第五章 多元函数微分学及其应用

第五节 多元向量值函数的导数与微分

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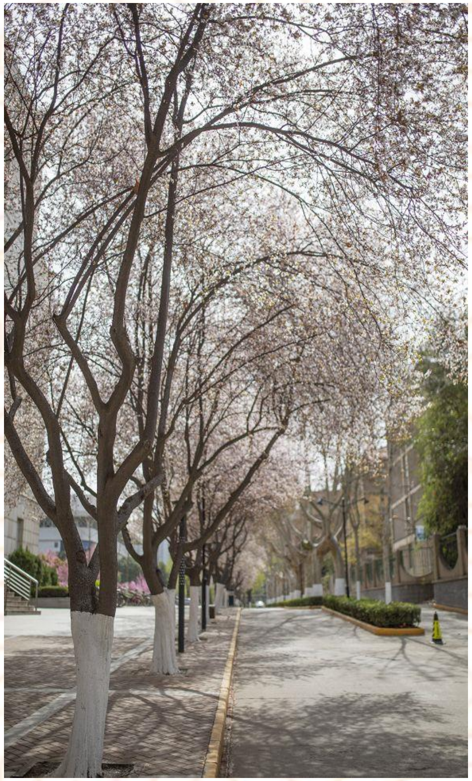

作业: 习题5.5 P95-96

(A) 1, 2, 3(3)(4) , 5, 6, 9

第一部分　一元向量值函数的导数与微分

一、问题的提出

神州13号载人飞船在太空飞行时的位置是由多元变量所表示的向量确定，每一分量是时间t的函数。如何刻画每一时刻飞船的飞行速度向量，加速度向量?

归结为向量值函数的导数与微分问题。

二、n元向量值函数

一般地，称映射f：$A\subseteq R^n \to R^m$为一个n元向量值函数。

即：
$$f(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2,\cdots,x_n) \\ f_2(x_1,x_2,\cdots,x_n) \\ \vdots \\ f_m(x_1,x_2,\cdots,x_n) \end{bmatrix}$$

其中 $\boldsymbol{x}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\cdots,\boldsymbol{x}_n)^{\mathrm{T}}\in \boldsymbol{A}\subseteq \boldsymbol{R}^n$,

$f_i(x)$为n元数量值函数。

三、一元向量值函数

映射f：$U(x_0)\subseteq R\to R^m$为一个一元向量值函数.

$$f(x)=\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

其中　$\boldsymbol{x}\in \boldsymbol{U}(\boldsymbol{x}_0)\subseteq \boldsymbol{R}$，　$\boldsymbol{f}_i(\boldsymbol{x})$为一元数量值函数。

数量值、向量值函数的统一定义

设$A \subseteq R^n$,映射 $f: A \to R^m$

若$m = 1, n = 1$, 则称f为一元数量值函数.

若$m = 1, n > 1$, 则称f为n元数量值函数.

若$m > 1, n = 1$, 则称f为一元向量值函数.

若$m > 1, n > 1$, 则称f为n元向量值函数.

统一记为

$$y = f(x) \qquad x \in A$$

定义1 （极限） 设一元向量值函数$f(x)=\left(f_1(x),f_2(x),\cdots,f_m(x)\right)^{\mathrm{T}}$

在x_0的某一去心领域$\overset{0}{U}(x_0)$内有定义，$a=\left(a_1,a_2,\cdots,a_m\right)^{\mathrm{T}}\in R^m$.

如果$\forall\varepsilon>0$，$\exists\delta>0$，使得当$\|x-x_0\|<\delta$时，

$$\|f(x)-a\|=\sqrt{\sum_{i=1}^{m}\left(f_i(x)-a_i\right)^2}<\varepsilon$$

则称当$x\to x_0$时，$f(x)$以a为极限. 记为$\lim\limits_{x\to x_0}f(x)=a$.

定理1 （等价命题）

$$\lim_{x\to x_0}f(x)=a\Leftrightarrow\lim_{x\to x_0}f_i(x)=a_i\quad(i=1,2,\cdots,m)$$

连续： 如果$f(x)$在$U(x_0)$有定义，且$\lim\limits_{x\to x_0}f(x)=f(x_0)$

$$f(x)\text{在}x_0\text{连续}\Leftrightarrow f_i(x)\text{在}x_0\text{连续}\quad(i=1,2,\cdots,m)$$

定义2 （导数）

设$f: U(x_0) \subseteq R \to R^m, x_0 + \Delta x \in U(x_0)$.

如果 $\lim\limits_{x \to x_0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 存在，

则 称f 在x_0处可导.并称此极限值为f 在x_0处的导数.

