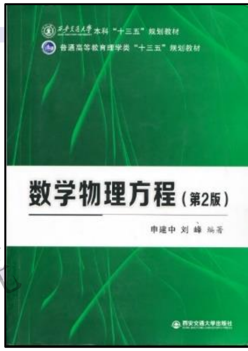
数学物理方程

9 习题

张 丹 副教授/博导
能源与动力工程学院
数学与统计学院（兼）
动力工程多相流国家重点实验室

习题



例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u_x(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=x,\ u_t(x,0)=0,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
2齐	2齐

解 第1步 确定特征值/特征函数

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$$

边界已经齐次, [2, 2]型边界

齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(l)=0 \end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2, X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$
[1,2]	$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X'(l)=0 \end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$
[2,1]	$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X(l)=0 \end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$
[2,2]	$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=X'(l)=0 \end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2, X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$
周期	$\begin{cases} \Phi''(\theta)+\lambda\Phi(\theta)=0 \\ \Phi(0)=\Phi(2\pi), \\ \Phi'(0)=\Phi'(2\pi) \end{cases}$	$\lambda_0=0,\ \Phi_0(\theta)=1,\ (n=0)$ $\lambda_n=n^2,\ \Phi_n(\theta)=C_1\cos n\theta+C_2\sin n\theta,\ (n=1,2,3\cdots)$

习题

习题2, 5(1)

例

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \ 0 < x < l, \ t > 0 \\ u_x(0,t) = 0, \ u_x(l,t) = 0, \ t \ge 0 \\ u(x,0) = x, \ u_t(x,0) = 0, \ 0 \le x \le l \end{cases}$$

波动/一维/齐次	
2齐	2齐

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \cos\left(\frac{n\pi}{l}x\right), (n = 0,1,2,3\cdots)$$

第2步 正交分解

设原问题有分离变量的形式解

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} T_n(t) \cos\left(\frac{n\pi}{l}x\right)$$

Fourier级数展开

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx, \quad (n = 0,1,2,3\cdots)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx, \quad (n = 1,2,3\cdots)$$

自由项/初始条件按特征函数系展开

$$u(x,0) = x = \sum_{n=0}^{\infty} \varphi_n X_n(x) = \sum_{n=0}^{\infty} \varphi_n \cos\left(\frac{n\pi}{l}x\right)$$

$$\begin{cases} n = 0 & \varphi_0 = \dfrac{1}{2}\cdot\dfrac{2}{l}\displaystyle\int_0^l x\cdot 1dx = \dfrac{l}{2} \\ n > 0 & \varphi_n = \dfrac{2}{l}\displaystyle\int_0^l x\cos\dfrac{n\pi x}{l}\cdot dx = \dfrac{2}{l}\left[\dfrac{l}{n\pi}x\sin\dfrac{n\pi}{l}x + \dfrac{l^2}{n^2\pi^2}\cos\dfrac{n\pi}{l}x\right]_0^l = \dfrac{2l}{n^2\pi^2}\left[(-1)^n - 1\right] \end{cases}$$

不能兼容$n=0$

习题

习题2, 5(1)

例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u_x(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=x,\ u_t(x,0)=0,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
2齐	2齐

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$$

第3步 建立初值问题ODE

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\cos\left(\frac{n\pi}{l}x\right)$$

假设形式解可逐项求导, 代入原PDE

$$\sum_{n=0}^{\infty}T_n''(t)X_n(x)-a^2\sum_{n=0}^{\infty}T_n(t)X_n''(x)=0$$

$$\sum_{n=0}^{\infty}T_n''(t)X_n(x)-a^2\sum_{n=0}^{\infty}T_n(t)\left[-\lambda_nX_n(x)\right]=0$$

$$\sum_{n=0}^{\infty}\left[T_n''(t)+a^2\lambda_nT_n(t)\right]X_n(x)=0$$

$$T_n''(t)+a^2\lambda_nT_n(t)=0$$

用原PDE初始条件→初值问题ODE初始条件

$$u(x,0)=\sum_{n=1}^{\infty}T_n(0)X_n(x)=x=\sum_{n=1}^{\infty}\varphi_nX_n(x)$$

$$u_t(x,0)=\sum_{n=1}^{\infty}T_n'(0)X_n(x)=0=\sum_{n=1}^{\infty}\psi_nX_n(x)$$

建立了初值问题ODE

$$\begin{cases} T_n''(t)+a^2\lambda_nT_n(t)=0 \\ T_n(0)=\varphi_n,\ T_n'(0)=0 \end{cases}$$

习题

例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u_x(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=x,\ u_t(x,0)=0,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
2齐	2齐

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$$

第4步 求初值问题

$$\begin{cases} T_n''(t)+a^2\lambda_nT_n(t)=0 \\ T_n(0)=\varphi_n,\ T_n'(0)=0 \end{cases}$$

二阶线性齐次ODE

$$T_n(t)=C_{1n}\cos\left(a\sqrt{\lambda_n}t\right)+C_{2n}\sin\left(a\sqrt{\lambda_n}t\right)$$

$$T_n'(t)=-C_{1n}a\sqrt{\lambda_n}\sin\left(a\sqrt{\lambda_n}t\right)+C_{2n}a\sqrt{\lambda_n}\cos\left(a\sqrt{\lambda_n}t\right)$$

$$\begin{cases} T_n(0)=C_{1n}=\varphi_n \\ T_n'(0)=C_{2n}a\sqrt{\lambda_n}=0 \end{cases} \Rightarrow \begin{cases} C_{1n}=\varphi_n \\ C_{2n}=0 \end{cases}$$

$$T_n(t)=\varphi_n\cos\left(\frac{n\pi a}{l}t\right)=\begin{cases} \dfrac{l}{2},\quad (n=0) \\ \dfrac{2l}{n^2\pi^2}\left[(-1)^n-1\right]\cos\left(\dfrac{n\pi a}{l}t\right),\quad (n>0) \end{cases}$$

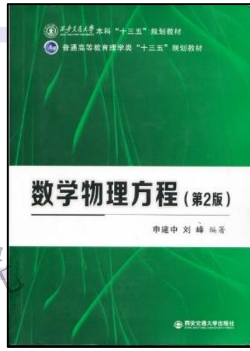
原定解问题的解可表示为

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\cos\left(\frac{n\pi}{l}x\right)$$
$$=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left[(-1)^n-1\right]\cos\left(\frac{n\pi a}{l}t\right)\cos\left(\frac{n\pi}{l}x\right)$$

特征函数有1时→其平方模与sin/cos不同→正交分解需单独讨论n=0

习题

习题2, 5(1)



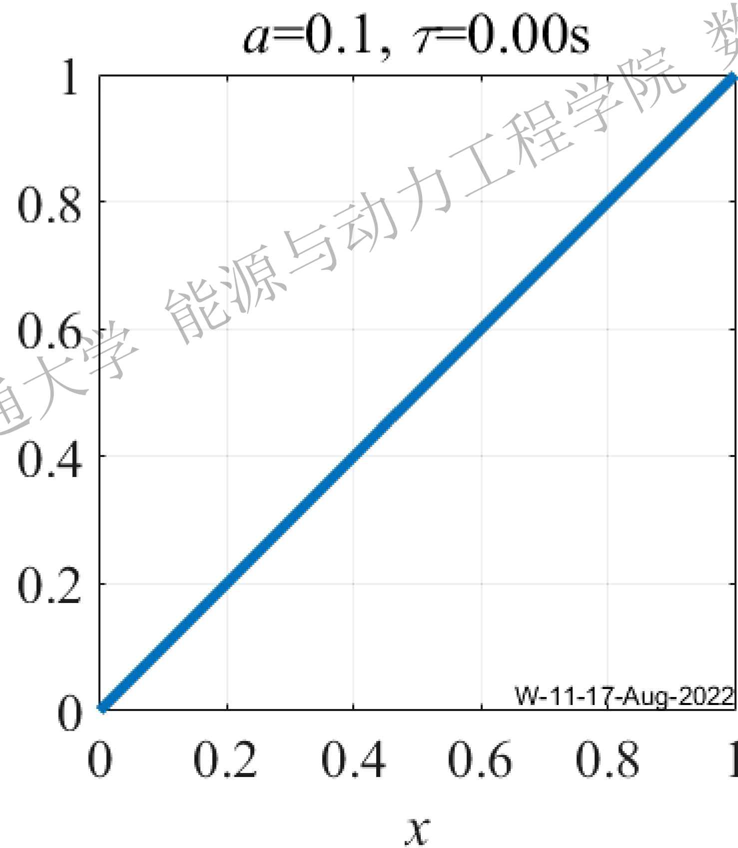
例

$$
\begin{cases}
u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0\\
u_x(0,t)=0,\ u_x(l,t)=0,\ t\ge 0\\
u(x,0)=x,\ u_t(x,0)=0,\ 0\le x\le l
\end{cases}
$$

波动/一维/齐次	
2齐	2齐

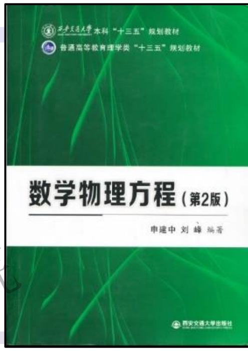
讨论

$$
u(x,t)=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left[(-1)^n-1\right]\cos\left(\frac{n\pi a}{l}t\right)\cos\left(\frac{n\pi}{l}x\right)
$$



习题

习题2, 5(2)



例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=\sin\dfrac{3\pi}{2l}x,\ u_t(x,0)=\sin\dfrac{5\pi}{2l}x,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
1齐	2齐

解 第1步 确定特征值/特征函数

边界已经齐次, [1, 2]型边界

$$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,\ X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$$

齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$
[1,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X'(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$
[2,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$
[2,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X'(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$
周期	$\begin{cases}\Phi''(\theta)+\lambda\Phi(\theta)=0\\\Phi(0)=\Phi(2\pi),\\\Phi'(0)=\Phi'(2\pi)\end{cases}$	$\lambda_0=0,\ \Phi_0(\theta)=1,\ (n=0)$ $\lambda_n=n^2,\ \Phi_n(\theta)=C_1\cos n\theta+C_2\sin n\theta,\ (n=1,2,3\cdots)$

习题

习题2, 5(2)

例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=\sin\dfrac{3\pi}{2l}x,\ u_t(x,0)=\sin\dfrac{5\pi}{2l}x,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
1齐	2齐

第2步 正交分解

$$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,\ X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

设原问题有分离变量的形式解

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\sin\left[\frac{(2n+1)\pi}{2l}x\right]$$

自由项/初始条件按特征函数系展开

初始分布与特征函数同型

$$u(x,0)=\sin\frac{3\pi}{2l}x \qquad n=1$$

$$=\sum_{n=0}^{\infty}\varphi_n X_n(x)=\sum_{n=0}^{\infty}\varphi_n\sin\left[\frac{(2n+1)\pi}{2l}x\right] \rightarrow \varphi_n=\begin{cases}\dfrac{2}{l}\displaystyle\int_0^l\sin\frac{3\pi}{2l}x\sin\frac{3\pi}{2l}x\cdot dx=1, & (n=1) \\ 0, & (n\ne 1)\end{cases}$$

$$u_x(x,0)=\sin\frac{5\pi}{2l}x \qquad n=2$$

$$=\sum_{n=0}^{\infty}\psi_n X_n(x)=\sum_{n=0}^{\infty}\psi_n\sin\left[\frac{(2n+1)\pi}{2l}x\right] \rightarrow \psi_n=\begin{cases}\dfrac{2}{l}\displaystyle\int_0^l\sin\frac{5\pi}{2l}x\sin\frac{5\pi}{2l}x\cdot dx=1, & (n=2) \\ 0, & (n\ne 2)\end{cases}$$

习题

习题2, 5(2)

例

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0,\ 0 < x < l,\ t > 0 \\ u(0,t) = 0,\ u_x(l,t) = 0,\ t \ge 0 \\ u(x,0) = \sin\dfrac{3\pi}{2l}x,\ u_t(x,0) = \sin\dfrac{5\pi}{2l}x,\ 0 \le x \le l \end{cases}$$

波动/一维/齐次	
1齐	2齐

$$\lambda_n = \left[\frac{(2n+1)\pi}{2l}\right]^2,\ X_n(x) = \sin\left[\frac{(2n+1)\pi}{2l}x\right],\ (n = 0,1,2,3\cdots)$$

第3步 建立初值问题ODE

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} T_n(t) \sin\left[\frac{(2n+1)\pi}{2l}x\right]$$

