第七节　空间曲线的曲率与挠率

- Frenet 标架
- 曲率
- 挠率

习题5.7

(A)2，3，4，5，8，10

(B)4

若曲线 Γ 参数方程为：$\vec{r}=\vec{r}(t)=\big(x(t),y(t),z(t)\big),(\alpha\le t\le\beta)$

设 $\dot{\vec{r}}(t)=\big(\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)\big)$ 连续，且 $\dot{\vec{r}}(t)\neq\vec{0}$ $\begin{cases}x=x(t)\\ y=y(t),\\ z=z(t)\end{cases}$

曲线长度 $s=\int_{\alpha}^{\beta}\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}dt.$

弧长函数 $s=s(t)=\int_{\alpha}^{t}\sqrt{x'^2(\tau)+y'^2(\tau)+z'^2(\tau)}d\tau.$

$$\frac{ds}{dt}=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}=\left\|\dot{r}(t)\right\|>0$$

弧长函数单调，存在反函数 $t=t(s)$

曲线 Γ 方程可写为：$\vec{r}=\vec{r}(t(s))$， s 称为自然参数.

Γ 的自然参数方程为：$\vec{r}=\vec{r}(s)=\big(x(s),\ y(s),\ z(s)\big),(c\le s\le d)$

自然参数：以弧长为曲线方程的参数则称为自然参数，这时方程称为自然参数方程.

Γ 的自然参数方程为：$\vec{r}=\vec{r}(s)=\big(x(s),\ y(s),\ z(s)\big),(c\le s\le d)$

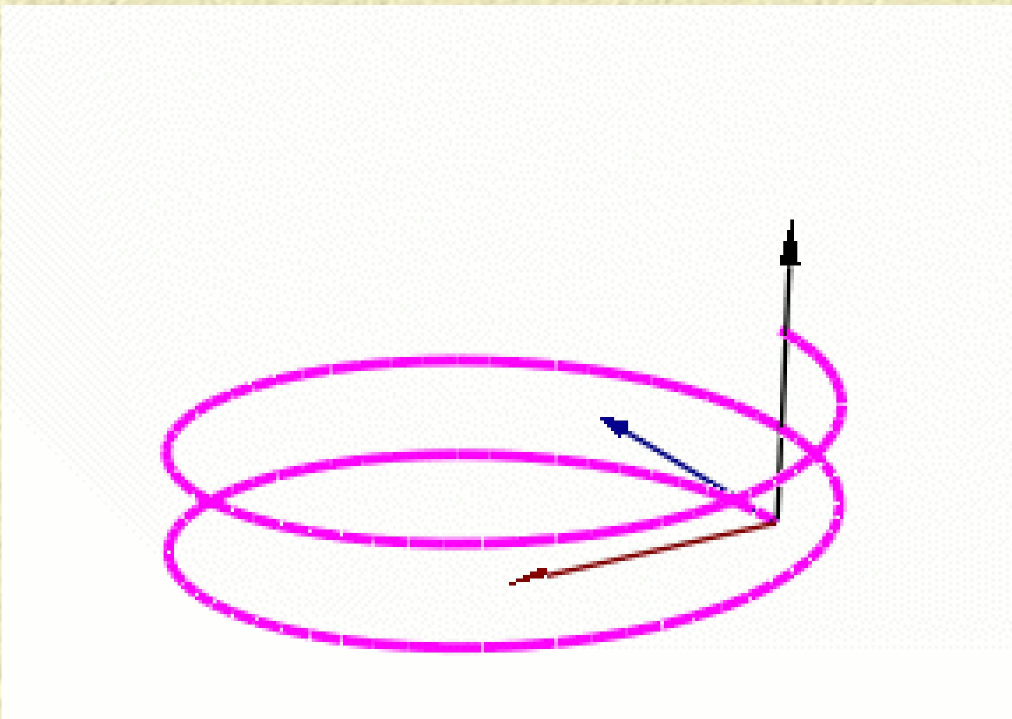
注：s是自然参数的充分必要条件是 $|r'(s)|=\left|\frac{dr}{ds}\right|=1$

$$ds=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}dt$$

$$=\sqrt{(\dot{x}(t)dt)^2+(\dot{y}(t)dt)^2+(\dot{z}(t)dt)^2}=\sqrt{(dx)^2+(dy)^2+(dz)^2}$$

$$\therefore (\frac{dx}{ds})^2+(\frac{dy}{ds})^2+(\frac{dz}{ds})^2=1 \quad 而 \frac{dr}{ds}=(\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds})$$

即 $\vec{r}'(s)=(x'(s),y'(s),z'(s))$ 是一个单位切向量.



The Frenet-Serret frame moving along a helix. The **T** is represented by the blue arrow,**N** is represented by the red vector while **B** is represented by the black vector.

7.1 Frenet 标架

(1) 法平面和切线

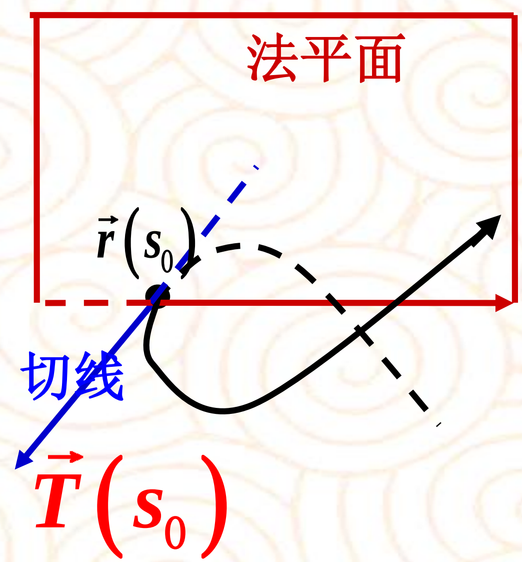
曲线Γ：$\vec{r}=\vec{r}(s)$, s为自然参数

Tangent

设$\vec{r}'(s_0)\neq 0$, $\vec{T}(s_0)=\vec{r}'(s_0)$是与曲线正向一致的单位**切向量**

则 **切线**: $\vec{\rho}=\vec{r}(s_0)+\lambda\vec{r}'(s_0)$

割线趋近于极限位置——切线



法平面: $(\vec{\rho}-\vec{r}(s_0))\cdot\vec{r}'(s_0)=0$

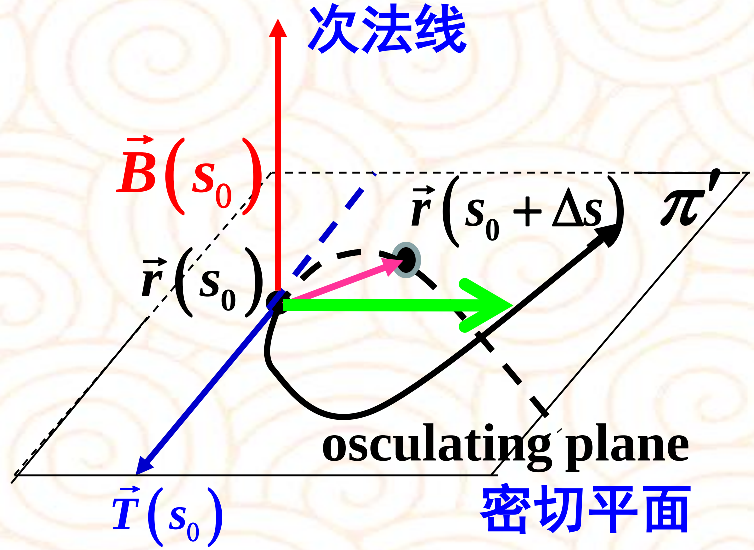
过该点且与切线垂直的所有法线决定的平面.

(2) 密切平面与次法线

$$\pi':\vec{T}(s_0),[\vec{r}(s_0+\Delta s)-\vec{r}(s_0)]$$

$$\Delta s\to 0,\pi'\to\pi$$

$$\vec{n}_{\pi'}=\vec{r}'(s_0)\times[\vec{r}(s_0+\Delta s)-\vec{r}(s_0)]$$



$$\vec{r}(s_0+\Delta s)-\vec{r}(s_0)\approx\vec{r}'(s_0)\Delta s+\frac{1}{2!}\vec{r}''(s_0)\Delta s^2+o(\Delta s^2)$$

$$\Delta s\to 0,\vec{n}_{\pi}=\vec{r}'(s_0)\times\vec{r}''(s_0)\longleftrightarrow \boxed{\text{次法线的方向向量}}$$

P96 Ex4. $\vec{r}'(s_0)\perp\vec{r}''(s_0)$ 单位次法线向量 $\vec{B}(s_0)$

Binormal

$$\vec{B}(s_0)=\frac{\vec{r}'(s_0)\times\vec{r}''(s_0)}{\|\vec{r}'(s_0)\times\vec{r}''(s_0)\|}=\frac{\vec{r}'(s_0)\times\vec{r}''(s_0)}{\|\vec{r}''(s_0)\|}$$

e is spanned by T

设$\vec{r}=\vec{r}(t)$为空间R^3中动点$\left(x(t),y(t),z(t)\right)^T$的向径.

