第三章 一元函数积分学及其应用

第三节 两种基本积分法

之一——换元积分法

- 不定积分的第一换元法
- 不定积分的第二换元法
- 定积分的换元法
- 小结

作业:Page213　1，3，4

第一部分 不定积分的第一换元法

问题 $\int \cos 2x dx = \sin 2x + C,$ ？

解决方法 利用复合函数，设置中间变量.

过程 令 $u = 2x$ 则 $dx = \frac{1}{2} du$ 从而

$$\int \cos 2x dx = \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C.$$

一般地， 设$F'(u) = f(u)$, 即 $\int f(u)du = F(u) + C.$

如 $u = \varphi(x)$可导且$\varphi'(x)$连续，

则 $dF[\phi(x)] = \underline{f[\phi(x)]\phi'(x)dx}$

即：$\int \underline{f[\phi(x)]\phi'(x)dx} = F[\phi(x)] + C = [\int f(u)du]_{u=\varphi(x)}$

由此可得如下换元法定理：

定理1 设 $f(u)$ 是连续函数，$u=\varphi(x)$ 可导，

且 $\varphi'(x)$ 连续，则有换元公式

$$\int f[\varphi(x)]\varphi'(x)dx = \left[\int f(u)du\right]_{u=\varphi(x)}$$

第一类换元公式（**凑微分法**）

说明 使用此公式的关键在于将

$$\int f[\varphi(x)]\varphi'(x)dx. \text{ 化为 } \int g(x)dx$$

观察重点不同，所得结论不同.

例1　求$\int \sin 2x dx$.

解（一）$\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

解（二）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C;$

解（三）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C.$

例2 求$\int \frac{1}{3+2x}dx$.

解 $\frac{1}{3+2x} = \frac{1}{2} \cdot \frac{1}{3+2x} \cdot (3+2x)'$,

$$\int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x} \cdot (3+2x)'dx$$

$$\underline{\underline{u=(3+2x)}}\ \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|3+2x| + C.$$

一般地 $\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$

例3 求 $\int \frac{1}{x(1+2\ln x)}dx.$

解 $\int \frac{1}{x(1+2\ln x)}dx = \int \frac{1}{1+2\ln x}d(\ln x)$

$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$

$\xlongequal{u=1+2\ln x} \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u|+C$

$= \frac{1}{2}\ln|1+2\ln x|+C.$

定理1 设 $f(u)$ 具有原函数，$u=\varphi(x)$ 可导，

且 $\varphi'(x)$ 连续，则有换元公式

$$\int f[\varphi(x)]\varphi'(x)dx = [\int f(u)du]_{u=\varphi(x)}$$

第一类换元公式（**凑微分法**）

说明 使用此公式的关键在于将

$$\int f[\varphi(x)]\varphi'(x)dx.\ \text{化为}\ \int g(x)dx$$

观察重点不同，所得结论不同.

常用的几种配元形式:

$$(1)\ \int f(ax+b)\mathrm{d}x=\frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$\left.\begin{aligned}&(2)\ \int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\ \mathrm{d}x^n\\&(3)\ \int f(x^n)\frac{1}{x}\mathrm{d}x=\frac{1}{n}\int f(x^n)\ \frac{1}{x^n}\ \mathrm{d}x^n\end{aligned}\right\}\text{凑幂法}$$

$$(4)\ \int f(\sin x)\cos x\mathrm{d}x=\int f(\sin x)\,\mathrm{d}\sin x$$

$$(5)\ \int f(\cos x)\sin x\mathrm{d}x=-\int f(\cos x)\ \mathrm{d}\cos x$$

$$(6)\ \int f(\tan x)\sec^2 x\mathrm{d}x=\int f(\tan x)\ \mathrm{d}\tan x$$

$$(7)\ \int f(e^x)e^x\mathrm{d}x=\int f(e^x)\ \mathrm{d}e^x$$

$$(8)\ \int f(\ln x)\frac{1}{x}\mathrm{d}x=\int f(\ln x)\ \mathrm{d}\ln x$$

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例4 求$\int \frac{x}{(1+x)^3}dx.$