记为 $\left.\dfrac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0} = \mathrm{D}f(x_0) = f'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

结论：

$\lim\limits_{x \to x_0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 存在. $\Longleftrightarrow$ $\lim\limits_{x \to x_0} \dfrac{f_i(x_0 + \Delta x) - f_i(x_0)}{\Delta x}$ 存在.

$(i = 1, 2, \cdots, m)$

即 $f'(x_0) = \left(f_1'(x_0), f_2'(x_0), \cdots, f_m'(x_0)\right)^{\mathrm{T}}$

导函数与高阶导函数

如果f在区间I中每一点都可导，则称f在I上可导，此时，f在I中每一点x都有导数$f'(x)$与之对应，称$f'(x)$为f的导函数.

类似于数量值函数，可定义向量值函数的二阶导数，高阶导数如下：

$$\left.\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right|_{x=x_0} = \frac{\mathrm{d}}{\mathrm{d}x}\left.\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right]\right|_{x=x_0} = f''(x_0) = \mathrm{D}^2 f(x_0)$$

$$\mathrm{D}^n f(x_0) = \mathrm{D}\left(\mathrm{D}^{n-1} f(x)\right)\Big|_{x=x_0}$$

显然 $f''(x_0) = \left(f_1''(x_0), f_2''(x_0), \cdots, f_m''(x_0)\right)^{\mathrm{T}}$

例1

已知 3维空间一运动质点在 t 时刻的位置为向径

$$r(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}=\begin{bmatrix} \sin(2t) \\ \ln(t^2+1) \\ 2t^3+3t-1 \end{bmatrix}.$$

试求质点的速度向量，加速度向量及初始加速度向量。

定义3 （一元向量值函数的微分）

设$f:U(x_0)\subseteq R\to R^m, x_0+\Delta x\in U(x_0)$. 若存在一个与$\Delta x$无关的$m$维列向量$a=(a_1,a_2,\cdots,a_m)^T$，使 $f(x_0+\Delta x)-f(x_0)=a\Delta x+o(\rho)$

其中$\rho=|\Delta x|$, $o(\rho)$是关于ρ的高阶无穷小向量，则称f在x_0处可微.并称$a\Delta x$为f在x_0处的微分. 记作：$\mathrm{d}f(x_0)=a\Delta x.$

定理5. 1（可微的充要条件）设$f(x)$为一元向量值函数, 则

$$f(x)\text{在}x_0\text{处可微}\Leftrightarrow f_i(x)\text{在}x_0\text{处可微}\ (i=1,2,\cdots,m)$$

且 $\mathrm{d}f(x_0)=f'(x_0)\Delta x.$

若记$\mathrm{d}x=\Delta x$，则 $\mathrm{d}f(x_0)=f'(x_0)\mathrm{d}x.$

证明思路：转化为分量可微，再利用一元数量值函数可微的定义

一元向量值函数在某点可微，等价于在该点处可导

第二部分　二元向量值函数的导数与微分

定义3（二元向量值函数的微分与导数）

设二元向量值函数$f: U\left(x_{01}, x_{02}\right) \subseteq R^2 \to R^m$，其中

$$f\left(x_1, x_2\right)=\begin{bmatrix} f_1\left(x_1, x_2\right) \\ f_2\left(x_1, x_2\right) \\ \vdots \\ f_m\left(x_1, x_2\right) \end{bmatrix},\quad f_i\left(x_1 x_2\right)\text{为二元数量值函数.}$$

$$\left(i = 1, 2, \cdots, m\right)$$

$$\lim_{x \to x_0} \frac{f\left(x_0+\Delta x\right)-f\left(x_0\right)}{\Delta x}$$ 无意义！ 分子分母维数不同！

如果f的每一分量f_i都在$x_0=\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$处可微,

则称 f 在 x_0 处可微（也称可导）.

其微分为:

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}_0)=\begin{bmatrix}\mathrm{d}f_1(\boldsymbol{x}_0)\\ \mathrm{d}f_2(\boldsymbol{x}_0)\\ \vdots\\ \mathrm{d}f_m(\boldsymbol{x}_0)\end{bmatrix}=\begin{bmatrix}\dfrac{\partial f_1(\boldsymbol{x}_0)}{\partial x_1}\mathrm{d}x_1+\dfrac{\partial f_1(\boldsymbol{x}_0)}{\partial x_2}\mathrm{d}x_2\\ \dfrac{\partial f_2(\boldsymbol{x}_0)}{\partial x_1}\mathrm{d}x_1+\dfrac{\partial f_2(\boldsymbol{x}_0)}{\partial x_2}\mathrm{d}x_2\\ \vdots\\ \dfrac{\partial f_m(\boldsymbol{x}_0)}{\partial x_1}\mathrm{d}x_1+\dfrac{\partial f_m(\boldsymbol{x}_0)}{\partial x_2}\mathrm{d}x_2\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{\partial f_1(\boldsymbol{x}_0)}{\partial x_1} & \dfrac{\partial f_1(\boldsymbol{x}_0)}{\partial x_2}\\ \dfrac{\partial f_2(\boldsymbol{x}_0)}{\partial x_1} & \dfrac{\partial f_2(\boldsymbol{x}_0)}{\partial x_2}\\ \vdots & \vdots\\ \dfrac{\partial f_m(\boldsymbol{x}_0)}{\partial x_1} & \dfrac{\partial f_m(\boldsymbol{x}_0)}{\partial x_2}\end{bmatrix}\begin{bmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\end{bmatrix}=\boldsymbol{A}\,\mathrm{d}\boldsymbol{x}$$

