1. (a). Denote $(X,Y) = (X_i, Y_i)_{i=1}^n$, then

$$p_\theta(X,Y) = (2\pi)^{-n}(1-\theta^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\left(\sum_{i=1}^n (X_i^2+Y_i^2)\frac{1}{1-\theta^2} + \sum_{i=1}^n X_iY_i \cdot \frac{-2\theta}{1-\theta^2}\right)\right)$$

Thus by factorization theorem, $T(X,Y) = \left(\sum_{i=1}^n X_i^2+Y_i^2, \sum_{i=1}^n X_iY_i\right)$ is sufficient statistics.

Furthermore,

$$\left\{(X,Y,X',Y') : \frac{p_\theta(X,Y)}{p_{\theta'}(X,Y)} = \frac{p_\theta(X',Y')}{p_{\theta'}(X',Y')}, \forall \theta,\theta' \in \Theta\right\}$$

$$= \left\{(X,Y,X',Y') : \left\langle T(X,Y)-T(X',Y'), \eta(\theta)-\eta(\theta')\right\rangle, \forall \theta,\theta' \in \Theta\right\}$$

where $\eta(\theta) = \left(1/(1-\theta^2), -2\theta/(1-\theta^2)\right)$.

Notice that

$$\left(\frac{-8}{15}, \frac{16}{15}\right), \left(-\frac{4}{3}, \frac{4}{3}\right), \left(\frac{4}{3}, \frac{4}{3}\right) \in \left\{\eta(\theta)~~\eta(\theta)~~, \theta \in (-1,1)\right\}$$

It immediately follows that $\text{span}\{\eta(\Theta)-\eta(\theta)\} = \mathbb{R}^2$.

Thus

$$\left\{(X,Y,X',Y') : \frac{p_\theta(X,Y)}{p_{\theta'}(X,Y)} = \frac{p_\theta(X',Y')}{p_{\theta'}(X',Y')}, \forall \theta,\theta' \in \Theta\right\}$$

$$\subseteq \left\{(X,Y,X',Y') : T(X,Y) = T(X',Y')\right\}$$

i.e. $T(X,Y)$ is minimal. $\square$

(b). Let $f(t) = t-2n$, then

$$\mathbb{E}\, f(T(X,Y)) = \mathbb{E}\left(\sum_{i=1}^n X_i^2+Y_i^2\right)-2n$$

$$= 0 \qquad (\text{since } \mathbb{E}X_i^2 = \mathbb{E}Y_i^2 = 1)$$

However, f is not equal to zero a.s. Thus T is not complete.

$\square$

(c). Notice that $X_i^2 \sim_{iid} \chi^2(1)$, $Y_i^2 \sim_{iid} \chi^2(1)$

Thus $Z_1 \sim \chi^2(n)$, $Z_2 \sim \chi^2(n)$. They are not dependent on θ, and so Z_1, Z_2 are ancillary.

Notice that

$$\mathbb{E}[Z_1 Z_2] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i^2 Y_j^2\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 Y_i^2 + \sum_{i\neq j} X_i^2 Y_j^2\right]$$

$$= n\cdot\left(1+2\theta^2\right) + n(n-1), \qquad (*)$$

where the last step uses $X_i^2 \perp\!\!\!\perp Y_j^2$ $(i\neq j)$ and the formula for the 4-th-order moment:

$\mathbb{E}\, x_i^2 x_j^2 = \sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2$ for $X = (x_1, \cdots x_n) \sim N(\mu, \Sigma)$.

It thus follows from $(*)$ that $\mathbb{E}[Z_1 Z_2]$ is dependent on θ. Therefore (Z_1, Z_2) is dependent on θ, so it is not ancillary.

$\square$

2. (a). We show that $T(x) = \max\left(X_{(n)}/2,\ -X_{(1)} \right)$ is complete sufficient.

Notice that

$$p_\theta(x) = \left(\frac{1}{3\theta}\right)^n \mathbb{1}\left(-\theta \le x_1, \cdots, x_n \le 2\theta\right)$$

$$= \left(\frac{1}{3\theta}\right)^n \cdot \mathbb{1}(T \le \theta) \cdot \mathbb{1}\left(-T \le x_1, \cdots, x_n \le 2T\right)$$

thus by factorization theorem, T is sufficient.

