第五章 多元函数微分学及其应用

第三节 多元数量值函数的导数与微分

- 高阶偏导数和高阶全微分
- 多元复合函数的偏导数和全微分
- 一阶全微分的形式不变性
- 由一个方程确定的隐函数的微分法

作业：习题5.3   Page57
19, 20, 22,18,
23,24(2)(4),26单号,
28, 31, 32,34 , 36

第一部分　多元复合函数的偏导数与全微分

一、多元复合函数偏导数的链式法则

定理3.1　如果$u=u(x,y)$及$v=v(x,y)$都在点(x,y)处可微，

且函数$z=f(u,v)$在对应的点(u,v)处也可微，

则复合函数$z=f[u(x,y),v(x,y)]$在点(x,y)处必可微.且

其全微分为:

$$dz=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\cdot dx+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\cdot dy$$

两个偏导数为:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

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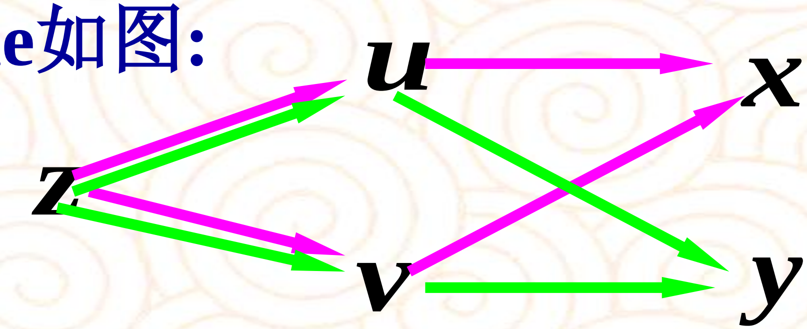
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

链式法则Chain Rule如图:



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

证明: 设自变量x有一改变量Δx,则相应地，u和v有改变量$\Delta u,\Delta v$

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\sqrt{\Delta u^2+\Delta v^2})$$

$$\Delta u=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+o_1(\rho)\qquad \Delta v=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+o_2(\rho)$$

其中 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$

$$\Delta z=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\Delta y+\frac{\partial z}{\partial u}o_1(\rho)$$

$$+\frac{\partial z}{\partial v}o_2(\rho)+o(\sqrt{\Delta u^2+\Delta v^2})$$

复合函数也可微的含义是?

$$\Delta z=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\Delta y+\frac{\partial z}{\partial u}o_1(\rho)$$

$$+\frac{\partial z}{\partial v}o_2(\rho)+o(\sqrt{\Delta u^2+\Delta v^2})$$

只需证: $$\lim_{\rho\to 0}\frac{\frac{\partial z}{\partial u}o_1(\rho)+\frac{\partial z}{\partial v}o_2(\rho)+o(\sqrt{\Delta u^2+\Delta v^2})}{\rho}=0$$

$$\frac{o(\sqrt{\Delta u^2+\Delta v^2})}{\rho}=\frac{o(\sqrt{\Delta u^2+\Delta v^2})}{\sqrt{\Delta u^2+\Delta v^2}}\cdot\frac{\sqrt{\Delta u^2+\Delta v^2}}{\rho}$$

无穷小量　　有界变量？？

$$\therefore \frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{o(\sqrt{\Delta u^2+\Delta v^2})}{\rho}=\frac{o(\sqrt{\Delta u^2+\Delta v^2})}{\sqrt{\Delta u^2+\Delta v^2}}\cdot\frac{\sqrt{\Delta u^2+\Delta v^2}}{\rho}$$

$$\Delta u=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+o_1(\rho)\qquad \rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$\boxed{\rho\to 0\Leftrightarrow \Delta x\to 0,\Delta y\to 0}$$

当ρ充分小时,

$$\left|\frac{\Delta u}{\rho}\right|\le\left|\frac{\partial u}{\partial x}\right|\frac{|\Delta x|}{\rho}+\left|\frac{\partial u}{\partial y}\right|\frac{|\Delta y|}{\rho}+\left|\frac{o_1(\rho)}{\rho}\right|\le\left|\frac{\partial u}{\partial x}\right|+\left|\frac{\partial u}{\partial y}\right|+2023$$

