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Lorenz Like Attractors in Models like Game Theory

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Submitted in part fulfilment of the requirements for the degree of
Master of Science (M.Sc.) in Applied Mathematics of
Imperial College, September 2021

Abstract

Several models of four species replicator dynamics are studied. We consider the case where a model is symmetric with respect to a cyclic permutation of species. In a sense, this is a 4-dimensional generalisation of the "rock-paper-scissors" game. It is shown that bifurcations of a symmetric inner equilibrium can lead to the birth of a Lorenz attractor in such models. The result is based on the study of bifurcations of triple-zero eigenvalues. Such bifurcations are known to lead to Lorenz attractor under certain symmetry conditions [**22**]. It is crucial for the system be non-degenerate. We check which of the model under consideration satisfies the non-degeneracy conditions and show that the asymptotic normal form for the system near the bifurcation we consider happens, in the non-degenerate case, to be the Shimizu-Morioka system, which has the Lorenz Attractor.

Keywords: Asymptotic Normal Form, Replicator Dynamics, Generalized Replicator Dynamics, Lorenz Attractors, Lotka-Volterra like system, Shimizu-Morioka.

Acknowledgements

I would like to express my sincere gratitude to:

- My supervisor, **Dr. Dmitri Turaev**, professor of Bifurcation Theory at Imperial College London, for his guidance and support for giving me the freedom to do my own research while giving me advice for the technical part.

- My senior, and recent doctoral graduate from Iowa State University, Dr. Manas Bhatnagar for valuable suggestions and inputs.

- My parents for simply being patient with me, encouraging me and loving me.

Declaration

I confirm that this submission is my own work. In it, I give references and citations whenever I refer to or use the published or unpublished work of other people.

Dedication

Dedicated to ***Dr. Steven Strogatz***, who is the reason I am studying mathematics today. Dr. Strogatz has inspired me to look mathematics in a completely amusing way, that is just going to compound over time.

”One of the pleasures of looking at the world through mathematical eyes is that you can see certain patterns that would otherwise be hidden”

Steven H. Strogatz

Contents

Chapter 1

Introduction

Study of dynamics began in mid-1600s, when Newton invented differential equations, discovered his laws of motion and universal gravitation, but it was not until the 1800s, that the subject saw a breakthrough with Poincaré's work. Unlike before, Poincaré introduced a new point of view that emphasized more on the qualitative questions rather than the quantitative questions [**24**]. He emphasized that in order to study the evolution of a physical system over time (like planetary motion), one has to construct a model which is based on necessary and sufficient parameters that characterize the system [**14**]. This powerful geometric approach soon flourished the subject of dynamics. Poincaré's was also the first person to notice chaos, but it did not gain much attention those days, and remained in the background until the invention of high speed computers in 1950s. This groundbreaking discovery of high speed computers allowed one to experiment with equations in a way that enabled one to develop some intuition about the non-linear systems. Such experiments led to Lorenz's discovery in 1963 of chaotic motion on a strange attractor. By 1970s, chaos had stole the spotlight, and by 1980s, dynamics had become prominent and since then, many important discoveries have been made. One such discovery that caught my fascination was the chaotic behaviour of strange attractors. Therefore, I decided to work on this project and explore the possibility and behaviour of such attractors in various non-linear models.

Primary goal of this project is to study several four species replicator dynamics and see whether they exhibit Lorenz attractor or not. We will consider the case of a model which is symmetric with respect to a cyclic permutation of species. It is shown that the flow Lorenz attractors can be born as result of local bifurcations of an equilibrium with three zero eigenvalues [**1**, **3**]. Therefore, we attempt to find an equilibrium state for this model with 3 zero eigenvalues. Once we find the parameters for which the system has 3 zero eigenvalues, we would diagonalize the linearization matrix and differentiate our space variables to see how they behave. We must verify if the system is symmetric with respect to the transformation, and that the system is non-degenerate, i.e., the system should not have a continuous family of equilibrium solutions. Therefore, it is essential for the system to be non-degenerate.

Once non-degeneracy is achieved, one can proceed with non linear transformations and transform the system into asymptotic normal form [**4**]. It is well known that a local bifurcation analysis is based upon a consideration of a normal form on the centre manifold [**6**]. Normal forms which can be reduced to the Lorenz model in some canonical notation, are considered here. One of the advantages of the normal form method is that the normal-form system is determined by the character of the bifurcation rather than the specific features of the equation itself [**23**]. The main idea behind transforming a dynamical system to normal form is to derive new sets of simple differential equations using consecutive non-linear transformation by removing as many non-linear terms as possible in the new system [**30**]. Therefore, one could reduce the system to asymptotic normal form and then transform it into Shimizu-Morioka model [**21**].

It is shown by L. P. Shil'nikov [**22**] that bifurcations of Lorenz-like attractors are studied in the Shimizu-Morioka model, as a result of which the parameter space (α, η, B) in Shimizu-Morioka Model (equation 5.2) which corresponds to formation of a homoclinic butterfly to a saddle with zero saddle value guarantees the existence of the Lorenz attractor under certain conditions [**27**]. Therefore, transforming asymptotic normal form into Shimizu-Morioka model, with appropriate parameters, guarantees the existence of a Lorenz attractor under certain conditions [**22**, **5**].

We perform theoretical investigations of Lorenz attractors on three dynamical systems:

(i) *Replicator Dynamics*,

(ii) *Generalized Replicator Dynamics*, and

(iii) *Lotka-Volterra like Dynamics.*

In case of replicator dynamics, the system at equilibrium ends up being *degenerate*, and hence, shows no possibility of an attractor. In case of of generalized replicator dynamics, the system at equilibrium turns out to be *non-degenerate*, and could therefore be reduced to asymptotic normal form and hence, into the Shimizu-Morioka model, but *did not exhibit a Lorenz attractor*. In case of of Lotka-Volterra like dynamics, the system at equilibrium turns out to be *non-degenerate*, could be reduced to asymptotic normal form and then into Shimizu-Morioka model, and *exhibited a Lorenz attractor*.

The content of this project is formatted into three essential components: Fundamentals, Motivation, and Computation. *Fundamentals* introduces readers to historical background, definitions, equations, and formulae of the topics that are essential for readers to know before they could proceed onto the main research. *Motivation* defines the purpose for our research and provides readers with an objective for every assumption and computation that is made. *Computation* covers all theoretical investigations made, that contains pure research content. It makes up more than 60% of the entire project which is new, creative and comprised of original ideas.

I have tried my best to convey this project in the most comprehensible manner, to make it digestible to the non-expert readers and to make the process of reading mathematics more enjoyable. Road-map to historical discoveries and story telling has been used extensively in order to ignite the curiosity of readers and to inspire the readers to explore the questions they want to be looking for. An attempt has been made to link these historical discoveries to this project, so that one understands the flow and evolution of this project.

Chapter 2

Replicator Dynamics

2.1 Background

The are two main types of dynamical systems: *Differential Equations* and *Difference Equations*. Differential equations describe the evolution of systems in continuous a time, whereas difference equations (also called iterated maps) arise in problems where time is discrete [**24**]. In this project, we will primarily focus on continuous dynamical systems, i.e., we will be working with set of differential equations, primarily non linear in nature.

2.1.1 Non-Linear Dynamics

A nonlinear system is basically a system of two or more equations in two or more variables containing at least one equation that is not linear, i.e., the change in the output is not proportional to the change in the input. Most differential equations and systems of differential equations one encounters in practice are of nonlinear nature. For example, a biologist might model the populations $x(t)$ and $y(t)$ of two interacting species of animals by the following nonlinear system:

$$\begin{aligned} \frac{dx}{dt} &= x(y-1) \\ \frac{dy}{dt} &= y(2-x-y) \end{aligned} \tag{2.1}$$

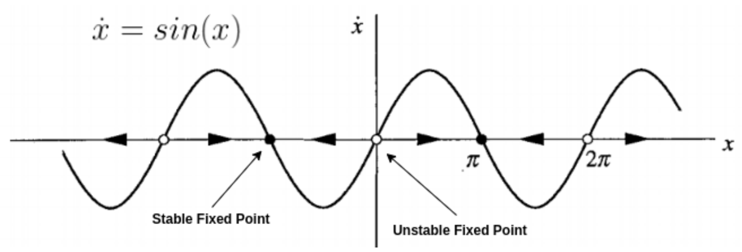

Figure 2.1: Vector field lines for $sin(x)$

where the populations are measured in thousands. Such non-linear dynamical systems, whose variables changes over time, may appear chaotic, unpredictable, or counterintuitive, in comparison to the linear systems which are much simple to study. Problems involving nonlinear differential equations are extremely diverse, and methods for solution of such equations are dependent on the problem itself. A nonlinear system that we will be dealing in this chapter is the Replicator Dynamics, used extensively in biology and economics.

2.1.2 Stability & Fixed Points

Before moving on to replicator dynamics, let us understand how one can find equilibrium points and determine the stability of a dynamical system.

Example 2.1: Consider a flow with differential equation

$$\dot{x} = sin(x)$$

Let $\dot{x} = sin(x) = f(x)$. To find the fixed points, set $f(x^*) = 0$ and solve for x^*, where x^* is called the *equilibrium point*. To determine stability, we plot the vector field line for $sin(x)$ (Figure 2.1). We see that, at points where $\dot{x} = 0$, there is no flow; such points are therefore called *fixed points*. Hence, fixed point for this equation occurs when $sin(x^*) = 0$, or $x^* = k\pi$, where k is an integer. As we can see in Figure 2.1 that there are two kinds of fixed points: solid black dots and open circles. Solid black dots represent *stable fixed points* (often called *attractors* or sinks, because the flow is toward them) and open circles represent *unstable fixed points* (also known as *repellers* or sources). The flow is to the right where $sin(x) > 0$ and to the left where $sin(x) < 0$. Therefore, x^* is stable when k is odd, and unstable when k is even.

2.1.3 Bifurcation

As we have seen from the previous example that dynamics of vector field on a line either settles down to equilibrium or heads out to infinity. Given this wide generalization, an interesting feature of such one-dimensional systems is its dependence on parameters. As these parameters are varied, the qualitative structure of the flow may change dramatically. In particular, fixed points may be created or destroyed, or their stability may change [**24**]. These qualitative changes in dynamics are called *bifurcations* [**8**], and the values of parameter at which they occur are called *bifurcation points*.

2.2 Replicator Dynamics

2.2.1 Evolutionary Game Theory

When you hang out with your friends, you really do not think too hard about the math behind the decisions you're making. But there is a whole field of math and science that applies to social interactions, called Game Theory. Game Theory was pioneered in 1950s by a mathematician John Nash. A game is any interaction between multiple people in which each person's payoff is affected by decision made by others. This could be surely applied to games like poker, but it could also be applied to practically any situation where people or species, or group of people or species directly or indirectly affects each other's decisions. Over the years it has been used by scientists, biologists, military tacticians, economists, and psychologists, to name just a few. Game theory has two main branches: Cooperative and Non-cooperative [**18**]. Non-cooperative covers social interactions where there will be some winners and some losers. This is also sometimes called, competitive game theory. Then there are cooperative games, where every player has agreed to work together towards a common goal. This could be anything from a group of friends deciding how to split up the cost to pay the bill at a restaurant or coalition of nations deciding how to divide up the burden of stopping the climate change.

In 1970s number of biologists started to recognize how similar the games being studied were to the interaction between animals within ecosystems. Originally evolutionary game theory was simply the application of game theory to evolving populations in biology asking how cooperative systems could have evolved over time from various strategies the biological creatures might have adopted. However, the development of evolutionary game theory has produced a theory which holds great promise as a general theory of games. Evolutionary game theory [**20**] differs from classical game theory in focusing more on dynamics of strategy change,i.e., how strategies evolve over time and which kind of dynamic strategies are most successful in this evolutionary process. Unlike static game theory evolutionary game theory does not require players (or species) to act rationally. Note that evolutionary games are dynamic, meaning, that agent strategies change over time, i.e., what is best for one agent to do often depends on what others are doing. It is then reasonable to ask that are there any strategies within a given game that are stable and resistant to invasion. One thing that biologists have been particularly interested in is the idea of evolutionary stability [**18**], which are the evolutionary games that lead to stable solutions for contending strategies. The central idea in dynamic games is that if the evolutionary stable strategies will endure over time [**11**, **18**]. An evolutionary stable strategy is a state of game dynamics where in a very large population of competitors another mutant strategy cannot successfully enter the population to disturb the existing dynamics [**11**]. In this view, equilibrium is the end story of how strategic thinking, optimization, competition, and learning work, and not the beginning or middle of one-shot-game.

2.2.2 Replicator Equation

The replicator equation is the first and most important game dynamic studied in connection with evolutionary game theory. Replicator equation and other deterministic game dynamics have become essential tools over the past forty years in applying evolutionary game theory to behavioral models in biology and social sciences. These models try to show the growth rate of proportion of agents using a certain strategy as will illustrate in this section. This growth rate is equal to the difference between the average payoff of that strategy and the average payoff of the population as a whole [**13**]. Consider a population of n types or species, and let x_i be the frequency of type i. The state of the population is given by $\mathbf{x} \in \mathbf{S}_n$. Now assume that the population is infinitely large and x_i are expected values for an ensemble of populations, that means x_i are differentiable functions of time t. If individuals or species interact randomly and then engage in a symmetric game with payoff matrix A, then $(Ax)_i$ is the expected payoff for a specie of type i and $\mathbf{x}^T A\mathbf{x}$ is the average payoff in the population state $\mathbf{x}$. The growth rate of proportion of agents $\Big($also called the per capita rate of growth or logarithmic derivative $\dfrac{(logx_i)}{dt} = \dfrac{1}{x_i}\dfrac{dx_i}{dt}\Big)$[**13**] is given by the difference between the average payoff of that strategy $(Ax)_i$, and the average payoff of the population $\mathbf{x}^T A\mathbf{x}$:

$$\frac{dx_i}{dt} = x_i((Ax)_i - \mathbf{x}^T A\mathbf{x}) \qquad i = 1, .., n \tag{2.2}$$

This is called the *Replicator Equation* [**13**]. Since the hyperplanes $\sum x_i = 1$ and $x_i = 0$ are invariant, it follows that the unit simplex $\mathbf{S}_n$ is invariant.

