第二章 一元函数微分学及其应用

第六节 Taylor定理（2学时）

- 问题的提出
- P_n和R_n的确定
- 泰勒(Taylor）定理
- 应用

作业

习题2.5 2.(2)(4), 3.(2)(4)，5，6，7

一、问题的提出

1. 设 $f(x)$ 在 x_0 处连续, 则有

$$\lim_{x\to x_0} f(x)=f(x_0) \quad [f(x)=f(x_0)+\alpha] \quad f(x)\approx f(x_0)$$

2. 设 $f(x)$ 在 x_0 处可导, 则有

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0) \quad [f(x)=f(x_0)+f'(x_0)(x-x_0)+o(x-x_0)]$$

$$f(x)\approx f(x_0)+f'(x_0)(x-x_0)$$

3. 设 $f(x)$ 在 x_0 处二阶可导, 则有?

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2}$$

$$=\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{2(x-x_0)}$$

$$=\frac{1}{2}f''(x_0)$$ 需$f(x)$在x_0附近二阶可导? $f''(x_0)$定义!

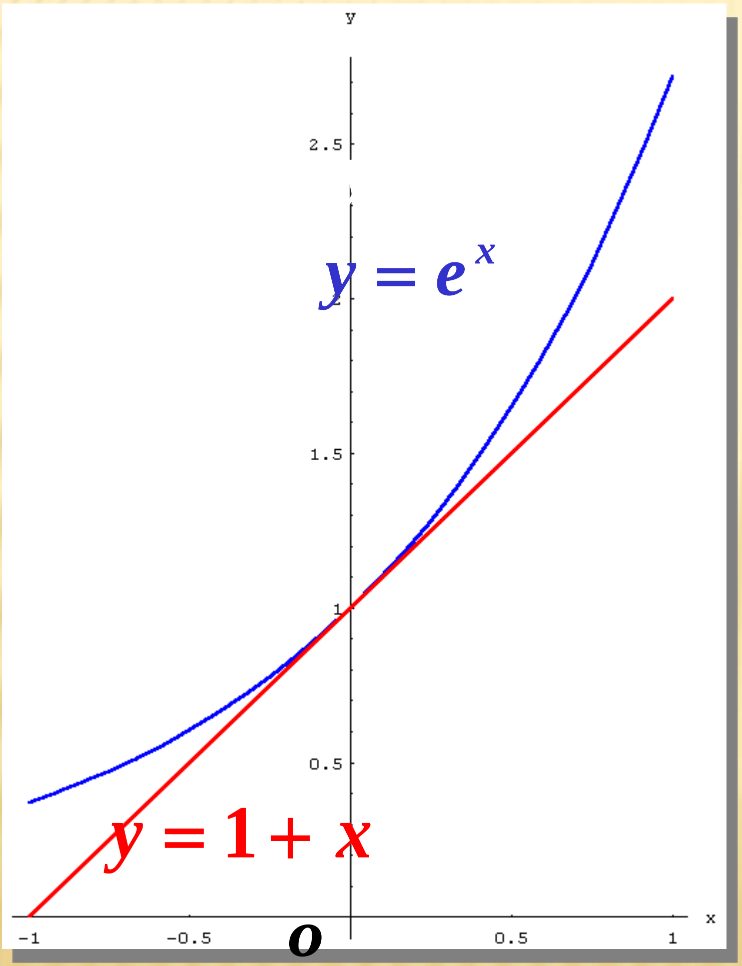
$$\therefore \frac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2}=\frac{1}{2}f''(x_0)+\alpha$$