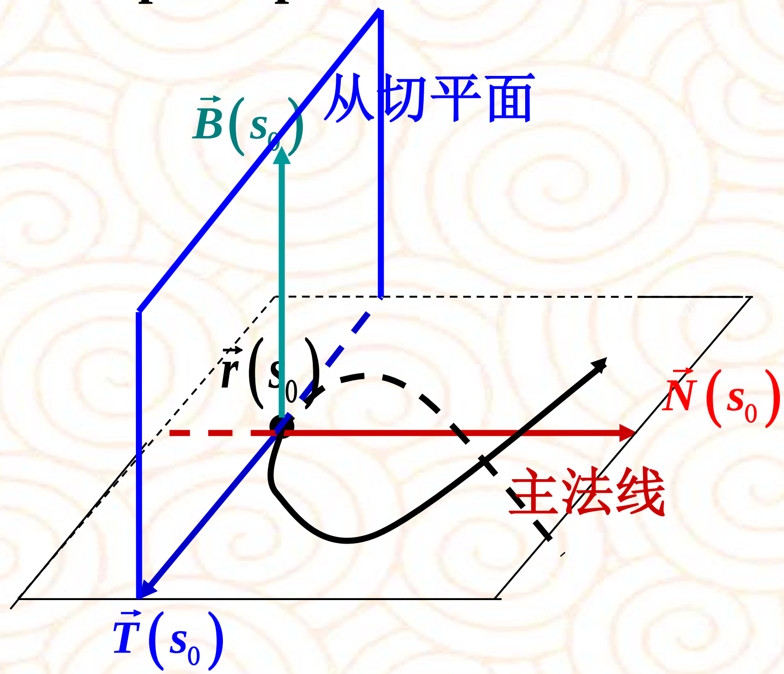
证明: $\|\vec{r}(t)\|=c\Leftrightarrow$ 内积$\left\langle\vec{r}'(t),\vec{r}(t)\right\rangle=0$. ($c$为常数)

(3) 从切平面和主法线

主法线向量 $\vec{N}(s_0)=\vec{B}(s_0)\times\vec{T}(s_0)$

从切平面 $(\vec{\rho}-\vec{r}(s_0))\cdot\vec{N}(s_0)=0$

主法线 $\vec{\rho}=\vec{r}(s_0)+\lambda\vec{N}(s_0).$



$$\vec{N}(s_0)=\vec{B}(s_0)\times\vec{T}(s_0),$$

$$\vec{T}(s_0)=\vec{r}'(s_0),\ \vec{B}(s_0)=\frac{\vec{r}'(s_0)\times\vec{r}''(s_0)}{\|\vec{r}''(s_0)\|}$$

$$\vec{N}(s_0)=\frac{1}{\|\vec{r}''(s_0)\|}(\vec{r}'(s_0)\times\vec{r}''(s_0))\times\vec{r}'(s_0)=\frac{\vec{r}''(s_0)}{\|\vec{r}''(s_0)\|}.$$

$\because \vec{r}'(s_0)\perp\vec{r}''(s_0)$

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

以上是曲线方程由自然参数s表示，如由一般参数t表示, 则:

$$\vec{r} = \vec{r}(t)$$

$$\vec{T}(t_0) = \frac{\dot{\vec{r}}(t_0)}{\left\|\dot{\vec{r}}(t_0)\right\|}$$

$$\vec{B}(t_0) = \frac{\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)}{\left\|\dot{\vec{r}}(t_0) \times \ddot{\vec{r}}(t_0)\right\|}$$

$$\vec{N}\left(t_0\right) = \vec{B}\left(t_0\right) \times \vec{T}\left(t_0\right)$$

$$T = \vec{r}'(s) = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{\dot{r}(t)}{\frac{ds}{dt}} = \frac{\dot{r}(t)}{\left\|\dot{r}(t)\right\|},$$

$$T = \vec{r}'(s) = \dot{r}(t) \cdot \frac{dt}{ds}$$

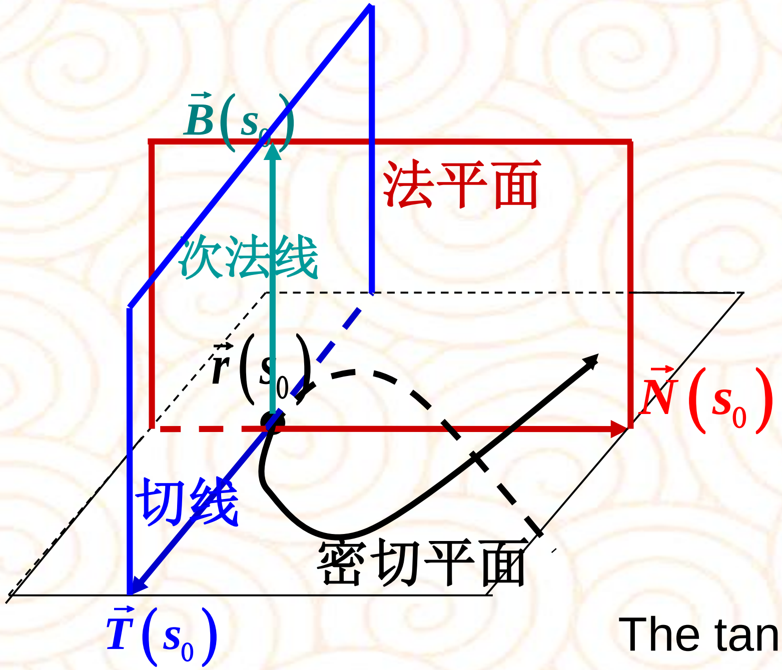
$$\vec{r}''(s) = \frac{dT}{ds} = \frac{d\left(\frac{d\vec{r}}{dt} \cdot \frac{dt}{ds}\right)}{ds} = \frac{d^2\vec{r}}{dt^2} \cdot \left(\frac{dt}{ds}\right)^2 + \frac{d\vec{r}}{dt} \cdot \frac{d^2t}{ds^2}$$

$$= \ddot{r}(t) \cdot \left(\frac{dt}{ds}\right)^2 + \dot{r}(t) \cdot \frac{d^2t}{ds^2}.$$

$$\vec{B}(s_0) = \frac{\vec{r}'(s_0) \times \vec{r}''(s_0)}{\left\|\vec{r}'(s_0) \times \vec{r}''(s_0)\right\|}$$

$$\vec{r}'(s) \times \vec{r}''(s) = \left(\dot{\vec{r}} \times \ddot{\vec{r}}\right)\left(\frac{dt}{ds}\right)^3$$

$$\frac{ds}{dt} = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} = \left\|\dot{r}(t)\right\| > 0$$



Frenet 标架

$$\vec{T}(s_0), \vec{N}(s_0), \vec{B}(s_0)$$

$$\vec{T}(t_0), \vec{N}(t_0), \vec{B}(t_0)$$

The tangent, normal, and binormal unit vectors, often called **T**, **N**, and **B**, or collectively the **Frenet–Serret frame** or **TNB frame**, together form an orthonormal basis spanning $\mathbb{R}^3$ and are defined as follows:

T is the unit vector tangent to the curve, pointing in the direction of motion.

N is the normal unit vector, the derivative of **T** with respect to the arclength parameter of the curve, divided by its length.

B is the binormal unit vector, the cross product of **T** and **N**.

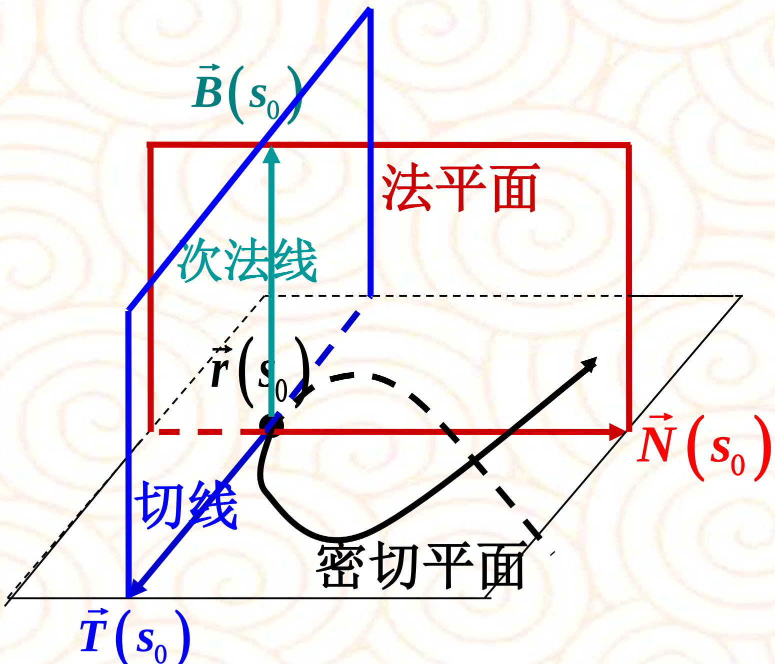
例7.1 求螺旋线 $\mathbf{r}=(a\cos t, a\sin t, kt)$ 的Frenet标架、密切平面以及从切平面的方程.