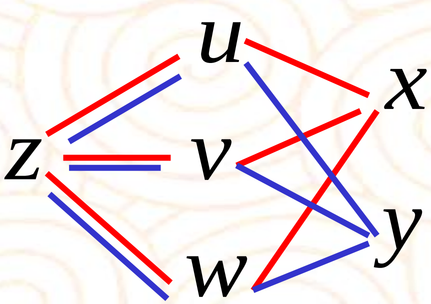
$\therefore \dfrac{\Delta u}{\rho}$有界, 同理$\dfrac{\Delta v}{\rho}$有界, $\therefore \dfrac{\sqrt{\Delta u^2+\Delta v^2}}{\rho}$有界

类似地，设$u=u(x,y)$、$v=v(x,y)$、$w=w(x,y)$都在点(x,y)可微，则复合函数$z=f[u(x,y),v(x,y),w(x,y)]$在对应点$(x,y)$处可微.

两个偏导数可用下列公式计算:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$



$$dz=\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}\cdot dy$$

特殊地 $z = f(u, x, y)$ 其中 $u = u(x, y)$

即 $z = f[u(x, y), x, y]$, 令 $v = x$, $w = y$,

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial w}{\partial y} = 1.$$

$$\boxed{\frac{\partial z}{\partial x}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \boxed{\frac{\partial f}{\partial x}}, \qquad \boxed{\frac{\partial z}{\partial y}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \boxed{\frac{\partial f}{\partial y}}.$$

区别类似

两者的区别

把复合函数 $z = f[u(x, y), x, y]$ 中的 y 看作不变而对 x 的偏导数

把 $z = f(u, x, y)$ 中的 u 及 y 看作不变而对 x 的偏导数

又如, $z = f(x, v),\ v = \psi(x, y)$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = \boldsymbol{f_1 + f_2 \psi_1}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = \boldsymbol{f_2 \psi_2}$$

$$\begin{tikzcd}[column sep=small] & z = f \arrow[dl] \arrow[dr] & & \\ x & & v \arrow[dl] \arrow[dr] & \\ & x & & y \end{tikzcd}$$

注意: 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

$\frac{\partial z}{\partial x}$ 表示固定 y 对 x 求导, $\frac{\partial f}{\partial x}$ 表示固定 v 对 x 求导

口诀: 分段用乘, 分叉用加, 单路全导, 叉路偏导

$$z = f(u,v),\ u=\varphi(x), v=\psi(x)$$

当u,v都可微时，有

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}$$

称为z对x的全导数

例 2 设$z=e^{u}\sin v$，而$u=xy$，$v=x+y$，

求 $\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=e^{u}\sin v\cdot y+e^{u}\cos v\cdot 1$$

$$=e^{u}(y\sin v+\cos v),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=e^{u}\sin v\cdot x+e^{u}\cos v\cdot 1$$

$$=e^{u}(x\sin v+\cos v).$$

例3 设$w = f(x+y+z, xyz)$，f具有二阶连续偏导数

求$\dfrac{\partial w}{\partial x}$和$\dfrac{\partial^2 w}{\partial x \partial z}$

解 令 $\boxed{u = x+y+z,}$ $\boxed{v = xyz}$

记 $f_1 = \dfrac{\partial f(u,v)}{\partial u}$，$f_{12} = \dfrac{\partial^2 f(u,v)}{\partial u \partial v} = \dfrac{\partial^2 f(u,v)}{\partial v \partial u}$

同理有 f_2 f_{11}，f_{22}

$$\frac{\partial w}{\partial x} = \boxed{\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}} = f_1 + yzf_2;$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}\left(f_1 + yzf_2\right) = \frac{\partial f_1}{\partial z} + yf_2 + yz\frac{\partial f_2}{\partial z};$$ 和的求导

$w = f(x+y+z, xyz)$ $\boxed{u = x+y+z,}$ $\boxed{v = xyz}$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}\left(f_1 + yzf_2\right) = \frac{\partial f_1}{\partial z} + yf_2 + yz\frac{\partial f_2}{\partial z};$$ 和的求导

$$\boxed{f_1 = \frac{\partial f(u,v)}{\partial u},}$$

而 $$\frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{11} + xyf_{12};$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{21} + xyf_{22};$$

于是 $$\frac{\partial^2 w}{\partial x \partial z} = f_{11} + xyf_{12} + yf_2 + yz\left(f_{21} + xyf_{22}\right)$$

