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Problem Chosen

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2016 Mathematical Contest in Modeling (MCM/ICM) Summary Sheet

Are we heading towards a thirsty planet?

Summary

Water shortage has become a global problem and has attracted universal attention for recent decades. Will the world run out of clean water?

Firstly, we develop a fuzzy comprehensive evaluation model, in which we select six indicators, such as the recovery cycle of water resources and the capacity of regeneration. The weight is determined by AHP and entropy weight method. And we sort the water shortage degree into 5 grades I, II, III, IV, V (I means no scarcity problem, while V means heavily overloaded.). Besides, we carry out the sensitiveness analysis of the model to prove the reliability of the model. Then we take China for example to estimate the drought extent and analyze the reason. Using the fuzzy comprehensive evaluation method, we get the drought degree of China for the IV grade, proving that water is moderately overloaded.

Secondly, we use GM (1,1) model to predict China's per capita consumption of water in the next 15 years. In the process of collecting data, we find that drought will happen when the rainfall of a year less than 26500 mm. Based on that, we build a disaster prediction model and predict that there will be two droughts in 2020 and 2030. Therefore, we give suggestions to the Chinese government to reserves sufficient fresh water to deal with the coming drought by using desalination technology and rainwater collection technology.

Finally, we put forward the intervention measures from the aspects of sewage purification and population. In the model of sewage purification, we analyze the urgency of sewage purification and list the annual amount of sewage purification in next decade. In the population control models, we select the Leslie model which satisfies China's population growth model better to predict the future population of China. Ultimately, we recommend that the Chinese government should relax the policy of family planning and maintain the current urbanization process to control the population in order to buffer the water using pressure.

Keywords: water scarcity, Grey Theory, Fuzzy Comprehensive Evaluation

1 Introduction

1.1 Problem Background

We cannot live without water. However, water problems are becoming increasingly serious in recent years, especially water scarcity. Water use has been growing at twice the rate of population over the last century. For the fact that there are many areas of severe water shortage on the earth, effective measures should be taken to prevent water resources from drying up.

1.2 Our Work

Firstly, we are asked to build a model that provides a measure of the ability of a region to provide clean water to meet the demands of its population, of which the dynamic nature of the factors is taken into consideration. Secondly, we take China for example to explain why and how water is scarce. We then use our model to predict what the water situation will be in 15 years and analyze how this situation impact the lives of citizens in China, in which the environmental drivers' effects on the model components is incorporated. Next, an intervention plan taking all the drivers of water scarcity into account is built. Finally, we use the intervention and model designed to project water availability into the future.

2 Model One: Water demand and supply analysis

2.1 Introduction

The water supply ability is determined by many factors and differs from region to region. To describe the ability of a region to provide clean water to meet the demand of its residents，

In view of the fuzziness and uncertainty of the indexes of water shortage risk evaluation, we build up a water resources comprehensive evaluation model based on entropy weight, with entropy theory applied into it.

2.2 Fuzzy Synthetic Evaluation Model

2.2.1 The analytic hierarchy process

We select five indicators[1,2] to evaluate the quality of water. According to paired comparisons, we can get a pairwise comparison matrix A, which is shown following table.

Table 1. Pairwise comparison matrix

	Risk Ratio	Vulnerability	Recoverability	Recurrence Period	Risk Degree
Risk Ratio	1.0000	1.1445	0.9066	0.8461	1.1153
Vulnerability	0.8738	1.0000	0.7921	0.7393	0.9745
Recoverability	1.1338	1.2976	1.0000	0.9593	1.2645
Recurrence Period	1.1484	1.3143	1.0411	1.0000	1.2808
Risk Degree	0.8966	1.0261	0.8129	0.7587	1.0000

$$A=\begin{bmatrix} 1.0000 & 1.1445 & 0.9066 & 0.8461 & 1.1153 \\ 0.8738 & 1.0000 & 0.7921 & 0.7393 & 0.9745 \\ 1.1338 & 1.2976 & 1.0000 & 0.9593 & 1.2645 \\ 1.1484 & 1.3143 & 1.0411 & 1.0000 & 1.2808 \\ 0.8966 & 1.0261 & 0.8129 & 0.7587 & 1.0000 \end{bmatrix}$$

It is obviously that this matrix is a consistent matrix, so there is no need for consistency check. After that, we used the method as follows to calculate weight:

a. normalize Each column vector

$$\widetilde{w}_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}} \ ;$$

b. sum according to row for $\widetilde{w}_{ij}$

$$\widetilde{w}_i = \sum_{j=1}^{n} \widetilde{w}_{ij} \ ;$$

c. normalize $\widetilde{w}_i$

$$w_i = \frac{\widetilde{w}_i}{\sum_{i=1}^{n} \widetilde{w}_i} \ ,$$

$w = (w_1, w_2, \cdots, w_n)^T$ is used to approximate the feature vector;

d. calculate $\lambda = \frac{1}{n}\sum_{i=1}^{n}\frac{(Aw)_i}{w_i}$ as the approximate value of maximum eigenvalue.

The weight of each evaluation index schemed out is

$$w^{\#} = (0.198, 0.173, 0.224, 0.227, 0.177) \ .$$

2.3 Entropy Method

2.3.1Modeling Process

The steps of comprehensive evaluation are to establish factor set U =

$\{u_1, u_2, \cdots, u_n\}$ for evaluation, to establish evaluation set $\mathrm{V} = \{V_1, V_2, \cdots, V_n\}$ and set a fuzzy relationship matrix to represent the relationship between evaluation set and factor set. The matrix is

$$\mathrm{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix}.$$

The key of the fuzzy comprehensive evaluation model is to get a result set, where we can choose the optimal one as the final result. The result set is

$$\mathrm{B} = \left(b_j\right)_{1xm} = w \cdot R,$$

where:

1) W is the weight of each factors on the water shortage risk indicators,

$$\mathrm{W} = (w_1, w_2, \cdots, w_n)$$

and the elements satisfy

$$\sum_{i=1}^{n} w_i = 1\,;$$

2) "$\cdot$" is the fuzzy synthetic operator, of which we choose the weighted average operator for comprehensive evaluation;
3) B is the result sets of water shortage risk evaluation,

$$b_j = \sum_{i=1}^{n} w_i r_{ij} (j = 1,2,\cdots,m),$$

and the maximum of b_j was selected as the final evaluation results.