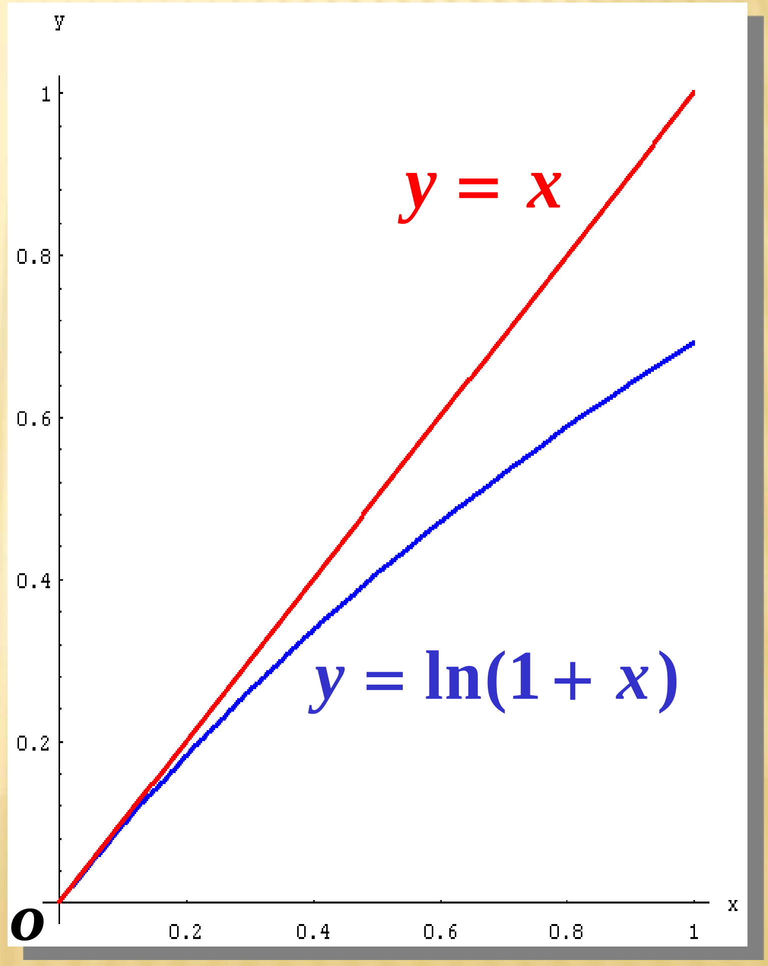
$$\therefore f(x)=$$

$$f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+o((x-x_0)^2)$$

例如，当$|x|$很小时，$e^x \approx 1+x$ ，$\ln(1+x) \approx x$

（如下图）





不足: 1、精确度不高； 2、误差不能估计.

问题: 寻找函数$P(x)$, 使得$f(x) \approx P(x)$

误差 $R(x) = f(x) - P(x)$ 可估计

设函数$f(x)$在含有x_0的开区间(a,b)内具有直到$(n+1)$阶导数, $P(x)$为n次多项式函数:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

误差 $R_n(x) = f(x) - P_n(x)$

二、P_n和R_n的确定

寻找函数$P_n(x)$, 使得$f(x) \approx P_n(x)$

分析:

近似程度越来越好

1.若在x_0点相交

$$P_n(x_0) = f(x_0)$$

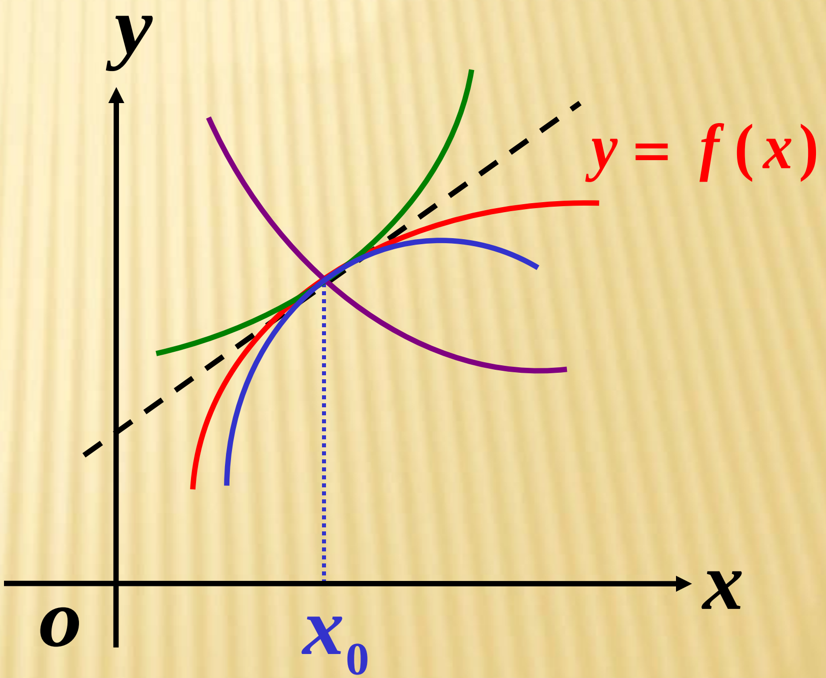
2.若有相同的切线

$$P_n'(x_0) = f'(x_0)$$

3.若弯曲方向相同

$$P_n''(x_0) = f''(x_0)$$

$\cdots\ \cdots \quad \cdots\ \cdots$



$$P_n(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n$$

假设 $P_n^{(k)}(x_0)=f^{(k)}(x_0) \quad k=1,2,\cdots,n$

则 $a_0=f(x_0), \quad 1\cdot a_1=f'(x_0), \quad 2!\cdot a_2=f''(x_0)$

$\cdots \quad \cdots, \quad n!\cdot a_n=f^{(n)}(x_0)$

得 $a_k=\dfrac{1}{k!}f^{(k)}(x_0) \qquad (k=0,1,2,\cdots,n)$

代入$P_n(x)$中得

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

三、泰勒(*Taylor*) 定理

定理 6. 1 (带 Peano 余项的 Taylor 公式)　如果函数 $f(x)$在x_0处 n 阶可微, 则 $f(x)$可以表示为$(x-x_0)$的一个n次多项式与一个高阶无穷小量 $o(x-x_0)^n$ 之和. 即:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o\left((x-x_0)^n\right)$$

该公式称为带*Peano*余项的*Taylor*公式.

令 $$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

又记: $R_n(x)=f(x)-P_n(x)$　只需证: $R_n(x)=o\left((x-x_0)^n\right)$

令 $P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2$

$$+\cdots+\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

该多项式称为 $f(x)$ 在 x_0 处的Taylor多项式.