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]d(1+x)$$

$$= -\frac{1}{1+x} + C_1 + \frac{1}{2(1+x)^2} + C_2$$

$$= -\frac{1}{1+x} + \frac{1}{2(1+x)^2} + C.$$

$$\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1)$$

例5 求$\int \frac{1}{a^2+x^2}dx.$

解 $$\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a}+C.$$

例3. 3. 1−5 求 $\int \frac{1}{a^2-x^2}dx.$

例3. 3. 1−4 求 $\int \frac{1}{\sqrt{a^2-x^2}}dx.$

例6 求$\displaystyle\int \frac{1}{x^2-8x+25}dx.$

解 $$\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

例7 求$\int \frac{1}{1+e^{x}} dx.$

解 $\int \frac{1}{1+e^{x}} dx = \int \frac{1+e^{x}-e^{x}}{1+e^{x}} dx$

$$= \int \left(1 - \frac{e^{x}}{1+e^{x}} \right) dx = \int dx - \int \frac{e^{x}}{1+e^{x}} dx$$

$$= \int dx - \int \frac{1}{1+e^{x}} d(1+e^{x})$$

$$= x - \ln(1+e^{x}) + C.$$

例8 求$\int(1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx.$

解 $\because \left(x+\frac{1}{x}\right)'=1-\frac{1}{x^2},$

$\therefore \int(1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx$

$=\int e^{x+\frac{1}{x}}d(x+\frac{1}{x})=e^{x+\frac{1}{x}}+C.$

例9 求 $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}dx.$

解 原式 $=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{\left(\sqrt{2x+3}+\sqrt{2x-1}\right)\left(\sqrt{2x+3}-\sqrt{2x-1}\right)}dx$

$$=\frac{1}{4}\int \sqrt{2x+3}\,dx-\frac{1}{4}\int \sqrt{2x-1}\,dx$$

$$=\frac{1}{8}\int \sqrt{2x+3}\,d(2x+3)-\frac{1}{8}\int \sqrt{2x-1}\,d(2x-1)$$

$$=\frac{1}{12}\sqrt{(2x+3)^3}-\frac{1}{12}\sqrt{(2x-1)^3}+C.$$

$$\int x^{\mu}dx=\frac{x^{\mu+1}}{\mu+1}+C\ (\mu\neq -1)$$

例10 求$\int \frac{1}{1+\cos x}dx.$

解 $\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\sin^2 x}d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

例11 求$\int \sin^2 x \cdot \cos^5 x dx.$

解 $\int \sin^2 x \cdot \cos^5 x dx = \int \sin^2 x \cdot \cos^4 x d(\sin x)$

$= \int \sin^2 x \cdot (1-\sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

说明 当被积函数是三角函数相乘时，拆开奇次项去凑微分.

例11−1 求$\int \sin^4 x \cdot \cos^2 x dx.$

求不定积分 $\int \sin^m x \cos^n x \, \mathrm{d}x$

① 若 m 为奇数，则

$$\int \sin^m x \cos^n x \, \mathrm{d}x = \int \sin^{m-1} x \cos^n x \sin x \, \mathrm{d}x$$

$$= -\int (1-\cos^2 x)^{\frac{m-1}{2}} \cos^n x \, \mathrm{d}\cos x \, .$$

② 若 n 为奇数，

$$\int \sin^m x \cos^n x \, \mathrm{d}x = \int \sin^m x (1-\sin^2 x)^{\frac{n-1}{2}} \, \mathrm{d}\sin x$$

$$= \int u^m (1-u^2)^{\frac{n-1}{2}} \, \mathrm{d}u \, .$$ 其中 $u = \sin x$

求不定积分 $\int \sin^m x \cos^n x \mathrm{d}x$

③ 若 m、n 均为偶数，

则一般先利用三角恒等式例如，$2\sin x\cos x=\sin 2x$，

$$\sin^2 x=\frac{1-\cos 2x}{2}，\quad \cos^2 x=\frac{1+\cos 2x}{2} 等$$

进行降次(在 m、n 较大时需要多次降次) 然后再进行换元

例12　求 $\int \cos 3x \cos 2x dx.$

解　$\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)],$

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x)dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

和差化积公式:

(1) $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

(2) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

(3) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

(4) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

积化和差公式:

(5) $$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

(6) $$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

(7) $$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

(8) $$\sin\alpha\sin\beta=-\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$$

例13 求$\int \csc x dx$.

