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Introduction

COMAP is proud to announce the Doug Faires Award. The purpose of the award is to encourage and recognize efforts to start modeling teams at both the high school and college levels. COMAP wishes to encourage current faculty advisors to reach out, recruit, and mentor new faculty advisors at either the college or high school level, particularly local schools. The goal is to form local groups with a common interest in mathematical modeling.

We dedicate this award to Doug Faires, who provided us with the perfect example of the goals we wish to attain. First, a snapshot of Doug Faires:

About Doug Faires

Doug graduated from Sharpsville High School in Pennsylvania, class of 1959. He pursued an undergraduate degree at Youngstown State University, graduating in 1963, and left YSU to complete a Ph.D. in mathematics at the University of South Carolina before returning to YSU as a faculty member in 1969. He has been associated with Youngstown State University ever since, retiring as professor of mathematics in 2006.

During his tenure at YSU, Doug was the recipient of numerous awards, including the Outstanding College–University Teacher of Mathematics by the Ohio Section of the Mathematical Association of America and five Distinguished Faculty awards from Youngstown State University, which also awarded him an Honorary Doctor of Science in 2006. For nearly two decades, he was a member of the council of Pi Mu Epsilon National Mathematics Honorary Society, including a term as president. In addition, he

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was awarded the MacDuffee Award by Pi Mu Epsilon for lifetime service in 2005. Doug was a Co-Director of examinations for the American Mathematics Competitions for 8 years and has been a long-term judge for the COMAP Interdisciplinary Contest in Modeling (ICM)™. He authored or co-authored more than 20 books, including 10 editions of the classic *Numerical Analysis* (Cengage Learning, 2015).

Dr. Faires, as he was known to his students, will be remembered not for his awards, but as a dedicated teacher. He was a master at getting the best out of those around him, encouraging them to recognize their own potential, and mentoring them to achievements far beyond their expectations. A tireless champion of undergraduate research in mathematics, Doug was a driving force behind the establishment in 2006 of the Center for Undergraduate Research in Mathematics at Youngstown State. Doug served first as a Faculty Advisor for the Mathematical Contest in Modeling™ (MCM). He gave talks to local high schools, inviting them to form modeling teams to compete in the HiMCM™ (COMAP's high school modeling contest). He recruited and mentored high school faculty advisors and invited the advisors and their teams to Youngstown State where the teams met one another, and the experienced members of the college teams mentored the high school students. Long-term bonds were formed, and each year college and high school teams were encouraged to participate in the modeling contests. Additionally, teams were given feedback by Doug and others after the contest was over. Later, Doug served as a Final Judge for the Mathematical Contest in Modeling, where he again was a true leader.

Our goal is to emulate Doug's success at the local level.

Award to Marie Vanisko

In the words of Marie Vanisko, a long-time supporter of both the MCM and the HiMCM:

> I think this is a great idea and a wonderful way to honor Doug Faires. I agree that the best way to get the HiMCM going at a high school is to have an MCM mentor from a local college. Recently, I invited both Helena local high schools, Capital High and Helena High, to the Montana Learning Center at Canyon Ferry. Both schools have had teams participating in the HiMCM. Additionally the coaches for those teams also help me out as triage judges for the HiMCM. This year was amazing—we had one Finalist, three Meritorious, five Honorable Mention, and two Successful Participants. The Finalist team are only sophomores, so next year should be even better. Last week, I went to Capital High to hand out the certificates with the principal present, and we are scheduled to honor the teams at an upcoming school board meeting.

We are proud to present the Doug Faires Award to Marie Vanisko for her outstanding achievement on behalf of the ideals that Doug strove for.

Award to Rick Spellerberg

Another example is provided by Rick Spellerberg of Simpson College: Year after year (approaching a dozen years now), Simpson College fields more ICM/MCM teams than any other U.S. institution. A big reason is the energy and work supplied by Rick Spellerberg and his colleagues in the Mathematics Dept. The Simpson College teams do very well—Finalist and Meritorious teams have come from this small college. Again, the reason is the work of Simpson Professor Spellerberg, who notes:

> The fact that the highly competitive event attracts mathematics teams from around the world doesn't intimidate the Simpson students. Success has bred success in the recruiting of students into the program—a significant number of high school seniors who decide to attend Simpson do so because of our success in the modeling competition. All of the students who participate are aware of student outcomes in terms of securing internships, full-time employment, undergraduate research experiences, and graduate school acceptance as a direct result of their success with the ICM/MCM.
>
> Simpson College's Student Government Association recognized the impact that this competition was having across campus and pays every team's registration fee.
>
> The success and experience our students have had competing in the ICM is now being replicated in interdisciplinary undergraduate research at Simpson College through the Bryan Summer Research Program in Mathematics.

Rick Spellerberg's work has energized colleagues across campus as well. The college has reached a critical mass of students from many departments and academic majors consistently participating, especially in the ICM. Spellerberg exports the contests' success to local high schools. The Simpson Mathematics Dept. recently sponsored 16 high school students, the majority from urban Chicago charter schools, to take part in a mathematical modeling workshop on the Simpson campus in Indianola, IA. Rick Spellerberg recruited seven of his current Simpson students to mentor the high school workshop participants.

We are proud to present the Doug Faires Award to Rick Spellerberg for his outstanding achievement on behalf of the ideals that Doug strove for.

The Future

The Doug Faires Award will be given for individuals who achieve great results in a particular year or cumulative excellent results over a period of time. Marie Vanisko and Rick Spellerberg exemplify both annual and cumulative achievement reflecting the goals of the Doug Faires award. Awardees receive a certificate of appreciation that expresses our enduring gratitude.

We would like you to nominate someone in your community (including possibly yourself!) who is promoting mathematical modeling at the local level.

About the Author

Solomon Garfunkel is the founder and Executive Director of COMAP and Executive Publisher of this *Journal*.

He served on the mathematics faculties of Cornell University and the University of Connecticut at Storrs, but he has dedicated the last 35 years to research and development efforts in mathematics education. He was project director for the Undergraduate Mathematics and Its Applications (UMAP) and the High School Mathematics and Its Applications (HiMAP) Projects funded by NSF, and directed three telecourse projects, including *Against All Odds: Inside Statistics* and *In Simplest Terms: College Algebra*, for the Annenberg/CPB Project. He has been the Executive Director of COMAP, Inc. since its inception in 1980.

Dr. Garfunkel was the project director and host for the video series *For All Practical Purposes: Introduction to Contemporary Mathematics*. He was the Co-Principal Investigator on the ARISE Project, and Co-Principal Investigator of the CourseMap, ResourceMap, and WorkMap projects. In 2003, Dr. Garfunkel was Chair of the National Academy of Sciences and Mathematical Sciences Education Board Committee on the Preparation of High School Teachers.

MCM Modeling Forum

Results of the 2016 Mathematical Contest in Modeling

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Introduction

A total of 7,421 teams of undergraduates from hundreds of institutions and departments in 14 countries spent a weekend working on applied mathematics problems in the 32nd Mathematical Contest in Modeling (MCM)®.

The 2016 MCM began at 8:00 P.M. EST on Thursday, January 28, and ended at 8:00 P.M. EST on Monday, February 1. During that time, teams of up to three undergraduates researched, modeled, and submitted a solution to one of two open-ended modeling problems. Students registered, obtained contest materials, downloaded the problems and data, and entered completion data through COMAP's MCM Website. After a weekend of hard work, solution papers were sent to COMAP on Monday. Three of the top papers appear in this issue of *The UMAP Journal*, together with commentaries from the contest judges.

In addition to the customary two modeling problems calling upon continuous mathematics and discrete mathematics respectively, the 2016 MCM for the first time featured a third problem, on data insights and statistical analysis.

A companion contest, the Interdisciplinary Contest in Modeling (ICM)®, took place concurrently over the same weekend. The ICM offers modeling problems involving network science, human-environment interactions, and policy modeling. Details about the 2016 ICM Contest and its results are in Volume 37, Number 2 of this *Journal*.

The 2017 MCM/ICM Contests will take place January 19–23, 2017.

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This year's three MCM problems were engaging challenges for contestants, each offering a unique dimension of mathematical modeling. The author of both the A and C problems was David Olwell (St. Martin's University). The author of the B problem was Kelly Black (Clarkson University).

The A problem posed a simply-stated (yet subtly-complicated) problem involving a person faced with the dilemma of taking a bath in a standard tub in which the water temperature is cooling and gradually becoming uncomfortable. Teams were challenged to model this situation in both time and space in order to identify an effective strategy for a bather to add heated water to raise the temperature back to near starting levels while minimizing the overall use of water. The combination of thermodynamic heat transfer in discretized space involving a human body, dynamic evolution and distribution of temperature and human motion over time, and optimizing a strategy for water use proved to be a substantial challenge for all teams.

The B problem revisited a relatively old (yet ironically current) problem of what to do about the accumulation of space debris in orbital space, which poses an ongoing real threat to satellites and other technology insertions and operations. While requiring a well-researched report regarding a potential collection, disposal, or elimination solution, the problem added as a prime consideration whether such an activity could be financially viable enough to warrant and support a commercial endeavor. In this setting, defining what "maximizing benefits" means became a major decision for contest teams.

Finally, the C problem, newly introduced this year, asked teams to develop an effective investment strategy for a fictitious nonprofit philanthropic entity called the Goodgrant Foundation, which is interested in improving undergraduate education in the United States without duplicating efforts of ongoing foundations such as the Gates Foundation and Lumina Foundation. This "data insights" problem provided teams with a large real-world data set containing nearly 100 variables over 15 years, including missing data, as a starting point for modeling. Teams submitted amazingly sophisticated and insightful solutions, and the judges were well-challenged to identify the top papers.

An Outstanding paper from each of the problems is featured in this issue of the *Journal*, together with commentaries by the problem authors and judges.

All of the competing teams are to be congratulated for their excellent work and enthusiasm for mathematical modeling and interdisciplinary problem solving.

COMAP, whose educational philosophy is centered on mathematical modeling, supports the use of mathematical concepts, methods, and tools to explore real-world problems. COMAP serves society by developing students as problem solvers in order to become better informed and prepared as citizens, contributors, consumers, workers, and community leaders. The MCM is an example of COMAP's efforts in achieving these goals.

Resources

In addition to this special issue of *The UMAP Journal*, COMAP offers at `www.mathmodels.org` the press releases for the 2016 contests, their results, their problems, unabridged versions of all the Outstanding papers, and judges' commentaries.

Results and winning papers from previous contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–2015). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains the 20 problems used in the first 10 years of the contest and an Outstanding paper for each year. That volume and the special MCM issues of the *Journal* for the last few years are available from COMAP. The 1994 volume is also available on COMAP's special *Modeling Resource* CD-ROM. Also available is *The MCM at 21* CD-ROM, which contains the 20 problems from the second 10 years of the contest, an Outstanding paper from each year, and advice from advisors of Outstanding teams. These CD-ROMs can be ordered from COMAP at

`http://www.comap.com/product/cdrom/index.html`.

Contest problems and results of the MCM/ICM contests are on the COMAP Website at

`http://www.comap.com/undergraduate/contests`.

Finally, the volume *Mathematical Modeling for the MCM/ICM Contests Volume 1* is an exposition of the ideas, background knowledge, and modeling methodologies for solving problems in the MCM/ICM contests.

That volume also presents a brief history of the MCM/ICM contests, offers ideas to help students prepare for the MCM/ICM contests, presents general modeling framework and methodologies, describes the judging procedure of the MCM/ICM papers, explains how to write successful MCM/ICM papers, and presents a sample scheduling of tasks during the contest. A number of exercise problems are included to help students understand the materials presented in the book.

Details and ordering are at

`http://216.250.163.249//product/?idx=1465`.

COMAP also sponsors:

- The MCM/ICM Media Contest (see p. 250).
- The Interdisciplinary Contest in Modeling (ICM), noted above.
- The High School Mathematical Contest in Modeling (HiMCM)®, which offers high school students a modeling opportunity similar to the MCM. Further details are at

  `http://www.comap.com/highschool/contests`.

2015 MCM Statistics

- 7,421 teams participated (with 5,023 more in the ICM)
- 389 U.S. teams (5%)
- 7,032 foreign teams (95%), from Australia, Canada, China, Finland, Hong Kong SAR, Indonesia, Mexico, Scotland, Singapore, South Korea, Spain, and the United Kingdom
- 13 Outstanding Winners
- 22 Finalist Winners (1%)
- 594 Meritorious Winners (8%)
- 2,604 Honorable Mentions (34%)
- 4,182 Successful Participants (55%)

Problem A: A Hot Bath

A person fills a bathtub with hot water from a single faucet and settles into the bathtub to cleanse and relax. Unfortunately, the bathtub is not a spa-style tub with a secondary heating system and circulating jets, but rather a simple water containment vessel. After a while, the bath gets noticeably cooler, so the person adds a constant trickle of hot water from the faucet to reheat the bathing water. The bathtub is designed in such a way that when the tub reaches its capacity, excess water escapes through an overflow drain.

Develop a model of the temperature of the bathtub water in space and time to determine the best strategy the person in the bathtub can adopt to keep the temperature even throughout the bathtub and as close as possible to the initial temperature without wasting too much water.

Use your model to determine the extent to which your strategy depends upon the shape and volume of the tub, the shape/volume/temperature of the person in the bathtub, and the motions made by the person in the bathtub. If the person used a bubble bath additive while initially filling the bathtub to assist in cleansing, how would this affect your model's results?

In addition to the required one-page summary for your MCM submission, your report must include a one-page non-technical explanation for users of the bathtub that describes your strategy while explaining why it is so difficult to get an evenly-maintained temperature throughout the bath water.

Problem B: Space Junk

The amount of small debris in orbit around Earth has been a growing concern. It is estimated that more than 500,000 pieces of space debris, also called orbital debris, are currently being tracked as potential hazards to space craft. The issue itself became more widely discussed in the news media when the Russian satellite Kosmos-2251 and the USA satellite Iridium-33 collided on 10 February, 2009.

A number of methods to remove the debris have been proposed. These methods include small space-based water jets and high-energy lasers used to target specific pieces of debris and large satellites designed to sweep up the debris, among others. The debris ranges in size and mass from paint flakes to abandoned satellites. The high-velocity orbits of the debris make capture difficult.

Develop a time-dependent model to determine the best alternative or combination of alternatives that a private firm could adopt as a commercial opportunity to address the space debris problem. Your model should include quantitative and/or qualitative estimates of costs, risks, benefits, as well as other important factors. Your model should be able to assess independent alternatives as well as combinations of alternatives and be able to explore a variety of important "What if?" scenarios.

Using your model, determine whether an economically attractive opportunity exists or no such opportunity is possible. If a viable commercial opportunity exists as an alternative solution, provide a comparison of the different options for removing debris, and include a specific recommendation as to how the debris should be removed. If no such opportunity is possible, then provide innovative alternatives for avoiding collisions.

In addition to the required one-page summary for your MCM submission, your report must include a two-page Executive Summary that describes the options considered and major modeling results, and provides a recommendation for a particular action, combination of actions, or no action, as appropriate from your work. The Executive Summary should be written for high level policy makers and news media analysts who do not have a technical background.

Problem C: The Goodgrant Challenge

The Goodgrant Foundation is a charitable organization that wants to help improve educational performance of undergraduates attending colleges and universities in the United States. To do this, the foundation intends to donate a total of $100,000,000 (USD 100 million) to an appropriate group of schools per year, for five years, starting July 2016. In doing so, they do not want to duplicate the investments and focus of other large grant organizations such as the Gates Foundation and Lumina Foundation.

Your team has been asked by the Goodgrant Foundation to develop a model to determine an optimal investment strategy that identifies the schools, the investment amount per school, the return on that investment, and the time duration that the organization's money should be provided to have the highest likelihood of producing a strong positive effect on student performance. This strategy should contain a 1 to N optimized and prioritized candidate list of schools you are recommending for investment based on each candidate school's demonstrated potential for effective use of private funding, and an estimated return on investment (ROI) defined in a manner appropriate for a charitable organization such as the Goodgrant Foundation.

To assist your effort, the attached data file (`ProblemCDATA.zip`) contains information extracted from the U.S. National Center on Education Statistics (`http://www.nces.ed.gov/ipeds`), which maintains an extensive database of survey information on nearly all post-secondary colleges and universities in the United States, and the College Scorecard data set (`https://collegescorecard.ed.gov`), which contains various institutional performance data. Your model and subsequent strategy must be based on some meaningful and defendable subset of these two data sets.

In addition to the required one-page summary for your MCM submission, your report must include a letter to the Chief Financial Officer (CFO) of the Goodgrant Foundation, Mr. Alpha Chiang, that describes the optimal investment strategy, your modeling approach and major results, and a brief discussion of your proposed concept of a return-on-investment (ROI) that the Goodgrant Foundation should adopt for assessing the 2016 donation(s) and future philanthropic educational investments within the United States. This letter should be no more than two pages in length. Note: When submitting your final electronic solution DO NOT include any database files. The only thing that should be submitted is your electronic (Word or PDF) solution. The `ProblemCDATA.zip` data file contains:

- `Problem C - IPEDS UID for Potential Candidate Schools.xlsx`
- `Problem C - Most Recent Cohorts Data (Scorecard Elements) .xlsx`
- `Problem C - CollegeScorecardDataDictionary-09-08-2015.xlsx`
- `IPEDS Variables for Data Selection.pdf`

You can download the data (`ProblemCDATA.zip`) from :

- `http://www.comap-math.com/mcm/ProblemCDATA.zip`
- `http://www.mathismore.net/mcm/ProblemCDATA.zip`
- `http://www.mathportals.com/mcm/ProblemCDATA.zip`
- `http://www.immchallenge.org/mcm/ProblemCDATA.zip`

The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at one of seven triage session sites. At each triage site, two time-limited comprehensive readings of each paper take place in comparison to a set of scoring criteria that establish minimum and maximum point allocations against a standard baseline. The scoring criteria are created by each problem's Head Judge in collaboration with the Contest Director and subsequently distributed to each triage site prior to the start of judging. These point allocations, in concert with established MCM quality elements, determine a paper's positioning within the overall pool of submissions for a particular problem. In the case that two triage judges' scores on a paper differ by more than 3 points, a third judge scores the paper.

Final judging took place in Carmel, CA.

The judges classified the papers as follows:

	Outstanding	Finalist	Meritorious	Honorable Mention	Successful Participation	Total
Hot Bath Problem	6	7	355	1,410	2,313	4,094
Space Junk Problem	3	7	125	475	841	1,453
Goodgrant Challenge	4	8	114	719	1,028	1,874
	12	22	594	2,604	4,182	7,421

We list here the 13 teams that the judges designated as Outstanding; the list of all participating schools, advisors, and results is at the COMAP Website.

Outstanding Teams

Institution and Advisor	**Team Members**

Hot Bath Problem

Northwestern Polytechnical University Xi'an, Shaanxi, China Junfeng Zhao	Yihua Hu Xulong Yang Luyu Wang
Chongqing University Chongqing, China Jian Xiao	Zhu Fang Fu Zhang Guang Zhao
University of Colorado Boulder Boulder, CO Bengt Fornberg	Jordan Deitsch Matthew R. Hurst Nathan J. Yeo

Southwest Jiaotong University Chengdu, Sichuan, China Changhong Xue	Zhenchuan Zeng Daoxing Huang Hong Li
Shanghai Jiaotong University Shanghai, China Fan Wu	Yu Qin XinYao Wang JiaYi Zhang
Adlai E. Stevenson High School Lincolnshire, IL Paul Y. Kim	Emory Liu Emily Leng Liyang Zhang

Space Junk Problem

Southwestern University of Finance and Economics Chengdu, Sichuan, China Dai Dai	Jiawen Yan Siyu Huang Qi Huang
Wuhan University Wuhan, Hubei, China Junmin Liao	Hui Yang Yulei Li Zhaoqi Wang
Zheijiang University Hangzhou, Zheijiang, China Huajun Feng	Wei Ke Yinchuan Xu Chuankun Zheng

Goodgrant Challenge Problem

Tsinghua University Beijing, China Lin Zhu	Jiaheng Yu Di Tian Shuxi Zeng
Shanghai University of Finance and Economics Shanghai, China Chengdong Dong	Yizhang Li Xinyu Zhao Meng Chen
Tsinghua University Beijing, China Zhenbo Wang	Jingze Ren Haonan Run Kai Wang
Virginia Tech Blacksburg, VA Mark Embree	Zheng Wang Brennen Woodruff Nathan Wycoff

Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, recognized three teams as INFORMS Outstanding teams: the teams from Chongqing University (Hot Bath Problem), Zhejiang University (Space Junk Problem), and Tsinghua University (with advisor Zhenbo Wang) (Goodgrant Challenge Problem). INFORMS provided the following recognition:

- a letter of congratulations from the current president of INFORMS to each team member and to the faculty advisor;
- a one-year complimentary access to the full suite of INFORMS journals online for the faculty advisor;
- a crystal trophy for display at the team's institution, commemorating the team members' achievement;
- individual crystal trophies for the team members, as a personal commemoration of this achievement; and
- a one-year student membership in INFORMS for each team member, which includes their choice of a professional journal plus the *OR/MS Today* periodical and the INFORMS newsletter.

The Society for Industrial and Applied Mathematics (SIAM) designated two Outstanding teams as SIAM Winners. The SIAM Award teams were from Southwest Jiaotong University (Hot Bath Problem) and Wuhan University (Space Junk Problem). Each team member was awarded a $300 cash prize. Their schools were given framed hand-lettered certificates in gold leaf.

The Mathematical Association of America (MAA) designated two North American team as MAA Winners. One team was from the University of Colorado Boulder (Hot Bath Problem), and the other was from Virginia Tech (Goodgrant Challenge Problem). Each team member was presented a certificate by an official of the MAA Committee on Undergraduate Student Activities and Chapters.

Ben Fusaro Award

One Meritorious, Finalist, or Outstanding paper is selected for the Ben Fusaro Award, named for the Founding Director of the MCM and awarded for the 12th time this year. It recognizes an especially creative approach; details concerning the award, its judging, and Ben Fusaro are in Vol. 25 (3) (2004): 195–196. The Ben Fusaro Award Winner was the Outstanding team from Adlai E. Stevenson High School (Hot Bath Problem).