其系数称为 $f(x)$在 x_0 处的Taylor系数.

记: $R_n(x)=f(x)-P_n(x)$ 只需证: $R_n(x)=o\left((x-x_0)^n\right)$

证明:

$$R_n'(x)=f'(x)-f'(x_0)-f''(x_0)(x-x_0)-\frac{f^{(3)}(x_0)}{2!}(x-x_0)^2-\cdots$$

$$R_n''(x)=f''(x)-f''(x_0)-f^{(3)}(x_0)(x-x_0)-\frac{f^{(4)}(x_0)}{2!}(x-x_0)^2-\cdots$$

$$R_n^{(n-1)}(x)=f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)$$ 则易知:

$$\lim_{x\to x_0}R_n(x)=\lim_{x\to x_0}R_n'(x)=\lim_{x\to x_0}R_n''(x)=\cdots=\lim_{x\to x_0}R_n^{(n-1)}(x)=0$$

$$R_n^{(n-1)}(x)=f^{(n-1)}(x)-f^{(n-1)}(x_0)-f^{(n)}(x_0)(x-x_0)$$

连续使用n-1次L'Hospital法则,得:

$$\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=\lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}}=\cdots=\lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{n!(x-x_0)}$$

又因为函数 $f(x)$ 在 x_0处可微,则

$$\lim_{x\to x_0}\frac{R_n^{(n-1)}(x)}{(x-x_0)}=\lim_{x\to x_0}\left[\frac{f^{(n-1)}(x)-f^{(n-1)}(x_0)}{(x-x_0)}-f^{(n)}(x_0)\right]=0$$

则: $\lim\limits_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=0$ 即: $R_n(x)=o\left((x-x_0)^n\right)$

$R_n(x)=o\left((x-x_0)^n\right)$ 称为 $f(x)$的Peano余项.

定理 6.2(带 Lagrange 余项的 Taylor 公式)

设函数 $f(x)$在区间I上$(n+1)$阶可导, $x_0 \in I$

则$\forall x \in I$，至少存在一点$\xi \in (x_0, x)$, 或$\xi \in (x, x_0)$使

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

$f(x)$可以表示为关于$(x - x_0)$的一个n次多项式与一个余项$R_n(x)$之和.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

该公式称为带*Lagrange*余项的*Taylor*公式.

证明：令：$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

则：$R_n(x)=f(x)-P_n(x)$

由假设，$R_n(x)$在区间I上具有直到$(n+1)$阶导数，且

$$R_n'(x)=f'(x)-f'(x_0)-f''(x_0)(x-x_0)-\frac{f^{(3)}(x_0)}{2!}(x-x_0)^2-\cdots$$

$$R_n''(x)=f''(x)-f''(x_0)-f^{(3)}(x_0)(x-x_0)-\frac{f^{(4)}(x_0)}{2!}(x-x_0)^2-\cdots$$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

证明: **令:** $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

则: $R_n(x)=f(x)-P_n(x)$

由假设, $R_n(x)$在区间I上具有直到$(n+1)$阶导数, 且

$$R_n(x_0)=R_n'(x_0)=R_n''(x_0)=\cdots=R_n^{(n)}(x_0)=0$$

函数$R_n(x)$及$(x-x_0)^{n+1}$在以x_0及x为端点的区间上满足柯西中值定理的条件, 得

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-0}=\frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}$$

(ξ_1在x_0与x之间)

两函数 $R_n'(x)$ 及 $(n+1)(x-x_0)^n$ 在以 x_0 及 ξ_1 为端点的区间上满足柯西中值定理的条件, 得

$$\frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}=\frac{R_n'(\xi_1)-R_n'(x_0)}{(n+1)(\xi_1-x_0)^n-0}$$

$$=\frac{R_n''(\xi_2)}{n(n+1)(\xi_2-x_0)^{n-1}} \qquad (\xi_2\text{在}x_0\text{与}\xi_1\text{之间})$$

如此下去, 经过 $(n+1)$ 次后, 得

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{\boxed{R_n^{(n+1)}(\xi)}}{(n+1)!} \quad (\xi\text{在}x_0\text{与}\xi_n\text{之间, 也在}x_0\text{与}x\text{之间})$$

$\because P_n(x)$ 是 n 次式, $\quad P_n^{(n+1)}(x)=0, \quad R_n(x)=f(x)-P_n(x)$

$$\therefore \quad R_n^{(n+1)}(x)=f^{(n+1)}(x) \qquad R_n(x)=\frac{\boxed{f^{(n+1)}(\xi)}}{(n+1)!}(x-x_0)^{n+1}$$

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n^{(n+1)}(\xi)}{(n+1)!}$$

$$\boxed{R_n^{(n+1)}(x)=f^{(n+1)}(x)}$$

$(\xi$在x_0与ξ_n之间, 也在x_0与x之间$)$

则由上式得 $R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ $(\xi$在x_0与x之间$)$

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\quad(\xi\text{在}x_0\text{与}x\text{之间})$$

称为Lagrange形式的余项.

$$|R_n(x)|=\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right|\le\frac{M}{(n+1)!}(x-x_0)^{n+1}$$

若 $I=(a,b)$ $\quad\le\dfrac{M}{(n+1)!}(b-a)^{n+1}\to 0\quad(n\to\infty)$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

注意:

1. 当 $n = 0$ 时, Taylor 公式变成 Lagrange 公式

$$f(x) = f(x_0) + f'(\xi)(x - x_0) \quad (\xi\text{在}x_0\text{与}x\text{之间})$$

2. 取 $x_0 = 0$,

ξ在0与x之间, 令 $\xi = \theta x \quad (0 < \theta < 1)$

则余项 $$R_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}$$

这时,Taylor公式称为Maclaurin公式.

