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Exercises for Algorithmic Bioinformatics II

Assignment 11

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Exercise 3 (Conjugate prior distribution, 10P):

Let $X \sim \mathcal{N}(0,1)$. Show that $\mathcal{N}(0,1)$ is a conjugate prior distribution for μ and, thus, the posterior distribution is a normal distribution.

First, show that

$$P(\mu)P(x|\mu) = \frac{1}{2\pi}\exp(-\frac{x^2}{4})\exp(-(\mu - \frac{x}{2}^2))$$

Then calculate $P(\mu|x)$. Integrate over μ to calculate the denominator of the Bayes law. Use the fact the integral of a probability distribution is 1.

$$P(\mu) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{\mu^2}{2})$$

$$P(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x-\mu)^2)$$

$$P(\mu)P(x|\mu) = \frac{1}{2\pi} \exp(-\frac{\mu^2}{2}) \exp(-\frac{1}{2}(x-\mu)^2) = \frac{1}{2\pi} \exp(-\frac{\mu^2}{2} - \frac{1}{2}(x-\mu)^2)$$

$$= \frac{1}{2\pi} \exp(-\frac{x^2}{4} - (\mu^2 - x\mu + \frac{x^2}{4})) = \frac{1}{2\pi} \exp(-\frac{x^2}{4} - (\mu - \frac{x}{2})^2) = \frac{1}{2\pi} \exp(-\frac{x^2}{4}) \exp(-(\mu - \frac{x}{2}^2))$$

For calculating $P(\mu|x)$ we use Bayes law:

$$P(\mu|x) = \frac{P(\mu)P(x|\mu)}{P(x)}$$

Apply law of total probability to calculate $P(\mu|x)$:

$$\begin{aligned} P(x) &= \int P(\mu)P(x|\mu)d\mu \\ &= \frac{1}{2\pi} \exp(-\frac{x^2}{4}) \int exp(-(\mu - \frac{x}{2}^2))d\mu \\ &= \frac{1}{2\pi} \exp(-\frac{x^2}{4}) \end{aligned}$$

$$\begin{aligned} P(\mu|x) &= \frac{P(\mu)P(x|\mu)}{P(x)} \\ &= \frac{\frac{1}{2\pi} \exp(-\frac{x^2}{4}) \exp(-(\mu - \frac{x}{2}^2))}{\frac{1}{2\pi} \exp(-\frac{x^2}{4})} \\ &= \frac{1}{\sqrt{\frac{1}{2}}\sqrt{2\pi}} \exp(-\frac{(\mu - \frac{x}{2})^2}{2(\sqrt{\frac{1}{2}})^2}) \end{aligned}$$

It shows $P(\mu|x)$ follows a normal distribution with parameter $\mu = \frac{x}{2}$ and $\sigma = \sqrt{\frac{1}{2}}$ and thus, $\mathcal{N}(0,1)$ is a conjugate prior distribution for μ.