解（一）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$$= \ln\left|\tan\frac{x}{2}\right| + C$$

解（二）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \qquad u = \cos x$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C.$$

解（三）$\int \csc x dx = \int \frac{\csc x(\csc x + \cot x)}{(\csc x + \cot x)} dx$

$$= -\ln\left|\csc x + \cot x\right| + C.$$

$\int \csc x dx$

$$= \int \frac{\csc x(\csc x - \cot x)}{(\csc x - \cot x)} dx \quad \boxed{= \ln\left|\csc x - \cot x\right| + C.}$$

$$\boxed{\int \csc x dx = -\ln\left|\csc x + \cot x\right| + C.}$$

$$\boxed{\int \sec x dx = \ln\left|\sec x + \tan x\right| + C.}$$

$$\frac{1-\cos x}{1+\cos x}=\frac{(1-\cos x)^2}{(1+\cos x)(1-\cos x)}$$

$$=\frac{1-2\cos x+\cos^2 x}{\sin^2 x}$$

$$=\csc^2 x-2\cot x\csc x+\cot^2 x$$

$$=(\csc x-\cot x)^2$$

$$\ln\left|\tan\frac{x}{2}\right|+C$$

$$\frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+C$$

$$-\ln\left|\csc x+\cot x\right|+C$$

$$\ln\left|\csc x-\cot x\right|+C$$

$$\csc x-\cot x=\frac{1-\cos x}{\sin x}=\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}$$

例14 设 $f'(\sin^2 x)=\cos^2 x$, 求 $f(x)$.

解 令 $u=\sin^2 x \quad \Rightarrow \quad \cos^2 x=1-u$,

$$f'(u)=1-u,$$

$$f(u)=\int(1-u)du=u-\frac{1}{2}u^2+C,$$

$$f(x)=x-\frac{1}{2}x^2+C.$$

例15 求 $\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx.$

解 $\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}\arcsin\frac{x}{2}}d\frac{x}{2}$

$= \int \frac{1}{\arcsin\frac{x}{2}}d(\arcsin\frac{x}{2})$

$= \ln\left|\arcsin\frac{x}{2}\right| + C.$

第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x} = \underset{\text{易求}}{\boxed{\int f(u)\mathrm{d}u}}\Bigg|_{u=\varphi(x)}$$

若所求积分 $\int f(u)\mathrm{d}u$ 难求，$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求，

$$\underset{\text{难求}}{\boxed{\int f(u)\mathrm{d}u}} = \underset{\text{易求}}{\int f[\phi(x)]\phi'(x)\mathrm{d}x}$$

则得**第二类换元积分法**．

二、不定积分的第二类换元法

问题 $\int x^5\sqrt{1-x^2}dx = ?$

解决方法 改变中间变量的设置方法.

过程 令 $x=\sin t \Rightarrow dx=\cos t dt$,

$$\int x^5\sqrt{1-x^2}dx=\int(\sin t)^5\sqrt{1-\sin^2 t}\cos t dt$$

$$=\int \sin^5 t\cos^2 t dt = \cdots\cdots$$

（应用“凑微分”即可求出结果）

例16 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0)$.