麦克劳林(Maclaurin)公式

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

$$+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \qquad (0<\theta<1)$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)$$

例 1 求 $f(x)=e^x$ 的 n 阶麦克劳林公式.

解 $\because \quad f'(x)=f''(x)=\cdots=f^{(n)}(x)=e^x,$

$\therefore \quad f(0)=f'(0)=f''(0)=\cdots=f^{(n)}(0)=1$

注意到 $f^{(n+1)}(\theta x)=e^{\theta x}$ 代入公式,得

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1} \quad (0<\theta<1).$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \quad (0<\theta<1)$$

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)$$

常用高阶导数公式:

$$(1)\ (a^x)^{(n)} = a^x \cdot \ln^n a \quad (a>0) \qquad (e^x)^{(n)} = e^x$$

$$(2)\ (\sin kx)^{(n)} = k^n \sin(kx + n\cdot\frac{\pi}{2})$$

$$(3)\ (\cos kx)^{(n)} = k^n \cos(kx + n\cdot\frac{\pi}{2})$$

$$(4)\ (x^\alpha)^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$$

$$(5)\ (\ln x)^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

$$\left(\sin x\right)^{(n)} = \sin(x+\frac{n\pi}{2}) \qquad \left(\cos x\right)^{(n)} = \cos(x+\frac{n\pi}{2})$$

$$(4)\ (x^{\alpha})^{(n)}=\alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$$

$$\left(\frac{1}{x}\right)^{(n)}=(-1)^{n}\frac{n!}{x^{n+1}}$$

$$\left(\frac{1}{a+x}\right)^{(n)}=(-1)^{n}\frac{n!}{(a+x)^{n+1}}$$

$$\left(\frac{1}{a-x}\right)^{(n)}=\frac{n!}{(a-x)^{n+1}}$$

$$(5)\ (\ln x)^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^{n}}$$

$y=\ln(1+x)$, $\quad y^{(n)}=(-1)^{n-1}\dfrac{(n-1)!}{(1+x)^{n}}$

常用函数的Maclaurin公式

指数函数:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!} x^{n+1}$$

$$x \in (-\infty, +\infty), \theta \in (0,1).$$

正弦函数:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1} \frac{x^{2m-1}}{(2m-1)!} + (-1)^m \frac{\cos \theta x}{(2m+1)!} x^{2m+}$$

$$x \in (-\infty, +\infty), \theta \in (0,1).$$

余弦函数:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + (-1)^{m+1} \frac{\cos \theta x}{(2m+2)!} x^{2m+2}$$

$$x \in (-\infty, +\infty), \theta \in (0,1).$$

对数函数:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+(-1)^{n-1}\frac{x^n}{(n+1)(1+\theta x)^{n+1}}$$

$$x\in(-1,+\infty),\theta\in(0,1)$$

幂函数:

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

$$+\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\frac{x^{n+1}}{(1+\theta x)^{n+1-\alpha}}\qquad x\in(-1,+\infty),\theta\in(0,1)$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots+(-1)^n x^n+(-1)^{n+1}\frac{x^{n+1}}{(1+\theta x)^{n+2}}$$

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\frac{x^{n+1}}{(1+\theta x)^{n+2}}$$

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^{2}+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^{n}$$

$$+\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\frac{x^{n+1}}{(1+\theta x)^{n+1-\alpha}}\qquad x\in(-1,+\infty),\theta\in(0,1)$$

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^{2}+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2\cdot 4\cdot\cdots\cdot(2n)}x^{n}+(-1)^{n}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n+2)}\frac{x^{n+1}}{(1+\theta x)^{n+\frac{1}{2}}}$$

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{3}{8}x^{2}-\cdots+(-1)^{n}\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n)}x^{n}+(-1)^{n+1}\frac{1\cdot 3\cdot\cdots\cdot(2n+1)}{2\cdot 4\cdot\cdots\cdot(2n+2)}\frac{x^{n+1}}{(1+\theta x)^{n+\frac{3}{2}}}$$

四、Taylor公式的应用 （1）近似计算

例 2： 求 e 的近似值并估计误差.

