Practitioner's Commentary: The Outstanding Water Tank Papers

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The problem of estimating the flow from a municipal water tank originated on 2 January 1991, when Equipment Technology and Design (a small engineering firm in Spokane, WA) telephoned me for assistance. The firm was then bridging the time interval when the pump was in action by using a straight line segment, between single estimates of the flow immediately before and after the pump action. Since the firm did not use any regression software, and since I had but a few minutes, I suggested the minuscule improvement of one piece of a cubic spline through four estimates, *two* on each side of the gap.

The firm later provided data from Union, a town of 11,500 in northeastern Oregon, with measurements taken about every 10 min. through the month of January 1991, whence came the data excerpted for the Mathematical Contest in Modeling.

All three winning teams wisely realized that modeling directly the volume of water in the tank might present greater discontinuities than first modeling the flow and then integrating it. They also realized that estimating the flow involves discrete approximations of derivatives. The teams from Ripon College and the University of Alaska—Fairbanks had the further insight to apply differentiation schemes that use several points, to smooth out measurement inaccuracies divided by small time increments (which would otherwise yield large errors).

We do not need to estimate the derivative $f = F'$ of the volume F at all times t. Indeed, suppose we model f as a polynomial or a trigonometric polynomial, with coefficients $c_0, \ldots, c_n$, so that $f = f(c_0, \ldots, c_n; t)$. Then we may select measurements of the volume $F(t_i)$ which lie k intervals from one another and use ordinary least squares to fit the coefficients $c_0, \ldots, c_n$ to equations of the type

$$\int_{t_i}^{t_{i+k}} f(c_0, \ldots, c_n; s)\, ds = F(t_{i+k}) - F(t_i).$$

The right-hand sides are differences of data values not as close together as values for neighboring points, so this approach may avoid the rounding errors from subtractions of nearly equal numbers. The method may be applied conveniently for any f that is linear in the coefficients, because then the left-hand sides will be, too [Wahba 1990].

The three teams further considered various models for the flow and methods to fit the model to the data, specifically, one polynomial or several pieces of polynomials fitted by ordinary linear least-squares, or a cubic spline interpolating the data. Remarkably, all three resulting estimates of daily consumption lie within 4% of 330,000 gal/day.

Also, the three teams proved resourceful in testing their models against independent assumptions, such as that the pump flow remains approximately constant (though the pump may slow down slightly as the water level rises and offers greater resisting pressure), or that the model should also fit the team's local water utility.

Finally, the teams offered thoughtful recommendations, specifically:

- Measure the rate of the pump (though it might prove as difficult as measuring the flow out of the tank) (Hiram College).
- Correlate the estimates of the flow with the use of electricity in the same town (Hiram College).
- Correlate the estimates of the flow with other water utilities (University of Alaska—Fairbanks).
- Fit not with least-squares but with Chebyshev's method of least absolute values (which translates into a larger linear-programming problem, but is statistically more robust and could be done with additional time) (Ripon College).

For the second and third recommendations, we can correlate the sum of the estimates of the daily usages with sum of the quantities measured at consumers' meters (but with a delay of the interval between reading meters).

The winning teams demonstrated a professional level of maturity in applied mathematics, in

- constructing sets of assumptions about the behavior of the flow,
- describing such assumptions with various mathematical formulations,
- fitting the mathematical forms to the data through various methods, and
- testing their results against other sets of data.

Their entries have indeed greatly impressed Equipment Technology and Design.

Reference

Wahba, Grace. 1990. *Spline Models for Observational Data*. Philadelphia, PA: Society for Industrial and Applied Mathematics.

About the Author

Yves Nievergelt graduated in mathematics from the École Polytechnique Fédérale de Lausanne (Switzerland) in 1976, with concentrations in functional and numerical analysis of PDEs. He obtained a Ph.D. from the University of Washington in 1984, with a dissertation in several complex variables under the guidance of James R. King. He now teaches complex and numerical analysis at Eastern Washington University.

Prof. Nievergelt is an associate editor of *The UMAP Journal*. He is the author of several UMAP Modules, a bibliography of case studies of applications of lower-division mathematics (*The UMAP Journal* 6(2)(1985): 37–56), and *Mathematics in Business Administration* (Irwin, 1989).