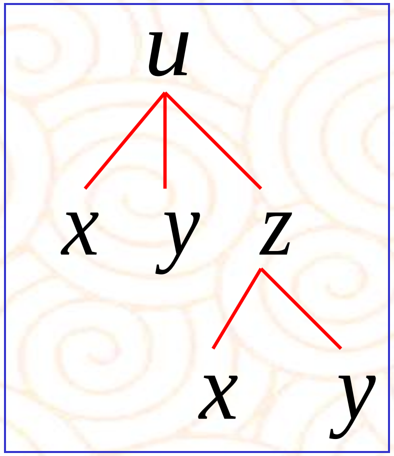
$$= f_{11} + y(x+z)f_{12} + xy^2zf_{22} + yf_2.$$

例4. $u = f(x, y, z) = e^{x^2+y^2+z^2},\ z = x^2 \sin y$，求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解: $$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= 2\boldsymbol{x}\boldsymbol{e}^{x^2+y^2+z^2} + 2\boldsymbol{z}\boldsymbol{e}^{x^2+y^2+z^2} \cdot 2\boldsymbol{x}\sin \boldsymbol{y}$$

$$= 2x(1+2x^2\sin^2 y)e^{x^2+y^2+x^4\sin^2 y}$$



$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2ye^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2} \cdot x^2\cos y$$

$$= 2(y + x^4 \sin y \cos y)e^{x^2+y^2+x^4\sin^2 y}$$

例5. 设 $u=f(x,y)$ 二阶偏导数连续，求下列表达式在极坐标系下的形式 (1) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$，(2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解: 已知 $x=r\cos\theta$, $y=r\sin\theta$, 则 $r=\sqrt{x^2+y^2}$，$\theta=\arctan\frac{y}{x}$

$$\boldsymbol{u=f(x,y)=f(r\cos\theta,r\sin\theta)=\mathrm{F}(r,\theta)}$$

(x,y)位于1,4象限时

Theta∈(-pi/2,pi/2)

$$(1)\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

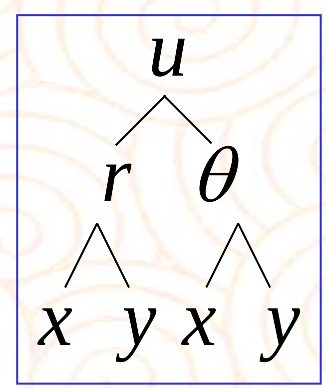
当点(x,y) 在2、3象限时

$$\theta=\arctan\frac{y}{x}+\pi$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta\ ,$$

$$\frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-y}{x^2+y^2}=-\frac{\sin\theta}{r}$$

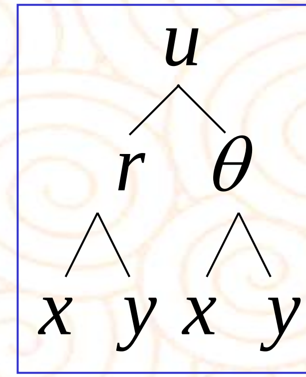
$$=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$



$x = r\cos\theta\,,\ y = r\sin\theta$

$$r = \sqrt{x^2 + y^2}\,,$$

$$\theta = \arctan\frac{y}{x}$$



$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}\,,\quad \frac{\partial \theta}{\partial y} = \frac{\frac{1}{x}}{1+(\frac{y}{x})^2} = \frac{x}{x^2+y^2}$$

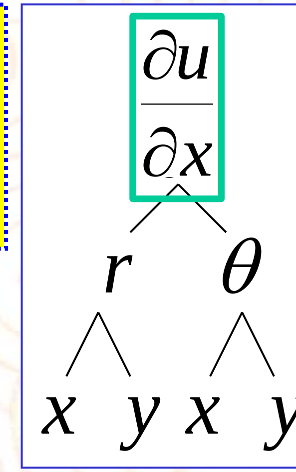
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\frac{x}{r^2} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$