解 Maclaurin公式 $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}}{(n+1)!} x^{n+1} \quad x \in (-\infty, +\infty), \theta \in (0,1).$

由公式可知 $e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$

估计误差 $|R_n(x)| = \left| \frac{e^{\theta x} x^{n+1}}{(n+1)!} \right| < \frac{e^x}{(n+1)!} |x|^{n+1} \quad (0 < \theta < 1).$

取 $x = 1$, $\quad e \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$

其误差 $|R_n| < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}$. 取n=8, $|R_8| < \frac{3}{(8+1)!} < 10^{-5}$

$$e \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!} \approx 2.71828$$

四、Taylor公式的应用

（2）求极限

例3 计算 $\lim\limits_{x\to 0}\dfrac{e^{x^2}+2\cos x-3}{x^4}$.

解 $\because \quad e^{x^2}=1+x^2+\dfrac{1}{2!}x^4+o(x^4)$

$$Maclaurin\text{公式}\ \ e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^m\frac{x^{2m}}{(2m)!}+(-1)^{m+1}\frac{\cos\theta x}{(2m+2)!}x^{2m+2}$$

$$\text{原式}=\lim_{x\to 0}\frac{\frac{7}{12}x^4+o(x^4)}{x^4}=\frac{7}{12}$$

例4 $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$

$$Maclaurin\text{公式}\quad \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+(-1)^{n-1}\frac{x^n}{(n+1)(1+\theta x)^{n+1}}$$

$$x\in(-1,+\infty),\theta\in(0,1)$$

例5 $\lim\limits_{x\to 0}\left[\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}\right]$

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots+(-1)^{n-1}\frac{1\cdot 3\cdot\cdots\cdot(2n-3)}{2\cdot 4\cdot\cdots\cdot(2n)}x^n+(-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n+2)}\frac{x^{n+1}}{(1+\theta x)^{n+\frac{1}{2}}}$$

四、Taylor公式的应用

(3) 证明不等式 已知：$f''(x)>0,\lim\limits_{x\to 0}\dfrac{f(x)}{x}=1.$

证明：$f(x)>x,(x\neq 0).$

将 $\tan x$ 展开到 x^5 的项.

解 在 $x=0$ 附近, $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6)$,

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) = 1 - \Delta$$

其中 $\Delta = \frac{x^2}{2} - \frac{x^4}{24} + o(x^5)$ 很小, 易见 $\Delta^2 = \frac{x^4}{4} + o(x^5)$.

$$\therefore \tan x = \frac{\sin x}{\cos x} = (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6))\frac{1}{1-\Delta}$$

$$=(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6))(1 + \Delta + \Delta^2 + o(x^4))$$

$$= (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^6))(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^4))$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$$

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln\frac{\tan x}{x}}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\left(1+\frac{\tan x}{x}-1\right)}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x^2}\left(\frac{\tan x}{x}-1\right)} = e^{\lim\limits_{x\to 0}\frac{\tan x - x}{x^3}}$$

$$= e^{\lim\limits_{x\to 0}\frac{\left(x+\frac{x^3}{3}+o(x^4)\right)-x}{x^3}} = e^{\frac{1}{3}}$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^5)$$

求极限 $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{\sin^2 x}}$.

解 $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{\sin^2 x}} = \lim\limits_{x\to 0} e^{\frac{1}{\sin^2 x}\ln\frac{\sin x}{x}} = e^{\lim\limits_{x\to 0}\frac{1}{\sin^2 x}\ln\frac{\sin x}{x}}$,

$$\ln\frac{\sin x}{x} = \ln\left(1+\frac{\sin x - x}{x}\right) \sim \frac{\sin x - x}{x} \sim \frac{1}{x}\left(-\frac{1}{3!}x^3\right) = -\frac{1}{6}x^2$$

$$\therefore \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{\sin^2 x}} = e^{\lim\limits_{x\to 0}\frac{-\frac{1}{6}x^2}{x^2}} = e^{-\frac{1}{6}}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + (-1)^m\frac{\cos\theta x}{(2m+1)!}x^{2m+1}$$

$x\in(-\infty,+\infty),\theta\in(0,1)$.

求极限$\lim\limits_{x\to 0}\dfrac{\tan(\tan x)-\sin(\sin x)}{x^3}$

$$\tan(\tan x)=\tan x+\frac{1}{3}\tan^3 x+o\left(x^3\right),\quad \sin(\sin x)=\sin x-\frac{1}{3!}\sin^3 x+o\left(x^3\right)$$

$$\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{x^3}=\lim_{x\to 0}\frac{\tan x+\frac{1}{3}\tan^3 x-\sin x+\frac{1}{3!}\sin^3 x+o\left(x^3\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}+\frac{1}{3}\lim_{x\to 0}\frac{\tan^3 x}{x^3}+\frac{1}{6}\lim_{x\to 0}\frac{\sin^3 x}{x^3}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x^3}{x^3}+\frac{1}{3}+\frac{1}{6}$$

$$=1.$$

$$x\to 0,\ \tan x-\sin x\sim\frac{1}{2}x^3$$

例 求 $\lim\limits_{x\to 0}\dfrac{tg(tgx)-\sin(\sin x)}{tgx-\sin x}$. ($tgx$ 即 $\tan x$)

分析：虽是 $\dfrac{0}{0}$ 型,但直接用洛必达法则或等价无穷小代替,都不能求得结果.