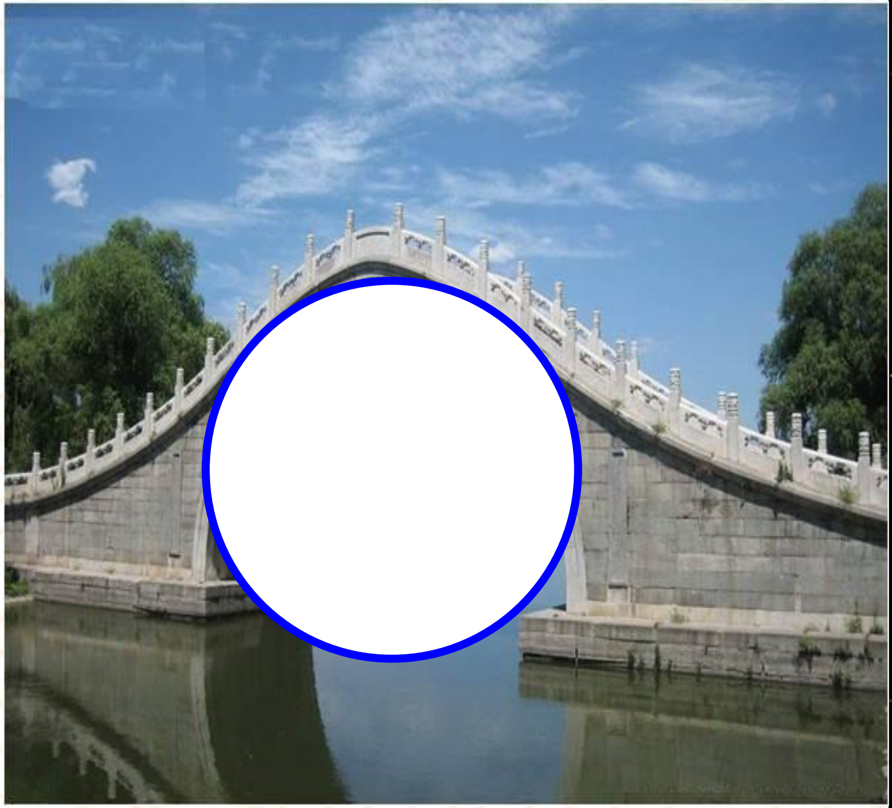
$$\vec{\mathrm{r}}=\vec{\mathrm{r}}(t) \qquad \vec{\mathrm{T}}(t_0)=\frac{\dot{\vec{r}}(t_0)}{\left\|\dot{\vec{r}}(t_0)\right\|} \qquad \vec{\mathrm{B}}(t_0)=\frac{\dot{\vec{r}}(t_0)\times\ddot{\vec{r}}(t_0)}{\left\|\dot{\vec{r}}(t_0)\times\ddot{\vec{r}}(t_0)\right\|}$$

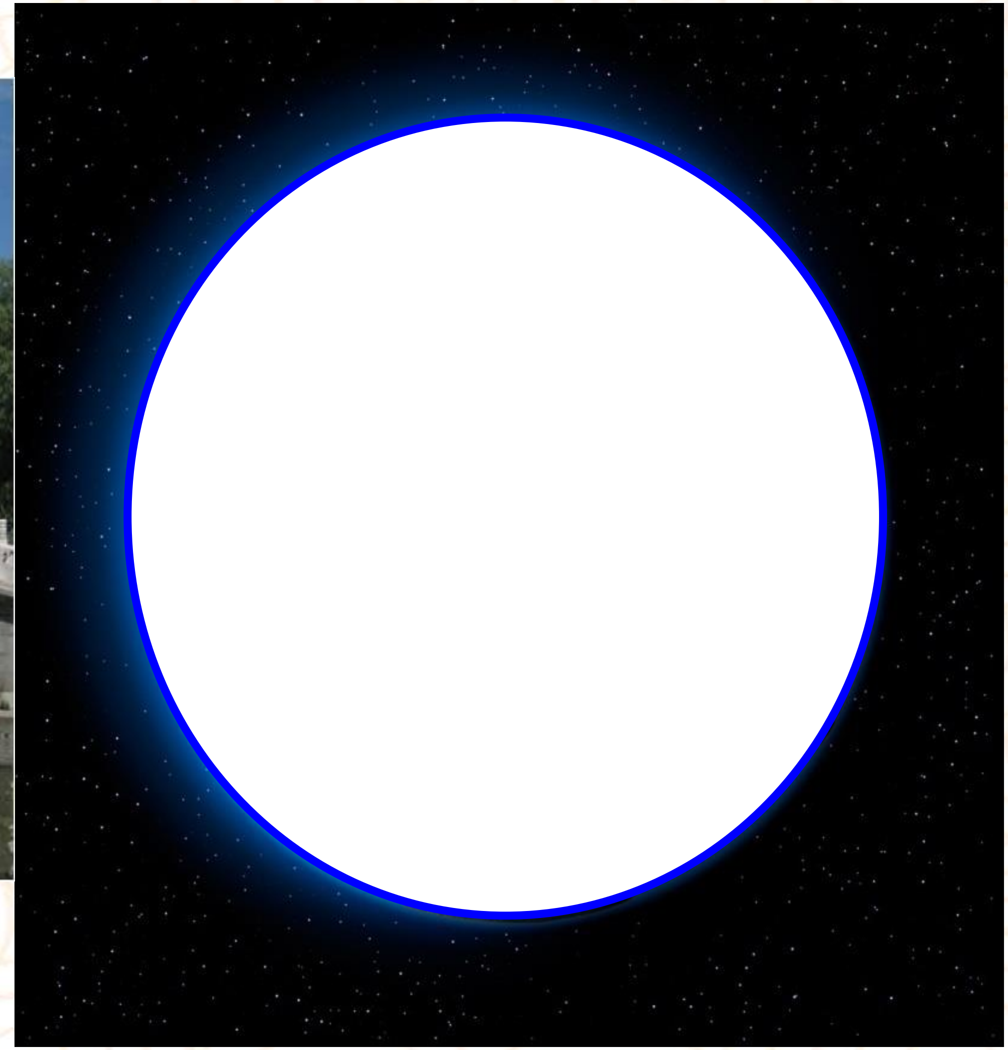
$$\vec{\mathrm{N}}(t_0)=\vec{B}(t_0)\times\vec{T}(t_0),$$

密切平面 $(\vec{\rho}-\vec{r}(t))\cdot\vec{B}(t)=0$

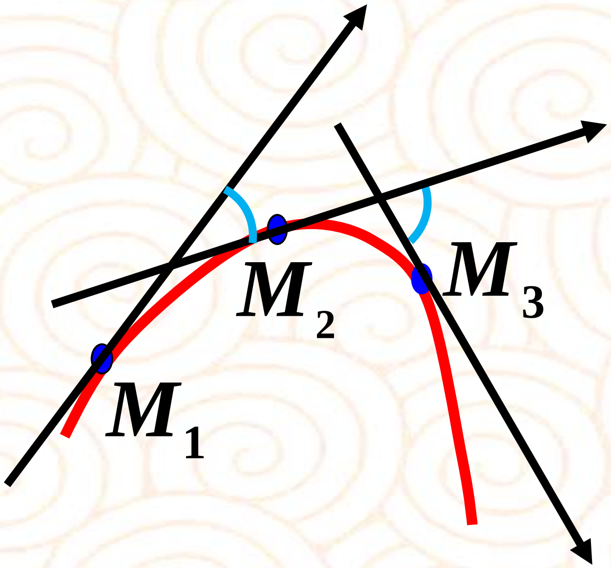
从切平面 $(\vec{\rho}-\vec{r}(t))\cdot\vec{N}(t)=0$

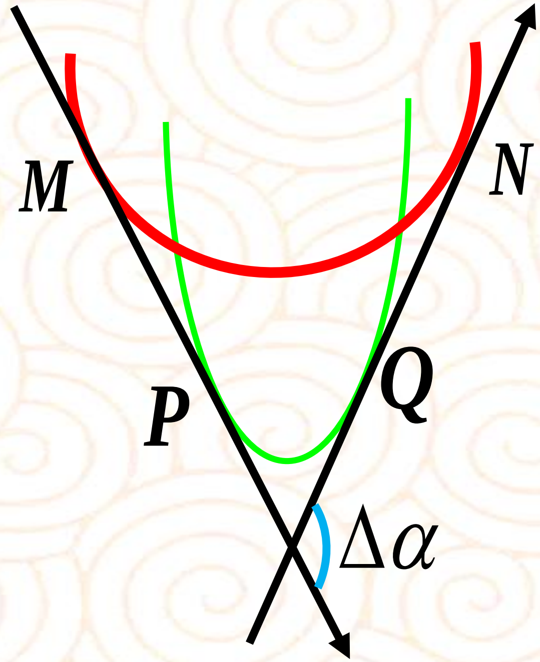






如何用数学描述这种差别？





曲线的弯曲程度 $\begin{cases} \text{与切线的转角有关} \\ \text{与曲线的弧长有关} \end{cases}$

7.2 曲率 Curvature of plane/space curves

在光滑曲线弧上自点 M 开始取弧段 MM', 其长为 Δs,

M、M' 两点处对应的切线之间夹角为 $\Delta\theta$,

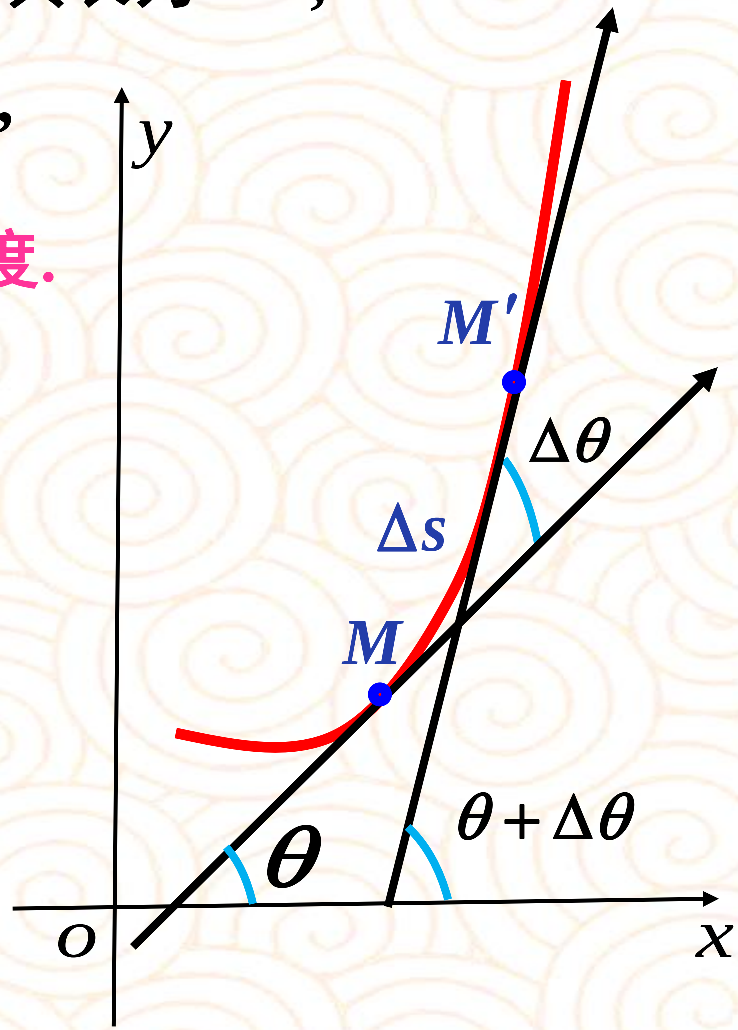
定义 $\bar{\kappa} = \left|\dfrac{\Delta\theta}{\Delta s}\right|$ 为弧段 MM' 的平均弯曲程度.