Frank Giordano Award

For the fourth time, the MCM is designating a paper with the Frank Giordano Award. This award goes to a paper that demonstrates a very good example of the modeling process in a problem featuring discrete mathematics—this year, the Lost Plane Problem. Having worked on the contest since its inception, Frank Giordano served as Contest Director for 20 years. The Frank Giordano Award for 2015 went to the Outstanding team from Southwestern University of Finance and Economics (Space Junk Problem).

Two Sigma Scholarship Award

The Two Sigma Scholarship Award went to the team from Virginia Tech (Goodgrant Challenge Problem). They received a total of $10,000. This was the second year of this award, for which the 480 U.S. teams competing in the MCM and ICM were eligible for one of two such scholarship prizes. We thank Two Sigma Investments for making this award possible.

Judging

Director
Patrick J. Driscoll, Dept. of Systems Engineering, U.S. Military Academy, West Point, NY
Associate Director
William C. Bauldry, Chair-Emeritus, Dept. of Mathematical Sciences, Appalachian State University, Boone, NC

Hot Bath Problem

Head Judge
Marvin Keener, Dept. of Mathematics, Oklahoma State University, Stillwater, OK
Associate Judges
Kelly Black, Dept. of Mathematics, Clarkson University, Potsdam, NY (SIAM Judge)
Karen Bolinger, Dept. of Mathematics, Clarion University, Clarion, PA (MAA Judge)
Jim Case, Baltimore, MD (Fusaro Award Judge)
Tim Elkins, Dept. of Systems Engineering, U.S. Military Academy, West Point, NY (INFORMS Judge)
Jerry Griggs, Dept. of Mathematics, University of South Carolina, Columbia, SC
Elizabeth "Libby" Schott, Dept. of Mathematics, Florida Southwestern State College, Fort Myers, FL

Kathleen M. Shannon, Dept. of Mathematics and Computer Science, Salisbury University, Salisbury, MD (MAA Judge)
Jai "Jed" Wang, Dept. of Computer Science, Univ. of Massachusetts Lowell, Lowell, MA
Bill Wilhelm, Lockheed Martin Corporation, Huntsville, AL

Space Junk Problem

Head Judge
Maynard Thompson, Mathematics Dept., University of Indiana, Bloomington, IN
Associate Judges
Thomas Fitzkee, Dept. of Mathematics, Francis Marion University, Florence, SC (Fusaro Award Judge)
Michael Jaye, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA
Ligang Lu, IBM T.J. Watson Research Center, Yorktown Heights, NY
Greg Mislick, Dept. of Operations Research, Naval Postgraduate School, Monterey, CA
Jack Picciuto, Director of Operations Analysis and Planning at IT Cadre, Ashburn, VA
Catherine Roberts, Dept. of Mathematics and Computer Science, College of the Holy Cross, Worcester, MA (MAA Judge)
Dan Solow, Weatherhead School of Management, Case Western Reserve University, Cleveland, OH (INFORMS Judge)
Rich West, Emeritus Professor of Mathematics, Francis Marion University, Florence, SC (Giordano Award Judge)

Goodgrant Challenge Problem

Head Judge
David H. Olwell, Professor and Dean, Hal and Inge Marcus School of Engineering, Saint Martin's University, Lacey, WA
Associate Judges
Robert Burks, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA
James Enos, Dept. of Systems Engineering, U.S. Military Academy, West Point, NY (INFORMS Judge)
Richard Marchand, Mathematics Dept., Slippery Rock University, Slippery Rock, PA
Veena Mendiratta, Lucent Technologies, Naperville, IL
Katie Oliveras, Mathematics Dept., Seattle University, Seattle, WA
Rodney Sturdivant, Dept. of Mathematics and Physics, Azusa Pacific University, Azusa, CA
Michael Tortorella, Dept. of Industrial and Systems Engineering,

Rutgers University, Piscataway, NJ
Wenbo Zhang, Beijing University of Posts and Telecommunications, Beijing, China

Acknowledgments

Major funding for the MCM is provided by COMAP. Additional support is provided by the Institute for Operations Research and the Management Sciences (INFORMS), the Society for Industrial and Applied Mathematics (SIAM), and the Mathematical Association of America (MAA). We are indebted to these organizations for providing judges and prizes.

We also thank Two Sigma Investments LLC for providing its prize. "This group of experienced, analytical, and technical financial professionals based in New York builds and operates sophisticated quantitative trading strategies for domestic and international markets. The firm is successfully managing several billion dollars using highly-automated trading technologies. For more information about Two Sigma, please visit `http://www.twosigma.com`."

Finally, we thank for their involvement and unflagging support the MCM judges and MCM Board members, as well as the advisors to the competing teams.

Cautions

To the reader of research journals:

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each paper here is the result of undergraduates working on a problem over a weekend. Editing (and usually substantial cutting) has taken place; minor errors have been corrected, wording has been altered for clarity or economy, and style has been adjusted to that of *The UMAP Journal*. The student authors have proofed the results. Please peruse these students' efforts in that context.

To the potential MCM advisor:

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

COMAP's Mathematical Contest in Modeling and Interdisciplinary Contest in Modeling are the only international modeling contests in which students work in teams. Centering its educational philosophy on mathematical modeling, COMAP serves the educational community as well as the world of work by preparing students to become better-informed and better-prepared citizens.

About the Author

Patrick J. Driscoll is Professor of Operations Research in the Dept. of Systems Engineering at the U.S. Military Academy. Formerly an Academy Professor in the Dept. of Mathematical Sciences, he has also served as the Director of the Mathematical Sciences Center of Excellence and the Associate Dean of Information and Educational Technology. He received both an M.S. in Operations Research and an M.S. in Engineering Economic Systems from Stanford University, and a Ph.D. in Industrial and Systems Engineering from Virginia Tech. He is a member of the Operational Research Society (ORS) of the United Kingdom, the Institute for Operations Research and the Management Sciences (INFORMS), the Military Operations Research Society (MORS), and the honor societies Phi Kappa Phi and Pi Mu Epsilon. He is the Director for the Mathematical Contest in Modeling (MCM) and one of the designers of the High School Mathematical Contest in Modeling (HiMCM). He serves on the Board of Directors for the Hudson Valley Shakespeare Festival, is a partner in Winemates & Company, LLC, has three cats, and is continuing to have more fun than should be legally allowed.

Media Contest

Over the years, contest teams have increasingly taken to various forms of documentation of their activities over the grueling 96 hours—frequently in video, slide, or presentation form. This material has been produced to provide comic relief and let off steam, as well as to provide some memories days, weeks, and years after the contest. We *love* it, and we want to encourage teams (outside help is allowed) to create media pieces and share them with us and the MCM/ICM community.

The media contest is *completely separate* from MCM and ICM. No matter how creative and inventive the media presentation, it has *no* effect on the judging of the team's paper for MCM or ICM. We do not want work on the media project to detract or distract from work on the contest problems in any way. This is a separate competition, one that we hope is fun for all.

Further information about the contest is at

`http://www.comap.com/undergraduate/contests/mcm/media.html.`

There were 20 entries—this year, only 10 of them were from Dalian Maritime University. (Come on, you other schools!)

Outstanding Winner of the 6th MCM/ICM Media Contest:

- NC School of Science and Mathematics, Durham, NC
  (Anna Hattle, Katherine Yang, and Sicheng Zheng)

Finalists:

- Dalian Maritime University, Dalian, Liaoning, China
  (Minmin Xiong, Jiajia Sun, Wen Ruan)
- Dalian Maritime University, Dalian, Liaoning, China
  (Jiaqiu Liu, Yuanchunyu Chen, Yiru Wang)
- Lawrence Technological University, Southfield, MI
  (Mohit Bansil, Aaron Craig, Nichola Paul)

The remaining entries were judged Successful Participants. Complete results, including links to the Outstanding video, are at

`http://www.comap.com/undergraduate/contests/mcm/contests/2016/solutions/index.html.`

Beneath the Surface: Thermal-Fluid Analysis of a Hot Bath

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Abstract

The overall temperature and uniformity of heat distribution are crucial to the enjoyment of a hot bath. Over time, a bath will lose heat to its surroundings; however, the temperature can be effectively controlled by adding a trickle of hot water. Quantifying the optimal method of restoring heat uniformly is our objective.

By making key assumptions, we reduce the problem to one spatial dimension with three-dimensional heat transfer considerations. We develop three independent generic mathematical models to describe the temperature distribution of the bathtub in space and time:

- **Steady-State Uniform Thermodynamic Model:** We characterize the energy and mass interactions between the system and the surroundings, deriving key physical constants used in the subsequent models.
- **Analytical Model**: Using *Fourier analysis* we derive an analytical solution to the one-dimensional heat equation with time dependent Neumann boundary conditions and temporally and spatially dependent heat generation/loss.
- **Numerical Model**: Using a *finite difference approximation* of the one-dimensional convection-diffusion equation, we implement an algorithm that estimates the temperature distribution over time based on bulk fluid motion and diffusion.

We analyze the results of each of these models, performing sensitivity analyses on the input parameters and demonstrating their flexibility. Ultimately, the numerical model proves the most physically accurate as it models convective fluid motion, which is found to dominate thermal diffusion. We then implement *closed-loop PID control* to regulate the total thermal energy of

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the system. We also model an intelligent strategy for inlet water distribution to achieve temperature uniformity.

Ultimately, our model demonstrates that using well-tuned PID control of input water temperature and evenly distributing this water upstream of the bather quickly achieves a uniform objective temperature while conserving water.

Introduction

Overview

Our specific scenario is that our friend Joseph Fourier (hereafter Joe) decides to take a warm bath and fills his bathtub to the desired temperature. After a while, he notices that the water is cooler, and he wishes to increase the temperature by adding a trickle of warm water. What should he do so as to most effectively regain the initial temperature while keeping the temperature uniform throughout and not wasting too much water? What if he starts adding water as soon as he gets in the bath?

We attempt to answer these questions for Joe. We present mathematical models describing the temperature distribution in a bathtub in space and time. We describe the relevant physical mechanisms, mathematical development, implementation, and results. Then we evaluate the accuracy and explore the best solution for Joe. Finally, we present the strengths and weaknesses of our model.

Simplifying Assumptions

- The faucet of the bathtub is exactly at one end of the length of the bathtub with the overflow drain exactly at the opposite end.
- When Joe is in the bathtub, it is full. Any addition of water results immediately in drainage of the same volume.
- The bathtub is free-standing, that is, it is surrounded by air on all sides.
- The bathtub itself is in thermal equilibrium with the water at all times. Thus, the interface with the ambient air on all sides can be described without separate knowledge of the temperature distribution within the bathtub material or the heat capacity of the bathtub.
- The only mechanism of stationary heat transfer is convection. We ignore all radiation effects.
- The ambient air temperature is constant in space and time, that is, the surroundings can be modeled as a thermal reservoir.
- The ambient air can be modeled as a stationary ideal gas at 1 bar $\approx$ 1 atm.

Table 1. Nomenclature

Abbreviation	Description
a	Boundary condition at faucet
b	Boundary condition at overflow drain
c_p	Specific heat
E_{system}	Energy of a system
f	Controls forcing function
g	Gravitational acceleration
Gr_L	Grashof number
h	Enthalpy
h_b	Convection heat transfer coefficient through bottom surface
h_{conv}	Convection heat transfer coefficient
h_t	Convection heat transfer coefficient through top surface
h_w	Convection heat transfer coefficient through the walls
H	Height of bathtub
i	Spatial index variable
j	Temporal index variable
k	Thermal conductivity
K_d	Derivative gain constant
K_i	Integral gain constant
K_p	Proportional gain constant
L	Length of bathtub
L_c	Characteristic length
m	Mass
n	Summation index variable
Nu	Nusselt number
p	Perimeter
Pe	Péclet number
Pr	Prandtl number
Q	Heat generation
Ra_L	Rayleigh number
Re_L	Reynolds number
t	Time
u	Temperature
u_{avg}	Average temperature
u_{in}	Inlet water temperature
u_{obj}	Desired water temperature
u_s	Surface temperature
$\overline{u}$	Difference between average and desired temperature
u_∞	Air temperature
U	Shifted temperature distribution
v	Flow velocity
W	Width of bathtub
x	Length direction
y	Width direction
z	Height direction
α	Thermal diffusivity
β	Volume expansion coefficient
ϵ	Porosity
η	Fraction of input heat to be redistributed
ν	Kinematic viscosity
σ	Standard deviation from average temperature
ϕ	Initial condition
ρ	Density

- No water loss occurs due to evaporation, that is, the volume of water in the tub remains constant and the surface of the water interacts directly with dry air.
- The presence of Joe in the tub provides only thermal effects. He causes no disturbance to the flow.
- The flow is uniform, laminar, incompressible, inviscid, and quasi one-dimensional.
- The water has constant density and constant specific heat and remains in liquid form.
- All material properties are isotropic.

Geometric Considerations

The bathtub is considered a rectangular prism, with dimensions L, W, and H (**Figure 1**).

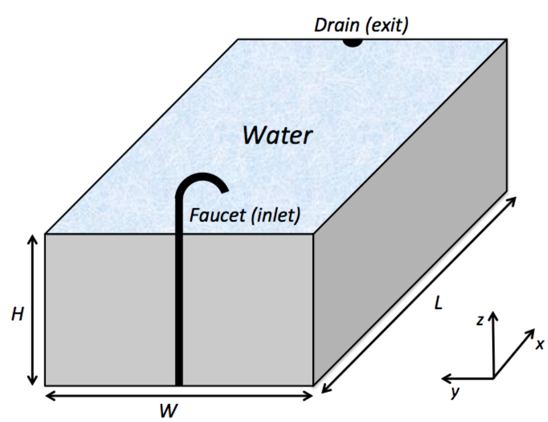


Figure 1. Bathtub geometry.

There are six rectangular faces: five composed of the bathtub material and one exposed to ambient air. The x-direction is along the length of the bathtub, and y and z are along the width and height. The faucet is at $x = 0$ and the overflow drain at $x = L$. When water is added through the faucet, an equal volume exits through the overflow drain.

The heat transfer and fluid flow within the water are assumed to be solely in the x-direction. The temperature within a cross-section parallel to the yz-plane is assumed to be uniform; it can be thought of as an average temperature over that cross-section. These two assumptions allow us to reduce the problem to one dimension.

Thermodynamic Considerations

We examine the key thermodynamic parameters of the problem: the specific enthalpy, specific heat, density, and thermal conductivity of water, as well as the convective heat transfer coefficients from the system to the surrounding air.

Properties of Water

We assume that the specific enthalpy of water varies linearly with temperature, with a slope equal to the specific heat. We validated this assumption using property values from the XSteam [1] Matlab function.

Enthalpy (h), the heat content, can be calculated from $h = c_p u$, where c_p is the specific heat and u is the temperature. We ignore the small y-intercept value and use a constant specific heat [2] of 4178 J/(kg-K).

The density and thermal conductivity of water are taken to be constant at 1000 kg/m^3 and 0.615 W/(mK) at an assumed average temperature of $30°$C [2].

Convective Heat Transfer

Because the bathtub itself is assumed to be in thermal equilibrium with the water, we consider only convective effects with the surrounding air. To appropriately estimate the heat transfer coefficient, this convection must be classified as natural (free), forced, or mixed. The situation depends on the ratio of the Grashof number (Gr_L) and the Reynolds number (Re_L). If $\mathrm{Gr}_L/\mathrm{Re}_L^2 \gg 1$, inertial forces prevail and natural convection dominates; if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \ll 1$, the opposite is true and forced convection dominates; if $\mathrm{Gr}_L/\mathrm{Re}_L^2 \approx 1$, mixed convection must be considered [2].

[Editor's Note: The authors calculate $\mathrm{Gr}_L/\mathrm{Re}_L^2$ for the system; the details are omitted.]

Since $\mathrm{Gr}_L/\mathrm{Re}_L^2 \approx 6 \times 10^5$, natural convection effects significantly dominate. Thus, we ignore forced convection.

Next, we estimate the convective heat transfer coefficients. We have six surfaces with three independent geometries. We model four of the surfaces (the sides) as vertical plates, one (the top) as a flat plate with the hotter side facing up, and one (the bottom) as a flat plate with the hotter side facing down. Due to the physics of natural convection, these three geometries have separate heat transfer coefficients. The coefficient depends on the respective Nusselt number, Nu, which in turn depends on the Rayleigh number, Ra_L, the product of the Grashof and Prandtl numbers. The Rayleigh number describes the ratio of buoyancy forces to thermal and momentum diffusivities. The Prandtl number, Pr, is a property of the air based on film temperature [2].

The Nusselt number can be determined as follows [2].

$$\text{Top:} \qquad \mathrm{Nu}_t = \begin{cases} 0.54\mathrm{Ra}_L^{1/4}, & 10^4 < \mathrm{Ra}_L < 10^7; \\ 0.15\mathrm{Ra}_L^{1/3}, & 10^7 < \mathrm{Ra}_L < 10^{11}. \end{cases}$$

$$\text{Bottom:} \qquad \mathrm{Nu}_b = 0.27\mathrm{Ra}_L^{1/4}, \quad 10^5 < \mathrm{Ra}_L < 10^{11}. \tag{1}$$

$$\text{Wall:} \qquad \mathrm{Nu}_w = \left(0.825 + \frac{0.387\mathrm{Ra}_L^{1/6}}{\left(1+(0.492/\mathrm{Pr})^{9/16}\right)^{8/27}}\right)^2, \quad \mathrm{Ra}_L > 0.$$

The convective heat transfer coefficient is

$$h_{\text{conv}} = \mathrm{Nu}\, k/L,$$

where k is the thermal conductivity of the air [2]. In practice, each thermodynamic property of air depends also upon the temperature, so there is no single explicit solution to these equations. Rather, a system temperature must be guessed and the validity of the guess evaluated. Assuming an air temperature of 20°C, the variation of h_t (the coefficient for the top surface) with water temperature is shown in **Figure 2**.

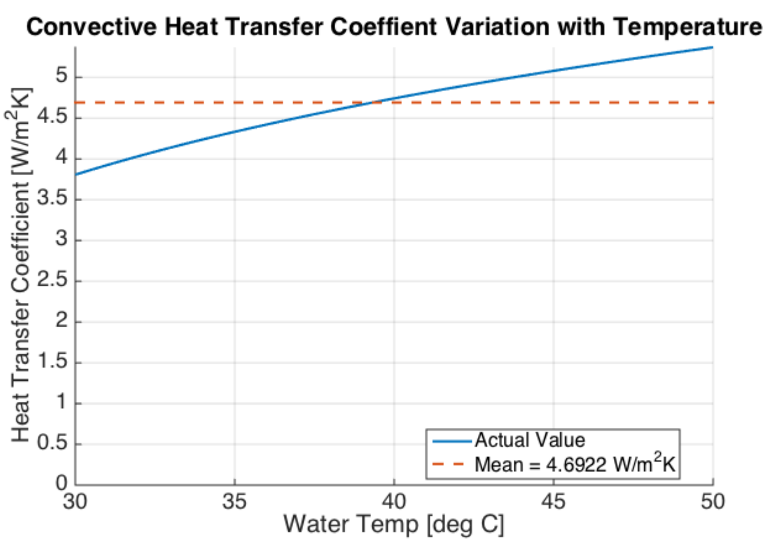


Figure 2. Variation of h_t with Water Temperature

The value of h_t rises slightly with temperature but very little over the range of temperatures that concern us. The same is true for h_b (for the bottom surface) and h_w (for the walls).

We have $h_t > h_w > h_b$, , demonstrating that natural convection is slightly more efficient from vertical walls than from face-down plates, but not as efficient as from face-up plates. For simplicity, we take the heat transfer coefficients to be constant at their mean values over the temperature range 30°–50° C.

Nominal Values for Parameters

Table 2 gives the nominal values for parameters that we use.

Table 2. Nominal values of parameters.

Parameter	Description	Nominal value
L	Bath Length	1.5 m
W	Bath Width	1 m
H	Bath Height	0.5 m
$\dot{m}$	Mass Flow Rate	0.05 kg/s
u_{in}	Input Water Temp.	45°C
L_{body}	Body Length	1 m
c_{body}	Body Circumference	0.5 m
u_{body}	Body Temperature	37°C
X_{body}	Position of Body Center	0.75 m
h_{body}	Heat Transfer Coeff. of Body	43 W/(m^{2}K) [3]
u_∞	Surroundings Temperature	20°C
u_0	Initial Water Temperature	37°C
k	Thermal Conductivity	0.615 W/(mK) [2]
ρ	Water Density	1000 kg/m^3 [2]
c_p	Specific Heat Capacity	4178 J/(kgK) [2]
h_t	Heat Transfer Coeff. of Top	4.692 W/(m^{2}K)
h_b	Heat Transfer Coeff. of Bottom	1.934 W/(m^{2}K)
h_w	Heat Transfer Coeff. of Wall	3.804 W/(m^{2}K)
t	Time of Bath	1 hr

Steady-State Uniform-Temperature Model

Theory

We move from defining the geometry and thermodynamic parameters of the problem to developing models for its solution.

For an initial model, we assume that the temperature in the bathtub is uniform and steady. This assumption allows the analysis to be simply derived from the First Law of Thermodynamics. The *system* is only the water in the bathtub, and the *surroundings* is composed of the inlet, the exit, the bather, and the ambient air.

Assuming steady state, the time derivatives of energy and mass in the system must be zero:

$$\frac{dE_{\text{system}}}{dt} = 0 \Rightarrow \dot{E}_{\text{in}} = \dot{E}_{\text{out}}, \qquad \frac{dm_{\text{system}}}{dt} = 0 \Rightarrow \dot{m}_{\text{in}} = \dot{m}_{\text{out}}. \tag{2}$$

Writing the first equation above using the First Law of Thermodynamics, we find [2]:

$$\dot{Q}_{\text{in}} + \dot{W}_{\text{in}} + \sum_{\text{inlets}} \dot{m}\left[h + \frac{v^2}{2} + gz\right] = \dot{Q}_{\text{out}} + \dot{W}_{\text{out}} + \sum_{\text{exits}} \dot{m}\left[h + \frac{v^2}{2} + gz\right]. \tag{3}$$

Physically, **(3)** is simply an energy balance. Energy transfer by heat, work, and mass are accounted for, with enthalpy, kinetic energy, and potential energy making up the mass transfer terms. We have only one inlet and one exit, which removes the summation from the equation. We also assume no work done on or by the system, no change in potential energy, and no change in kinetic energy. These are valid assumptions, since the inlet (faucet) and exit (overflow drain) should both be at the level of the water, and the kinetic energy terms will be negligible due to the assumed trickle of water. Furthermore, the mass flow entering and leaving should be the same, assuming $\dot{m}_{\text{in}} = \dot{m}_{\text{out}}$ from **(2)** above. Finally, we can combine $\dot{Q}_{\text{in}}$ and $\dot{Q}_{\text{out}}$ into one term representing the net heat transfer. Using these simplifications and the relationship $h = c_p u$, we get:

$$\dot{m} c_p u_{\text{in}} = \dot{m} c_p u_{\text{out}} + \dot{Q}_{\text{net,out}}. \tag{4}$$

Since we assume uniform temperature, the temperature at the exit is the temperature of the system ($u_{\text{out}} = u$).