We have $\mathbb{P}\left(X_{(1)} \le u,\ X_{(n)} \le v\right) = \left((v+\theta)^n - (v-u)^n\right)(3\theta)^{-n}$

Thus $p_{(X_{(1)}, X_{(n)})}(u,v) = n(n-1)(v-u)^n \cdot (3\theta)^{-n}$.

So,

$$\mathbb{P}(T(x) \le t) = \int_{-t}^{2t}\int_{v}^{2t} n(n-1)(v-u)^n \cdot (3\theta)^{-n}\, du\, dv$$

$$= (t/\theta)^n$$

Thus $p_{T(x)}(t) = n t^{n-1} \cdot \theta^{-n}$

It follows that

$$\mathbb{E}_\theta f(T(x)) = 0 \quad \forall \theta$$

$$\Rightarrow \int_0^\theta f(t)\, n t^{n-1} \cdot \theta^{-n}\, dt = 0 \quad \forall \theta$$

$$\Rightarrow f(t) \cdot t^{n-1} = 0 \text{ a.e.} \Rightarrow f = 0 \text{ a.e.}$$

Therefore $T(x)$ is complete sufficient. $\square$

(b). We have

$$\mathbb{E}_\theta T(x) = \int_0^\theta t \cdot n t^{n-1} \theta^{-n}\, dt = \frac{n}{n+1}\theta.$$

Thus $\frac{n+1}{n} T(X)$ is unbiased.

By Lehmann–Scheffé Theorem, $\frac{n+1}{n} T(x) = \mathbb{E}\left(\frac{n+1}{n} T(x) \middle| T\right)$ is the unique UMVUE for θ.

Therefore, $\frac{n+1}{n} T(x)$ is the UMVUE for θ. $\square$

3. (a). Lemma: Let $Z_1 = n(X_{(1)} - \mu)$, $Z_2 = (n-1)(X_{(2)} - X_{(1)})$, $\cdots$ $Z_n = X_{(n)} - X_{(n-1)}$, then $Z_1 \cdots Z_n \sim_{iid} Exp(\sigma)$.

Proof: Notice that the density function of $(X_{(1)} \cdots X_{(n)}) := Y$ is given by

$$p_{(X_{(1)} \cdots X_{(n)})}(y_1 \cdots y_n) = n!\,\sigma^{-n} \exp\left(-\sum_{i=1}^{n} (x_{(i)} - \mu)/\sigma\right) \cdot \mathbb{1}(\mu \le x_{(1)} \le \cdots \le x_{(n)})$$

Consider the Linear transform $Z = AY + b$ where

$$A = \begin{pmatrix} n & 0 & & & \\ -(n-1) & n-1 & 0 & & \\ 0 & -(n-2) & n-2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & & -2 & 2 & 0 \\ & & 0 & -1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} -n\mu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then $p_Z(z_1 \cdots z_n) = p_Y(A^{-1}Z) \cdot |J_{A^{-1}}(Z)|$

$$= \exp\left(-\sum_{i=1}^{n} z_i / \sigma\right) \cdot \sigma^{-n}$$

Thus $z_i \sim_{iid} Exp(\sigma)$. $\square$

Back to the original problem.
We have

$$p_\theta(X) = \sigma^{-n} \exp\left(-\sum_{i=1}^{n} (x_i - \mu)/\sigma\right) \cdot \mathbb{1}(\mu \le x_1 \cdots x_n)$$

$$= \sigma^{-n} \exp\left(-n(x_{(1)} - \mu)/\sigma\right) \cdot \exp\left(-\sum_{i=1}^{n} (x_i - x_{(1)})/\sigma\right) \cdot \mathbb{1}(x_{(1)} \ge \mu) \cdot \mathbb{1}(x_{(1)} \le x_1 \cdots x_n)$$

Thus factorization theorem implies that $\left(X_{(1)}, \sum_{i=1}^{n} X_i - X_{(1)}\right)$ is sufficient statistics.

By Lemma, $n(X_{(1)} - n) \sim Exp(\sigma)$, $\sum_{i=1}^{n} (X_i - X_{(1)}) \sim \Gamma(n-1, \sigma)$ and $X_{(1)} \perp\!\!\!\perp \sum_{i=1}^{n} (X_i - X_{(1)})$.