2.2.3 Linearization & Equilibrium Solution

In applications one is often interested in the behavior of a dynamical system near an equilibrium state. To study the behavior of a nonlinear dynamical system near an equilibrium point, we can *linearize* the system. Let us consider a system of replicator equations (2.2) with the case of a model where there is a symmetry with respect to a cyclic permutation of species. Consider

$$\frac{dx_i}{dt} = x_i((Ax)_i - \mathbf{S}) \qquad i = 1, 2, 3, 4 \tag{2.3}$$

with a payoff matrix $A = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$, given $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$, and $\mathbf{S} = \mathbf{x}^T A\mathbf{x}$:

$$\frac{dx_1}{dt} = x_1(ax_1 + bx_2 + cx_3 + dx_4 - \mathbf{S})$$

$$\frac{dx_2}{dt} = x_2(ax_2 + bx_3 + cx_4 + dx_1 - \mathbf{S})$$
$$\frac{dx_3}{dt} = x_3(ax_3 + bx_4 + cx_1 + dx_2 - \mathbf{S})$$
$$\frac{dx_4}{dt} = x_4(ax_4 + bx_1 + cx_2 + dx_3 - \mathbf{S})$$
$$\mathbf{S} = x_1(ax_1 + bx_2 + cx_3 + dx_4) + x_2(dx_1 + ax_2 + bx_3 + cx_4) + x_3(cx_1 + dx_2 + ax_3 + bx_4) + x_4(bx_1 + cx_2 + dx_3 + ax_4)$$

This is a system of 4 differential equations with 1conserved quantity - so it is a system of 3 differential equations that depends on 3 parameters. To study the behavior of a nonlinear dynamical system near an equilibrium point, we can linearize the system. Consider the system (2.3) with an equilibrium solution (n_1, n_2, n_3, n_4), that is, $f_1(n_1, n_2, n_3, n_4) = f_2(n_1, n_2, n_3, n_4) = f_3(n_1, n_2, n_3, n_4) = f_4(n_1, n_2, n_3, n_4) = 0$.

$$\frac{dx_1}{dt} = x_1(ax_1 + bx_2 + cx_3 + dx_4 - \mathbf{S}) = f_1(x_1, x_2, x_3, x_4) = 0$$
$$\frac{dx_2}{dt} = x_2(dx_1 + ax_2 + bx_3 + cx_4 - \mathbf{S}) = f_2(x_1, x_2, x_3, x_4) = 0$$
$$\frac{dx_3}{dt} = x_3(cx_1 + dx_2 + ax_3 + bx_4 - \mathbf{S}) = f_3(x_1, x_2, x_3, x_4) = 0$$
$$\frac{dx_4}{dt} = x_4(bx_1 + cx_2 + dx_3 + ax_4 - \mathbf{S}) = f_4(x_1, x_2, x_3, x_4) = 0$$

this gives us the equilibrium system

$$ax_1 + bx_2 + cx_3 + dx_4 = \mathbf{S}$$
$$dx_1 + ax_2 + bx_3 + cx_4 = \mathbf{S}$$
$$cx_1 + dx_2 + ax_3 + bx_4 = \mathbf{S}$$
$$bx_1 + cx_2 + dx_3 + ax_4 = \mathbf{S}$$

which gives us a symmetric stable equilibrium point $(1/4, 1/4, 1/4, 1/4)$ because $\sum_{i=1}^{4} x_i = 1$ and $x_1 = x_2 = x_3 = x_4 = 1/4$. To understand this formula, note that the rate of change of f in the x_i-direction near the point (n_1, n_2, n_3, n_4) is approximately $\frac{\partial f}{dx_i}(n_1, n_2, n_3, n_4)$. Thus

$$f(x_1, x_2, x_3, x_4) \cong f(n_1, n_2, n_3, n_4) + \frac{\partial f}{\partial x_1}(n_1, n_2, n_3, n_4).(x - n_1) + \frac{\partial f}{\partial x_2}(n_1, n_2, n_3, n_4).(x_2 - n_2) + \frac{\partial f}{\partial x_3}(n_1, n_2, n_3, n_4).(x_3 - n_3) + \frac{\partial f}{\partial x_4}(n_1, n_2, n_3, n_4).(x_4 - n_4)$$

To linearize the system near an equilibrium point (n_1, n_2, n_3, n_4) means to replace the functions $f_1(x_1, x_2, x_3, x_4)$, $f_2(x_1, x_2, x_3, x_4)$, $f_3(x_1, x_2, x_3, x_4)$, $f_4(x_1, x_2, x_3, x_4)$ by their linear approximations. Keeping in mind that $f_i(n_1, n_2, n_3, n_4) = 0$, this approximation is

$$\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial f_1}{\partial x_1}(n_1,n_2,n_3,n_4).(x_1-n_1) \ + \frac{\partial f_1}{\partial x_2}(n_1,n_2,n_3,n_4).(x_2-n_2) \ + \\
&\quad \frac{\partial f_1}{\partial x_3}(n_1,n_2,n_3,n_4).(x_3-n_3) \ + \frac{\partial f_1}{\partial x_4}(n_1,n_2,n_3,n_4).(x_4-n_4) \\
\frac{dx_2}{dt} &= \frac{\partial f_2}{\partial x_1}(n_1,n_2,n_3,n_4).(x_1-n_1) \ + \frac{\partial f_2}{\partial x_2}(n_1,n_2,n_3,n_4).(x_2-n_2) \ + \\
&\quad \frac{\partial f_2}{\partial x_3}(n_1,n_2,n_3,n_4).(x_3-n_3) \ + \frac{\partial f_2}{\partial x_4}(n_1,n_2,n_3,n_4).(x_4-n_4) \\
\frac{dx_3}{dt} &= \frac{\partial f_3}{\partial x_1}(n_1,n_2,n_3,n_4).(x_1-n_1) \ + \frac{\partial f_3}{\partial x_2}(n_1,n_2,n_3,n_4).(x_2-n_2) \ + \\
&\quad \frac{\partial f_3}{\partial x_3}(n_1,n_2,n_3,n_4).(x_3-n_3) \ + \frac{\partial f_3}{\partial x_4}(n_1,n_2,n_3,n_4).(x_4-n_4) \\
\frac{dx_4}{dt} &= \frac{\partial f_4}{\partial x_1}(n_1,n_2,n_3,n_4).(x_1-n_1) \ + \frac{\partial f_4}{\partial x_2}(n_1,n_2,n_3,n_4).(x_2-n_2) \ + \\
&\quad \frac{\partial f_4}{\partial x_3}(n_1,n_2,n_3,n_4).(x_3-n_3) \ + \frac{\partial f_4}{\partial x_4}(n_1,n_2,n_3,n_4).(x_4-n_4)
\end{aligned}$$

We have to make sure that we do not leave out S and compute $\frac{d\mathbf{S}}{dx_1}$, $\frac{d\mathbf{S}}{dx_2}$, $\frac{d\mathbf{S}}{dx_3}$, $\frac{d\mathbf{S}}{dx_4}$. Further, approximating the equation yields:

$$\begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_2}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_3}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_4}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_4}(n_1,n_2,n_3,n_4)
\end{pmatrix}
\begin{pmatrix} (x_1-n_1) \\ (x_2-n_2) \\ (x_3-n_3) \\ (x_4-n_4) \end{pmatrix}$$

We can use the substitution $u = x - a$ and $v = y - b$ to simplify further:

$$= \underbrace{\begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_1}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_2}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_2}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_3}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_3}{\partial x_4}(n_1,n_2,n_3,n_4) \\
\frac{\partial f_4}{\partial x_1}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_2}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_3}(n_1,n_2,n_3,n_4) & \frac{\partial f_4}{\partial x_4}(n_1,n_2,n_3,n_4)
\end{pmatrix}}_{\mathbf{J}}
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

The matrix $\mathbf{J}$ is called the Jacobian Matrix of the system at the point (n_1, n_2, n_3, n_4). Hence, at equilibrium point the Jacobian for our Replicator Dynamics equation $\mathbf{J}\ f(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is:

$$\mathbf{J}\ f(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) = \frac{1}{8}\begin{pmatrix} a-b-c-d & -a+b-c-d & -a-b+c-d & -a-b-c+d \\ -a-b-c+d & a-b-c-d & -a+b-c-d & -a-b+c-d \\ -a-b+c-d & -a-b-c+d & a-b-c-d & -a+b-c-d \\ -a+b-c-d & -a-b+c-d & -a-b-c+d & a-b-c-d \end{pmatrix}$$

Eigenvalues as follows:

$\lambda_1 = \frac{1}{4}(a - c + \sqrt{(b-d)^2}i)$

$\lambda_2 = \frac{1}{4}(a - c - \sqrt{(b-d)^2}i)$

$\lambda_3 = \frac{1}{4}(a + c - b - d)$

$\lambda_4 = -\frac{1}{4}(a + b + c + d)$

2.3 Three 0 Eigenvalues

2.3.1 Motivation

The aim of the present work is to illustrate how, in a physical system with a three zero eigenvalues, chaos occurs when the parameters are chosen to be arbitrarily close to values at which we encounter the simultaneous onset of triple-zero eigenvalues. It is shown in Arneodo et al. [**3**] that in the case of a bifurcation of an equilibrium with three zero eigenvalues and a complete Jordan block there can arise spiral chaos associated with a homo-clinic loop to a saddle-focus. Therefore, we begin by computing the asymptotic version of the normal form of replicator dynamics (2.3) such that it has three zero eigenvalues. The numerical investigation of this asymptotic normal form done in [**3**] strongly suggests that chaotic behavior occurs as close as one wants to the onset of the triple zero eigenvalues. Hence, finding values of parameters such that we have three zero eigenvalues, might lead us to a chaotic behaviour and possibly a Lorenz Attractor.

2.3.2 Computation

Based on eigenvalues computed in 2.2.3, we would now find if it is possible to have three 0 eigenvalues. In order for us to have three zero eigenvalues, the following conditions should hold, (which can be verified easily).

Case I : $a = b = c = d$ where is a positive real number (say α), then the payoff matrix

$$\mathbf{A} = \frac{1}{4}\begin{pmatrix} -\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & -\alpha \end{pmatrix}$$

In which case the Eigenvalues would be $\lambda_1 = -\alpha, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

Case II : $a = b = -c = -d$ where is a positive real number (say α), then the payoff matrix

$$\mathbf{A} = \frac{1}{4}\begin{pmatrix} -\alpha & \alpha & -\alpha & \alpha \\ \alpha & -\alpha & \alpha & -\alpha \\ -\alpha & \alpha & -\alpha & \alpha \\ \alpha & -\alpha & \alpha & -\alpha \end{pmatrix}$$

The Eigenvalues would then be $\lambda_1 = -\alpha, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

In either case, we observe that we have a negative real valued eigenvalue along with three 0 eigenvalues. Therefore, it is sufficient to say that there is a possibility of chaotic behaviour and hence, a Lorenz Attractor. We also observe that the matrix is circulant. Therefore, our next task will be to compute the asymptotic version of normal form (or asymptotic normal form), details of which will be elaborated in the next chapter.

Chapter 3

The Normal Form & Symmetry

3.1 Normal Form

3.1.1 Motivation

The study of stability and bifurcations of a system near an equilibrium point can often lead to a simple set of differential equations which is called a *normal form*. It is a way of reducing dynamical system into the simplest form which is topologically equivalent to the original system [**30**]. With the degree of degeneracy in the system, it becomes more difficult to study a dynamical system. Thus, the study of normal form becomes more pertinent when dealing with such systems. The core idea behind normal form is to derive new sets of differential equations using consecutive non-linear transformation in such a way that we could remove as many non-linear terms as possible in the new system. Generally, in a given dynamical system a locally invariant small dimensional manifold - center manifold, is obtained by applying the center manifold theory [**6**]. This center manifold is further simplified to a normal form by introducing additional non-linear transformations. In this project, a method is presented to perform various nonlinear transformations and compute the *Asymptotic Normal Forms*, for a system whose Jacobian has a triple zero eigenvalue at the origin. Asymptotic normal form differs from normal form in a sense that normal form is obtained by regular coordinate transformations, whereas asymptotic normal form is obtained after further re-scaling. In this project, there are many occasions where we re-scale our system to either make the system simpler, or to reduce the system to Shimizu-Morioka system [**21**]. Therefore, a step-by-step procedure and algorithm is followed to reduce our system to asymptotic normal form.

3.1.2 Algorithm

We will use the following algorithm to reduce our non-linear system of equations to Asymptotic Normal Form

Step 1: *Linearize* the non-linear system of equations $\mathbf{x}(t)$, and check if the system has three-zero eigenvalues. Compute the corresponding eigenvalues and eigenvectors.

Step 2: *Diagonalize* the linearization matrix and write $\mathbf{x}(t)$ as a linear combination of these eigenvectors and some real variables, called the *space variables* (see equation 3.1).

Step 3: Compute the evolution of space variables by differentiating these space variables with respect to the time variable. This evolution of space variables is proportional to small parameters (called the *control parameters*) corresponding to deviation from bifurcation.

Step 4: Check if the *symmetry* holds by rotating the co-ordinates $90°$ using transformation $(x_1, x_2, x_3, x_4) \to (x_4, x_1, x_2, x_3) \to (x_3, x_4, x_1, x_2) \to (x_2, x_3, x_4, x_1) \to (x_1, x_2, x_3, x_4)$.

Step 5: Verify if the system is *Degenerate* or *Non-Degenerate*. Proceed, if the system is Non-Degenerate.

Step 6: *Re-scale* the system in step 2 to make small parameters sufficiently non-small such that, when these parameters approaches 0, the coefficients of first and second order terms do not disappear.

Step 7: Perform *co-ordinate shift* for equilibrium points.

Step 8: Introduce a *new variable*, and set it equal to the right hand side of one of the equations calculated in step 2 and re-write your entire system in terms of this newly introduced variable.

Step 9: Omit terms of higher orders, as with higher order terms the control parameters are extremely small and tend to become zero.

The resulting system of equations will be the *Asymptotic Normal form.* We can now begin the process of reducing our system of replicator equation to asymptotic normal form.

3.2 Computation

3.2.1 Diagonalisation

In chapter 2, it was observed that in any case, our matrix A is a circulant, so its eigenvectors (for any choice of a, b, c, d) are

$$e_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \; e_2 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \; e_1 = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

(i.e., e_3 is complex conjugate of e_2). In order to diagonalise our system we just go to the basis of eigenvectors e_0, e_1, e_2, e_3, and take the projection to e_0 to be zero, as we want to stay at $x_1 + x_2 + x_3 + x_4 = 1$. So, we need to write our system in this basis, i.e., we represent our vector $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = x^* + z\mathbf{e_1} + (u + \mathbf{i}w)\mathbf{e_2} + (u - \mathbf{i}w)\mathbf{e_3} \tag{3.1}$$

$$= \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} + z \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + (u + iw) \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} + (u - iw) \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

where z, u, w are some real numbers (we do not add a projection to e_0, as we need x to lie in the invariant plane $x_1 + x_2 + x_3 + x_4 = 1$). Thus, the idea is to make this substitution into the replicator dynamics system and write the system for z, u, and w.

Note that x^* is invariant for any value of parameter as it can be seen in equation (2.3) that x^* does not change by varying our parameters. Now, in order to study the behaviour of our space variables u, w, z, we compute $\frac{du}{dt}, \frac{dw}{dt}$, and $\frac{dz}{dt}$.
It is easy to see from (3.1) that

$$\mathbf{x} \cdot \mathbf{e_1} = 4z$$
$$\mathbf{x} \cdot \mathbf{e_2} = 4u - (4w)i$$
$$\mathbf{x} \cdot \mathbf{e_3} = 4u + (4w)i$$

$$\frac{dz}{dt} = \tfrac{1}{4}\mathbf{e_1} \cdot \frac{dx}{dt} = \tfrac{1}{4}\mathbf{e_1} \cdot (x_i((\mathbf{A}x)_i - \mathbf{S}))$$

$$\frac{du}{dt} - i\frac{dw}{dt} = \tfrac{1}{4}\mathbf{e_2} \cdot \frac{dx}{dt} = \tfrac{1}{4}\mathbf{e_2} \cdot (x_i((\mathbf{A}x)_i - \mathbf{S}))$$

$$\frac{du}{dt} + i\frac{dw}{dt} = \tfrac{1}{4}\mathbf{e_3} \cdot \frac{dx}{dt} = \tfrac{1}{4}\mathbf{e_3} \cdot (x_i((\mathbf{A}x)_i - \mathbf{S}))$$

One can check that

$$x_1 = \tfrac{1}{4} + z + 2u$$
$$x_2 = \tfrac{1}{4} - z - 2w$$
$$x_3 = \tfrac{1}{4} + z - 2u$$
$$x_4 = \tfrac{1}{4} - z + 2w$$
$$x_1^2 = \tfrac{1}{16} + z^2 + 4u^2 + \tfrac{z}{2} + u + 4uz$$
$$\mathrm{x}_2^2 = \tfrac{1}{16} + z^2 + 4w^2 - \tfrac{z}{2} - w + 4wz$$
$$\mathrm{x}_3^2 = \tfrac{1}{16} + z^2 + 4u^2 + \tfrac{z}{2} - u - 4uz$$
$$\mathrm{x}_4^2 = \tfrac{1}{16} + z^2 + 4w^2 - \tfrac{z}{2} + w - 4wz$$
$$\mathrm{x}_1 x_2 = \tfrac{1}{16} + \tfrac{u}{2} - \tfrac{w}{2} - z^2 - 2wz - 2uz - 4uw$$
$$\mathrm{x}_2 x_3 = \tfrac{1}{16} - \tfrac{u}{2} - \tfrac{w}{2} - z^2 - 2wz + 2uz + 4uw$$
$$\mathrm{x}_3 x_4 = \tfrac{1}{16} - \tfrac{u}{2} + \tfrac{w}{2} - z^2 + 2wz + 2uz - 4uw$$
$$\mathrm{x}_4 x_1 = \tfrac{1}{16} + \tfrac{u}{2} + \tfrac{w}{2} - z^2 + 2wz - 2uz + 4uw$$
$$\mathrm{x}_1 x_3 = \tfrac{1}{16} + \tfrac{z}{2} + z^2 - 4u^2$$

$$\mathrm{x}_2x_4 = \tfrac{1}{16} - \tfrac{z}{2} + z^2 - 4w^2$$

$$\mathbf{S} = a(x_1^2+x_2^2+x_3^2+x_4^2) + b(x_1x_2+x_2x_3+x_3x_4+x_4x_1) + c(x_1x_3+x_2x_4+x_1x_3+x_2x_4) + d(x_1x_4+x_1x_2+x_2x_3+x_3x_4)$$

$$= a(x_1^2+x_2^2+x_3^2+x_4^2) + (b+d)((x_1+x_3)(x_2+x_4)) + c(2x_1x_3+2x_2x_4),$$

or

$$\mathbf{S} = a(\tfrac{1}{4}+4z^2+8u^2+8w^2) + (b+d)(\tfrac{1}{4}-4z^2) + c(\tfrac{1}{4}+4z^2-8u^2-8w^2)$$