The last piece of this model is to quantify $\dot{Q}_{\text{net,out}}$ using the heat transfer coefficients discussed in the previous section. We use Newton's Law of Cooling [2] to find an expression for the convective heat transfer:

$$\dot{Q}_{\text{conv,out}} = h_{\text{conv}} A (u - u_\infty). \tag{5}$$

Associating the appropriate surface areas with each coefficient, we arrive at the following expression for the total convective heat loss:

$$\dot{Q}_{\text{conv,out}} = (u - u_\infty)\left[(h_t + h_b)LW + h_w(2HW + 2HL)\right]. \tag{6}$$

The presence of Joe in the bathtub provides an additional heat transfer term with its own heat transfer coefficient. We assume the submerged part of the body is a perfect cylinder without end caps, with length L_{body}, circumference c_{body}, and external temperature u_{body}. The total heat loss from the system now includes a term to account for the body:

$$\dot{Q}_{\text{net,out}} = (u - u_\infty)\big[(h_t + h_b)LW + h_w(2HW + 2HL)\big] + (u - u_{\text{body}})h_{\text{body}}L_{\text{body}}c_{\text{body}}. \quad \textbf{(7)}$$

By substituting **(7)** into **(4)** and solving for u, we find the final expression for steady uniform temperature:

$$u = \frac{\dot{m}c_p u_{\text{in}} + u_\infty\left((h_t + h_b)LW + h_w(2HW + 2HL)\right) + u_{\text{body}}h_{\text{body}}L_{\text{body}}c_{\text{body}}}{(h_t + h_b)LW + h_w(2HW + 2HL) + h_{\text{body}}L_{\text{body}}c_{\text{body}}}. \quad \textbf{(8)}$$

Implementation and Results

We implemented **(8)** using Matlab. Using average values for the heat transfer coefficients and the nominal parameter values, we obtained the steady uniform temperature as a function of mass flow as shown in **Figure 3**.

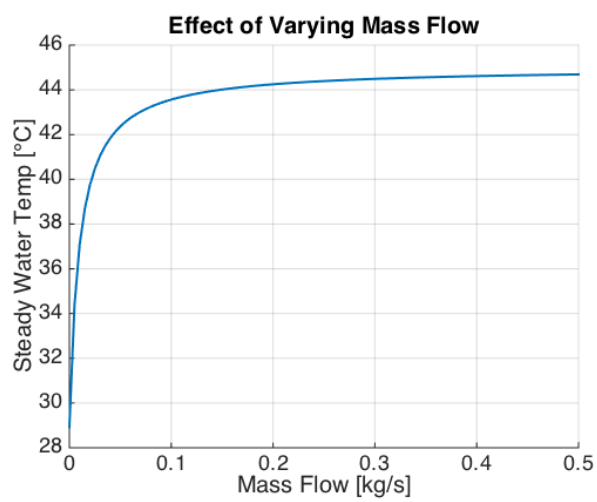


Figure 3. Sensitivity of final temperature to inlet mass flow.

The figure indicates that increasing the mass flow results in a final temperature that approaches the inlet temperature of 45°C. Also, as expected, the final steady temperature depends linearly on the inlet temperature.

When the inlet temperature is close to that of the ambient air (20°C), the steady temperature is higher than that of the inlet, due to the minimal convective heat losses and the effect of Joe at 37°C. However, when the inlet temperature is closer to body temperature, the steady temperature is slightly below the inlet temperature, due to the increased convective heat losses.

The final steady temperature depends linearly and relatively insensitively on the ambient air temperature. The model is clearly much less sensitive to this parameter, illustrating minimal heat losses to convection due to the relatively low heat transfer coefficients.

Ultimately, this model is too simple to capture the true dynamical nature of the bathtub system. Though more accurate models are described in the following sections, the qualitative results here serve as a good baseline for comparison with subsequent models.

Analytical Model of the Heat Equation

[EDITOR'S NOTE: This section is omitted, since the authors subsequently abandon the analytical model, which features only heat diffusion, in favor of a numerical model that includes also convection.]

Numerical Model of Convection-Diffusion

We use a finite difference approximation of the convection-diffusion equation. We briefly discuss the theoretical development and implementation, and then present initial results.

Theory

We introduce a convection term into the model, which allows the actual flow of the water—and not just diffusion of heat—to be modeled. Convection in this sense means the bulk motion of the water in the tub and not the heat loss to the surroundings. The governing PDE for a convection-diffusion problem is shown below, where the only new quantity is ϵ, the porosity of the medium:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} - \epsilon v \frac{\partial u}{\partial x} + \frac{q(x,t)}{\rho c_p} \tag{9}$$

In this equation,

- $\dfrac{\partial u}{\partial t}$ represents the transient,
- $\alpha \dfrac{\partial^2 u}{\partial x^2}$ represents diffusion,
- $\epsilon v \dfrac{\partial u}{\partial x}$ represents convection, and
- the final term represents heat generation or loss.

Eliminating the convection and generation terms yields the familiar heat equation. Prescribing an initial condition and enforcingappropriateboundary conditions fully defines this problem. Due to the difficulty in solving this equation analytically, we implemented a numerical solution using the finite difference method, as outlined by Majchrzak and Turcha [6].

The first step is to discretize the temporal and spatial variables. We let j index time and i index space, and Δt and Δx are the time step and grid spacing:

$$t_{j+1} = t_j + \Delta t, \qquad x_{i+1} = x_i + \Delta x. \tag{10}$$

Each derivative can be approximated using a Taylor series:

$$\frac{u_i^j - u_i^{j-1}}{\Delta t} = \alpha \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{(\Delta x)^2} - \epsilon v \frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{\Delta x} + \frac{q_i^{j-1}}{\rho c_p}. \tag{11}$$

Solving for the temperature at time j and location i yields:

$$\begin{aligned} u_i^j = \left(1 - \frac{2\alpha\Delta t}{(\Delta x)^2}\right) u_i^{j-1} + \left(\frac{\alpha \Delta t}{(\Delta x)^2} + \frac{\epsilon v \Delta t}{2\Delta x}\right) u_{i-1}^{j-1} \\ + \left(\frac{\alpha \Delta t}{(\Delta x)^2} - \frac{\epsilon v \Delta t}{2\Delta x}\right) + \frac{q_i^{j-1}}{c_p \rho}\, \Delta t. \end{aligned} \tag{12}$$

The porosity ϵ is the ratio of fluid volume to total volume; since the medium is water (entirely fluid), we take $\epsilon = 1$.

To complete this model, we must quantify q and enforce appropriate boundary conditions.

The fundamental source of heat transfer in this problem is convection from the water surface and from the bathtub surface. In **(12)** above, q is the heat transfer per unit volume, where the volume in consideration is a slice of the bathtub in the x direction. This slice has volume $WH\Delta x$. Thus,

$$q = \frac{Q}{WH\Delta x}, \tag{13}$$

where Q is the macroscopic heat loss of the slice. This heat loss is described by the convection terms and the conduction term as follows:

$$\begin{aligned} Q &= (u - u_\infty)\big[h_t A_t + h_w A_w + h_b A_b\big] \\ &= (u - u_\infty)\big[h_t (W\Delta x) + h_s (2H\Delta x) + h_b (W\Delta x)\big]. \end{aligned} \tag{14}$$

Plugging this into **(13)** gives

$$q = \frac{(u - u_\infty)}{WH}\, [W(h_t + h_b) + 2Hh_w]. \tag{15}$$

At location i at time step j, we use u_i^{j-1} to estimate u in the equation.

The same method can be used to model the presence of a person in the bathtub. [EDITOR'S NOTE: The details of the calculation are omitted.]

To enforce the boundary condition on the inlet side, we must specify the correct amount of heat flux into the system as a Neumann condition, increasing the energy content of the fluid element at the boundary based on the mass flow and temperature of the inlet.

The energy entering per unit volume, a q term, can be described as

$$q = \frac{Q}{WH\Delta x} = \frac{\dot{m}c_p(u_{\text{in}} - u)}{WH\Delta x}. \tag{16}$$

This can be translated directly into a change in temperature, as in **(12)** above, by dividing by ρc_p, multiplying by the time step Δt, and approximating u from the previous time step:

$$\Delta u_i^j = \frac{\dot{m}\left(u_{\text{in}} - u_i^{j-1}\right)}{\rho WH\Delta x}\Delta t \tag{17}$$

Equation **(17)** provides a direct method for changing the temperature of the boundary point in order to account for the heat entering the system.

At the other boundary point, we wish the water to exit the system naturally, which poses a challenge in prescribing a boundary condition. A realistic condition is that the flux out is equal to the flux between the two points adjacent to the boundary. Thus, the temperature at the end can be specified as follows:

$$\begin{aligned} u_{end}^j &= u_{end-1}^j + \frac{\partial u}{\partial x}\Delta x \\ &= u_{end-1}^j + \left(\frac{u_{end-1}^j - u_{end-2}^j}{\Delta x}\right)\Delta x = 2u_{end-1}^j - u_{end-2}^j. \end{aligned} \tag{18}$$

Now both boundary conditions have been specified and we can implement the finite difference method in Matlab.

Basic Results and Verification

We present basic results to demonstrate the functionality of our model.

First, we simulated the situation with no heat input (the faucet turned off) (**Figure 4**). The temperature in the tub remains uniform and slowly decreases due to heat loss to the surrounding air.

Next, we simulated the system with the faucet on with steady flow (**Figure 5**). In this case, the temperature away from the faucet (near $x = L$) drops steadily, as before, but the temperature near the faucet increases to the temperature of the incoming water. Through time, this wave of heat convects through the system at a rate determined by the mass flow and the geometry of the tub.

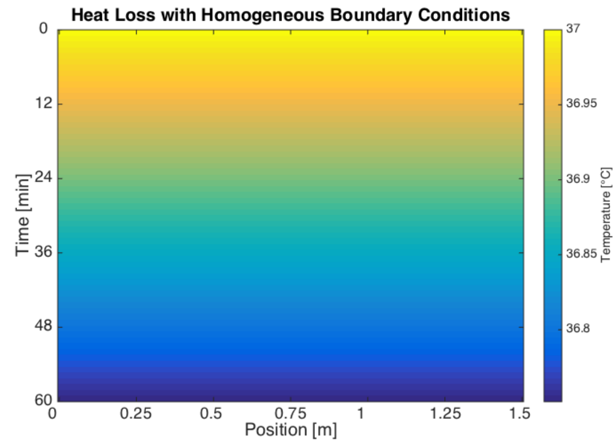


Figure 4. Heat loss over time with no heat input.

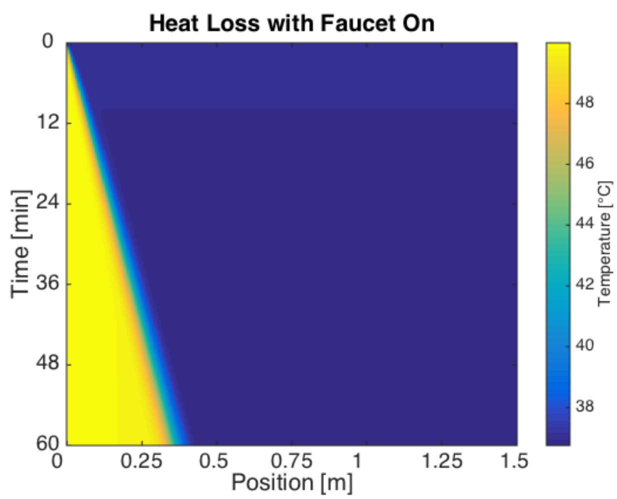


Figure 5. Heat map for steady flow from the faucet, together with heat loss.

We simulated adding bather Joe but found that the bather has almost no effect, which makes sense because both bath and body temperatures are nearly equal.

Modeling Bather Actions

We examine the effect of actions of the bather redistributing heat by moving, in the case of fixed inlet water temperature and mass flow.

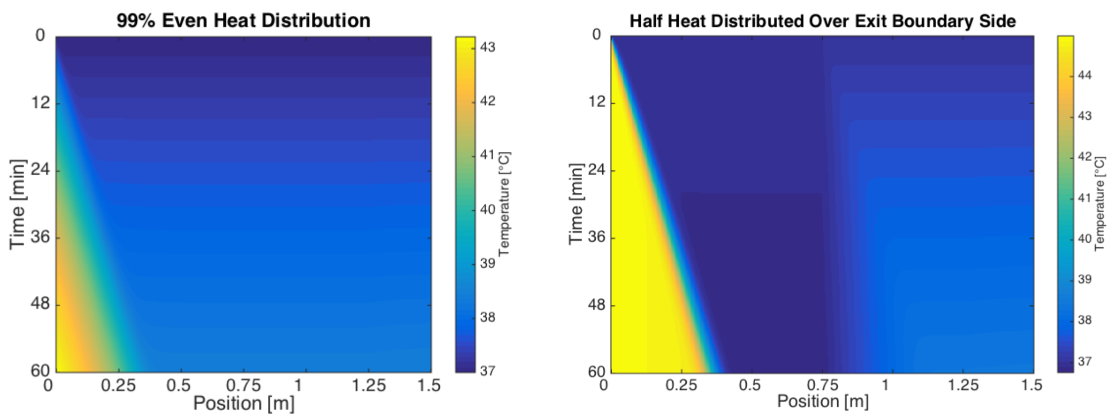
We extend the logic for the boundary condition on the inlet to the full domain. The input energy can be added to any locations in the domain, provided that the total energy is conserved. So we modify **(17)**, using a parameter η_i^j that is the fraction of available heat prescribed to enter grid point i at time j.

$$\Delta u_i^j = \eta_i^j \frac{\dot{m}\left(u_{in} - u_i^{j-1}\right)}{\rho W H \Delta x} \Delta t. \tag{19}$$

This temperature change term is in addition to that already calculated from the discretized derivatives and heat loss to the surroundings. Conservation of energy means

$$\sum_i \eta_i^j = 1. \tag{20}$$

Two interesting applications of bather effects are shown in **Figures 6a** and **6b**. In the left figure, 99% of the input water was evenly distributed over the whole domain and the remaining 1% was left to trickle through the $x = 0$ boundary. In the right figure, half of the heat was distributed evenly over the region from $x = 0.75$ to $x = 1.5$ and the other half trickled through the faucet.



(a) Nearly uniform heat redistribution by bather. **(b)** Partial redistribution to back half of tub.

Figure 6. Effects of Redistribution of Water

Model Comparison

Solution Metrics

First, we must develop a metric to evaluate our solutions. To do this, we return to the original objective: Joe wishes to attain a desired temperature uniformly distributed throughout the bath. We decouple the *desired temperature* constraint and the *uniform distribution* constraint and evaluate them separately.

Desired Temperature

The first metric evaluates the total heat within the bath, addressing the *desired temperature* constraint, with objective function $\overline{u}$:

$$\overline{u}(t) = u_{\text{avg}}(t) - u_{\text{obj}}. \tag{21}$$

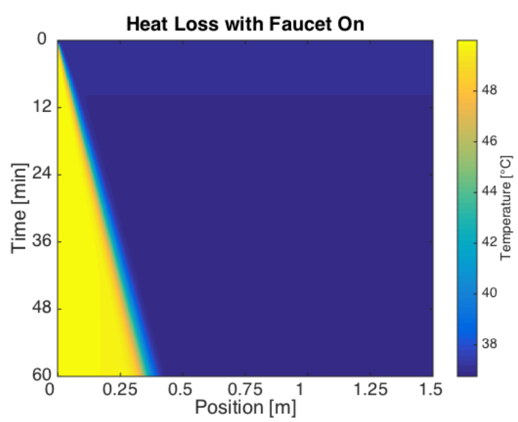
This metric will yield a time-dependent value proportional in magnitude to the difference between the average and objective, with a sign that denotes whether the bath is below or above the objective temperature. As a standard value, the objective temperature of the water will always be a comfortable 37°C.

Uniform Temperature

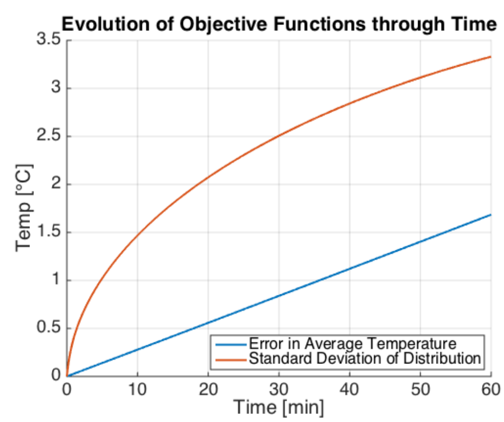
The second metric quantifies the *uniformity* of the temperature distribution over time. We use σ, the standard deviation of the temperature distribution at a given point in time:

$$\sigma(t) = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(u_n(t) - u_{avg}\right)^2} \tag{22}$$

We show in **Figure 7** the evolution of the objective functions (metrics) through time for the model with nominal parameters, along with the heat map describing the temperature distribution.



(a) Heat map with faucet on.



(b) Objective Functions From Simulation

Figure 7. Demonstration of the objective functions, $\overline{u}(t)$ and $\sigma(t)$ with nominal parameters.

The temperature becomes less uniform as we add too much heat to the system. For Joe to be happy with his bath, our goal is to drive both of these objective functions to zero. As a standard, we consider the temperature distribution at the end of a one-hour bath.

With metrics defined, we now run simulations to test the impact of each parameter. The following subsections will each describe the effect of a different parameter on the solution metrics.

Effect of Bathtub Geometry

Changing the dimensions of the tub changes the surface areas exposed to the air, the velocity of the water flow, and properties of the fluid element such as mass and volume. In **Figure 8**, we examine the effect of changing the length of the bathtub while holding other parameters constant.

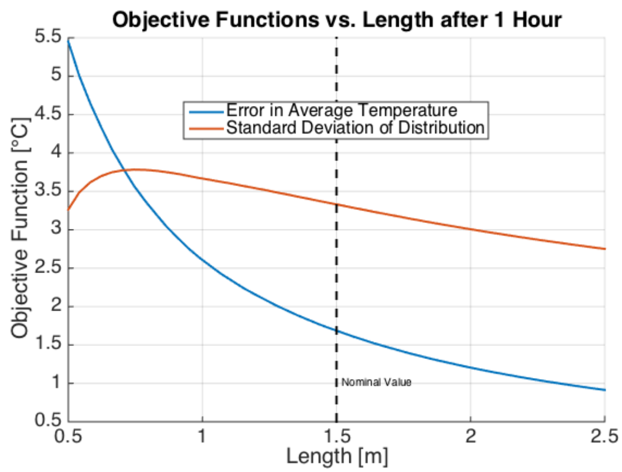


Figure 8. Effect of variation of length of tub.

Increasing the length of the tub decreases both the deviation from average temperature and the variability. The water is both more uniform and on average closer to the desired temperature. Since the flow rate is unchanged, a longer bathtub is less affected by the influx of the hot water.

We conducted similar simulations of the effect of tub width, tub height, and total volume, with similar results. [EDITOR'S NOTE: The details are omitted.]

Effect of Adding Soap

If Joe were to use a bubble bath additive or soap, the thermal diffusivity of the water would change. The primary ingredient of soap and bubble bath solutions is sodium stearate, whose density is nearly the same as that of water. Adding a solute such as sodium chloride to water reduces the thermal conductivity [9], so it is reasonable that sodium stearate has a similar effect.

To perform a sensitivity analysis of the diffusivity, we simulated a wide range of diffusivities—a full order of magnitude about the standard diffusivity of pure water, 1.472×10^{-7} m^2/s—to check for a noticeable change in solution behavior.

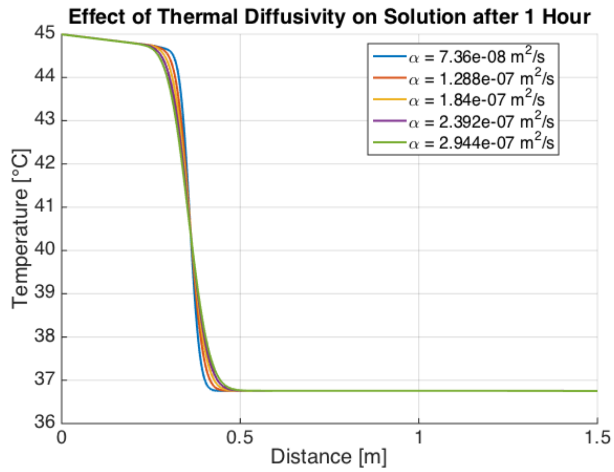


Figure 9. Effect of variation of diffusivity.