Thus

$$\mathbb{E}_{\mu,\sigma} f\left(X_{(1)}, \sum_{i=1}^{n} (X_i - X_{(1)})\right) = 0 \qquad \forall \mu, \sigma$$

$$\Rightarrow \int_{\mu}^{\infty} \int_{0}^{\infty} f(y_1, y_2) \exp\left(-\frac{y_1 + y_2 - n\mu}{\sigma}\right) \cdot \sigma^{-n} \cdot \frac{y_2^{n-1}}{(n-1)!}\, dy_2\, dy_1 = 0, \ \forall \mu, \sigma > 0$$

$$\Rightarrow \int_0^\infty f(y_1, y_2)\, y_2^{n-1} \exp\left(-\frac{y_2}{\sigma}\right) dy_2 = 0, \quad \forall y_1, \sigma > 0$$

$$\Rightarrow f(y_1, y_2) \cdot y_2^{n-1} = 0 \quad \forall y_2, y_1 \quad \text{(uniqueness of Laplace transform)}$$

$$\Rightarrow f(y_1, y_2) = 0 \quad \text{a.e.}$$

Thus $T = \left(X_{(1)}, \sum_{i=1}^n X_i - X_{(1)}\right)$ is complete sufficient $\square$

(b) We have

$$\mathbb{E} X_{(1)} = \frac{1}{n} \mathbb{E}\, n(X_{(1)} - \mu) + \mu = \frac{\sigma}{n} + \mu$$

$$\mathbb{E} \sum_{i=1}^n X_i - X_{(1)} = (n-1)\sigma.$$

Thus by Lehmann–Scheffé Theorem,

~~$X_{(1)} - \mathbb{E}X$~~ $\left(X_{(1)} - \frac{1}{n(n-1)} \sum_{i=1}^n X_i - X_{(1)}, \ \frac{1}{n-1} \sum_{i=1}^n X_i - X_{(1)} \right)$

is the unique UMVUE.

(Since $X_{(1)} - \frac{1}{n(n-1)} \sum_{i=1}^n X_i - X_{(1)} = \mathbb{E}\left[X_{(1)} - \frac{1}{n(n-1)} \sum_{i=1}^n X_i - X_{(1)} \,\middle|\, T(X) \right]$

$\frac{1}{n-1} \sum_{i=1}^n X_i - X_{(1)} = \mathbb{E}\left[\frac{1}{n-1} \sum_{i=1}^n X_i - X_{(1)} \,\middle|\, T(X) \right]$

and $T(X)$ is complete sufficient). $\square$

4 (a). $\mathcal{P}^T$ is complete family

$\Leftrightarrow$ $\int f(x)\, dP_\theta^T = 0 \ \forall \theta$ implies $f \overset{a.s.}{=} 0$ a.s. $\forall\theta$

$\Leftrightarrow$ $\int f(T(x))\, dP_\theta = 0 \ \forall\theta$ implies $f = 0$ a.s. $\forall\theta$

(notice that $\int f(x) dP_\theta^T = \int f(T) dP_\theta$)

$\Leftrightarrow$ T is complete statistic. $\square$

(b). If $\mathrm{Span}(v^\theta : \theta \in \Theta) = \mathbb{R}^d$, then

$\mathbb{E} f(T) = 0 \Rightarrow \sum_i f(x_i) P_\theta(x_i) = 0$

$\Rightarrow \langle f, v^\theta \rangle = 0$ where we denote $f = \langle f(x_1) \cdots f(x_n) \rangle \in \mathbb{R}^d$

$\Rightarrow f = 0$ (since v^θ spans $\mathbb{R}^d$.)

Thus T is complete.

If T is a complete statistics. We prove by contradiction.

Suppose $\exists\, v \in \mathbb{R}^d$ s.t. $v \notin \mathrm{span}(v^\theta)$.

Then let $f = v$ where $f(x_i) = v_i$

It follows that (we assume $v \in \mathrm{span}(v^\theta)^\perp$ WLOG)

$\mathbb{E} f(T(x)) = \sum_i f(x_i) p_\theta(x_i) = \langle f, v^\theta \rangle = 0$.

but $f \neq 0$. Contradiction!

Thus $\mathrm{span}(v^\theta)$ must be $\mathbb{R}^\theta$. $\square$

(c). We show $T(X) = \sum_{i=1}^n x_i$ is what we look for.

