

# 西安交通大学本科生课程考试试题标准答案与评分标准

课程名称： 数理方程 B 课时： 32 考试时间： 2017 年 1 月 10 日

一、(5 分/题×4 题=16 分)

1、C 2、C 3、D 4、B

二、(每小题 5 分，共 10 分)

$$\begin{aligned} 1. \int_0^{\mu_m^{(0)}} x^3 J_0(x) dx &= \int_0^{\mu_m^{(0)}} x^2 d(xJ_1(x)) = x^3 J_1(x) \Big|_0^{\mu_m^{(0)}} - \int_0^{\mu_m^{(0)}} 2x^2 J_1(x) dx \\ &= [\mu_m^{(0)}]^3 J_1(\mu_m^{(0)}) - 2 \int_0^{\mu_m^{(0)}} dx^2 J_2(x) = [\mu_m^{(0)}]^3 J_1(\mu_m^{(0)}) - 2 [\mu_m^{(0)}]^2 J_2(\mu_m^{(0)}) \\ &= \mu_m^{(0)} J_1(\mu_m^{(0)}) ([\mu_m^{(0)}]^2 - 4) \end{aligned}$$

$$2. \text{令 } t = x^2, \int_0^\infty x^3 e^{-x^2} dx = \int_0^\infty t^{\frac{3}{2}} e^{-t} d\sqrt{t} = \int_0^\infty \frac{1}{2} t e^{-t} dt = \frac{1}{2} \Gamma(2) = \frac{1}{2} \Gamma(1) = \frac{1}{2}$$

三、(10 分) 首先证明特征值非负： 方程两边同乘以  $X(x)$ ，即

$$X''(x)X(x) + \lambda X^2(x) = 0, \int_0^l X''(x)X(x) dx + \lambda \int_0^l X^2(x) dx = 0$$

$$X(x)X'(x) \Big|_0^l - \int_0^l (X'(x))^2 dx + \lambda \int_0^l X^2(x) dx = 0$$

$$\text{由于 } X'(0) = 0, X(l) = 0, \text{ 所以 } \lambda = \frac{\int_0^l (X'(x))^2 dx}{\int_0^l X^2(x) dx} \geq 0 \quad (6 \text{ 分})$$

①当  $\lambda = 0$  时， $X''(x) = 0$ ，则  $X(x) = C_1 + C_2 x$ ， $X'(0) = 0 \Rightarrow C_2 = 0$ ， $X(l) = 0 \Rightarrow C_1 = 0$ ，则  $X=0$ ，不是特征函数

②当  $\lambda > 0$  时，

$$X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x,$$

$$X'(0) = 0 \quad X(l) = 0 \Rightarrow C_2 = 0, \sqrt{\lambda} l = \frac{(2n+1)}{2} \pi \quad n \geq 0 \quad (10 \text{ 分})$$

$$\lambda_n = \left[ \frac{(2n+1)}{2l} \pi \right]^2 \quad X_n(x) = \cos \frac{(2n+1)}{2l} x$$

四、(每小题 15 分，共 30 分)

1.  $u(x, t) = X(x)T(x)$  代入齐次方程得：

$$XT'' - a^2 X''T = 0 \quad \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda,$$

特征值问题为： 
$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X(0) = 0, X'(l) = 0 \end{cases}$$

其特征值与特征函数为：  $\lambda_n = \left[ \frac{(2n+1)\pi}{2l} \right]^2 (n \geq 0), X_n(x) = \sin \frac{(2n+1)\pi}{2l} x$  (6 分)

$$u(x, t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

$$\sin \frac{\pi x}{2l} = \sum_{n=0}^{\infty} \varphi_n X_n(x) \quad \varphi_0 = 1 \quad \varphi_n = 0 (n \geq 1) \quad (9 \text{ 分})$$

$$\sin \frac{3\pi}{2l} x = \sum_{n=0}^{\infty} \psi_n X_n(x) \quad \psi_1 = 1 \quad \psi_n = 0 (n \geq 0 \text{ 且 } n \neq 1)$$

代入方程得关于  $T$  的定解问题

$$\begin{cases} T_n''(t) - \lambda_n T_n(t) = 0 \\ T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n \end{cases}$$