假设形式解可逐项求导, 代入原PDE

$$\sum_{n=0}^{\infty} T_n''(t) X_n(x) - a^2 \sum_{n=0}^{\infty} T_n(t) X_n''(x) = 0$$

$$\sum_{n=0}^{\infty} T_n''(t) X_n(x) - a^2 \sum_{n=0}^{\infty} T_n(t) \left[-\lambda_n X_n(x)\right] = 0$$

$$\sum_{n=0}^{\infty} \left[T_n''(t) + a^2 \lambda_n T_n(t)\right] X_n(x) = 0$$

$$T_n''(t) + a^2 \lambda_n T_n(t) = 0$$

用原PDE初始条件→初值问题ODE初始条件

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) X_n(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x)$$

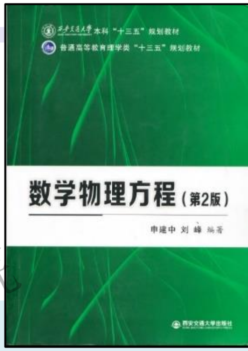
$$u_t(x,0) = \sum_{n=1}^{\infty} T_n'(0) X_n(x) = \sum_{n=1}^{\infty} \psi_n X_n(x)$$

建立了初值问题ODE

$$\begin{cases} T_n''(t) + a^2 \lambda_n T_n(t) = 0 \\ T_n(0) = \varphi_n,\ T_n'(0) = \psi_n \end{cases} (n = 1,2)$$

习题

习题2, 5(2)



例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=\sin\dfrac{3\pi}{2l}x,\ u_t(x,0)=\sin\dfrac{5\pi}{2l}x,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
1齐	2齐

$$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,\ X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$$

第4步 求初值问题

$$\begin{cases} T_n''(t)+a^2\lambda_nT_n(t)=0 \\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n \end{cases}\quad (n=1,2)$$

二阶线性齐次ODE

$$\varphi_n=\begin{cases}1, & (n=1)\\ 0, & (n\ne 1)\end{cases}\qquad \psi_n=\begin{cases}1, & (n=2)\\ 0, & (n\ne 2)\end{cases}$$

各阶导数

$$T_n(t)=C_{1n}\cos\left(a\sqrt{\lambda_n}t\right)+C_{2n}\sin\left(a\sqrt{\lambda_n}t\right)$$

$$T_n'(t)=-C_{1n}a\sqrt{\lambda_n}\sin\left(a\sqrt{\lambda_n}t\right)+C_{2n}a\sqrt{\lambda_n}\cos\left(a\sqrt{\lambda_n}t\right)$$

根据初始条件确定系数

$$\begin{cases} T_n(0)=C_{1n}=\varphi_n \\ T_n'(0)=C_{2n}a\sqrt{\lambda_n}=\psi_n \end{cases} \Rightarrow \begin{cases} C_{1n}=\varphi_n \\ C_{2n}=\dfrac{\psi_n}{a\sqrt{\lambda_n}} \end{cases}\quad (n=1,2)$$

$$T_n(t)=\varphi_n\cos\left(\frac{(2n+1)\pi a}{2l}t\right)+\psi_n\frac{2l}{(2n+1)\pi a}\sin\left(\frac{(2n+1)\pi a}{2l}t\right),\ (n=1,2)$$

习题

例

$$\begin{cases} u_{tt}-a^2u_{xx}=0,\ 0<x<l,\ t>0 \\ u(0,t)=0,\ u_x(l,t)=0,\ t\ge 0 \\ u(x,0)=\sin\dfrac{3\pi}{2l}x,\ u_t(x,0)=\sin\dfrac{5\pi}{2l}x,\ 0\le x\le l \end{cases}$$

波动/一维/齐次	
1齐	2齐

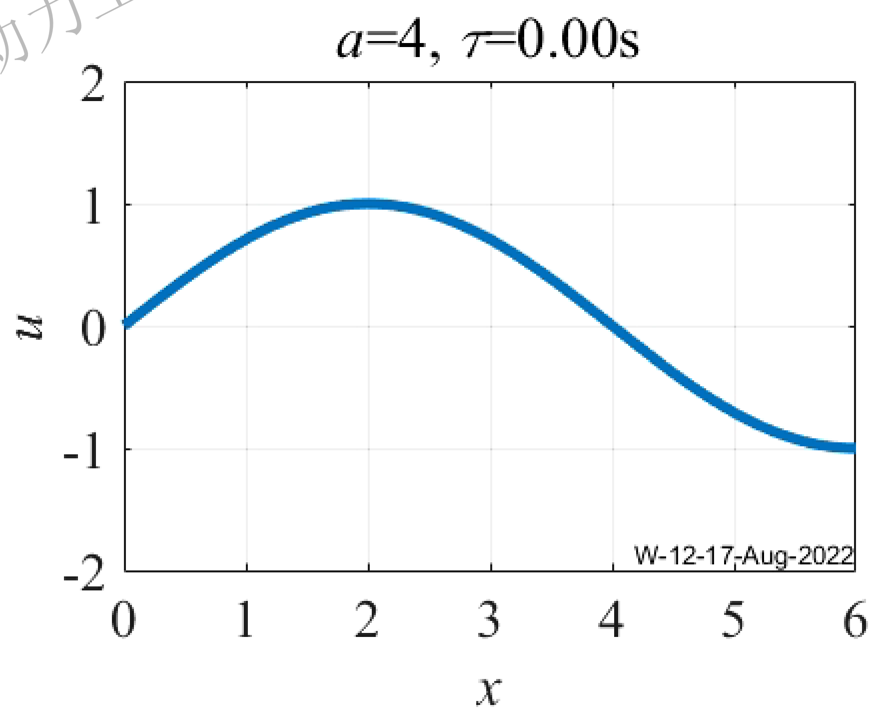
$$\varphi_n=\begin{cases}1, & (n=1)\\ 0, & (n\ne 1)\end{cases}\qquad \psi_n=\begin{cases}1, & (n=2)\\ 0, & (n\ne 2)\end{cases}$$

$$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,\ X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$$

$$T_n(t)=\varphi_n\cos\left(\frac{(2n+1)\pi a}{2l}t\right)+\psi_n\frac{2l}{(2n+1)\pi a}\sin\left(\frac{(2n+1)\pi a}{2l}t\right)$$

原定解问题的解可表示为

$$\begin{aligned} u(x,t)&=\sum_{n=0}^{\infty}T_n(t)X_n(x)\\ &=T_1(t)X_1(x)+T_2(t)X_2(x)\\ &=T_1(t)\sin\frac{3\pi}{2l}x+T_2(t)\sin\frac{5\pi}{2l}x\\ &=\cos\frac{3\pi a}{2l}t\cdot\sin\frac{3\pi}{2l}x+\frac{2l}{5\pi a}\sin\frac{5\pi a}{2l}t\cdot\sin\frac{5\pi}{2l}x \end{aligned}$$



若初始分布恰好为某个特征函数，可仅针对该n值对应的特征函数展开

习题

习题2, 5(4)

例

$$\begin{cases} u_{tt}-u_{xx}-4u=2\sin^2 x,\ 0<x<\pi,\ t>0 \\ u_x(0,t)=0,\ u_x(\pi,t)=0,\ t\geq 0 \\ u(x,0)=0,\ u_t(x,0)=0,\ 0\leq x\leq \pi \end{cases}$$

波动/一维/非齐	
2齐	2齐

解 边界已经齐次, [2, 2]型边界 $\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$

本例中 $l=\pi,\ \lambda_n=n^2,\ X_n(x)=\cos nx,\ (n=0,1,2,3\cdots)$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

正交分解, 原问题有分离变量的形式解

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\cos nx$$

自由项/初始条件按特征函数系展开

$$2\sin^2 x=\sum_{n=0}^{\infty}f_nX_n(x)=\sum_{n=0}^{\infty}f_n\cos nx$$

特征函数含有1, 需单独讨论

$$f_n=\frac{2}{l}\int_0^l 2\sin^2 x\cos nxdx=\frac{4}{\pi}\int_0^{\pi}\sin^2 x\cos nxdx=\begin{cases}1 & (n=0)\\ -1 & (n=2)\\ 0 & (n\neq 0,2)\end{cases}$$

其中

$$f_1=\frac{\int_0^{\pi}2\sin^2 x\cdot 1\cdot dx}{\int_0^{\pi}1^2\cdot dx}=\frac{2\int_0^{\pi}\sin^2 x\cdot 1\cdot dx}{\int_0^{\pi}1^2\cdot dx}=\frac{2\frac{\pi}{2}}{\pi}=1$$

习题

例

$$\begin{cases} u_{tt} - u_{xx} - 4u = 2\sin^2 x, \ 0 < x < \pi, \ t > 0 \\ u_x(0,t) = 0, \ u_x(\pi,t) = 0, \ t \ge 0 \\ u(x,0) = 0, \ u_t(x,0) = 0, \ 0 \le x \le \pi \end{cases}$$

波动/一维/非齐	
2齐	2齐

$l = \pi, \ \lambda_n = n^2, \ X_n(x) = \cos nx, \ (n = 0, 1, 2, 3\cdots)$

建立初值ODE, 将正交分解后的形式解/自由项代入原PDE

$$\sum_{n=0}^{\infty} T_n''(t) X_n(x) - \sum_{n=0}^{\infty} T_n(t) X_n''(x) - 4\sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} f_n X_n(x)$$

$$\sum_{n=0}^{\infty} T_n''(t) X_n(x) + \sum_{n=0}^{\infty} T_n(t) \lambda_n X_n(x) - 4\sum_{n=0}^{\infty} T_n(t) X_n(x) = \sum_{n=0}^{\infty} f_n X_n(x)$$

$$\sum_{n=0}^{\infty} \left[T_n''(t) + (\lambda_n - 4) T_n(t) \right] X_n(x) = \sum_{n=0}^{\infty} f_n X_n(x)$$

根据原初始条件

$$u(x,0) = \sum_{n=0}^{\infty} T_n(0) X_n(x) = 0$$

$$u_t(x,0) = \sum_{n=0}^{\infty} T_n'(0) X_n(x) = 0$$

初值ODE定解问题

$$\begin{cases} T_n''(t) + (n^2 - 4) T_n(t) = f_n \\ T_n(0) = 0, \ T_n'(0) = 0 \end{cases}$$

根据自由项取值不同分类讨论

$$f_n = \frac{2}{l}\int_0^l 2\sin^2 x \cos nx dx = \frac{4}{\pi}\int_0^{\pi} \sin^2 x \cos nx dx = \begin{cases} 1 & (n = 0) \\ -1 & (n = 2) \\ 0 & (n \ne 0, 2) \end{cases}$$

习题

例

$$\begin{cases} u_{tt}-u_{xx}-4u=2\sin^2 x,\ 0<x<\pi,\ t>0 \\ u_x(0,t)=0,\ u_x(\pi,t)=0,\ t\ge 0 \\ u(x,0)=0,\ u_t(x,0)=0,\ 0\le x\le \pi \end{cases}$$

波动/一维/非齐	
2齐	2齐

$l=\pi,\ \lambda_n=n^2,\ X_n(x)=\cos nx,\ (n=0,1,2,3\cdots)$

求解初值ODE

$$\begin{cases} T_n''(t)+(n^2-4)T_n(t)=f_n \\ T_n(0)=0,\ T_n'(0)=0 \end{cases}$$

特征方程定通解

$$s^2+(n^2-4)=0$$

自由项定特解

$$f_n=\begin{cases} 1 & (n=0) \\ -1 & (n=2) \\ 0 & (n\ne 0,2) \end{cases}$$

$n=0$

$$\begin{cases} T_0''(t)-4T_0(t)=1 \\ T_0(0)=0,\ T_0'(0)=0 \end{cases}$$

$$T_0(t)=C_{01}e^{2t}+C_{02}e^{-2t}+\bar{T}_0(t)$$

$$T_0(t)=\frac{1}{8}\left(e^{2t}+e^{-2t}\right)-\frac{1}{4}$$

$n=1$

$$\begin{cases} T_1''(t)-3T_1(t)=0 \\ T_1(0)=0,\ T_1'(0)=0 \end{cases}$$

$$T_1(t)=C_{01}e^{\sqrt{3}t}+C_{02}e^{-\sqrt{3}t}$$

$$T_1(t)=0$$

$n=2$

$$\begin{cases} T_2''(t)=-1 \\ T_2(0)=0,\ T_2'(0)=0 \end{cases}$$

$$T_2(t)=-\frac{t^2}{2}+C_{21}t+C_{02}$$

$$T_2(t)=-\frac{t^2}{2}$$

$n\ge 3$

$$\begin{cases} T_n''(t)+(n^2-4)T_n(t)=0 \\ T_n(0)=0,\ T_n'(0)=0 \end{cases}$$

$$T_n(t)=C_{n1}\cos\sqrt{n^2-4}t+C_{n2}\sin\sqrt{n^2-4}t$$

$$T_n(t)=0$$

本质: 根据叠加原理对自由项的拆分

$$u(x,t)=T_0(t)X_0(x)+T_1(t)X_1(x)+T_2(t)X_2(x)+\sum_{n=3}^{\infty}T_n(t)X_n(x)$$