Coming back to calculating $\frac{du}{dt}, \frac{dw}{dt}$, and $\frac{dz}{dt}$ in terms of a, b, c, d

$$\frac{dz}{dt} = \tfrac{1}{4}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\begin{pmatrix}\mathrm{x}_1(ax_1+bx_2+cx_3+dx_4-\mathbf{S})\\ \mathrm{x}_2(ax_2+bx_3+cx_4+dx_1-\mathbf{S})\\ \mathrm{x}_3(ax_3+bx_4+cx_1+dx_2-\mathbf{S})\\ \mathrm{x}_4(ax_4+bx_1+cx_2+dx_3-\mathbf{S})\end{pmatrix}$$

$$= \tfrac{1}{4}(a(x_1^2-x_2^2+x_3^2-x_4^2) + b(x_1x_2-x_2x_3+x_3x_4-x_4x_1) + c(x_1x_3-x_2x_4+x_1x_3-x_2x_4) + d(x_1x_4-x_1x_2+x_2x_3-x_3x_4) - \mathbf{S}(x_1-x_2+x_3-x_4))$$

$$= \tfrac{1}{4}(a(x_1^2-x_2^2+x_3^2-x_4^2) + (b-d)((x_1-x_3)(x_2-x_4)) + c(2x_1x_3-2x_2x_4) - \mathbf{S}(x_1-x_2+x_3-x_4))$$

Plugging in x_1, x_2, x_3, x_4 and $\mathbf{S}$, we achieve the requires result

$$\frac{dz}{dt} = \left((\tfrac{a-b+c-d}{4}) - 8(a-c)(u^2+w^2)\right)z - (4a-4b+4c-4d)z^3 + 2(a-c)(u^2-w^2) + (4d-4b)uw$$

$$\frac{du}{dt} - i\frac{dw}{dt} = \tfrac{1}{4}\begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}\begin{pmatrix}\mathrm{x}_1(ax_1+bx_2+cx_3+dx_4-\mathbf{S})\\ \mathrm{x}_2(ax_2+bx_3+cx_4+dx_1-\mathbf{S})\\ \mathrm{x}_3(ax_3+bx_4+cx_1+dx_2-\mathbf{S})\\ \mathrm{x}_4(ax_4+bx_1+cx_2+dx_3-\mathbf{S})\end{pmatrix}$$

$$= \tfrac{1}{4}((a(x_1^2-x_3^2) + b(x_1x_2-x_3x_4) + c(x_1x_3-x_1x_3) + d(x_1x_4-x_2x_3) - \mathbf{S}(x_1-x_3)) + i(\mathrm{a}(\mathrm{x}_2^2-x_4^2) + b(x_2x_3-x_1x_4) + c(x_2x_4-x_2x_4) + d(x_1x_2-x_3x_4) - \mathbf{S}(x_2-x_4)))$$

$$\frac{du}{dt} = \tfrac{1}{4}((a(x_1^2-x_3^2) + b(x_1x_2-x_3x_4) + c(x_1x_3-x_1x_3) + d(x_1x_4-x_2x_3) - \mathbf{S}(x_1-x_3))$$
$$\frac{dw}{dt} = \tfrac{1}{4}(a(x_4^2-x_2^2) + b(x_1x_4-x_2x_3) + c(x_2x_4-x_2x_4) + d(x_3x_4-x_1x_2) - \mathbf{S}(x_4-x_2)))$$

Plugging in x_1, x_2, x_3, x_4 and $\mathbf{S}$, we achieve the requires result

$$\frac{du}{dt} = \left((\tfrac{a-c}{4}) + (2a-b-d)z - 4(b+d-a-c)z^2 + 8(c-a)w^2\right)u + 8(c-a)u^3 + (\tfrac{d-b}{4})w + (d-b)wz$$

$$\frac{dw}{dt} = \left((\tfrac{a-c}{4}) + (-2a+b+d)z - 4(b+d-a-c)z^2 + 8(c-a)u^2\right)w + 8(c-a)w^3 + (\tfrac{b-d}{4})u + (d-b)uz$$

3.2.2 Checking the Symmetry

We will check if the system is symmetric. We can shift the co-ordinates and verify if symmetry holds. What we observe is that for shift $(x_1, x_2, x_3, x_4) \to (x_4, x_1, x_2, x_3)$, (z, u, w) transforms into $(-z, w, -u)$. Similarly $(x_4, x_1, x_2, x_3) \to (x_3, x_4, x_1, x_2)$ transforms $(-z, w, -u)$ into $(z, -u, -w)$, $(x_3, x_4, x_1, x_2) \to (x_2, x_3, x_4, x_1)$ transforms $(z, -u, -w)$ into $(-z, -w, u)$, and $(x_2, x_3, x_4, x_1) \to (x_1, x_2, x_3, x_4)$ transforms $(-z, -w, u)$ back into (z, u, w).

1. $(z, u, w) \to (-z, w, -u)$

$$\frac{dz}{dt} = -\left(\tfrac{1}{4}(a-b+c-d) + 8(a-c)(u^2+w^2)\right)z + (4a-4b+4c-4d)z^3 - 2(a-c)(u^2-w^2) - (4d-4b)uw = -\frac{dz}{dt}$$

$$\frac{du}{dt} = \left(\tfrac{1}{4}(a-c) + (-2a+b+d)z - 4(b+d-a-c)z^2 + 8(c-a)u^2\right)w + 8(c-a)w^3 + (\tfrac{b-d}{4})u + (d-b)uz = \frac{dw}{dt}$$

$$\frac{dw}{dt} = -\left((\tfrac{a-c}{4}) + (2a-b-d)z - 4(b+d-a-c)z^2 + 8(c-a)w^2\right)u - 8(c-a)u^3 - (\tfrac{d-b}{4})w - (d-b)wz = -\frac{du}{dt}$$

2. $(-z, w, -u) \to (z, -u, -w)$

$$\frac{dz}{dt} = \left(\tfrac{1}{4}(a-b+c-d) + 8(a-c)(u^2+w^2)\right)z - (4a-4b+4c-4d)z^3 + 2(a-c)(u^2-w^2) + (4d-4b)uw = \frac{dz}{dt}$$

$$\frac{du}{dt} = -\left(\tfrac{1}{4}(a-c) + (2a-b-d)z - 4(b+d-a-c)z^2 + 8(c-a)w^2\right)u - 8(c-a)u^3 - (\tfrac{d-b}{4})w - (d-b)wz = -\frac{du}{dt}$$

$$\frac{dw}{dt} = -\left(\left(\tfrac{a-c}{4}\right) + (-2a+b+d)z - 4(b+d-a-c)z^2 + 8(c-a)u^2 \right) w - 8(c-a)w^3 - \left(\tfrac{b-d}{4}\right)u - (d-b)uz = -\frac{dw}{dt}$$

3. $(z, -u, -w) \rightarrow (-z, -w, u)$

$$\frac{dz}{dt} = -\left(\tfrac{1}{4}(a-b+c-d) + 8(a-c)(u^2+w^2) \right) z + (4a-4b+4c-4d)z^3 - 2(a-c)(u^2-w^2) - (4d-4b)uw = -\frac{dz}{dt}$$

$$\frac{du}{dt} = -\left(\tfrac{1}{4}(a-c) + (-2a+b+d)z - 4(b+d-a-c)z^2 + 8(c-a)u^2 \right) w - 8(c-a)w^3 - \left(\tfrac{b-d}{4}\right)u - (d-b)uz = -\frac{dw}{dt}$$

$$\frac{dw}{dt} = \left(\left(\tfrac{a-c}{4}\right) + (2a-b-d)z - 4(b+d-a-c)z^2 + 8(c-a)w^2 \right) u + 8(c-a)u^3 + \left(\tfrac{d-b}{4}\right)w + (d-b)wz = \frac{du}{dt}$$

4. $(-z, -w, u) \rightarrow (z, u, w)$

$$\frac{dz}{dt} = \left(\tfrac{1}{4}(a-b+c-d) - 8(a-c)(u^2+w^2) \right) z - (4a-4b+4c-4d)z^3 + 2(a-c)(u^2-w^2) + (4d-4b)uw = \frac{dz}{dt}$$

$$\frac{du}{dt} = \left(\tfrac{1}{4}(a-c) + (2a-b-d)z - 4(b+d-a-c)z^2 + 8(c-a)w^2 \right) u + 8(c-a)u^3 + \left(\tfrac{d-b}{4}\right)w + (d-b)wz = \frac{du}{dt}$$

$$\frac{dw}{dt} = \left(\left(\tfrac{a-c}{4}\right) + (-2a+b+d)z - 4(b+d-a-c)z^2 + 8(c-a)u^2 \right) w + 8(c-a)w^3 + \left(\tfrac{b-d}{4}\right)u + (d-b)uz = \frac{dw}{dt}$$

Hence, we see that it is a symmetric system. We now find equilibria points for (u, w, z), such that $\left(\frac{du}{dt}, \frac{dw}{dt}, \frac{dz}{dt}\right) = 0$. We see that, when $\frac{du}{dt} = \frac{dw}{dt} = 0$, $(u, w) = 0$, and when

$$\frac{dz}{dt} = \left(\left(\tfrac{a-b+c-d}{4}\right) - 8(a-c)(u^2+w^2) \right) z - (4a-4b+4c-4d)z^3 + 2(a-c)(u^2-w^2) + (4d-4b)uw = 0$$

$$= \left(\tfrac{1}{4}(a-b+c-d) \right) z - (4a-4b+4c-4d)z^3 = 0,$$

$$z = 0, +\tfrac{1}{4}, -\tfrac{1}{4}$$

3.2.3 Degeneracy and Failure

This means that the equilibrium point z is independent of parameters a, b, c, d and has infinitely many points, which leads us to a ***degenerate system***. Because of this degeneracy, it is now not possible to study the behaviour of these small parameters near the equilibrium points as we have an entire line of equilibrium which is independent of any sort of parameter(s). Because we want a non-degenerate system that could be expressed in the normal form and then into Shimizu Morioka model in order for the system to have a Lorentz attractor, we cannot proceed with this system anymore. Consequently, what we could do is generalize our replicator equation, and see if the system is still degenerate.

Chapter 4

Generalized Replicator Dynamics

4.1 Generalized Replicator Dynamics

4.1.1 Definition

In chapter 2 we defined a replicator equation. Here, we will define a more general form of replicator equation and state the reason why we chose to do so. The general replicator equation was first introduced by Taylor and Jonker in 1978. In a general replicator equation, there are set of agents that represent a particular strategy and each type of strategy has a payoff associated with it, which is how well they are doing. There is also a parameter associated with how many of each type there is in the overall population. Each type represent a certain percentage of the overall population. The replicator model is one way of trying to capture the dynamics of this evolutionary game to see which strategies become more prevalent over time. One thing to note is that theory assumes large homogeneous populations with random interactions [**13**]. The general replicator equation differs from a replicator equation in a sense that it allows the fitness function to incorporate the distribution of population types rather than setting it to a particular type of fitness [**19**]. This important property allows the general equation the essence of selection but unlike other models, the generalized replicator equation does not incorporate mutations, so it is not able to innovate new types of pure strategies.

The generalized replicator dynamics stems from the simplest possible growth rate function in game dynamics, where the growth rate of individual's playing strategy in a population is directly proportional to the strategy's linear fitness $f_i(x)$ [**19**]. This leads to a continuous-time dynamical system [**2**]

$$\dot{y}_i = y_i f_i(x)$$

where y_i is the number of individuals in the population playing a certain strategy, and $\mathbf{x} \in \mathbf{S}_n$ is the strategic population state, just like before. This system now induces a game dynamics in the population shares. If we denote the population shares as $x_i = \dfrac{y_i}{\sum_{i=1}^n y_j}$ [**2**], then

$$\begin{aligned} \dot{x}_i &= \frac{\dot{y}_i \sum_{i=1}^n y_j - y_i \sum_{i=1}^n \dot{y}_j}{(\sum_{i=1}^n y_j)^2} \\ &= x_i f_i(\mathbf{x}) - x_i \textstyle\sum_{i=1}^n x_j f_j(\mathbf{x}) \\ &= x_i(f_i(\mathbf{x}) - \phi(\mathbf{x})) \end{aligned} \tag{4.1}$$

where $\phi(\mathbf{x}) = \sum_j x_j f_j(\mathbf{x})$ is the average fitness of a population in state $\mathbf{x}$, or equivalently, the average or expected payoff in a game between two randomly chosen members of the population [**19**]. It is interesting to observe that if we substitute the linear fitness function $f_i(x) = (Ax)_i$ into (4.1), we get exactly equation (2.2), without the loss of generality [**2**]. Where replicator dynamics (2.2) had set its fitness function to a particular fitness type $(Ax)_i$, the generalized replicator dynamics had the advantage of allowing its fitness function of any fitness type to incorporate the distribution of population. This is why it can be called the *simplest* or the most *general* form of replicator equation, and the dynamical system (4.1) could therefore be called the *Generalized Replicator Dynamics.*

4.1.2 Motivation

In chapter 2 we have seen that once we diagonalize the linearization matrix for replicator equation (2.3), we obtain an entire line of equilibrium with infinitely many solutions, and the system ends up being degenerate. Whereas, our goal is to obtain a non-degenerate system, which can then be reduced to asymptotic normal form in order to study the system at small values of control parameters and in small neighbourhood of $(u, w, z) = 0$. Therefore, we would now generalize our system of replicator equations to *generalized replicator dynamics*. Working with such generalized replicator dynamics gives us an advantage of introducing non-linearity which is not zero at the point of equilibrium. This means, for small parameter values at equilibrium point $(\frac{du}{dt}, \frac{dw}{dt}, \frac{dz}{dt}) = 0$, the system will have non-zero z and z^3 terms where z will depend on some control parameter(s), i.e., the z will not have a solution with infinitely many points. This way, one could obtain a non-degenerate system and carry further computations. From now on, we will consider a system of equations of the form (4.1) and check whether this system is degenerate or not.