通解为：  $T_n(t) = C_1 \cos \sqrt{\lambda_n} t + C_2 \sin \sqrt{\lambda_n} t$

$n = 0$  时,

$$T_0(0) = \varphi_0 = 1 \quad T_0'(0) = \psi_0 = 0,$$

所以  $C_1 = 1, C_2 = 0$ ,

$$T_0(t) = \cos \sqrt{\lambda_0} t = \cos \frac{\pi}{2l} t$$

当  $n = 1$  时,

$$T_1(0) = \varphi_1 = 0$$

$$T_1'(0) = \psi_1 = 1'$$

所以  $C_1 = 0, C_2 = \frac{1}{\sqrt{\lambda_1}} = \frac{2l}{3\pi}, T_1(t) = \frac{2l}{3\pi} \sin \frac{3\pi}{2l} t$

当  $n \neq 0, n \neq 1$  时,  $T_n(0) = 0, T_n'(0) = 0$ , 则  $C_1 = 0, C_2 = 0, T_n(t) = 0$

综上

$$u(x, t) = \cos \frac{\pi t}{2l} \sin \frac{\pi x}{2l} + \frac{2l}{3\pi} \sin \frac{3\pi t}{2l} \sin \frac{3\pi}{2l} x \quad (15 \text{ 分})$$

2. 令  $u(x,t) = X(x)T(t)$ . 代入齐次方程

$$XT' - a^2 X''T = 0$$

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda$$

特征值问题为:  $\begin{cases} X''(x) + \lambda X = 0, & 0 < x < \pi \\ X'(0) = 0, & X(\pi) = 0 \end{cases}$ , 解得

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2 \quad X_n = \cos\left(\frac{2n+1}{2}x\right) \quad n \geq 0 \quad (6 \text{ 分})$$

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

$$\cos \frac{3x}{2} = \sum_{n=0}^{\infty} f_n(t) X_n(x), \quad f_1 = 1 \quad f_n = 0 \quad (n \geq 0, n \neq 1)$$

代入原方程得  $T_n(t)$  的定解问题  $\begin{cases} T'_n(t) + \lambda_1 a^2 T_n(t) = f_n \\ T(0) = 0 \end{cases} \quad (9 \text{ 分})$

① 当  $n=1$  时,  $\begin{cases} T'_1(t) + \lambda_1 a^2 T_1(t) = 1 \\ T_1(0) = 0 \end{cases}$ , 齐次方程通解  $\bar{T}_1(t) = C_1 e^{-\lambda_1 a^2 t}$ , 令特解

$\tilde{T}_1(t) = d$ , 代入方程得  $d = \frac{1}{\lambda_1 a^2}$ , 则

$$T_1(t) = C_1 e^{-\lambda_1 a^2 t} + \frac{1}{\lambda_1 a^2}.$$

由  $T(0) = 0$  得,  $T_1(t) = \frac{1}{\lambda_1 a^2} (1 - e^{-\lambda_1 a^2 t})$

② 当  $n \neq 1$  时,  $\begin{cases} T'_n(t) + \lambda_n a^2 T_n(t) = 0 \\ T_n(0) = 0 \end{cases}$ ,

由  $T_n(t) = C_1 e^{-\lambda_n a^2 t}, T_n(0) = 0 \Rightarrow C_1 = 0, T_n(t) = 0$

$$u(x,t) = \frac{1}{\lambda_1 a^2} (1 - e^{-\lambda_1 a^2 t}) \cos \frac{3x}{2} \quad (15 \text{ 分})$$

五. 作变量代换  $x = \sqrt{\lambda} \rho$ , 原方程变为:

$$x^2 R''(x) + xR'(x) + (x^2 - n^2)R(x) = 0 \quad (5 \text{ 分})$$

为  $n$  阶贝塞尔方程, 其通解为:  $R(x) = C_1 J_n(x) + C_2 N_n(x)$ 。 (7 分)

$$\text{则 } R(\rho) = C_1 J_n(\sqrt{\lambda}\rho) + C_2 N_n(\sqrt{\lambda}\rho)$$