而 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$

$$\therefore \quad (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 = (\frac{\partial u}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

$$(2)\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}$$



$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$-\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = -\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$u = f(x, y)$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial^2 u}{\partial x^2}=\underline{\frac{\partial^2 u}{\partial r^2}\cos^2\theta}-\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\boxed{\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}}$$

$$+\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\underline{\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2}=\underline{\frac{\partial^2 u}{\partial r^2}\sin^2\theta}+\cancel{2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}}+\boxed{\frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}}$$

$$-\cancel{\frac{\partial u}{\partial\theta}\frac{2\sin\theta\cos\theta}{r^2}}+\underline{\frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial\theta^2}\right]$$

二、一阶全微分形式不变性

设复合函数 $z = f(u,v)$ 在 (u,v) 可微，

其中 $u = \phi(x, y)$、$v = \psi(x, y)$ 在 (x,y) 可微.

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \boxed{\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv}$$

二、一阶全微分形式不变性

设函数$z = f(u,v)$在(u,v)可微，则有全微分：

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

当$u = \phi(x, y)$、$v = \psi(x, y)$且在(x,y)可微时，有:

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy} = \cdots\cdots = \boxed{\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv}$$

全微分形式不变性的实质:

无论z是自变量u、v的函数或中间变量u、v的函数，它的全微分形式是一样的.

全微分的有理运算法则

设 $u(x)$, $v(x)$ 均可微，则

1. $\mathrm{d}(u \pm v) = \mathrm{d}u \pm \mathrm{d}v$

2. $\mathrm{d}(Cu) = C\mathrm{d}u$ (C 为常数).

3. $\mathrm{d}(uv) = v\mathrm{d}u + u\mathrm{d}v$

4. $\mathrm{d}(\frac{u}{v}) = \frac{v\mathrm{d}u - u\mathrm{d}v}{v^2} \quad (v \neq 0)$

例4 已知$e^{-xy}-2z+e^{z}=0$，求$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$.

解 $\because d(e^{-xy}-2z+e^{z})=0,$

$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

第二部分 由一个方程确定的隐函数的微分法

1. $F(x,y)=0$

隐函数存在定理1

设函数$F(x,y)$在点$P(x_0,y_0)$的某一邻域内具有连续的偏导数，且$F(x_0,y_0)=0$，$F_y(x_0,y_0)\neq 0$，则方程$F(x,y)=0$在点$P(x_0,y_0)$的某一邻域内唯一确定了一个具有连续导数的一元函数$y=f(x)$，它满足条件$y_0=f(x_0)$，$F[x,f(x)]=0$

并有 $$\frac{dy}{dx}=-\frac{F_x}{F_y}.$$

隐函数的求导公式

证明从略，仅就求导公式推导如下：

设 $y=f(x)$ 为方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,f(x))\equiv 0$$

两边对 x 求导

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\equiv 0$$

在 (x_0,y_0) 的某邻域内 $F_y\neq 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

可推广到多元函数情形：

若 $z=f(x,y)$ 为方程 $F(x,y,z)=0$ 所确定的隐函数，则：

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

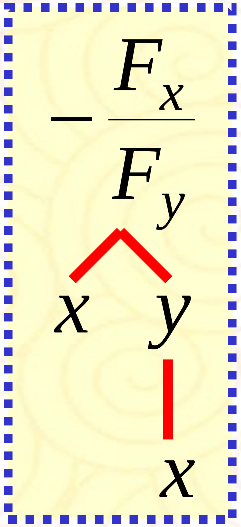
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

若还要求二阶导数，怎么办？

设 $y = f(x)$ 为方程 $F(x, y) = 0$ 所确定的隐函数 ,则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

若$F(x, y)$的二阶偏导数也都连续，则还有



二阶导数：

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right) + \frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= -\frac{F_{xx}F_y - F_{yx}F_x}{F_y^2} - \frac{F_{xy}F_y - F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$= -\frac{F_{xx}F_y{}^2 - 2F_{xy}F_xF_y + F_{yy}F_x{}^2}{F_y^3}$$

例5 验证方程$x^2+y^2-1=0$在点$(0,1)$的某邻域内能唯一确定一个可导、且$x=0$时$y=1$的隐函数$y=f(x)$，并求这函数的一阶和二阶导数在$x=0$的值.