4.2 Computation

In our original setting we have the following system

$$
\begin{aligned}
\dot{x}_1 &= x_1(ax_1 + bx_2 + cx_3 + dx_4 - S(x)),\\
\dot{x}_2 &= x_2(dx_1 + ax_2 + bx_3 + cx_4 - S(x)),\\
\dot{x}_3 &= x_3(cx_1 + dx_2 + ax_3 + bx_4 - S(x)),\\
\dot{x}_4 &= x_4(bx_1 + cx_2 + dx_3 + ax_4 - S(x)),
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{gathered}
S(x) = x_1(ax_1 + bx_2 + cx_3 + dx_4) + x_2(dx_1 + ax_2 + bx_3 + cx_4) + x_3(cx_1 + dx_2 + ax_3 + bx_4)\\
+x_4(bx_1 + cx_2 + dx_3 + ax_4).
\end{gathered}
\tag{4.3}
$$

Denote $\beta = b-a$, $\gamma = c-a$, $\delta = d-a$. Since we consider our system in the space $x_1+x_2+x_3+x_4 = 1$, we have

$$
\begin{aligned}
ax_1 + bx_2 + cx_3 + dx_4 &= a + \beta x_2 + \gamma x_3 + \delta x_4,\\
dx_1 + ax_2 + bx_3 + cx_4 &= a + \beta x_3 + \gamma x_4 + \delta x_1,\\
cx_1 + dx_2 + ax_3 + bx_4 &= a + \beta x_4 + \gamma x_1 + \delta x_2,
\end{aligned}
$$

$$bx_1 + cx_2 + dx_3 + ax_4 = a + \beta x_1 + \gamma x_2 + \delta x_3.$$

Substitution in (6.6),(6.2) gives

$$\begin{aligned}
\dot{x}_1 &= x_1(\beta x_2 + \gamma x_3 + \delta x_4 - \sigma(x)),\\
\dot{x}_2 &= x_2(\delta x_1 + \beta x_3 + \gamma x_4 - \sigma(x)),\\
\dot{x}_3 &= x_3(\gamma x_1 + \delta x_2 + \beta x_4 - \sigma(x)),\\
\dot{x}_4 &= x_4(\beta x_1 + \gamma x_2 + \delta x_3 - \sigma(x)),
\end{aligned} \tag{4.4}$$

where $\sigma = x_1(\beta x_2+\gamma x_3+\delta x_4)+x_2(\delta x_1+\beta x_3+\gamma x_4)+x_3(\gamma x_1+\delta x_2+\beta x_4)+x_4(\beta x_1+\gamma x_2+\delta x_3)$. By scaling time (i.e., by division of the right-hand sides to $\sqrt{\beta^2+\gamma^2+\delta^2}$) we achieve

$$\beta^2 + \gamma^2 + \delta^2 = 1.$$

Thus, our parameter space is the two-dimensional sphere.

However, we need three independent parameters to have the 3 zero eigenvalues needed to create the Lorenz attractor. This means we need to modify our system. The general form of the 4 species replicator dynamics is this:

$$\dot{x}_i = x_i(f_i(x) - \phi(x)), \quad i = 1, \ldots, 4,$$

where f_i are any functions of $x = (x_1, x_2, x_3, x_4)$ and

$$\phi(x) = x_1 f_1(x) + x_2 f_2(x) + x_3 f_3(x) + x_4 f_4(x).$$

One can see, that the space

$$x_1 + x_2 + x_3 + x_4 = 1$$

is invariant, for any choice of the functions f_i. We want to consider the case where the system is symmetric with respect to the cyclic permutation $(x_1, x_2, x_3, x_4) \to (x_2, x_3, x_4, x_1)$. So we take $f_1(x) = f(x_1, x_2, x_3, x_4)$, $f_2(x) = f(x_2, x_3, x_4, x_1)$, $f_3(x) = f(x_3, x_4, x_1, x_2)$,

$f_4(x) = f(x_4, x_1, x_2, x_3)$, with the same function f. For any choice of f, the symmetry implies that $(1/4, 1/4, 1/4, 1/4)$ is an equilibrium state. Our goal is to study dynamics in a small neighborhood of this equilibrium.

The simplest case is when $f(x_1, x_2, x_3, x_4)$ is linear in (x_2, x_3, x_4) and nonlinear in x_1:

$$f(x_1, x_2, x_3, x_4) = \hat{\beta}(x_2 - \tfrac{1}{4}) + \hat{\gamma}(x_3 - \tfrac{1}{4}) + \hat{\delta}(x_4 - \tfrac{1}{4}) + \psi(x_1),$$

where we expand

$$\psi = \psi_0 + \psi_1(x_1 - \tfrac{1}{4}) - \psi_2(x_1 - \tfrac{1}{4})^2 - \psi_3(x_1 - \tfrac{1}{4})^3 + O((x_1 - \tfrac{1}{4})^4).$$

Since we consider the system on the invariant subspace

$$(x_1 - \tfrac{1}{4}) + (x_2 - \tfrac{1}{4}) + (x_3 - \tfrac{1}{4}) + (x_4 - \tfrac{1}{4}) = 0,$$

we have

$$f(x_1, x_2, x_3, x_4) = \psi_0 + \beta(x_2 - \tfrac{1}{4}) + \gamma(x_3 - \tfrac{1}{4}) + \delta(x_4 - \tfrac{1}{4}) + -\psi_2(x_1 - \tfrac{1}{4})^2 - \psi_3(x_1 - \tfrac{1}{4})^3 + O((x_1 - \tfrac{1}{4})^4),$$

where we denote $\beta = \hat{\beta} - \psi_1,\ \gamma = \hat{\gamma} - \psi_1,\ \delta = \hat{\delta} - \psi_1$.

Denote $y_i = x_i - \frac{1}{4}$, $i = 1, \ldots, 4$. The system takes the form

$$\begin{aligned}
\dot{y}_1 &= (\tfrac{1}{4} + y_1)(\beta y_2 + \gamma y_3 + \delta y_4 - \psi_2 y_1^2 - \psi_3 y_1^3 - \sigma(y)) + O(\|y\|^4),\\
\dot{y}_2 &= (\tfrac{1}{4} + y_2)(\delta y_1 + \beta y_3 + \gamma y_4 - \psi_2 y_2^2 - \psi_3 y_2^3 - \sigma(y)) + O(\|y\|^4),\\
\dot{y}_3 &= (\tfrac{1}{4} + y_3)(\gamma y_1 + \delta y_2 + \beta y_4 - \psi_2 y_3^2 - \psi_3 y_3^3 - \sigma(y)) + O(\|y\|^4),\\
\dot{y}_4 &= (\tfrac{1}{4} + y_4)(\beta y_1 + \gamma y_2 + \delta y_3 - \psi_2 y_4^2 - \psi_3 y_4^3 - \sigma(y)) + O(\|y\|^4),
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
\sigma = {} & (\tfrac{1}{4} + y_1)(\beta y_2 + \gamma y_3 + \delta y_4 - \psi_2 y_1^2 - \psi_3 y_1^3) + (\tfrac{1}{4} + y_2)(\delta y_1 + \beta y_3 + \gamma y_4 - \psi_2 y_2^2 - \psi_3 y_2^3) + (\tfrac{1}{4} + \\
& y_3)(\gamma y_1 + \delta y_2 + \beta y_4 - \psi_2 y_3^2) + (\tfrac{1}{4} + y_4)(\beta y_1 + \gamma y_2 + \delta y_3 - \psi_2 y_4^2 - \psi_3 y_4^3)\\
= {} & -\tfrac{\psi_2}{4}(y_1^2 + y_2^2 + y_3^2 + y_4^2) + (\beta + \delta)(y_1 + y_3)(y_2 + y_4) + 2\gamma(y_1 y_3 + y_2 y_4) + O(\|y\|^3).
\end{aligned}$$

(we used in this computation that we consider the system in restriction to the invariant subspace $y_1 + y_2 + y_3 + y_4 = 0$; as a result, the linear terms in σ vanished).

4.2.1 Diagonalisation

The linearization matrix at the equilibrium $y = 0$ is a circulant:

$$A = \frac{1}{4}\begin{pmatrix} 0 & \beta & \gamma & \delta \\ \delta & 0 & \beta & \gamma \\ \gamma & \delta & 0 & \beta \\ \beta & \gamma & \delta & 0 \end{pmatrix}.$$

Its eigenvectors are

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{e_3} = \mathbf{e_4}^* = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}.$$

We diagonalise this matrix by making the coordinate change

$$y = z\mathbf{e_2} + (u + iw)\mathbf{e_3} + (u - iw)\mathbf{e_4}. \tag{4.6}$$

Note that this sum does not include the vector $\mathbf{e_1}$, as we are interesting only in such y for which $y_1 + y_2 + y_3 + y_4 = 0$.

We can write (4.5) as

$$y_1 = z + 2u, \qquad y_2 = -z - 2w, \qquad y_3 = z - 2u, \qquad y_4 = -z + 2w.$$

Substituting y_1, y_2, y_3, y_4 in terms of u, w, z we have

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 4z^2 + 8u^2 + 8w^2,$$

$$(y_1 + y_3)(y_2 + y_4) = -4z^2,$$

$$y_1 y_3 + y_2 y_4 = 2z^2 - 4u^2 - 4w^2.$$

$$\sigma(y) = -\,\psi_2(z^2 + 2u^2 + 2w^2) - 4(\beta+\delta)z^2 + 4\gamma(z^2 - 2u^2 - 2w^2) + O(|z|^3 + |u|^3 + |w|^3).$$

It would be helpful to use the following identities:

$$y_1^2 - y_2^2 + y_3^2 - y_4^2 = 8(u^2 - w^2),$$

$$y_1^3 - y_2^3 + y_3^3 - y_4^3 = 4z(z^2 + 6(u^2 + w^2)),$$

$$(y_1 - y_3)(y_2 - y_4) = -16uw,$$

$$y_1 y_3 - y_2 y_4 = 4(w^2 - u^2),$$

$$y_1^2 - y_3^2 = 8uz,$$

$$y_4^2 - y_2^2 = -8wz,$$

$$y_1 y_2 - y_3 y_4 = -4z(u + w),$$

$$y_1 y_4 - y_3 y_2 = -4z(u - w).$$

One can check that

$$y \cdot \mathbf{e_2} = 4z,$$

$$y \cdot \mathbf{e_3} = 4u - (4w)i, \;,$$

$$y \cdot \mathbf{e_4} = 4u + (4w)i$$

or

$$z = \frac{1}{4}(y_1 - y_2 + y_3 - y_4), \qquad u = \frac{1}{4}(y_1 - y_3), \qquad w = \frac{1}{4}(y_4 - y_2).$$

Hence, using the above identities, we obtain

$$\frac{dz}{dt} = \frac{1}{4}(y_1 - y_2 + y_3 - y_4)\Big(\frac{\gamma - \beta - \delta}{4} - \sigma(y)\Big) - \frac{\psi_2}{16}(y_1^2 - y_2^2 + y_3^2 - y_4^2) - \frac{4\psi_2 + \psi_3}{16}(y_1^3 - y_2^3 + y_3^3 - y_4^3) +$$

$$+\frac{\beta - \delta}{4}(y_1 - y_3)(y_2 - y_4) + \frac{\gamma}{2}(y_1 y_3 - y_2 y_4) + O(\|y\|^4),$$

$$= \frac{\gamma - \beta - \delta}{4}z - \frac{\psi_2 + 4\gamma}{2}(u^2 - w^2) - 4(\beta - \delta)uw - \frac{\psi_3 - 16(\beta + \delta - \gamma)}{4}z^3 - (4\psi_2 + \frac{3}{2}\psi_3 - 8\gamma)z(u^2 + w^2) +$$

$$+O(z^4 + u^4 + w^4),$$

$$\frac{du}{dt} = -\,\frac{\gamma}{16}(y_1 - y_3) - \frac{\beta - \delta}{16}(y_4 - y_2) - \frac{\psi_2}{16}(y_1^2 - y_3^2) + \frac{\beta}{4}(y_1 y_2 - y_3 y_4) + \frac{\delta}{4}(y_1 y_4 - y_3 y_2) + O(\|y\|^3)$$

$$= -\frac{\gamma}{4}u - \frac{\beta-\delta}{4}w - \frac{\psi_2 + 2(\beta+\delta)}{2}uz - (\beta-\delta)wz + O(|z|^3 + |u|^3 + |w|^3),$$

$$\frac{dw}{dt} = -\frac{\gamma}{16}(y_4 - y_2) + \frac{\beta-\delta}{16}(y_1 - y_3) - \frac{\psi_2}{16}(y_4^2 - y_2^2) + \frac{\beta}{4}(y_1 y_4 - y_2 y_3) + \frac{\delta}{4}(y_3 y_4 - y_1 y_2) + O(\|y\|^3).$$

$$= -\frac{\gamma}{4}w + \frac{\beta-\delta}{4}u + \frac{\psi_2 + 2(\beta+\delta)}{2}wz - (\beta-\delta)uz + O(|z|^3 + |u|^3 + |w|^3).$$

4.2.2 Symmetry

The symmetry $(y_1, y_2, y_3, y_4) \to (y_2, y_3, y_4, y_1)$ means that the system in the (z, u, w)-coordinates is symmetric with respect to the transformation

$$S : (u, w, z) \to (-w, u, -z).$$

In particular, it implies the symmetry with respect to

$$S^2 : (u, w) \to (-u, -w),$$

i.e., the right-hand sides of the equations for $\frac{du}{dt}$ and $\frac{dw}{dt}$ are odd functions of u and w. Hence, they vanish at $u = w = 0$ and the $O(|z|^3 + |u|^3 + |w|^3)$-terms in these equations can be written as

$$O((|u| + |v|)(z^2 + u^2 + w^2)).$$

One could verify symmetry in a similar fashion (as done in section 3.2.2):

1. $(z, u, w) \to (-z, w, -u)$

$$\frac{dz}{dt} = -\frac{\gamma-\beta-\delta}{4}z - \frac{\psi_2 + 4\gamma}{2}(w^2 - u^2) + 4(\beta-\delta)wu + \frac{\psi_3 - 16(\beta+\delta-\gamma)}{4}z^3$$
$$+(4\psi_2 + \frac{3}{2}\psi_3 - 8\gamma)z(u^2 + w^2) + O(z^4 + u^4 + w^4) = -\frac{dz}{dt}$$

$$\frac{du}{dt} = -\frac{\gamma}{4}w + \frac{\beta-\delta}{4}u + \frac{\psi_2+2(\beta+\delta)}{2}wz - (\beta-\delta)uz + O(|z|^3 + |u|^3 + |w|^3) = \frac{dw}{dt}$$

$$\frac{dw}{dt} = \frac{\gamma}{4}u + \frac{\beta-\delta}{4}w + \frac{\psi_2+2(\beta+\delta)}{2}uz + (\beta-\delta)wz + O(|z|^3 + |u|^3 + |w|^3) = -\frac{du}{dt}$$

2. $(-z, w, -u) \to (z, -u, -w)$

$$\frac{dz}{dt} = \frac{\gamma-\beta-\delta}{4}z - \frac{\psi_2+4\gamma}{2}(u^2-w^2) - 4(\beta-\delta)uw - \frac{\psi_3-16(\beta+\delta-\gamma)}{4}z^3$$
$$-(4\psi_2+\frac{3}{2}\psi_3-8\gamma)z(u^2+w^2)+O(z^4+u^4+w^4) = \frac{dz}{dt}$$

$$\frac{du}{dt} = \frac{\gamma}{4}u + \frac{\beta-\delta}{4}w + \frac{\psi_2+2(\beta+\delta)}{2}uz + (\beta-\delta)wz + O(|z|^3+|u|^3+|w|^3) = -\frac{du}{dt}$$

$$\frac{dw}{dt} = \frac{\gamma}{4}w - \frac{\beta-\delta}{4}u - \frac{\psi_2+2(\beta+\delta)}{2}wz + (\delta-\beta)uz + O(|z|^3+|u|^3+|w|^3) = -\frac{dw}{dt}$$

3. $(z, -u, -w) \to (-z, -w, u)$

$$\frac{dz}{dt} = -\frac{\gamma-\beta-\delta}{4}z - \frac{\psi_2+4\gamma}{2}(w^2-u^2) + 4(\beta-\delta)wu + \frac{\psi_3-16(\beta+\delta-\gamma)}{4}z^3$$
$$+(4\psi_2+\frac{3}{2}\psi_3-8\gamma)z(u^2+w^2)+O(z^4+u^4+w^4) = -\frac{dz}{dt}$$

$$\frac{du}{dt} = \frac{\gamma}{4}w - \frac{\beta-\delta}{4}u - \frac{\psi_2+2(\beta+\delta)}{2}wz + (\delta-\beta)uz + O(|z|^3+|u|^3+|w|^3) = -\frac{dw}{dt}$$

$$\frac{dw}{dt} = -\frac{\gamma}{4}u - \frac{\beta-\delta}{4}w - \frac{\psi_2+2(\beta+\delta)}{2}uz - (\beta-\delta)wz + O(|z|^3+|u|^3+|w|^3) = \frac{du}{dt}$$

4. $(-z, -w, u) \to (z, u, w)$

$$\frac{dz}{dt} = \frac{\gamma-\beta-\delta}{4}z - \frac{\psi_2+4\gamma}{2}(u^2-w^2) - 4(\beta-\delta)uw - \frac{\psi_3-16(\beta+\delta-\gamma)}{4}z^3$$
$$-(4\psi_2+\frac{3}{2}\psi_3-8\gamma)z(u^2+w^2)+O(z^4+u^4+w^4) = \frac{dz}{dt}$$

$$\frac{du}{dt} = -\frac{\gamma}{4}u - \frac{\beta-\delta}{4}w - \frac{\psi_2+2(\beta+\delta)}{2}uz - (\beta-\delta)wz + O(|z|^3+|u|^3+|w|^3) = \frac{du}{dt}$$

$$\frac{dw}{dt} = -\frac{\gamma}{4}w + \frac{\beta-\delta}{4}u + \frac{\psi_2+2(\beta+\delta)}{2}wz - (\beta-\delta)uz + O(|z|^3+|u|^3+|w|^3) = \frac{dw}{dt}$$

Hence, our system is symmetric.