其中, $\mathrm{d}\boldsymbol{x}=\begin{bmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\end{bmatrix}$.

称$m\times 2$矩阵 $$\boldsymbol{A}=\begin{bmatrix} \dfrac{\partial f_1(\mathbf{x}_0)}{\partial x_1} & \dfrac{\partial f_1(\mathbf{x}_0)}{\partial x_2} \\ \dfrac{\partial f_2(\mathbf{x}_0)}{\partial x_1} & \dfrac{\partial f_2(\mathbf{x}_0)}{\partial x_2} \\ \vdots & \vdots \\ \dfrac{\partial f_m(\mathbf{x}_0)}{\partial x_1} & \dfrac{\partial f_m(\mathbf{x}_0)}{\partial x_2} \end{bmatrix}$$ (Jacobi矩阵)

为 f 在x_0处的导数.记为$\mathrm{D}f(x_0)$. 即

$$\mathrm{D}f(x_0)=\begin{bmatrix} \dfrac{\partial f_1(x_0)}{\partial x_1} & \dfrac{\partial f_1(x_0)}{\partial x_2} \\ \dfrac{\partial f_2(x_0)}{\partial x_1} & \dfrac{\partial f_2(x_0)}{\partial x_2} \\ \vdots & \vdots \\ \dfrac{\partial f_m(x_0)}{\partial x_1} & \dfrac{\partial f_m(x_0)}{\partial x_2} \end{bmatrix}=\begin{bmatrix} \mathrm{grad} f_1(x_0) \\ \mathrm{grad} f_2(x_0) \\ \vdots \\ \mathrm{grad} f_m(x_0) \end{bmatrix}=\begin{bmatrix} \nabla f_1(x_0) \\ \nabla f_2(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{bmatrix}$$

从而 $\mathrm{d}f(x_0)=\mathrm{D}f(x_0)\mathrm{d}x$

例2 设 $\vec{f}(x,y)=\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}=\begin{pmatrix} x^2-y^2 \\ 2xy \end{pmatrix}$

求：$\vec{f}$在$(1,1)$点处的导数($Jacobi$矩阵)及微分。

解

$$D\vec{f}(x,y)\Big|_{(1,1)}=\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(1,1)}=\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}\Bigg|_{(1,1)}=\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

$$d\vec{f}(x,y)=D\vec{f}(1,1)\vec{dx}=\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}=2\begin{pmatrix} dx-dy \\ dx+dy \end{pmatrix}$$

一般地, 对于n元向量值函数 f: $U(x_0)\subseteq R^n \to R^m$,

如果 f 的每一分量 f_i 都在 x_0处可微,

则定义f 在x_0处的微分为:

$$\mathrm{d}f(\mathbf{x}_0)=\begin{bmatrix}\mathrm{d}f_1(\mathbf{x}_0)\\ \mathrm{d}f_2(\mathbf{x}_0)\\ \vdots\\ \mathrm{d}f_m(\mathbf{x}_0)\end{bmatrix}=\begin{bmatrix}\frac{\partial f_1(\mathbf{x}_0)}{\partial x_1} & \frac{\partial f_1(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x}_0)}{\partial x_n}\\ \frac{\partial f_2(\mathbf{x}_0)}{\partial x_1} & \frac{\partial f_2(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x}_0)}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_m(\mathbf{x}_0)}{\partial x_1} & \frac{\partial f_m(\mathbf{x}_0)}{\partial x_2} & \cdots & \frac{\partial f_m(\mathbf{x}_0)}{\partial x_n}\end{bmatrix}\begin{bmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \vdots\\ \mathrm{d}x_n\end{bmatrix}$$

即 $\mathrm{d}f(x_0)=\mathrm{D}f(x_0)\mathrm{d}x$

当$m=1$时, 退化为n元数量值函数 f 在该点处的全微分!