称为弧段 MM' 的平均曲率.

当 $\Delta s \to 0$ 时（即 $M' \to M$ 时）,

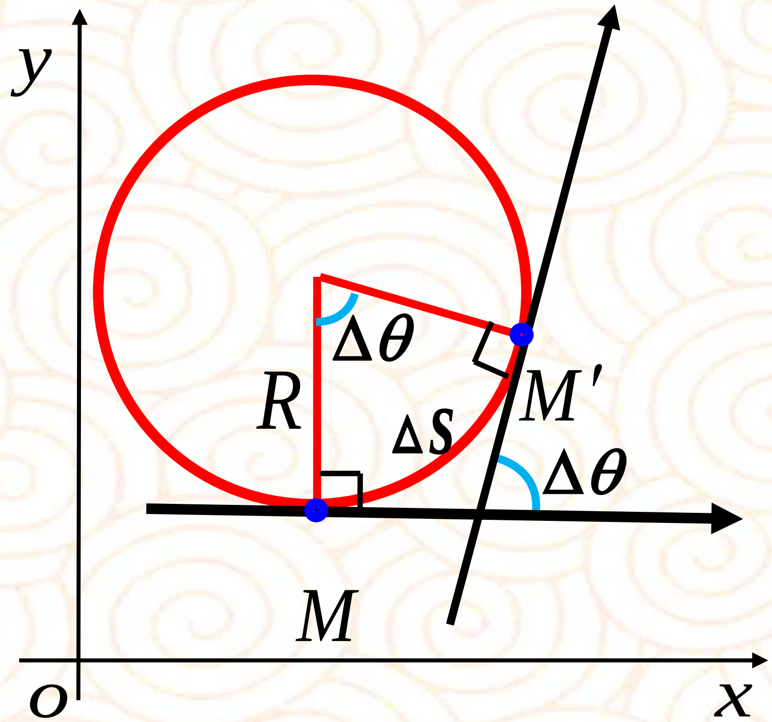
该平均曲率的极限值 称为点 M 处的曲率,

记作: $\kappa = \lim\limits_{\Delta s\to 0}\left|\dfrac{\Delta\theta}{\Delta s}\right| = \left|\dfrac{\mathrm{d}\theta}{\mathrm{d}s}\right|$



例1 求半径为R 的圆上任意点处的曲率.

解：如图所示，



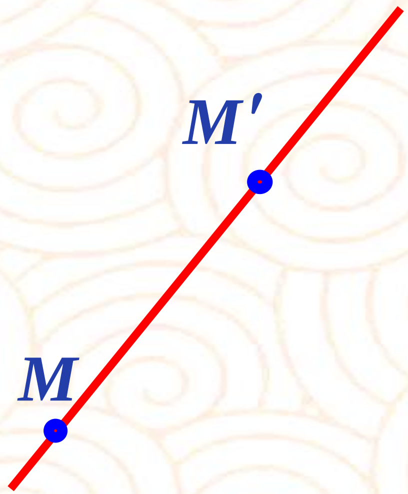
$$\Delta s = R\Delta\theta$$

$$\therefore \quad \kappa = \lim_{\Delta s \to 0}\left|\frac{\Delta\theta}{\Delta s}\right| = \frac{1}{R}，$$ 为一常数.

R越大曲率越小，圆弧越平直；R越小曲率越大，弯曲得越厉害.

地球半径6371km，曲率 $= \dfrac{1}{6371000} \approx 0.000000157$(rad/m)

例2 求直线上任一点处的曲率.



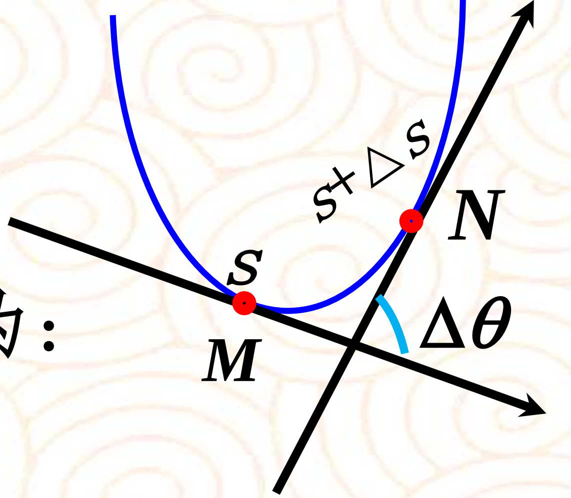
解：如图所示，

$$\kappa = \lim_{\Delta s \to 0} \left| \frac{\Delta \theta}{\Delta s} \right| = \lim_{\Delta s \to 0} 0 = 0.$$

直线上任意点处的曲率为 0, ——— 直线不弯！

对一般的曲线，曲率的计算公式？

1. 当曲线方程的参数为自然参数s 时



定义7.1（曲率） 设空间光滑曲线方程为:

$\Gamma:\vec{r}=\vec{r}(s)\ (a\le s\le b)$ s为自然参数,

曲线上点M对应的参数为s,点M附近的点N对应的参数为$s+\Delta s$,点M处的切向量$r'(s)$与点N处的切向量$r'(s+\Delta s)$夹角为$\Delta\theta$,

称$\lim\limits_{\Delta s\to 0}\left|\dfrac{\Delta\theta}{\Delta s}\right|$为曲线$\Gamma$在点M处的曲率，记作$\kappa$.

$$\text{i.e. } \kappa=\lim_{\Delta s\to 0}\left|\frac{\Delta\theta}{\Delta s}\right|.$$

定理7.1 设空间光滑曲线Γ方程为$\vec{r}=\vec{r}(s)$，s为自然参数

$r(s)$二阶可导,则Γ上点$r(s)$处的曲率为:$\kappa(s)=\|r''(s)\|$.

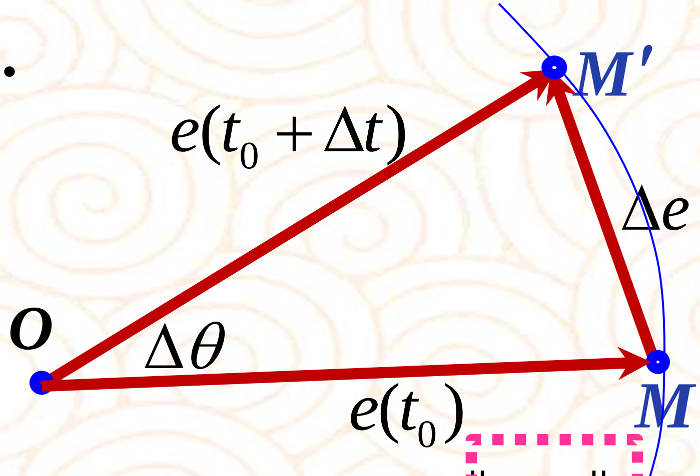
引理 设$e(t)$为定义在$U(t_0)\subseteq R$上的单位向量值函数，

$\Delta e=e(t_0+\Delta t)-e(t_0)$, $\Delta\theta$为$e(t_0)$与$e(t_0+\Delta t)$的夹角，则：$\lim\limits_{\Delta t\to 0}\left\|\dfrac{\Delta e}{\Delta\theta}\right\|=1.$

分析

$$\|\Delta e\|=2\|e(t_0)\|\sin\frac{|\Delta\theta|}{2}=2\sin\frac{|\Delta\theta|}{2}.$$

$$\lim_{\Delta t\to 0}\left\|\frac{\Delta e}{\Delta\theta}\right\|=\lim_{\Delta\theta\to 0}\frac{2\sin\frac{|\Delta\theta|}{2}}{|\Delta\theta|}=1.$$



进一步

$\because T(s)=r'(s)$是与Γ正向一致的单位切向量 $\therefore \boxed{\lim\limits_{\Delta t\to 0}\left\|\dfrac{\Delta T}{\Delta\theta}\right\|=1}$

T就是e的特殊情况

$$\kappa(s)=\lim_{\Delta s\to 0}\left|\frac{\Delta\theta}{\Delta s}\right|=\lim_{\Delta s\to 0}\boxed{\left\|\frac{\Delta\vec{T}}{\Delta\theta}\right\|}\left|\frac{\Delta\theta}{\Delta s}\right|=\lim_{\Delta s\to 0}\left\|\frac{\Delta\vec{T}}{\Delta s}\right\|=\|\vec{T}'(s)\|=\|\vec{r}''(s)\|$$

定理7.1 设空间光滑曲线Γ方程为$\vec{r}=\vec{r}(s)$，s为自然参数，$r(s)$二阶可导，则Γ上点$r(s)$处的曲率为：$\kappa(s)=\|r''(s)\|.$

