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Exercises for Algorithmic Bioinformatics II

Assignment 13

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Exercise 3 (Markov-Chains II, 10P):

A woman has three umbrellas, which are either in her office or her flat. Let X_n be the number of umbrellas at the place where she currently is. If she leaves her flat in the morning, or the office in the afternoon, and it is currently raining, she will take an umbrella with her, if there is one. Otherwise she has to take the bus. Consider that the probability of rain for each of her walks is 0.2 (independent of the previous walk).

(a) Create a Markov chain for X_n and determine the respective transition probabilities.

- The set of all states $S := \{0, 1, 2, 3\}$
- The probabilities from state $x_n = 0$ to state $x_n = \{0, 1, 2\}$ are 0, 0, 0, as there are 3 umbrellas in total which meas $x_n + x_{n+1} \geq 3$. Thus the probability from state $x_n = 0$ to state $x_n = 3$ is 1. Obviously, $P(x_n = 1, x_{n+1} = 0) = P(x_n = 1, x_{n+1} = 1) = P(x_n = 2, x_{n+1} = 0) = 0$.
- $P(x_n = 2, x_{n+1} = 3) = P(x_n = 3, x_{n+1} = 2) = P(x_n = 3, x_{n+1} = 3) = 0$ as the woman can only take one umbrella with her which means $x_n + x_{n+1} \leq 4$.

- $P(x_n = 1, x_{n+1} = 2) = P(x_n = 2, x_{n+1} = 1) = P(x_n = 3, x_{n+1} = 0) = 0.8$, as these imply that the woman has not taken a umbrella on a no-rainy day with $P = 1 - 0.2 = 0.8$.
- $P(x_n = 1, x_{n+1} = 3) = P(x_n = 2, x_{n+1} = 2) = P(x_n = 3, x_{n+1} = 1) = 0.2$, as these imply that the woman took a umbrella on a rainy day with $P = 0.2$.

Thus, the state transition matrix A is defined as:

$$A = \begin{pmatrix} x_n & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0.8 & 0.2 \\ 2 & 0 & 0.8 & 0.2 & 0 \\ 3 & 0.8 & 0.2 & 0 & 0 \end{pmatrix}$$

(b) Calculate the stationary distribution of the Markov chain and derive from that, with which probability the woman has to take the bus.

Let $\pi = (a, b, c, d)$ for stationary distribution where a, b, c, d refer to the probability of $x = \{0, 1, 2, 3\}$.

$$(a, b, c, d) = (a, b, c, d) \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{pmatrix}$$

$$a \cdot 0 + b \cdot 0 + c \cdot 0 + d \cdot 0.8 = 0.8d = a \tag{1}$$

$$a \cdot 0 + b \cdot 0 + c \cdot 0.8 + d \cdot 0.2 = 0.8c + 0.2d = b \tag{2}$$

$$a \cdot 0 + b \cdot 0.8 + c \cdot 0.2 + d \cdot 0 = 0.8b + 0.2c = c \tag{3}$$

$$a \cdot 1 + b \cdot 0.2 + c \cdot 0 + d \cdot 0 = a + 0.2b = d \tag{4}$$

- from (3): $b = c$
- from (2): $b = c = d$
- thus, $a : b : c : d = 4 : 5 : 5 : 5$, $\pi = (\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19})$
- therefore, the probability of taking bus (there is no umbrella AND it is rainy outside) is $P(x = 0|rain) = \frac{4}{19} \times 0.2 \approx 0.042$