$$\mathrm{d}f(\mathbf{x}_0)=\frac{\partial f(\mathbf{x}_0)}{\partial x_1}\mathrm{d}x_1+\frac{\partial f(\mathbf{x}_0)}{\partial x_2}\mathrm{d}x_2+\cdots+\frac{\partial f(\mathbf{x}_0)}{\partial x_n}\mathrm{d}x_n$$

一般地，对于n元向量值函数 f：$U(x_0)\subseteq R^n \to R^m$，

如果 f 的每一分量 f_i 都在 x_0处可微，

则定义f在x_0处的导数（Jacobi矩阵）为：

$$Df(x_0)=\begin{bmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \frac{\partial f_1(x_0)}{\partial x_2} & \cdots & \frac{\partial f_1(x_0)}{\partial x_n} \\ \frac{\partial f_2(x_0)}{\partial x_1} & \frac{\partial f_2(x_0)}{\partial x_2} & \cdots & \frac{\partial f_2(x_0)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(x_0)}{\partial x_1} & \frac{\partial f_m(x_0)}{\partial x_2} & \cdots & \frac{\partial f_m(x_0)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla f_1(x_0) \\ \nabla f_2(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{bmatrix}$$

$m=1$时 f：$R^n \to R$，$Df(x_0)=\left(\frac{\partial f(x_0)}{\partial x_1}, \frac{\partial f(x_0)}{\partial x_2}, \cdots, \frac{\partial f(x_0)}{\partial x_n}\right)$

n元数量值函数的导数为梯度行向量!

f 在x_0处的导数(**Jacobi矩阵**)为:

$$\mathrm{D}f\left(x_0\right)=\begin{bmatrix} \dfrac{\partial f_1\left(x_0\right)}{\partial x_1} & \dfrac{\partial f_1\left(x_0\right)}{\partial x_2} & \cdots & \dfrac{\partial f_1\left(x_0\right)}{\partial x_n} \\ \dfrac{\partial f_2\left(x_0\right)}{\partial x_1} & \dfrac{\partial f_2\left(x_0\right)}{\partial x_2} & \cdots & \dfrac{\partial f_2\left(x_0\right)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m\left(x_0\right)}{\partial x_1} & \dfrac{\partial f_m\left(x_0\right)}{\partial x_2} & \cdots & \dfrac{\partial f_m\left(x_0\right)}{\partial x_n} \end{bmatrix}$$

当$m=n$时,将Jacobi矩阵的行列式称为f 在x_0处的Jacobi行列式. 记为:

$$J_f\left(x_0\right)=\left.\frac{\partial\left(f_1, f_2,\cdots f_n\right)}{\partial\left(x_1, x_2,\cdots, x_n\right)}\right|_{x_0}$$

- 向量值函数的偏导数

设$f:U(x_0)\subseteq R^n \to R^m$. 若 $\lim\limits_{\Delta x_i\to 0}\dfrac{f(x_0+\Delta x_i e_i)-f(x_0)}{\Delta x_i}$

存在, $e_i=(0,\cdots,0,1,0,\cdots,0)^T, i=1,2,\cdots,n$ 称该极限

为 f 在 x_0 处关于 x_i 的偏导函数. 记作$\dfrac{\partial f(x_0)}{\partial x_i}$, 或$f_{x_i}(x_0)$

$$\lim_{x\to x_0} f(x)=a \Leftrightarrow \lim_{x\to x_0} f_i(x)=a_i \quad (i=1,2,\cdots,m)$$

$\dfrac{\partial f(x_0)}{\partial x_i}$在$x_0$处存在 $\Leftrightarrow$ f_i 在x_0处关于x_i 的偏导数存在

此时, $\dfrac{\partial f(x_0)}{\partial x_i}=\left(\dfrac{\partial f_1(x_0)}{\partial x_i},\dfrac{\partial f_2(x_0)}{\partial x_i},\cdots,\dfrac{\partial f_m(x_0)}{\partial x_i}\right)^{\mathrm{T}}$

设$f:U\left(x_0\right)\subseteq R^n\to R^m$.

定理 5.2（可微的充分条件） 如果向量值函数 f 的所有分量对各变量的偏导数都在点 x_0 连续，则 f 在点 x_0 处可微.

$$\text{即}:\frac{\partial f_i\left(x_0\right)}{\partial x_j}\text{在点}x_0\text{连续}, i=1,2,\cdots,m, j=1,2,\cdots,n.$$

$$\Downarrow$$

f 在点 x_0 处可微.