从极限式构成形式,可尝试借助微分中值定理.

$$\frac{tg(tgx)-\sin(\sin x)}{tgx-\sin x}=\frac{tg(tgx)-tg(\sin x)+tg(\sin x)-\sin(\sin x)}{tgx-\sin x}$$

$$=\frac{tg(tgx)-tg(\sin x)}{tgx-\sin x}+\frac{tg(\sin x)-tg(\sin x)\cos(\sin x)}{tgx-tgx\cos x}$$

$$=\frac{tg(tgx)-tg(\sin x)}{tgx-\sin x}+\frac{tg(\sin x)}{tgx}\cdot\frac{1-\cos(\sin x)}{1-\cos x}$$

由微分中值定理知 $\dfrac{tg(tgx)-tg(\sin x)}{tgx-\sin x}=\sec^2\theta_x,\theta_x\epsilon(\sin x,tgx)$.

$$\therefore \lim_{x\to 0}\frac{tg(tgx)-tg(\sin x)}{tgx-\sin x}=\lim_{x\to 0}\sec^2\theta_x=\lim_{\theta_x\to 0}\sec^2\theta_x=1.$$

又 $\lim\limits_{x\to 0}\dfrac{tg(\sin x)}{tgx}=\lim\limits_{x\to 0}\dfrac{tg(\sin x)}{\sin x}\cdot\dfrac{\sin x}{x}\cdot\dfrac{x}{tgx}=1$

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例 求$\lim\limits_{x\to 0}\dfrac{tg(tgx)-\sin(\sin x)}{tgx-\sin x}$. ($tgx$ 即$\tan x$)

$$\frac{tg(tgx)-\sin(\sin x)}{tgx-\sin x}=\frac{tg(tgx)-tg(\sin x)}{tgx-\sin x}+\frac{tg(\sin x)}{tgx}\cdot\frac{1-\cos(\sin x)}{1-\cos x}$$

由微分中值定理知$\dfrac{tg(tgx)-tg(\sin x)}{tgx-\sin x}=\sec^2\theta_x,\theta_x\epsilon(\sin x,tgx)$.

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又$\lim\limits_{x\to 0}\dfrac{tg(\sin x)}{tgx}=\lim\limits_{x\to 0}\dfrac{tg(\sin x)}{\sin x}\cdot\dfrac{\sin x}{x}\cdot\dfrac{x}{tgx}=1$

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{1-\cos x}=\lim_{x\to 0}\frac{\frac{1}{2}(\sin x)^2}{\frac{1}{2}x^2}=1$$

$$\therefore \lim_{x\to 0}\frac{tg(tgx)-\sin(\sin x)}{tgx-\sin x}=2.$$

写法二 求极限 $\lim\limits_{x\to 0}\dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$

$$\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{\tan x(1-\cos x)}=2\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{x^3}$$

$$=2\lim_{x\to 0}\frac{\tan(\tan x)-\tan x+\tan x-\sin x+\sin x-\sin(\sin x)}{x^3}$$

$$=2\lim_{x\to 0}\frac{\tan(\tan x)-\tan x}{x^3}+2\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$$

$$+2\lim_{x\to 0}\frac{\sin x-\sin(\sin x)}{x^3}$$

求极限 $\lim\limits_{x\to 0}\dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}$

$$\frac{1}{2}$$

$$=2\lim_{x\to 0}\frac{\tan(\tan x)-\tan x}{x^3}+2\boxed{\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}}$$

$$\tan(\tan x)=\tan x+\frac{1}{3}\tan^3 x+o\left(x^3\right),$$

$$\sin(\sin x)=\sin x-\frac{1}{3!}\sin^3 x+o\left(x^3\right)$$

$$+2\lim_{x\to 0}\frac{\sin x-\sin(\sin x)}{x^3}$$

$$\lim_{x\to 0}\frac{\tan(\tan x)-\tan x}{x^3}=\lim_{x\to 0}\frac{\tan(\tan x)-\tan x}{(\tan x)^3}=\lim_{t\to 0}\frac{\tan(\mathrm{t})-\mathrm{t}}{\mathrm{t}^3}=\frac{1}{3}$$

$$\lim_{x\to 0}\frac{\sin x-\sin(\sin x)}{x^3}=\lim_{x\to 0}\frac{\sin x-\sin(\sin x)}{(\sin x)^3}=\lim_{t\to 0}\frac{\mathrm{t}-\sin(\mathrm{t})}{\mathrm{t}^3}=\frac{1}{6}$$

当 $\square \to 0$ 时

$\sin \square \sim \square$

$\tan \square \sim \square$

$\ln(1+\square) \sim \square$

$e^{\square} - 1 \sim \square$

$\arcsin \square \sim \square$

$\arctan \square \sim \square$

$\log_a(1+\square) \sim \dfrac{\square}{\ln a}$

$a^{\square} - 1 \sim \square \ln a$

$1-\cos \square \sim \dfrac{1}{2}\square^2$

$\sqrt[n]{1+\square} - 1 \sim \dfrac{\square}{n}$

$\square - \sin \square \sim \dfrac{1}{6}\square^3$

$\tan \square - \square \sim \dfrac{1}{3}\square^3$

$(1+\square)^{\alpha} - 1 \sim \alpha\square$

$\arcsin \square - \square \sim \dfrac{1}{6}\square^3$

$\square - \arctan \square \sim \dfrac{1}{3}\square^3$

$\tan \square - \sin \square \sim \dfrac{1}{2}\square^3$

例 设 $f(x)$ 在(0,1)内二阶可导，且 $\max\limits_{0<x<1} f(x)=1,\ \min\limits_{0<x<1} f(x)=0$