2.3.2 The determination of relative membership degree

The value to assess the risk of water scarcity has no obvious boundaries, thus, we can use fuzzy set theory to describe the continuous changes of the indicators. According to the theory of fuzzy mathematics, the evaluation indexes can be divided into several levels. We sorted the evaluations into five levels, corresponding to the five standard values respectively, namely lower, low, medium, high and higher.

The water shortage risk evaluation indexes can be described by the following function:

$$r_{ij}(x) = \begin{cases} 1 - \frac{max\{a_{i1} - x, x - a_{i2}\}}{max\{a_{i1} - minx, max - a_{i2}\}} & x \notin [a_{i1}, a_{i2}] \\ 1 & x \epsilon\ [a_{i1}, a_{i2}] \end{cases} i = 1,2,\cdots,n; j = 1,2,\cdots,m \quad (3.1)$$

Table 2. The evaluation index and classification

Risk level	u_1	u_2	u_3	u_4	u_5
$v_1(lower)$	≤ 0.200	≤ 0.200	≥ 0.800	≥ 0.900	≤ 0.200
$v_2(low)$	0.200~0.400	0.200~0.400	0.600~0.800	6.000~9.000	0.200~0.600
$v_3(medium)$	0.400~0.600	0.400~0.600	0.400~0.600	3.000~6.000	0.200~0.600
$v_4(high)$	0.600~0.800	0.600~0.800	0.200~0.400	1.000~3.000	1.000~1.200
$v_5(higher)$	≥ 0.800	≥ 0.800	≤ 0.200	≤ 1.000	≥ 2.000

2.3.3 The entropy value method to determine the weight coefficient

In information theory, the weight of index is determined by the judgment matrix made up of evaluation index, instead of subjective factors, which makes the evaluation more approximate to physical truth. The process is demonstrated below:

(1) build a judgement matrix of several evaluation objects, the number is m, and several judgment matrixes of evaluation indexes, the number is n:

$$\mathrm{R} = \left(r_{ij}\right)_{m \cdot n} (i = 1,2,\cdots,m)$$

(2) normalize the judgment matrix, then a normalized matrix B was got. The elements of B is defined by following equation:

$$b_{ij} = \frac{r_{ij} - r_{min}}{r_{max} - r_{min}}$$

where:

r_{max} is the optimal value of different objects under the same indicators;

r_{min} is the worst value of different objects under the same indicators;

(3) The entropy of evaluation indexes can be written as

$$H_i = -\frac{1}{\ln m}\sum_{j=1}^{m} f_{ij}\, lnf_{ij} \tag{3.2}$$

where,

$$f_{ij} = \frac{b_{ij}}{\sum_{j=1}^{m} b_{ij}} \quad i = 1,2,\cdots,m; j = 1,2,\cdots,m$$

$$0 \leq H_i \leq 1$$

obviously, $\ln f_{ij}$ makes no sense when $f_{ij} = 0$, thus, f_{ij} need to be redefine as

$$f_{ij} = \frac{1 + b_{ij}}{\sum_{j=1}^{m}\left(1 + b_{ij}\right)}$$

(4) Calculate the entropy weight of each evaluation index by using the entropy;

$$w^* = {w_i}^* = \frac{1 - H_i}{n - \sum_{i-1}^{n} H_i}$$

$$(3.3)$$

where, $\mathrm{i} = 1,2,\cdots,\mathrm{n}$, also, $\sum_{i=1}^{n} w_i{}^* = 1$;

It can be seen from equations above that the smaller the entropy value is, the bigger the entropy weight will be.

(5) Analyze the comprehensive weights of evaluation indexes. The comprehensive weight of an evaluation index can be written as,

$$w_i = \frac{w_i{}^* w_i^{\#}}{\sum_{i=1}^{n} w_i^* w_i^{\#}} \tag{3.4}$$

where, $w_i^{\#}$ is weight determined by the AHP method.

2.4 Model calculation

Figures about the national water consumption we getting from China Statistical Yearbook in 2014[3] are listed in Table 3.

Table 3. Figures on the national water consumption

Year	**Water consumption**	**Agriculture water consumption**	**Industrial water consumption**	**Domestic water consumption**
2000	5497.6	3783.5	1139.1	574.9
2001	5567.4	3825.7	1141.8	599.9
2002	5497.3	3736.2	1142.4	618.7
2003	5320.4	3432.8	1177.2	630.9
2004	5547.8	3585.7	1228.9	651.2
2005	5633.0	3580.0	1285.2	675.1
2006	5795.0	3664.4	1343.8	693.8
2007	5818.7	3599.5	1403.0	710.4
2008	5910.0	3663.5	1397.1	729.3
2009	5965.2	3723.1	1390.9	748.2
2010	6022.0	3689.1	1447.3	765.8
2011	6107.2	3743.6	1461.8	789.9
2012	6141.8	3880.3	1423.9	728.8
2013	6183.4	3921.5	1406.4	750.1

Then we can get another table.

Table 4. Water shortage risk evaluation index value of some provinces in China.