$$=\frac{1}{8}\left(e^{2t}+e^{-2t}\right)-\frac{1}{4}-\frac{1}{2}t^2\cos 2x$$

习题

例

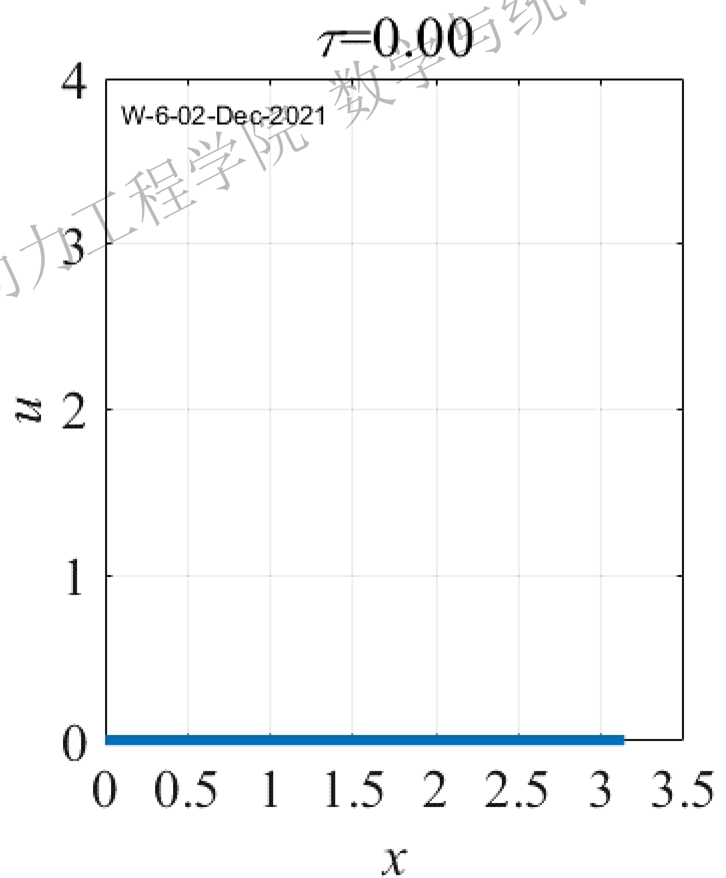
$$\begin{cases} u_{tt}-u_{xx}-4u=2\sin^2 x,\ 0<x<\pi,\ t>0 \\ u_x(0,t)=0,\ u_x(\pi,t)=0,\ t\ge 0 \\ u(x,0)=0,\ u_t(x,0)=0,\ 0\le x\le\pi \end{cases}$$

波动/一维/非齐	
2齐	2齐

$l=\pi,\ \lambda_n=n^2,\ X_n(x)=\cos nx,\ (n=0,1,2,3\cdots)$

原定解问题的解

$$u(x,t)=\frac{1}{8}\left(e^{2t}+e^{-2t}\right)-\frac{1}{4}-\frac{1}{2}t^2\cos 2x$$



习题

例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\dfrac{x}{2}, \ 0<x<\pi, \ t>0 \\ u_x(0,t)=1, \ u(\pi,t)=\pi, \ t\ge 0 \\ u(x,0)=0, \ 0\le x\le \pi \end{cases}$$

导热/一维/齐次	
2非齐	1非齐

边界条件齐次化常用辅助函数 $u(x,t)=v(x,t)+w(x,t)$

边界类型	边界条件	辅助函数
[1,1]	$u(0,t)=g_1(t), \ u(l,t)=g_2(t)$	$w(x,t)=\dfrac{g_2(t)-g_1(t)}{l}x+g_1(t)$
[2,1]	$u_x(0,t)=g_1(t), \ u(l,t)=g_2(t)$	$w(x,t)=g_1(t)(x-l)+g_2(t)$
[1,2]	$u(0,t)=g_1(t), \ u_x(l,t)=g_2(t)$	$w(x,t)=g_2(t)x+g_1(t)$
[2,2]	$u_x(0,t)=g_1(t), \ u_x(l,t)=g_2(t)$	$w(x,t)=\dfrac{g_2(t)-g_1(t)}{2l}x^2+g_1(t)x$

解 第1步 边界条件齐次化

设 $u(x,t)=v(x,t)+w(x,t)$

可取 $w(x,t)=1\cdot(x-\pi)+\pi=x$

各阶导数 $w_t=0$

$w_x=1, \ w_{xx}=0$

改造原PDE

$$u_t-a^2u_{xx}=\cos\frac{x}{2}$$
$$v_t+w_t-a^2v_{xx}-a^2w_{xx}=\cos\frac{x}{2}$$
$$v_t-a^2v_{xx}=\cos\frac{x}{2}$$

改造初始条件

$$u(x,0)=0$$
$$v(x,0)+w(x,0)=0$$
$$v(x,0)+x=0$$
$$v(x,0)=-x$$

原定解问题可转化为

$$\begin{cases} v_t - a^2 v_{xx} = \cos\dfrac{x}{2}, \ 0<x<\pi, \ t>0 \\ v_x(0,t)=0, \ v(\pi,t)=0, \ t\ge 0 \\ v(x,0)=-x, \ 0\le x\le \pi \end{cases}$$

习题

导热/一维/齐次

2非齐 | 1非齐

习题2, 6(1)

例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\dfrac{x}{2}, & 0<x<\pi,\ t>0 \\ u_x(0,t)=1,\ u(\pi,t)=\pi, & t\ge 0 \\ u(x,0)=0, & 0\le x\le \pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\dfrac{x}{2}, & 0<x<\pi,\ t>0 \\ v_x(0,t)=0,\ v(\pi,t)=0, & t\ge 0 \\ v(x,0)=-x, & 0\le x\le \pi \end{cases}$$

第2步 确定特征值/特征函数

$$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$$

本例中 $l=\pi$

$$\lambda_n=\left(\frac{2n+1}{2}\right)^2, X_n(x)=\cos\left(\frac{2n+1}{2}x\right), (n=0,1,2,3\cdots)$$

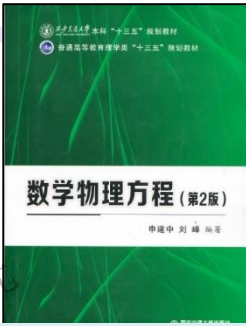
齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$
[1,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X'(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$
[2,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$
[2,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X'(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$
周期	$\begin{cases}\Phi''(\theta)+\lambda\Phi(\theta)=0\\\Phi(0)=\Phi(2\pi),\\\Phi'(0)=\Phi'(2\pi)\end{cases}$	$\lambda_0=0,\ \Phi_0(\theta)=1,\ (n=0)$ $\lambda_n=n^2,\ \Phi_n(\theta)=C_1\cos n\theta+C_2\sin n\theta,\ (n=1,2,3\cdots)$

习题

导热/一维/齐次

2非齐 | 1非齐

习题2, 6(1)



例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\dfrac{x}{2}, & 0<x<\pi,\ t>0 \\ u_x(0,t)=1,\ u(\pi,t)=\pi, & t\ge 0 \\ u(x,0)=0, & 0\le x\le \pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\dfrac{x}{2}, & 0<x<\pi,\ t>0 \\ v_x(0,t)=0,\ v(\pi,t)=0, & t\ge 0 \\ v(x,0)=-x, & 0\le x\le \pi \end{cases}$$

$$\lambda_n=\left(\frac{2n+1}{2}\right)^2,\ X_n(x)=\cos\left(\frac{2n+1}{2}x\right),\quad (n=0,1,2,3\cdots)$$

第3步 正交分解

设原问题有分离变量的形式解

$$u(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\cos\left(\frac{2n+1}{2}x\right)$$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

自由项/初始条件按特征函数系展开

$$\cos\frac{x}{2}=\sum_{n=1}^{\infty}f_nX_n(x)=\sum_{n=1}^{\infty}f_n\cos\left(\frac{2n+1}{2}x\right)$$

$$f_n=\frac{2}{l}\int_0^l\cos\frac{x}{2}\cos\frac{2n+1}{2}x\cdot dx=\begin{cases}1, & (n=0)\\ 0, & (n\ne 0)\end{cases}$$

$$v(x,0)=-x=\sum_{n=0}^{\infty}\varphi_n\cos\left(\frac{2n+1}{2}x\right)$$

$$\varphi_n=\frac{2}{l}\int_0^l(-x)\cos\left(\frac{2n+1}{2}x\right)\cdot dx=-\frac{2}{\pi}\left[\frac{2}{2n+1}x\sin\frac{2n+1}{2}x+\frac{4}{(2n+1)^2}\cos\frac{2n+1}{2}x\right]_0^{\pi}=\frac{4(-1)^{n+1}}{2n+1}+\frac{8}{(2n+1)^2\pi}$$

习题

导热/一维/齐次

2非齐 1非齐

习题2, 6(1)

例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\frac{x}{2}, \ 0<x<\pi,\ t>0 \\ u_x(0,t)=1,\ u(\pi,t)=\pi,\ t\ge 0 \\ u(x,0)=0,\ 0\le x\le\pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\frac{x}{2}, \ 0<x<\pi,\ t>0 \\ v_x(0,t)=0,\ v(\pi,t)=0,\ t\ge 0 \\ v(x,0)=-x,\ 0\le x\le\pi \end{cases}$$

$$\lambda_n=\left(\frac{2n+1}{2}\right)^2, X_n(x)=\cos\left(\frac{2n+1}{2}x\right),\ (n=0,1,2,3\cdots)$$

第4步 建立初值问题ODE

$$v(x,t)=\sum_{n=0}^{\infty}T_n(t)X_n(x)=\sum_{n=0}^{\infty}T_n(t)\cos\left(\frac{2n+1}{2}x\right)$$

假设形式解可逐项求导, 代入原PDE

$$\sum_{n=0}^{\infty}T_n'(t)X_n(x)-a^2\sum_{n=0}^{\infty}T_n(t)X_n''(x)=\sum_{n=0}^{\infty}f_nX_n(x)$$

$$\sum_{n=0}^{\infty}T_n'(t)X_n(x)-a^2\sum_{n=0}^{\infty}T_n(t)\left[-\lambda_nX_n(x)\right]=\sum_{n=0}^{\infty}f_nX_n(x)$$

$$\sum_{n=0}^{\infty}\left[T_n'(t)+a^2\lambda_nT_n(t)\right]X_n(x)=\sum_{n=0}^{\infty}f_nX_n(x)$$

$$T_n'(t)+a^2\lambda_nT_n(t)=f_n$$

用原PDE初始条件→初值问题ODE初始条件

$$v(x,0)=\sum_{n=1}^{\infty}T_n(0)X_n(x)=-x=\sum_{n=1}^{\infty}\varphi_nX_n(x)$$

建立了初值问题ODE

$$\begin{cases} T_n'(t)+a^2\lambda_nT_n(t)=f_n \\ T_n(0)=\varphi_n \end{cases}$$

习题

导热/一维/齐次

2非齐 | 1非齐

习题2, 6(1)

例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\dfrac{x}{2},\ 0<x<\pi,\ t>0 \\ u_x(0,t)=1,\ u(\pi,t)=\pi,\ t\ge 0 \\ u(x,0)=0,\ 0\le x\le \pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\dfrac{x}{2},\ 0<x<\pi,\ t>0 \\ v_x(0,t)=0,\ v(\pi,t)=0,\ t\ge 0 \\ v(x,0)=-x,\ 0\le x\le \pi \end{cases}$$

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, X_n(x)=\cos\left(\frac{2n+1}{2}x\right),\ (n=0,1,2,3\cdots)$$

第5步 求初值问题

$$\begin{cases} T_n'(t)+a^2\lambda_n T_n(t) = f_n \\ T_n(0)=\varphi_n \end{cases}$$

$$f_n = \frac{2}{l}\int_0^l \cos\frac{x}{2}\cos\frac{2n+1}{2}x\cdot dx = \begin{cases} 1,\ (n=0) \\ 0,\ (n\ne 0) \end{cases}$$

二阶线性非齐次ODE, 通解的形式为

$$\varphi_n = \frac{2}{l}\int_0^l (-x)\cos\left(\frac{2n+1}{2}x\right)\cdot dx = \frac{4(-1)^{n+1}}{2n+1}+\frac{8}{(2n+1)^2\pi}$$

$$T_n(t) = C_{1n}\exp\left(-a^2\lambda_n t\right)+C_{2n}$$

代入ODE

$$T_n'(t) = -a^2\lambda_n C_{1n}\exp\left(-a^2\lambda_n t\right)$$

$$-a^2\lambda_n C_{1n}\exp\left(-a^2\lambda_n t\right)+a^2\lambda_n C_{1n}\exp\left(-a^2\lambda_n t\right)+a^2\lambda_n C_{2n} = f_n$$

