ \omega_n^{(n)} \end{bmatrix}, \Gamma = \begin{bmatrix} \mathbf{F}_1^{(0)} \\ \tau_1^{(1)} \\ \vdots \\ \mathbf{F}_n^{(0)} \\ \tau_n^{(n)} \end{bmatrix}, \mathbf{M} = \begin{bmatrix} m_1\mathbf{I} & \mathbf{0} & \ldots & & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_1^{(1,1)} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & m_n\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \ldots & \ldots & \mathbf{0} & \mathbf{J}_n^{(n,n)} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots\ldots & & \mathbf{0} \\ \mathbf{0} & \Omega^{(1,1)} & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ldots & \ldots & \mathbf{0} & \Omega^{(n,n)} \end{bmatrix}. \tag{39}$$

Constraint equations describe how the bodies are connected and also include motion limitations, such as no-slip conditions. The constraints are written in velocity form, so that both holonomic (positional) constraints and nonholonomic (velocity) constraints are easily enforced. For n bodies with d positional constraints the number of degrees of freedom is $m = 6n - d$. Thus, all essential information about the system can be captured by a vector $\mathbf{q}$ with m independent (generalized) coordinates. Its time derivative is called the essential generalized velocity. The constraints relating the generalized velocities in $\dot{\mathbf{X}}$ to the essential generalized velocities in $\dot{\mathbf{q}}$ are expressed as follows [11][3]

$$\dot{\mathbf{X}} = \mathbf{B}(\mathbf{q})\,\dot{\mathbf{q}}. \tag{40}$$

For the pendulum example in Section 2 this relationship is given by Eqn. (13). Using a similar approach as in Section 2, we can quickly write down the equations of motion for a constrained multi-body system. As an alternative, we also provide a projection-based approach, which directly uses Eqn. (40).

For a quick 'derivation' of the constrained equations of motion we use the following analogy: equation of motion (38) (augmented with a constraint force $\mathbf{Q}^c$) and constraint equation (40)

[3]For simplicity, we have assumed that the constraints are time-independent.

are similar to Eqns. (11) and (13). Thus, using Eqn. (16), we obtain the constrained equation of motion:

$$\mathbf{B}^T\mathbf{M}\mathbf{B}\,\ddot{\mathbf{q}}+\mathbf{B}^T\mathbf{M}\dot{\mathbf{B}}\,\dot{\mathbf{q}}=\mathbf{B}^T\mathbf{Q}, \tag{41}$$

or,

$$\mathbf{B}^T\mathbf{M}\mathbf{B}\,\ddot{\mathbf{q}}+\mathbf{B}^T(\mathbf{M}\dot{\mathbf{B}}+\mathbf{D}\mathbf{M}\mathbf{B})\dot{\mathbf{q}}=\mathbf{B}^T\Gamma. \tag{42}$$

3.2.1 Projection-based derivation

The correct motion of a multi-body system under constraints is obtained by augmenting the equations of unconstrained rigid body motion, Eqn. (38), with a constraint force $\mathbf{Q}^c$, such that the constraints (40) are satisfied. The (augmented) equation of motion and constraint equation can be written together as

$$\mathbf{M}\ddot{\mathbf{X}}=\mathbf{Q}(\mathbf{X},\dot{\mathbf{X}},t)+\mathbf{Q}^c(\mathbf{X},\dot{\mathbf{X}},t), \tag{43a}$$
$$\dot{\mathbf{X}}=\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}. \tag{43b}$$

Following Gauss' principle of least constraint [4, 15, 14], we scale the velocity and forces as follows

$$\dot{\mathbf{X}}_s=\mathbf{M}^{1/2}\dot{\mathbf{X}},\;\;\mathbf{Q}_s=\mathbf{M}^{-1/2}\mathbf{Q},\;\;\mathbf{Q}_s^c=\mathbf{M}^{-1/2}\mathbf{Q}^c. \tag{44}$$

where $\mathbf{M}^{1/2}$ follows from the square root factorization of $\mathbf{M}$ ($\mathbf{M}=\mathbf{M}^{1/2}\mathbf{M}^{1/2}$). Thus, we obtain

$$\ddot{\mathbf{X}}_s=\mathbf{Q}_s+\mathbf{Q}_s^c,\text{ and} \tag{45a}$$
$$\dot{\mathbf{X}}_s=\mathbf{B}_s\dot{\mathbf{q}}, \tag{45b}$$

where $\mathbf{B}_s=\mathbf{M}^{1/2}\mathbf{B}$. Differentiating the constraints (40) twice we obtain[4]

$$\ddot{\mathbf{X}}_s=\mathbf{Q}_s+\mathbf{Q}_s^c,\text{ and} \tag{46a}$$
$$\ddot{\mathbf{X}}_s=\dot{\mathbf{B}}_s\dot{\mathbf{q}}+\mathbf{B}_s\ddot{\mathbf{q}}. \tag{46b}$$