解 令 $F(x,y)=x^2+y^2-1=0$

则 $F_x=2x,\ \ F_y=2y,$

$$F(0,1)=0,\qquad F_y(0,1)=2\neq 0,$$

依定理知方程$x^2+y^2-1=0$在点$(0,1)$的某邻域内能唯一确定一个可导、且$x=0$时$y=1$的函数$y=f(x)$.

$$F(x,y)=x^2+y^2-1=0$$

隐函数的一阶和二阶导数为：

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x}{y}, \qquad \left.\frac{dy}{dx}\right|_{x=0}=0,$$

$$\frac{d^2y}{dx^2}=-\frac{y-xy'}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{y^2+x^2}{y^3},$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=-1.$$

例6 已知 $\ln\sqrt{x^2+y^2}=\arctan\dfrac{y}{x}$，求 $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$.

解 令 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\dfrac{y}{x}=0$

则 $F_x(x,y)=\dfrac{x+y}{x^2+y^2}$, $F_y(x,y)=\dfrac{y-x}{x^2+y^2}$,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}=\frac{x+y}{x-y}.$$

或由全微分形式不变性，两边取微分,整理为
(   )dx+(   )dy=0也可

$$\frac{d^2y}{dx^2}=\frac{d\left(\dfrac{x+y}{x-y}\right)}{dx}=\frac{(1+y')(x-y)-(x+y)(1-y')}{(x-y)^2}$$

$$=\frac{2(x^2+y^2)}{(x-y)^3}$$

求下列方程所确

(1) $\ln\sqrt{x^2 +$

(2) $y = 2x \cdot$ a

$(2)\ y=2x\arctan\frac{y}{x}$

$F(x,y)=y-2x\arctan\frac{y}{x}$

$F_x=-2\arctan\frac{y}{x}+2x\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{y}{x^2}$

$=-2\arctan\frac{y}{x}+\frac{2xy}{x^2+y^2}=-\frac{y}{x}+\frac{2xy}{x^2+y^2}$

$F_y=1-2x\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=1-\frac{2x^2}{x^2+y^2}$

则 $\frac{dy}{dx}=-\frac{F_x}{F_y}=\frac{y}{x}$

$\frac{d^2y}{dx^2}=\frac{y'x-y}{x^2}=\frac{\frac{y}{x}x-y}{x^2}=0$

解 (2) 方程两端

$$\frac{dy}{dx} = 2 \cdot \text{arctan} \quad x \quad 2x \quad 1+\left(\frac{y}{x}\right)^2 \quad x^2$$

$$= \frac{y}{x} + \frac{2x}{x^2+y^2}\left(x\frac{dy}{dx} - y\right)，所以\frac{dy}{dx} = \frac{y}{x}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dx} - y}{x^2} = \frac{x\frac{y}{x} - y}{x^2} = 0.$$

2. $F(x, y, z) = 0$

隐函数存在定理2

设函数$F(x, y, z)$在点$P(x_0, y_0, z_0)$的某一邻域内有连续的偏导数，且$F(x_0, y_0, z_0) = 0$，$F_z(x_0, y_0, z_0) \neq 0$，则方程$F(x, y, z) = 0$在点$P(x_0, y_0, z_0)$的某一邻域中唯一确定了一个具有连续偏导数的函数$z = f(x, y)$，它满足条件$z_0 = f(x_0, y_0)$，

并有

$$\boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.}$$

例7 设 $x^2+y^2+z^2-4z=0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解1 令 $F(x,y,z)=x^2+y^2+z^2-4z$,

则 $F_x=2x$, $F_z=2z-4$, $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=\dfrac{x}{2-z}$,

$$\frac{\partial^2 z}{\partial x^2}=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)+x\cdot\dfrac{x}{2-z}}{(2-z)^2}$$

$$=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

设 $x^2+y^2+z^2-4z=0$，求 $\dfrac{\partial^2 z}{\partial x^2}$.

解2 利用隐函数求导

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \boxed{\frac{\partial z}{\partial x}=\frac{x}{2-z}}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解3 利用全微分形式不变性求一阶导数，再求二阶

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(\quad)dx+(\quad)dy$$

例8 设$z=f(x+y+z,\ xyz)$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial x}{\partial y}$，$\dfrac{\partial y}{\partial z}$.