比较系数得 $C_{2n} = \dfrac{f_n}{a^2\lambda_n}$

初值ODE的解

由初始条件 $T_n(0)=\varphi_n = C_{1n}+\dfrac{f_n}{a^2\lambda_n}$ ➡ $C_{1n} = \varphi_n - \dfrac{f_n}{a^2\lambda_n}$

$$T_n(t) = \left(\varphi_n - \frac{f_n}{a^2\lambda_n}\right)\exp\left(-a^2\lambda_n t\right)+\frac{f_n}{a^2\lambda_n}$$

习题

导热/一维/齐次

2非齐 | 1非齐

习题2, 6(1)

例

$$\begin{cases} u_t - a^2 u_{xx} = \cos\frac{x}{2}, & 0<x<\pi,\ t>0 \\ u_x(0,t)=1,\ u(\pi,t)=\pi, & t\ge 0 \\ u(x,0)=0, & 0\le x\le \pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\frac{x}{2}, & 0<x<\pi,\ t>0 \\ v_x(0,t)=0,\ v(\pi,t)=0, & t\ge 0 \\ v(x,0)=-x, & 0\le x\le \pi \end{cases}$$

$$\lambda_n = \left(\frac{2n+1}{2}\right)^2,\ X_n(x)=\cos\left(\frac{2n+1}{2}x\right),\ (n=0,1,2,3\cdots)$$

第6步 回代

$$f_n = \frac{2}{l}\int_0^l \cos\frac{x}{2}\cos\frac{2n+1}{2}x\cdot dx = \begin{cases} 1, & (n=0) \\ 0, & (n\ne 0) \end{cases}$$

$$\varphi_n = \frac{2}{l}\int_0^l (-x)\cos\left(\frac{2n+1}{2}x\right)\cdot dx = \frac{4(-1)^{n+1}}{2n+1}+\frac{8}{(2n+1)^2\pi}$$

n是否=0

$$T_n(t) = \left(\varphi_n - \frac{f_n}{a^2\lambda_n}\right)\exp\left(-a^2\lambda_n t\right)+\frac{f_n}{a^2\lambda_n}$$

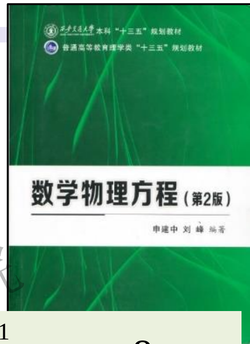
边界条件齐次化辅助函数

n=0单列

$$u(x,t)=w(x,t)+v(x,t)$$

$$= x + T_0(t)\cos\frac{x}{2}+\sum_{n=1}^{\infty}T_n(t)\cos\left(\frac{2n+1}{2}x\right)$$

$$= x + \left[\left(\varphi_0-\frac{4}{a^2}\right)\exp\left(-\frac{a^2}{4}t\right)+\frac{4}{a^2}\right]\cos\frac{x}{2}+\sum_{n=1}^{\infty}\varphi_n\exp\left(-a^2\lambda_n t\right)\cos\left(\frac{2n+1}{2}x\right)$$



例

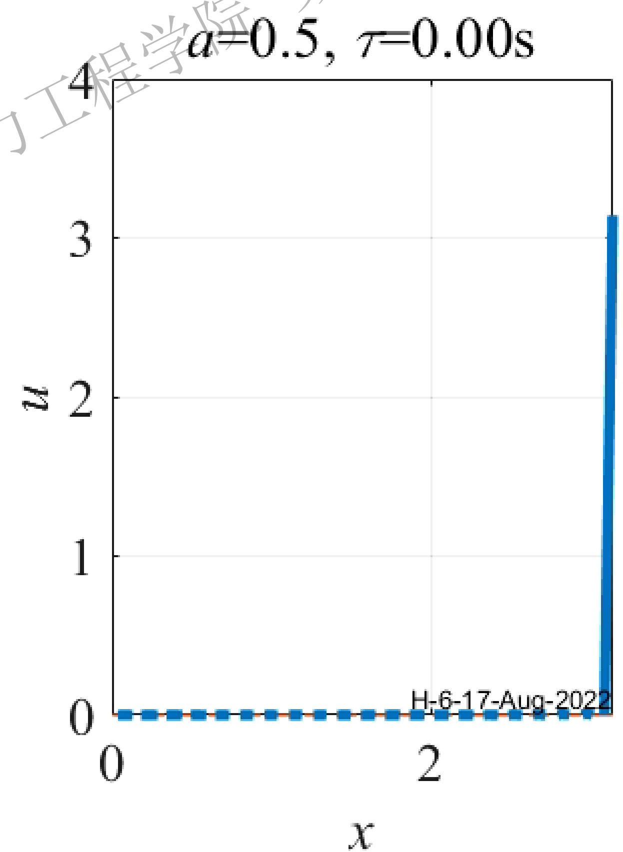
$$\begin{cases} u_t - a^2 u_{xx} = \cos\dfrac{x}{2}, & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = 1,\ u(\pi,t) = \pi, & t \ge 0 \\ u(x,0) = 0, & 0 \le x \le \pi \end{cases}$$

$$\begin{cases} v_t - a^2 v_{xx} = \cos\dfrac{x}{2}, & 0 < x < \pi,\ t > 0 \\ v_x(0,t) = 0,\ v(\pi,t) = 0, & t \ge 0 \\ v(x,0) = -x, & 0 \le x \le \pi \end{cases}$$

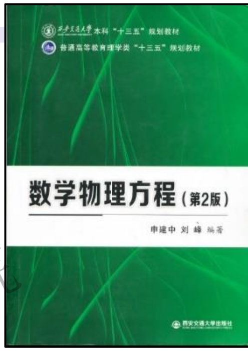
$$\lambda_n = \left(\frac{2n+1}{2}\right)^2, X_n(x) = \cos\left(\frac{2n+1}{2}x\right),\ (n = 0,1,2,3\cdots)$$

$$\varphi_n = \frac{2}{l}\int_0^l (-x)\cos\left(\frac{2n+1}{2}x\right)dx = \frac{4(-1)^{n+1}}{2n+1} + \frac{8}{(2n+1)^2\pi}$$

$$u(x,t) = x + \left[\left(\varphi_0 - \frac{4}{a^2}\right)\exp\left(-\frac{a^2}{4}t\right) + \frac{4}{a^2}\right]\cos\frac{x}{2} + \sum_{n=1}^{\infty}\varphi_n \exp\left(-a^2\lambda_n t\right)\cos\left(\frac{2n+1}{2}x\right)$$



习题



例

$$\begin{cases} u_t - a^2 u_{xx} + b^2 u = 0, \ 0 < x < l, \ t > 0 \\ u(0,t) = 0, \ u(l,t) = 0, \ t \geq 0 \\ u(x,0) = \varphi(x), \ 0 \leq x \leq l \end{cases}$$

导热/一维/齐次	
1齐	1齐

解 第1步 确定特征值/特征函数

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \sin\left(\frac{n\pi}{l}x\right), (n = 1, 2, 3\cdots)$$

齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \sin\left(\frac{n\pi}{l}x\right), (n = 1, 2, 3\cdots)$
[1,2]	$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X'(l) = 0 \end{cases}$	$\lambda_n = \left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x) = \sin\left[\frac{(2n+1)\pi}{2l}x\right], (n = 0, 1, 2, 3\cdots)$
[2,1]	$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X(l) = 0 \end{cases}$	$\lambda_n = \left[\frac{(2n+1)\pi}{2l}\right]^2, X_n(x) = \cos\left[\frac{(2n+1)\pi}{2l}x\right], \ (n = 0, 1, 2, 3\cdots)$
[2,2]	$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X'(0) = X'(l) = 0 \end{cases}$	$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \cos\left(\frac{n\pi}{l}x\right), (n = 0, 1, 2, 3\cdots)$
周期	$\begin{cases} \Phi''(\theta) + \lambda\Phi(\theta) = 0 \\ \Phi(0) = \Phi(2\pi), \\ \Phi'(0) = \Phi'(2\pi) \end{cases}$	$\lambda_0 = 0, \ \Phi_0(\theta) = 1, \ (n = 0)$ $\lambda_n = n^2, \ \Phi_n(\theta) = C_1 \cos n\theta + C_2 \sin n\theta, \ (n = 1, 2, 3\cdots)$

习题

习题2, 6(3)

例

$$\begin{cases} u_t - a^2 u_{xx} + b^2 u = 0, \ 0 < x < l, \ t > 0 \\ u(0,t) = 0, \ u(l,t) = 0, \ t \ge 0 \\ u(x,0) = \varphi(x), \ 0 \le x \le l \end{cases}$$

导热/一维/齐次	
1齐	1齐

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \sin\left(\frac{n\pi}{l}x\right), (n = 1, 2, 3\cdots)$$

Fourier级数展开

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx, \quad (n = 0, 1, 2, 3\cdots)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx, \quad (n = 1, 2, 3\cdots)$$

第2步 正交分解

设原问题有分离变量的形式解

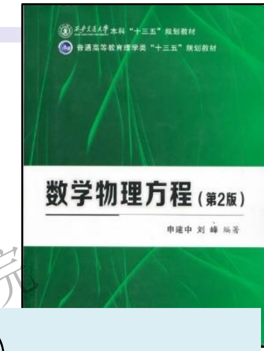
$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

自由项/初始条件按特征函数系展开

$$u(x,0) = \varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x) = \sum_{n=1}^{\infty} \varphi_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\varphi_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\cdot dx$$

习题2, 6(3)



例

$$\begin{cases} u_t - a^2 u_{xx} + b^2 u = 0, \ 0<x<l, \ t>0 \\ u(0,t)=0, \ u(l,t)=0, \ t\ge 0 \\ u(x,0)=\varphi(x), \ 0\le x\le l \end{cases}$$

导热/一维/齐次	
1齐	1齐

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \ X_n(x) = \sin\left(\frac{n\pi}{l}x\right), (n=1,2,3\cdots)$$

第3步 建立初值问题ODE

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{l}x\right)$$

假设形式解可逐项求导, 代入原PDE

$$\sum_{n=1}^{\infty} T_n'(t) X_n(x) - a^2 \sum_{n=1}^{\infty} T_n(t) X_n''(x) + b^2 \sum_{n=1}^{\infty} T_n(t) X_n(x) = 0$$

$$\sum_{n=1}^{\infty} T_n'(t) X_n(x) - a^2 \sum_{n=1}^{\infty} T_n(t) \left[-\lambda_n X_n(x)\right] + b^2 \sum_{n=1}^{\infty} T_n(t) X_n(x) = 0$$

$$\sum_{n=1}^{\infty} \left[T_n'(t) + \left(a^2\lambda_n + b^2\right) T_n(t)\right] X_n(x) = 0$$

$$T_n'(t) + \left(a^2\lambda_n + b^2\right) T_n(t) = 0$$

用原PDE初始条件→初值问题ODE初始条件

$$u(x,0) = \sum_{n=1}^{\infty} T_n(0) X_n(x) = \varphi(x) = \sum_{n=1}^{\infty} \varphi_n X_n(x)$$

建立了初值问题ODE

$$\begin{cases} T_n'(t) + \left(a^2\lambda_n + b^2\right) T_n(t) = 0 \\ T_n(0) = \varphi_n \end{cases}$$

习题

例

$$\begin{cases} u_t - a^2 u_{xx} + b^2 u = 0,\ 0<x<l,\ t>0 \\ u(0,t)=0,\ u(l,t)=0,\ t\ge 0 \\ u(x,0)=\varphi(x),\ 0\le x\le l \end{cases}$$

导热/一维/齐次	
1齐	1齐

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$$

第4步 求初值问题

$$\begin{cases} T_n'(t) + \left(a^2\lambda_n + b^2\right)T_n(t) = 0 \\ T_n(0)=\varphi_n \end{cases}$$

二阶线性齐次ODE, 通解的形式为

$$T_n(t) = C_{1n}\exp\left[-\left(a^2\lambda_n+b^2\right)t\right]$$

由初始条件 $T_n(0)=\varphi_n = C_{1n}$ ➡ $C_{1n}=\varphi_n$

初值ODE的解 $T_n(t)=\varphi_n\exp\left[-\left(a^2\lambda_n+b^2\right)t\right]$

原定解问题的解可表示为

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\left(\frac{n\pi}{l}x\right)$$
$$=\sum_{n=1}^{\infty}\varphi_n\exp\left[-\left(a^2\lambda_n+b^2\right)t\right]\sin\left(\frac{n\pi}{l}x\right)$$

$$\varphi_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\cdot dx$$

习题

例

$$\begin{cases} u_t - a^2 u_{xx} + b^2 u = 0, & 0 < x < l,\ t > 0 \\ u(0,t) = 0,\ u(l,t) = 0, & t \ge 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \end{cases}$$

导热/一维/齐次	
1齐	1齐

讨论

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n \exp\left[-\left(a^2\lambda_n + b^2\right)t\right] \sin\left(\frac{n\pi}{l}x\right)$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2,\ X_n(x) = \sin\left(\frac{n\pi}{l}x\right), (n = 1, 2, 3 \cdots)$$