Notice

$$P_\theta(x) = \prod_{i=1}^n \frac{e^{-\theta}}{x_i!} \theta^{x_i} = \frac{e^{-n\theta}}{\prod x_i!} \cdot \theta^{\sum x_i}$$

Thus by factorization theorem, T is sufficient.

Furthermore,

$$\left\{ (X,Y) : \frac{p_\theta(x)}{p_{\theta'}(x)} = \frac{p_\theta(Y)}{p_{\theta'}(Y)}, \ \forall \theta, \theta' \in \Theta \right\}$$

$$= \left\{ (X,Y) : \left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)\left(\log\theta - \log\theta' \right) = 0, \ \forall \theta,\theta' \in \Theta \right\}$$

$$\subset \{ (X,Y) : T(x) = T(Y) \}$$

Thus T is minimal sufficient statistics.

Finally we show T is not complete. We prove by contradiction.
Suppose T is complete, then

$$\{ f: \mathbb{N} \to \mathbb{R} \text{ s.t. } \mathbb{E}_\theta f(T) = 0 \ \forall \theta \in \Theta \} = \{0\}$$

Notice $\mathbb{E}_\theta f(T) = \sum_{k=0}^{\infty} \frac{n^k}{k!} \theta^k f(k) e^{-n\theta}$ (since $T \sim \text{poisson}(n\theta)$)

We can let $f(k) = \frac{k!}{n^k} \cdot a_k$ where a_k are coefficients of $\prod_{\theta_i \in \Theta} (\theta - \theta_i)$, i.e. $\sum_k a_k \theta^k = \prod_{\theta_i \in \Theta} (\theta - \theta_i)$.

Thus $\mathbb{E}_\theta f(T) = 0$ but $f \neq 0$, contradiction!
Therefore T is not complete. $\square$

(d). Let $\begin{cases} f(2n+1) = \frac{(-1)^n}{(2n+1)!} \cdot \frac{(2n+1)!}{n^{2n+1}} = \frac{(-1)^n}{n^{2n+1}} \neq 0 \\ f(2n) = 0 \end{cases}$

Then $\mathbb{E}_\theta f(T)$

$$= \sum_{k=0}^{\infty} \frac{n^k}{k!} \theta^k f(k) e^{-n\theta}$$

$$= \sum_{k=0}^{\infty} e^{-n\theta} \cdot \frac{(-1)^k}{(2k+1)!} \theta^{2k+1} = \sin\theta = 0,$$

$\forall \theta \in n\mathbb{Z}_+$

Thus $T(x) = \sum_{i=1}^{n} x_i$ is not complete.

By exactly the same proof of (c) (just choosing a finite subset of $\Theta = n\mathbb{Z}_+$), T is minimal sufficient.

Thus $T(x)$ is minimal but not complete. $\square$

5 (a). Let p_θ^T be the push-forward density of $T(x)$, and denote $\{t_1 \cdots t_d\}$ all the values that T can take.

Then if $g(\theta)$ is U-estimable, $g(\theta)$ must be written as

$$g(\theta) = \mathbb{E}_\theta \delta(T(x))$$
$$= \sum_{i=1}^{d} p_\theta^T(t_i)\, \delta_i \qquad \text{where } \delta_i = \delta(t_i).$$

Let $A = \text{span}(v^\theta)$ where $v^\theta = (p_\theta^T(t_1) \cdots p_\theta^T(t_d)) \in \mathbb{R}^d$

Then $\mathcal{G}$ is the dual space of A.

It follows that

$$\dim(\mathcal{G}) = \dim(A).$$

Therefore

$$\dim(\mathcal{G}) = d \Leftrightarrow \dim(A) = d \overset{4.(b)}{\Leftrightarrow} T \text{ is complete.}$$

$\square$

(b). Notice that

$$p_\theta(x) = \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n - \sum_{i=1}^n x_i} = \exp\left(\log\frac{\theta}{1-\theta} T(x) + n\log(1-\theta)\right)$$

(where $T(x) = \sum_{i=1}^n x_i$).

is an exponential family full rank case.

Thus $T(x) = \sum_{i=1}^n x_i$ is complete sufficient statistics.

It follows that:

$g(\theta)$ is U-estimable

$$\Leftrightarrow \quad g(\theta) = \mathbb{E}_\theta \delta(T(x)) = \sum_{k=0}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k} \delta(k)$$

($T(x) \sim \text{Binomial}(n,\theta)$)

$\Leftrightarrow$ g is a polynomial of degree $\leq n$.