4.2.3 Non-Degeneracy

We can now check if the system is Degenerate or not. We find equilibria points for (u, w, z), such that $\left(\frac{du}{dt}, \frac{dw}{dt}, \frac{dz}{dt}\right) = 0$. We see that, when $\frac{du}{dt} = \frac{dw}{dt} = 0$, $(u, w) = 0$, and when

$$\frac{dz}{dt} = \frac{\gamma - \beta - \delta}{4} z - \frac{\psi_2 + 4\gamma}{2}(u^2 - w^2) - 4(\beta - \delta)uw - \frac{\psi_3 - 16(\beta + \delta - \gamma)}{4} z^3$$
$$-(4\psi_2 + \frac{3}{2}\psi_3 - 8\gamma)z(u^2 + w^2) + O(z^4 + u^4 + w^4) = 0,$$

$$z = 0, \pm\sqrt{\frac{(\gamma - \beta - \delta)}{\psi_3 + 16(\gamma - \beta - \delta)}}$$

Thus we achieved our z which depends on parameters γ, β, δ, and ψ_3 which means we do not have a line of equilibrium anymore. Instead, we have three equilibrium points

$$O(0,0,0), O^+\left(0, 0, +\sqrt{\frac{(\gamma - \beta - \delta)}{\psi_3 + 16(\gamma - \beta - \delta)}}\right), \text{ and } O^-\left(0, 0, -\sqrt{\frac{(\gamma - \beta - \delta)}{\psi_3 + 16(\gamma - \beta - \delta)}}\right),$$

lying on this line. This was only achievable due to the fact that non-linearity was introduced in case of generalized replicator equation, which resulted in z not having infinitely many points on a line of equilibrium. Thus, it can be concluded that our system of *generalized* replicator dynamics is ***non-degenerate***.

Chapter 5

The Shimizu-Morioka System

We already know from introduction (Chapter 1) about the breakthrough of Poincaré in the late 1800s when he introduced a new point of view that emphasized qualitative rather than quantitative questions. Poincaré was also the first person to hint the possibility of *chaos*, in which an unpredictable, but a deterministic system exhibited *aperiodic* behavior that depended sensitively on the initial conditions, thereby making it impossible to perform long-term predictions accurately [**14**, **24**].

5.1 Lorenz System

5.1.1 Lorenz's Discovery & Chaos

Chaos really came into focus in 1960s after invention of high-speed computers in the 1950s, when meteorologist Ed Lorenz tried to make a basic computer simulation on earth's atmosphere. He had 12 equations and 12 variables, like temperature, pressure etc., and the computer would print out each time step as a row of 12 numbers, so you could watch how they evolved over time. The breakthrough though came when Lorenz wanted to redo a run, but as a shortcut he entered the numbers from halfway through a previous printout and then he set the computer

calculating. When he saw the results, the new run followed the old one for a short while but then it diverged and pretty soon it was describing a total different state of atmosphere. The real reason for the difference came down to that when Lorenz entered the initial conditions, the printer had rounded to three decimal places whereas the computer calculated with six decimal places. Lorenz kept simplifying his equations down to just three equations where only three variables change over time. But again he got the same kind of behaviour, i.e., if he changed the numbers tiny bit, results diverged dramatically. Lorenz's system displayed what's become known as *sensitive dependence on initial conditions* [**14**, **16**]. No matter how close two initial conditions were chosen, they ended up diverging on totally different trajectories and the results were completely different. It exhibited *chaos* [**10**]. Although there is no universally accepted mathematically definition of chaos, but according to Robert L. Devaney [**7**], for a dynamical system to be chaotic it must satisfy three conditions:
(i) it must be sensitive on initial conditions,
(ii) it must be topologically transitive,
(iii) it must have dense periodic orbits.

Lorenz showed that there was a structure in chaos - when plotted in three dimensions, his solutions took the shape of a butterfly (as one can see in Figure 5.1), later came to be known as the *butterfly effect* [**16**]. Some dynamical systems like logistic map $x \to 4x(1-x)$ are chaotic everywhere, whereas many dynamical systems are chaotic only in a subset of phase space. Of these, the most interesting dynamical system is where trajectories of a large set of initial conditions converged to this chaotic region. Such dynamical systems are said to have an *attractor* [**16**]. For example, in Figure (5.1) we see that trajectories over large set of initial conditions are attracted to this butterfly. Although, we would not go into the details of attractors, but we would discuss a particular type of attractor called the Lorenz attractor.

5.1.2 Lorenz Attractor

Since Lorenz was working with three variables, we can plot the phase of this system in three dimensions. We can pick any point in our state and watch how it evolves. On observing

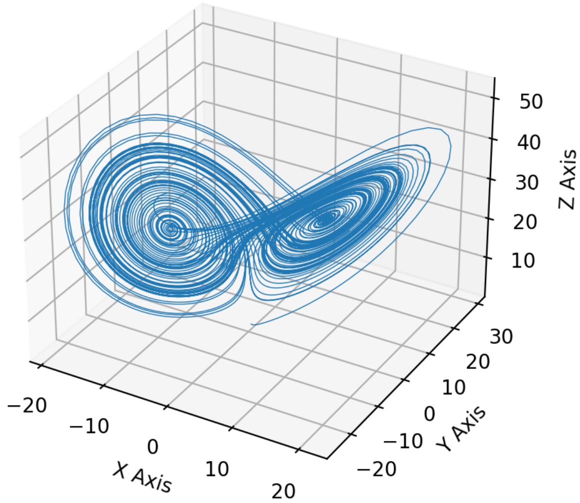

Figure 5.1: Lorenz Attractor ($r = 28$, $\sigma = 10$, $b = 8/3$; $x(0) = 0, y(0) = 1, z(0) = 1.05$)

the trajectories, there was nothing random about the system at all. In fact, the system was completely deterministic. In 1963, Lorenz found a three-dimensional system from a simplified version of this deterministic system used to model convection rolls in atmosphere, called the *Lorenz Equation* [**15**]:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - x$$
$$\dot{z} = y - bz$$

where $\sigma, r, b > 0$ are parameters: σ the *Prandtl number*, r is the *Rayleigh number*, and b has no name. This system has only three variables that could change over time over a wide range of parameters. When Lorenz took the three parameters to be $r = 28$, $\sigma = 10$, and $b = 8/3$, the system exhibited *chaotic behaviour* [**9**]. This is shown on a three-dimensional plot (Figure 5.1), which I plotted using python simulation by taking parameters $r = 28$, $\sigma = 10$, $b = 8/3$; initial values as $x(0) = 0, y(0) = 1, z(0) = 1.05$; simulation time $t = 100[s]$ and step size $\Delta t = 0.01$.

5.1.3 The Shimizu-Morioka System

Our goal is to somehow connect the dots between asymptotic normal form and Lorenz attractors, and Shimizu-Morioka model is one feasible system that could possibly lead us to a Lorenz attractor. The asymptotic normal form for our system near the bifurcation we consider happens to be this *Shimizu-Morioka system* [**21**]. It is represented by a set of three ordinary differential equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-z) - \eta y \\ \dot{z} &= -\alpha z + x^2 \end{aligned} \tag{5.1}$$

This system has one fixed point P_0, located at the origin of the phase space and two fixed points $P_\pm$ located at $(\pm\sqrt{\alpha}, 0, 1)$. It was found that for different parameters (α, η), the system displayed different trajectories. The system was extensively studied [**1**, **21-23**], and it was shown that there exists a region of positive values of (α, η) for which Shimizu-Morioka model (5.1) has a Lorenz attractor [**1**]. When paramters took the values $\alpha = 0.75$, and $\eta = 0.45$, the system produced a "Lorenz-like" chaotic attractor [**21**]. Figure 5.2. illustrates the chaotic behaviour of equation (5.1), which I plotted using python simulation by taking initial values $x(0) = -1, y(0) = 2, z(0) = 1$, parameters as $\alpha = 0.75$, $\eta = 0.45$, simulation time $t = 100[s]$, and step size $\Delta t = 0.01$. Thus, our goal should be to reduce our system to Shimizu-Morioka model. When we reduce our generalized replicator dynamics to asymptotic normal form, we shall show that asymptotic normal forms for such bifurcations can be reduced by re-scaling the phase space and time variables to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x(1-z) - Bx^3 - \eta y \\ \dot{z} &= -\alpha(z - x^2). \end{aligned} \tag{5.2}$$

At $B = 0$, this system will result in the Shimizu-Morioka system itself. To verify this, one can make the transformations $x \to \dfrac{x}{\sqrt{\alpha}}$, $y \to \dfrac{y}{\sqrt{\alpha}}$ in system (5.2) [**25**].

5.1.4 Motivation

The real reason behind bringing Shimizu Morioka into the picture is that once we achieve the asymptotic normal form (section 3.1.2), we could then re-scale our system to Shimizu-

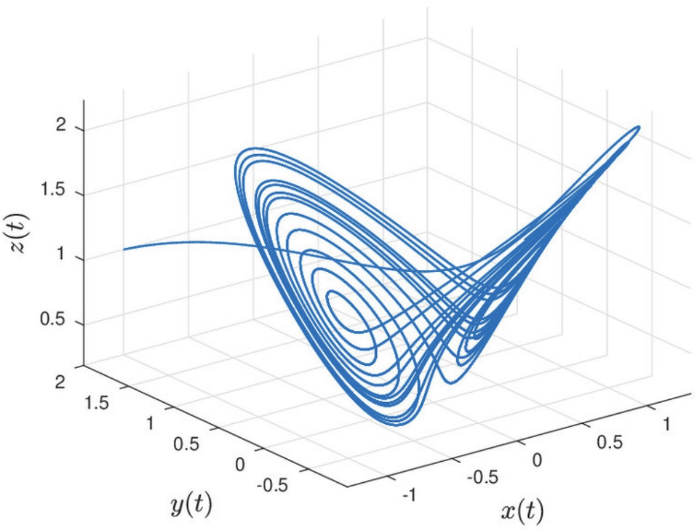

Figure 5.2: Shimizu Morioka Model ($\alpha = 0.75$, $\eta = 0.45$; $x(0) = -1, y(0) = 2, z(0) = 1$)

Morioka system, which is the asymptotic normal form near the bifurcation that we consider. To understand the complex behaviour of trajectories of Shimizu-Morioka model, extensive studies were done by means of computer simulations [**1**, **4**, **21-23**] that investigates the bifurcation of Lorenz-like attractors in Shimizu-Morioka model. In fact, one can see in [**5**] that for open set of parameter values, Shimizu-Morioka system has a Lorenz attractor. Therefore, under certain conditions, Shimizu-Morioka model guarantees a Lorenz Attractor [**22**, **4**]. This way we know if our system has a Lorenz Attractor or not. One thing that we need to ensure is that the signs of parameters in our Shimizu-Morioka model should be same as the signs of parameters in (5.2) or (5.1). Any discrepancies in the sign of parameters will not lead us to a Lorenz attractor.

5.2 Computation

5.2.1 Re-Scaling

By now we have obtained a set of equations (section 4.2.1)

$$\frac{dz}{dt} = \frac{\gamma - \beta - \delta}{4} z - \frac{\psi_2 + 4\gamma}{2}(u^2 - w^2) - 4(\beta - \delta)uw - \frac{\psi_3 - 16(\beta + \delta - \gamma)}{4} z^3 \\ -(4\psi_2 + \frac{3}{2}\psi_3 - 8\gamma)z(u^2 + w^2) + O(z^4 + u^4 + w^4)$$

$$\frac{du}{dt} = -\frac{\gamma}{4}u - \frac{\beta-\delta}{4}w - \frac{\psi_2 + 2(\beta+\delta)}{2}uz - (\beta-\delta)wz + O(|z|^3 + |u|^3 + |w|^3)$$

$$\frac{dw}{dt} = -\frac{\gamma}{4}w + \frac{\beta-\delta}{4}u + \frac{\psi_2 + 2(\beta+\delta)}{2}wz - (\beta-\delta)uz + O(|z|^3 + |u|^3 + |w|^3)$$

We will study the system at small values of γ, β, and δ and in a small neighbourhood of $(u, w, z) = 0$. We, therefore, scale the variables to a small factor: a small neighbourhood in the original variables corresponds to an order 1 neighbourhood in the rescaled variables.

Thus, we consider the region

$$\gamma - \beta - \delta > 0$$

in the parameter space, assume

$$\psi_2 \neq 0 \ \text{ and } \ \psi_3 > 0$$

from now on, and make the following scaling:

$$t \to \frac{2\sqrt{\psi_3}}{\psi_2\sqrt{\gamma-\beta-\delta}}\, t,$$

$$(u, w, z) \to \sqrt{\frac{\gamma-\beta-\delta}{\psi_3}}\, (u, w, z),$$

On rescaling, the system can be written as

$$\frac{du}{dt} = -\left(\frac{\sqrt{\psi_3}\gamma}{2\psi_2\sqrt{\gamma-\beta-\delta}}\right)u - \left(\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}}\right)w - uz + O(\mu(|u| + |v|)),$$

$$\frac{dw}{dt} = -\left(\frac{\sqrt{\psi_3}\gamma}{2\psi_2\sqrt{\gamma-\beta-\delta}}\right)w + \left(\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}}\right)u + wz + O(\mu(|u| + |v|)),$$

$$\frac{dz}{dt} = (w^2 - u^2)(1 + O(\mu)) +$$
$$\frac{\sqrt{\psi_3(\gamma-\beta-\delta)}}{2\psi_2}\left(z - z^3 - \frac{32\psi_2}{\psi_3}\left(\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}}\right)uw - \frac{16\psi_2 + 6\psi_3}{\psi_3}z(u^2 + w^2)\right) + O(\mu^2),$$

Note that β, δ, γ are very small numbers, which means $\frac{\sqrt{\psi_3}\gamma}{2\psi_2\sqrt{\gamma-\beta-\delta}}$ and $\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}}$ will be non-small and when β, δ, and γ tends to 0, $\frac{\sqrt{\psi_3}\gamma}{2\psi_2\sqrt{\gamma-\beta-\delta}}$ and $\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}}$ converge to some non-small constant, say $\hat{a}$ and $\hat{b}$ respectively. Denoting

$$\frac{\sqrt{\psi_3(\gamma-\beta-\delta)}}{2\psi_2} = \mu$$

$$\frac{\sqrt{\psi_3}\gamma}{2\psi_2\sqrt{\gamma-\beta-\delta}} = \hat{a}$$

$$\frac{\sqrt{\psi_3}(\beta-\delta)}{2\psi_2\sqrt{\gamma-\beta-\delta}} = \hat{b}$$

as $(\beta, \delta, \gamma) \to (0, 0, 0)$, our system takes the form

$$\begin{aligned} \frac{dz}{dt} &= (w^2-u^2)(1+O(\mu)) + \ \mu(z - z^3 - \tfrac{32\psi_2}{\psi_3}\hat{b}uw - \tfrac{16\psi_2+6\psi_3}{\psi_3}z(u^2+w^2)) + O(\mu^2), \\ \frac{du}{dt} &= -\hat{a}u - \hat{b}w - uz + O(\mu(|u|+|v|)), \\ \frac{dw}{dt} &= -\hat{a}w + \hat{b}u + wz + O(\mu(|u|+|v|)), \end{aligned} \tag{5.3}$$

Note that $\mu > 0$ is small, while $\hat{a}$ and $\hat{b}$ can be arbitrary (we assume that they stay uniformly bounded as $\mu \to 0$)

5.2.2 Equilibrium Points

Now, we find an equilibria points for z, where $u = w = 0$

$$\frac{dz}{dt} = (w^2-u^2)(1+O(\mu)) + \ \mu(z - z^3 - \tfrac{32\psi_2}{\psi_3}\hat{b}uw - \tfrac{16\psi_2+6\psi_3}{\psi_3}z(u^2+w^2)) + O(\mu^2) = 0,$$

we get:

$$z = 0, +1, -1$$

We have three equilibrium $O(0,0,0), O^+(0,0,1)$, and $O^-(0,0,-1)$.