2. 当曲线方程的参数为一般参数 t 时

$\Gamma:\vec{r}=\vec{r}(t)$, $r(t)$二阶可导且$\dot{r}(t)\neq 0$,则 $\kappa(t)=\dfrac{\|\dot{r}(t)\times\ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$

分析 $\vec{r}=\vec{r}(t)$, $\quad \vec{r}'(s)=\dfrac{d\vec{r}}{ds}=\dfrac{d\vec{r}}{dt}\cdot\dfrac{dt}{ds}=\dot{r}(t)\cdot\dfrac{dt}{ds}$

$$\vec{r}''(s)=\frac{d^2\vec{r}}{ds^2}=\frac{d\left(\frac{d\vec{r}}{dt}\cdot\frac{dt}{ds}\right)}{ds}=\frac{d^2\vec{r}}{dt^2}\cdot\left(\frac{dt}{ds}\right)^2+\frac{d\vec{r}}{dt}\cdot\frac{d^2t}{ds^2}=\ddot{r}(t)\cdot\left(\frac{dt}{ds}\right)^2+\dot{r}(t)\cdot\frac{d^2t}{ds^2}$$

$$\|r'(s)\|=1\Leftrightarrow r''(s)\perp r'(s)$$

$$\kappa(s)=\|r''(s)\|=\|r'(s)\times r''(s)\|$$

$$=\left\|\dot{r}(t)\times\ddot{r}(t)\left(\frac{dt}{ds}\right)^3+\dot{r}(t)\times\dot{r}(t)\frac{dt}{ds}\cdot\frac{d^2t}{ds^2}\right\|$$

$$=\left\|\left(\dot{\vec{r}}\times\ddot{\vec{r}}\right)\left(\frac{dt}{ds}\right)^3\right\|=\frac{\|\dot{r}(t)\times\ddot{r}(t)\|}{\|\dot{r}(t)\|^3}$$

$$\frac{ds}{dt}=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)+\dot{z}^2(t)}$$
$$=\|\dot{r}(t)\|>0$$

2. 当曲线方程的参数为一般参数 t 时

设空间曲线由一般的参数方程表示为 $\Gamma: \vec{r} = \vec{r}(t)$,

$\Gamma: \vec{r} = \vec{r}(t)$, $r(t)$二阶可导且$\dot{r}(t) \neq 0$, 则 $\boxed{\kappa(t) = \dfrac{\|\dot{r}(t) \times \ddot{r}(t)\|}{\|\dot{r}(t)\|^3}}$

特殊情况:平面曲线

$(1)\ \vec{r} = (x(t), y(t), 0)$

曲线弧方程为参数方程

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

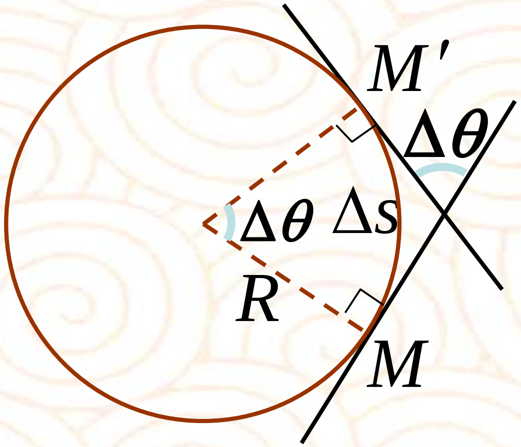
$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}.$$

$(2)\ y = y(x)$

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

例 求半径为R 的圆上任意点处的曲率 .

解: 如图所示， $\Delta s = R\Delta\theta$



$$\therefore \quad \kappa = \lim_{\Delta s \to 0} \left| \frac{\Delta\theta}{\Delta s} \right| = \frac{1}{R}$$

或：$\vec{r} = (x(t), y(t), 0) =$（$Rcost, Rsint, 0$）

$$\begin{cases} x = Rcost \\ y = Rsint \end{cases} \qquad \kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[(\dot{x})^2 + (\dot{y})^2]^{3/2}} = \cdots = \frac{1}{R}.$$

可见: R 愈小, 则K 愈大 , 圆弧弯曲得愈厉害；

R 愈大, 则K 愈小 , 圆弧弯曲得愈小 .

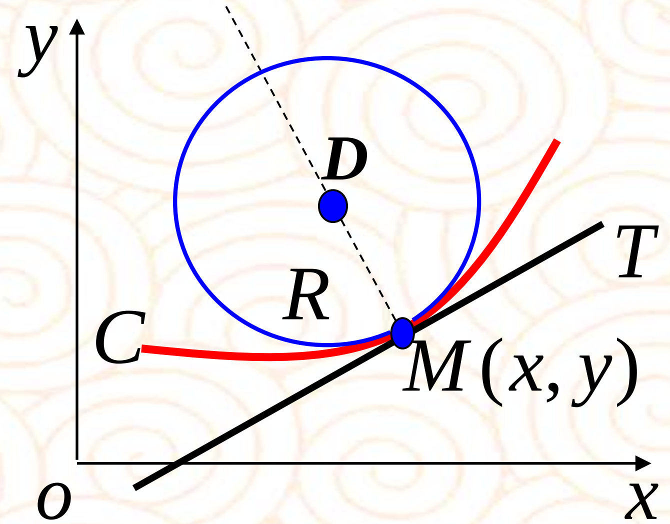
3. 曲率中心与曲率半径

设M 为平面曲线C 上任一点，在点

M 处作曲线的切线和法线, 在曲线

的凹向一侧法线上取点D 使

$$|DM| = \frac{1}{\kappa} = R$$



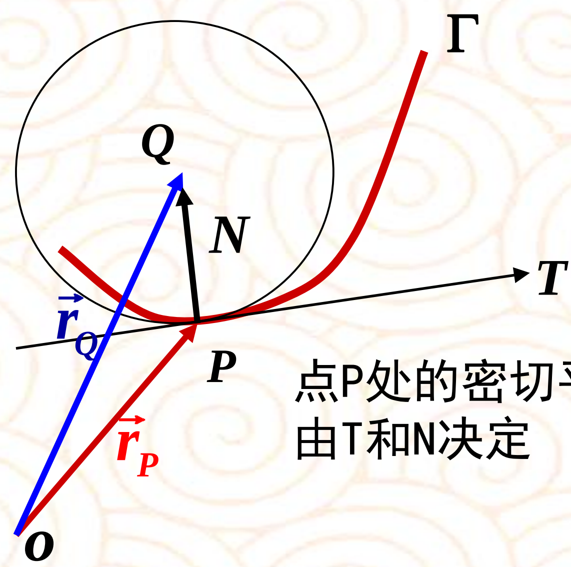
称以D 为中心, R 为半径的圆为曲线在点M 处的曲率圆,

R 叫做曲率半径, D 叫做曲率中心.

在点M 处曲率圆与原曲线C 有下列密切关系:

(1) 有公切线; (2) 凹向一致; (3) 曲率相同 .

3. 曲率中心与曲率半径



$$\left\|\overrightarrow{PQ}\right\| = \frac{1}{\kappa}$$

点P处的密切平面
由T和N决定

当曲线C 为空间曲线时，

借助于点P处的密切平面与主法线, 可同样作出曲率圆

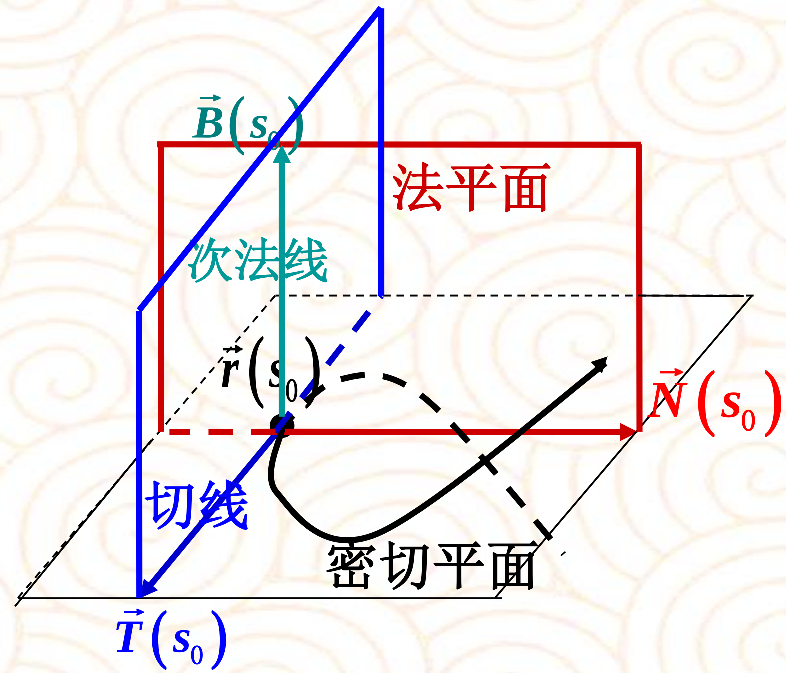
称为曲率圆**或密切圆**

曲率中心的向径为：

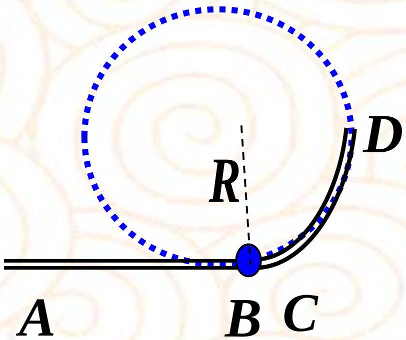
$$\vec{r}_Q = \vec{r}_P(s) + R(s)\vec{N}(s)$$

$\vec{r}_P(s)$为点 P 的向径

曲率半径为： $R = \dfrac{1}{\kappa}$

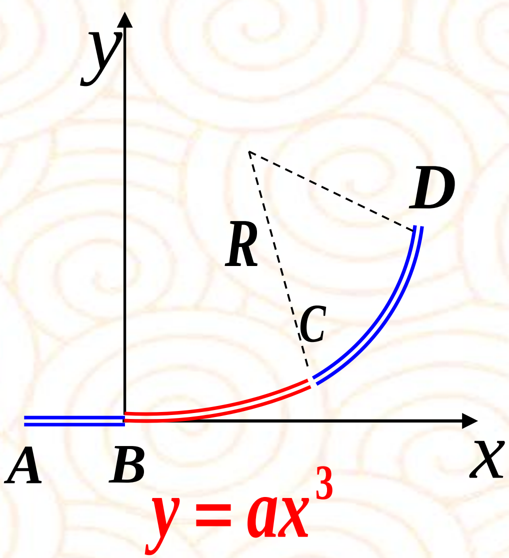


例1 火车从直道进入圆弧弯道(半径为R)时，为什么常常会产生摇晃震动？怎样减小这种晃动？



向心力 $F = \dfrac{mv^2}{R}$ 有突变(间断).