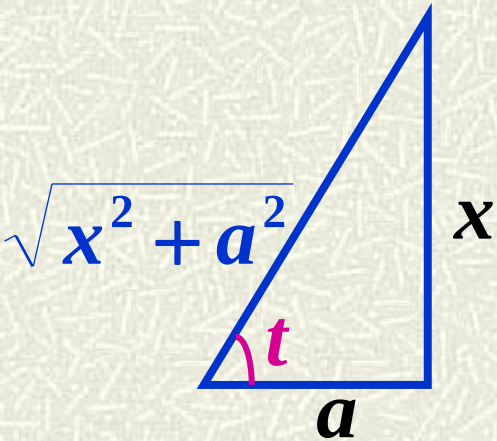
解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t}\cdot a\sec^2 t dt \boxed{= \int \sec t dt}$$

$$= \ln|\sec t + \tan t| + C_1$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C_1.$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$



$$\boxed{\int \csc x dx = -\ln|\csc x + \cot x| + C.}$$

$$\boxed{\int \sec x dx = \ln|\sec x + \tan x| + C.}$$

定理2．设 $f(x)$连续，$x=\psi(t)$有连续的导数，且$\psi'(t)$定号，

则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $x=\psi(t)$ 的反函数．

证: 两边对x 导数，相等即可，

注意右边为t的函数

例17. 求$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0)$.

$\boxed{\int \sec x dx = \ln|\sec x + \tan x| + C.}$

解: 当$x>a$时, 令 $x = a\sec t, t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = a\tan t$$

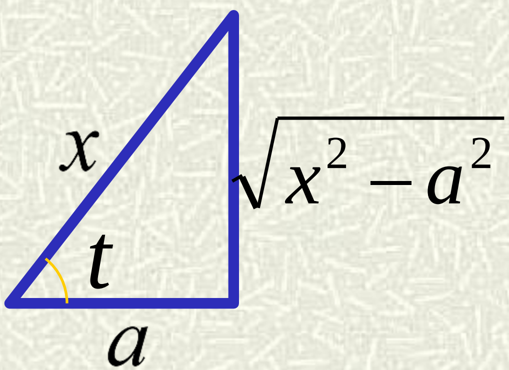
$$\mathrm{d}x = a\sec t\tan t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec t\tan t}{a\tan t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

$$= \ln|\sec t + \tan t| + C_1$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C \quad (C = C_1 - \ln a)$$



当$x<-a$时, 令 $x=-u$, 则$u>a$, 于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=-\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\quad(C=C_1-2\ln a)$$

$x>a$时, $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

例18 求 $\int x^3\sqrt{4-x^2}dx.$

解 令 $x=2\sin t \quad dx=2\cos t dt \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

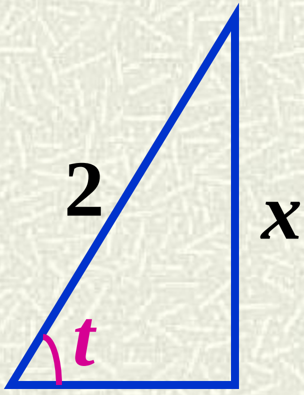
$$\int x^3\sqrt{4-x^2}dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=32\int\sin^3 t\cos^2 t dt=32\int\sin t(1-\cos^2 t)\cos^2 t dt$$

$$=-32\int(\cos^2 t-\cos^4 t)d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$



$\sqrt{4-x^2}$

说明(1) 以上几例所使用的均为三角代换.

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2 - x^2}$ 可令 $x = a\sin t$;

(2) $\sqrt{a^2 + x^2}$ 可令 $x = a\tan t$;

(3) $\sqrt{x^2 - a^2}$ 可令 $x = a\sec t$.