Final temperature distributions after one hour are shown in **Figure 9** for various values of thermal diffusivity. Thermal diffusivity has very little effect on the solution, supporting the hypothesis that the heat transfer is primarily due to bulk flow and not diffusion. By examining the Péclet number (Pe) [10], we can confirm this hypothesis. The Péclet number is the ratio of the convective transport rate to the diffusive transport rate. We calculate it with our nominal parameter values:

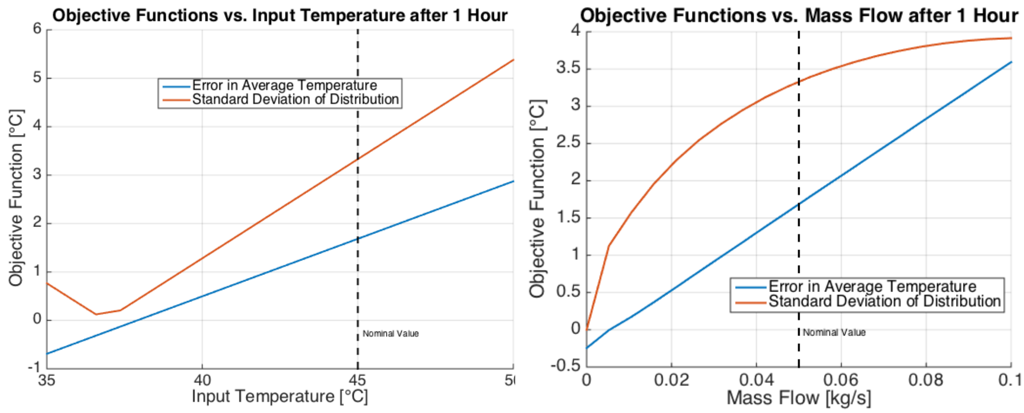
$$\mathrm{Pe} = \frac{vL}{\alpha} = \frac{(1.2 \times 10^{-4}\ \mathrm{m/s})(1.5\ \mathrm{m})}{1.472 \times 10^{-7}\ \mathrm{m}^2/\mathrm{s}} \approx 1223 \gg 1. \qquad \textbf{(23)}$$

The large ratio confirms that diffusion effects are negligible in comparison with bulk motion effects.

Effect of Input Temperature and Flow Rate

The effects of varying the input temperature and input rate are shown in **Figure 10**.

Increasing either parameter corresponds to increasing net energy in the system. A large flow rate results in a more uniform distribution than using instead a high temperature at the inlet, because increasing the mass flow rate increases the convective term, which is more efficient than the diffusion. The vertical lines in the plots are the nominal values, but the x-intercepts of the straight blue lines represent the correct amount of heat to the system so as to maintain the desired temperature.



(a) Variation of input temperature.

(b) Variation of mass flow rate.

Figure 10. Effects of varying input temperature and mass flow at input.

Effect of Varied Size and Position of Bather

Neither Joe's body length nor his circumference has a significant effect on the temperature distribution, because his body temperature is close to the temperature of the bath. The body temperature itself has a slight impact. [EDITOR'S NOTE: The details are omitted.]

Control Theory and Implementation

Now we consider the case where the temperature is uniform, but noticeably lower than desired, a potentially more realistic application.

PID Control Theory

Joe's ultimate goal is a bath that maintained at a uniform desired temperature. We develop a time-dependent strategy for Joe using Proportional-Integral-Derivative (PID) control.

The first portion of the control system is designed to achieve and maintain the target temperature by using the average temperature metric detailed previously. This segment of the control system focuses on optimization of the inlet temperature and flow rate of the water from the faucet.

PID control is implemented through the selection of the *gains* associated with each component of the control. Equation **(24)** shows the general gain equation of a PID controller as a function of the objective function $\overline{u}(t)$ with variable gains K for each stage, where $f(t)$ represents the calculated forcing (input).

$$f(t) = K_p \overline{u}(t) + K_i \int_0^t \overline{u}(\tau)\, d\tau + K_d \frac{\partial \overline{u}(t)}{\partial t}. \qquad \textbf{(24)}$$

Each of the three components of a PID controller corresponds primarily to a specific response of the system:

- The proportional control element is based on the difference between actual temperature and objective temperature; this control is what physically drives the system to the objective.
- The integral component defines the response on accumulated error from the objective instead of current error, decreasing the rise time and removing the steady state error from the system.
- The derivative component predicts the response error based on the temporal derivative of the response, damping any oscillations.

Properly tuning each gain results in the optimal responses of a small rise time with minimal oscillation and no steady-state error.

Implementation of PID Control in Our Model

We implemented PID control by specifying the forcing as a function of $\overline{u}(t)$, as shown in **(24)**. In particular, we set

$$f(t) = \dot{m}[u_{\text{in}} - u(0,t)], \tag{25}$$

a term in the inlet Neumann condition.

We needed to determine the gain coefficients. They should weight the response with the ultimate goal of achieving a response with minimal rise time, little-to-no overshoot, and long-term stability. A rise time in the bath corresponds to the time it takes to raise the bath temperature from the initial temperature to the target temperature. The rise time should be short so that Joe does not have to wait long for his bath to reach the target temperature. Too much overshoot could result in Joe being scalded. Finally, long-term stability reflects maintaining the desired temperature over the course of the entire bath.

Methods for determining optimal gains include PID control software and algorithms such as the Ziegler-Nichols tuning process; but since we lack a well-defined transfer function, neither option is applicable and manual tuning is required.

We manually tuned our gains to obtain a roughly optimal response. **Figure 11** shows system response using only proportional control, starting at 30°C and attempting to reach 37°C.

There are noticeable oscillations about 0 (representing zero error in the temperature) in the objective function (the blue curve starting below at left), and there is an overshoot of about 4°C after one hour. Since the forcing is a function of both mass flow and inlet temperature, we fix mass flow and present the result as the required Δu (the red curve starting above at left). The Δu peaks at around 20°C, meaning that the inlet water temperature must be 20°C hotter than the current water temperature at the inlet.

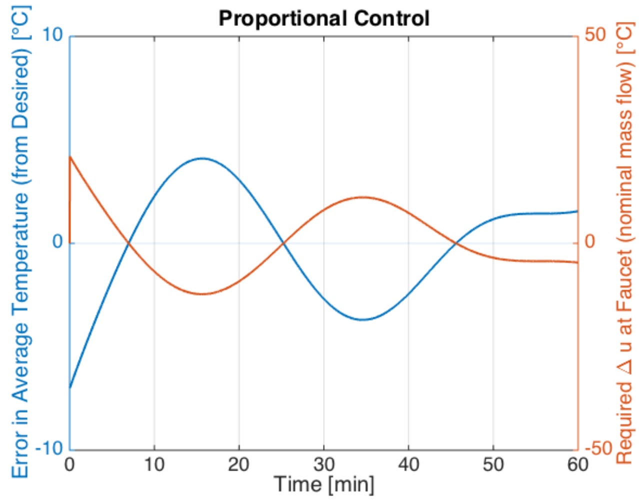


Figure 11. Effect of proportional control.

The results from using full PID are shown in **Figure 12**. The objective function is driven to nearly zero, as desired. More importantly, the response is achieved with a relatively short rise time, minimal overshoot, and maximal oscillation of less than 1°C. The associated forcing function is more reasonable, requiring a maximum Δu of about 3°C.

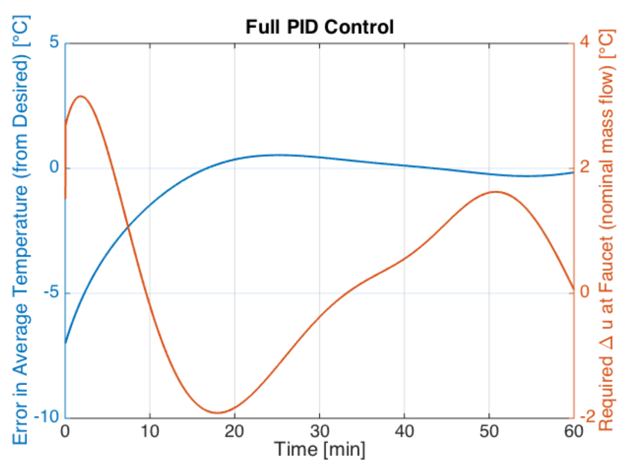


Figure 12. Optimal control of average bath temperature.

Control of Uniformity of Water Temperature Distribution

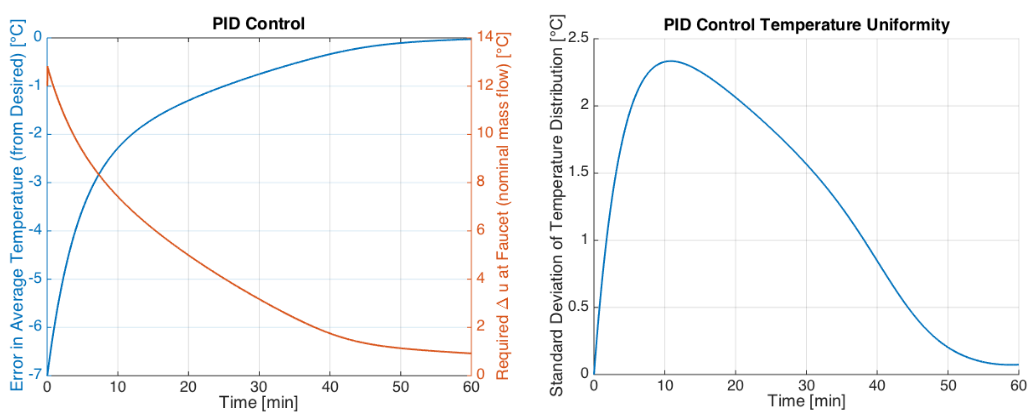
With the average temperature driven to the target temperature using PID control, the only remaining factor is the uniformity of the temperature. To maintain uniformity is very simple: Instead of adding heat from the faucet to the bathtub at $x = 0$, we evenly distribute the hot water over the whole bath. Physically, this method corresponds to Joe continuously redirecting the water from the faucet, using his hands, a cup, or another redistribution system. Our simulations confirm that uniform heat distribution is optimally effective and results in a perfect bath, using either the optimal input temperature or the optimal mass flow rate. [EDITOR'S NOTE: The details are omitted.]

Final PID Control with With Manual Heat Distribution

We close with a simultaneous consideration of average temperature (PID) control and uniformity control. Up to this point, we have analyzed these control methods separately and under different initial conditions; but to complete our solution to the problem, we need to implement them simultaneously. The initial condition is a 30°C uniform bath (Joe realizes that his bath has become cold) with target 37°C.

We use PID control to determine the required temperature of the input water (given a constant nominal mass flow rate) and using intelligent redistribution to maintain uniformity. We determined that distributing inlet water over the region from $x = 0$ (the faucet) to where the body starts, $x = 0.25$, is optimal. Doing so allows the body heat to convect downstream and the inlet heat to warm the water upstream of the body.

After manually tuning the gains and implementing full PID in addition to intelligent inlet water distribution, we realized the results shown in **Figure 13**.



(a) PID Control with Manual Heat Distribution **(b)** Uniformity over Time

Figure 13. System response using PID and intelligent heat distribution.

On the left, the average temperature objective function rises to zero relatively quickly with no overshoot, an optimal result. Additionally, the required Δu with a nominal flow rate is physically reasonable. On the right, the temperature uniformity initially increases (due to Joe's body) but settles to zero as the inlet water distribution begins to have an effect.

The heat map corresponding to this control is shown in **Figure 14**.

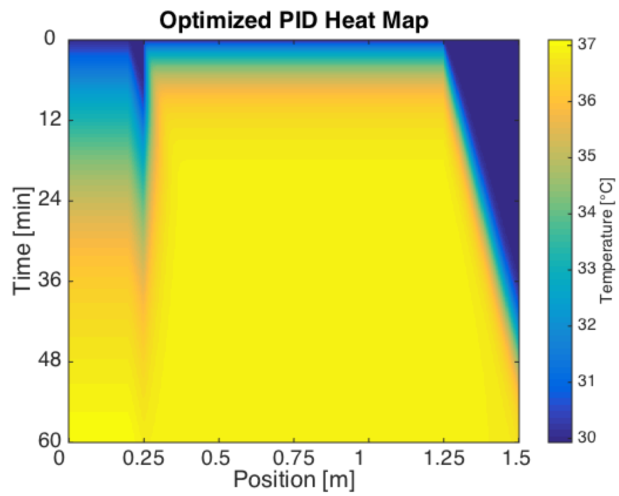


Figure 14. Heat map of system response using PID and intelligent heat distribution.

Joe should use PID control to vary the input temperature and simultaneously use a cup or other redirecting device to distribute the inlet water over the region of the bathtub upstream from his body.

Final Remarks

Strengths and Weaknesses of Solution Model

Strengths

- The greatest strength is generality, allowing for a wide range of parameter customization, from dimensions of the bath, types of boundaries enforced on the system, heat transfer coefficients, effects of Joe on the system, and final control application.
- The model incorporates effects of both convection and diffusion, yielding a realistic fluid thermodynamic simulation.
- Time-dependent boundaries and a spatially- and temporally-dependent heat source can be customized.
- The straightforward implementation of simulations allows for a variety of tests of parameter sensitivity as well as the implementation of control.
- The numerical approach allows for the possibility of expanding the simulations into a multidimensional model.

Weaknesses

- The main weakness is that the model considers only one-dimensional heat flow.

- The mathematics involved in this numerical implementation is highly sophisticated.

Future Model Development

As discussed in the weaknesses of the models, many possibilities exist for the development of a more precise model. In the future, a more comprehensive and definitive model would be developed in the following ways:

- Expand the models to incorporate more detailed information through increasing the dimensionality of each model, more accurately reflecting a physical three-dimensional system.
- Conduct experimental trials and extensive research to ensure the accuracy of certain physical parameters, such as heat transfer coefficients for humans or the precise effects of soluble materials on water's thermal properties.
- Apply geometry and coordinate transformations to explore the effect of various geometries on the system, such as cylindrical and trapezoidal geometries.
- Analyze the steady-state distribution of the thermal energy with high accuracy across three dimensions through implementation of Laplace's equation, spatially dependent boundary conditions, and realistic heat loss.

Conclusions

We modeled the temperature distribution in a bathtub in space and time so as to determine the most effective method of maintaining the temperature. Three models were developed to describe this temperature distribution: a steady-state uniform model, an analytical heat equation model, and a numerical convection-diffusion equation model. Ultimately, the numerical model proved most accurate because it alone took into account bulk fluid motion, the dominant heat transfer effect. Using this model, we conducted sensitivity analyses on different parameters to explore the effects of different bathtub geometries, the addition of soap, and other variations to the problem at hand. We also implemented PID control of the inlet water temperature and intelligent distribution of inlet water. By optimizing this process, we developed a strategy for quickly re-heating the bath while maintaining uniformity.

Overall, our model shows that well-tuned PID control combined with distributing the inlet water upstream of the bather is the most effective strategy.

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Superior Products*:
Applied Mathematical Consulting Firm

Superior Products, LLC[1]
Letter to International Society for the Promotion of Comfort and Cleanliness
2/1/16

As many of you are likely aware, there has been a singular problem facing avid bath users for many generations. Though we have come quite far as a society with the invention of running water, synthetic bathtub materials, and even advances in soap and shampoo solutions, the issue of keeping your bath at a comfortable temperature without pockets of cold or hot water has not been improved upon for generations.

Solutions range from self-circulating baths, expensive water jets, or even the deferring of a magnificent bath to a common shower, but an explanation has slowly presented itself to the difficulty shrouding this dilemma. The underlying factor of the cooling bath problem is a combination of being able to maintain heat within the bath over the course of your soak and also to distribute any hot water you introduce to the system. This is difficult due to the fact that water diffuses heat very poorly. Your relaxing afternoon bath is in reality a complex dynamical system, controlled by the physical phenomena of thermodynamics, heat transfer, and fluid mechanics.

Fortunately for the bathing community, a small percentage of our fellow bathers are not only aspiring to the ideals of cleanliness, but also ardently pursuing an understanding of applied mathematics. This question of the fluid dynamics of the bath tub has intrigued your friends here at Superior Products, and we have attempted to find a solution to the age old question of how to achieve the perfect bath.

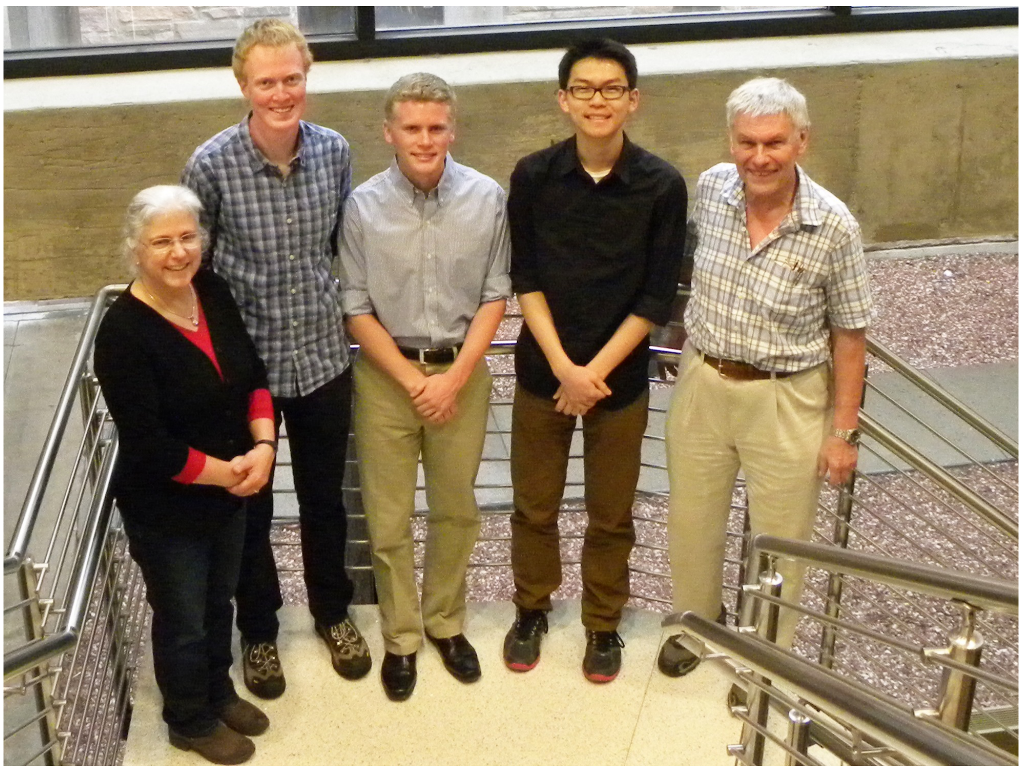
Despite the daunting complexities of the common bathtub, the advent of modern computational methods and simulation tools has allowed for an extensive understanding of heat flow within the bath. Through careful studying of the heat loss from the bath, diffusion of energy from warm to cold regions, and the bulk flow of water from faucet to drain, a final solution to your issues has been found.

Ultimately, the solution involves precision control theory to define the optimal temperature of the incoming water at every given time of your bath. This temperature is chosen to achieve your ideal temperature quickly, without the risk of scalding yourself. The second fundamental portion of the strategy is to spread the incoming water continuously over the surface of your bath

[1] This company is purely a work of fiction.

as it enters through the faucet, possibly through the use of a cup or customized bathwater distribution device. Our generic solution can take into account any uniqueness in your specific bath, such as your desired bath temperature and your bathtub shape and size.

This simple yet refined strategy will ensure that your bath is always at the perfect, uniform temperature. Feel free to use whatever soaps, shampoos, or even enhanced bubble solutions to improve your bathing experience.



Anne Dougherty (Associate Mathematics Department Chair), team members Matthew Hurst, Jordan Deitsch, and Nathan Yeo, and team advisor Bengt Fornberg.

Judges' Commentary: Hot Bath Problem

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Introduction

The teams who chose the Hot Bath Problem in the 2016 MCM were asked to find an optimal strategy for taking a hot bath in a traditional tub, that is, one with no circulating jets. This was a deceptively difficult problem to address. Of course, there are many established models for heat transfer and fluid flow, and we saw a number of papers using the heat equation and Navier-Stokes equations; but also, as one would expect in an undergraduate competition, we saw papers that simply used Newton's law of cooling. Many undergraduates do not see partial differential equations in the course of their study, and most of the judges took this into account when evaluating the papers.

The judging process itself should be of interest to teams and advisors; but since it has been described in detail in previous judges' commentaries (see Black [2009; 2011; 2013]), I do not provide an overview here. In particular, the process sheds light on the importance of various components in solution papers. In this commentary, I will focus primarily on specifics as they pertain to this problem and on general advice to teams.

Graphics, Simulations, and Models

We saw a number of simulations and graphical representations of heat flow. The better papers explained well what the graphics were showing and how they related to the model and the team's recommendations. These papers also gave a good description of the algorithm used to create the graphics. All too many papers present graphics, and sometimes include code in an appendix, without giving the judges sufficient information to

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evaluate them. Judges have neither the time nor the interest in reading code in order to discern the algorithm. Descriptive captions, along with clear exposition of how the graphics were created, are a must.

The higher-ranked papers included mathematics that was appropriate and justified. If there was a simulation, it was clear how the simulation was created and there was a clear justification of the model used. Better papers limited the number of models that they used in order to flesh out fully the details on the ones they did use.

Communication

The value of clear communication in this competition cannot be overly stressed. Judges have limited time to read the paper, and the most brilliant modeling will fall by the wayside if the details are not communicated clearly and efficiently.

In particular, the project summary should outline the model used and the results. For this problem, it should have indicated the best strategy for keeping the water hot, what model was used, and how the model led to this strategy.

When data, models, equations, or graphics are taken from other sources, appropriate documentation is required! But teams also need to discuss why this model, equation, data, etc., is relevant and exactly how they adapted it to fit this problem.

The Non-Technical Explanation for Users

The non-technical explanations that are frequently asked for in modeling competitions are important and often not given enough attention by participants. The best mathematics is not going to be helpful in the "real world" unless you can convince a non-technical audience that the results are valid. And the results will be totally useless if they cannot be translated into indications of what the "client" should actually do with them.

In this case, what should the bather do to optimize enjoyment of the bath? There was some room for originality and creativity here. Many of the best papers specifically recognized that you somehow need to stir the water, and that adding hot water to the top of the tub while the overflow valve is also at the top is going to present problems. Some papers recommended draining part of the cooler water before adding new hot water, and others devised mechanisms for moving the hot water to the bottom of the tub.

Bathtubs

Teams that looked at different sizes and shapes of tubs were more highly regarded than those that didn't. It was also expected that the tub would be modeled in three dimensions, not just as a series of two-dimensional slices. The best papers had three-dimensional models and discussed the effect of the shape and size of the tub had on their models.

Modeling vs. Applying Models Correctly

One of the more interesting tensions in this year's judging was between developing and/or adapting models as opposed to finding and appropriately using existing models.

As mentioned previously, there are many accepted models and differential equations available to describe heat transfer and fluid flow. Among papers that used these, the better ones were explicit in stating where the models came from, how and why they were applied, and what results were obtained from their use.