Province	Risk Ratio	Vulnerability	Recoverability	Recurrence Period	Risk Degree
Hebei	0.734	0.384	0.384	3.624	0.297
Shandong	0.984	0.403	0.000	3.104	0.457
Sichuan	0.615	0.295	0.451	3.721	0.235
Zhejiang	0.428	0.127	0.507	3.887	0.196
Xinjiang	0.892	0.315	0.106	3.419	0.425
Shanxi	0.402	0.113	0.619	4.079	0.127
Fujian	0.547	0.287	0.473	4.000	0.224
Jilin	0.782	0.294	0.338	3.542	0.315

After normalization, we can get a matrix F:

$$F = \begin{bmatrix} 0.53444 & 0.8103 & 0.6204 & 0.5322 & 0.4855 \\ 1.0000 & 1.0000 & 0.0000 & 0.0000 & 0.9483 \\ 0.3600 & 0.6276 & 0.7286 & 0.6315 & 0.3103 \\ 0.0447 & 0.0483 & 0.8191 & 0.8014 & 0.1983 \\ 0.8419 & 0.6966 & 0.1712 & 0.3224 & 0.8563 \\ 0.0000 & 0.0000 & 1.0000 & 0.9980 & 0.0000 \\ 0.2491 & 0.6000 & 0.7641 & 0.6868 & 0.2787 \\ 0.6529 & 0.6241 & 0.5460 & 0.4483 & 0.5042 \end{bmatrix}$$

After normalize the Equation (3.1), we can get the entropy of each evaluation index,

$$H = (0.9870\ \ 0.9886\ \ 0.9899\ \ 0.9915\ \ 0.9896).$$

Besides, according to Equation (3.2), we are able to figure out the weight of each evaluation index,

$$w^* = (0.2428\ \ 0.2138\ \ 0.1896\ \ 0.1590\ \ 0.1949)$$

The weights calculated by the AHP method are

$$w^{\#} = (0.198, 0.173, 0.224, 0.227, 0.177)$$

Finally, according to Equation (3.3), the combination weight vector is figured out as

$$w = (0.181\ \ 0.183\ \ 0.233\ \ 0.237\ \ 0.165\,)$$

Based on the data of Table 3 and Founction (3.1), we can schemed out the relative membership degree and estabilish the fuzzy relation matrix R,

$$R=\begin{bmatrix} 0.3457 & 0.4640 & 1.0000 & 0.7057 & 0.7814 \\ 0.2708 & 1.0000 & 0.4825 & 0.8188 & 0.3412 \\ 0.3840 & 0.4088 & 1.0000 & 0.6400 & 0.3796 \\ 0.0000 & 0.0882 & 0.1478 & 1.0000 & 0.1115 \\ 0.6226 & 1.0000 & 0.1947 & 0.3594 & 0.0908 \end{bmatrix},$$

and the final comprehensive evaluation results are:

$$\mathrm{B} = (0.317\ \ 0.587\ \ 0.562\ \ 0.713\ \ 0.335\ ).$$

For the reason that the maximum is 0.731, so the degree of drought in China is grade IV, namely overload. For limits of economic scarcity, China is still moderately loaded though has abundant water resources,

2.5 Sensitivity analysis

The fuzzy membership matrix is related to people's subjectivity for the fact that it is voted by experts. Therefore, sensitivity analysis should be taken with the fuzzy membership matrix.

Based on data we have got, we can draw a table as shown below:
After calculation, we were able to get a weight vector

$$\mathrm{B} = (0.317\ \ 0.587\ \ 0.562\ \ 0.713\ \ 0.335\ ).$$

As can be seen from the result that, water shortage degree in China belongs to the fourth grade, namely overloaded.

Assume that analysis of the risk ratio is far from sufficient for some reason, if the risk ratio of sensitivity matrix R reduced by 10%, the new weight vector will be

$$\mathrm{B} = (0.340\ \ 0.601\ \ 0.605\ \ 0.690\ \ 0.327).$$

There is no obvious change in value of the matrix B and China is still in the high degree. Then we analyze the sensitivity of the other four indicators in the same way, therefore, we can come to the conclusion that some small fluctuations in a single index will not produce significant effect on the results of fuzzy comprehensive evaluation, suggesting the validity of our model.

3 Model Two: Forecasting

3.1 Assumptions

(1) Assume that China’s population and related policies have not changed dramatically.
(2) Assume that China’s economy is growing steadily.

3.2 DGM(2,1) Model[4]

Defining X^0 as the original data sequence of water consumption per capita:

$$x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$$

where:

$x^{(1)}$ is 1-AGO data sequence;

$\alpha^{(1)}x^{(0)}$ is 1-IAGO data sequence.

Then we define $\alpha^{(1)}x^{(0)}(k) + ax^{(0)}(k) = b$ as DGM(2,1) model and $\frac{d^2x^{(1)}}{dt} +$

a $\frac{dx^{(1)}}{dt} = \mathrm{b}$ is the albinism differential equation of DGM(2,1) model.

The least squares criterion of the parameters in the DGM(2,1) model satisfy

$$\hat{u} = \left[\hat{a}\,, \hat{b}\right]^T = (B^T B)^{-1} B^T Y.$$

The solution to the albinism differential equation $\frac{d^2x^{(1)}}{dt} + \mathrm{a}\frac{dx^{(1)}}{dt} = \mathrm{b}$ is

$$\hat{x}^{(1)}(t) = \left(\frac{\hat{b}}{\hat{a}^2} - \frac{x^{(0)}(1)}{\hat{a}}\right)e^{-\hat{a}t} + \frac{\hat{b}}{\hat{a}}t + \frac{1+\hat{a}}{\hat{a}}x^{(0)}(1) - \frac{\hat{b}}{\hat{a}^2}.$$

And the reduced value is $\hat{x}^{(0)}(k+1) = \alpha^{(1)}\hat{x}^{(1)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k)$.