In order to determine the constraint force $\mathbf{Q}_s^c$, it is convenient to split it up into an nonideal component, $\mathbf{Q}_s^{c,ni}$, that is in the *column space* of $\mathbf{B}_s$, and a ideal component, $\mathbf{Q}_s^{c,i}$, in the *null space* of $\mathbf{B}_s^T$, i.e.

$$\mathbf{Q}_s^c=\mathbf{Q}_s^{c,ni}+\mathbf{Q}_s^{c,i},\text{ where }\mathbf{Q}_s^{c,ni}=\mathbf{B}_s\mathbf{r},\;\mathbf{B}_s^T\mathbf{Q}_s^{c,i}=\mathbf{0}. \tag{47}$$

[4]Note that integration of the differentiated constraint equation with the correct initial condition yields Eqn. (40).

The ideal component of the constraint force, $\mathbf{Q}_s^{c,i}$, satisfies d'Alembert's principle, since the power P^i due to the allowable velocities $\dot{\mathbf{X}}=\mathbf{B}\dot{\mathbf{q}}$ is zero:

$$P^i=\dot{\mathbf{X}}^T\mathbf{Q}^{c,i}=\dot{\mathbf{X}}_s^T\mathbf{Q}_s^{c,i}=\dot{\mathbf{q}}^T\mathbf{B}_s^T\mathbf{Q}_s^{c,i}=0. \tag{48}$$

Defining $\mathbf{P}_{\mathbf{B}_s}=\mathbf{B}_s(\mathbf{B}_s^T\mathbf{B}_s)^{-1}\mathbf{B}_s^T$ as the projector onto the *column space* of $\mathbf{B}_s$ [5], we also have

$$\mathbf{Q}_s^{c,ni}=\mathbf{P}_{\mathbf{B}_s}\mathbf{Q}_s^c, \tag{49a}$$
$$\mathbf{Q}_s^{c,i}=(\mathbf{I}-\mathbf{P}_{\mathbf{B}_s})\,\mathbf{Q}_s^c. \tag{49b}$$

Next, using Eqn. (46) and projecting onto the the *column space* of $\mathbf{B}_s$ and the *null space* of $\mathbf{B}_s^T$ respectively, we obtain the requirements on the nonideal and ideal components of the constraint force

$$\mathbf{P}_{\mathbf{B}_s}\ddot{\mathbf{X}}_s=\mathbf{P}_{\mathbf{B}_s}\dot{\mathbf{B}}_s\dot{\mathbf{q}}+\mathbf{B}_s\ddot{\mathbf{q}}=\mathbf{P}_{\mathbf{B}_s}\mathbf{Q}_s+\mathbf{Q}_s^{c,ni} \tag{50a}$$
$$(\mathbf{I}-\mathbf{P}_{\mathbf{B}_s})\ddot{\mathbf{X}}_s=(\mathbf{I}-\mathbf{P}_{\mathbf{B}_s})\dot{\mathbf{B}}_s\dot{\mathbf{q}}=(\mathbf{I}-\mathbf{P}_{\mathbf{B}_s})\mathbf{Q}_s+\mathbf{Q}_s^{c,i}. \tag{50b}$$

When the nonideal component of the constraint force is zero ($\mathbf{Q}_s^{c,ni}=\mathbf{0}$), which is true when when d'Alembert's principle holds, the constrained motion is directly obtained from integration of Eqn. (50a). Subsequently, the (ideal) constraint force can be calculated from Eqn. (50b). A nonideal constraint is present when the power or work due to the constraint force is nonzero (d'Alembert's principle does not hold). In this case, the work needs to be explicitly prescribed by the modeler of the system, such that the nonideal constraint force can be derived as in [14]. Then, the motion is obtained as in the ideal case: integrate Eqn. (50a) with the $\mathbf{Q}_s^{c,ni}$ derived from the prescribed work, and, optionally, calculate the ideal constraint force from Eqn. (50b).

When d'Alembert's principle is satisfied, we can further simplify the equations of motion (50a) by taking the product with ${\mathbf{B}_s}^T$, which yields

$$\mathbf{B}^T\mathbf{M}(\dot{\mathbf{B}}\dot{\mathbf{q}}+\mathbf{B}\ddot{\mathbf{q}})=\mathbf{B}^T\mathbf{Q}, \tag{51}$$

and is the same as equation of motion (41).