第三部分　微分运算法则

定理5.3　（微分运算法则）

设向量值函数 f 与 g 都在 x 处可微，u 是在 x 处可微的数量值函数，则有

(1) $f+g$ 在 x 处可微，且 $\mathrm{D}(f+g)(x)=\mathrm{D}f(x)+\mathrm{D}g(x)$

(2) $\langle f,g\rangle$ 在 x 处可微，且

$$\mathrm{D}\langle f,g\rangle(x)=\left(f(x)\right)^{\mathrm{T}}\mathrm{D}g(x)+\left(g(x)\right)^{\mathrm{T}}\mathrm{D}f(x)$$

(3) uf 在 x 处可微，且 $\mathrm{D}(uf)(x)=u\mathrm{D}f(x)+f(x)\mathrm{D}u(x)$

(4) 若 $f:R\to R^3,g:R\to R^3$，则向量积 $f\times g$ 在 x 处可微，且

$$\mathrm{D}(f\times g)(x)=\mathrm{D}f(x)\times g(x)+f(x)\times\mathrm{D}g(x)$$

(3) uf 在x处可微,且$\mathrm{D}(uf)(x)=u\mathrm{D}f(x)+f(x)\mathrm{D}u(x)$

证明:

$$\mathrm{D}(uf)(x)=\begin{bmatrix} \frac{\partial(uf_1)}{\partial x_1} & \frac{\partial(uf_1)}{\partial x_2} & \cdots & \frac{\partial(uf_1)}{\partial x_n} \\ \frac{\partial(uf_2)}{\partial x_1} & \frac{\partial(uf_2)}{\partial x_2} & \cdots & \frac{\partial(uf_2)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(uf_m)}{\partial} & \frac{\partial(uf_m)}{\partial} & \cdots & \frac{\partial(uf_m)}{\partial} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{\partial u}{\partial x_1}f_1+u\frac{\partial f_1}{\partial x_1} & \frac{\partial u}{\partial x_2}f_1+u\frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n}f_1+u\frac{\partial f_1}{\partial x_n} \\ \frac{\partial u}{\partial x_1}f_2+u\frac{\partial f_2}{\partial x_1} & \frac{\partial u}{\partial x_2}f_2+u\frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n}f_2+u\frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_1}f_m+u\frac{\partial f_m}{\partial x_1} & \frac{\partial u}{\partial x_2}f_m+u\frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n}f_m+u\frac{\partial f_m}{\partial x_n} \end{bmatrix}_x$$

$$=\begin{bmatrix} \frac{\partial u}{\partial x_1} f_1 + u\frac{\partial f_1}{\partial x_1} & \frac{\partial u}{\partial x_2} f_1 + u\frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} f_1 + u\frac{\partial f_1}{\partial x_n} \\ \frac{\partial u}{\partial x_1} f_2 + u\frac{\partial f_2}{\partial x_1} & \frac{\partial u}{\partial x_2} f_2 + u\frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} f_2 + u\frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_1} f_m + u\frac{\partial f_m}{\partial x_1} & \frac{\partial u}{\partial x_2} f_m + u\frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} f_m + u\frac{\partial f_m}{\partial x_n} \end{bmatrix}_x$$

$$=\begin{bmatrix} \frac{\partial u}{\partial x_1} f_1 & \frac{\partial u}{\partial x_2} f_1 & \cdots & \frac{\partial u}{\partial x_n} f_1 \\ \frac{\partial u}{\partial x_1} f_2 & \frac{\partial u}{\partial x_2} f_2 & \cdots & \frac{\partial u}{\partial x_n} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u}{\partial x_1} f_m & \frac{\partial u}{\partial x_2} f_m & \cdots & \frac{\partial u}{\partial x_n} f_m \end{bmatrix}_x + \begin{bmatrix} u\frac{\partial f_1}{\partial x_1} & u\frac{\partial f_1}{\partial x_2} & \cdots & u\frac{\partial f_1}{\partial x_n} \\ u\frac{\partial f_2}{\partial x_1} & u\frac{\partial f_2}{\partial x_2} & \cdots & u\frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u\frac{\partial f_m}{\partial x_1} & u\frac{\partial f_m}{\partial x_2} & \cdots & u\frac{\partial f_m}{\partial x_n} \end{bmatrix}_x$$

$$\left(f_1, f_2, \cdots, f_m\right)^T \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

$$\mathbf{D}(uf)(x) = f(x)\mathbf{D}u(x) + u\mathbf{D}f(x)$$

设$\vec{r}=\vec{r}(t)$为空间R^3中动点$\left(x(t),y(t),z(t)\right)^T$的向径.