Table 5. Annual water consumption per capita in China

Year	2000	2001	2002	2003	2004	2005	2006
Water consumption	435.4	437.7	429.3	412.9	428.0	432.1	442.0
Year	2007	2008	2009	2010	2011	2012	2013
Water consumption	441.5	446.2	448.0	450.2	454.4	454.7	455.5

Based on figures in the table, the original data sequence can be written as

$$\hat{x}^{(0)} = (443.4, 437.5, 437.8, 438.2, 438.6, 439.2, 439.9, 440.9, 442, 443.5, 445.4, 447.8, 450.9, 454.8)$$

The respective time response sequence of DGM model is

$$\hat{x}^{(1)}(k+1) = 436.469\text{*k}+3.90908\text{*}e^{0.238146\text{k}}+439.491$$

therefore,

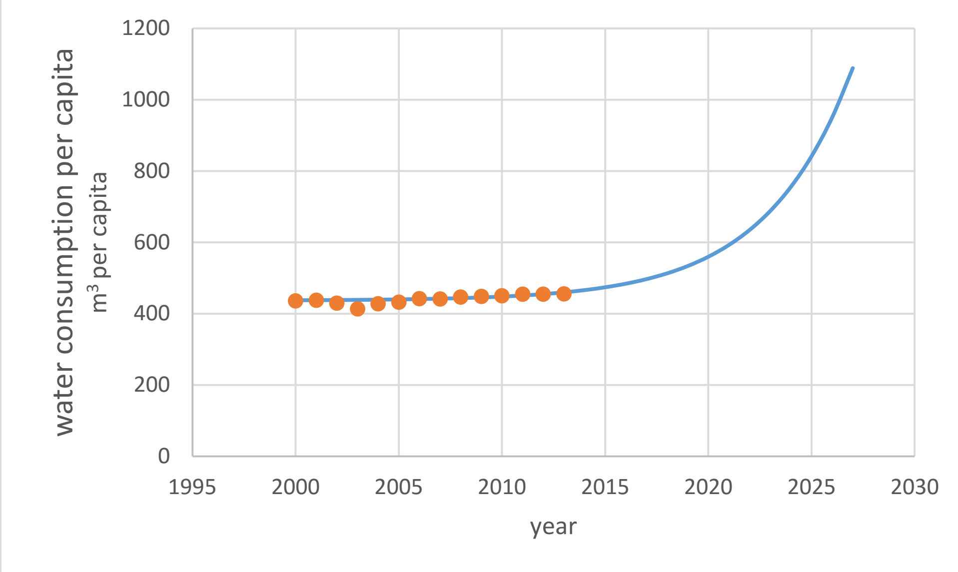
$$\hat{x}^{(1)} = (443.4, 880.9, 1318.7, 1756.9, 2195.5, 2634.7, 3074.6, 3515.5, 3957.5, 4401, 4846.5, 5294.3, 5745.2, 6200)$$

And the reduced can be obtained by repeated decreasing:

$$\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$$

By applying the figures into equation above, we can get annual water consumption per capita:

$$\hat{x}^{(0)} = (443.4, 437.5, 437.8, 438.2, 438.6, 439.2, 439.9, 440.9, 442, 443.5, 445.4, 447.8, 450.9, 454.8)$$



Graph 1. Predicted water consumption per capital by DGM(2,1).

3.3 GM(1,1) Model

The reference sequence has been known as

$$x^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)\right).$$

Besides, 1-AGO is a once accumulation generation sequence, in which

$$z^{(1)} = 0.5x^{(1)}(k) + 0.5x^{(1)}(k-1), k = 2,3, \cdots, n.$$

So the estimated value can be got $\frac{dx^{(1)}}{dt} + \mathrm{a}x^{(1)}(\mathrm{t}) = \mathrm{b}$. Let $\mathrm{Y} = \left[x^{(0)}(2), x^{(0)}(3), \cdots, x^{(0)}(n)\right]^T$, $\mathrm{B} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix}$, we can get an estimated value of u to obtain the minimum of $\mathrm{J(u)}$ through the least square method, $\mathrm{J(u)} = (\mathrm{Y} - \mathrm{Bu})^T(Y - Bu)$. At last, the estimated value of u is $\hat{u} = \left[\hat{a}, \hat{b}\right]^T = (B^TB)^{-1}B^TY$. Then solve the albinism differential equation $\frac{dx^{(1)}}{dt} + \mathrm{a}x^{(1)}(\mathrm{t}) = \mathrm{b}$, the result is $\hat{x}^{(1)}(k+1) = \left(x^{(0)}(1) - \frac{\hat{b}}{\hat{a}}\right)e^{-\hat{a}t} + \frac{\hat{b}}{\hat{a}}t, k = 0,1, \cdots, n-1, \cdots$. And we can get the reduced $\hat{x}^{(0)}$ by repeated decreasing: $\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k), \quad k = 1, 2, \cdots, n-1, \cdots$.

Time sequence is as follows

$$x^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(7)\right)$$

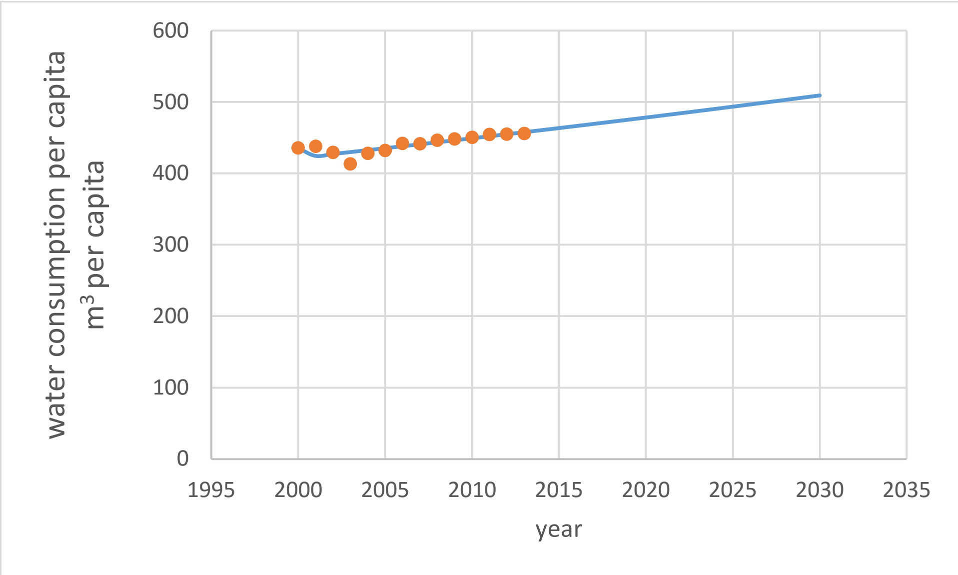
$= (435.4, 437.7, 429.3, 412.9, 428.0, 432.1, 442.0, 441.5, 446.2, 448.0, 450.2, 454.4, 454.7, 455.5)$.