One of the impressive things that we saw in one of the Outstanding papers was a creative solution to the need for appropriate parameters for these models—in particular the proportionality constant k in Newton's law of cooling. The team found sources online that gave the values for k for water in beakers; but they needed to model water in a variety of shapes of bathtubs. They performed experiments to measure the proportionality constant for a particular bathtub, then used a variety of curve-fitting techniques to estimate the parameter value for tubs of other sizes. (Experiments in general, when appropriate, are always a good idea and provide a welcome respite for the judges from textbook approaches.)

The curve-fitting that the team used was informed by analyzing the dimensions involved, making assumptions about which dimensions would impact the value of k, and making sure that the units would cancel out in the curves that they applied. Unfortunately, this team was not perfect. In fact, they had Newton's law of cooling wrong! Because of this, their initial model's prediction was contrary to common sense—the temperature of the bath-water went to freezing instead of to room temperature. Of course, the team should have realized that this was because they had the rate of change of temperature proportional to the temperature of the water rather than proportional to the difference between the temperature of the water and the temperature of the room. They did realize that the predictions were off and adjusted the output of the model appropriately, arriving at the solution to the differential equation that they should have been solving in the first place.

None of the papers that we see are ever perfect. Teams, after all, have only a weekend to come up with a model, solve it, enhance it, and write a clear and coherent paper presenting their results. Anyone who has ever written a textbook will tell you about still finding mistakes (with hope that they are not substantive!) after years of use.

So one of the most difficult, and sometimes contentious, decisions that the judges have to make each year is what constitutes a "fatal" flaw in a paper. What error is so grave that, in and of itself, it would eliminate an otherwise Outstanding paper from receiving the Outstanding designation? This year it was the error noted above in Newton's law of cooling that led to the most heated debate. Some judges felt that the team had redeemed themselves sufficiently, in how they handled the break from reality and the other modeling that they did, to allow us to overlook this flaw; while others disagreed, believing that the team should have recognized that the differential equations model used did not make good sense. This was a rare case where, after lengthy discussion, a consensus was not reached, and the paper was awarded the Outstanding designation by majority vote. However, judges on both sides of the controversy recognized the validity of the argument for the other side.

The take-away from this for teams in future competitions is to double- or triple-check your established models. In particular, when an established model seems to predict something that you know is wrong, question whether you have made a mistake in the model. This small error almost cost this team the Outstanding designation.

Conclusion

The Hot Bath Problem allowed for a variety of approaches and a fair amount of creativity, even though most papers used established models for heat and fluid flow. Teams who thought about the actual situation of taking a bath rose above those who merely took the established models and presented results. As always, communication was key in determining the top papers.

Overall, there has been an improvement in the quality of the papers that we see in final judging. Some of this is certainly due to the increased popularity of the contest and the smaller percentage of papers that make it this far. However, it also seems that teams are making better use of the advice and materials available to them.

In particular, we are seeing more teams perform meaningful sensitivity analysis. We are also seeing better use of assumptions in the modeling process. More are making assumptions that simplify their models and later testing the sensitivity of their results to those assumptions.

All of the teams who successfully participated in this year's competition should be proud of their accomplishment. For those who wish to improve for the future, paying attention to the advice in this and other judges' commentaries would be a good step.

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About the Author

Kathleen Shannon is Professor of Mathematics at Salisbury University in Maryland. She earned her bachelor's degree from The College of the Holy Cross with majors in both Mathematics and Physics. She received Master's and Ph.D. degrees in Applied Mathematics from Brown University. Her primary focus is in teaching undergraduate mathematics, from liberal arts courses through calculus and discrete mathematics to numerical analysis and real analysis. She is active in the mathematical modeling community as a regular judge of both the MCM and Moody's Mega Math Challenge. She is a member of the Mathematical Association of America and of the Society for Industrial and Applied Mathematics.

Will We Survive the Space Junk?

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Abstract

We devise a model to evaluate the performance of an ODR (Orbital Debris Removal) system and propose the best alternative method. We offer a conclusion about whether a profitable business opportunity exists. We use the Analytic Hierarchy Process (AHP) to assess the overall performance of ODR methods. We also come up with a new method for collision avoidance.

Our first model simulating the debris removal process is built on ordinary differential equations. This model serves as a basis for the profit model, where profit is total revenue minus ODR cost. Currently, there is no profit to be made; it would take a private firm more than 50 years to benefit economically from this industry.

We extend the profit model to include recycling the debris. Doing so has smaller benefits than the non-recycled mode, due to the high recycling cost. Further extensions take varying sizes of debris and different types of cleaners into account. We conclude that using a single type of cleaner makes more profits than using different types.

Our third model utilizes AHP to choose the best ODR method in terms of overall performance based on five factors. The SpaDE ODR system is evaluated to be the most favorable one.

Our sensitivity analysis demonstrates that our first debris removal model is robust to changes in parameters. The AHP model is sensitive to the relative weights value of the five factors.

We additionally contrive a new method of collision avoidance. The advantages of the traditional "box" method and the "collisional probability" method are incorporated in our new method.

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Introduction

Background

Since the first spacecraft Sputnik 1 was sent into space in 1957, there have been about 7,100 spacecraft launched into space [1]. The problem that arises with this increasing number is the leftover debris in space resulting from defunct satellites, collisions between spacecraft, and so on. Currently, Earth is surrounded by approximately 21,000 bits of debris larger than 10 cm and much more in smaller sizes [2]. **Figure 1** shows the debris distribution in low Earth orbit (LEO).

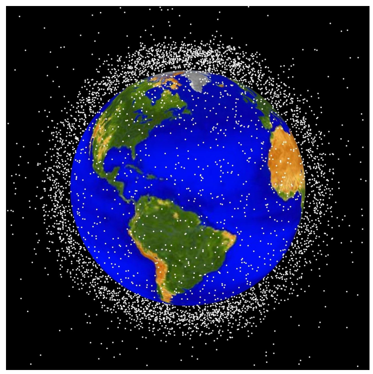

Figure 1. View of debris in low Earth orbit.
Source: `https://orbitaldebris.jsc.nasa.gov/photo-gallery.html`.

The large amount of debris poses an non-negligible threat to operating spacecraft, since collisions with even a small piece of such debris would involve considerable energy and destroy the satellite [3]. **Figure 2** illustrates the number of different types of debris over time.

With the space debris issue becoming more and more alarming as more satellites are sent into space, a potentially profitable opportunity exists for private firms to commercialize space debris removal.

Restatement of the Problem

We are to build a mathematical model to judge whether a private firm can benefit from removing space debris. The problem has three parts:

- Build a model to simulate the debris elimination process and predict the effect of the ODR system.
- Propose the best method using a mathematical criterion.
- Provide innovative schemes for collision avoidance if no viable commercial opportunity exists.

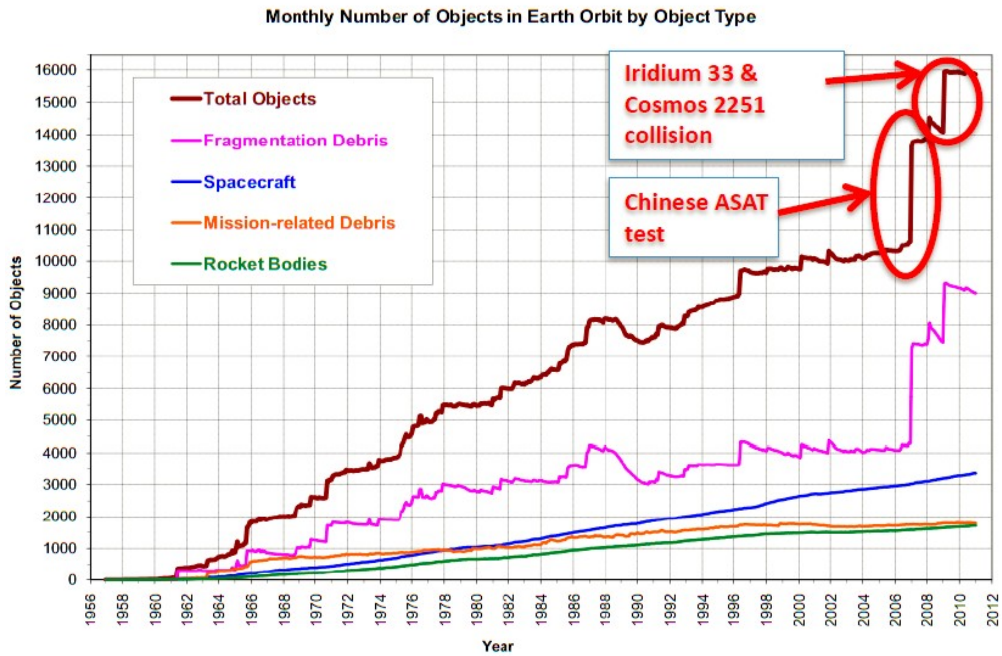

Figure 2. Numbers of objects in orbit around the Earth. Source: [8].

We note:

- We consider only debris larger than 10 cm that can be traced and processed by a "cleaner." By "debris," we refer to human-made objects in orbit around the Earth that no longer serve a useful purpose [3] (so we do not consider pieces of meteorites).
- We consider solely the benefits from the business opportunity of debris removal. Other related opportunities, such as satellite launching, are not included.

Literature Review

Donald Kessler advanced a mathematical model in 1978 to predict the rate at which the debris belt might form. It was concluded that the debris flux would increase exponentially with time even if zero net input rate is maintained [4]; this phenomenon is referred to as the "Kessler Syndrome" [5].

Numerous solutions have been proposed to deal with this issue. These technologies are mainly categorized as follows:

- Ground-based approaches: High-powered laser systems can remove debris in low Earth orbit.
- Passive de-orbit systems that are deployed at the end of the life of a satellite or rocket: Inflatable balloons, inflatable tube membranes, suspended

tethers, etc. are all simple and easily deployable drag systems that can hasten the speed of the debris de-orbiting process [6].

Results

We use a system of ordinary differential equations to simulate the environment of the space from qualitative and quantitative perspectives, with ODR present or not. The results of modeling generate approximate amounts of debris and remaining spacecraft based on their initial numbers.

We conclude that ODR can help eliminate debris while protecting spacecraft effectively. The profits of different methods are calculated by subtracting the ODR systems' cost from the total value of protected satellites.

Next, we use the Analytic Hierarchy Process (AHP) to evaluate the performance of different ODR systems in regards to pre-set essential factors. The result indicates that the SpaDE ODR system is the most favorable one. Our results show that given the best model, the debris removal industry will not make money in the current situation, but will be profitable more than 50 years from now on.

Models

Overview

Assumptions and Justifications

- There are only two types of satellites, those in high orbit and those in low orbit.
- All the destructive space debris is the same and evenly distributed around the earth.
- The collision between a satellite and a piece of debris is assumed to be a catastrophic one in which the satellite is destroyed. The number and size of debris pieces generated in a collision are constant, regardless of the size of the spacecraft.
- The Earth is a closed system. No space junk from outer space enters into the system.
- The debris does not decompose nor spontaneously fall into Earth's atmosphere. It will always orbit the Earth unless there is an external impact.
- Among all types of space debris, we concentrate on the relatively large objects (10 cm across or larger). There are hundreds of millions of smaller pieces not deadly to satellites.

Table 1.
Symbols.

Symbol	Meaning	Value
t	time (measured in years)	
S_h	satellite population in high orbit	0.482 (thousands) [1]
S_l	satellite population in low orbit	0.589 (thousands) [1]
D	debris population	21 (thousands) [Wikipedia]
β_h	collision coefficient, high-orbit satellite and debris	2.47×10^{-5} (from recent data)
β_l	collision coefficient, low-orbit satellite and debris	1.213×10^{-4} (from recent data)
α_h	number of high-orbit satellites launched per year	0.034 (thousands) [1]
α_l	number of high-orbit satellites launched per year	0.112 (thousands) [1]
γ_h	number of high-orbit satellites expired per year	0.027 (thousands) [1]
γ_l	number of low-orbit satellites expired per year	0.086 (thousands) [1]
ρ	proportion of retired satellites that become debris	0.1 [10]
N_1	number of fragments when a satellite crashes	1 (thousands)
N_2	number of fragments when a satellite crashes	0.5 (thousands)
λ	natural growth rate of space debris	0.05 [Wikipedia]

Model without ODR

The Satellites

Change of satellite population can occur in three ways:

- new satellites are launched into space,
- satellites are retired, and
- satellites are destroyed in a collision.

Over the past several decades, the numbers of satellites launched has not varied much. So we treat the number launched as constant and do so also for the number retired satellites. Therefore, the dynamics of the satellite populations in high orbit and in low orbit can be depicted as

$$\frac{dS_h}{dt} = -\beta_h D S_h + \alpha_h - \gamma_h, [3pt] \tag{1}$$

$$\frac{dS_l}{dt} = -\beta_l D S_l + \alpha_l - \gamma_l. \tag{2}$$

The Debris

Similarly, we assume that space debris is created in three ways:

- collision between satellite and debris,
- defunct satellites, and
- natural growth rate of debris (caused by collisions among debris pieces or with booster rockets and dead spacecraft).

For simplicity, we assume that the number of fragments produced in a collision is the same, and a fixed percentage of satellites transform into space junk in the process of dying. We also hold the annual natural growth rate of space debris λ to be constant. So the dynamics of the debris population can be described as

$$\frac{dD}{dt} = (\beta_h S_h + \beta_l S_l)DN + (\gamma_h + \gamma_l)\rho N + \lambda D. \tag{3}$$

Our system of ordinary differential equations with initial values is:

$$\begin{cases} \dfrac{dS_h}{dt} = -\beta_h D S_h + \alpha_h - \gamma_h, \\ \dfrac{dS_l}{dt} = -\beta_l D S_l + \alpha_l - \gamma_l, \\ \dfrac{dD}{dt} = (\beta_h S_h + \beta_l S_l)DN_1 + (\gamma_h + \gamma_l)\rho N_2 + \lambda D, \\ S_h(0) = S_{h0}, S_l(0) = S_{l0}, \qquad D(0) = D_0. \end{cases}$$

Test of the Model

We use data found on Websites to test the validity of our model. Based on the initial values and the value of parameters shown in **Table 1**, we can predict the number of satellites and debris in the future and see the consequence of increasing space debris.

Analysis and Conclusions

Figure 3 shows that the satellite population reaches its peak about 50 years in the future. The satellite population in low orbit grows faster in the next 40 years but gradually decreases, with few satellites left in the orbit. Similarly, the satellite population in high orbit gradually drops to zero at a relatively small rate. Debris grows exponentially.

Model with ODR

The model of the last section shows that with no removal measures, “wandering” debris will continue to propagate. This result is consistent with the Kessler syndrome, which means that the density of the objects around the earth will become so high in the future that the resulting debris cascade could make prospects for long-term viability of satellites in low Earth orbit extremely low. The result also suggests that to make sure

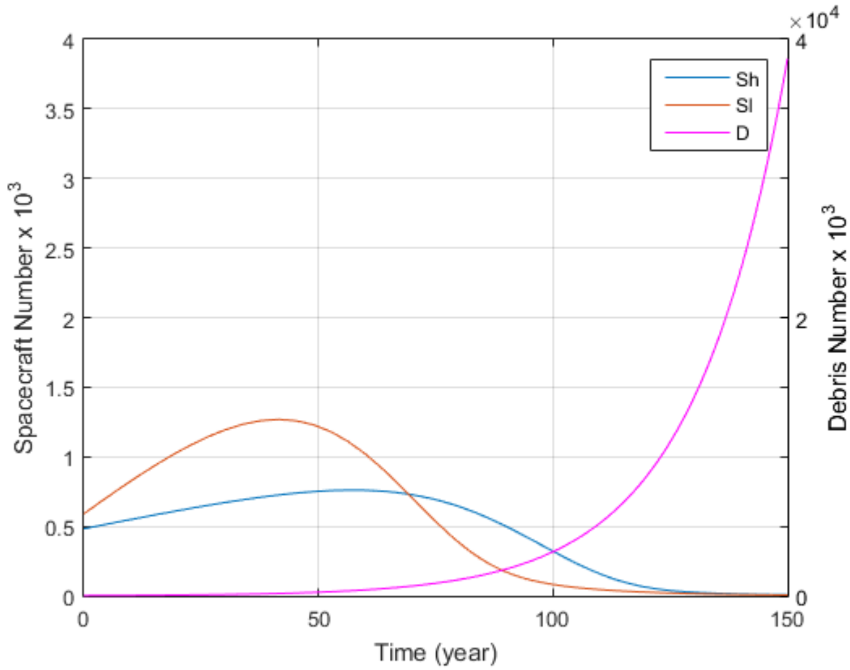

Figure 3. The satellite and debris population without ODR.

that satellites can survive in the next few decades, mitigating measures are indispensable.

First, we analyze the effects that a cleaning system can have. We assume that the system is composed of several cleaning machines operating in space to de-orbit or recycle objects. For now, we treat all cleaners as identical. We introduce the parameters W and θ to denote the *cleaner population* and its *cleaning efficiency*.

Similarly to satellites, cleaners also face the risk of collision with space debris, or undergo machinery damage in the process of working. We let δ denote the *machinery damage rate*.

Additional Assumptions

We make additional assumptions as follows:

- The machines are not fully occupied, so the number of debris items removed per unit time is positively correlated with the debris population in the space.
- The more debris that a machine has removed, the larger the possibility of machinery damage, which means that the damage frequency is relevant to the debris population.
- We neglect the debris produced by the cleaners themselves, since the number of cleaners is very small compared to the satellite population, and the collision with debris can be ascribed to machinery damage.

Equations and Test

ODR affects the satellite population by removing the debris in the space, thus the differential equations on the satellite part remain unchanged. The debris removed per unit time can be calculated as θWD, so on the debris part, **(3)** can be modified to

$$\frac{dD}{dt} = (\beta_h S_h + \beta_l S_l)DN + (\gamma_h + \gamma_l)\rho N + \lambda D - \theta WD. \tag{4}$$

Considering the machinery damage of the cleaners, the cleaner population can be depicted as

$$\frac{dW}{dt} = -\delta WD. \tag{5}$$

So the system of ordinary differential equations with ODR is

$$\begin{cases} \dfrac{dS_h}{dt} = -\beta_h D S_h + \alpha_h - \gamma_h, \\ \dfrac{dS_l}{dt} = -\beta_l D S_l + \alpha_l - \gamma_l, \\ \dfrac{dD}{dt} = (\beta_h S_h + \beta_l S_l)DN_1 + (\gamma_h + \gamma_l)\rho N_2 + \lambda D - \theta WD, \\ \dfrac{dW}{dt} = -\delta WD, \\ S_h(0) = S_{h0}, S_l(0) = S_{l0}, \qquad D(0) = D_0. \end{cases}$$

We illustrate the structure of the system in **Figure 4**.

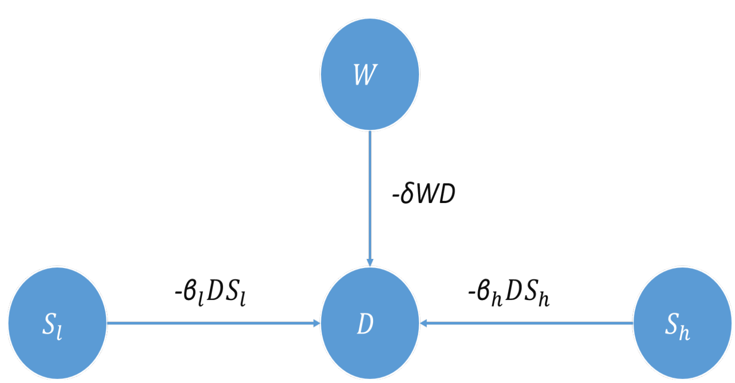

Figure 4. The structure of the ODR system.

Based on the values of parameters, we can simulate the satellite and debris population in the future and see the effect of ODR. The parameter values related to cleaners, calculated from [11], are:

- $\theta \approx 4$,
- $W \approx 0.02$,
- $\delta \approx 0.0011905$.

Running the model on Matlab, we have the results in **Figure 5**.

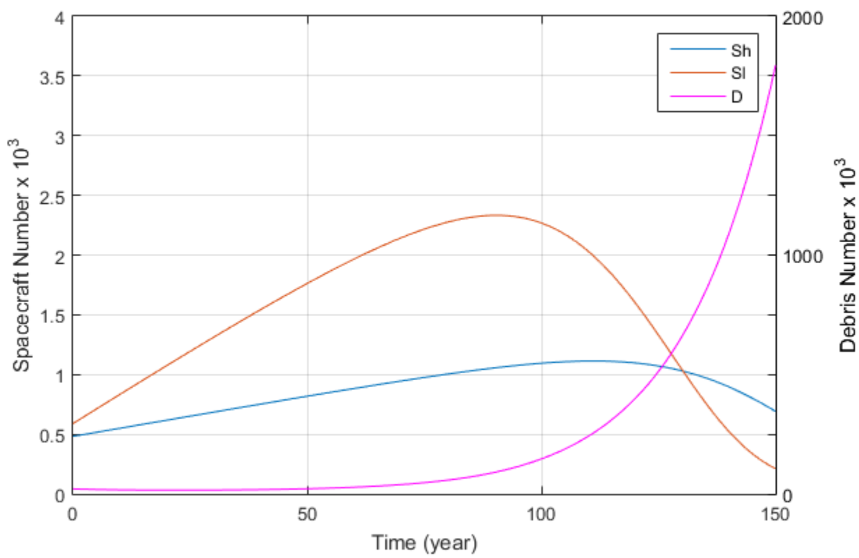

Figure 5. The satellite and debris population with ODR.

Analysis and Conclusions

With ODR, the satellite population remains at a high level for more than 100 years. Compared to the "do-nothing" system, ODR has controlled the amount of debris to some extent and assured that more satellites can survive in the next few decades. From **Figure 5**, we also see that in the long run the debris population still grows exponentially. This may be due to the fact that the cleaners will all eventually be damaged and the growth of debris is thereafter unrestricted.

The result of this model also shows that it is wise for us to launch cleaners continuously in order to keep the debris at a low level. Here we don't take the specific number of cleaners into account; nevertheless, there is the question of how many cleaners should be launched, which we treat later.

Model Extension

What if We Launch Different Kinds of Cleaners?