For $g(\theta) = \sum_{k=0}^{n} a_k \theta^k$, it suffices to let $\delta(k)$ match the coefficients, i.e.

$$\sum_{k=0}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k} \delta(k) = \sum_{k=0}^{n} a_k \theta^k.$$

Solving this, we have $\delta(k) = \sum_{i=0}^{n} a_i \frac{\binom{n-i}{k-i}}{\binom{n}{k}} \mathbb{1}(k \geq i)$

Thus $\delta(T(x)) = \sum_{i=1}^{n} a_i \mathbb{1}(T \geq i) \frac{\binom{n-i}{T-i}}{\binom{n}{T}}$ is UMVUE.

$\square$

6. (a) Since $p_\mu(x_1, x_2) = \frac{\phi(x_1-\mu)\mathbb{1}(x_1>c)}{1-\Phi(c-\mu)} \cdot \phi(x_2-\mu)$

We have $f_\mu(x_2) = \phi(x_2-\mu)$. Thus $\mathbb{E}_\mu X_2 = \mu$ (*).

Furthermore, $\mathbb{E}_\mu X_1 = \mathbb{E}_\mu[X|X>c] > \mu$

Thus $\mathbb{E}\bar{X} = \frac{1}{2}\mathbb{E}_\mu X_1 + \frac{1}{2}\mathbb{E}_\mu X_2 > \mu$. $\square$

(b) We have proved in (*) that X_2 is unbiased for μ.
For admissibility, Lehmann–Scheffé Theorem shows that $\delta(X) = \mathbb{E}[X_2|\bar{X}]$ is the unique UMVUE for μ.
[Here we use the fact that $\bar{X}$ is complete sufficient because

$$p_\mu(x_1, x_2) = \mathbb{1}(x_1>c) \cdot \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2+x_2^2-2\mu(x_1+x_2)) - \mu^2 - \log(1-\Phi(c-\mu))\right)$$

is a full-rank case of exponential family.]

It ~~also~~ follows that

$$\mathbb{E}_\mu L(\mu;\delta) < \mathbb{E}_\mu L(\mu; X_2) \quad \forall L, \mu.$$

(c). By (b), it suffices to show that

$$\mathbb{E}_\mu[X_2|\bar{X}] = \bar{X} - \frac{1}{\sqrt{2}}\zeta(\sqrt{2}(c-\bar{X})). \quad (*)$$

We have

$$\delta(\bar{x}) = \bar{x} + \frac{1}{2}\mathbb{E}[X_2 - X_1|\bar{x}]$$

$$= \bar{x} + \frac{\int_c^\infty \int_{-\infty}^{+\infty} (x_2-x_1)\mathbb{1}(\frac{x_1+x_2}{2}=\bar{x})\, p_\mu(x_1,x_2)\, dx_1 dx_2}{2\int_c^{+\infty}\int_{-\infty}^{\infty} \mathbb{1}(\frac{x_1+x_2}{2}=\bar{x})\, p_\mu(x_1,x_2)\, dx_1 dx_2}$$

$$= \bar{x} + \frac{\int_c^\infty (\bar{x}-x)\frac{\phi(x-\mu)}{1-\Phi(c-\mu)}\phi(2\bar{x}-x-\mu)\,dx}{\int_c^\infty \frac{\phi(x-\mu)}{1-\Phi(c-\mu)}\phi(2\bar{x}-x-\mu)\,dx}$$

$$= \bar{X} + \frac{\int_c^\infty (\bar{x}-x)\exp\left(-\frac{2(x-\bar{x})^2+2(\bar{x}-\mu)^2}{2}\right)dx}{\int_c^\infty \exp\left(-\frac{2(x-\bar{x})^2+2(\bar{x}-\mu)^2}{2}\right)dx}$$

$$= \bar{x} + \int_{\sqrt{2}(c-\bar{x})}^{\infty} -\frac{u}{\sqrt{2}} \phi(x)\,dx \Big/ \int_{\sqrt{2}(c-\bar{x})}^{\infty} \phi(x)\,dx \quad \left(\text{letting } x = \sqrt{2}x - \bar{x}\right)$$

$$= \bar{x} - \frac{1}{\sqrt{2}} \int_{\sqrt{2}(c-\bar{x})}^{+\infty} u\,\phi(u)\,du \Big/ \left(1 - \Phi(\sqrt{2}(c-\bar{x}))\right)$$

$$= \bar{x} - \zeta\left(\sqrt{2}(c-\bar{x})\right)/\sqrt{2}.$$

This completes the proof of $(*)$. $\square$

(d). We know that

$$1 - \Phi(\sqrt{2}(c-\bar{x})) \longrightarrow 1 \quad (\bar{x} \to +\infty)$$

$$\int_{\sqrt{2}(c-\bar{x})}^{+\infty} u\,\phi(u)\,du \longrightarrow 0 \quad (\bar{x} \to +\infty)$$