By applying data into the albinism differential equation,

$$\hat{x}^{(1)}(k+1) = \left(x^{(0)}(1) - \frac{\hat{b}}{\hat{a}}\right)e^{-\hat{a}t} + \frac{\hat{b}}{\hat{a}}t = 67459.3 \times e^{0.00627374 \times k} - 67023.9$$

Then, we can get annual water consumption per capita:

$$x^{(0)} = \left(x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(7)\right)$$
$$= (435.4, 424.5, 427.2, 429.9, 432.6, 435.3, 438.0, 440.8, 443.6, 449.2, 452.0, 454.8, 457.7, 460.6)$$



Graph 2. Water consumption per capita predicted by GM(1,1)

It can be seen from the graph 1 and graph 2 that the data predicted by DGM(2,1) agree with the actual data well at the beginning, but the growth trend changes too dramatic and it is far from actual situation. While, the GM(1,1) model fit the actual situation well.

Then we analyze the Residual Error, Relative Error of GM(1,1) prediction, and the following table shows the results.

Table 6. Analysis outcome

year	Actual consumption per capital	Predicted Consumption per capital	Residual Error	relative Error
2000	435.4027	435.4000	0.0000	0.000%
2001	437.7427	424.5525	-13.1902	3.013%
2002	429.3408	427.2244	-2.1164	0.493%
2003	412.9463	429.9131	16.9668	4.109%
2004	428.0000	432.6187	4.61874	1.079%
2005	432.0698	435.3414	3.2716	0.757%
2006	442.0199	438.0812	-3.9386	0.891%
2007	441.5157	440.8383	-0.6774	0.153%
2008	446.1501	443.6127	-2.5374	0.569%
2009	448.0430	446.4045	-1.6384	0.366%
2010	450.1736	449.214	-0.9596	0.213%
2011	454.4000	452.0411	-2.3589	0.519%
2012	454.7142	454.8860	0.1717	0.038%
2013	455.5430	457.7488	2.2057	0.484%

As we can see from table 6, we can prove the consistency between GM(1,1) prediction and the actual water consumption per capital.

Table 7. Water consumption per capital prediction

year	2015	2016	2017	2018	2019	2020	2021	2022
water consumption per capital	463.5	466.4	469.3	472.3	475.3	478.2	481.3	484.3
year	2023	2024	2025	2026	2027	2028	2029	2030
water consumption per capital	487.3	490.4	493.5	496.6	499.7	502.9	506.0	509.2

3.4 Disaster Prediction model

In order to predict the time of the next drought, we have gathered a lot of information about annual rainfall in each province in China from 2003 to 2013. The national annual rainfall can be revealed by the sum of annual rainfall of each province. (unit: mm)

Table 8. Annual rainfall

Year	2003	2004	2005	2006	2007	2008
Annual rainfall	27345.3	28149.8	24421.9	29706.9	27445.2	30162.5
Year	2009	2010	2011	2012	2013	
Annual rainfall	28613	24395.8	30345.4	28981	26443.4	

According to the situation in China, we found that drought will happen when the rainfall of a year less than 26500 mm. In order to forecast the drought years,
Rainfall sequence is written as follows:

$$x^{(0)} = (27345, 28149, 24421, 29706, 27445, 30162, 28613, 24395, 30345, 28981, 26443).$$

If annual rainfall satisfies $x^{(0)}(i) < 26500$, we can regard the year as a drought year. With 2003 marked as the first year, it is not difficult to get the dry year sequence

$$\mathrm{t} = (3, 8, 11).$$

we developed a GM(1,1) model to predict the drought years. The result is showed in the following table:

Table 9. The result of disaster prediction model

3.0003	7.921962	10.86366	14.89778	20.42994	28.01641

From the figures in Table 8, we are easy to draw a conclution that the next two droughts will take place in 20 years and 28 years later after 2003. In other words, the next two troughts will occur in 2022 and 2030 in the next 15 years. While, if annual rainfall is over than 29000mm, we will say that this year's rainfall is abundant.

After taking all these into account, we suggest the Chinese government to collect rainwater in 2021 by the rainwater collecting technology. Then desalination technology should be adopted to get enough fresh water to provent China from the drought in 2030.

The results show that the per capita water consumption will increase and drought year and flood year will occur frequently, which will make some consquences.

4. Model Three: Intervention Plan

4.1 Pollution intervention plan

4.1.1 Introduction

The Yangtze river is the longest river in China and third longest in the world, which has been seriously polluted. After consideration, we take the Yangtze river for example.

4.1.2 One-dimensional water quality mathematical model

The differential equation of one dimensional steady decline rule was used to describe the concentration of river pollution. The equation is shown as follows:

$$u\frac{dc}{dx}=-Kc$$

in which the discrete function is ignored., after integration, $C=C_0 \cdot e^{-Kx/u}$.

where:

u is the average flow velocity distribution in the cross section of the river (the unit is meters per second);

x is the distance along the river (the unit is km);

K is the comprehensive degradation coefficient (the unit is 1/d);

C is the pollutants concentration along the river (the unit is mg/L);

$c_0$is the concentration of the pollutants after a node (the unit is mg/L).

4.1.3: The river diluted mixture model under the condition of constant design

In terms of point source, the dilution equation of the mixture of water and wastewater is

$$C=\frac{C_P \cdot Q_P + C_E \cdot Q_E}{Q_P+Q_E} ,$$

where:

C is the concentration of completely mixed water (the unit is mg/L);

Q_p, C_p are the water designed from upstream and water quality concentration designed respectively. (The units are cubic meters per second and milligrams per liter.)

Q_E, C_E are sewage flow designed and wastewater discharge concentration designed respectively. (The units are cubic meters per second and milligrams per liter.)