说明(2) 积分中为了化掉根式除采用三角代换外还可用双曲代换.

$$\because \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$$

$\therefore\ x = a\operatorname{sh}t,\quad x = a\operatorname{ch}t$ 也可以化掉根式

例 $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

令 $x = a\operatorname{sh}t$ $\boxed{dx = a\operatorname{ch}t\,dt}$

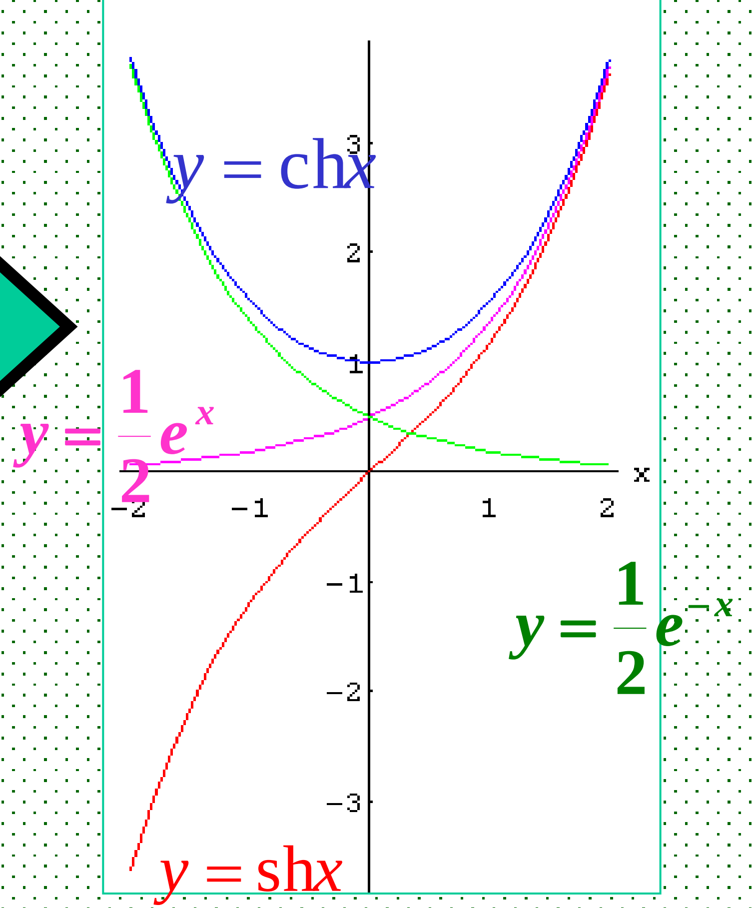
$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{a\operatorname{ch}t}{a\operatorname{ch}t}dt = \int dt = t + C$$

$$= \operatorname{arsh}\frac{x}{a} + C = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

双曲函数

双曲正弦 $\operatorname{sh}x=\dfrac{e^{x}-e^{-x}}{2}$

$D:(-\infty,+\infty)$, 奇函数.

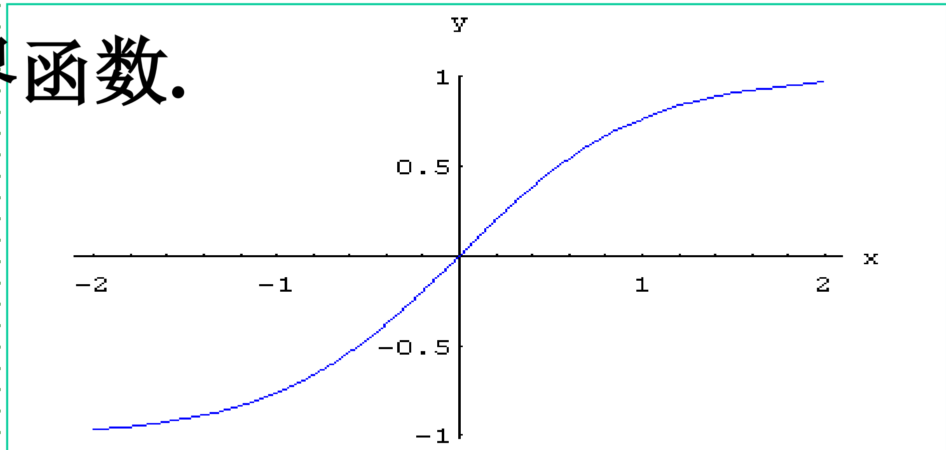


双曲余弦 $\operatorname{ch}x=\dfrac{e^{x}+e^{-x}}{2}$

$D:(-\infty,+\infty)$, 偶函数.