When $(\beta, \delta, \gamma) = (0,0,0)$

$$\frac{dz}{dt} = (w^2-u^2),$$

$$\frac{du}{dt} = -\hat{a}u - \hat{b}w - uz,$$
$$\frac{dw}{dt} = -\hat{a}w + \hat{b}u + wz,$$

5.2.3 Co-ordinate Shift

Following the algorithm (section 3.1.2), we now shift the co-ordinates from $z_{old} \to z_{new} + 1$. On performing the shift we obtain:

$$\frac{dz}{dt} = (w^2 - u^2)(1 + O(\mu)) + \mu((z+1) - (z+1)^3 - \frac{32\psi_2}{\psi_3}\hat{b}uw - \frac{16\psi_2 + 6\psi_3}{\psi_3}(z+1)(u^2 + w^2)) + O(\mu^2),$$

$$\frac{du}{dt} = -\hat{a}u - \hat{b}w - u(z+1) + O(\mu(|u| + |v|)),$$
$$\frac{dw}{dt} = -\hat{a}w + \hat{b}u + w(z+1) + O(\mu(|u| + |v|)),$$

Note that $\mu > 0$ is small, while $\hat{a}$ and $\hat{b}$ can be arbitrary (we assume that they stay uniformly bounded as $\mu \to 0$)

Separating u, w, z, uz, wz terms

$$\frac{du}{dt} = -(\hat{a}+1)u - \hat{b}w - uz + O(\mu(|u| + |v|))$$
$$\frac{dw}{dt} = -(\hat{a}-1)w + \hat{b}u + wz + O(\mu(|u| + |v|))$$
$$\frac{dz}{dt} = -2\mu z + (w^2 - u^2)(1 + O(\mu)) - \mu(3z^2 + z^3 + \frac{32\psi_2}{\psi_3}\hat{b}uw + \frac{16\psi_2 + 6\psi_3}{\psi_3}(u^2 + w^2)(z+1)) + O(\mu^2)$$

5.2.4 Linearization and Three 0 Eigenvalues

Considering only the linear terms for u, w, z in the obtained equation, consider a linearised matrix M

$$M = \begin{pmatrix} -(\hat{a}+1) & -b & 0 \\ b & -(\hat{a}-1) & 0 \\ 0 & 0 & -2\mu \end{pmatrix}$$

Eigenvalues are as follows:

$$\lambda_1 = -\hat{a} - \sqrt{1 - \hat{b}^2}$$
$$\lambda_2 = -\hat{a} + \sqrt{1 - \hat{b}^2}$$
$$\lambda_3 = -2\mu$$

For the system to have three 0 Eigenvalues, $\lambda_1, \lambda_2, \lambda_3 = 0$, and $Tr(M) = 0$. Which is satisfied when:

$$\mu = 0 \qquad \hat{a} = 0 \qquad \hat{b} = \pm 1$$

Note that our system has now been reduced to Z_2 from Z_4 system. Let us now take

$$\hat{a} = \alpha \qquad \hat{b} = 1 + \zeta$$

where α and ζ are very small numbers. Substituting these values in our system and separating u, w, z, uz, wz terms, we get

$$\begin{aligned}
\frac{du}{dt} &= -(\alpha+1)u - (1+\zeta)w - uz + O(\mu(|u| + |v|)) \\
\frac{dw}{dt} &= -(\alpha-1)w + (1+\zeta)u + wz + O(\mu(|u| + |v|)) \\
\frac{dz}{dt} &= -2\mu z + (w^2 - u^2)(1 + O(\mu)) \ - \mu(3z^2 + z^3 + \frac{32\psi_2}{\psi_3}(1+\zeta)uw \\
&\quad + \frac{16\psi_2 + 6\psi_3}{\psi_3}(u^2 + w^2)(z+1)) + O(\mu^2)
\end{aligned}$$

Note that we are considering terms upto second order, and all other higher order terms are irrelevant as they are very small and tend to disappear. For now, we do not want to completely neglect these higher order terms, so we keep them in the equation to avoid any small errors that might depend on these higher order terms.

5.2.5 Introducing a New Variable

We now assign a new variable where we let $\dfrac{du}{dt} = v$

$$v = -(\alpha+1)u - (1+\zeta)w - uz + O(\mu(|u|+|w|))$$

$$\frac{dv}{dt} = -(\alpha+1)v - (1+\zeta)\frac{dw}{dt} - \left(u\frac{dz}{dt} + zv\right) + O(\mu(|u|+|w|))$$

One can check the following derivations in terms of u, v, z:

$$w = \frac{-u - \alpha u - v - uz + O(\mu(|u|+|v|)}{1+\zeta}$$

$$\frac{dw}{dt} = -(\alpha-1)\left(\frac{-u-\alpha u - v - uz + O(\mu(|u|+|w|)}{1+\zeta}\right) + (1+\zeta)u + \left(\frac{-u-\alpha u - v - uz + O(\mu(|u|+|w|)}{1+\zeta}\right) z + O(\mu(|u|+|w|))$$

$$= \frac{(\alpha^2+\zeta^2+2\zeta)u + (\alpha-1)v - 2uz - vz - uz^2 + O(\mu(|u|+|v|)}{\zeta+1}$$

$$\frac{dz}{dt} = -2\mu z + \left(\left(\frac{-u-\alpha u - v - uz + O(\mu(|u|+|v|)}{1+\zeta}\right)^2 - u^2\right)(1+O(\mu)) - \frac{32\psi_2}{\psi_3}\left(\frac{-u-\alpha u - v - uz + O(\mu(|u|+|v|)}{1+\zeta}\right)(1+\zeta)\mu u \; - \mu(3z^2 + z^3 + \frac{16\psi_2 + 6\psi_3}{\psi_3}\left(u^2 + \left(\frac{-u-\alpha u - v - uz + O(\mu(|u|+|v|))}{1+\zeta}\right)^2\right)(z+1)) + O(\mu^2)$$

$$= -2\mu z + \frac{1}{(1+\zeta)^2}\left((\alpha+1)^2 - (\zeta+1)^2 + \frac{32\psi_2\mu}{\psi_3}(-\tfrac{\alpha}{2} + \zeta + 2\alpha\zeta + \tfrac{\zeta^2}{2} + \alpha\beta^2) - 12\mu(1+\zeta+\tfrac{3}{2}\alpha + \tfrac{\zeta^2}{2})\right)u^2 + \frac{1}{(1+\zeta)^2}\left(1 + \frac{32\psi_2\mu}{\psi_3}(\tfrac{1}{2} + \zeta^2 + 2\zeta) - 6\mu\right)v^2 + \frac{1}{(1+\zeta)^2}\left(2 + 2\alpha + \frac{32\psi_2\mu}{\psi_3}(1+\alpha+4\zeta+4\alpha\zeta+2\zeta^2+2\alpha\zeta^2) - 12\mu - 12\mu\alpha\right)uv - \mu\left(3z^2 + z^3 + O(|z|(u+v))\right) + O(\mu^2)$$

Plugging in $w, \frac{dz}{dt}, \frac{dw}{dt}$ in terms of u, v, z, we have $\frac{dv}{dt}$ in terms of u, v, z:

$$\frac{dv}{dt} = -(\alpha^2+\zeta^2+2\zeta)u - 2\alpha v - 2(1-\mu)uz + O(\mu(|u|+|v|)$$

Putting $\frac{du}{dt}, \frac{dv}{dt}, \frac{dz}{dt}$ together, we achieve our *Asymptotic Normal Form*:

$$
\begin{aligned}
\frac{du}{dt} &= v \\
\frac{dv}{dt} &= -2\zeta u - 2\alpha v - 2uz + O(\mu(|u| + |v|)) \\
\frac{dz}{dt} &= -2\mu z - 12\mu u^2 + 2uv + O(\mu(z^2)) + O(v^2) + O(\mu^2)
\end{aligned}
\tag{5.4}
$$

We have tried to eliminate the higher order terms while keeping only linear and quadratic terms. We would also get rid of quadratic parameter if there is a linear form of the same as they would be really small to be considered and make our equation unnecessary tedious.

5.3 Results

We could say that $\frac{d}{dt}u^2 = 2uv$. Therefore, we could shift our z to $-u^2$ which in return can help us eliminate uv term from $\frac{dz}{dt}$. Let us now consider co-ordinate shift of $z_{new} = z_{old} - u^2$ and $\frac{dz_{new}}{dt} = \frac{dz_{old}}{dt} - 2uv$. Then we have our system as

$$
\begin{aligned}
\frac{du}{dt} &= v \\
\frac{dv}{dt} &= -2\zeta u - 2\alpha v - 2uz - 2u^3 + O(\mu(|u| + |v|)) \\
\frac{dz}{dt} &= -2\mu z - 14\mu u^2 + O(\mu(z^2)) + O(v^2) + O(\mu^2)
\end{aligned}
\tag{5.5}
$$

Upon further re-scaling, (5.5) will transform into Shimizu Morioka system but the parameters will have wrong signs in order to exhibit an attractor. It important to note that for the Shimizu Morioka system to exhibit a Lorenz Attractor, the coefficient of u and v in $\frac{dv}{dt}$, and z and u^2 in $\frac{dz}{dt}$ should have opposite signs. Upon comparing (5.2) and (5.5), we observe that the coefficient of u and v in $\frac{dv}{dt}$ (i.e., -2ζ and -2α respectively), and coefficients of z and u^2 in $\frac{dz}{dt}$ (i.e., -2μ and -14μ respectively) holds the same sign, despite re-scaling. Therefore, we have wrong parameters which leads us to conclude that *Lorenz Attractor is **not** present in our Generalized Replicator Dynamics.*

Chapter 6

Lotka-Volterra Type System

6.1 Definition

The starting point of our analysis of multiple species growth (or decay) was given by an Italian mathematician called Vito Volterra [**29**]. In 1924, after the first world war, the proportion of predator fishes caught in the Upper Adriatic was found to be higher than they were in the years before the war, whereas the proportion of prey fishes was found to be considerably lower. Seeing this, an Italian biologist D'Ancona, who later become Volterra's son in law, introduced Volterra to this problem. Volterra then proposed a differential equation for the growth and decay of predator and prey fishes, which later became a part of classic literature in mathematical ecology. He assumed that the rate of prey fishes decreases linearly as a function of the predator fish density, leading the prey equation [**26**] to be

$$\dot{x}/x = a - by, \qquad a, b > 0.$$

and the rate of predator fishes increases linearly as a function of the prey fish density, leading the predator equation [**26**] to be

$$\dot{y}/y = -c + dx, \qquad c, d > 0.$$

where x is the density of prey fishes and y is the density of predator fishes. Volterra combined these two differential equations to get a dynamical system:

$$\begin{aligned}\dot{x} &= x(a - by)\\ \dot{y} &= y(-c + dx).\end{aligned} \tag{6.1}$$

called the *Lotka Volterra Model* for predator-prey interactions [**28**, **26**]. Not just the predator-prey, this model could be extended to multiple species interaction [**17**] as well.

Now consider a population growth of n species and denote x_i as the density of i-th species, where $i = 1, ..., n$. Let b_i be the constant intrinsic growth (or decay) rate for $i-$th species if there exists no predator. If $(b_i + a_{ii}x_i)$ describes growth of $i-$th sepcies in the absence of other species, and $a_{ij}x_j$ represents effects on the growth of $i-$th species from $j-$th species (i.e., $i \neq j$), then the rate of growth of proportion of $i-$th species can be determined by:

$$\begin{aligned}\frac{\dot{x}_i}{x_i} &= b_i + a_{ii}x_i + \textstyle\sum_{i\neq j} a_{ij}x_j, \quad i = 1, ..., n,\\ \dot{x}_i &= x_i(b_i + \textstyle\sum_{j=1}^{n} a_{ij}x_j), \quad i = 1, ..., n.\end{aligned} \tag{6.2}$$

This is called the *General Lottka-Volterra System* for population of n species [**26**]. Matrix $A = (a_{ij})$ is called the interaction matrix.

6.2 Motivation

In chapter 5, we proved the Shimizu-Morioka system, and saw that the control parameters of our system (5.5) does not hold same values as control parameters in (5.2), which meant that there was no sign of a Lorenz Attractor. Therefore, we further generalize our replicator equation to something called the *Lotka-Volterra equation*. Replicator equation is basically a derivation of the Lotka-Volterra equation. In 1998, Hofbauer and Sigmund showed that replicator equation (2.2) is formally equivalent to the Lotka-Volterra equations of competition [**12**]. It is known that the replicator equation is a cubic equation on the compact set $\mathbf{S}_n$, while the Lotka-Volterra equation is quadratic on $\mathbb{R}^n_+$, therefore, the replicator equation (2.2) in n variables $x_1, ..., x_n$ is equivalent to the Lotka-Volterra equation (6.2) in $n-1$ variables $x_1, ..., x_{n-1}$, and results about Lotka-Volterra equations can therefore be carried over to the replicator equation and vice-versa [**13**]. Hence, we consider a system of *Lotka-Volterra-type* equations of the form:

$$\dot{x}_i = x_i(f(x_i) - \textstyle\sum_{i \neq j} a_{ij} x_j), \quad i = 1, \ldots, n, \tag{6.3}$$

and check whether it exhibits a Lorenz Attractor. The reason I say Lotka-Volterra 'type' is because in our original system f is non-linear, whereas f in (6.2) is linear ($b_i + a_{ii}x_i$). This is because in our original system, interaction between two species is linear but its overall evolution is non-linear, and sum of all the species is 1. As a result of which we do not have the standard Lotka-Volterra system (6.2), but we do have a system which is Lotka-Volterra *type*. Therefore, from now on, we will be working with a system of the form (6.3).

6.3 Computation

Consider a 4 species *Lotka-Volterra like system* :

$$\dot{x}_i = x_i(f(x_i) - \sum_{i \neq j} a_{ij} x_j), \quad i = 1, \ldots, 4,$$

where f is a nonlinear function, and the coefficients a_{ij} are chosen such that the matrix $A = (a_{ij})_{i,j=1,\ldots,4}$ is a circulant

$$A = \begin{pmatrix} 0 & \beta & \gamma & \delta \\ \delta & 0 & \beta & \gamma \\ \gamma & \delta & 0 & \beta \\ \beta & \gamma & \delta & 0 \end{pmatrix}.$$

This system is symmetric with respect to the cyclic permutation $(x_1, x_2, x_3, x_4) \to (x_2, x_3, x_4, x_1)$.