For the fact that the pollution source function fit linear superposition well, we set Single point source discharge formula as

$$W_C = S \cdot (Q_p + Q_E) - Q_p \cdot C_p \quad ;$$

where:

w_c is water load (the unit is g/L);

S is water quality standards of the control section (the unit is mg/L).

While multipoint sources discharge formula is shown as follows:

$$W_C = S \cdot (Q_p + \sum_{i=1}^{n} Q_{Ei}) - Q_p \cdot C_p$$

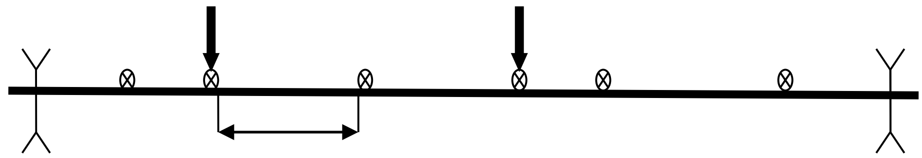
where:

Q_{Ei} is the sewage flow designed from sewage outlet i;

n is the number of sewage outlet.

4.1.4 Water environment capacity

(1) overview of the river



(2) Environmental capacity calculation

By applying $C = C_i + \frac{Wi/31.54}{Q_i + Q_j}$ into the model, we can get a formula of one-dimensional water environmental capacity:

$$Wi = 31.54 * (C * e^{Kx/86.4*u} - C_i) * (Q_i + Q_j)$$

where:

w_i is the emissions from sewage outlet which is number i. (t/a)

c_i is background concentration around node i in the river. (mg/L)

C is the concentration along the river. (mg/L)

Q_i is flow after the node. (Cubic meters per second)

Q_j is the quantity of waste water into the river around node i. (Cubic meters per second)

u is flow velocity of river i. (meters per second)

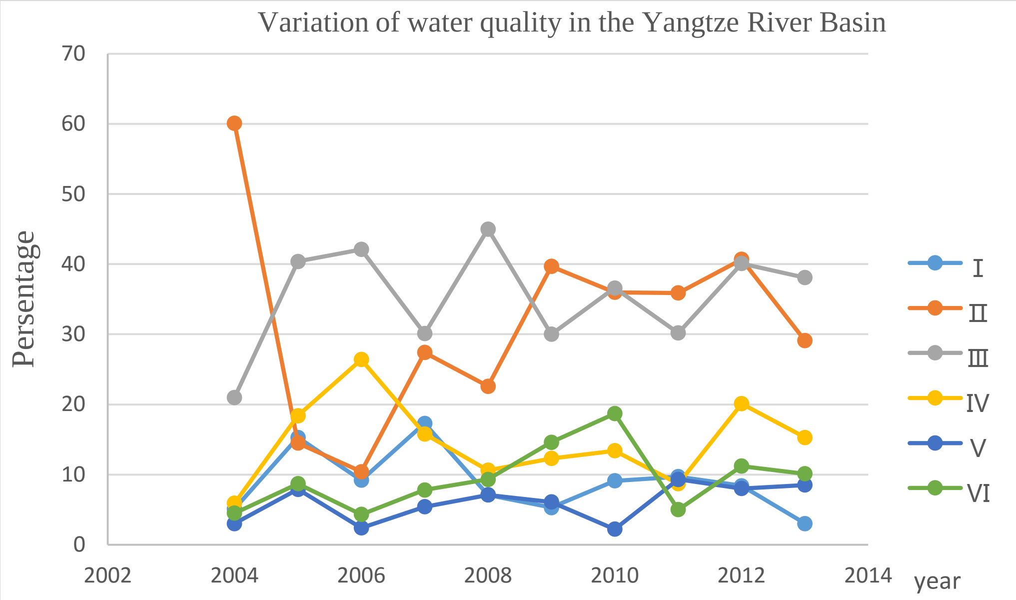
x is the distance between calculated point and node i.

The water environment capacity value of environment function is

$$W = \sum_{i}^{n} Wi.$$

According to the analysis above, measures should be taken to prevent the Yangtze river from being polluted more seriously.

4.1.5 Water category proportion in the total basin analysis



Graph 3. Variation of water quality in the Yangtze River Basin

If no effective measures is taken, water in level I would tend to scarcity. While, water in level IV would fluctuate to 14%. At the same time water in level V would get further increased. Therefore the treatment of Yangtze river is of great urgency.

4.1.6 Analysis of the changes of total sewage[5]

Table 8. The changes of total sewage

Year	2004	2005	2006	2007	2008
Total discharge of waste water(One hundred million tons)	174	179	183	189	207
Year	2009	2010	2011	2012	2013
Total discharge of waste water(One hundred million tons)	234	220.5	256	270	285

The simulation is accomplished by the least square method,

$$Q_F = 167.375 + 3.68 \times t + 0.835 \times t^2$$

in which, t is increase in the number of years.

Wastewater emissions in the next 10 years are listed in the follow table：

Year	2004	2005	2006	2007	2008
Total discharge of waste water(One hundred million tons)	308.9	331.8	356.3	382.5	410.4
Year	2009	2010	2011	2012	2013
Total discharge of waste water(One hundred million tons)	440.0	471.3	504.2	538.7	575.0

By far, we have gathered all the information that we need to predict the quantity of sewage should be treated every year, with the proportion of water belonging to level IV and V stay below 20%.

We simulate the percentage of water of IV, V and VI respectively, and get three prediction model.

$$IV_F = 4.3 - 2.18 \times t + 0.06 \times t^2 + 10.5 \times \ln t$$
$$V_F = 2.15 + 0.23 \times t + 0.001 \times t^2 - 0.3 \times \ln t$$
$$VI_F = 2.3 + 0.5 \times t - 0.005 \times t^2 - 1.14 \times \ln t$$

in which, t is the year.