由于  $|R(0)| < +\infty$ , 则  $C_2 = 0$ ; 由于  $R(2) = 0$ , 则  $J_n(2\sqrt{\lambda}) = 0$

$$\lambda_m = \left(\frac{\mu_m^{(n)}}{2}\right)^2, R_m(\rho) = J_n\left(\frac{\mu_m^{(n)}}{2}\rho\right) \quad (10 \text{ 分})$$

六. (10 分) 设  $\Omega = \{(x, y) | y > 0\}$ ,  $P_0(x_0, y_0) \in \Omega$ , 则  $P_0(x_0, y_0)$  关于  $y = 0$  的对称点为  $P_1(x_0, -y_0)$ , 于是可以构造格林函数为

$$\begin{aligned} G(P, P_0) &= \frac{1}{2\pi} \left( \ln \frac{1}{r_0} - \ln \frac{1}{r_1} \right) \\ &= \frac{1}{4\pi} \left( -\ln((x-x_0)^2 + (y-y_0)^2) + \ln((x-x_0)^2 + (y+y_0)^2) \right) \\ &= \frac{1}{4\pi} \ln \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2}, \end{aligned}$$

显然有  $G(P, P_0) = 0$ ,  $P \in \partial\Omega$ . (8 分)

直接计算可得

$$\begin{aligned} \left. \frac{\partial G}{\partial n} \right|_{\partial\Omega} &= - \left. \frac{\partial G}{\partial y} \right|_{y=0} = - \frac{1}{4\pi} \left( - \frac{2(y-y_0)}{(x-x_0)^2 + (y-y_0)^2} + \frac{2(y+y_0)}{(x-x_0)^2 + (y+y_0)^2} \right) \Bigg|_{y=0} \\ &= - \frac{y_0}{\pi} \left( \frac{1}{(x-x_0)^2 + y_0^2} \right), \end{aligned}$$

(9 分)

所以原定解问题的解为

$$\begin{aligned} u(x_0, y_0) &= - \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} ds + \iint_{\Omega} G f d\sigma \\ &= \frac{y_0}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(x)}{(x-x_0)^2 + y_0^2} dx + \frac{1}{4\pi} \int_0^{+\infty} dy \int_{-\infty}^{+\infty} f(x, y) \ln \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2} dx. \end{aligned}$$

(10 分)

七. (12 分) 该问题特征方程为:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} - 3 = 0 \text{ 可得 } \frac{dy}{dx} = 3 \text{ 和 } \frac{dy}{dx} = -1$$

特征线为:

$$x - \frac{y}{3} = c_1 \text{ 和 } x + y = c_2$$

做变量代换:

$$\begin{cases} \xi = x - \frac{y}{3} \\ \eta = x + y \end{cases}$$

原方程化简为:

$$\frac{16}{3} u_{\xi\eta} = \eta \quad (7 \text{ 分})$$

解之得:

$$\frac{16}{3} u_{\xi} = \frac{\eta^2}{2} + f_1(\xi)$$

$$u(\xi, \eta) = \frac{3}{32} \xi \eta^2 + \int f_1(\xi) d\xi + g(\eta) = \frac{3}{32} \xi \eta^2 + f(\xi) + g(\eta)$$

$$u(x, y) = \frac{3}{32} \left(x - \frac{y}{3}\right)(x + y)^2 + f\left(x - \frac{y}{3}\right) + g(x + y)$$

由初始条件得:

$$\begin{cases} u(x, 0) = \frac{3x^3}{32} + f(x) + g(x) = 0 \\ u_y(x, 0) = \frac{5x^2}{32} - \frac{1}{3} f'(x) + g'(x) = 0 \Rightarrow -\frac{1}{3} f(x) + g(x) + \frac{5x^3}{96} = c \end{cases}$$

$$f(x) = -\frac{3}{4}c - \frac{1}{32}x^3, g(x) = \frac{3}{4}c - \frac{1}{16}x^3$$

$$u(x, y) = \frac{3}{32} \left(x - \frac{y}{3}\right)(x + y)^2 - \frac{1}{32} \left(x - \frac{y}{3}\right)^3 - \frac{1}{16} (x + y)^3 \quad (10 \text{ 分})$$