3.2.2 Equation of motion for constrained system

We provide a brief summary for obtaining the constrained equation of motion of a mechanical system:

1. Decide on the generalized coordinates / velocities for the problem.

[5]Thus, $\mathbf{I}-\mathbf{P}_{\mathbf{B}_s}$ is the projector onto the *null space* of $\mathbf{B}_s^T$.

2. Formulate the constraints in essential generalized velocity form.
3. Calculate the projected equation of motion.
4. Integrate the projected equation of motion and obtain the generalized velocities.
5. (Optional) Compute the ideal constraint force.

4 Results

Using the methods of Sec. 3, we illustrate how the equations of motion can be obtained for a modified Chaplygin sleigh and a rotary inverted pendulum.

4.1 Modified Chaplygin sleigh

This section illustrates the projection-based derivation of the equation of motion of a modified Chaplygin sleigh as studied in [5]. The system consists of a vehicle with a no side-slip condition and an internal rotor ('a pendulum'), which is connected by a torsional spring, see Fig. 3. Interestingly, an initial excitation of the rotor can move the vehicle along a straight line or circular path [5]. In the referenced study, the equation of motion was derived using a Lagrangian approach. We derive the equation of motion using a projection based approach as described in Section Section 3.2.1.

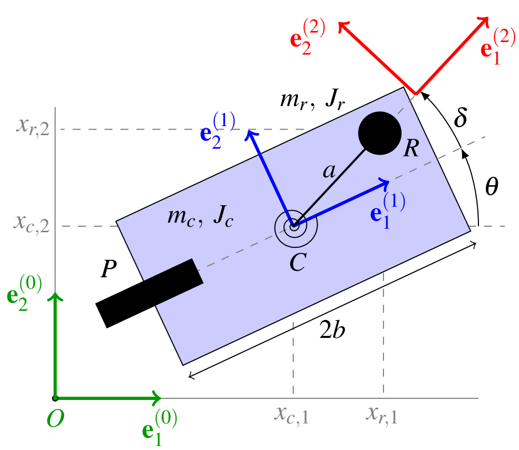

FIGURE 3. Modified Chaplygin sleigh setup.

The mass of the sleigh is denoted by m_c and the moment of inertia about its center of mass C by J_c. An internal rotor of mass m_r and moment of inertia J_r about its own center of mass is hinged to the sleigh at C, such that the rotor can rotate freely without any friction. The rotor and sleigh are connected by a torsional spring that resists a change in the relative motion of the two bodies. From the given potential energy of the spring in [5], $V(\delta) = k_1\delta^2 + k_2\delta^4$, follows the spring torque on the rotor

$$\tau_s = -\frac{\partial V}{\partial \delta} = -2k_1\delta - 4k_2\delta^3. \tag{52}$$

Next, we derive the equation of motion following the steps in Section 3.2.2.

First, we write down the generalized velocities of the system. The system consists of two rigid bodies in the plane, each of which has three degrees of freedom (two translational and one orientational). Referring to Fig. 3, we use the following generalized coordinates,

$$\mathbf{X} = \left[x_{c,1}\ x_{c,2}\ \theta\ x_{r,1}\ x_{r,2}\ (\theta+\delta)\right]^T, \tag{53}$$

with corresponding *generalized velocities*

$$\dot{\mathbf{X}} = \left[\dot{x}_{c,1}\ \dot{x}_{c,2}\ \dot{\theta}\ \dot{x}_{r,1}\ \dot{x}_{r,2}\ (\dot{\theta}+\dot{\delta})\right]^T. \tag{54}$$

Second, we note all constraints and choose the essential generalized velocities. The rotor is attached to the sleigh by a revolute joint, which reduces the rotor degrees of freedom to one (δ). The associated constraints are positional or holonomic. The rear wheel or runner at P is not allowed to slip the transverse direction. Thus, the velocity component of P in the $\mathbf{e}_2^{(1)}$ direction is zero, which is a velocity or nonholonomic constraint. Alternatively, P only has a velocity component (u) in the longitudinal $\mathbf{e}_1^{(1)}$ direction. Based on these considerations, we show below that the *essential generalized velocity* can be defined as

$$\dot{\mathbf{q}} = \left[u\ \dot{\theta}\ \dot{\delta}\right]^T. \tag{55}$$

We now proceed to write the *constraints* in the form $\dot{\mathbf{X}} = \mathbf{B}\dot{\mathbf{q}}$, see Eqn. (40). We start with holonomic constraint and show that the velocities of the sleigh, $\dot{x}_{c,1}$ and $\dot{x}_{c,2}$, can be written in terms of u and $\dot{\theta}$ only. Noting again that P can only move in the longitudinal direction, the velocity of P in moving frame (1) is given by

$$\mathbf{v}_P = \mathbf{e}_1^{(1)}u = \left[\mathbf{e}_1^{(1)}\mathbf{e}_2^{(1)}\right]\begin{bmatrix}u\\0\end{bmatrix} = \mathbf{E}^{(1)}\begin{bmatrix}u\\0\end{bmatrix}. \tag{56}$$