The final predicted results are listed in the following table:

Year	2004	2005	2006	2007	2008
Total discharge of waste water(One hundred million tons)	308.9	331.8	356.3	381.5	410.4
Year	2009	2010	2011	2012	2013
Total discharge of waste water(One hundred million tons)	440.0	471.3	504.2	538.7	575.0

The extra sewage need to treat every year is showed in the following table:

Year	2004	2005	2006	2007	2008
Extra sewage need to treat(One hundred million tons)	24	47	71	98	125
Year	2009	2010	2011	2012	2013
Extra sewage need to treat(One hundred million tons)	155	186	219	254	195

4.2 Population intervention plan

4.2.1 Logistic model

Assuming that the population in a particular time t is $x(t)$ and environment capacities of population is x_m, the net population growth rate can be written as

$$r(t) = r.(1 - \frac{x}{x_m})$$

Establish the differential equation of the block type population:

$$\frac{dx}{dt} = r(1 - \frac{x}{x_m}).x, x(0) = x_0$$

Therefore,

$$x(t) = \frac{x_m}{1 + \left(\frac{x_m}{x_0} - 1 \right).e^{-r.t}},$$

which is regarded as Logistic model.

The result is obtained by simulating the data,

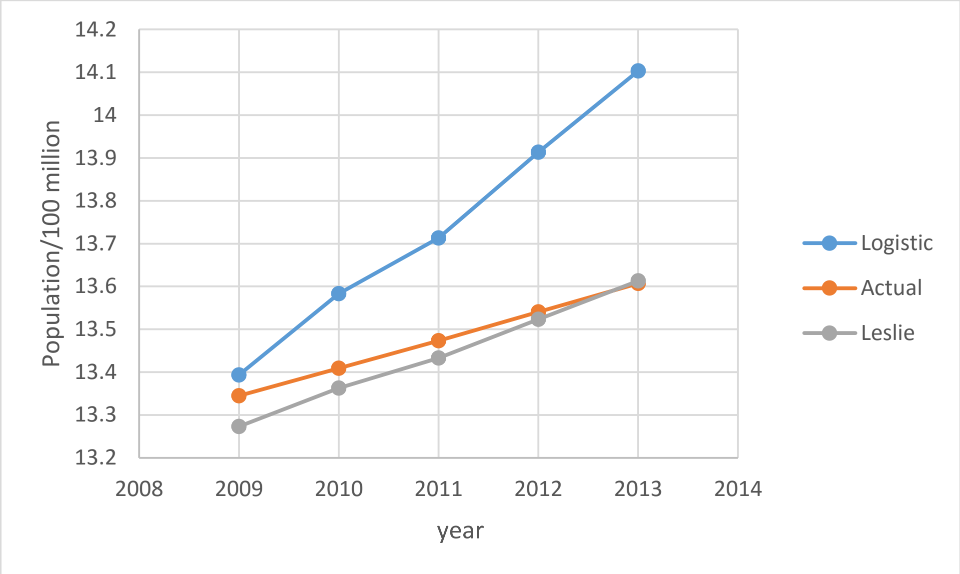
$$y = \frac{18.2939}{1 + 2.5656 e^{-0.0342.t}}.$$

The root-mean-square-error $RMSE = 0.1346$.

4.2.2 Leslie model

The outcome we get by estimating the parameters in the Leslie model are shown as follows:

$$X(t+1)=\begin{bmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_n & 0 \\ 1-\mu_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-\mu_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & 1-\mu_n & 0 \end{bmatrix} X(t)$$



Graph 4. The comparison of Logistic and Leslie model.

As showed in graph 4, the Leslie model match China's population growth model well, so we select the Leslie model to predict the future population of China.

Prediction of population growth in China

The long term forecast of population growth must take the impact of policy into consideration. Policies that can be taken containing Family Planning and Urbanization.

Next we analyze the impact on population of five policies.

Policy one: keep the existing policy not change;

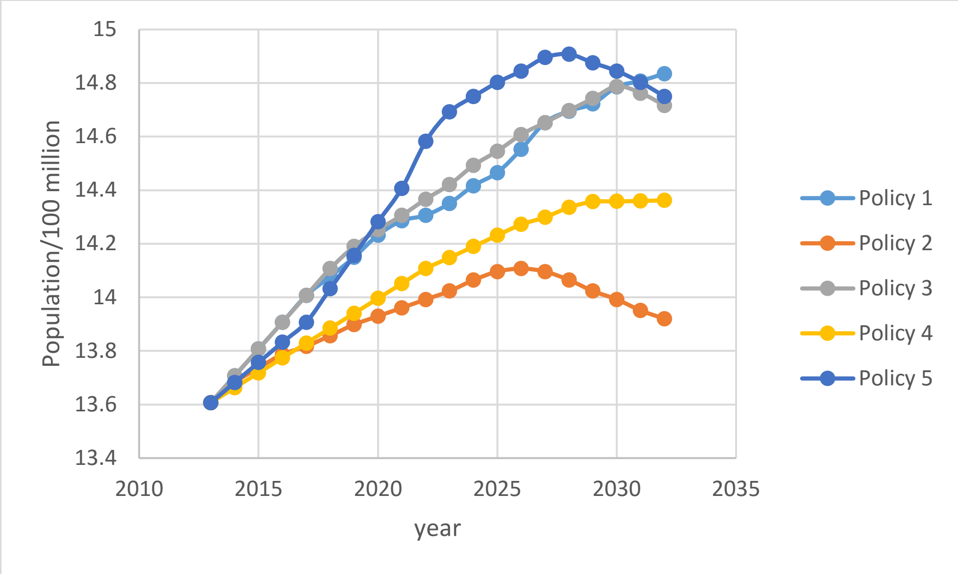
Policy two: Strengthen the work of family planning and limit the process of Urbanization;

Policy three: unwind the work of family planning and limit the process of Urbanization;

Policy four: unwind the work of family planning and keep the process of Urbanization;

Policy five: unwind the work of family planning and limit the process of Urbanization strictly.

In this paper, we carry on predict and estimate aiming at the five policies, in order to provide useful information to the government of China.