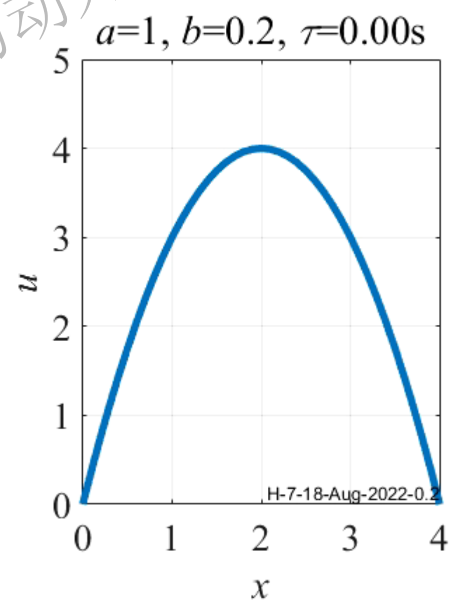
设初始分布

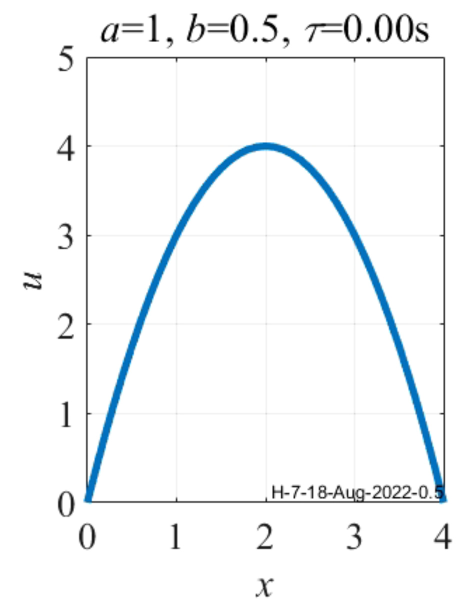
$$\varphi(x) = x(l-x) = \sum_{n=1}^{\infty} \varphi_n X_n(x) = \sum_{n=1}^{\infty} \varphi_n \sin\left(\frac{n\pi}{l}x\right)$$

$$\varphi_n = \frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left[l\int_0^l x\sin\frac{n\pi x}{l}\cdot dx - \int_0^l x^2 \sin\frac{n\pi x}{l}\cdot dx\right]$$

$$= \frac{4l^2}{\pi^3}\cdot\frac{1-\cos n\pi}{n^3} = \frac{4l^2}{\pi^3}\cdot\frac{1-(-1)^n}{n^3}$$



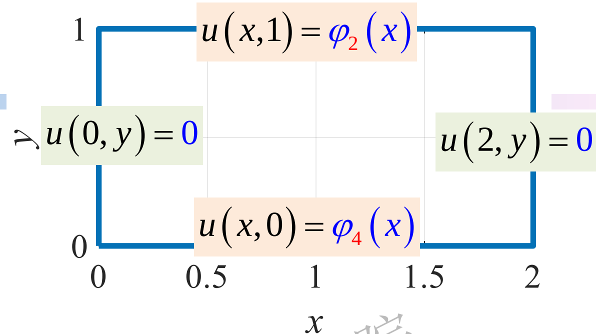


习题

例

$$\begin{cases} u_{xx}+u_{yy}=A,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(l,y)=0,\ 0\le y\le h \\ u(x,0)=\varphi_4(x),\ u(x,h)=\varphi_2(x),\ 0\le x\le l \end{cases}$$

Laplace/二维/齐次		
x边界	1齐	1齐
y边界	1非齐	1非齐



解 第1步 特征值问题

方程非齐次/x边界[1,1]型→特征值/特征函数为

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),\ (n=1,2,3\cdots)$$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

第2步 正交分解

方程的解/自由项/ y边界按照特征函数系展开

$$u(x,y)=\sum_{n=1}^{\infty}Y_n(y)X_n(x)=\sum_{n=1}^{\infty}Y_n(y)\sin\left(\frac{n\pi}{l}x\right)$$

$$A=\sum_{n=1}^{\infty}f_nX_n(x)\qquad f_n=\frac{2}{l}\int_0^l A\sin\left(\frac{n\pi}{l}x\right)dx=\frac{2A}{n\pi}[1-\cos n\pi]$$

$$\varphi_4(x)=\sum_{n=1}^{\infty}\varphi_{4n}(x)X_n(x)\qquad \varphi_{2n}=\frac{2}{l}\int_0^l\varphi_2(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

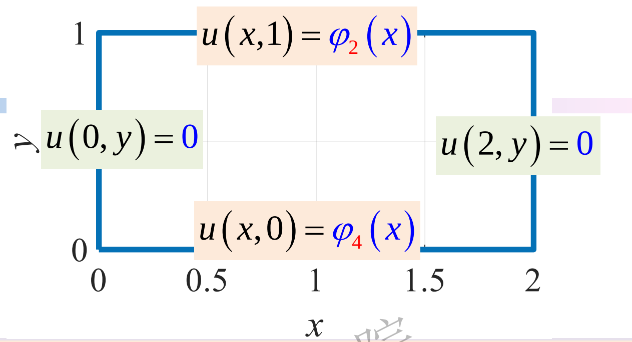
$$\varphi_2(x)=\sum_{n=1}^{\infty}\varphi_{2n}(x)X_n(x)\qquad \varphi_{4n}=\frac{2}{l}\int_0^l\varphi_4(x)\sin\frac{n\pi x}{l}\cdot dx$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=A,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(l,y)=0,\ 0\le y\le h \\ u(x,0)=\varphi_4(x),\ u(x,h)=\varphi_2(x),\ 0\le x\le l \end{cases}$$

Laplace/二维/齐次		
x边界	1齐	1齐
y边界	1非齐	1非齐



$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),\ (n=1,2,3\cdots)$$

第3步 建立y方向ODE

正交分解形式代入原方程

$$\sum_{n=1}^{\infty}Y_n(y)X_n''(x)+\sum_{n=1}^{\infty}Y_n''(y)X_n(x)=\sum_{n=1}^{\infty}f_nX_n(x)$$

$$\sum_{n=1}^{\infty}Y_n(y)\left[-\lambda_nX_n(x)\right]+\sum_{n=1}^{\infty}Y_n''(y)X_n(x)=\sum_{n=1}^{\infty}f_nX_n(x)$$

$$\sum_{n=1}^{\infty}\left[Y_n''(y)-\lambda_nY_n(y)\right]X_n(x)=\sum_{n=1}^{\infty}f_nX_n(x)$$

$$Y_n''(y)-\lambda_nY_n(y)=f_n$$

根据下底面边界 $u(x,0)=\varphi_4(x)$

$$\sum_{n=1}^{\infty}Y_n(0)X_n(x)=\sum_{n=1}^{\infty}\varphi_{4n}X_n(x)$$

$$Y_n(0)=\varphi_{4n}$$

根据上底面边界 $u(x,h)=\varphi_2(x)$

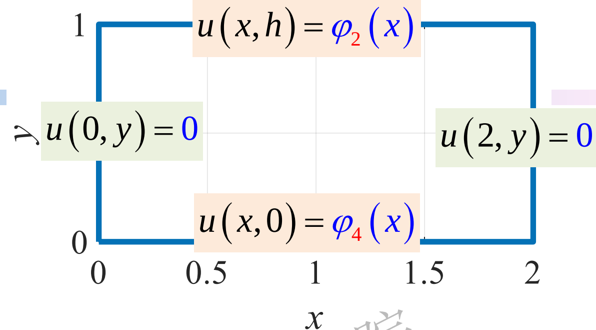
$$\sum_{n=1}^{\infty}Y_n(h)X_n(x)=\sum_{n=1}^{\infty}\varphi_{2n}X_n(x)$$

$$Y_n(h)=\varphi_{2n}$$

$$\begin{cases} Y_n''(y)-\lambda_nY_n(y)=f_n\quad 0<y<h \\ Y_n(0)=\varphi_{4n},Y_n(h)=\varphi_{2n},0\le y\le h \end{cases}$$

习题

Laplace/二维/齐次		
x边界	1齐	1齐
y边界	1非齐	1非齐



例

$$\begin{cases} u_{xx}+u_{yy}=A,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(l,y)=0,\ 0\le y\le h \\ u(x,0)=\varphi_4(x),\ u(x,h)=\varphi_2(x),\ 0\le x\le l \end{cases}$$

第4步 解y方向ODE

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),\ (n=1,2,3\cdots)$$

$$\begin{cases} Y_n''(y)-\lambda_n Y_n(y)=f_n\ \ 0<y<h \\ Y_n(0)=\varphi_{4n},Y_n(h)=\varphi_{2n},0\le y\le h \end{cases}$$

通解形式为

$$Y(y)=C_{1n}\sinh\left(\frac{n\pi}{l}y\right)+C_{2n}\cosh\left(\frac{n\pi}{l}y\right)+\bar{Y}$$

假设特解为常数 $\bar{Y}=C$ 代入ODE

$$0-\lambda_n C=f_n \qquad \bar{Y}=C=-\frac{f_n}{\lambda_n}$$

通解可表示为

$$Y(y)=C_{1n}\sinh\left(\frac{n\pi}{l}y\right)+C_{2n}\cosh\left(\frac{n\pi}{l}y\right)-\frac{f_n}{\lambda_n}$$

根据边界条件

$$Y(0)=\varphi_{4n}=C_{2n}-\frac{f_n}{\lambda_n}$$

$$Y(h)=\varphi_{2n}=C_{1n}\sinh\left(\frac{n\pi}{l}h\right)+C_{2n}\cosh\left(\frac{n\pi}{l}h\right)-\frac{f_n}{\lambda_n}$$

解得 $C_{2n}=\varphi_{4n}+\dfrac{f_n}{\lambda_n}$

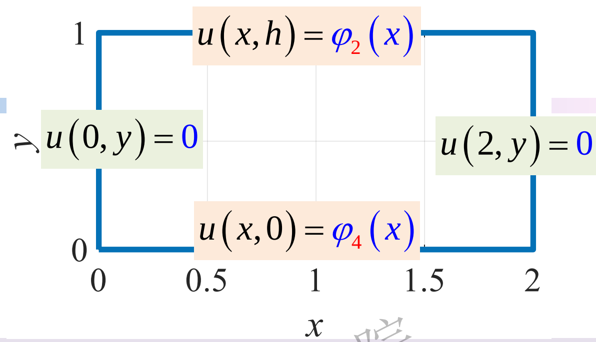
$$C_{1n}=\frac{\varphi_{2n}-\varphi_{4n}\cosh\left(\frac{n\pi}{l}h\right)+\frac{f_n}{\lambda_n}\left[1-\cosh\left(\frac{n\pi}{l}h\right)\right]}{\sinh\left(\frac{n\pi}{l}h\right)}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=A,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(l,y)=0,\ 0\le y\le h \\ u(x,0)=\varphi_4(x),\ u(x,h)=\varphi_2(x),\ 0\le x\le l \end{cases}$$

Laplace/二维/齐次		
x边界	1齐	1齐
y边界	1非齐	1非齐



第5步 回代

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),\ (n=1,2,3\cdots)$$

$$u(x,y)=\sum_{n=1}^{\infty}\left[C_{1n}\sinh\left(\frac{n\pi}{l}y\right)+C_{2n}\cosh\left(\frac{n\pi}{l}y\right)-\frac{f_n}{\lambda_n}\right]\sin\left(\frac{n\pi}{l}x\right)$$

$$C_{1n}=\frac{\varphi_{2n}-\varphi_{4n}\cosh\left(\frac{n\pi}{l}h\right)+\frac{f_n}{\lambda_n}\left[1-\cosh\left(\frac{n\pi}{l}h\right)\right]}{\sinh\left(\frac{n\pi}{l}h\right)},\quad C_{2n}=\varphi_{4n}+\frac{f_n}{\lambda_n}$$

$$f_n=\frac{2}{l}\int_0^l A\sin\left(\frac{n\pi}{l}x\right)dx=\frac{2A}{n\pi}[1-\cos n\pi]$$

$$\varphi_{2n}=\frac{2}{l}\int_0^l\varphi_2(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

$$\varphi_{4n}=\frac{2}{l}\int_0^l\varphi_4(x)\sin\frac{n\pi x}{l}\cdot dx$$

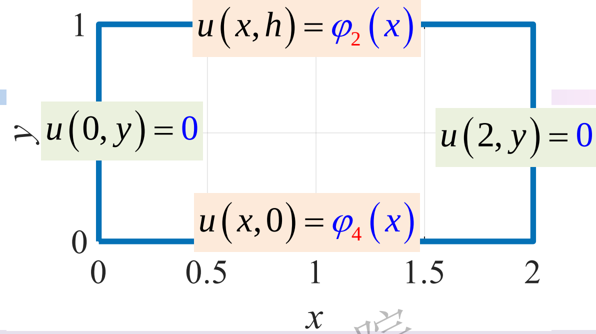
可被视为矩形区域上
Laplace方程的通解

习题

例

$$\begin{cases} u_{xx}+u_{yy}=A,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(l,y)=0,\ 0\le y\le h \\ u(x,0)=\varphi_4(x),\ u(x,h)=\varphi_2(x),\ 0\le x\le l \end{cases}$$