证明: $\left\|\vec{r}(t)\right\|=c \Leftrightarrow$ 内积$\left\langle \vec{r}'(t),\vec{r}(t)\right\rangle=0$. ($c$为常数)

$$\frac{d}{dx}\langle f,g\rangle(x)=\langle f,g'\rangle(x)+\langle f',g\rangle(x)$$

(2)$\langle f,g\rangle$在x处可微,且

$$\mathrm{D}\langle f,g\rangle(x)=\left(f(x)\right)^{\mathrm{T}}\mathrm{D}g(x)+\left(g(x)\right)^{\mathrm{T}}\mathrm{D}f(x)$$

向量值函数求导的链式法则

定理5.4 设❶ 向量值函数 $u = g = \{g_1, g_2, \cdots, g_p\}^{\mathrm{T}}$ 在点 $x_0 \in R^n$ 处可微，

❷ 向量值函数 $w = f = \{f_1, f_2, \cdots, f_m\}^{\mathrm{T}}$ 在对应的点 $u_0 = g(x_0) \in R^p$ 处可微，

则 ⇒：复合函数 $w = f(g)$ 在点 x_0 处可微，

且 $Dw(x_0) = Df(u_0)|_{u_0 = g(x_0)} \cdot Dg(x_0)$.

$$= Df(g(x_0)) \cdot Dg(x_0).$$

证明思路： 转化为分量的可微，再利用数量值函数复合后的可微性，及链导法则

$$Dw(x_0) = Df(g(x_0)) \cdot Dg(x_0)$$

$$\underset{(m\times n)}{} \qquad \underset{(m\times p)}{} \qquad \underset{(p\times n)}{}$$

即

$$\begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} & \cdots & \frac{\partial w_1}{\partial x_n} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} & \cdots & \frac{\partial w_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial w_m}{\partial x_1} & \frac{\partial w_m}{\partial x_2} & \cdots & \frac{\partial w_m}{\partial x_n} \end{pmatrix}_{x = x_0}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_p} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial u_1} & \frac{\partial f_m}{\partial u_2} & \cdots & \frac{\partial f_m}{\partial u_p} \end{pmatrix}_{\substack{u_0 \\ \| \\ g(x_0)}} \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_p}{\partial x_2} & \cdots & \frac{\partial g_p}{\partial x_n} \end{pmatrix}_{x_0}$$

- 特别地, 当 $n = m = p = 3$ 时, 采用Jacobi行列式的记法，便于记忆以上公式：

$$\frac{\partial(w_1, w_2, w_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)} \cdot \frac{\partial(g_1, g_2, g_3)}{\partial(x_1, x_2, x_3)}$$

而这与一元复合函数求导公式

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

的形式完全类似。

例3 设有向量值函数 $\overrightarrow{w}=\overrightarrow{f}(\overrightarrow{u})=\begin{pmatrix} u_1^2-u_2u_3 \\ u_1u_3-u_2^2 \end{pmatrix}$，

$$\overrightarrow{u}=\overrightarrow{g}(\overrightarrow{x})=\begin{pmatrix} x_1\cos x_2 \\ x_2\sin x_1 \\ x_1^2e^{x_2} \end{pmatrix},$$

求：$D(f\circ g)_{(1,0)}$，及 $\left.\dfrac{\partial w_1}{\partial x_1}\right|_{(1,0)}$ 和 $\left.\dfrac{\partial(w_1,w_2)}{\partial(x_1,x_2)}\right|_{(1,0)}$.

解 当 $\overrightarrow{x_0}=(1,0)$，对应的 $\overrightarrow{u_0}=(u_1,u_2,u_3)_{x0}=(1,0,1)$

$$D(f\circ g)\Big|_{(1,0)}=D\overrightarrow{f}(\overrightarrow{u_0})\cdot D\overrightarrow{g}(\overrightarrow{x_0})\Big|_{(1,0)}$$

$$D(f\circ g)\Big|_{(1,0)} = D\vec{f}(\vec{u_0})\cdot D\vec{g}(\vec{x_0})\Big|_{(1,0)}$$

$$= \begin{pmatrix} 2u_1 & -u_3 & -u_2 \\ u_3 & -2u_2 & u_1 \end{pmatrix}_{2\times 3}\Bigg|_{(1,0,1)} \cdot \begin{pmatrix} \cos x_2 & -x_1\sin x_2 \\ x_2\cos x_1 & \sin x_1 \\ 2x_1 e^{x_2} & x_1^2 e^{x_2} \end{pmatrix}_{3\times 2}\Bigg|_{(1,0)}$$

$$= \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \sin 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -\sin 1 \\ 3 & 1 \end{pmatrix}$$

$$\frac{\partial w_1}{\partial x_1}\Big|_{(1,0)} = 2, \quad \frac{\partial(w_1,w_2)}{\partial(x_1,x_2)}\Big|_{(1,0)} = \begin{vmatrix} 2 & -\sin 1 \\ 3 & 1 \end{vmatrix} = 2+3\sin 1.$$

$$\vec{w} = \vec{f}(\vec{u}) = \begin{pmatrix} u_1^2 - u_2u_3 \\ u_1u_3 - u_2^2 \end{pmatrix}, \quad \vec{u} = \vec{g}(\vec{x}) = \begin{pmatrix} x_1\cos x_2 \\ x_2\sin x_1 \\ x_1^2 e^{x_2} \end{pmatrix},$$

由方程组所确定的隐函数的微分法

在许多问题的研究中还会遇到由方程组确定的隐函数求导问题.