证明：至少存在一点 $\zeta\in(0,1)$，使 $f''(\zeta)>2$

证： 令 $f(x_1)=\min\limits_{0<x<1} f(x)=0,\ f(x_2)=\max\limits_{0<x<1} f(x)=1$

$$\therefore f(x)=f(x_1)+\frac{1}{2}f''(\xi)(x-x_1)^2,\xi\in(x_1,x)$$

取 $x=x_2$,则

$$f(x_2)=f(x_1)+\frac{1}{2}f''(\zeta)(x_2-x_1)^2,\zeta\in(x_1,x_2)$$

$$\therefore 1-0=\frac{1}{2}f''(\zeta)(x_2-x_1)^2,\zeta\in(x_1,x_2)$$

即至少存在一点 $\zeta\in(0,1)$,使 $f''(\zeta)=\dfrac{2}{(x_2-x_1)^2}>2$

例 设 $f(x)$在$[a,b]$上一阶可导,在(a,b)内二阶可导,且 $f'(a)=f'(b)=0$

证明：存在一点 $\xi\in(a,b)$, 使$\left|f''(\xi)\right|\geq\frac{4}{(b-a)^2}\left|f(b)-f(a)\right|$

证: 由$Taylor$定理知:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(\xi)}{2!}(x-x_0)^2$$

$$\therefore f\left(\frac{a+b}{2}\right)=f(a)+f'(a)\left(\frac{a+b}{2}-a\right)+\frac{1}{2!}f''(\xi_1)\left(\frac{a+b}{2}-a\right)^2,\xi_1\in\left(a,\frac{a+b}{2}\right)$$

$$=f(a)+\frac{1}{8}f''(\xi_1)(b-a)^2$$

$$\therefore f\left(\frac{a+b}{2}\right)=f(b)+f'(b)\left(\frac{a+b}{2}-b\right)+\frac{1}{2!}f''(\xi_2)\left(\frac{a+b}{2}-b\right)^2,\xi_2\in\left(\frac{a+b}{2},b\right)$$

$$=f(b)+\frac{1}{8}f''(\xi_2)(b-a)^2$$

二式相减, 得$f(b)-f(a)=\frac{(b-a)^2}{8}\left[f''(\xi_1)-f''(\xi_2)\right]$

二式相减，得$f(b)-f(a)=\frac{(b-a)^2}{8}\left[f''(\xi_1)-f''(\xi_2)\right]$

$$\therefore \left|f(b)-f(a)\right| \le \frac{(b-a)^2}{8}\left[\left|f''(\xi_1)\right|+\left|f''(\xi_2)\right|\right] \le \frac{(b-a)^2}{4}\left|f''(\xi)\right|$$

其中，$\left|f''(\xi)\right| = \max\left\{\left|f''(\xi_1)\right|, \left|f''(\xi_2)\right|\right\}$