解法1： 把z看成x, y的函数对x求偏导数得$\dfrac{\partial z}{\partial x}$，

令 $u = x + y + z,\quad v = xyz,\quad z = f(u,v),$

则 $$\frac{\partial z}{\partial x} = f_u \cdot \left(1 + \frac{\partial z}{\partial x}\right) + f_v \cdot \left(yz + xy\frac{\partial z}{\partial x}\right),$$

整理得 $$\frac{\partial z}{\partial x} = \frac{f_u + yzf_v}{1 - f_u - xyf_v},$$

例8 设$z = f(x+y+z,\ xyz)$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial x}{\partial y}$，$\dfrac{\partial y}{\partial z}$.

令 $u = x+y+z,\quad v = xyz,\quad z = f(u,v),$

把x看成z, y的函数对y求偏导数得

$$0 = f_u \cdot \left(\frac{\partial x}{\partial y}+1\right) + f_v \cdot \left(\frac{\partial x}{\partial y} yz + xz\right),$$

整理得 $$\frac{\partial x}{\partial y} = -\frac{f_u + xzf_v}{f_u + yzf_v},$$

把y看成x, z的函数对z求偏导数得$\dfrac{\partial y}{\partial z}$.

得 $$\frac{\partial y}{\partial z} = \frac{1-f_u - xyf_v}{f_u + xzf_v}.$$

解法2. 利用全微分形式不变性同时求出各偏导数.

思考题1

设$z = f(u, v, x)$，而$u = \phi(x)$，$v = \psi(x)$，

则$$\frac{dz}{dx} = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx} + \frac{\partial f}{\partial x},$$

试问$\frac{dz}{dx}$与$\frac{\partial f}{\partial x}$是否相同？为什么？

思考题2　已知$\dfrac{x}{z}=\varphi(\dfrac{y}{z})$，其中$\varphi$为可微函数，

求$x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=?$

习题选讲

1. 已知 $f(x,y)\Big|_{y=x^2}=1$, $f_1'(x,y)\Big|_{y=x^2}=2x$,

求 $f_2'(x,y)\Big|_{y=x^2}$.

2. 设函数 $z=f(x,y)$ 在点$(1,1)$处可微，且

$$f(1,1)=1,\quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)}=2,\quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)}=3,$$

$$\varphi(x)=f(x,\underline{f(x,x)}),\ \text{求}\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}.$$

3. 设 $u=f(x,y,z)$ 有连续的一阶偏导数，

又函数 $y=y(x)$ 及 $z=z(x)$ 分别由下列两式确定：

$$e^{xy}-xy=2,\ e^{x}=\int_{0}^{x-z}\frac{\sin t}{t}\,\mathrm{d}t,\ 求\ \frac{\mathrm{d}u}{\mathrm{d}x}.$$

4. 设 $y = y(x),\ z = z(x)$ 是由方程 $z = x f(x+y)$ 和 $F(x, y, z) = 0$ 所确定的函数，求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

4、设函数 $u = f(x, y)$ 具有二阶连续偏导数，且满足

$$4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0.$$ 确定 a, b 的值，使等式在变换

$\xi = x + ay, \eta = x + by$ 下简化为 $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0.$

7. 设函数$f(x, y)$有二阶连续偏导数, $\dfrac{\partial f}{\partial y} \neq 0$,

证明： $\forall C, f(x, y) = C$为一条直线的充要条件是:

$$\left(f_2\right)^2 f_{11} - 2 f_1 f_2 f_{12} + \left(f_1\right)^2 f_{22} = 0.$$

8. 设 $z = f(e^{x+y}, \frac{x}{y})$, f 具有二阶连续偏导数, 求 $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x \partial y}$

9. 若$\forall t>0$,有$f(tx,ty)=t^n f(x,y)$,则函数$f(x,y)$为n次齐次函数.

证明：若$f(x,y)$可微,则$f(x,y)$是n次齐次函数的充要条件是

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y).$$

11.讨论极限 $\lim\limits_{\substack{x\to 1\\ y\to 0}}(x+y)^{\frac{1}{\sin(x-1)}}$

12.讨论极限 $\lim\limits_{(x,y)\to(\infty,\infty)}\dfrac{x+y}{x^2-xy+y^2}$

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7. 设 $u=u(\sqrt{x^2+y^2})$ 具有连续二阶偏导数，且满足

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{1}{x}\left(\frac{\partial u}{\partial x}\right)+u=x^2+y^2$$

求函数 u 的表达式.