双曲正切 $\operatorname{th}x=\dfrac{\operatorname{sh}x}{\operatorname{ch}x}=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$D:(-\infty,+\infty)$ 奇函数, 有界函数.



双曲函数常用公式

$$\operatorname{sh}(x \pm y) = \operatorname{sh}x\operatorname{ch}y \pm \operatorname{ch}x\operatorname{sh}y;$$

$$\operatorname{ch}(x \pm y) = \operatorname{ch}x\operatorname{ch}y \pm \operatorname{sh}x\operatorname{sh}y;$$

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1;$$

$$\operatorname{sh}2x = 2\operatorname{sh}x\operatorname{ch}x;$$

$$\operatorname{ch}2x = \operatorname{ch}^2 x + \operatorname{sh}^2 x.$$

双曲函数的导数

$$(1) y = \text{sh}x = \frac{e^x - e^{-x}}{2} \qquad (2) y = \text{ch}x = \frac{e^x + e^{-x}}{2}$$

$$(3) y = \text{th}x = \frac{\text{sh}\, x}{\text{ch}\, x}$$

$$(1)(\text{sh}x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \text{ch}x$$

$$(2)(\text{ch}x)' = \frac{e^x - e^{-x}}{2} = \text{sh}x$$

$$(3)(\text{th}x)' = \left[\frac{\text{sh}x}{\text{ch}x}\right]' = \frac{\text{ch}^2 x - \text{sh}^2 x}{\text{ch}^2 x} = \frac{1}{\text{ch}^2 x}$$

说明(3) 积分中为了化掉根式是否一定采用三角代换（或双曲代换）并不是绝对的，需根据被积函数的情况来定.

例19 求 $\int \frac{x^5}{\sqrt{1+x^2}}dx$ （三角代换很繁琐）

解 令 $t=\sqrt{1+x^2} \Rightarrow x^2=t^2-1, \quad xdx=tdt,$

$$\int \frac{x^5}{\sqrt{1+x^2}}dx=\int \frac{\left(t^2-1\right)^2}{t}tdt=\int\left(t^4-2t^2+1\right)dt$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例20 求 $\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令 $t=\sqrt{1+e^x} \Rightarrow e^x = t^2-1,$

$$x=\ln(t^2-1), \quad dx=\frac{2t}{t^2-1}dt,$$

$$\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{2}{t^2-1}dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=2\ln\left(\sqrt{1+e^x}-1\right)-x+C.$$

半角代换法(万能代换法)

$\int R(\sin x, \cos x)dx$ 可用半角代换法化为有理函数的积分。

令 $\tan\frac{x}{2}=t$，则 $x=2\arctan t$，$dx=\frac{2}{1+t^2}dt$，

$$\sin x=\frac{2t}{1+t^2},\quad \cos x=\frac{1-t^2}{1+t^2},$$

推导

$$\therefore \int R(\sin x, \cos x)dx=\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt。$$

例. 求 $\displaystyle\int \frac{\cot x}{\sin x+\cos x+1}dx$

解：令 $\tan\frac{x}{2}=t$，则 $dx=\frac{2}{1+t^2}dt$， $\sin x=\frac{2t}{1+t^2}$ ，

$\cos x=\frac{1-t^2}{1+t^2}$， $\cot x=\frac{1-t^2}{2t}$，

$$\int\frac{\cot x}{\sin x+\cos x+1}dx=\int\frac{\frac{1-t^2}{2t}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+1}\frac{2}{1+t^2}dt=\int\frac{1-t}{2t}dt$$

$$=\frac{1}{2}[\int\frac{1}{t}dt-\int dt]=\frac{1}{2}[\ln|t|-t]+C=\frac{1}{2}[\ln\left|\tan\frac{x}{2}\right|-\tan\frac{x}{2}]+C.$$