Laplace/二维/齐次		
x边界	1齐	1齐
y边界	1非齐	1非齐



讨论 $u(x,0)=\varphi_4(x)=0,\ u(x,h)=\varphi_2(x)=x(x-l),\ f=0$

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),\ (n=1,2,3\cdots)$$

解可简化为

$$Y(y)=C_{1n}\sinh\left(\frac{n\pi}{l}y\right)$$

$$C_{1n}=\frac{\varphi_{2n}}{\sinh\left(\frac{n\pi}{l}h\right)}$$

$$\varphi_{2n}=\frac{2}{l}\int_0^l\varphi_2(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

$$=\frac{2}{l}\int_0^l x(x-l)\sin\frac{n\pi x}{l}\cdot dx=\underbrace{\frac{2}{l}\int_0^l x^2\sin\frac{n\pi x}{l}\cdot dx}_{1}-\underbrace{2\int_0^l x\sin\frac{n\pi x}{l}\cdot dx}_{2}=\frac{4l^2}{n^3\pi^3}\left[(-1)^n-1\right]$$

$$\frac{2}{l}\int_0^l x^2\sin\frac{n\pi x}{l}\cdot dx \quad 1$$

$$=\frac{2}{l}\left[-\frac{l}{n\pi}x^2\cos\frac{n\pi}{l}x+2\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi}{l}x\right]_0^l$$

$$=-\frac{2l^2}{n\pi}\cos(n\pi)+\frac{4l^2}{n^3\pi^3}\cos(n\pi)-\frac{4l^2}{n^3\pi^3}$$

$$-2\int_0^l x\sin\frac{n\pi x}{l}\cdot dx \quad 2$$

$$=-2\left[-\frac{l}{n\pi}x\cos\frac{n\pi}{l}x+\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi}{l}x\right]_0^l=\frac{2l^2}{n\pi}\cos n\pi$$

$$u(x,y)=\sum_{n=1}^{\infty}\left[\begin{array}{l}C_{1n}\sinh\left(\frac{n\pi}{l}y\right)\\ +C_{2n}\cosh\left(\frac{n\pi}{l}y\right)-\frac{f_n}{\lambda_n}\end{array}\right]\sin\left(\frac{n\pi}{l}x\right)$$

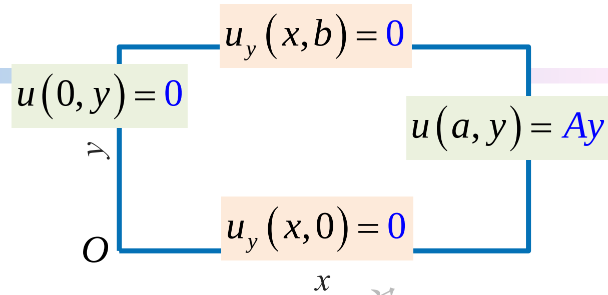
$$u(x,y)=\frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}\cdot\frac{\sinh\left(\frac{n\pi}{l}y\right)}{\sinh\left(\frac{n\pi}{l}h\right)}\sin\left(\frac{n\pi}{l}x\right)$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



解 第1步 特征值问题　方程非齐次/y边界[2,2]型→特征值/特征函数为

习题2, 7(1)

特征函数是y向的

$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\ Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),\ (n=0,1,2,3\cdots)$$

特征函数含有 **1**

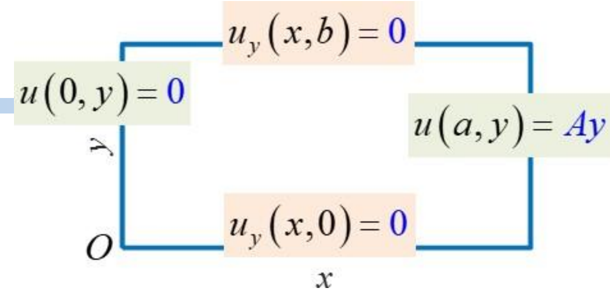
齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$
[1,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X'(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$
[2,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$
[2,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X'(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$
周期	$\begin{cases}\Phi''(\theta)+\lambda\Phi(\theta)=0\\\Phi(0)=\Phi(2\pi),\\\Phi'(0)=\Phi'(2\pi)\end{cases}$	$\lambda_0=0,\ \Phi_0(\theta)=1,\ (n=0)$ $\lambda_n=n^2,\ \Phi_n(\theta)=C_1\cos n\theta+C_2\sin n\theta,\ (n=1,2,3\cdots)$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



第2步 正交分解

$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\ Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),(n=0,1,2,3\cdots)$$

设原定解问题有分离变量的形式解

$$u(x,y)=\sum_{n=0}^{\infty}X_n(x)Y_n(y)=\sum_{n=0}^{\infty}X_n(x)\cos\left(\frac{n\pi}{b}y\right)$$

自由项按特征函数系展开

$$x=\sum_{n=0}^{\infty}f_nY_n(y)=\sum_{n=0}^{\infty}f_n\cos\left(\frac{n\pi}{b}y\right)$$

$$\begin{cases} n=0 & f_0=\frac{1}{2}\cdot\frac{2}{b}\int_0^b x\cdot 1\cdot dy=\frac{x}{b}\cdot b=x \\ n>0 & f_n=\frac{2}{b}\int_0^b x\cos\left(\frac{n\pi}{b}y\right)dy=\frac{2x}{b}\frac{b}{n\pi}\sin\left(\frac{n\pi}{b}y\right)\Big|_0^b=0 \end{cases}$$

x方向右边界按特征函数系展开

$$Ay=\sum_{n=0}^{\infty}\psi_nY_n(y)=\sum_{n=0}^{\infty}\psi_n\cos\left(\frac{n\pi}{b}y\right)$$

$$\begin{cases} n=0 & \psi_0=\frac{1}{2}\cdot\frac{2}{b}\int_0^b Ay\cdot 1\cdot dy=\frac{A}{b}\cdot\frac{b^2}{2}=\frac{Ab}{2} \\ n>0 & \psi_n=\frac{2}{b}\int_0^b Ay\cos\left(\frac{n\pi}{b}y\right)dy \end{cases}$$

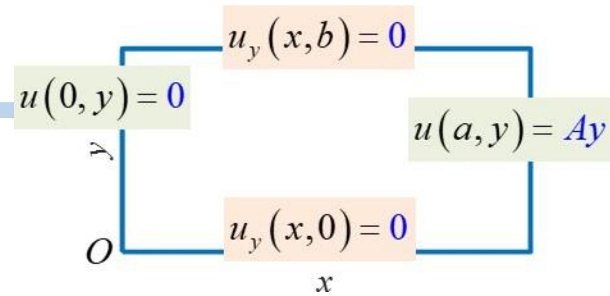
$$=\frac{2A}{b}\left[\frac{b}{n\pi}y\sin\frac{n\pi}{b}y+\frac{b^2}{n^2\pi^2}\cos\frac{n\pi}{b}y\right]_0^b=\frac{2\left[(-1)^n-1\right]}{n^2\pi^2}Ab$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



第3步 建立x方向ODE

$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\ Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),(n=0,1,2,3\cdots)$$

$$u(x,y)=\sum_{n=0}^{\infty}X_n(x)Y_n(y)=\sum_{n=0}^{\infty}X_n(x)\cos\left(\frac{n\pi}{b}y\right)$$

设形式解可逐项求导

$$\sum_{n=0}^{\infty}X_n''(x)Y_n(y)+\sum_{n=0}^{\infty}X_n(x)Y_n''(y)=\sum_{n=0}^{\infty}f_nY_n(y)$$

$$\sum_{n=0}^{\infty}X_n''(x)Y_n(y)+\sum_{n=0}^{\infty}X_n(x)\left[-\lambda_nY_n(y)\right]=\sum_{n=0}^{\infty}f_nY_n(y)$$

$$\sum_{n=0}^{\infty}\left[X_n''(x)-\lambda_nX_n(x)\right]Y_n(y)=\sum_{n=0}^{\infty}f_nY_n(y)$$

PDE的x方向边界条件→ODE边界条件

$$u(0,y)=\sum_{n=0}^{\infty}X_n(0)Y_n(y)=0$$

$$u(a,y)=\sum_{n=0}^{\infty}X_n(a)Y_n(y)=Ay=\sum_{n=0}^{\infty}\psi_nY_n(y)$$

建立初值定解问题

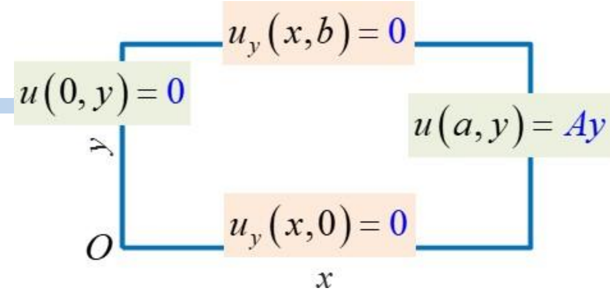
$$\begin{cases} X_n''(x)-\lambda_nX_n(x)=f_n \\ X_n(0)=0,\ X_n(a)=\psi_n \end{cases}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\ Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),(n=0,1,2,3\cdots)$$

第4步 解x方向ODE

$$\begin{cases} X_n''(x)-\lambda_n X_n(x)=f_n \\ X_n(0)=0,\ X_n(a)=\psi_n \end{cases}$$

$$f_n=\begin{cases} x,\ (n=0) \\ 0,\ (n>0) \end{cases}\qquad \psi_n=\begin{cases} \dfrac{Ab}{2},\ (n=0) \\ \dfrac{2\left[(-1)^n-1\right]}{n^2\pi^2}Ab,\ (n>0) \end{cases}$$

ODE类型随n不同而不同→分类讨论

❶ 当 $n=0$ 时

$$\lambda_0=0,\ Y_0(x)=1,\ f_0=x\neq 0$$

ODE可化简为

$$\begin{cases} X_0''(x)=f_0=x \\ X_0(0)=0,\ X_0(a)=\psi_0 \end{cases}$$ 直接积分法

通解为 $X_0(x)=\dfrac{x^3}{6}+\bar{C}_{n1}x+\bar{C}_{n2}$

根据边界条件

$$X_0(0)=\frac{x^3}{6}+\bar{C}_{n1}x+\bar{C}_{n2}=0$$

$$X_0(a)=\frac{a^3}{6}+\bar{C}_{n1}a+\bar{C}_{n2}=\psi_0=\frac{Ab}{2}$$

➡ $\bar{C}_{n1}=\dfrac{Ab}{2a}-\dfrac{a^2}{6}$，$\bar{C}_{n2}=0$

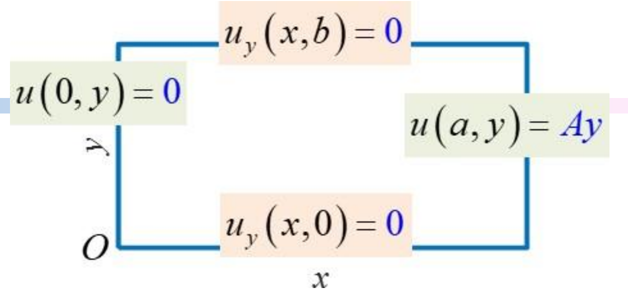
$n=0$ 时x方向ODE的解为 $X_0(x)=\dfrac{x^3}{6}+\left(\dfrac{Ab}{2a}-\dfrac{a^2}{6}\right)x$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\ Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),(n=0,1,2,3\cdots)$$

第4步 解x方向ODE

$$\begin{cases} X_n''(x)-\lambda_n X_n(x)=f_n \\ X_n(0)=0,\ X_n(a)=\psi_n \end{cases}$$

$$f_n=\begin{cases} x,\ (n=0) \\ 0,\ (n>0) \end{cases}\qquad \psi_n=\begin{cases} \dfrac{Ab}{2},\ (n=0) \\ \dfrac{2\left[(-1)^n-1\right]}{n^2\pi^2}Ab,\ (n>0) \end{cases}$$

❷ 当 $n>0$ 时 $\lambda_0\ne 0,\ f_n=0$

ODE可化简为

$$\begin{cases} X_n''(x)-\lambda_n X_n(x)=0 \\ X_n(0)=0,\ X_n(a)=\psi_n \end{cases}$$