The velocity of P can be expressed as follows

$$\mathbf{v}_P = \mathbf{v}_C + \mathbf{v}_{P/C}, \tag{57}$$

where $\mathbf{v}_C$ is the velocity of C and $\mathbf{v}_{P/C}$ is the velocity of P with respect to C. The latter is determined as follows. The relative position of P with respect to C is

$$\mathbf{r}_{P/C} = -b\mathbf{e}_1^{(1)} = \mathbf{E}^{(1)} \begin{bmatrix} -b \\ 0 \end{bmatrix}, \tag{58}$$

and differentiation yields the relative velocity

$$\mathbf{v}_{P/C} = \dot{\mathbf{E}}^{(1)} \begin{bmatrix} -b \\ 0 \end{bmatrix} = \mathbf{E}^{(1)}\Omega^{(1,1)} \begin{bmatrix} -b \\ 0 \end{bmatrix} = \mathbf{E}^{(1)} \begin{bmatrix} 0 \\ b \end{bmatrix} \dot{\theta}, \tag{59}$$

where $\Omega^{(1,1)}$ is given by Eqn. (5). Solving for $\mathbf{v}_C$, using Eqns. (57), (56), (59) and (2), in the inertial frame (0), yields

$$\begin{aligned} \mathbf{v}_C = \mathbf{E}^{(0)} \begin{bmatrix} \dot{x}_{c,1} \\ \dot{x}_{c,2} \end{bmatrix} &= \mathbf{E}^{(0)}\mathbf{R}^{(0,1)} \left(\begin{bmatrix} u \\ 0 \end{bmatrix} + \Omega^{(1,1)} \begin{bmatrix} -b \\ 0 \end{bmatrix} \right) \\ &= \mathbf{E}^{(0)} \begin{bmatrix} \cos\theta & -b\sin\theta \\ \sin\theta & b\cos\theta \end{bmatrix} \begin{bmatrix} u \\ \dot{\theta} \end{bmatrix}. \end{aligned} \tag{60}$$

Next, we derive the holonomic constraint equations for the rotor in velocity form. The position of the rotor (R) can be written as follows

$$\mathbf{r}_R = \mathbf{r}_C + \mathbf{r}_{R/C}, \tag{61}$$

where $\mathbf{r}_C$ is the position of C and $\mathbf{r}_{R/C}$ is the position of R with respect to C. Expressing this relationship in the inertial frame (0), we obtain

$$\mathbf{r}_R = \mathbf{E}^{(0)} \begin{bmatrix} x_{r,1} \\ x_{r,2} \end{bmatrix} = \mathbf{E}^{(0)} \left(\begin{bmatrix} x_{c,1} \\ x_{c,2} \end{bmatrix} + \begin{bmatrix} a\cos(\theta+\delta) \\ a\sin(\theta+\delta) \end{bmatrix} \right). \tag{62}$$

Differentiating with respect to time, yields the velocity

$$\mathbf{v}_R = \mathbf{E}^{(0)} \begin{bmatrix} \dot{x}_{r,1} \\ \dot{x}_{r,2} \end{bmatrix} = \mathbf{E}^{(0)} \left(\begin{bmatrix} \dot{x}_{c,1} \\ \dot{x}_{c,2} \end{bmatrix} + \begin{bmatrix} -a\sin(\theta+\delta) \\ a\cos(\theta+\delta) \end{bmatrix} (\dot{\theta}+\dot{\delta}) \right). \tag{63}$$

Substituting Eqn. (60), we obtain

$$\mathbf{v}_R = \mathbf{E}^{(0)} \left(\begin{bmatrix} \cos\theta & -b\sin\theta \\ \sin\theta & b\cos\theta \end{bmatrix} \begin{bmatrix} u \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} -a\sin(\theta+\delta) \\ a\cos(\theta+\delta) \end{bmatrix} (\dot{\theta}+\dot{\delta}) \right). \tag{64}$$

We observe from Eqns. (60) and (64) that indeed all generalized velocities in $\dot{\mathbf{X}}$, see Eqn. (54), can be in expressed in u, $\dot{\theta}$ and $\dot{\delta}$. By comparison of Eqns. (40), (54), (55), (60), and (64), it follows that the constraint matrix is given by

$$\mathbf{B} = \begin{bmatrix} \cos\theta & -b\sin\theta & 0 \\ \sin\theta & b\cos\theta & 0 \\ 0 & 1 & 0 \\ \cos\theta & -a\sin(\delta+\theta) - b\sin\theta & -a\sin(\delta+\theta) \\ \sin\theta & a\cos(\delta+\theta) + b\cos\theta & a\cos(\delta+\theta) \\ 0 & 1 & 1 \end{bmatrix}. \tag{65}$$