半角代换对三角函数有理式的积分总是有效的,

但并非在任何情况下都是简便的。

例 求下列不定积分:

(1) $\int \frac{1}{1+\cos^2 x}dx = \int \frac{1}{\cos^2 x(\sec^2 x+1)}dx$

$= \int \frac{1}{(\tan^2 x+2)}d(\tan x) = \frac{1}{\sqrt{2}}\arctan(\frac{\tan x}{\sqrt{2}})+C$

(2) $\int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x(1-\sin x)}{\cos^2 x}dx = \int \frac{\sin x}{\cos^2 x}dx - \int \tan^2 x dx$

$= \int \sec x \cdot \tan x dx - \int (\sec^2 x - 1)dx = \sec x - \tan x + x + C$

基本积分表

(16) $\int \tan x dx = -\ln|\cos x| + C$

(17) $\int \cot x dx = \ln|\sin x| + C$

(18) $\int \sec x dx = \ln|\sec x + \tan x| + C$

(19) $\int \csc x dx = \ln|\csc x - \cot x| + C$

(20) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C;$

(21) $$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C;$$

(22) $$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C;$$

(23) $$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C;$$

(24) $$\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C.$$

三、定积分的换元公式

定理 假设

（1）$f(x)$在$[a,b]$上连续；

（2）函数$x=\varphi(t)$在$[\alpha,\beta]$上有连续导数；

（3）当t在区间$[\alpha,\beta]$上变化时，$x=\varphi(t)$的值在$[a,b]$上变化，且$\varphi(\alpha)=a$、$\varphi(\beta)=b$，

则 有$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt.$

注意 当$\alpha>\beta$时，换元公式仍成立.

设$F'(x)=f(x)$

$$\int_a^b f(x)dx = F(b) - F(a)$$

$$\because F'\left[\phi(t)\right]=f[\phi(t)]\phi'(t)$$

$$\therefore \int_\alpha^\beta f[\phi(t)]\phi'(t)dt=F\left[\phi(\beta)\right]-F\left[\phi(\alpha)\right]$$

$$= F(b) - F(a)$$

则 有$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt.$

例24 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx.$

解 令 $t = \cos x,\qquad dt = -\sin x dx,$

$$x = \frac{\pi}{2} \Rightarrow t = 0, \qquad x = 0 \Rightarrow t = 1,$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx$$

$$= -\int_1^0 t^5 dt = \frac{t^6}{6}\Big|_0^1 = \frac{1}{6}.$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_1^0 t^5 dt = \left.\frac{t^6}{6}\right|_0^1 = \frac{1}{6}.$$

应用换元公式时应注意:

（1）用$x=\varphi(t)$把变量x换成新变量t时，积分限也相应的改变.

（2）求出$f[\varphi(t)]\varphi'(t)$的一个原函数$\Phi(t)$后，不必象计算不定积分那样再把$\Phi(t)$变换成原变量x的函数，而只需把新变量t的上、下限分别代入$\Phi(t)$然后相减即可.

例27 计算 $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx. \quad (a>0)$

解 令 $x = a\sin t$, 则 $dx = a\cos t dt$,

$$x = a \Rightarrow t = \frac{\pi}{2}, \quad x = 0 \Rightarrow t = 0,$$

$$\text{原式} = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + \sqrt{a^2(1-\sin^2 t)}}dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 + \frac{\cos t - \sin t}{\sin t + \cos t}\right)dt$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\left[\ln\left|\sin t + \cos t\right|\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

例28 当 $f(x)$ 在 $[-a,a]$ 上连续，且有

① $f(x)$ 为偶函数，则

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx;$$

② $f(x)$ 为奇函数，则 $\int_{-a}^{a} f(x)dx = 0.$

证 $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx,$

在 $\int_{-a}^{0} f(x)dx$ 中令 $x=-t$，$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt$

$$= \int_{0}^{a} f(-t)dt$$

① $f(x)$ 为偶函数，则 $f(-t) = f(t)$，

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(t)dt;$$

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt = \int_{0}^{a} f(-t)dt,$$

① $f(x)$为偶函数，则 $f(-t) = f(t)$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(t)dt;$$