例如，在多元函数微分学几何应用中将要讨论空间曲线

$$\begin{cases} 2x^2+y^2+z^2=45 \\ x^2+2y^2=z \end{cases} \Longrightarrow \begin{aligned} x&=x \\ y&=y(x) \\ z&=z(x) \end{aligned}$$

的切线和法平面问题，就属于这类问题.

$$\begin{cases} 4x+2y\cdot y_x+2z\cdot z_x=0 \\ 2x+4y\cdot y_x=z_x \end{cases} \Longrightarrow y_x=-\frac{2x(1+z)}{y(1+4z)},\quad z_x=-\frac{6x}{y(1+4z)}.$$

第四部分 由方程组确定的隐函数的微分法

对函数方程组
$$\begin{cases} F_1(x_1,\cdots,x_n,y_1,\cdots,y_m)=0 \\ F_2(x_1,\cdots,x_n,y_1,\cdots,y_m)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1,\cdots,x_n,y_1,\cdots,y_m)=0 \end{cases}$$

若存在m个函数

$y_i = f_i(x_1,\cdots,x_n), i=1,2,\cdots,\mathrm{m}$ 使每个等式成立

则称这m个函数是方程组的解，或称为由该方程组确定的隐函数。

定理5.4 隐函数存在定理 对函数方程组$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$

若$F(x,y,u,v)$、$G(x,y,u,v)$在点$P(x_0,y_0,u_0,v_0)$的某邻域内有对各个变量的连续偏导数，且

$F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$，且偏导数所组成的函数行列式（或称雅可比(Jacobi)行列式）

$$J=\left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P(x_0,y_0,u_0,v_0)}=\left.\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}\right|_{P(x_0,y_0,u_0,v_0)}\neq 0$$

则方程组$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$在点$P(x_0,y_0,u_0,v_0)$的某邻域内唯一确定了两个单值且有连续偏导数的二元函数$u=u(x,y),v=v(x,y)$

它们满足：$u_0=u(x_0,y_0)$,

$$v_0=v(x_0,y_0),$$

$$F(x,y,u(x,y),v(x,y))\equiv 0,$$

$$G(x,y,u(x,y),v(x,y))\equiv 0.$$

恒等式两边对x求导得：
$$\begin{cases} F_x+F_u\dfrac{\partial u}{\partial x}+F_v\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\dfrac{\partial u}{\partial x}+G_v\dfrac{\partial v}{\partial x}=0 \end{cases}$$

$$\therefore \frac{\partial u}{\partial x}=\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}\Bigg/\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

二元一次线性非齐次方程组求解公式

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

解: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ 时, $x = \dfrac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

$$y = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

并有 $$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}\Bigg/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}\Bigg/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}\Bigg/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix},$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}\Bigg/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}.$$

在点$P(x_0,y_0,u_0,v_0)$处成立.

例 设 $y = y(x), z = z(x)$ 是由方程 $z = x f(x+y)$ 和

$F(x, y, z) = 0$ 所确定的函数，求 $\dfrac{\mathrm{d} z}{\mathrm{d} x}$.

解法1 分别在各方程两端对 x 求导，得

$$\begin{cases} z' = f + x \cdot f' \cdot (1 + y') \\ F_x + F_y \cdot y' + F_z \cdot z' = 0 \end{cases} \Longrightarrow \begin{cases} -x f' \cdot y' + z' = f + x f' \\ F_y \cdot y' + F_z \cdot z' = -F_x \end{cases}$$

$$\therefore \quad \frac{\mathrm{d} z}{\mathrm{d} x} = \frac{\begin{vmatrix} -x f' & f + x f' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -x f' & 1 \\ F_y & F_z \end{vmatrix}} = \frac{(f + x f') F_y - x f' \cdot F_x}{F_y + x f' \cdot F_z}$$

$$(F_y + x f' \cdot F_z \neq 0)$$

例 设 $y = y(x), z = z(x)$ 是由方程 $z = x f(x+y)$ 和

$F(x, y, z) = 0$ 所确定的函数，求 $\dfrac{\mathrm{d} z}{\mathrm{d} x}$.