二阶线性齐次ODE, 通解为

$$X_n(x)=\bar{C}_{n1}\sinh\frac{n\pi}{b}x+\bar{C}_{n2}\cosh\frac{n\pi}{b}x$$

根据边界条件

$$X_n(0)=\bar{C}_{n1}\sinh\frac{n\pi}{b}x+\bar{C}_{n2}\cosh\frac{n\pi}{b}x=0$$

$$X_n(a)=\bar{C}_{n1}\sinh\frac{n\pi}{b}a+\bar{C}_{n2}\cosh\frac{n\pi}{b}a=\psi_n=\frac{2\left[(-1)^n-1\right]}{n^2\pi^2}Ab$$

$$\bar{C}_{n1}=\frac{\left[(-1)^n-1\right]}{n^2\pi^2}\frac{2Ab}{\sinh\dfrac{n\pi a}{b}},\quad \bar{C}_{n2}=0$$

$n>0$ 时x方向ODE的解为

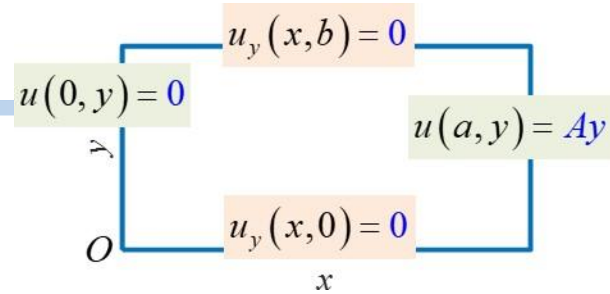
$$X_n(x)=\frac{2Ab\left[(-1)^n-1\right]}{n^2\pi^2}\cdot\frac{\sinh\dfrac{n\pi}{b}x}{\sinh\dfrac{n\pi}{b}a}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=x,\ 0<x<l,\ 0<y<h \\ u(0,y)=0,\ u(a,y)=Ay,\ 0\le y\le b \\ u_y(x,0)=0,\ u_y(x,b)=0,\ 0\le x\le a \end{cases}$$

Poisson/二维/非齐次		
x边界	1齐	1非齐
y边界	2齐	2齐



$$\lambda_n=\left(\frac{n\pi}{b}\right)^2,\quad Y_n(x)=\cos\left(\frac{n\pi}{b}y\right),(n=0,1,2,3\cdots)$$

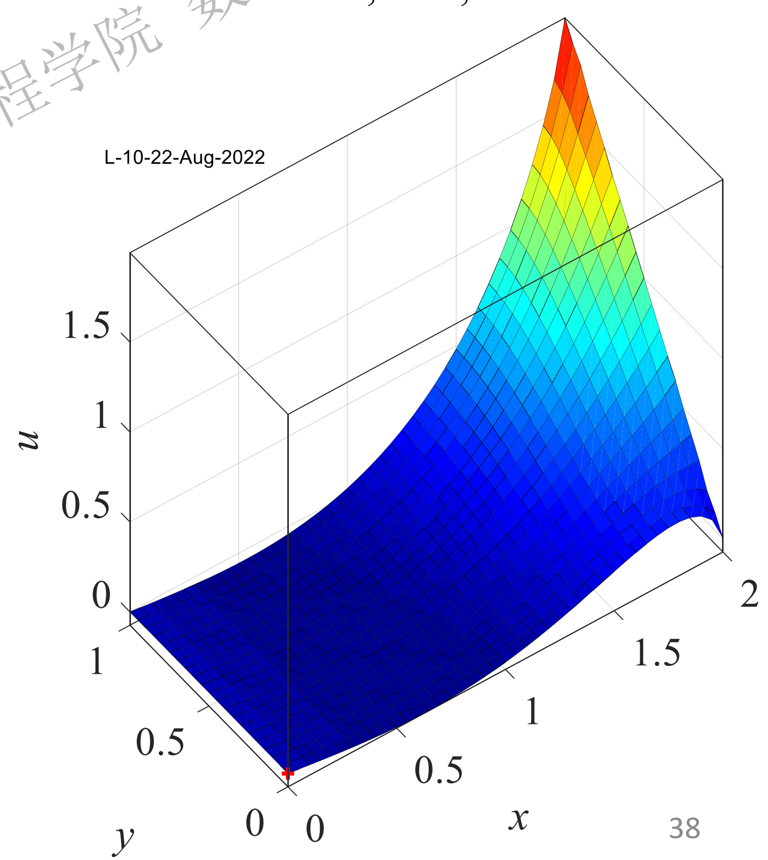
第5步 回代 原定解问题的解为

$$u(x,y)=\sum_{n=0}^{\infty}X_n(x)Y_n(y)=X_0(x)\cdot 1+\sum_{n=1}^{\infty}X_n(x)\cos\left(\frac{n\pi}{b}y\right)$$

$$=\frac{x^3}{6}+\left(\frac{Ab}{2a}-\frac{a^2}{6}\right)x$$

$$+2Ab\cdot\sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2\pi^2}\frac{\sinh\frac{n\pi}{b}x}{\sinh\frac{n\pi}{b}a}\cos\left(\frac{n\pi}{b}y\right)$$

a=2, b=1, A=2



习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

解 第1步 坐标转换 扇形区域→直角系无齐次边界→向极坐标转换

$$x=\rho\cos\theta,\ y=\rho\sin\theta$$

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_{\rho}+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

第2步 特征值/特征函数

周向周期性边界

$$\lambda_0=0,\ \ \Phi_0(\theta)=1,\ (n=0)$$

$$\lambda_n=n^2,\ \ \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\ (n=1,2,3\cdots)$$

特征函数含有**1**, 需分类讨论

算子 $A=-\dfrac{d^2}{dx^2}$ 不同齐次边界条件下的特征值/特征函数

齐次边界类型	特征值问题	特征值/特征函数
[1,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\sin\left(\frac{n\pi}{l}x\right),(n=1,2,3\cdots)$
[1,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X(0)=X'(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\sin\left[\frac{(2n+1)\pi}{2l}x\right],(n=0,1,2,3\cdots)$
[2,1]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X(l)=0\end{cases}$	$\lambda_n=\left[\frac{(2n+1)\pi}{2l}\right]^2,X_n(x)=\cos\left[\frac{(2n+1)\pi}{2l}x\right],\ (n=0,1,2,3\cdots)$
[2,2]	$\begin{cases}X''(x)+\lambda X(x)=0\\X'(0)=X'(l)=0\end{cases}$	$\lambda_n=\left(\frac{n\pi}{l}\right)^2,\ X_n(x)=\cos\left(\frac{n\pi}{l}x\right),(n=0,1,2,3\cdots)$
周期	$\begin{cases}\Phi''(\theta)+\lambda\Phi(\theta)=0\\\Phi(0)=\Phi(2\pi),\\\Phi'(0)=\Phi'(2\pi)\end{cases}$	$\lambda_0=0,\ \Phi_0(\theta)=1,\ (n=0)$ $\lambda_n=n^2,\ \Phi_n(\theta)=C_1\cos n\theta+C_2\sin n\theta,\ (n=1,2,3\cdots)$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第3步 正交分解 设原定解问题有形式解

$$u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

自由项按特征函数系展开

$$\frac{1}{2}\rho\sin 2\theta=\sum_{n=0}^{\infty}f_n\Phi_n(\theta)$$

$n=0$

$$f_0=\frac{1}{2}\cdot\frac{2}{2\pi}\int_0^{2\pi}\frac{1}{2}\rho\sin 2\theta\cdot 1\cdot d\theta=\frac{\rho}{4\pi}\left[-\frac{1}{2}\cos 2\theta\right]_0^{2\pi}=0$$

$n>0$

$$f_{n1}=\frac{2}{2\pi}\int_0^{2\pi}\frac{1}{2}\rho\sin 2\theta\cdot\cos n\theta\cdot d\theta=\begin{cases}\frac{\rho}{2\pi}\left[-\frac{\cos 4\theta}{8}\right]_0^{2\pi}=0,\ (n=2) \\ \frac{\rho}{2\pi}\left[-\frac{\cos(2+n)\theta}{2(2+n)}-\frac{\cos(2-n)\theta}{2(2-n)}\right]_0^{2\pi}=0,\ (n\ne 2)\end{cases}$$

$$f_{n2}=\frac{2}{2\pi}\int_0^{2\pi}\frac{1}{2}\rho\sin 2\theta\cdot\sin n\theta\cdot d\theta=\begin{cases}\frac{\rho}{2\pi}\left[\frac{1}{4}(2\theta-\sin 2\theta\cos 2\theta)\right]_0^{2\pi}=\frac{\rho}{2},\ (n=2) \\ \frac{\rho}{2\pi}\left[-\frac{\sin(2+n)\theta}{2(2+n)}+\frac{\sin(2-n)\theta}{2(2-n)}\right]_0^{2\pi}=0,\ (n\ne 2)\end{cases}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$

$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$

第3步 正交分解 $u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

径向外边界按照特征函数系展开

$$a(\cos\theta+\sin\theta)=\sum_{n=0}^{\infty}\varphi_n\Phi_n(\theta)$$

$n=0$

$$\varphi_0=\frac{1}{2}\cdot\frac{2}{2\pi}\int_0^{2\pi}a(\cos\theta+\sin\theta)\cdot 1\cdot d\theta=\frac{a}{2\pi}[\sin\theta-\cos\theta]_0^{2\pi}=0$$

$n>0$

$$\varphi_{n1}=\frac{2}{2\pi}\int_0^{2\pi}a(\cos\theta+\sin\theta)\cdot\cos n\theta\cdot d\theta$$

$$=\frac{a}{\pi}\left[\int_0^{2\pi}\cos\theta\cdot\cos n\theta\cdot d\theta+\int_0^{2\pi}\sin\theta\cdot\cos n\theta\cdot d\theta\right]$$

$$=\begin{cases}\frac{a}{\pi}\left[\frac{1}{2}(\theta+\sin\theta\cos\theta)-\frac{1}{4}\cos 2\theta\right]_0^{2\pi}=a,\quad (n=1)\\ \frac{a}{\pi}\left[\frac{\sin(1+n)\theta}{2(1+n)}+\frac{\sin(1-n)\theta}{2(1-n)}-\frac{\cos(1+n)\theta}{2(1+n)}-\frac{\cos(1-n)\theta}{2(1-n)}\right]_0^{2\pi}=0,\quad (n\ne 1)\end{cases}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_{\rho}+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第3步 正交分解 $u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$

Fourier级数展开

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\cdot dx,\quad (n=0,1,2,3\cdots)$$

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\cdot dx,\quad (n=1,2,3\cdots)$$

径向外边界按照特征函数系展开

$$a(\cos\theta+\sin\theta)=\sum_{n=0}^{\infty}\varphi_n\Phi_n(\theta)$$

$n>0$

$$\varphi_{n2}=\frac{2}{2\pi}\int_0^{2\pi}a(\cos\theta+\sin\theta)\cdot\sin n\theta\cdot d\theta$$

$$=\frac{a}{\pi}\left[\int_0^{2\pi}\cos\theta\cdot\sin n\theta\cdot d\theta+\int_0^{2\pi}\sin\theta\cdot\sin n\theta\cdot d\theta\right]$$

$$=\begin{cases} \frac{a}{\pi}\left[-\frac{1}{4}\cos 2\theta+\frac{1}{2}(\theta-\sin\theta\cos\theta)\right]_0^{2\pi}=a,\quad (n=1) \\ \frac{a}{\pi}\left[-\frac{\cos(1+n)\theta}{2(1+n)}+\frac{\cos(1-n)\theta}{2(1-n)}-\frac{\sin(1+n)\theta}{2(1+n)}+\frac{\sin(1-n)\theta}{2(1-n)}\right]_0^{2\pi}=0,\quad (n\ne1) \end{cases}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\ (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\ (n=1,2,3\cdots)$$

第3步 正交分解 $u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$

汇总

自由项

$$\frac{1}{2}\rho\sin 2\theta=\sum_{n=0}^{\infty}f_n\Phi_n(\theta)$$

$$f_n=\begin{cases} (n=0),\ 0,\ (\Phi_0=1) \\ (n>0),\ \begin{cases} f_{n1}=0,\ (\Phi_n=\cos n\theta) \\ f_{n2}=\begin{cases}\frac{\rho}{2},\ (n=2) \\ 0,\ (n\ne 2)\end{cases},\ (\Phi_n=\sin n\theta) \end{cases} \end{cases}$$

径向外边界

$$a(\cos\theta+\sin\theta)=\sum_{n=0}^{\infty}\varphi_n\Phi_n(\theta)$$

$$\varphi_n=\begin{cases} (n=0),\ 0,\ (\Phi_0=1) \\ (n>0),\ \begin{cases} \varphi_{n1}=\begin{cases}a,\ (n=1) \\ 0,\ (n\ne 1)\end{cases},\ (\Phi_n=\cos n\theta) \\ \varphi_{n2}=\begin{cases}a,\ (n=1) \\ 0,\ (n\ne 1)\end{cases},\ (\Phi_n=\sin n\theta) \end{cases} \end{cases}$$