$y = ax^3\ (a > 0)$, $\quad \kappa = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}}$



曲率半径 $R = \dfrac{1}{\kappa} = \dfrac{\left(1+9a^2x^4\right)^{3/2}}{6a|x|}$,

使BC段端点C处的的曲率半径等于圆弧弯道CD的半径R

$x \to 0$(B点),曲率半径$R \to \infty$;

$x \neq 0$ 时,R是连续变化的.

例1. 我国铁路常用立方抛物线 $y=\dfrac{1}{6Rl}x^3$ 作缓和曲线,

其

$$\kappa=\frac{|y''|}{(1+y'^2)^{3/2}}$$

的半径, l 是缓和曲线的长度, 且 $l \ll R$.

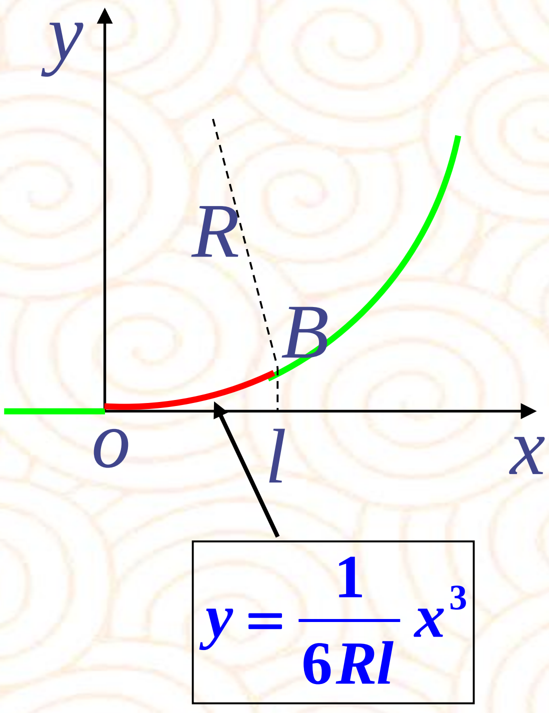
求 两个端点 $O(0,0), B(l,\dfrac{l^2}{6R})$ 处的曲率.

解: 当 $x\in[0,l]$ 时,

$$\because\ y'=\frac{1}{2Rl}x^2\le\frac{l}{2R}\approx 0 \qquad y''=\frac{1}{Rl}x$$

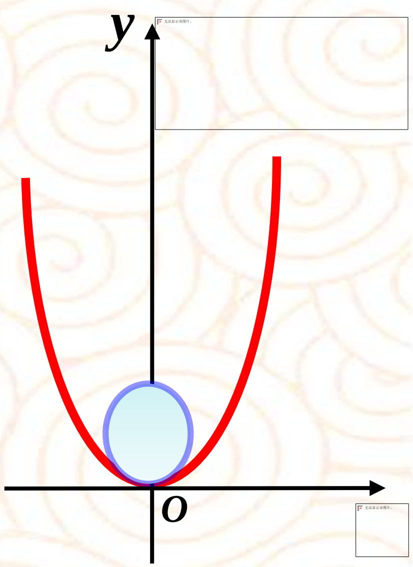
$$\therefore\ \kappa\approx|y''|=\frac{1}{Rl}x$$

显然 $\kappa\big|_{x=0}=0;\ \kappa\big|_{x=l}\approx\dfrac{1}{R}$



例2 设工件内表面的截线为 $y=0.4x^2$（单位：CM），现要用砂轮磨光其内表面，问用直径多大的砂轮才比较合适？

解 问题实质：如何做一个砂轮，使得砂轮的半径在任意点处都不超过该点曲率圆半径的最小值.



$$y'=0.8x,\ \ y''=0.8,\quad R=\frac{1}{\kappa}=\frac{\left(1+0.64x^2\right)^{3/2}}{0.8}$$

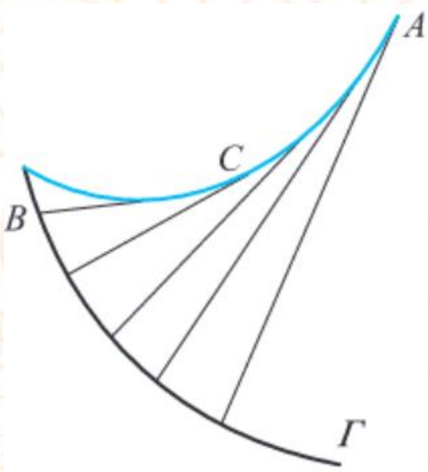
$$\min R(x)=\frac{1}{0.8}=1.25,\quad \left(x=0\text{时}\right)$$

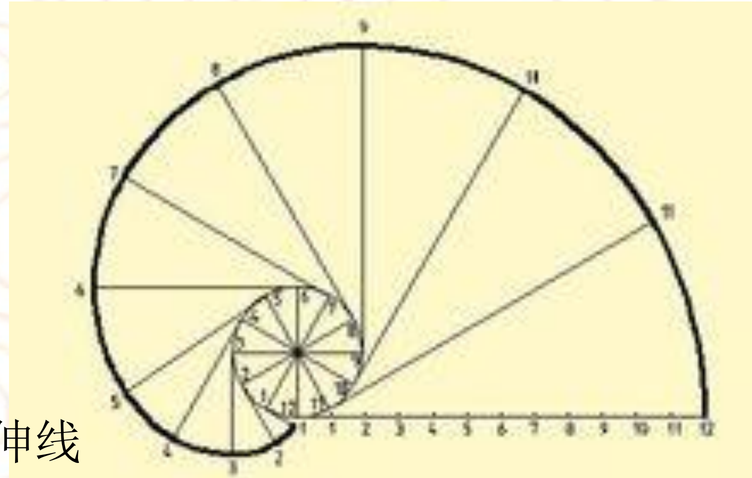
故砂轮的直径应该略小于1.25×2=2.5 cm.

$$\kappa=\frac{|y''|}{[1+(y')^2]^{3/2}}.$$

渐屈线与渐伸线

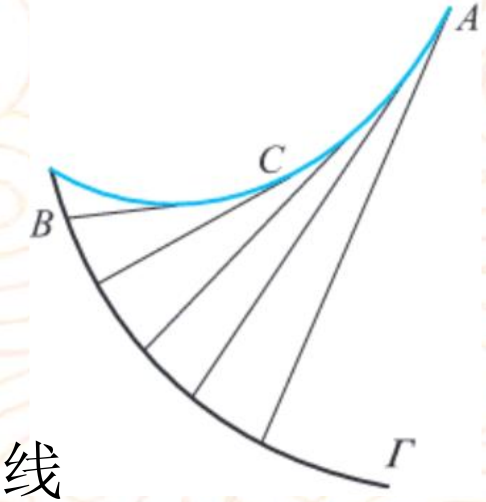
- 渐伸线(evolvent)，也称渐近线。与一条曲线C的所有切线相交成直角的曲线Γ，称为曲线C的渐伸线。
- 当一根绳正沿着另一曲线绕上或脱下时，它描出一条渐伸线.
- 渐伸线的形状见于鹰嘴、鲨鱼背鳍和棕榈树悬叶尖端。机器齿轮，齿两侧曲线(齿廓曲线）大多采用渐伸线。
- 渐屈线与渐伸线是一对相对的概念，若曲线A是曲线B的渐屈线，曲线B即为曲线A的渐伸线。每条曲线的渐屈线唯一确定,但却可以有无穷多条渐伸线。
- 任何两条渐伸线对应点的距离是常数.





圆的渐伸线

渐伸线的方程



设曲线C方程为 $\boldsymbol{r} = \boldsymbol{r}(s)$,

曲线C的渐伸线方程为 $\boldsymbol{\rho} = \boldsymbol{\rho}(s)$,

曲线C上任意一点的切线均为其渐伸线相应点的法线

$\boldsymbol{\rho}(s) = \boldsymbol{r}(s) + \alpha(s)\boldsymbol{T}(s)$, 求出$\alpha(s)$即可

$\boldsymbol{\rho}'(s) = \boldsymbol{r}'(s) + \alpha'(s)\boldsymbol{T}(s) + \alpha(s)\boldsymbol{T}'(s)$, $\quad \boldsymbol{T}(s) = \boldsymbol{r}'(s), \boldsymbol{T}'(s) = \boldsymbol{r}''(s)$

$\boldsymbol{\rho}'(s) = \boldsymbol{r}'(s) + \alpha'(s)\boldsymbol{r}'(s) + \alpha(s)\boldsymbol{r}''(s)$,

两边用$\boldsymbol{r}'(s)$作内积, $\boldsymbol{r}'(s) \cdot \boldsymbol{r}''(s) = 0, \|\boldsymbol{r}'(s)\| = 1$

$\boldsymbol{\rho}'(s) \cdot \boldsymbol{r}'(s) = 1 + \alpha'(s)$,

$\because \boldsymbol{\rho}'(s)$是渐伸线的切向量, $\quad \therefore \boldsymbol{\rho}'(s) \cdot \boldsymbol{r}'(s) = 0 \quad \alpha'(s) = -1$,

$\alpha'(s) = -s + c$,

渐伸线方程为: $\boldsymbol{\rho}(s) = \boldsymbol{r}(s) + (-s + c)\boldsymbol{T}(s)$.