Analysis of the Problem

In the last section, we assumed that all the debris is the same and there is only one type of cleaner. In fact, debris comes in different sizes. We assume that there are two kinds of debris: *large* and *small*, with numbers denoted D_1 and D_2, and each has the same probability to hit a satellite. However, because of the difference in size, the efforts needed to de-orbit them are not the same.

Let p_1 and p_2 denote the *removal coefficient* (i.e., efforts needed) for each type. The number of pieces of debris removed per unit time is $p\theta WD$. (In the previous section, we did not take this parameter into account, since we normalize average value $\overline{p} = 1$.) Thus we have $0 < p_1 < 1$ and $p_2 > 1$.

Now we consider the specialization in cleaners. "Heavy" cleaners have higher power and "light" cleaners have lower power. So the cleaning efficiency θ for these two types is different, $\theta_1 > \theta_2$ (1 for heavy and 2 for light).

Our goal is to see whether this strategy of categorization of both debris and cleaners is more effective. Again, we assume that each size of debris has the same distribution in the space. Then with the same damage rate, the system of ordinary differential equations with specialization in cleaners is:

$$
\begin{cases}
\dfrac{dS_h}{dt} = -\beta_h(D_1 + D_2)S_h + \alpha_h - \gamma_h, \\
\dfrac{dS_l}{dt} = -\beta_l(D_1 + D_2)S_l + \alpha_l - \gamma_l, \\
\dfrac{dD_1}{dt} = (\beta_h S_h + \beta_l S_l)(D_1 + D_2)N_1 + (\gamma_h + \gamma_l)\rho_1 N_2 + \lambda D_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (\theta_1 W_1 + \theta_2 W_2)p_1 D_1, \\
\dfrac{dD_2}{dt} = (\beta_h S_h + \beta_l S_l)(D_1 + D_2)N_1 + (\gamma_h + \gamma_l)\rho_2 N_2 + \lambda D_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (\theta_1 W_1 + \theta_2 W_2)p_2 D_2, \\
\dfrac{dW_1}{dt} = -\delta W_1(D_l + D_2), \\
\dfrac{dW_2}{dt} = -\delta W_2(D_l + D_2).
\end{cases}
$$

We illustrate the structure in **Figure 6**.

Test and Conclusions

- $W_1 = 0.005$,
- $W_2 = 0.015$,
- $D_1 = 5000$ (consistent with the current distribution),
- $D_2 = 15000$,
- $p_1 = 0.7$ (larger debris is difficult to remove),

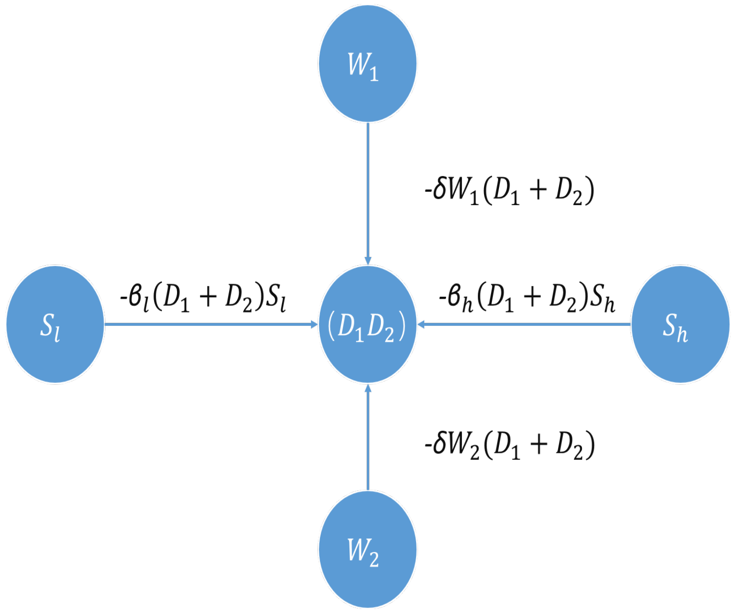

Figure 6. The structure of the system with different cleaners.

- $p_2 = 1.2$,
- $\theta_1 = 10$ (heavy cleaner has higher cleaning efficiency),
- $\theta_2 = 3$,
- $\rho_1 = 0.25$ (consistent with the current distribution),
- $\rho_2 = 0.75$.

The result is shown in **Figure 7**.

Specialization of cleaners indeed has a somewhat positive effect on restricting the debris. The peak of the satellite population occurs more than 100 years in the future, and the maximum is also larger. However, this result is not surprising. Practically speaking, it is better to design machines for special use or aimed at different types of debris.

Is There a Commercial Opportunity?

The Profit Motive

How can anyone make money with space debris? Space trash has little value compared to the tens or hundreds of millions of dollars needed to retrieve it. However, an insurance premium can be imposed to finance debris removal. We make several assumptions about this industry:

- All space-faring countries invest in the removal system and share the same protection.

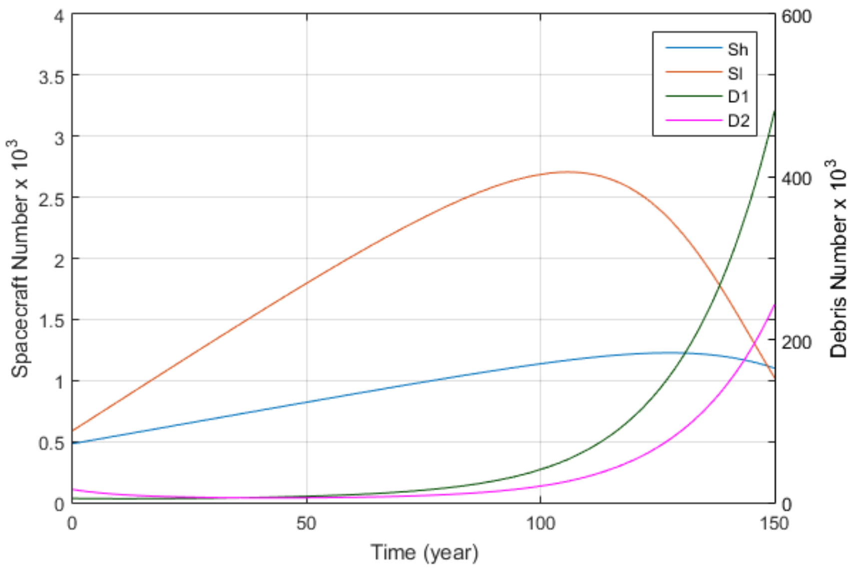

Figure 7. The satellite and debris population with specialization of cleaners.

- Income from cleaning the debris must equal the value that a firm produces, measured in the number of satellites protected.
- All the firms in this industry are identical.

We measure the profit by subtracting cost from revenue. Cost is composed of research and development, production, and maintenance.

The Results

The profit of ODR can be calculated as

$$\pi(t) = [N_2(t) - N_1(t)]V - C(t)$$

where N_2 is the satellite population with ODR, N_1 is the satellite population without ODR, V is the average value of a satellite, and $C(t)$ is the cost function.

We set the number the cleaners to be $W = 0.02$, the same as before. Additionally, \$100-\$200 million per year is required for maintenance [9]. We assume an additional cost of \$20 billion is required for research and \$2 billion is required to produce and launch the cleaners. The average replacement cost for the approximately 1,071 active satellites in orbit today is around \$109 million each [ESA/ESOC, 2013]. We thus set $V = 0.101$.

In the near future, the research cost is so high that it's impossible to make a profit. When $t = 50$, we have $N_1 = 1935$, $N2 = 2679$, and $\pi(50) \approx -\$5$ billion; when $t = 60$, we have $N1 = 1771$, $N2 = 2877$, and $\pi(60) \approx +\$26$ billion.

So the firms can finally make a profit after 50 to 60 years.

What about Recycling the Debris?

[EDITOR'S NOTE: The authors conclude that recycling the debris is impractical in terms of cost. We omit the details.]

Sensitivity Analysis

Collision Rate β

The collision rate between the debris and spacecraft is calculated according to data available from NASA. Currently, collisions take place nearly once per year. We varied β by a factor of 2 in each direction ($\beta = 0.5$, $\beta = 2$). For the lower value, the number of satellites reaches a higher peak later; for the higher value, the number reaches a lower peak later. The values of the peaks do not change much.

Number N of Fragments Produced Per Collision

We estimated the amount of debris produced in a collision from current data reported by detectors in space. These data are inevitably inaccurate since many pieces are lost because of inability to trace them. We varied the amount of generated debris in one collision from the original 1000 to 500,000 pieces. Results show that this factor exerts little influence on the final predicted number of satellites.

Evaluation Model for Debris Cleaners

Metrics

We evaluate a space debris cleaner on five aspects:

1. cost (including research, production, and launch);
2. cleaning efficiency;
3. production cycle;
4. (environmental) disturbance; and
5. life expectancy.

We collect data and rank the four cleaners:

- **Space-based laser radiation** is aimed at ablating the objects and provides an impulse to the debris, causing it to de-orbit.
- **NASA's SpaDE (Space Debris Elimination)** is a method to push satellites into a lower orbit by using air bursts.

- **Space-based water jets** use water to make the debris deviate from its orbit.
- **Tethered space tug** is another promising technology to de-orbit the space debris.

Analytic Hierarchy Process (AHP)

In evaluating different aspects of those cleaners, we need to consider the five criteria and should try to avoid subjective judgment. So we use the Analytical Hierarchy Process (AHP) to make the decision.

We compare the importance of the five criteria pairwise to determine the pairwise-comparison criteria judging matrix. In AHP, integers 1 to 9 and their reciprocals are used to represent the relative importance between criteria. For example, we find life expectancy (criterion 5) to be more important than disturbance (criterion 4) , so we set $a_{54} = 3$ and correspondingly $a_{45} = 1/3$. In the judging matrix $A = (a_{ij})$, each element must satisfy the constraints:

$$a_{ij} > 0, \qquad a_{ii} = 1, \qquad a_{ij} = \frac{1}{a_{ji}}.$$

From applying our judgments, we arrive at

$$A = \begin{bmatrix} 1 & 1 & 5 & 7 & 2 \\ 1 & 1 & 7 & 5 & 1 \\ 1/5 & 1/7 & 1 & 1/5 & 1/9 \\ 1/7 & 1/5 & 5 & 1 & 1/3 \\ 1/2 & 1 & 9 & 3 & 1 \end{bmatrix}.$$

We calculate the eigenvalues of A and select the eigenvector u corresponding to the maximum eigenvalue $\lambda_{\max}$ as the weight vector for the criteria, normalizing it to

$$u'_i = \frac{u_i}{\sum_{j=1}^{n} u_j}.$$

The weight vector for A is $u = (0.3522, 0.2895, 0.0356, 0.0841, 0.2386)$.

We need to be consistent in our relative rankings. We can check if that is so by calculating the consistency ratio. [EDITOR'S NOTE: The authors do so and are satisfied that their rankings are consistent.]

We grade the four cleaners on each of the criteria, arriving at the result shown in **Figure 8**.

The final scores of the four alternatives, based on our criteria judging matrix, are shown in **Table 2**. Cost plays the most significant role in evaluating the debris cleaners, followed by cleaning efficiency. SpaDE gets the highest score.

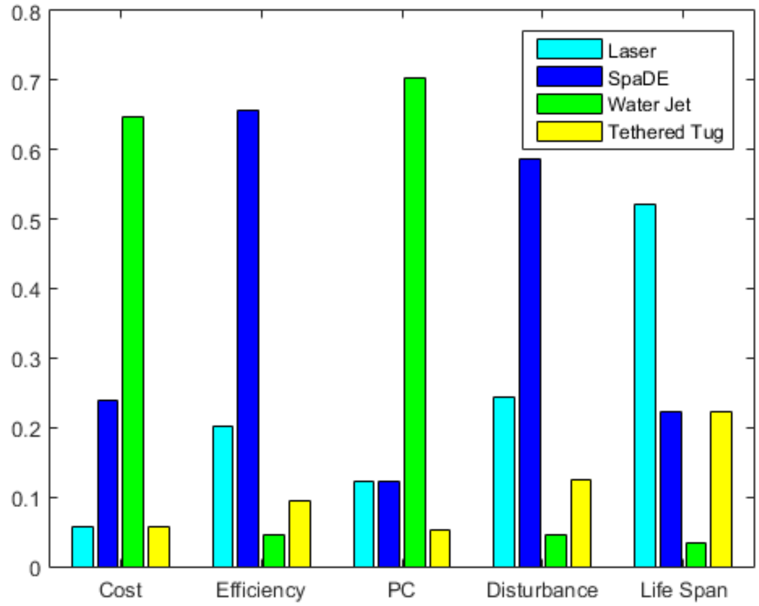

Figure 8. Ratings of the four cleaners on the five criteria.

Table 2.
Rankings of the four cleaner methods.

Rank	Method	Score
1	SpaDE	0.3812
2	Water Jet	0.2778
3	Laser	0.2279
4	Tethered Tug	0.1131

Sensitivity Analysis

We analyze how modifying the pairwise-comparison matrix influences the ranking of cleaners.

Suppose in the future, when de-debris business is more profitable, the cost of the cleaner plays a less significant role in assessing a cleaner. We modify the pairwise-comparison criteria matrix to become

$$A = \begin{bmatrix} 1 & 1/3 & 3 & 1/5 & 1/2 \\ 3 & 1 & 5 & 1/3 & 1 \\ 1/3 & 1/5 & 1 & 1/9 & 1/5 \\ 5 & 3 & 9 & 1 & 3 \\ 2 & 1 & 5 & 1/3 & 1 \end{bmatrix}$$

The weight vector becomes $u = (0.0920, 0.2012, 0.0396, 0.4841, 0.1831)$, and the rankings are as in **Table 3**. The ranking is different from the original one (though SpaDE still comes out on top); so the selections in the criteria matrix have considerable influence on the final result.

Table 3.
Rankings of the four cleaner methods.

Rank	Method	Score
1	SpaDE	0.4834
2	Laser	0.2639
3	Tethered Tug	0.1280
4	Water Jet	0.1248

A New Approach to Avoid Collisions

[EDITOR'S NOTE: We omit the authors' analysis.]

Strengths and Weaknesses

Strengths

- We approximate our model parameters from existing data.
- Our debris removal models are insensitive to parameter changes.
- We evaluate the ODR system on the basis of profit potential over time.

Weaknesses

- The exponential model simulating debris growth may be unrealistic.
- The categorization of Earth orbits into low-level and high-level is rough.
- The rankings from the AHP method are somewhat sensitive to the judgments of the pairwise-comparisons of the criteria.

Conclusions

According to our model, if no action is taken, the turning point of the increasing debris and number of collisions will come in few decades. On the contrary, if we start cleaning space debris now, the turning point can be postponed significantly. The collision rate of spacecraft and debris is still at a low level currently, so it is hard for one private company to make a money through ODR business in the short term. Moreover, our model predicts that the business will not be profitable for at least 50 years later.

Our cleaner assessment model, based on the Analytic Hierarchy Process and five criteria, concludes that SpaDE, conducted by NASA, is the best alternative.

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Executive Summary

Space debris is becoming increasingly alarming as nations send more satellites into space.

We constructed several models to study the effect of space junk, on several basic assumptions.

According to our models, the debris threat now is at a low level and the increasing rate of debris is at a low level. But the turning point will occur after about 50 years, when debris start to grow exponentially and collisions between spacecraft and debris become increasingly frequent.

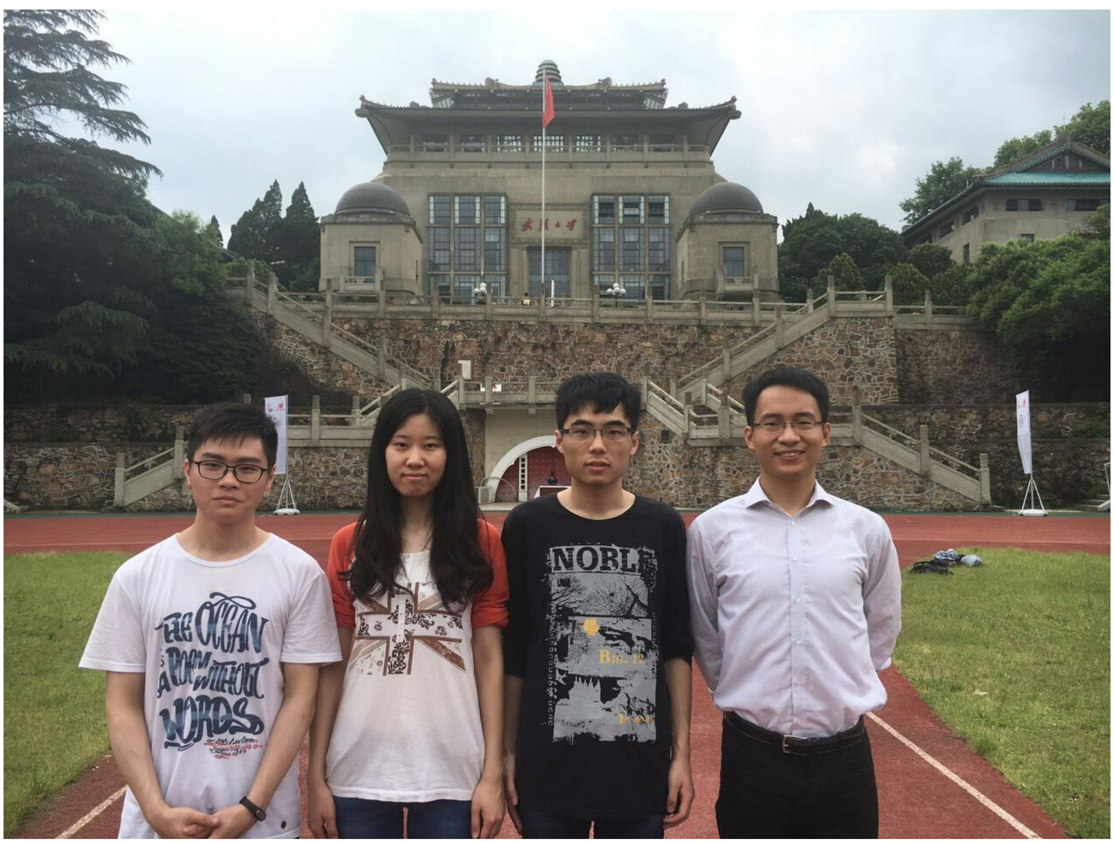
We find that with an ODR (Orbital Debris Removal) system, it will take about 50–60 years for the numbers of satellites to decrease. Additionally, launching specialized cleaners may be a better idea.

Our calculations also show that it is hard for a private company to make profit in the near future, but there exist opportunities to make money several decades after. At that time, ODR may begin to make money through finance from governments or companies that own satellites.

We construct another model to evaluate different types of space debris cleaners. This model takes five properties of the cleaner into consideration: cost, cleaning efficiency, production cycle, ecological disturbance, and life expectancy. We treat cost as the most important factor, followed by cleaning efficiency, life expectancy, disturbance, production cycle.

Of the four cleaning methods considered—NASA's SpaDE, water jets, lasers, and tethered tugs—we find that SpaDe is the best approach.

We strongly recommend that governments finance research in debris removal.



Team members Yulei Li, Hui Yang, and Zhaoqi Wang with advisor Junmin Liao.

Judges' Commentary: The Space Junk Papers

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Introduction

The number of teams choosing to participate in Problem B for the Mathematical Contest in Modeling continues to climb: In 2016, we judged more than 30 times the number of submissions compared to 1990. This year, for the first time, all of the winning papers were from teams in China.

The judges for Problem B have several observations to share that might aid teams in preparing for future competitions.

Suggestions

Answer the Question

Our main suggestion is to make sure that you answer the question(s) posed in the problem. In Problem B, you were asked to "develop a time-dependent model to determine the best alternative or combination of alternatives that a private firm could adopt as a commercial opportunity to address the space debris problem." This is your main goal!

While the problem statement went on to elaborate on what your model should include and be able to do, the judges understand that in the limited time allowed, you may not be able to comprehensively address each and every point in the problem statement. The MCM problems are intentionally open-ended and offer you the ability to expand your solution in interesting directions—but before you head off into unchartered territory, **please make sure that you have addressed the main elements of the problem statement**.

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The better papers thoroughly handled the main requirement to develop a time-dependent model and examined multiple alternatives—all with the goal of arriving at a commercially viable procedure for dealing with space debris. The better papers considered the mathematics and the economics in equal measure.

Use Sources but Be Original

Unlike many previous MCM problems, a tremendous amount of information on the topic of space debris was available on the Internet. The judges found this situation to be a bit problematic. Please realize that every team has access to the Internet and is very likely considering the same links that you found. This contest is not a report on all that the Internet says about a topic! We are seeking unique and creative new models, not an exhaustive report on existing information. Moreover, the judges noted identical charts and graphs in several papers—yet only some of the papers included the relevant citations from the Internet.

It is important to be very careful about what you use from other sources and that you reference them. The judges knew that some equations pertaining to space debris could also be found on the Internet—again, this is not interesting to us. **We would prefer to see your own equations that you derive based on your own logic and understanding of the problem.** If you do end up building your model from existing equations, you must cite them. This also applies to equations that you may have encountered in a textbook or math course. Cite, cite, cite!

But if all you are doing is citing, then your approach is misguided—we want you to be creative as you wrestle with an open-ended problem, so try not to get persuaded that the information that you track down is the best or only way to approach the problem. (Meritorious paper 47676 from Shanghai Jiao Tong University was the exemplar in regard to citations.)

Keep It Simple

This is a competition for undergraduate students, and although a handful of teams from high schools participate in the MCM, no students beyond the undergraduate level are permitted to participate. This means that **the judges are not looking to be wowed by advanced mathematical equations**. Neither are we interested in a list of variables and parameters that stretches over multiple pages. **It is okay to be simple and straightforward.** What is important is that you define your variables and parameters and that you explain the logic of your model. We are seeking creativity in the modeling process. Sometimes it may be appropriate for you to utilize an existing package or model in your work. If so, take some time to describe it for the judges and explain why you felt that it was an appropriate choice in this instance (and cite it!).

The careful analysis of your model and any solutions it generates is essential to the MCM. Be particularly attentive to its strengths and weaknesses. For any parameters, let the judges know where you found numerical values to use or how you estimated them. (This was particularly relevant in the Space Junk Problem if you used subjective approaches such as Analytic Hierarchy Process models.) An appropriate sensitivity analysis is essential, particularly if you are unable to obtain accurate values for any parameters in your problem. (This suggestion is relevant if you used a weighted matrix method in your model.)