Thus $\delta(\bar{x}) - \bar{x} = \frac{1}{\sqrt{2}} \cdot \frac{\int_{\sqrt{2}(c-\bar{x})}^{+\infty} u\phi(u)\,du}{1 - \Phi(\sqrt{2}(c-\bar{x}))} \to 0 \quad (\bar{x} \to +\infty)$

This completes the proof of $\lim_{\bar{x} \to \infty} \delta(\bar{x}) - \bar{x} = 0$.

It makes sense because if $\bar{x} \gg c$, then $\mathbb{P}(X_1 < c)$ is very small and can be neglected. Thus the selection bias is very small, and the naive estimator is very close to the UMVUE. $\square$

7. (a) Let $S(x) = (S_1(x) \cdots S_m(x))$ where $S_i(x) = \begin{cases} n - \sum_{i=1}^{m-1} S_i(x) & i = m \\ \sum_{j=1}^{n} \mathbb{1}(x_j = y_i) & i \neq m \end{cases}$

Then we can map $T(x)$ one-to-one to $S(x)$

Notice that

$$p_\theta(x) = \prod_{i=1}^{m} \theta_i^{S_i(x)} = \exp\left(\sum_{i=1}^{m} S_i(x) \log \theta_i\right)$$

$$= \exp\left(\sum_{i=1}^{m-1} S_i(x) \log \theta_i/\theta_m + n \log \theta_m\right)$$

is a full-rank case of exponential family.

Thus $S(x)$ is complete sufficient, which means that $T(x)$ is complete sufficient. □

(b) Let $\mathcal{Y} = \{y_1 \cdots y_m\}$ be an arbitrary subset of $\mathbb{R}$

then (a) indicates that $T(x)$ is complete sufficient for $P^{\mathcal{Y}}$,

Thus $\mathbb{E}_p f(T) = 0 \quad \forall p \in P$

$\Rightarrow \mathbb{E}_{p^*} f(T) = 0 \quad \forall p^* \in P^{\mathcal{Y}}, \quad \forall P^{\mathcal{Y}}$

$\Rightarrow f(T) = 0, \quad \forall \mathcal{Y}$

$\Rightarrow f(T) \equiv 0 \quad \forall x \in \mathbb{R}$

It follows that T is complete

For sufficieny, we have

$$p(x|T) = \mathbb{1}(T(x) = T) \cdot \frac{\prod_{i=1}^{n} p(x_i)}{\prod_{i=1}^{n} p(x_{(i)}) \cdot \binom{n}{n_1 \cdots n_k}}$$

$$= \frac{\mathbb{1}(T(x) = T)}{\binom{n}{n_1 \cdots n_k}}$$ (where $n_1 \cdots n_k$ are the counts of distinct values in $x_{(i)}$)

is not dependent on p.

Combining, we have that T is complete sufficient. □

(c) By the same proof in (b), $T(x) = (X_{(1)} \cdots X_{(n)})$ is complete sufficient.

Obviously, $\bar{x}$ and s^2 are unbiased.

$\left(\mathbb{E}\bar{x} = \frac{1}{n} \mathbb{E} \sum X_i = \mathbb{E}X, \quad \mathbb{E}S^2 = \frac{1}{n-1}\left(\mathbb{E}\sum_i x_i^2 - n\mathbb{E}\bar{x}^2\right) = \mathbb{E}(X - \mathbb{E}X)^2 \right)$

Thus by Lehmann-Scheffé Theorem,

$$\bar{x} = \mathbb{E}[\bar{x} | T(x)], \quad S^2 = \mathbb{E}[S^2 | T(x)]$$

are UMVUE of $\mathbb{E}[x]$ and $Var(x)$ respectively. □