Let $x = (x^*, x^*, x^*, x^*)$ with $x^* > 0$ be an equilibirum of this system, i.e.,

$$f(x^*) = (\beta + \gamma + \delta)x^*.$$

By scaling x, we can always assume $x^* = 1$. Put the equilibrium at the origin, i.e., let $y = x - (1, 1, 1, 1)$. The system takes the form

$$\begin{aligned}
\dot{y}_1 &= (1 + y_1)(f_1 y_1 + \beta y_2 + \gamma y_3 + \delta y_4 + f_2 y_1^2 + f_3 y_1^3 + O(y_1^4)),\\
\dot{y}_2 &= (1 + y_2)(\delta y_1 + f_1 y_2 + \beta y_3 + \gamma y_4 + f_2 y_2^2 + f_3 y_2^3 + O(y_2^4)),\\
\dot{y}_3 &= (1 + y_3)(\gamma y_1 + \delta y_2 + f_1 y_3 + \beta y_4 + f_2 y_3^2 + f_3 y_3^3 + O(y_2^4)),\\
\dot{y}_4 &= (1 + y_4)(\beta y_1 + \gamma y_2 + \delta y_3 + f_1 y_4 + f_2 y_4^2 + f_3 y_4^3 + O(y_2^4)),
\end{aligned} \tag{6.4}$$

where f_k are the Taylor coefficients of f at 1.

6.3.1 Linearization & Diagonalization

The equilibrium is at zero, and the linearization matrix at $y = 0$ is a circulant:

$$\begin{pmatrix} f_1 & \beta & \gamma & \delta \\ \delta & f_1 & \beta & \gamma \\ \gamma & \delta & f_1 & \beta \\ \beta & \gamma & \delta & f_1 \end{pmatrix}.$$

Its eigenvectors are

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{e_3} = \mathbf{e_4}^* = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}.$$

We diagonalise this matrix by making the coordinate change

$$y = Y\mathbf{e_1} + z\mathbf{e_2} + (u + iw)\mathbf{e_3} + (u - iw)\mathbf{e_4}. \tag{6.5}$$

We can write (6.5) as

$$y_1 = Y + z + 2u, \qquad y_2 = Y - z - 2w, \qquad y_3 = Y + z - 2u, \qquad y_4 = Y - z + 2w.$$

The inverse transformation is

$$Y = \frac{1}{4}(y_1 + y_2 + y_3 + y_4), \qquad z = \frac{1}{4}(y_1 - y_2 + y_3 - y_4), \qquad u = \frac{1}{4}(y_1 - y_3), \qquad w = \frac{1}{4}(y_4 - y_2).$$

Thus

$$\begin{aligned}
\frac{dY}{dt} &= \sigma Y + \tfrac{\psi_2}{4}(y_1^2 + y_2^2 + y_3^2 + y_4^2) + \tfrac{\beta+\delta}{4}(y_1 + y_3)(y_2 + y_4) + \tfrac{\gamma}{2}(y_1y_3 + y_2y_4) + O(\|y\|^3),\\
\frac{dz}{dt} &= \lambda z + \tfrac{\psi_2}{4}(y_1^2 - y_2^2 + y_3^2 - y_4^2) + \tfrac{\psi_3}{4}(y_1^3 - y_2^3 + y_3^3 - y_4^3) + \tfrac{\beta-\delta}{4}(y_1 - y_3)(y_2 - y_4) + \tfrac{\gamma}{2}(y_1y_3 - y_2y_4) + O(\|y\|^4),\\
\frac{du}{dt} &= \alpha u - \omega w + \tfrac{\psi_2}{4}(y_1^2 - y_3^2) + \tfrac{\beta}{4}(y_1y_2 - y_3y_4) + \tfrac{\delta}{4}(y_1y_4 - y_3y_2) + O(\|y\|^3),\\
\frac{dw}{dt} &= \omega u + \alpha w + \tfrac{\psi_2}{4}(y_4^2 - y_2^2) + \tfrac{\beta}{4}(y_1y_4 - y_3y_2) + \tfrac{\delta}{4}(y_3y_4 - y_1y_2) + O(\|y\|^3),
\end{aligned}$$

where $\psi_2 = f_1 + f_2$, $\psi_3 = f_1 + f_3$, and $\sigma = f_1 + \beta + \gamma + \delta$, $\lambda = f_1 - \beta + \gamma - \delta$, $\alpha = f_1 - \gamma$, $\omega = \beta - \delta$.

One can check that

$$\begin{aligned}
y_1^2 + y_2^2 + y_3^2 + y_4^2 &= 4Y^2 + 4z^2 + 8u^2 + 8w^2,\\
(y_1 + y_3)(y_2 + y_4) &= 4Y^2 - 4z^2,\\
y_1y_3 + y_2y_4 &= 2Y^2 + 2z^2 - 4u^2 - 4w^2,\\
y_1^2 - y_2^2 + y_3^2 - y_4^2 &= 8(Yz + u^2 - w^2),\\
y_1^3 - y_2^3 + y_3^3 - y_4^3 &= 12Y^2z + 24Y(u^2 - w^2) + 4z(z^2 + 6(u^2 + w^2)),\\
(y_1 - y_3)(y_2 - y_4) &= -16uw,\\
y_1y_3 - y_2y_4 &= 4(Yz + w^2 - u^2),\\
y_1^2 - y_3^2 &= 8u(Y + z),\\
y_4^2 - y_2^2 &= 8w(Y - z),\\
y_1y_2 - y_3y_4 &= 4Y(u - w) - 4z(u + w),\\
y_1y_4 - y_3y_2 &= 4Y(u + w) - 4z(u - w).
\end{aligned}$$

Plugging in these identities give us

$$
\begin{aligned}
\frac{dY}{dt} &= \sigma Y + (\psi_2+\beta+\delta+\gamma)Y^2 + (\psi_2-\beta-\delta+\gamma)z^2 + 2(\psi_2-\gamma)(u^2+2w^2) \\
&\quad +O(|Y|^3+|z|^3+|u|^3+|w|^3) \\
\frac{dz}{dt} &= \lambda z + 2(\psi_2+\gamma)Yz + 2(\psi_2-\gamma)(u^2-w^2) + \psi_3 z(z^2+6(u^2+w^2)) - 4(\beta-\delta)uw \\
&\quad +O(|Y|^3+|Y|(u^2+w^2)+Y^2|z|+z^4+u^4+w^4) \\
\frac{du}{dt} &= \alpha u - \omega w + (2\psi_2-\beta-\delta)uz - (\beta-\delta)zw + O(|Y|^3+|Y|(|u|+|w|)+|z|^3+|u|^3+|w|^3) \\
\frac{dw}{dt} &= \omega u + \alpha w - (2\psi_2-\beta-\delta)wz - (\beta-\delta)zu + O(|Y|^3+|Y|(|u|+|w|)+|z|^3+|u|^3+|w|^3)
\end{aligned}
\tag{6.6}
$$

yielding the following linearization matrix

$$
\begin{pmatrix}
\sigma & 0 & 0 & 0 \\
0 & \alpha & -\omega & 0 \\
0 & \omega & \alpha & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix}
$$

with eigenvalues:

$$\lambda_1 = \lambda, \qquad \lambda_2 = \sigma, \qquad \lambda_3 = \alpha + i\omega, \qquad \lambda_4 = \alpha - i\omega.$$

In order for the system to have three zero eigenvalues, $\lambda = \omega = \alpha = 0$ and σ be some real number. Since, we want stability, $\sigma < 0$. It can be verified using substitution that:

$$\sigma = 4f_1 < 0$$

Thus, we have three zero and one negative eigenvalue $\sigma < 0$. The system has an attractive invariant centre manifold in this case, which can be written as

$$Y = cz^2 + d(u^2+w^2) + O(|z|^3+|u|^3+|w|^3), \tag{6.7}$$

where c and d are coefficients to be determined. We did not write other quadratic terms in the expansion (6.7) because the system has a symmtery, and the centre manifold is known to inherit the symmetry of the system. The symmetry with respect to the permutation $(y_1, y_2, y_3, y_4) \to (y_2, y_3, y_4, y_1)$ means that the system in the (Y, z, u, w)-coordinates is symmetric with respect to the transformation

$$S : (u, w, z) \to (-w, u, -z).$$

the right-hand side of (6.7) is the general form of the expansion invariant with respect to S.

We find c and d by substituting (6.7) into the equation for $\frac{dY}{dt}$: since the center manifold is invariant with respect to our system, the result of such substitution must be an identity. We have

$$\begin{gathered}
2cz\frac{dz}{dt} + 2d(u\frac{du}{dt} + w\frac{dw}{dt}) = \\
\sigma(cz^2 + d(u^2 + w^2)) + (\psi_2 - \beta - \delta + \gamma)z^2 + 2(\psi_2 - \gamma)(u^2 + 2w^2) + O(|z|^3 + |u|^3 + |w|^3), \\
2c\lambda z^2 + 2d\alpha(u^2 + w^2) = \\
\sigma(cz^2 + d(u^2 + w^2)) + (\psi_2 - \beta - \delta + \gamma)z^2 + 2(\psi_2 - \gamma)(u^2 + 2w^2) + O(|z|^3 + |u|^3 + |w|^3),
\end{gathered}$$

which implies

$$c = -\frac{\psi_2 - \beta - \delta + \gamma}{\sigma - 2\lambda}, \qquad d = -2\frac{\psi_2 - \gamma}{\sigma - 2\alpha}.$$

Now, we can substitute (6.7) into the equations for $\frac{dz}{dt}$, $\frac{du}{dt}$, and $\frac{dw}{dt}$ to obtain the restriction of our system to the centre manifold:

$$\begin{aligned}
\frac{dz}{dt} &= \lambda z + p(u^2 - w^2) + z(hz^2 + g(u^2 + w^2)) - 4\omega uw + O(z^4 + u^4 + w^4), \\
\frac{du}{dt} &= \alpha u - \omega w + quz - 4\omega zw + O(|z|^3 + |u|^3 + |w|^3), \\
\frac{dw}{dt} &= \omega u + \alpha w - qwz - 4\omega zu + O(|z|^3 + |u|^3 + |w|^3),
\end{aligned}$$

where

$$h = \psi_3 - 2\frac{(\psi_2 + \gamma)(\psi_2 - \beta - \delta + \gamma)}{\sigma - 2\lambda},$$

$$g = 6\psi_3 - 4\frac{(\psi_2 + \gamma)(\psi_2 - \gamma)}{\sigma - 2\alpha},$$

$$p = 2(\psi_2 - \gamma), \qquad q = 2\psi_2 - \beta - \delta.$$

Recall that at the bifurcation moment we have $f_1 = \beta = \gamma = \delta$ and $\sigma = 4f_1$. Therefore, since $\psi_3 = f_1 + f_3$ and $\psi_2 = f_1 + f_2$, we have, when $\lambda = \alpha = \omega = 0$,

$$\begin{aligned} h &= f_1 + f_3 - f_2 - \tfrac{f_2^2}{2f_1}, \\ g &= 6f_1 + 6f_3 - 2f_2 - \tfrac{f_2^2}{f_1}, \\ p &= q = 2f_2. \end{aligned} \tag{6.8}$$

6.3.2 Symmetry

Note that the symmetry with respect to $S : (u, w, z) \to (-w, u, -z)$ implies the symmetry with respect to

$$S^2 : (u, w) \to (-u, -w),$$

i.e., the right-hand sides of the equations for $\frac{du}{dt}$ and $\frac{dw}{dt}$ are odd functions of u and w. Hence, they vanish at $u = w = 0$ and the $O(|z|^3 + |u|^3 + |w|^3)$-terms in these equations can be written as

$$O((|u| + |w|)(z^2 + u^2 + w^2)).$$

More precisely, taking into account the symmetry with respect to S, we write these terms in the form

$$s_1 z^2 u + s_2 z^2 w + O(|u|^3 + |w|^3 + |z|^3(|u| + |w|))$$

in the equation for $\frac{du}{dt}$ and

$$-s_2 z^2 u + s_1 z^2 w + O(|u|^3 + |w|^3 + |z|^3(|u| + |w|))$$

in the equation for $\frac{dv}{dt}$, where $s_{1,2}$ are some coefficients.

Also, the symmetry with respect to S implies that the right-hand side of the equation for $\frac{dz}{dt}$ is an odd function of z at $(u, v) = 0$, hence it does not contain the term z^4. Therefore, the $O(u^4 + w^4 + z^4)$ term in this equation can be written as

$$O(u^4 + w^4 + (|u| + |v|)|z|^3 + |z|^5).$$

Summarizing: our permutation-symemtric 4-species dynamics near the symmetric equilibrium undergoing the triple-zero bifurcation is described by the equation of the form

$$
\begin{aligned}
\frac{dz}{dt} &= \lambda z + p(u^2 - w^2) + z(hz^2 + g(u^2 + w^2)) - 4\omega uw + O(u^4 + w^4 + (|u| + |v|)|z|^3 + |z|^5),\\
\frac{du}{dt} &= (\alpha + s_1 z^2)u - (\omega - s_2 z^2)w + quz - 4\omega zw + O(|u|^3 + |w|^3 + |z|^3(|u| + |w|)),\\
\frac{dw}{dt} &= (\alpha + s_1 z^2)w + (\omega - s_2 z^2)u - qwz - 4\omega zu + O(|u|^3 + |w|^3 + |z|^3(|u| + |w|)).
\end{aligned}
\tag{6.9}
$$

6.3.3 Reducing to Shimizu-Morioka System

We begin by making the following scaling to (6.9):

$$
t \to \frac{\sqrt{|h|}}{\sqrt{\lambda}}\, t, \qquad\qquad (u, w, z) \to \sqrt{\frac{\lambda}{|h|}}\,(u, w, z).
$$

The system takes the form

$$
\begin{aligned}
\frac{dz}{dt} &= p(u^2 - w^2) + \ \mu(z - z^3 - \tfrac{4\hat{b}}{|h|}uw + \tfrac{g}{|h|}z(u^2 + w^2)) + \mu^2 O(\mu + |u| + |v|),\\
\frac{du}{dt} &= (\hat{a} + \mu\hat{s}_1 z^2)u - (\hat{b} + \mu\hat{s}_2 z^2)w + quz + \mu O(|u|^3 + |w|^3) + \mu^2 z^3 O(|u| + |w|),\\
\frac{dw}{dt} &= (\hat{a} + \mu\hat{s}_1 z^2)w + (\hat{b} + \mu\hat{s}_2 z^2))u - qwz + \mu O(|u|^3 + |w|^3) + \mu^2 z^3 O(|u| + |w|),
\end{aligned}
\tag{6.10}
$$

where we denote

$$
\mu = \sqrt{\lambda|h|},
$$

$$
\hat{a} = \frac{\alpha\sqrt{|h|}}{\sqrt{\lambda}},
$$

$$
\hat{b} = \frac{\omega\sqrt{|h|}}{\sqrt{\lambda}}.
$$

Note that $\mu > 0$ is small, while $\hat{a}$ and $\hat{b}$ can be arbitrary (we assume that they stay uniformly bounded as $\mu \to 0$).