② $f(x)$为奇函数，则 $f(-t) = -f(t)$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

例 设$f(x)$为连续的周期函数,其周期为T,

利用积分的换元法证明:

$$\boxed{\int_a^{a+T} f(x)\mathrm{d}x=\int_0^T f(x)\mathrm{d}x,\ (a\text{为常数}).}$$

证明:

$$\int_a^{a+T} f(x)\mathrm{d}x=\int_a^0 f(x)\mathrm{d}x+\int_0^T f(x)\mathrm{d}x+\int_T^{a+T} f(x)\mathrm{d}x,$$

而$$\int_T^{a+T} f(x)\,dx \xlongequal[x=T+u]{x-T=u} \int_0^a f(u)\mathrm{d}u=-\int_a^0 f(x)\mathrm{d}x$$

所以 $$\int_a^{a+T} f(x)\mathrm{d}x=\int_0^T f(x)\mathrm{d}x.$$

例29 计算 $\displaystyle\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}dx.$

解 原式 $\displaystyle=\int_{-1}^{1}\boxed{\frac{2x^2}{1+\sqrt{1-x^2}}}dx+\int_{-1}^{1}\boxed{\frac{x\cos x}{1+\sqrt{1-x^2}}}dx$

偶函数　　奇函数

$$=4\int_{0}^{1}\frac{x^2}{1+\sqrt{1-x^2}}dx=4\int_{0}^{1}\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}dx$$

$$=4\int_{0}^{1}(1-\sqrt{1-x^2})dx=4-4\boxed{\int_{0}^{1}\sqrt{1-x^2}dx}$$

1/4单位圆的面积

$$=4-\pi.$$

例30 若 $f(x)$ 在[0,1]上连续，证明

（1）$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$;

（2）$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

证 （1）设 $x = \frac{\pi}{2} - t \quad \Rightarrow dx = -dt,$

$x = 0 \Rightarrow t = \frac{\pi}{2}, \qquad x = \frac{\pi}{2} \Rightarrow t = 0,$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2} - t\right)\right]dt$$

$$= \int_0^{\frac{\pi}{2}} f(\cos t)dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

证明: $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

（2）设 $x = \pi - t \Rightarrow dx = -dt,$

$$x = 0 \Rightarrow t = \pi, \qquad x = \pi \Rightarrow t = 0,$$

$$\int_0^{\pi} xf(\sin x)dx = -\int_{\pi}^{0}(\pi - t) f[\sin(\pi - t)]dt$$

$$= \int_0^{\pi}(\pi - t) f(\sin t)dt$$

$$= \pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

$$= \pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx,$$

$$\therefore \int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}d(\cos x)$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

三、小结

1. 两类积分换元法:

$$\begin{cases} \text{（一）凑微分} \\ \text{（二）三角代换、倒代换、根式代换} \end{cases}$$

（实现有理化）

基本积分表(2)

2. 定积分的换元法(导函数连续)

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

不定积分换元法(导函数定号)

万能公式

$$\sin\alpha=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}},\ \cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}},\quad \tan\alpha=\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$$

若令$t=\tan\frac{\alpha}{2}$,则

$$\boxed{\sin\alpha=\frac{2t}{1+t^2},\qquad \cos\alpha=\frac{1-t^2}{1+t^2},\quad \tan\alpha=\frac{2t}{1-t^2}.}$$

该式将“三角”与“代数”沟通起来，故称为“万能公式”。

第三章 一元函数积分学及其应用

第三节 两种基本积分法

之一——换元积分法

- 不定积分的第一换元法
- 不定积分的第二换元法
- 定积分的换元法
- 小结

作业:Page213　1，3，4

万能公式 （利用二倍角公式推导）

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}} \boxed{= \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}$$

$$\cos\alpha = \cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} \boxed{= \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}$$

$$\boxed{\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}}$$