Third, we calculate the *projected equation of motion*. Assuming ideal constraint forces, see Section 3.2.2, Eqn. (51), the equation of motion is

$$\mathbf{B}^T\mathbf{M}(\dot{\mathbf{B}}\dot{\mathbf{q}} + \mathbf{B}\ddot{\mathbf{q}}) = \mathbf{B}^T\mathbf{Q},$$

with $\mathbf{Q} = -\mathbf{D}\mathbf{M}\dot{\mathbf{X}} + \Gamma$, where the mass matrix is given by the diagonal matrix

$$\mathbf{M} = \operatorname{diag}\left[m_c\ m_c\ J_c\ m_r\ m_r\ J_r\right], \tag{66}$$

the product $\mathbf{D}\mathbf{M}\dot{\mathbf{X}}$ is zero for planar problems (see e.g. [11]), and the force/torque vector is given by

$$\Gamma = \left[0\ 0\ -\tau_s\ 0\ 0\ \tau_s\right]^T, \tag{67}$$

where τ_s is the torsional spring torque on the rotor as defined by Eqn. (52) and, according to Newton's third law, $-\tau_s$ is the spring torque on the sleigh. From the projected mass matrix

$$\mathbf{M}^* = \mathbf{B}^T\mathbf{M}\mathbf{B} = \begin{bmatrix} m_c + m_r & -am_r\sin\delta & -am_r\sin\delta \\ -am_r\sin\delta & J_c + J_r + a^2m_r + 2abm_r\cos\delta + b^2m_c + b^2m_r & J_r + a^2m_r + abm_r\cos\delta \\ -am_r\sin\delta & J_r + a^2m_r + abm_r\cos\delta & J_r + a^2m_r \end{bmatrix} \tag{68}$$

and the expression $\mathbf{B}^T\left(-\mathbf{M}\dot{\mathbf{B}}\dot{\mathbf{q}} + \Gamma\right)$ we obtain the equation of motion:

$$\mathbf{M}^*\ddot{\mathbf{q}} = \begin{bmatrix} am_r\cos\delta\dot{\delta}^2 + 2am_r\cos\delta\dot{\delta}\dot{\theta} + am_r\cos\delta\dot{\theta}^2 + bm_c\dot{\theta}^2 + bm_r\dot{\theta}^2 \\ abm_r\sin\delta\dot{\delta}^2 + 2abm_r\sin\delta\dot{\delta}\dot{\theta} - am_ru\cos\delta\dot{\theta} - bm_cu\dot{\theta} - bm_ru\dot{\theta} \\ -abm_r\sin\delta\dot{\theta}^2 - am_ru\cos\delta\dot{\theta} - 2k_1\delta - 4k_2\delta^3 \end{bmatrix}, \tag{69}$$

which is the same as Eqn. (2.12) in [5].

4.2 Rotary pendulum

The stabilization of an inverted pendulum is a benchmark problem in control literature, and commonly studied in control system textbooks (see e.g. [6, 2]) and labs. In Fig. 4, a rotary or Furuta inverted pendulum, made by Quanser [13], is shown. The equations of motion of this system are typically obtained using a Lagrangian approach, see e.g. [3, 1]. Here, we provide an alternative derivation using the projection based approach as described in Section 3.2.1.

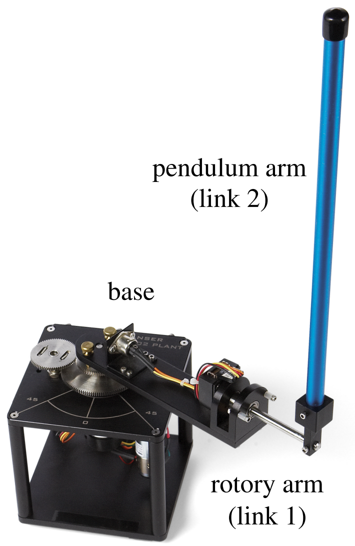

FIGURE 4. Quanser Rotary Inverted Pendulum [13].