渐屈线的方程

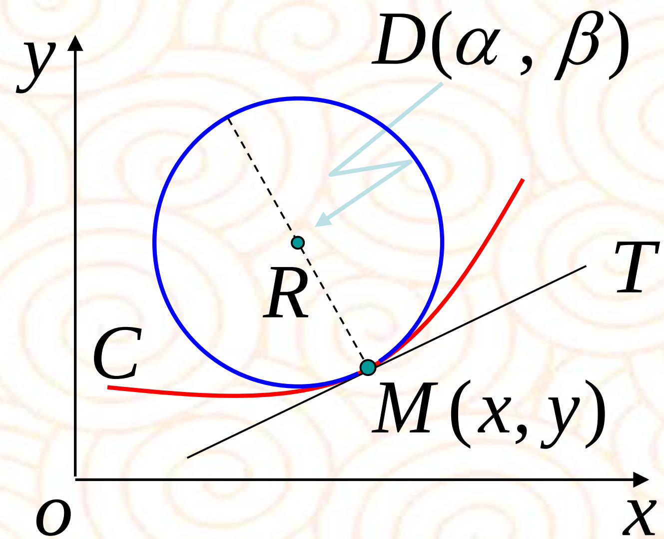
设平面上的曲线方程为 $y=f(x)$，且 $y''\neq 0$，求曲线上点M处的曲率半径及曲率中心 $D(\alpha,\beta)$的坐标公式 .

设点M处的曲率圆方程为

$$(\xi-\alpha)^2+(\eta-\beta)^2=R^2$$

故曲率半径公式为

$$\boxed{R=\frac{1}{\kappa}=\frac{(1+y'^2)^{3/2}}{|y''|}}$$

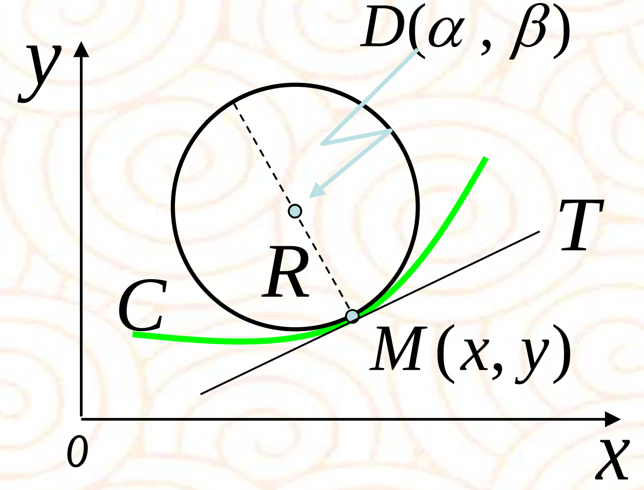


α,β 满足方程组

$$\begin{cases}(x-\alpha)^2+(y-\beta)^2=R^2 & (M(x,y)\text{在曲率圆上})\\ y'=-\dfrac{x-\alpha}{y-\beta} & (DM\perp MT)\end{cases}$$

由此可得曲率中心公式

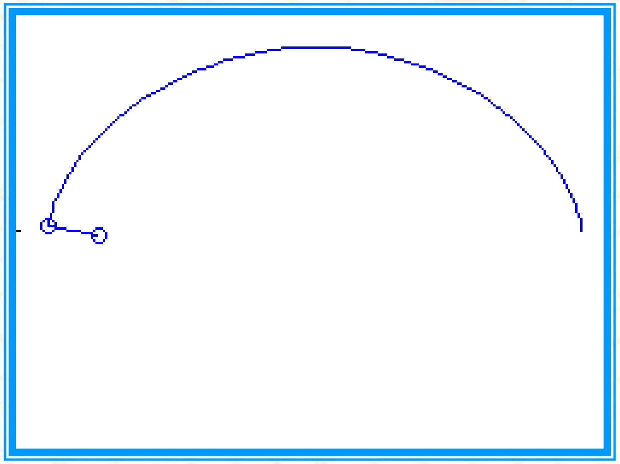
$$\begin{cases} \alpha = x - \dfrac{y'(1+y'^2)}{y''} \\ \beta = y + \dfrac{1+y'^2}{y''} \end{cases}$$



(注意 $y-\beta$ 与 y'' 异号)

当点 $M(x, y)$ 沿曲线 $\boldsymbol{y = f(x)}$ 移动时，相应的曲率中心的轨迹 G 称为曲线 C 的**渐屈线**，曲线 C 称为曲线 G 的**渐伸线**.

曲率中心公式可看成渐屈线的参数方程(参数为x).

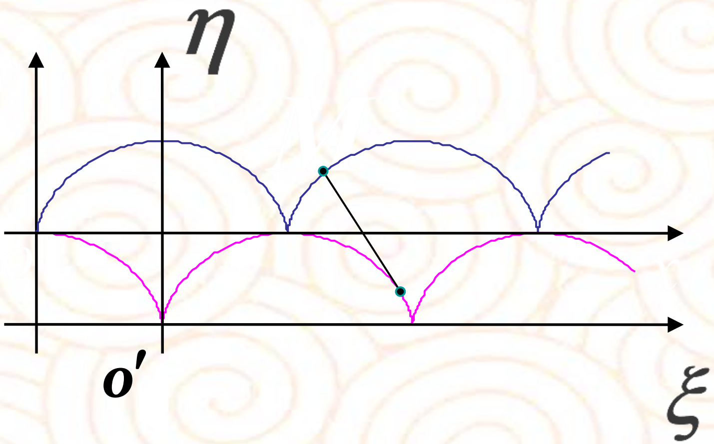


点击图中任意点动画开始或暂停

例. 求摆线 $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$ 的渐屈线方程 .

解: $y' = \dfrac{\dot{y}}{\dot{x}} = \dfrac{\sin t}{1 - \cos t}$, $y'' = \dfrac{\frac{\mathrm{d}}{\mathrm{d}t}(y')}{\dot{x}} = \dfrac{-1}{a(1 - \cos t)^2}$

代入曲率中心公式 ,得



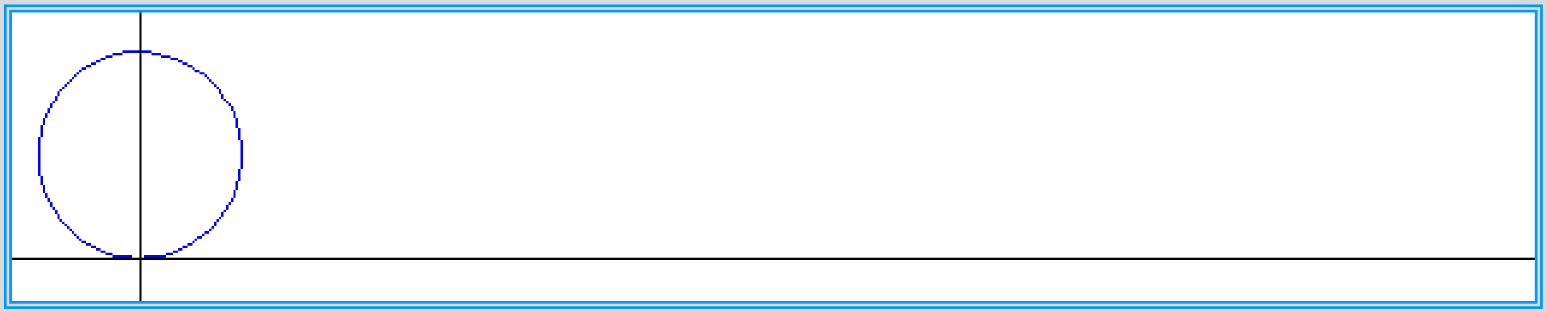
$$\begin{cases} \alpha = a(t + \sin t) \\ \beta = a(\cos t - 1) \end{cases}$$

令 $t = \pi + \tau$, $\begin{cases} \xi = \alpha - \pi a \\ \eta = \beta + 2a \end{cases}$

$$\begin{cases} \xi = a(\tau - \sin \tau) \\ \eta = a(1 - \cos \tau) \end{cases}$$ (仍为摆线)

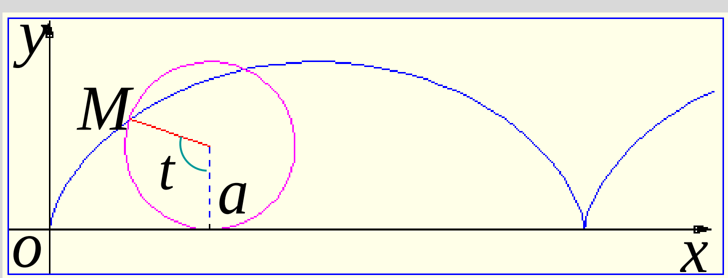
摆线 $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$

半径为 a 的圆周沿直线无滑动地滚动时，其上定点 M的轨迹为摆线 .