Theorem. *For any sufficiently small $\mu > 0$ there exists an open region in the $(\hat{a}, \hat{b})$-plane such that for $(\hat{a}, \hat{b})$ from this region the system has Lorenz attractor.*

For a proof, we note, first, that system (6.10) has an equilibrium at $(u, w) = 0$ and $z = z^* = 1 + O(\mu^2)$. We shift the coordinate origin to this point, i.e., make the transformation $z \to z + z^*$. The resulting system is

$$\begin{aligned}
\frac{dz}{dt} &= p(u^2 - w^2) + \mu(-2z - 4\tfrac{b+q}{|h|}uw + \tfrac{g}{|h|}(u^2 + w^2)) + O(\mu^2 + \mu z^2 + \mu|z|(u^2 + w^2)),\\
\frac{du}{dt} &= (a - q)u - (b + q)w + quz + \mu O((|u| + |w|)(u^2 + w^2 + \mu + |z|)),\\
\frac{dw}{dt} &= (a + q)w + (b + q)u - qwz + \mu O((|u| + |w|)(u^2 + w^2 + \mu + |z|)),
\end{aligned} \tag{6.11}$$

where we denote

$$a = \hat{a} + \hat{s}_1\mu, \qquad b = \hat{b} + \hat{s}_2\mu - q.$$

We will consider the case of small a and b from now on. Let us introduce a new variable v (just like we did in chapter 5.2.5), such that

$$v = -(q - a)u - (q + b)w + quz + \mu O((|u| + |w|)(u^2 + w^2 + \mu + |z|)) = \frac{du}{dt}.$$

Then

$$w = -\frac{q - a}{q + b}u - \frac{1}{q + b}v + \frac{q}{q + b}uz + \mu O((|u| + |v|)(u^2 + v^2 + \mu + |z|)).$$

So

$$\frac{dv}{dt} = -(q - a)v - (q + b)\frac{dw}{dt} + q(u\frac{dz}{dt} + zv) + \mu O((|u| + |v|)(u^2 + v^2 + \mu + |z|)) =$$

$$-(q-a)v - w(q+b)(q+a) - u(q+b)^2 + q(q+b)wz + qzv + pqu(u^2 - w^2) + \mu O((|u| + |v|)(u^2 + v^2 + \mu + |z|)).$$

Substituting the expression for w as a function of (u, v, z) in the right-hand sides, we find that the system takes the *Asymptotic Normal Form*:

$$\begin{aligned}
\frac{du}{dt} &= v,\\
\frac{dv}{dt} &= (2a + O(\mu^2))v - u(2qb + O(a^2 + b^2 + \mu^2)) - (2q^2 + O(|a| + |b| + \mu))uz + O(|v|(|z| + u^2) +\\
&\qquad |u|v^2 + uz^2 + \mu|u|^3),\\
\frac{dz}{dt} &= -(2p + O(|a| + |b| + \mu))uv + \mu(-2z + (\tfrac{4q+2g}{|h|} + O(|a| + |b| + \mu))u^2) + O(\mu^2 + \mu z^2 + \mu|z|u^2 + v^2),
\end{aligned}$$

which, after the transformation $z \to z - (p + O(|a| + |b| + \mu))u^2$, takes the form

$$\frac{du}{dt} = v,$$
$$\frac{dv}{dt} = (2a + O(\mu^2))v + u\zeta - (2q^2 + O(|a| + |b| + \mu))uz + (2q^2 p + O(|a| + |b| + \mu))u^3 + O(|v|(|z| + u^2) + |u|v^2 + uz^2 + \mu|u|^3),$$
$$\frac{dz}{dt} = -2\mu(z - (p + \tfrac{2q+g}{|h|} + O(|a| + |b| + \mu))u^2) + O(\mu^2 + \mu z^2 + \mu|z|u^2 + v^2),$$

where

$$\zeta = -2qb + O(a^2 + b^2 + \mu^2)$$

is a small parameter (we can vary it by varying b).

Now we take $\zeta > 0$ and do the following scaling:

$$t \to t/\sqrt{\zeta}, \qquad z \to \frac{\zeta}{2q^2},$$

$$u \to \sqrt{\frac{\zeta}{2q^2(p + \frac{2q+g}{|h|})}}u, \qquad v \to \sqrt{\frac{\zeta^2}{2q^2(p + \frac{2q+g}{|h|})}}v;$$

The resulting system is our Shimizu Morioka system

$$\begin{aligned} \frac{du}{dt} &= v, \\ \frac{dv}{dt} &= u(1 - z) - \eta v - Bu^3 + O(\sqrt{\zeta}), \\ \frac{dz}{dt} &= -\alpha(z - u^2) + O(\sqrt{\zeta}), \end{aligned} \tag{6.12}$$

where

$$\eta = -\frac{2a}{\sqrt{\zeta}}, \qquad B = -\frac{p|h|}{p|h| + 2q + g}, \qquad \alpha = 2\frac{\mu}{\sqrt{\zeta}}.$$

Recall chapter 5, where we concluded that Lorenz Attractor was not present in a generalized replicator dynamics because the control parameters in Shimizu Morioka system were wrong (see equation 5.5). But in case of Lotka-Volterra like system, we have reduced our system to Shimizu-Morioka system with appropriate control parameters (compare (5.2) and (6.12)). This guarantees the existence of a Lorenz Attractor. Therefore, it can be concluded that a *Lorenz*

Attractor is born as a result of local bifurcation of equilibrium in a Lotka-Volterra like system, under certain conditions.

6.3.4 Conditions for a Lorenz Attractor

The following conditions (highlighted in bold) must be taken into account in order for Lotka-Volterra like system to exhibit a Lorenz Attractor:

1. Since our linearized system (6.6) with linearization matrix

$$\begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & \alpha & -\omega & 0 \\ 0 & \omega & \alpha & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

   has eigenvalues

$$\lambda_1 = \lambda, \qquad \lambda_2 = \sigma, \qquad \lambda_3 = \alpha + i\omega, \qquad \lambda_4 = \alpha - i\omega.$$

   it is quite evident that the eigenvalues λ and $\alpha \pm i\omega$ at zero vanish when $\beta = \gamma = \delta = f_1$. Because we have to consider the dynamics of the system for the values of parameters close to this moment, the eigenvalue σ equals to $4f_1$ then. Since we want stability, σ must be negative. Hence,

$$\mathbf{f_1 < 0}$$

2. When our system near the symmetric equilibrium undergoing triple-zero bifurcation takes the form (6.9), our goal is to study the system at small values of α, λ, and ω and in a small neighbourhood of $(u, w, z) = 0$. We therefore, scaled the variables to a small factor (equation 6.10): a small neighbourhood in the original variables corresponds to an order 1 neighbourhood in the rescaled variables. Thus, we consider the region

$$\lambda > \mathbf{0}$$

in the parameter space.

We know that at equilibrium $(u, w) = 0, z$ is some real constant. Therefore, at $(u, w) = 0$

$$\frac{dz}{dt} = \lambda z + p(u^2 - w^2) + z(hz^2 + g(u^2 + w^2)) - 4\omega uw + O(u^4 + w^4 + (|u| + |v|)|z|^3 + |z|^5) = 0$$
$$\lambda z + hz^3 = 0$$

And $\lambda > 0$ implies

$$\mathbf{h < 0}$$

Additionally, Shimizu Morioka holds that coefficients of uz in $\frac{du}{dt}$, wz in $\frac{dw}{dt}$, and $(u^2 - w^2)$ in $\frac{dz}{dt}$ must be non-zero. Therefore,

$$\mathbf{p \neq 0}, \mathbf{q \neq 0}$$

3. We know that for Shimizu Morioka system to exhibit a Lorenz attractor, coefficient of u^2 in $\frac{dz}{dt}$ must be positive. Therefore,

$$\mathbf{p + \frac{2q + g}{|h|} > 0}$$

4. Once we achieved the Shimizu-Morioka form (6.12), parameters η and α can take any positive values. Parameter B is found from (6.8):

$$B = \frac{2f_2(f_1 + f_3 - f_2 - \frac{f_2^2}{2f_1})}{2f_2(1 - f_1 + f_3 - f_2 - \frac{f_2^2}{2f_1}) + 6f_1 + 6f_3 - \frac{f_2^2}{f_1}}.$$

It is known that if $B > -1/3$, there exist values of η and $\alpha > 0$ when the system has Lorenz attractor [**23**]. Hence,

$$\mathbf{B \in (-1/3, \infty)};\ \eta > \mathbf{0}, \alpha > \mathbf{0}$$

Chapter 7

Conclusion

7.1 Summary of Thesis Achievements

The project began with shedding some light on the history of evolution of differential equations. We saw how complex non-linear differential equations became and that how unpredictable the dynamics were for one to study. With invention of high speed computers in 1950s, it became possible for one to develop intuition about the non linear systems, which further led to discovery of Lorenz Attractor. We then began this project with an intention to study the presence of Lorenz Attractor in a Replicator Dynamics.

In *chapter 2*, we laid the foundation of evolutionary dynamics (i.e., study of how species evolve over time) and introduced Replicator Dynamics. We considered a 4 species replicator equation, which is a system of 4 non-linear differential equations with 1 conserved quantity. We linearized this system and found the Jacobian Matrix at quilibrium points. Eigenvalues for this linearized matrix were computed. We then tested if it was possible to have a system with triple-zero eigenvalues. We found out that it was possible and computed the eigenvalues for which we have three zero eigenvalues and one negative real eigenvalue. Given that a Lorenz attractor will be born as result of local bifurcation of equilibrium with three zero eigenvalues, we were motivated to proceed onto computing a normal form to study the presence of an attractor.

In *chapter 3*, we introduced what a normal form is. We stated reasons why a normal form is very critical and plays an extremely important role in simplifying complex dynamical systems. Difference between Normal and Asymptotic Normal form was highlighted too. I laid down the algorithm that we would use throughout the project to compute the Asymptotic Normal form. We then continued our computation and diagonalized our linearization matrix, and verified the symmetry of our system. The system happened to be symmetric, but it had infinitely many points at equilibrium. It had a continuous family of equilibrium solutions and thus, the system landed up being degenerate. Therefore, we had to now look at a different system, rather, a more generalized system.

In *chapter 4* we defined a generalized replicator dynamics, and highlighted our primary motive behind choosing a general form of replicator dynamics. A generalized replicator dynamics with a system of 4 non-linear equations were considered and similar computations were carried out, i.e., first we linearized our system and confirmed that the system has three-zero eigenvalues; then we diagonalized our linearization matrix and computed $\frac{dz}{dt}, \frac{du}{dt}, \frac{dw}{dt}$ in terms of control parameters β, δ, γ ; and finally checked that the system is symmetric. At equilibrium points $(\frac{dz}{dt}, \frac{du}{dt}, \frac{dw}{dt}) = 0$, we saw that $\frac{du}{dt}, \frac{dw}{dt}$ vanished at $(u, w) = 0$, whereas z turned out be dependent on β, δ, γ. This means we did not have an entire line of equilibrium anymore, and concluded that the system was non-degenerate.

In *chapter 5*, we shed light on the concepts of chaos and attractors. We introduced the classical Shimizu-Morioka system, and highlighted its role in studying the possibility of a Lorenz attarctor. This motivated us to carry further computations, and transform our system into Shimizu-Morioka system. We began by re-scaling our system: small parameters were made sufficiently non-small such that, when these parameters approached 0, the coefficients of first and second order terms did not disappear. Equilibrium point for this re-scaled system was found to be $(u, w, z) = (0, 0, \pm 1)$, and co-ordinate shift for this equilibrium point was performed. Then, we introduced a new variable $v = \frac{du}{dt}$ and computed $\frac{du}{dt}, \frac{dv}{dt}$, and $\frac{dz}{dt}$ in terms of u, v, z. This new system of $\frac{du}{dt}, \frac{dv}{dt}, \frac{dz}{dt}$ that we achieved (5.4) was the Asymptotic Normal Form. This system was then re-scaled again, and transformed into the Shimizu Morioka system. What we observed was, that even though this system (5.5) could be tranformed into the Shimizu-Morioka system, it exhibited wrong parameters when compared to the system (5.2). Consequently, it was reported that lorenz attractor is not present in a generalized replicator dynamics. This meant, we had to now work with a different system. This time, we chose a more generalized system, called the Lotka-Volterra type system.

Chapter 6 began with the history of evolutionary dynamics and predator-prey interaction which entails the foundation of Lotka-Volterra dynamics. Building upon the fact that replicator equation is equivalent to Lotka-Volterra equation [**12**], we introduced the Lotka-Volterra type equation, and pointed how it differs from the Lotka-Volterra system. We began the same computation: checked that the system had three zero eigenvalues; computed $\frac{du}{dt}, \frac{dw}{dt}, \frac{dz}{dt}$ (6.9), this time in terms of small parameters $\alpha, \beta, \delta, \gamma, \omega, \lambda, \sigma$; assigned a new variable $v = \frac{du}{dt}$ and computed the asymptotic normal form $\frac{du}{dt}, \frac{dv}{dt}, \frac{dz}{dt}$ in terms of u, v, z. On further re-scaling, we were able to tranform this system into the Shimizu-Morioka system (6.12), but this time with correct parameters. This guaranteed the presense of a Lorenz attractor. We laid down four important conditions (section 6.3.4) under which a lorenz attractor would be present in our system, and concluded that a *Lorenz Attractor is born as a result of local bifurcation of equilibrium in a Lotka-Volterra like system.*

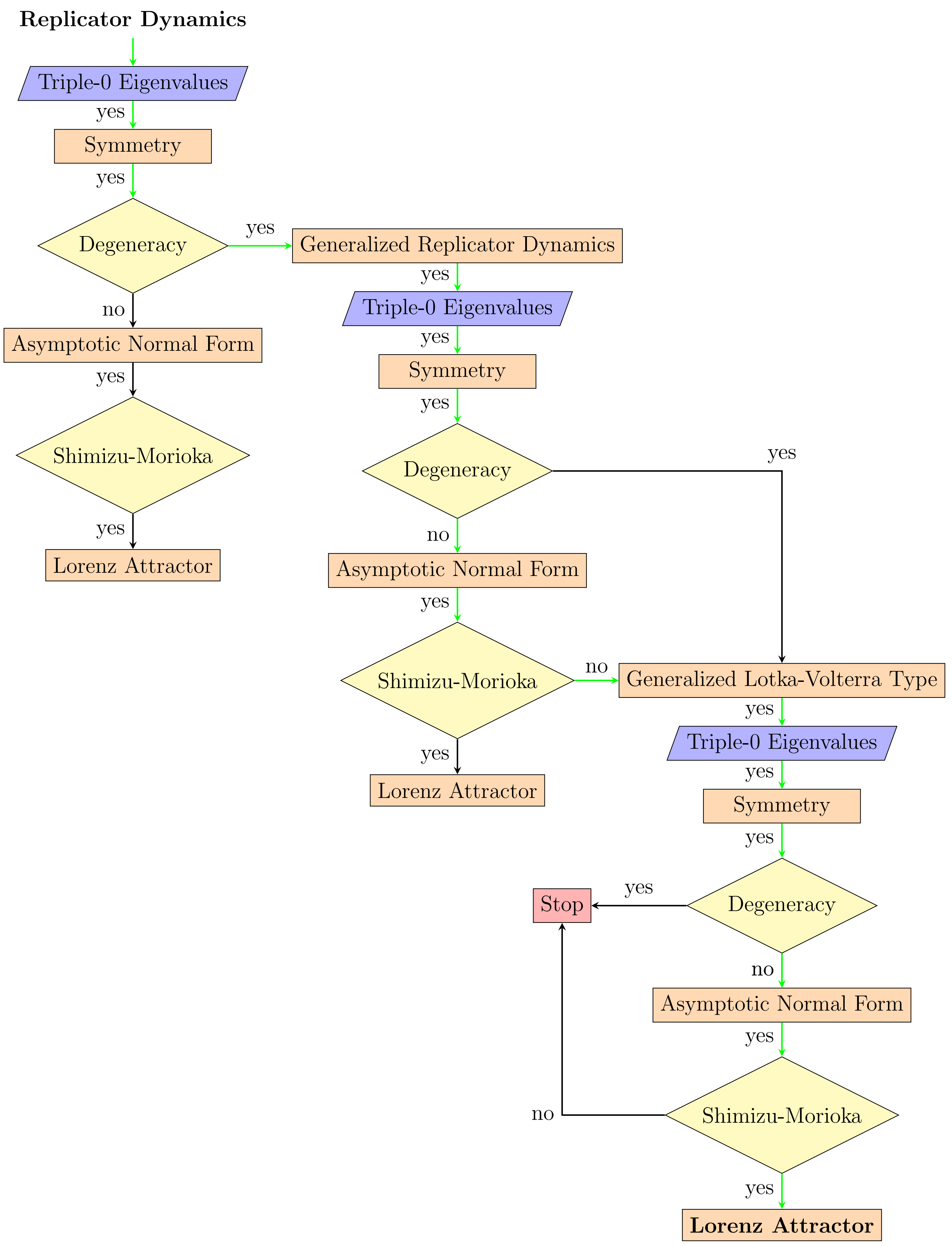

Figure 7.1: *Flowchart summarizing the sequence of this project. Green arrows represents the sequence of results obtained.*

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