Referring to Fig. 4, we see that that the rotary arm of the pendulum (link 1) is attached to a base by a revolute joint which is driven by a DC motor. The pendulum arm (link 2) is connected by another revolute joint to the end of the rotary arm. A corresponding sketch is shown in Fig. 5. We introduce three reference frames: fixed frame (0) with origin O at the base, moving frame (1) aligned with the principal directions of the first link with origin at C_1, and moving frame (2) aligned with the principal direction of the second link with origin at C_2. Frame (1) and frame (0) are related by a rotation matrix

$$\mathbf{E}^{(1)} = \mathbf{E}^{(0)}\mathbf{R}^{(0,1)}, \text{ where } \mathbf{R}^{(0,1)} = \mathbf{R}_3(\theta) \tag{70}$$

is the rotation matrix corresponding to a rotation of θ around the 3-axis ($\mathbf{e}_3^{(0)}$ direction). Similarly,

$$\mathbf{E}^{(2)} = \mathbf{E}^{(1)}\mathbf{R}^{(1,2)}, \text{ where } \mathbf{R}^{(1,2)} = \mathbf{R}_1(\alpha) \tag{71}$$

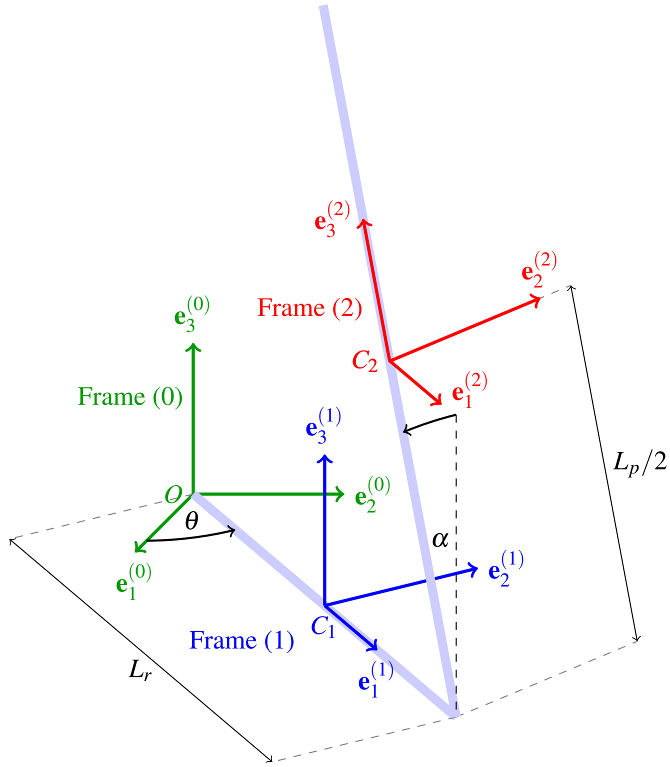

FIGURE 5. Rotary inverted pendulum convention [1].

is the rotation matrix corresponding to a rotation of α around the 1-axis ($\mathbf{e}_1^{(1)}$ direction). The rotary arm (link 1) has mass m_r and mass moment of inertia matrix with respect to moving frame (1) given by the diagonal matrix[6]

$$\mathbf{J}^{(1,1)} = \text{diag}\left[J_{r,1}\ J_{r,2}\ J_{r,3}\right]. \tag{72}$$

The pendulum link (link 2) is connected to the end of the rotary arm. It has mass m_p and mass moment of inertia matrix with respect to moving frame (2) given by the diagonal matrix

$$\mathbf{J}^{(2,2)} = \text{diag}\left[J_{p,1}\ J_{p,2}\ J_{p,3}\right]. \tag{73}$$

Next, we derive the equation of motion following the steps in Section 3.2.2.

First, we write down the generalized velocities of the system. The system consists of two connected rigid bodies, each of which (independently) has six degrees of freedom (three translational and three orientational). Referring to Eq. (38) and (39),

[6]Frame (1) is aligned with the principal axes of the inertia matrix.

we use the following *generalized velocities*

$$\dot{\mathbf{X}} = \left[\dot{\mathbf{x}}_1 \ \omega_1^{(1)} \ \dot{\mathbf{x}}_2 \ \omega_2^{(2)}\right]^T, \tag{74}$$

where $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ are the translational velocities of the center of masses of link 1 and 2, expressed in the fixed frame (0); and $\omega_1^{(1)}$ and $\omega_2^{(2)}$ are the angular velocities of frames (1) and (2) with respect to frame (0), expressed in frame (1) and frame (2) respectively. We note that $\dot{\mathbf{X}}$ is a 12-vector.