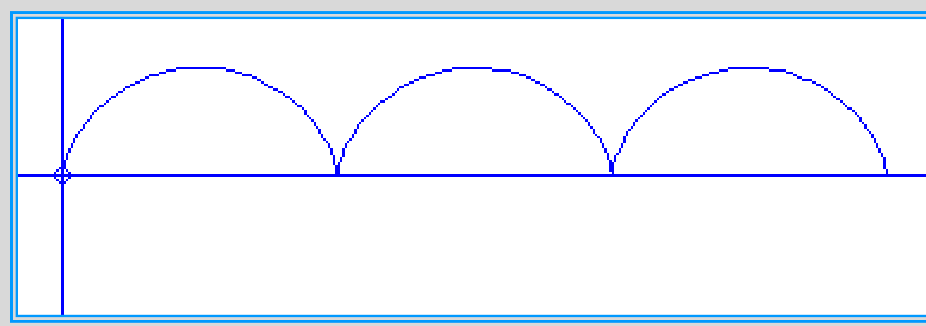


点击图中任意点动画开始或暂停

参数的几何意义

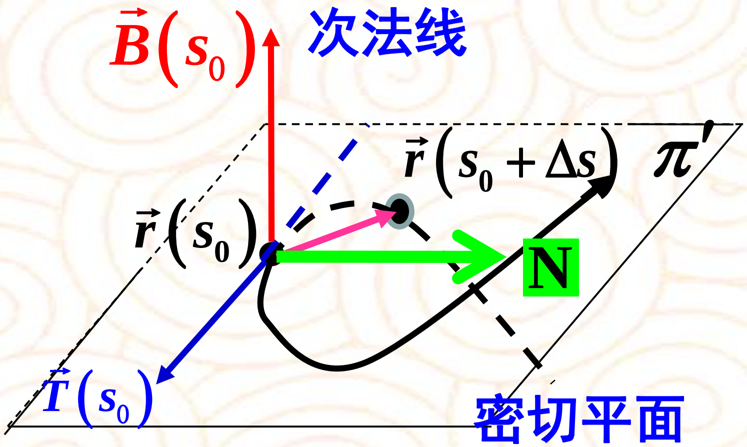


摆线的渐屈线



点击图中任意点动画开始或暂停

7.3 挠率 Torsion of a curve



如何描述曲线偏离密切平面的弯曲程度?

$\mathbf{B}(s)$ 是单位向量，故 $\mathbf{B}'(s)\perp\mathbf{B}(s)$

$\mathbf{B}(s)=\mathbf{T}(s)\times\mathbf{N}(s)$

得 $\mathbf{B}'=\mathbf{T}'\times\mathbf{N}+\mathbf{T}\times\mathbf{N}'$

因 $\mathbf{N}=\dfrac{r''}{\|r''\|}=\dfrac{\mathbf{T}'}{\|\mathbf{T}'\|}$，从而有 $\mathbf{T}'\times\mathbf{N}=\vec{0}$，代入得 $\boxed{\mathbf{B}'=\mathbf{T}\times\mathbf{N}'}$

故 $\mathbf{B}'\perp\mathbf{T}$，$\mathbf{B}'(s)\perp\mathbf{B}(s)$ 故 $\mathbf{B}'$ 必与 $\mathbf{B}\times\mathbf{T}$，即 $\mathbf{N}$ 是共线的，不妨设

$$\mathbf{B}'=-\tau\mathbf{N}\qquad \tau=-\mathbf{B}'\cdot\mathbf{N}$$

而且 $\|\mathbf{B}'\|=|\tau|$

定义7.2（挠率） 称 $\tau(s)=-\mathbf{B}'(s)\cdot\mathbf{N}(s)$ 为曲线 $\mathbf{r}=\mathbf{r}(s)$ 在点 $\mathbf{r}(s)$ 处的挠率. $|\tau|=\|\mathbf{B}'(s)\|$

$$\tau(s) = -\mathbf{B}' \cdot \mathbf{N} = -(\mathbf{T}\times\mathbf{N}')\cdot\mathbf{N}$$

$$= (\mathbf{T}\times\mathbf{N})\cdot\mathbf{N}'$$ （混合积改变顺序的性质）

$$\boxed{\mathbf{B}' = \mathbf{T}\times\mathbf{N}'}$$

$$\boxed{\mathbf{N} = \frac{r''}{\|r''\|} = \frac{1}{\kappa}\mathbf{T}'}$$

$$D(uf)(x) = f(x)Du(x) + uDf(x)$$

$$= (\mathbf{T}\times\frac{1}{\kappa}\mathbf{T}')\cdot\left[(\frac{1}{\kappa})'\mathbf{T}' + \frac{1}{\kappa}\mathbf{T}''\right]$$

$$\kappa(s) = \|r''(s)\|$$

$$= (\mathbf{T}\times\frac{1}{\kappa}\mathbf{T}')\cdot\frac{1}{\kappa}\mathbf{T}'' = \frac{1}{\kappa^2}\left[\mathbf{T}\ \ \mathbf{T}'\ \ \mathbf{T}''\right]$$

$$\tau(s) = \frac{\left[r'(s)\ \ r''(s)\ \ r'''(s)\right]}{\|r''(s)\|^2}, \quad s\text{为自然参数}$$

$$\Gamma:\ r = r(t) = (x(t), y(t), z(t)),\ t_1 \le t \le t_2.$$

$$\tau(t) = \frac{\left[\dot{r}(t)\ \ \ddot{r}(t)\ \ \dddot{r}(t)\right]}{\|\dot{r}(t)\times\ddot{r}(t)\|^2}.$$

例 求圆柱螺线$r=\{a\cos t, a\sin t, bt\}$($a>0$, $b>0$均为常数) 的曲率、挠率.

解
$$\dot{\vec{r}}=\{-a\sin t,\ a\cos t,\ b\},$$
$$\ddot{\vec{r}}=\{-a\cos t,\ -a\sin t,\ 0\},$$
$$\dddot{\vec{r}}=\{a\sin t,\ -a\cos t,\ 0\}.$$
$$\kappa=\frac{\left\|\dot{\vec{r}}\times\ddot{\vec{r}}\right\|}{\left\|\dot{\vec{r}}\right\|^3}=\frac{a}{a^2+b^2}$$
$$\tau=\frac{\left[\dot{\vec{r}},\ddot{\vec{r}},\dddot{\vec{r}}\right]}{\left\|\dot{\vec{r}}\times\ddot{\vec{r}}\right\|^2}=\frac{b}{a^2+b^2}$$

所以圆柱螺线的曲率和挠率都是常数.

例：证明空间曲线$\vec{r}=\vec{r}(s)$为直线的充要条件是曲率$\kappa(s)=0$

证明：若为直线,设其方程 $\vec{r}(s)=s\vec{e}+\vec{b}$, 其中 $\vec{e}$为单位方向向量. $\vec{b}$为直线上某定点的向径，$\|\vec{e}\|=1$，则:

$$\kappa(s)=\|\vec{r}''(s)\|=\|\vec{e}'\|=0$$

反之, 若 $\kappa(s)=0$, 则 $\kappa(s)=\|\vec{r}''(s)\|=0, \vec{r}''(s)=\vec{0}$

故 $\vec{r}=s\vec{a}+\vec{b}$,即该曲线是直线.

例：证明: 空间曲线 $\vec{r}=\vec{r}(s)$为平面曲线的充要条件是其上任一点处挠率$\kappa(s)=0$

参P128，例7.8

4. 求曲线 $\begin{cases} x+\operatorname{sh} x=y+\sin y, \\ z+e^z=x+1+\ln(x+1) \end{cases}$ 在点 $O(0,0,0)$ 处的曲率和 Frenet 标架.

$\begin{cases} x+\operatorname{sh}x=y+\sin y \\ z+e^z=x+1+\ln(x+1) \end{cases}$ 以x为参数. 该曲线可写成 $r=(x, y(x), z(x))$

对x求导. $\begin{cases} 1+\operatorname{ch}x=\dot y(x)+\cos y\cdot \dot y(x) = (1+\cos y)\dot y(x) \\ \dot z(x)+e^z\dot z(x)=1+\frac{1}{x+1} \end{cases}$

$\dot r(x)=(\dot x(x)\ \dot y(x)\ \dot z(x))$, $\dot r(0)=(1,1,1)$

$\begin{cases} \dot x=1 \\ \dot y=\frac{1+\operatorname{ch}x}{1+\cos y} \\ \dot z=\frac{2+x}{(1+x)(1+e^z)} \end{cases}$

$T(0)=\frac{\dot r(0)}{\|\dot r(0)\|}=\frac{1}{\sqrt3}(1,1,1)$

再对x求导.

$\begin{cases} \operatorname{sh}x=-\sin y\cdot(\dot y)^2+(1+\cos y)\ddot y \\ e^z(\dot z)^2+(1+e^z)\ddot z=-\frac{1}{(x+1)^2} \end{cases}$

$\ddot y(0)=0$

$1+2\ddot z=-1$, $\ddot z(0)=-1$

$\ddot r(x)=(0,\ddot y(x),\ddot z(x))$, $\ddot r(0)=(0,0,-1)$

该曲线在O(0,0,0)的曲率 $k=\frac{\|\dot r(0)\times\ddot r(0)\|}{\|\dot r(0)\|^3}=\frac{\left\|\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix}\right\|}{(\sqrt3)^3}=\frac{\|(-1,1,0)\|}{3\sqrt3}=\frac{\sqrt6}{9}$

$B(0)=\frac{\dot r(0)\times\ddot r(0)}{\|\dot r(0)\times\ddot r(0)\|}=\frac{(-1,1,0)}{\sqrt2}=\frac{1}{\sqrt2}(-1,1,0)$. $N(0)=B(0)\times T(0)=\begin{vmatrix} i & j & k \\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{vmatrix}=\left(\frac{1}{\sqrt6},\frac{1}{\sqrt6},\frac{-2}{\sqrt6}\right)$.