In our view, the exemplar paper for the Space Junk Problem was the Outstanding paper from Zhejiang University. The main idea in this paper is to establish an insurance company and use the premiums collected to clean up some space debris, thereby reducing the overall risk of future claims. The team presented a very creative mathematical model with an economic perspective, which helped focus on the main task of developing a plan that would be commercially viable for a private company. The exposition was extremely clear; it was a pleasure to read. The scenarios to illustrate how the company could make money were solid. Although the Executive Summary discussed free riders, the judges felt it would have been helpful if the team had discussed the issue in greater detail in the body of the report and taken into account the potential for government involvement to require insurance of all satellite entities (companies or countries) in order to eliminate free riders. Another weakness in this paper was that the parameter choices for the decision-making matrix were not explained. The judges appreciated how the team used light humor, with comments such as "This page will not focus on this," and "Here comes the figure."

Conclusion

The judges feel that the better papers come from teams that do not rely too much on material from one mathematics course or from Internet sources. The better papers attempted to address all the points from the problem statement. They included a sensitivity analysis and a discussion of strengths, weaknesses, and next steps that illuminated a deeper understanding of the problem.

About the Author

Catherine Roberts is the Executive Director of the American Mathematical Society. Previously, she was the chair of Mathematics and Computer Science at College of the Holy Cross. She is also editor-in-chief of the journal *Natural Resource Modeling*. She has an A.B. from Bowdoin College in mathematics and art history, as well as a Ph.D. from Northwestern University in applied mathematics. She has been an Associate Editor of this journal.

An Educational Funding Mechanism Based on Data Insight

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Abstract

In recent years, Big Data has become increasingly popular and the guidance of Big Data is required in many fields, including philanthropy. We construct a new return on investment (ROI) evaluation system for a charitable organization, using data mining methods to process data, using which we succeed in determining an optimal investment strategy for the Goodgrant Foundation.

First, we operate on the data. We do data screening, deleting data with information less than a threshold and merging different attributes using linear fitting and principal components analysis (PCA). For the reserved attributes and schools that are kept, we do data imputation to fill in missing data, based on k-means clustering. Then we normalize all the data to make them comparable in the following analysis.

Second, we construct an ROI evaluation criterion, which is a ratio of output and input multiplied by an adjustment coefficient, named "urgency." The ratio reflects the benefits related to the cost, while urgency reflects the demand for funding, an important factor considered by charitable organizations. We use PCA to select attributes, letting salary, educational quality, and some other aspects represent output, while tuition represents input and federal loan, debt, and some other factors represent urgency. Then we use the Analytical Hierarchy Process (AHP) to measure the importance of different factors and allocate weights.

Third, we put forward two models: a basic model for one year, and a time series model for five years. Seeing the ROI as benefits from investment, we introduce the fluctuation of output as "risk," imitating the concept of modern portfolio theory in the financial sector to solve the problems. In the basic model, we apply a mixed integer linear programming (MILP) algorithm and succeed in finding 14 schools to invest in. Further, we model the time factor and improve the model into a time series model, using MILP and Grey

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prediction to determine the long-term investment strategy. Sixteen schools are chosen, with different time duration and different amounts of money.

Finally, we do sensitivity analysis for our model, changing the number of schools, the funding restrictions, whether to allocate the money equally or not, and so on, to analyze the results and to find better parameters for ideal results.

Our model is a feasible and reasonable model with technical and data support. Because of the subjectivity, this model can be used flexibly after data training.

Introduction

Background

In the commercial, economic, and other areas, decisions will increasingly be made based on data and analysis, rather than on experience and intuition. It is the same in philanthropy. In the past, it was more difficult to give money away intelligently than to earn it in the first place [1]. Now new and faster information could make charitable giving more effective and efficient; moreover, it provides the possibility of linking charitable-giving issues with the investment issues in the financial sector.

This report is about charitable funding of ("investment" in) colleges and universities in the U.S. We design a measure of return on investment (ROI) based on large quantities of data through data mining methods. We use portfolio theory, mixed integer linear programming, Grey theory, and other methods to determine the optimal funding strategy over time.

Overview of Our Work

First, we note a few key points:

- The volume of data is large and of different types. We must determine how to normalize the data.
- In the data, there are many missing values, which makes batch processing difficult.
- We need to decide how to classify the large number of attributes of colleges and universities.
- Different attributes focus on different aspects; we must discern how to judge their importance.
- We must choose schools from the candidate list and decide how to allocate funding.
- The funding process lasts for five years, so time is an important factor that will influence our ROI criteria.

To determine the optimal investment strategy, we boil down the tasks above to the following four steps:

- We do data screening. For the attributes and schools that we keep, we use k-means clustering to fill in missing data. Then we normalize all the data.
- Second, we use principal components analysis (PCA) to choose and classify different attributes. Then we use the Analytical Hierarchy Process (AHP) and knowledge of finance to construct an ROI concept. We use the ROI concept to rank the candidate schools.
- We introduce modern portfolio theory and construct two models. In the basic model, we use mixed integer linear programming (MILP) to determine an optimal funding strategy. Further, we consider the time factor and use a more complex MILP and Grey prediction to determine a long-term funding strategy.
- We do sensitivity analysis on some parameters and constraint to further analyze and discuss the model.

Assumptions

- We ignore inflation and other time value of money.
- As a charitable organization, our aim is to improve educational performance and expect more social benefits rather than gain profit.
- Our funding is holistic, not a reward for outstanding contributions in specific fields.
- Having different goals and strategies from other charitable organizations can reduce the possibility of duplication of investment to a great extent.
- The object for funding is the school, not the individual students, though some individual information is included in our criteria. And we leave it to the school to allocate the funds wisely.
- We focus on fairness in our strategy so as to fund schools regardless of their reputations.
- Because of marginal utility, we try not to invest a large amount of money in a single school.
- In the model over time, we assume that the influence of exogenous factors can be ignored. That's to say, the future is predictable.

Data Processing

Data Screening

First, we do data screening on the 7,805 schools:

- We consider only the 2,978 candidate schools in the file `Problem C - IPEDS UID for Potential Candidate Schools`. We match them with their 95 attributes in the file `Problem C - Most Recent Cohorts Data (Scorecard Elements).xlsx`.
- We delete the schools that are not currently operating, as well as those on Heightened Cash Monitoring 2 by the Dept. of Education, meaning that they suffer financial difficulties, may lack students, and for which there may be no or very limited information about percentage of degrees awarded. It is meaningless to invest in these schools.
- We delete schools for whom at least 50% of the attributes are "NULL," since any imputation might result in great error [2].

We do data screening on 95 attributes:

- We combine some binary variables. For example, we delete the net price for different income classes and combine the total net price for public schools and private schools together, named as "Net Price." And we combine (with weighting) the retention rates for full-time and part-time students together, named as "Retention Rate".
- Though there are large quantities of missing data in "SAT scores" and "ACT scores," we keep the midpoints of them as categorical variables. High entrance requirements with SAT or ACT scores would indicate higher education quality, which we treat differently from schools with low entrance requirements in our ROI evaluation system.
- We keep all the flag attributes such as "flag for Historically Black College and University" and "flag for women-only college," plus all the subject attributes such as "percentage of degrees awarded in Architecture And Related Services," using them for the clustering and data imputation in the next step.
- We delete location, age of students, and some other items that contribute less to our ROI evaluation system.

After data screening, there are about 2,700 schools in the candidate list, each with about 60 attributes.

Data Imputation

Missing data are a common occurrence and can have a significant influence on conclusions that can be drawn from the data. We impute values for missing data.

Many approaches are available for data imputation, such as listwise deletion, mean imputation, regression imputation, and multiple imputation. We use k-means clustering imputation.

- **k-means Clustering** We first group similar schools, and then use the mean of schools in the group with complete data to fill in the missing data for others in the group [2].

  Given observations x_i, where each observation is a d-dimensional real vector, k-means clustering aims to partition the n observations into k sets $S_1, \ldots, S_k$ so as to minimize the within-cluster sum of distance functions of each point in the cluster to its center [3]. In other words, the objective is to find:

$$\min \sum_{i=1}^{k} \sum_{x \in S_i} \left(x - \mu_i\right)^2, \tag{1}$$

  where μ_i is the mean of points in S_i.

  For the k-means algorithm, k must be specified in advance; but selecting the value of k is very difficult. According to the theory of variance, we often determine k based on the R^2 statistic [3]. That statistic is the ratio of the sum of squares between clusters and the total sum of squares; the higher the ratio, the better the value of k. So we choose the value of k that maximizes R^2.

- **Choosing Variables** The missing data columns that need to be filled are listed in **Table 1**. These are important variables in our ROI evaluation system. For example, "Net Price" is a measure of input of education, and "md_earn_wne_p10" means median earnings of students working and not enrolled 10 years after entry and is a measure of output of education.

Table 1.
Missing-data columns.

Variable Name	Variable Name
Net Price	gt_25k_p6
Retention Rate	md_earn_wne_p10
GRAD_DEBT_MDN_SUPP	sd_earn_wne_p10
GRAD_DEBT_MDN10YR_SUPP	md_earn_wne_p6
RPY_3YR_RT_SUPP	sd_earn_wne_p6

Having determined what variables need data imputation, we determine variables on which to base the imputation. **Table 2** gives some integral data columns for us to use.

By contrast, these variables are more integral and have significance to guide the imputation. For example, "PCIP_XXXXX" means percentage of degrees awarded in different subjects (departments) (including

Table 2.
Integral data columns.

Variable Name	Variable Name
PREDDEG	MENONLY
CONTROL	WOMENONLY
HBCU	PCIP_XXXXX
PBI	DISTANCEONLY
ANNHI	UGDS
TRIBAL	UGDS_XXXXX
AANAPII	PPTUG_EF
HSI	PCTFLOAN
NANTI	PCTPELL

38 subjects), indicating the subjects covered by a school; "UGDS" means the size of the school and "UGDS_XXXXX" means the percentage of students who are WHITE, BLACK, ASIAN and so on (including 9 different attributes), indicating the composition of students at the schools; and "PCTFLOAN" means percent of all undergraduate students receiving a federal student loan, indicating the students' financial situation.

- **Clustering Output** To simplify the model, we do k-means clustering only once, using all the data from the integral data columns. We assume that the classification using all the integral variables is applicable to all of the missing variables. And we try values of k from 3 to 8, ending up finding that the optimal value is 5.

  After clustering, we can believe that the schools in the same cluster have maximum similarity. So, we use the means of no-missing-data schools for each missing variable column to fill in the corresponding blanks of missing-data schools in the same cluster.

  Table 3 shows the results of k-means clustering and an example of the imputation value of "md_earn_wne_p10."

Table 3.
Clustering output and imputation value.

Cluster	Size	md_earn_wne_p10
1	27	19655.56
2	5	24980.11
3	1151	31176.46
4	1499	40780.52
5	11	42063.64

In the table, the value of "md_earn_wne_p10" for each of the five clusters is the mean value of no-missing-data schools; we use that value to fill in any blanks in the cluster.

Data Normalization

Normalization can provide an approach for comparison of different kinds of data and reflect the combined results of different factors.

We use min-max normalization, doing linear transformation of the raw data, mapping data set $x = \{x_1, x_2, \ldots, x_n\}$ into $[0, 1]$. The normalized value is

$$x' = \frac{x - x_{\min}}{x_{\max} - x_{\min}}, \tag{2}$$

with $x'_{\min} = 0$ and $x'_{\max} = 1$.

ROI Evaluation System

Concept of ROI

We focus our attention on how to determine the return on investment evaluation system. In the financial sector, ROI is the benefit to the investor resulting from investment of some resource. We construct an ROI evaluation system appropriate for our charitable organization. The formula is:

$$\text{ROI} = \frac{\text{Output}}{\text{Input}} \times \text{Urgency}. \tag{3}$$

Output

Output is the measure of the quantity of education, in other words, effectiveness. We use four factors to measure output: Salary After Graduation, Retention Rate, Repayment Ability, and Education Enhance Rate. But why these four factors?

- First, there is low correlation among the four factors, each of which can be separated out as an impact indicator of Output. To illustrate the independence, and the relationship between different variables, we use Principal Component Analysis (PCA), a statistical procedure that uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of values of linearly uncorrelated variables [4]. When some of the variables can explain a great percentage of the results, we can say these variables are strongly correlated and we can use one to represent others.
- Second, they are strongly related to Output of a school:
  - Salary After Graduation(SAG) measures the overall level of students. More salary means the students as graduates make more valuable

output. So they can contribute more to society when they receive the same amount of funding. There are three factors representing salary, "md_earn_wne_p10," "md_earn_wne_p6," and "gt_25k_p6'," meaning midpoint earning in 10 years, midpoint earning in 6 years, and share of students earning over $25,000 per year after 6 years. Using PCA to study the correlation between them and assign weights, we find that the three factors are strongly related and one of them, "md_earn_wne_p6," can explain 95% of the result. So we use only the value of "md_earn_wne_p6" to represent Salary After Graduation.

– Retention Rate (RR) measures the utilization of the investment. On the one hand, the larger the retention rate is, the more students retained in schools will enjoy the investment. And on the other hand, higher retention means greater school spirit and learning atmosphere, which is worthy of investment.

– Repayment Ability (RA) measures the ability to contribute to society. There is a difference between Salary After Graduation and this factor. The former emphasize the potential ability, while higher Repayment Ability shows that the graduate is no longer subject to debts or other burdens in life and has real ability to contribute to society. Here we use "RPY_3YR_RT_SUPP" to represent Repayment Ability.

– Education Enhance Rate (EER) measures the education quality of schools. Here we use normalized data of "md_earn_wne_p6" minus normalized data of "SAT scores" and then normalize it again to represent the Education Enhance Rate. Rationale: Salary can reflect the level of students after graduation and "SAT scores" can reflect the level before entry. As for the schools without "SAT scores," the weight for SAT is assigned to three other factors.

In order to get the overall evaluation index, we need to compute the weight of the four factors. In the following, the Analytical Hierarchy Process (AHP) is applied to judge the importance of the four factors. We compare the four factors in pairs and give the following matrix according to our experience, intuition, and subjective opinion. The columns from left to right are Salary After Graduation, Retention Rate, Repayment Ability, and Education Enhance Rate.

$$\begin{bmatrix} 1 & 5 & 6 & 3 \\ 3 & 1 & 2 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{5} & 1 \end{bmatrix}.$$

The weight vector, the normalized eigenvector corresponding to the largest eigenvalue, is $V_1 = [0.576, 0.125, 0.077, 0.222]$. We do a consistency check, finding that the the consistency index CI $= 0.0113 < 0.10$, meaning that the weight determined by AHP is very reasonable [5].

For the schools without "SAT scores," we assign the weight of EER to three other factors and use the same AHP method. The weight vector is $V_2 = [0.741, 0.161, 0.098]$, also passing the consistency check.

AHP, which we adopt here, is the most widely used method to evaluate weights of factors, though its objectivity is pretty poor. Although defining weights through AHP involves subjective judgement in the pairwise comparisons, the method nevertheless possesses some scientific rationality.

Finally, the comprehensive calculation formula is described as:

$$\text{Output} = \begin{cases} V_1 \cdot [\ \text{SAG} \quad \text{RR} \quad \text{RA} \quad \text{EER}\ ]^T \\ V_2 \cdot [\ \text{SAG} \quad \text{RR} \quad \text{RA}\ ]^T \end{cases} . \qquad \textbf{(4)}$$

Input

Adding Input into the ROI formula, we get a ratio considering profits in relation to capital invested, in other words, efficiency. Schools with low Input and high Output can create more value. Here we have only one factor, "Net Price" (NP), to represent Input.

There is a problem in that the sole factor in Input will have a larger influence than the factors in the Output, resulting in schools with lower tuition getting a higher ROI. So we add a correction factor $\alpha \in [0, 1]$:

$$\text{Input} = (1 - \alpha) + \alpha \times \text{Net Price}. \qquad \textbf{(5)}$$

We let $\alpha = 0.25$.

Urgency

For a commercial firm, the definition of ROI with Input and Output is enough. But for this charitable agency, we add a correction index, "Urgency," because charitable organizations should donate money to the people who are the most eager for it. Here we use three factors to measure Urgency: Pell Grants, Federal Loan, and Debt. We choose these three factors because of independence and meaningfulness.

- Pell Grants (PG) measures students' degree of demand for funds in different schools. We use "PCTPELL," percentage of undergraduates who receive a Pell Grant, to represent PG.
- Federal Loan (FL) has a similar meaning as PG. We use "PCTFLOAN," percent of all federal undergraduate students receiving a federal student loan, to represent it.
- Debt measures the degree of indebtedness of graduates. Two factors, "GRAD_DEBT_MDN_SUPP" and "GRAD_DEBT_MDN10YR_SUPP" related to Debt, but they are strongly related. So we choose "GRAD_DEBT_MDN_SUPP", median debt of completers, to represent Debt.

Though the three factors have similar meaning, the PCA indicates they are independent: They respectively explain 31%, 13%, and 56% of the result. We use these three numbers as their importance in AHP, and get the final weight vector $V_3 = [0.25, 0.25, 0.50]$, with consistency index $\mathrm{CI} = 0.0089 < 0.10$.

$$\text{Urgency} = V_3 \cdot [\ \text{PG} \quad \text{FL} \quad \text{Debt}\ ]^T \tag{6}$$

On the basis of the discussion above, we draw a diagram to visually represent our ROI evaluation system (**Figure 1**).

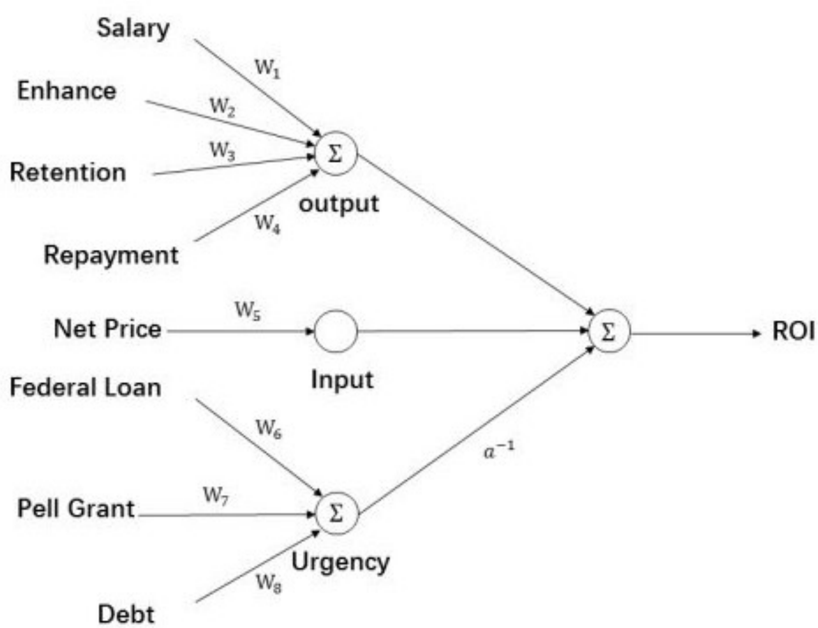


Figure 1. Representation of the ROI evaluation system.

Using Grey Theory to Predict ROI

We have determined what the ROI should be, and how to calculate it for a specific year using the given data. But the funding will be given over five years. In the long run, we can use Grey theory to predict the future ROI using the previous data.

Grey theory focuses on model uncertainty and information insufficiency in analyzing and understanding systems via research on conditional analysis, prediction and decision-making [7]. This theory is applied mainly on incomplete and indeterminate problems. One of models of Grey theory is GM(1,1), which uses an accumulated generation operation (AGO) and some equations to produce an infinite Grey sequence to predict subsequent data from observed data:

- Step 1. Start from the sequence of observed data

$$X^{(0)} = \left(X^{(0)}(1), X^{(0)}(2), ..., X^{(0)}(n)\right), \tag{7}$$

  where $X^{(0)}(i)$ represents the observed data at time i.

- Step 2. Generate the first-order accumulated generating operation (1-AGO) sequence $X^{(1)}(i)$ based on the observed sequence $X^{(0)}(i)$:

$$X^{(1)} = \left(X^{(1)}(1), X^{(1)}(2), ..., X^{(1)}(n)\right), \tag{8}$$

  where $X^{(1)}(k)$ is formed using the formula

$$X^{(1)}(k) = \sum_{i=1}^{k} X^{(0)}(i) = X^{(1)}(k-1) + X^{(0)}(k). \tag{9}$$

- Step 3. Establish the first-order differential equation for sequence $X^{(1)}(k)$:

$$\frac{dX^{(1)}}{dt} + aX^{(1)} = u. \tag{10}$$

- Step 4. Derive the estimated first-order AGO sequence

$$\hat{X}^{(1)}(k+1) = \left(X^{(0)}(1) - \frac{u}{a}\right) e^{-ak} + \frac{u}{a}. \tag{11}$$

- Step 5. Let $\hat{a}$ be a sequence of parameters where $\hat{a} = [a, u]^T$:

$$\hat{a} = (B^T B)^{-1} B^T Y_n \tag{12}$$

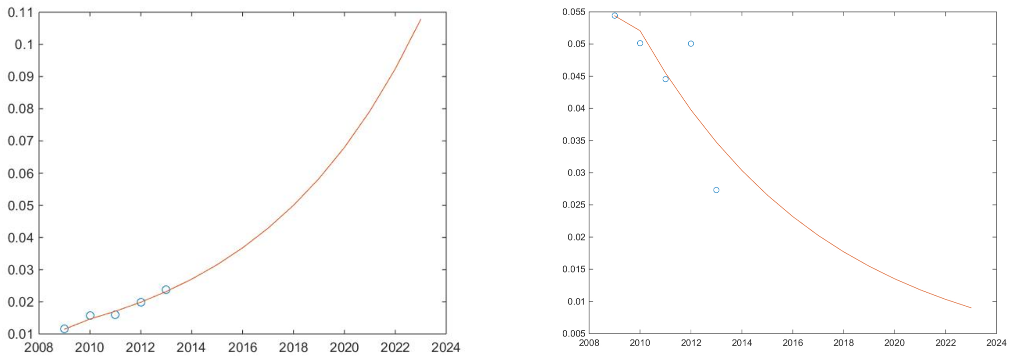
  where

$$B = \begin{bmatrix} -\frac{1}{2}(X^{(1)}(1) + X^{(1)}(2)) & 1 \\ -\frac{1}{2}(X^{(1)}(2) + X^{(1)}(3)) & 1 \\ \cdots & \\ -\frac{1}{2}(X^{(1)}(n-1) + X^{(1)}(n)) & 1 \end{bmatrix} \tag{13}$$

  and

$$Y_n = [X^{(0)}(2), X^{(0)}(3), \ldots, X^{(0)}(n)]^T. \tag{14}$$

Applying Grey theory as above, we use the data of the past five years, from 2009 to 2013, to predict the future ROI from 2016 to 2020. **Figure 2** shows some outcomes of the Grey prediction for two schools. The complex trends are fit well by Grey prediction, better than with linear fitting.