Second, we consider all constraints and choose the essential generalized velocities. Link 1 is attached to a fixed base by a revolute joint, and link 2 is connected to link 1 by another revolute joint, resulting in two degrees of freedom, θ and α, see Fig. 5.Therefore, the *essential generalized velocity* is a 2-vector given by

$$\dot{\mathbf{q}} = \left[\dot{\theta} \ \dot{\alpha}\right]^T. \tag{75}$$

We now relate the generalized velocities to the essential generalized velocities, i.e. we write the *constraints* in the form $\dot{\mathbf{X}} = \mathbf{B}\dot{\mathbf{q}}$ (see Eqn. (40)), where $\mathbf{B}$ is a 12×2 matrix. We start with the angular velocities of both links. Using Eq. (70), we find the rate of change of the (1) frame expressed in frame (1) is

$$\dot{\mathbf{E}}^{(1)} = \mathbf{E}^{(0)}\dot{\mathbf{R}}^{(0,1)} = \mathbf{E}^{(1)}\mathbf{R}^{(1,0)}\dot{\mathbf{R}}^{(0,1)} = \mathbf{E}^{(1)}\Omega_1^{(1,1)}, \tag{76}$$

where the absolute angular velocity matrix of frame (1) is given by

$$\Omega_1^{(1,1)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}. \tag{77}$$

Likewise, from Eq. (71), we obtain

$$\begin{aligned} \dot{\mathbf{E}}^{(2)} &= \dot{\mathbf{E}}^{(1)}\mathbf{R}^{(1,2)} + \mathbf{E}^{(2)}\Omega^{(2,2)} \\ &= \mathbf{E}^{(2)}\left(\mathbf{R}^{(2,1)}\Omega_1^{(1,1)}\mathbf{R}^{(1,2)} + \Omega_{1,2}^{(2,2)}\right) \\ &= \mathbf{E}^{(2)}\Omega_2^{(2,2)}, \end{aligned} \tag{78}$$

where $\mathbf{R}^{(2,1)} = \left(\mathbf{R}^{(1,2)}\right)^T$, and where the relative angular velocity matrix of frame (2) with respect to frame (1) is given by

$$\Omega_{1,2}^{(2,2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \dot{\alpha}. \tag{79}$$

Using the fact that angular velocity matrices are uniquely related to the components of the angular velocity vector, see Eq. (34), we can write

$$\Omega_1^{(1,1)} \leftrightarrow \omega^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix}. \tag{80}$$

Similarly, we obtain

$$\Omega_2^{(2,2)} \leftrightarrow \omega^{(2)} = \mathbf{R}^{(2,1)}\omega^{(1)} + \begin{bmatrix} \dot{\alpha} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin(\alpha) & 0 \\ \cos(\alpha) & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix}. \tag{81}$$

Next, we relate the translational velocities to the essential generalized velocity vector. Referring to Fig. 5, the position of the center of mass C_1 of link 1 is

$$\mathbf{r}_1 = 0.5L_r\mathbf{e}_1^{(1)} = \mathbf{E}^{(1)} \begin{bmatrix} 0.5L_r \\ 0 \\ 0 \end{bmatrix}, \tag{82}$$

and, differentiation, followed by substitution of Eqs. (76) (70), yields the velocity

$$\begin{aligned} \mathbf{v}_1 = \dot{\mathbf{E}}^{(1)} \begin{bmatrix} 0.5L_r \\ 0 \\ 0 \end{bmatrix} &= \mathbf{E}^{(1)}\Omega^{(1,1)} \begin{bmatrix} 0.5L_r \\ 0 \\ 0 \end{bmatrix} \\ &= \mathbf{E}^{(0)}\mathbf{R}^{(0,1)}\Omega^{(1,1)} \begin{bmatrix} 0.5L_r \\ 0 \\ 0 \end{bmatrix} = \mathbf{E}^{(0)}\dot{\mathbf{x}}_1. \end{aligned} \tag{83}$$

From this expression, the velocity components of C_1 in fixed frame (0) follow

$$\dot{\mathbf{x}}_1 = \begin{bmatrix} -0.5L_r\sin(\theta) & 0 \\ 0.5L_r\cos(\theta) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix}. \tag{84}$$

For link 2, a similar procedure is followed. The position of the center of mass C_1 of link 2 is

$$\mathbf{r}_2 = L_r\mathbf{e}_1^{(1)} + 0.5L_p\mathbf{e}_3^{(2)} = \mathbf{E}^{(1)} \begin{bmatrix} L_r \\ 0 \\ 0 \end{bmatrix} + \mathbf{E}^{(2)} \begin{bmatrix} 0 \\ 0 \\ 0.5L_p \end{bmatrix}. \tag{85}$$

Differentiation, followed by substitution of Eqs. (76), (78) (70), (71), yields

$$\mathbf{v}_2 = \mathbf{E}^{(0)} \begin{bmatrix} -L_r\sin(\theta)+0.5L_p\cos(\theta)\sin(\alpha) & 0.5L_p\sin(\theta)\cos(\alpha) \\ L_r\cos(\theta)+0.5L_p\sin(\theta)\sin(\alpha) & -0.5L_p\cos(\theta)\cos(\alpha) \\ 0 & -0.5L_p\sin(\alpha) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \end{bmatrix} = \mathbf{E}^{(0)}\dot{\mathbf{x}}_2. \quad (86)$$

Using Eqs. (74), (75), (80), (81), (84), (86), it is straightforward to set up the constraint matrix $\mathbf{B}$ in $\dot{\mathbf{X}} = \mathbf{B}\mathbf{q}$.