Graph 5 The comparison of 5 policies

The graph reveals the total population in China under different policies in nearly two decades. Comparing the trends under different policies, we are able to find that the policy of family planning has great influence on the trend of population.

The death rate in suburb is different from that in cities. Generally speaking, the death rate in villages is higher than that in cities. All of these suggest the validity of our model. In short, it is suggested that the government adopt policy four. Under this policy, the total population will enter a stable period.

The treatment of sewage and control of the population can have a positive impact on surroundings and ease the pressure of water shortage. But water will sill be a critical issue in the future. If we intervene in accordance with our policy and improve the economic scarcity, water scarcity will not happen in a short term.

5. Strengths and Weakness

5.1 Strengths

(1) The weight of each index is obtained by using entropy weight method and AHP method comprehensively.

(2) After the comparison of DGM (2,1) and GM (1,1) model, the GM(1,1) model which has better effects is finally selected to predict the per capita water consumption. Moreover, we choose disaster prediction model to predict the rainfall, basing on the characteristics of the data. Different methods have been applied for different purpose, improving validity.

(3) We select sewage purification and population as the interventions to plan for intervention policy. When considering the population factor, we test the Logistic model and Leslie model and choose the Leslie to select effective police to control the number

of population.

5.2 Weakness

(1) We only take per capita water consumption and rainfall into consideration when consider how the situation of water resources impact the lives of citizens in China, which has effect on the integrity.

(2) The current affairs policy are always decided by the national government when the Intervention policy being considered.

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Appendix and Supporting Data

1. DGM(2,1) Model

```
clc,clear
  x0=[435.4    437.7      429.3      412.9      428.0      432.1      442.0      441.5...
       446.2      448.0      450.2      454.4      454.7      455.5]
x0=[29706.9 28149.8 30162.5 25345.3 24421.9 28613      26443.4 24395.8 28981
27445.2];
n=length(x0);
a_x0=diff(x0)';
B=[-x0(2:end)',ones(n-1,1)];
u=B\a_x0
x=dsolve('D2x+a*Dx=b','x(0)=c1,Dx(0)=c2');
x=subs(x,{'a','b','c1','c2'},{u(1),u(2),x0(1),x0(1)});
yuce=subs(x,'t',0:n-1);
x=vpa(x,6)
```

2. GM(1,1) Model

```
clc,clear
  x0=[435.4    437.7      429.3      412.9      428.0      432.1      442.0      441.5...
       446.2      448.0      450.2      454.4      454.7      455.5]';
n=length(x0);
lamda=x0(1:n-1)./x0(2:n)
range=minmax(lamda')
x1=cumsum(x0);
B=[-0.5*(x1(1:n-1)+x1(2:n)),ones(n-1,1)];
Y=x0(2:n);
u=B\Y
x=dsolve('Dx+a*x=b','x(0)=x0');
x=subs(x,{'a','b','x0'},{u(1),u(2),x0(1)});
yuce1=subs(x,'t',[0:n-1]);
y=vpa(x,6)
```

3 Disaster Prediction Model

```
clc,clear
a=[27345.3    28149.8 24421.9 29706.9 27445.2 30162.5 28613      24395.8 30345.4
28981      26443.4]';
t0=find(a<=26500);n=length(t0);
t0=find(a>=29000);n=length(t0);
```

```
t1=cumsum(t0);
B=[-0.5*(t1(1:end-1)+t1(2:end)),ones(n-1,1)];Y=t0(2:end);
r=B\Y
y=dsolve('Dy+a*y=b','y(0)=y0');
y=subs(y,{'a','b','y0'},{r(1),r(2),t1(1)});
yuce1=subs(y,'t',[0:n+1])
y=vpa(y,6)
```

Table one: Variation of water quality in the Yangtze River Basin

year	Ⅰ	Ⅱ	Ⅲ	Ⅳ	Ⅴ	Ⅵ
2004	5.1	60.1	21	5.9	3	4.5
2005	15.3	14.5	40.4	18.4	7.9	8.7
2006	9.2	10.4	42.1	26.4	2.4	4.3
2007	17.3	27.4	30.1	15.8	5.4	7.8
2008	7.1	22.6	45.0	10.6	7.1	9.3
2009	5.3	39.7	30.0	12.3	6.1	14.6
2010	9.1	36.4	36.6	13.4	2.2	18.7
2011	9.7	35.9	30.2	8.7	9.3	5.0
2012	8.4	40.7	40.1	20.1	8.0	11.2
2013	3.0	29.1	38.1	15.3	8.5	10.1

Table two: The predicted population in five policies. (Unit: 100 million)

year	Policy one	Policy two	Policy three	Policy four	Policy five
2013	13.6072	13.6072	13.6072	13.6072	13.6072
2014	13.7072	13.6822	13.7072	13.66276	13.6822
2015	13.8072	13.7372	13.8072	13.71831	13.7572
2016	13.9072	13.7872	13.9072	13.77387	13.8322
2017	14.0072	13.8172	14.0072	13.82942	13.9072
2018	14.0772	13.8572	14.1072	13.88498	14.0322
2019	14.1497	13.89845	14.1897	13.94053	14.1572
2020	14.2322	13.9297	14.2522	13.99609	14.2822
2021	14.2857	13.96095	14.3057	14.05164	14.4072
2022	14.3061	13.9922	14.3661	14.1072	14.5822
2023	14.3507	14.02345	14.4207	14.14845	14.6922
2024	14.4167	14.0647	14.4922	14.1897	14.7497
2025	14.4657	14.09595	14.5457	14.23195	14.8022
2026	14.5528	14.1072	14.6072	14.2722	14.8447
2027	14.6522	14.09595	14.6522	14.29945	14.8952
2028	14.6952	14.0647	14.6972	14.3357	14.9072
2029	14.7222	14.02345	14.7422	14.3572	14.8752
2030	14.7852	13.9922	14.7872	14.3582	14.8447
2031	14.8072	13.95095	14.7622	14.3602	14.8022