(a) Grey Predicting for Future ROI　　(b) Grey Predicting for Future ROI

Figure 2. Examples of Grey prediction for two different schools.

Model Construction

The allocation strategy of the Goodgrant Foundation given ROI and risk can be modeled as a mixed integer linear programming (MILP) problem. At first, in the basic model, we ignore the time factor and suppose that we allocate all of the funding in one year. The objective is to maximize the total return of the Foundation, and constraints include the control of the risk, the minimum and maximum number of schools awarded, the minimum and maximum funding a school can get, and the size of school. We introduce a continuous decision variable to denote the funding that each school receives and a binary variable to indicate whether the school is invested in or not.

Then, in the complicated model, we take time into consideration.

Definition of Risk

In the financial sector, modern portfolio theory is used to measure risk and benefits, drawing the efficient frontier of all the risky assets and finding the tangency portfolio [6]. For an investment, risk is certainly attached to benefits.

For a charitable investment, there also exists risk, which will forbid investment reaching the optimal point though it does not bring any tangible loss. Above we have discussed the ROI for charitable investment, which can be seen as benefits. Here we define "risk" r for our investment:

$$r = \frac{\sigma_e}{\mu_e}. \tag{15}$$

In the formula, we use the concept of coefficient of variation: σ_e indicates the standard deviation of "md_earn_wne_p6," and μ_e indicates the mean of "md_earn_wne_p6." Larger fluctuation of Output leads to larger risk. Using the ratio to indicate risk prevents the influence of different amounts of Output itself.

Then we use modern portfolio theory, drawing risk-benefit plots and using mixed integer linear programming (MILP) to solve the problem.

Basic Model

The constants and variables are as in **Table 4**.

Table 4.

Constants	
ROI_i	ROI of school i
r_i	risk of school i
$A_i^{\max}$	maximum investment in school i
$A_i^{\min}$	minimum investment in school i
s_i	number of student in school i
$N^{\max}$	maximum investment school number
$N^{\min}$	minimum investment school number
$r^{\max}$	maximum acceptable risk
$S^{\min}$	minimum acceptable student number
A	sum of the fund
Variables	
x_i	Investment in school i
y_i	State of school i: 0 for award, 1 recommended

The objective is to maximize the total return:

$$\max \sum_i \mathrm{ROI}_i x_i. \tag{16}$$

The funding for schools should not exceed the total funds available, and the average risk of the allocation strategy should not go beyond the maximum acceptable risk, so we have:

$$\sum_i x_i \leq A, \tag{17}$$

$$\sum_i r_i x_i \leq A \cdot r^{\max}. \tag{18}$$

The number of schools that get funding should be less than the maximum number $N^{\max}$ and greater than minimum number $N^{\min}$:

$$N^{\min} \leq \sum_i y_i \leq N^{\max}. \tag{19}$$

The basic logical constraints between x_i and y_i—that $x_i \geq 0$ if and only if $y_i = 1$, and if school i is funded, the funding it gets should lie between $A_i^{\min}$ and $A_i^{\max}$—yields:

$$A_i^{\min} y_i \leq x_i \leq A_i^{\max} y_i. \tag{20}$$

Finally, if the school is too small, we wouldn't choose it, since funding it wouldn't benefit enough students:

$$s_i y_i \leq S^{\min}. \tag{21}$$

Results of Basic Model

The values of ROI_i and r_i are derived from the previous work. We set the minimum acceptable student number $S^{\min}$ as 1,000, and the minimum funding $A_i^{\min}$ at \$20,000,000. Since the maximum funding a school can get is related to the school's student population, we set

$$A_i^{\max} = A_i^{\min} + b s_i,$$

where $b = 3000$, corresponding to the minimum funding plus \$3,000 per student. Finally, we require that funding be awarded to at least 10 but at most 30 schools.

If the decision maker doesn't care about risk so much, that is, the highest return is the only purpose, then using ILOG CPLEX 12.6 with python API, we get in **Table 5** a list of schools that should be recommended according to our model, with ranking according to the ROI.

Table 5.
Schools to be funded if the highest return is the only goal.

Uta Mesivta of Kiryas Joel
Florida Memorial University
Everglades University
United Talmudical Seminary
Johnson Wales University–Charlotte
Johnson Wales University–North Miami
University of Maine at Farmington
Washington Jefferson College
The Sage Colleges
Baker College of Port Huron
Juniata College
SUNY College at Potsdam
Knox College
Johnson Wales University–Denver

The funds are distributed fairly evenly among the schools, with each receiving between 5% and 13% of the total.

Time Series Model

In the real situation, according to the requirement of the Goodgrant foundation, the total \$1 billion[1] will be distributed to the schools over five years, so we modify our previous mixed integer problem and take the time series into consideration. So, we have the time series $t = 1$–5, corresponding to years 2016–2020. The whole model is as follows:

$$\max \sum_i \sum_t \mathrm{ROI}_i^t x_i^t \tag{22}$$

subject to:

$$\sum_i \sum_t x_i^t \le A, \tag{23}$$

$$\sum_i \sum_t r_i^t x_i^t \le A \cdot r^{\max}\ ; \tag{24}$$

$$\text{and for all } t, \tag{25}$$

$$N^{\min} \le \sum_i y_i^t \le N^{\max}, \tag{26}$$

$$A_i^{\min} y_i^t \le x_{it} \le A_i^{\max} y_i^t, \tag{27}$$

$$s_i y_i^t \le S^{\min}. \tag{28}$$

Results of the Time Series Model

The constants in the time series model are similar to the previous model with some minor adjustment: We set $A_i^{\min} = \$5$ million and $b = 8{,}000$. The values of ROI_i^t and r_i^t are determined by the previous calculation. By using IBM ILOG CPLEX, we solve the model, get the global optimal solution and the allocation in **Figure 3**.

We don't require that the funds be distributed evenly over the five years—instead, we set a upper bound and a lower bound for each year. We find that actually we should increase the funding as time passes.

Sensitivity Analysis and Validation

Risk–Return

In the basic model, we vary the maximum acceptable risk and analyze the maximum return, with results as in **Figure 4**. As the maximum acceptable risk increases, so does the maximum return, a result that conforms

[1][EDITOR'S NOTE: The problem statement specifies (a nominal) \$500 million.]

INSTNM	Investment in year i				
	2016	2017	2018	2019	2020
United Talmudical Seminary	5000000	0	0	0	0
Whitworth University	23696000	5000000	23696000	23696000	23696000
Moravian College	5000000	5000000	0	0	0
Clark University	5000000	5000000	5760000	23496000	23496000
Juniata College	14224000	17024000	17024000	17024000	17024000
Hampshire College	5000000	16744000	16744000	16744000	16744000
Florida Memorial University	5000000	0	0	0	0
Ithaca College	0	31232000	54616000	54616000	54616000
Uta Mesivta of Kiryas Joel	5000000	0	0	0	0
Roger Williams University	0	5000000	40224000	40224000	40224000
Fairfield University	0	0	0	0	11656000
Heidelberg University	13792000	5000000	5000000	11656000	0
Pitzer College	0	0	13648000	13648000	13648000
The Sage Colleges	18288000	5000000	18288000	18288000	18288000
Western Connecticut State University	0	0	0	46472000	46472000
Brandman University	0	5000000	5000000	34136000	34136000
sum of the money in the year	100000000	100000000	200000000	300000000	300000000

Figure 3. Allocations over five years.

to our expectation and to reality. But when the maximum acceptable risk index is greater than 0.82, the maximum return does not increase further.

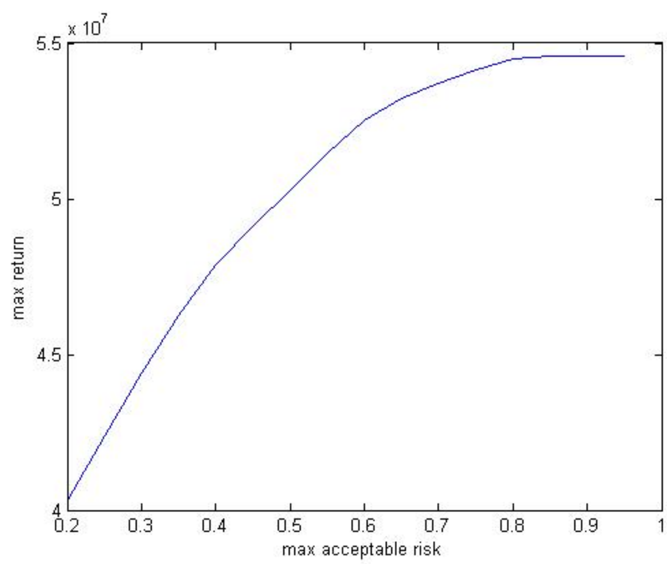


Figure 4. Return vs. risk.

Number of Schools

In the time series model, we change the number of schools to be chosen. The result are as in **Figure 5**. As the number of schools increases, the maximum return rises first, then falls. The optimal school number is about 12, which lies in our model's school number constraint between 10 and 30.

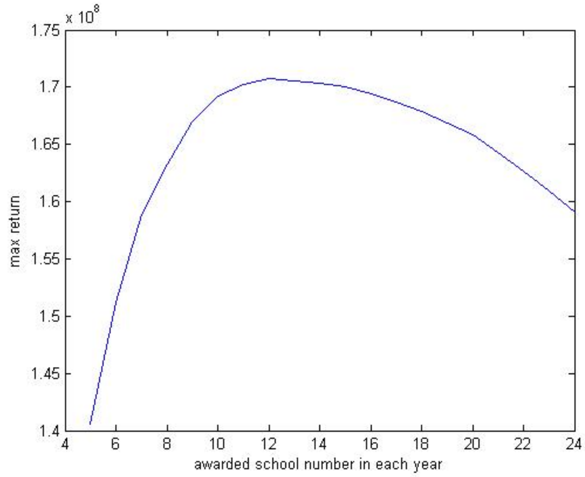


Figure 5. Return vs. number of schools awarded.

Policy of Distribution

When considering time, how to allocate the fund properly is important, we test four basic policies and compare their results.

1. All the funds are distributed in the first year.
2. The funding is distributed evenly over the five years, that is, $200 million dollars per year.
3. The funds are distributed unevenly over the years, but there is a constraint that confines the minimum and maximum amounts distributed in each year.
4. We distribute funds each year without any constraint.

Our previous recommendation list is based on policy 3. **Table 3** compares the four policies in terms of outcome.

Table 6.

Return vs. policy.

Policy	Maximum return
1	54, 579, 267
2	157, 020, 989
3	169, 677, 616
4	175, 745, 328

Policy 4 gives the best result. But under that policy, only $5 million dollars is distributed in the first year and in the second year, which is not practical. We also find that the difference between our policy (policy 3) and policy 4 is only about 3.5%, and our policy avoids the extreme bias among the years, so we recommend policy 3.

Future Work

- **Invite experts to do the subjective evaluation.** In our model, there are many subjective aspects, such as using AHP, choosing variables, and setting constraints for the variables. Some of the variables are put forward from our own experience and intuition but may not be very credible. For some parameters, we do sensitivity analysis but cannot decide on a good value. Maybe experts can have a better view of the evaluation, verify our model, and make a better use of it.

- **Focus on different subjects and different students.** In our model, the school is our object and we treat the school as a whole. We use the information about departments only to do the clustering, not for evaluating of the ROI. But nowadays, we should judge the level of a school by the levels of its different departments, not by the overall level. An investment in an outstanding department may gain more valuable benefits and contribute to the development of society. For example, in future work, we give more weight to departments such as Medical Science, IT, Ecology, and so on. Besides, it's more important to treat students as the object. We should focus more on the individual differences and the level of demand for different students.

- **More data is needed.** First, more real data should fill in the blanks of the missing data, which can reduce the error resulting from the data imputation.

  Second, we should find more data for more information of schools. For example, in our model, using just tuition to present the input of schools is not credible. Having more data means that we can rethink whether the attributes we have selected are suitable and replace improper ones. Having more data means that we can do more detailed analysis along the dimension of time.

  Third, find more data on other charitable foundations. The Lumina Foundation's goal is to increase the proportion of Americans with what it calls "high-quality" degrees and credentials to 60% by 2025; the Gates Foundation aims to help minority students and poor students enjoy a good education. Because we have different goals and strategies from these two organizations, we believe we reduce the possibility of duplication of funding. But relying just on belief is not credible. We need more data on other charitable organizations, on the one hand to study their funding strategy to improve ours, and on the other hand to reduce the possibility of duplication of funding to a greater extent.

Conclusion

We have done lots of data processing, put forward the ROI evaluation system for a charitable organization, and constructed two kinds of models, a basic model for one year and a time series model for five years. We use many methods such as AHP, PCA, k-means clustering, MILP, Grey theory and other financial theory to complete our work.

Like any model, our models have strengths and weaknesses.

Strengths

- **Integrity** We comprehensively use all of the data, either for evaluation or for clustering, making evaluation criteria more credible.
- **Technical Support** We use many theory-based methods to support our work, and each one is used reasonably and properly.
- **Cross Fusion** We get inspiration from the financial sector and use financial knowledge to support our mathematical model.
- **Flexible and Extendable** The models could be extended to include additional factors, to apply to many situations. And everyone can make different subjective assessments based on the specific circumstances.

Weaknesses

- **Lack of Data Support** The models require a large amount of data, some of which is hard to obtain. We have no data to validate our model and our result is not very ideal.
- **Subjective** There are many subjective methods in the model and some of the variables and parameter values are put forward from our own experience and intuition.
- **Simplifying Assumptions** Simplifying assumptions had to be made in order to create a solvable model. Thus, some valuable data and information could not be used.
- **Neglect of Future Dynamics** We ignore the very fact that the distributed funding will affect the schools' performance and our evaluation in the following years. In reality, the fund allocation strategy should be based on the schools' feedback and response to previous funding, so it is actually a dynamic process.

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Judges' Commentary: The Goodgrant Challenge Papers

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Introduction: New Problem Type

The 2016 MCM introduced a new modeling challenge—Problem C—that is best described as data insights. Problem C focuses on mathematical modeling challenges associated with large, messy data sets. In this sense, techniques stemming from statistics and pattern classification will play a larger role in creating a mathematical model on this problem than in Problems A and B.

While not a "big data" challenge, in the sense that teams need to develop specialized computer science-based data-handling algorithms and analysis techniques, or have access to high performance computing platforms, Problem C provides teams with an opportunity to encounter real-world, challenging data sets that have interesting characteristics.

Naturally-occurring complicating factors such as data set size, blend of data types, breadth of representation in data elements, cross-disciplinary

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sources, time series dependencies, censored or missing data, and others present themselves, depending on the specifics of the problem.

Problem C was intended to motivate the same high levels of team exploration, discussions, and decisions as have occurred in past MCM challenges.

The Goodgrant Challenge Problem

(The text of the problem is given in the report on the 2016 contest, and we do not repeat it here.)

Overall, this year's problem appeared to present a challenge for teams. While there is an extensive literature on collegiate success, several of the questions posed in the problem are not directly represented in these sources. The data set was large, and as with most data sets, contained missing data. Additionally, the problem required students to define two key elements:

- "improve(d) undergraduate performance of undergraduates in colleges and universities in the United States," and
- "an estimated return on investment (ROI) defined in a manner appropriate for a charitable organization."

The most successful teams carefully addressed the problem of defining what constituted improved performance. They prepared their data, addressing missing data and in many cases reducing the dimensionality of the problem. Once they defined their response, they used mathematical and statistical modeling techniques to determine which predictors influence the response, and how. They then further modeled how funding affected these predictors, as necessary. Once they had a model that allowed funding to predict the improvement in student success, they then used optimization techniques to allocate the available funding to schools over the five years. The most successful teams carefully defined the return on the Goodgrant investment, and showed how they had maximized it.

As always, judges valued well-written and well-illustrated papers that carefully followed the contest directions.

The Judging Process

For this inaugural problem, many new judges were introduced, including a new head judge (Dr. Olwell), a new regional judging site (Saint Martin's University), and a new final judging team. The judging process for Problem C followed the usual scheme of triage, screening rounds, and final judging. That process is described well in earlier judges' commentaries [Black 2009; Black 2011; Black 2013].

Defining the Problem

The instructions for the contest require teams to restate and clarify the problem. The best papers submitted for Problem C identified the required elements of the problem first. The best papers carefully established the goals of their model before moving on to actual data analysis or model fitting. They thought about the context of the problem, possible motivations of the Goodgrant Foundation, and whether the focus of the model would be on improving students or institutions. Of these issues, defining Return on Investment presented the greatest challenge. There were many interesting and creative approaches in the better papers.

The team's proposal was required to have the "highest likelihood of producing a strong positive effect on student performance." Very few papers considered the probabilistic elements of the problem or its solution; those that did so tended to be ranked higher. Again, the problem statement identified that student performance was an important element. How did teams model student performance? Economically? Academically? Did they consider the quality of the student at entry, or just the value added by the collegiate experience?

One of the design criteria for MCM problems is to avoid presenting a problem where there is a single "correct" solution. In the inaugural Problem C, the teams were left the task of defining the objective of their modeling effort. Accordingly, there was no "correct" answer, but there were many good analyses.

The problem was worded so that the extensive literature on measuring educational performance would not provide an apparent analysis template. Many of the better papers did review the literature in their papers, which is always helpful, and used elements of it in their modeling.

Preparing the Data Set for Analysis

The data set was taken from U.S. government statistics and had many of the challenging aspects often associated with such data. There were missing fields in many of the institutional records. Some were missing purposefully: For example, a school might use SAT scores or ACT scores for admission, but not always both. Other data might be modeled as missing at random. Very few of the institutional records were complete.

The better papers attempted to deal with missing data. Some imputed values for the missing data using means or medians, while others attempted to predict the missing values directly based on other variables. Papers that merely dropped institutions with incomplete data from consideration were not evaluated highly.

There was also a large number of variables, with extensive linear correlation as well as nonlinear patterns. A large number of papers attempted some dimension-reduction techniques. These frequently included principal component analysis and factor analysis, as well as other techniques.

These dimension reduction efforts produced new variables for the modeling effort. Teams that attempted to interpret these new variables in the context of the problem were viewed very favorably by the judges.

Depending on the definition of the problem, some variables could be considered responses and others predictors. It was considered better technique to separate responses and predictors when doing dimension reduction.

Judges did not assign any extra weight to teams that attempted to supplement the provided data set with information from other sources, either longitudinally or with extra variables for each school. The expectation was that teams would use the provided data. Two reasons drove this:

- The first was equity— having everyone with the same starting point; and
- to avoid the challenge of judges having to verify and validate external data sources.

Building the Model(s)

There were a variety of techniques possible for modeling, and even more were used. Judges had no preferred technique, but rather looked for clarity of rationale, internal coherence in the techniques chosen, and thoroughness in the validation and interpretation of the model.

An important element of this was the careful listing of assumptions. The best papers not only included the assumptions that they had made about the problem and data, but also addressed the assumptions implicit in the techniques they were using. For example, those using regression addressed the corresponding standard assumptions; these in turn were checked during the model validation.

The data set provided for the problem did not include many inherently financial variables. Teams that developed a model that said a non-financial predictor variable improved student performance were expected to develop a second model that showed how funding affected that predictor. Many teams identified candidates for funding without explicitly modeling how the funding would affect the predictors that in turn affect performance.

Judges looked for a model that explained how funding affected whatever measure(s) of student performance the team developed, and how the team explicitly calculated return on investment for its proposed funding.

Model Analysis

Teams that used statistical modeling methods were expected to include standard diagnostic plots, and to use those plots to explore if the assumptions of the models looked reasonable.

Careful consideration of ways to illustrate the methodology and results made it easier to use visualization to analyze and validate the model. Two of the Outstanding papers were illustrated particularly well.

The problem instructions included the required element of identifying strengths and weaknesses. Higher-ranked papers carefully identified both strengths and weaknesses. Many papers mitigated shortcomings in their approach or models by frankly identifying them in the weaknesses, and were ranked higher as a result.

Model analysis also includes model interpretation. What does the recommended solution mean in the context of the original problem?

Model Justification/Validation

When a team selects its final model, it should carefully justify why that model is appropriate and valid. That justification can be based on the literature, on the analysis of the model and its results, or first principles from the underlying context of the problem. Whatever the justification, it needs to be explicit.

Sensitivity analysis is a useful way to build confidence in the model and to identify the effects of departure from the model assumptions. Most statistical techniques have standard diagnostic tests that can be used to support arguments for model validity.

Communicating Results

As papers pass through the various stages of judging, the importance of good communication increases. A highly-ranked paper is well organized, well illustrated, well documented, and includes all the required elements.

The summary of results should present the results, not just a description of the methods to be used. Even this year's Outstanding papers could have been improved by explicitly including the return on investment in their results.

Even papers with brilliant mathematics cannot overcome poor communication of results. Previous judges' commentaries have identified specific ways to improve communication when writing an MCM paper [Olwell 2013]. They apply as well to Problem C!

Conclusions

The inaugural Problem C proved to be challenging, and the 1,875 student teams that submitted papers produced many interesting approaches. Of those 1,875 teams, four were recognized as Outstanding, and another eight as Finalists.

The reader will see that the commentary for this year's Problem C addresses many of the contest topics and issues from MCM problems in previous years. There are also some new issues that arise from the data based nature of this problem.

The judges anticipate that Problem C will provide a challenge in the future. As Problem C develops its own history, we expect team performance to grow even better.

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