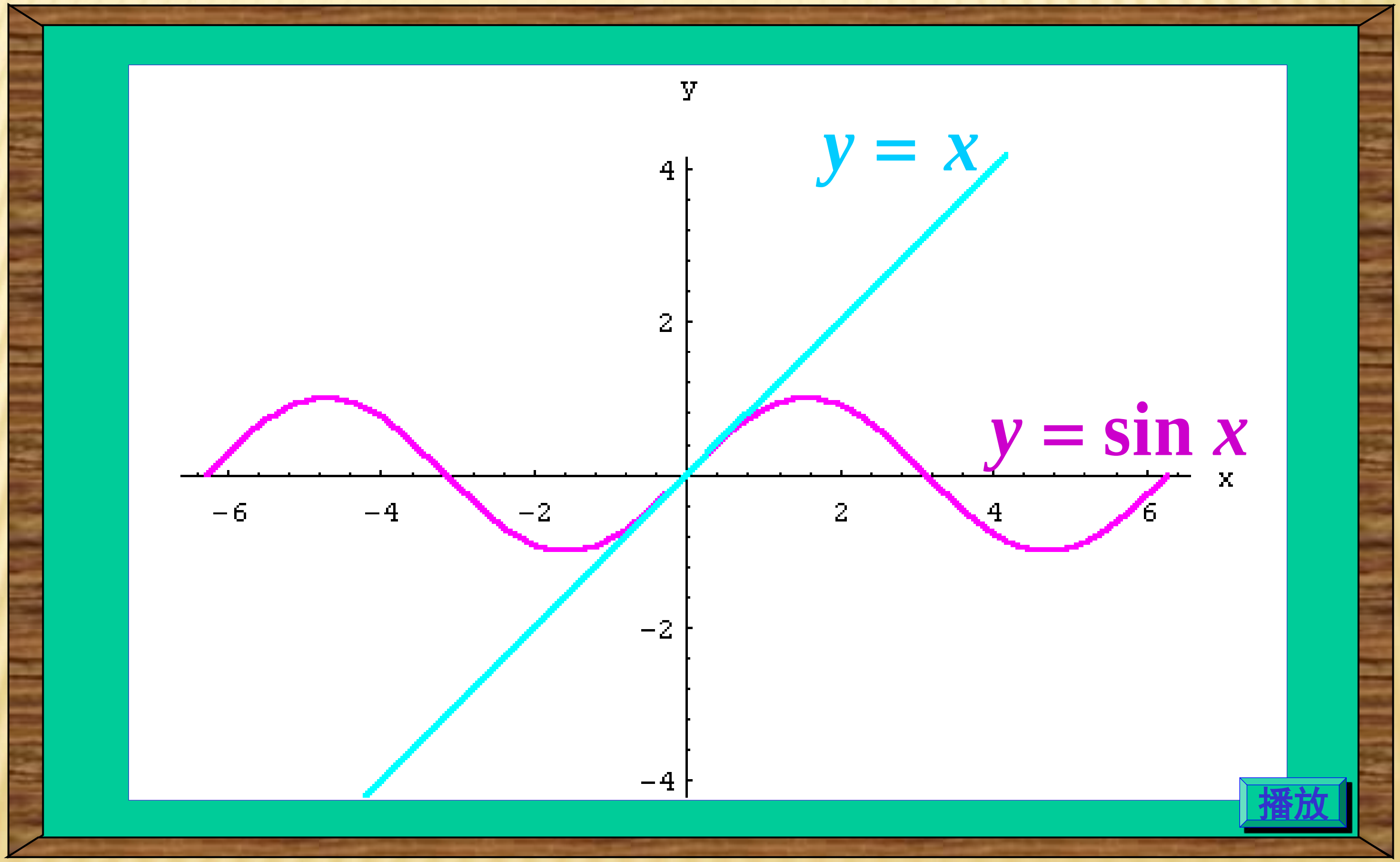
$$\therefore \exists \xi \in (a,b), \text{ 使} \left|f''(\xi)\right| \ge \frac{4}{(b-a)^2}\left|f(b)-f(a)\right|$$

*Taylor*定理中的x_0，通常要选信息量较多的点，本题中

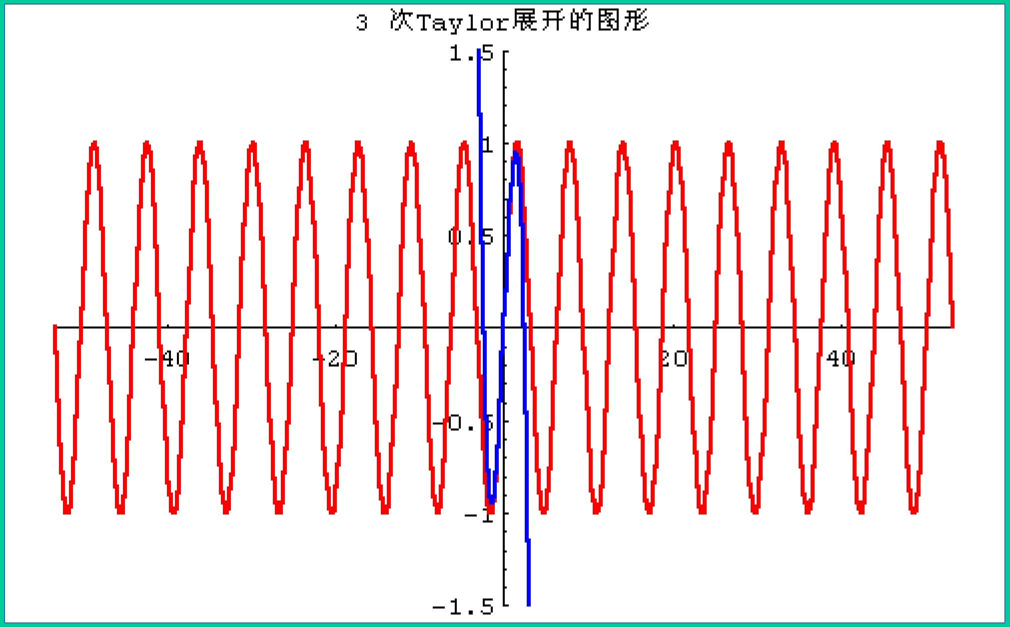
$f'(a)=f'(b)=0$，故x_0分别取a和b.

五、小结

1. Taylor 公式在近似计算中的应用;



2. Taylor 公式的数学思想---局部逼近.



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思考题

利用泰勒公式求极限 $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x(1+x)}{x^3}$

思考题解答

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\therefore \lim_{x\to 0}\frac{e^x\sin x - x(1+x)}{x^3} =$$

$$\lim_{x\to 0}\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)\right)\left(x - \frac{x^3}{3!} + o(x^3)\right) - x(1+x)}{x^3}$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{2!} - \frac{x^3}{3!} + o(x^3)}{x^3} = \frac{1}{3}$$