Third, we calculate the *projected equation of motion*. Assuming ideal constraint forces, see Section 3.2.2, Eqn. (51), the equation of motion is

$$\mathbf{B}^T\mathbf{M}(\dot{\mathbf{B}}\dot{\mathbf{q}}+\mathbf{B}\ddot{\mathbf{q}}) = \mathbf{B}^T\mathbf{Q},$$

with $\mathbf{Q} = -\mathbf{D}\mathbf{M}\dot{\mathbf{X}}+\Gamma$, where $\mathbf{D}$ is the following block diagonal matrix

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_3 & & & \\ & \Omega_1^{(1,1)} & & \\ & & \mathbf{0}_3 & \\ & & & \Omega_2^{(2,2)} \end{bmatrix}, \quad (87)$$

with $\mathbf{0}_3$ the 3×3 zero matrix, the mass matrix is given by the block diagonal matrix

$$\mathbf{M} = \begin{bmatrix} m_r\mathbf{I}_3 & & & \\ & \mathbf{J}^{(1,1)} & & \\ & & m_p\mathbf{I}_3 & \\ & & & \mathbf{J}^{(2,2)} \end{bmatrix}, \quad (88)$$

with $\mathbf{I}_3$ the 3×3 identity matrix, and the force/torque vector is given by

$$\Gamma = \begin{bmatrix} 0 & 0 & -m_rg & B_p\dot{\alpha} & 0 & \tau-B_r\dot{\theta} & \dots \\ 0 & 0 & -m_pg & -B_p\dot{\alpha} & 0 & 0 \end{bmatrix}^T, \quad (89)$$

with $-m_rg$ and $-m_pg$ the graviational force components of both links (in $\mathbf{e}_3^{(0)}$ direction), $-B_r\dot{\theta}$ and $-B_p\dot{\alpha}$ viscous damping torque components of the revolute joints (in $\mathbf{e}_3^{(1)}$ and $\mathbf{e}_1^{(2)}$ direction, respectively), and τ the applied torque by the DC motor (in the $\mathbf{e}_3^{(1)}$ direction). Performing all matrix multiplications to obtain the equation of motion, Eqn. (51), yields

$$\begin{aligned} &\frac{L_pL_rm_p\sin(\alpha)}{2}\dot{\alpha}^2 - \frac{L_pL_rm_p\cos(\alpha)}{2}\ddot{\alpha} \\ &+\left(J_{p,2}-J_{p,3}+\frac{L_p^2m_p}{4}\right)\sin(2\alpha)\,\dot{\alpha}\dot{\theta} \\ &+\left(J_{p,2}\sin^2(\alpha)+J_{p,3}\cos^2(\alpha)+J_{r,3}+\frac{L_p^2m_p\sin^2(\alpha)}{4}\right. \\ &\left.\quad +L_r^2m_p+\frac{L_r^2m_r}{4}\right)\ddot{\theta} = -B_r\dot{\theta}+\tau \qquad (90a) \\ &-\frac{L_p^2m_p\sin(2\alpha)}{8}\dot{\theta}^2 - \frac{L_pL_rm_p\cos(\alpha)}{2}\ddot{\theta} \\ &+\left(J_{p,1}+\frac{L_p^2m_p}{4}\right)\ddot{\alpha} = -B_p\dot{\alpha}+\frac{L_pm_pg}{2}\sin(\alpha), \qquad (90b) \end{aligned}$$

which are the same as Eqs. (2.2) and (2.3) in [1] given the simplications that $J_{p,2} = J_{p,3} = 0$.

5 Conclusions

The projection-based derivation of the equations of motion for multi-body systems only requires knowledge of projection concepts in linear algebra. It avoids variational principles and therefore is more accessible to undergraduate students. The derivation also clearly shows that (ideal) constraint forces can be eliminated by projection onto the subspace spanned by the constraint matrix. The constraint matrix is the key to deriving the equations of motion: Given the generic form of the equation of motion (Euler's laws), see Eqs. (38), (42), applied forces and moments, the equation of motion for a particular multi-body system is fully determined by the constraint matrix (Eq. (40)). We illustrate the derivation of the constraint matrix, followed by projection of Euler's laws, to obtain the equation of motion for two examples: (1) a modified Chaplygin sleigh, and (2) a rotary inverted pendulum. The derivation of the constraint matrix can be easily systemized for automated derivation of equations of motion. This will be considered in a future publication.

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