后续径向ODE的求解需分段讨论 $n=0,\ n=1,\ n=2,\ n\ge 3$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_\rho+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第4步 建立径向ODE

设原定解问题有形式解 $u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$

$$\sum_{n=0}^{\infty}R_n''(\rho)\Phi_n(\theta)+\frac{1}{\rho}\sum_{n=0}^{\infty}R_n'(\rho)\Phi_n(\theta)+\frac{1}{\rho^2}\sum_{n=0}^{\infty}R_n(\rho)\Phi_n''(\theta)=\sum_{n=0}^{\infty}f_n\Phi_n(\theta)$$

$$\sum_{n=0}^{\infty}R_n''(\rho)\Phi_n(\theta)+\sum_{n=0}^{\infty}\frac{1}{\rho}R_n'(\rho)\Phi_n(\theta)+\sum_{n=0}^{\infty}\frac{1}{\rho^2}R_n(\rho)\left[-\lambda_n\Phi_n(\theta)\right]=\sum_{n=0}^{\infty}f_n\Phi_n(\theta)$$

$$\sum_{n=0}^{\infty}\left[R_n''(\rho)+\frac{1}{\rho}R_n'(\rho)+\frac{-\lambda_n}{\rho^2}R_n(\rho)\right]\Phi_n(\theta)=\sum_{n=0}^{\infty}f_n\Phi_n(\theta)$$

$$\begin{cases} \rho^2R_n''(\rho)+\rho R_n'(\rho)-\lambda_nR_n(\rho)=f_n \\ R_n(b)=0,R_n(a)=\varphi_n \end{cases}$$

径向边界条件

$$u(b,\theta)=\sum_{n=0}^{\infty}R_n(b)\Phi_n(\theta)=0$$

$$u(a,\theta)=\sum_{n=0}^{\infty}R_n(a)\Phi_n(\theta)=a(\cos\theta+\sin\theta)=\sum_{n=0}^{\infty}\varphi_n\Phi_n(\theta)$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$

$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$

第5步 解径向ODE

$$\begin{cases} \rho^2R_n''(\rho)+\rho R_n'(\rho)-\lambda_nR_n(\rho)=f_n \\ R_n(b)=0, R_n(a)=\varphi_n \end{cases}$$

❶ 当 $n=0$ 时 $\lambda_0=0, f_0=0, \varphi_0=0$

$$\begin{cases} \rho^2R_0''(\rho)+\rho R_0'(\rho)=0 \\ R_0(b)=0, R_0(a)=0 \end{cases}$$

齐次Euler方程,通解为 $R_0(\rho)=C_{01}\ln\rho+C_{02}$

由径向边界条件

$R_0(b)=C_{01}\ln b+C_{02}=0$

$R_0(a)=C_{01}\ln a+C_{02}=0$

$C_{01}=C_{02}=0$

Euler 方程的解 $t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+a_nx=0$

$$\begin{cases} a_n<0,\ x=C_1t^{\sqrt{-a_n}}+C_2t^{-\sqrt{-a_n}} \\ a_n=0,\ x=C_1\ln t+C_2 \\ a_n>0,\ x=C_1\cos\left(\sqrt{a_n}\ln t\right)+C_2\sin\left(\sqrt{a_n}\ln t\right) \end{cases}$$

$n=0$ 时径向ODE解为 $R_0(\rho)=0$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_\rho+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第5步 解径向ODE

$$\begin{cases} \rho^2R_n''(\rho)+\rho R_n'(\rho)-\lambda_nR_n(\rho)=f_n \\ R_n(b)=0, R_n(a)=\varphi_n \end{cases}$$

❷ 当 $n=1$ 时 $\lambda_1=1,\ f_1=0,\ \varphi_{1,1}=\varphi_{1,2}=a$

$$\begin{cases} \rho^2R_1''(\rho)+\rho R_1'(\rho)-\lambda_1R_1(\rho)=0 \\ R_1(b)=0, R_1(a)=\varphi_{1,1}=\varphi_{1,2}=a \end{cases}$$

齐次Euler方程, 通解为 $R_1(\rho)=C_{11}t+C_{12}t^{-1}$

由径向边界条件

$$R_1(b)=C_{11}b+C_{12}b^{-1}=0$$

$$R_1(a)=C_{11}a+C_{12}a^{-1}=a$$

$$C_{11}=\frac{a^2}{a^2-b^2}$$

$$C_{12}=-\frac{a^2b^2}{a^2-b^2}$$

Euler 方程的解 $t^2\dfrac{d^2x}{dt^2}+t\dfrac{dx}{dt}+a_nx=0$

$$\begin{cases} a_n<0,\ x=C_1t^{\sqrt{-a_n}}+C_2t^{-\sqrt{-a_n}} \\ a_n=0,\ x=C_1\ln t+C_2 \\ a_n>0,\ x=C_1\cos\left(\sqrt{a_n}\ln t\right)+C_2\sin\left(\sqrt{a_n}\ln t\right) \end{cases}$$

$n=1$ 时径向ODE解为

$$R_1(\rho)=\frac{a^2}{a^2-b^2}\rho-\frac{a^2b^2}{a^2-b^2}\rho^{-1}$$

该解同时适用于特征函数

$$\Phi_{1,1}(\theta)=\cos\theta,\ \Phi_{1,2}(\theta)=\sin\theta$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=\frac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$

$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$

第5步 解径向ODE

$$\begin{cases} \rho^2R_n''(\rho)+\rho R_n'(\rho)-\lambda_nR_n(\rho)=f_n \\ R_n(b)=0,R_n(a)=\varphi_n \end{cases}$$

❸ 当 $n=2,\Phi_2(\theta)=\cos 2\theta$ 时

$\lambda_2=2^2,\ f_{2,1}=0,\quad \varphi_{2,1}=\varphi_{2,2}=0$

$$\begin{cases} \rho^2R_2''(\rho)+\rho R_2'(\rho)-\lambda_2R_2(\rho)=f_{2,1}=0 \\ R_2(b)=0,R_2(a)=0 \end{cases}$$

齐次Euler方程, 通解为 $R_2(\rho)=C_{21}t^2+C_{22}t^{-2}$

根据径向边界条件

$R_2(b)=C_{21}b^2+C_{22}b^{-2}=0$

$R_2(a)=C_{21}a^2+C_{22}a^{-2}=0$

$C_{21}=C_{22}=0$

Euler 方程的解 $\quad t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}+a_nx=0$

$$\begin{cases} a_n<0,\ x=C_1t^{\sqrt{-a_n}}+C_2t^{-\sqrt{-a_n}} \\ a_n=0,\ x=C_1\ln t+C_2 \\ a_n>0,\ x=C_1\cos\left(\sqrt{a_n}\ln t\right)+C_2\sin\left(\sqrt{a_n}\ln t\right) \end{cases}$$

$n=2$ 径向ODE的解为

$$R_2(\rho)=0$$

该解仅适于特征函数 $\Phi_{2,1}(\theta)=\cos 2\theta$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_\rho+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第5步 解径向ODE

$$\begin{cases} \rho^2R_n''(\rho)+\rho R_n'(\rho)-\lambda_n R_n(\rho)=f_n \\ R_n(b)=0, R_n(a)=\varphi_n \end{cases}$$

❸ 当 $n=2,\Phi_2(\theta)=\sin 2\theta$ 时

$$\lambda_2=2^2,\ f_2=\frac{\rho}{2},\quad \varphi_{2,1}=\varphi_{2,2}=0$$

$$\begin{cases} \rho^2R_2''(\rho)+\rho R_2'(\rho)-\lambda_2 R_2(\rho)=f_{2,2}=\dfrac{\rho}{2} \\ R_n(b)=0, R_n(a)=0 \end{cases}$$

非齐次Euler方程 令 $\rho=\exp s$，则 $s=\ln\rho$

$$R'(\rho)=\frac{dR}{ds}\frac{ds}{d\rho}=\frac{1}{\rho}\frac{dR}{ds}\qquad R''(\rho)=-\frac{1}{\rho^2}\frac{dR}{ds}+\frac{1}{\rho^2}\frac{d^2R}{ds^2}$$

$$\rho^2\left(-\frac{1}{\rho^2}\frac{dR}{ds}+\frac{1}{\rho^2}\frac{d^2R}{ds^2}\right)+\rho\left(\frac{1}{\rho}\frac{dR}{ds}\right)-\lambda_2R_2(\rho)=\frac{\exp s}{2}$$

$$\frac{d^2R_2(s)}{ds^2}-\lambda_2R_2(s)=\frac{1}{2}\exp s$$

$$R(s)=C_{21}\exp(2s)+C_{21}\exp(-2s)+\bar{R}$$

特解形式为 $\bar{R}=\bar{C}_1\exp s$

各阶导数 $\bar{R}'=\bar{R}''=\bar{C}_1\exp s$

代入ODE $\bar{C}_1\exp s-\lambda_2\bar{C}_1\exp s=\dfrac{1}{2}\exp s$

得 $\bar{C}_1=\dfrac{1}{2(1-\lambda_2)}=-\dfrac{1}{6}$

特解为 $\bar{R}=-\dfrac{1}{6}\exp s=-\dfrac{\rho}{6}$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_\rho+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
径向ρ	1齐	1非齐

$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$

$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$

第5步 解径向ODE

❸ 当 $n=2,\Phi_2(\theta)=\sin 2\theta$ 时

$$\begin{cases} \rho^2R_2''(\rho)+\rho R_2'(\rho)-\lambda_2R_2(\rho)=f_2=\dfrac{\rho}{2} \\ R_2(b)=0,R_2(a)=0 \end{cases}$$

非齐次Euler方程通解为

$$R(s)=C_{21}\exp(2s)+C_{22}\exp(-2s)-\frac{1}{6}\exp s$$

还原变量 $\rho=\exp s,\ s=\ln\rho$

$$R(\rho)=C_{21}\rho^2+C_{22}\rho^{-2}-\frac{\rho}{6}$$

根据径向边界条件

$$R(b)=C_{21}b^2+C_{22}b^{-2}-\frac{b}{6}=0$$

$$R(a)=C_{21}a^2+C_{22}a^{-2}-\frac{a}{6}=0$$

解得

$$C_{21}=\frac{1}{6}\left(\frac{a^3-b^3}{a^4-b^4}\right),\quad C_{22}=\frac{a^3b^3}{6}\left[\frac{a-b}{a^4-b^4}\right]$$

$n=2,\Phi_2(\theta)=\sin 2\theta$ 时, 径向ODE通解为

$$R_2(\rho)=\frac{1}{6}\left(\frac{a^3-b^3}{a^4-b^4}\right)\rho^2+\frac{a^3b^3}{6}\left[\frac{a-b}{a^4-b^4}\right]\rho^{-2}-\frac{\rho}{6}$$

习题

例

$$\begin{cases} u_{xx}+u_{yy}=xy,\ b^2<x^2+y^2<a^2 \\ u(x,y)=0,\ x^2+y^2=b^2 \\ u(x,0)=x+y,\ x^2+y^2=a^2 \end{cases}$$

Poisson/二维/非齐次		
x边界		
y边界		

$$\begin{cases} u_{\rho\rho}+\dfrac{1}{\rho}u_{\rho}+\dfrac{1}{\rho^2}u_{\theta\theta}=\dfrac{1}{2}\rho\sin 2\theta,\ a<\rho<b,\ 0\le\theta<2\pi \\ u(b,\theta)=0,\ \rho=b \\ u(a,\theta)=a(\cos\theta+\sin\theta),\ \rho=a \end{cases}$$

Laplace/二维/齐次		
周向θ	周期	周期
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$$\lambda_0=0,\quad \Phi_0(\theta)=1,\quad (n=0)$$

$$\lambda_n=n^2,\quad \Phi_n(\theta)=C_{n1}\cos n\theta+C_{n2}\sin n\theta,\quad (n=1,2,3\cdots)$$

第6步 回代

原定解问题的解可表示为

$$u(\rho,\theta)=\sum_{n=0}^{\infty}R_n(\rho)\Phi_n(\theta)$$

$$=\underbrace{R_0(\rho)\Phi_0(\theta)}_{=0}+R_1(\rho)\Phi_1(\theta)+R_2(\rho)\Phi_2(\theta)+\underbrace{\sum_{n=3}^{\infty}R_n(\rho)\Phi_n(\theta)}_{=0}$$

$$=\left(\frac{a^2}{a^2-b^2}\rho-\frac{a^2b^2}{a^2-b^2}\rho^{-1}\right)(\cos\theta+\sin\theta)$$

$$+\left[\frac{\rho^2}{6}\cdot\frac{a^3-b^3}{a^4-b^4}+\frac{a^3b^3}{6\rho^2}\cdot\frac{a-b}{a^4-b^4}-\frac{\rho}{6}\right]\sin 2\theta$$