解法2 微分法.

$$z = x f(x+y),\ \ F(x, y, z) = 0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + x f' \cdot (\mathrm{d}x + \mathrm{d}y) \\ F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z = 0 \end{cases}$$

化简得 $\begin{cases} (f + x f')\,\mathrm{d}x + x f'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1\,\mathrm{d}x + F_2\,\mathrm{d}y + F_3\,\mathrm{d}z = 0 \end{cases}$

消去 $\mathrm{d}\,y$ 可得 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

例9 设 $xu-yv=0, yu+xv=1$，求 $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x}$和$\dfrac{\partial v}{\partial y}$.

解1 直接代入公式；

解2 运用公式推导的方法，

将所给方程的两边对 x 求导并移项

$$\begin{cases} x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=-u \\ y\dfrac{\partial u}{\partial x}+x\dfrac{\partial v}{\partial x}=-v \end{cases},\qquad J=\begin{vmatrix} x & -y \\ y & x \end{vmatrix}=x^2+y^2,$$

在 $J\neq 0$ 的条件下，

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}=-\frac{xu+yv}{x^2+y^2},\qquad \frac{\partial v}{\partial x}=\frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}=\frac{yu-xv}{x^2+y^2},$$

例9 设 $xu - yv = 0, yu + xv = 1$,求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ 和 $\frac{\partial v}{\partial y}$.

在 $J \neq 0$ 的条件下,

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = -\frac{xu + yv}{x^2 + y^2}, \quad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{yu - xv}{x^2 + y^2},$$

将所给方程的两边对 y 求导，用同样方法得

$$\frac{\partial u}{\partial y} = \frac{xv - yu}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2}.$$

解3 两边同时微分法.

多元向量值函数的导数和微分

与数量值函数相应概念的比较

对于 n 元 m 维向量值函数：

$$\vec{y} = \vec{f}(\vec{x}),\ \vec{x} \in U(\vec{x}_0) \subseteq R^n,\ \vec{y} \in R^m$$

$y = f(x)$	$n=1, m=1$	$n \geq 2,\ m=1$	$n \geq 1,\ m \geq 2$
	数量	数量	m 维向量
方向导数	$f'_{\pm}(x)$	$\dfrac{\partial f(\vec{x})}{\partial \vec{l}}$	$\dfrac{\partial \vec{f}(\vec{x})}{\partial \vec{l}} = D_{\vec{l}}\vec{f}(\vec{x}) = \left\{\dfrac{\partial f_1}{\partial \vec{l}}, \dfrac{\partial f_2}{\partial \vec{l}}, \cdots, \dfrac{\partial f_m}{\partial \vec{l}}\right\}^{\mathrm{T}}$

$y=f(x)$	$n=1, m=1$	$n\ge 2,\ m=1$	$n\ge 1,\ m\ge 2$
偏导数	无	在$\overrightarrow{x}$点沿R^n各标准正交基方向$\overrightarrow{e_k}$的方向导数(n个) $\frac{\partial f(\overrightarrow{x})}{\partial x_k},(k=1,\cdots,n)$	在$\overrightarrow{x}$点沿R^n各标准正交基方向$\overrightarrow{e_k}$的方向导数($m\times n$个) $\frac{\partial f(\overrightarrow{x})}{\partial x_k}=\left\{\frac{\partial f_1}{\partial x_k},\frac{\partial f_2}{\partial x_k},\cdots\frac{\partial f_m}{\partial x_k}\right\}^{\mathrm{T}}$ $(k=1,\cdots,n)$
(假设可微) 导函数	数量	n维向量	$(m\times n)$ *Jacobi*矩阵
	$y'=\frac{df}{dx}$	$\nabla f=grad(f)=\left\{\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\cdots\frac{\partial f}{\partial x_n}\right\}$	$D f(\overrightarrow{x})=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}$

$y=f(x)$	$n=1, m=1$	$n\geq 2,\ m=1$	$n\geq 1,\ m\geq 2$
	数量	数量	m 维向量
可微定义	$\Delta f = f'\Delta x + o(\Delta x)$ $(\Delta x \to 0)$	$\Delta f = <\overrightarrow{\nabla f}, \overrightarrow{\Delta x}> + o(\rho)$ $(\rho = \Vert\overrightarrow{\Delta x}\Vert \to 0)$	$\overrightarrow{\Delta f} = Df(x)\overrightarrow{\Delta x} + \overrightarrow{o(\rho)}$ $(\rho = \Vert\overrightarrow{\Delta x}\Vert \to 0)$
微分	$df = f'dx$	$df = <\overrightarrow{\nabla f}, \overrightarrow{dx}> = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot dx_k$	$\overrightarrow{df} = A\overrightarrow{dx} = Df(\overrightarrow{x})\overrightarrow{dx}$ $= \underset{m\times n}{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}} \underset{n\times 1}{\begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}}$