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T1 ________________	**52888**	F1 ________________
T2 ________________		F2 ________________
T3 ________________	Problem Chosen	F3 ________________
T4 ________________	**A**	F4 ________________

Mathematical Contest in Modeling (MCM/ICM) Summary Sheet

Summary

It's pleasant to go home to take a bath with the evenly maintained temperature of hot water throughout the bathtub. This beautiful idea, however, can not be always realized by the constantly falling water temperature. Therefore, people should continually add hot water to keep the temperature even and as close as possible to the initial temperature without wasting too much water. This paper proposes a partial differential equation of the heat conduction of the bath water temperature, and an object programming model. Based on the Analytic Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), this paper illustrates the best strategy the person in the bathtub can adopt to satisfy his desires.

First, a spatiotemporal partial differential equation model of the heat conduction of the temperature of the bath water is built. According to the priority, an object programming model is established, which takes the deviation of temperature throughout the bathtub, the deviation of temperature with the initial condition, water consumption, and the times of switching faucet as the four objectives. To ensure the top priority objective—homogenization of temperature, the discretization method of the Partial Differential Equation model (PDE) and the analytical analysis are conducted. The simulation and analytical results all imply that the top priority strategy is: The proper motions of the person making the temperature well-distributed throughout the bathtub. Therefore, the Partial Differential Equation model (PDE) can be simplified to the ordinary differential equation model.

Second, the weights for the remaining three objectives are determined based on the tolerance of temperature and the hobby of the person by applying Analytic Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS). Therefore, the evaluation model of the synthesis score of the strategy is proposed to determine the best one the person in the bathtub can adopt. For example, keeping the temperature as close as the initial condition results in the fewer number of switching faucet while attention to water consumption gives rise to the more number.

Third, the paper conducts the analysis of the diverse parameters in the model to determine the best strategy, respectively, by controlling the other parameters constantly, and adjusting the parameters of the volume, shape of the bathtub and the shape, volume, temperature and the motions and other parameters of the person in turns. All results indicate that the differential model and the evaluation model developed in this paper depends upon the parameters therein. When considering the usage of a bubble bath additive, it is equal to be the obstruction between water and air. Our results show that this strategy can reduce the dropping rate of the temperature

effectively, and require fewer number of switching.

The surface area and heat transfer coefficient can be increased because of the motions of the person in the bathtub. Therefore, the deterministic model can be improved as a stochastic one. With the above evaluation model, this paper present the stochastic optimization model to determine the best strategy. Taking the disparity from the initial temperature as the suboptimum objectives, the result of the model reveals that it is very difficult to keep the temperature constant even wasting plentiful hot water in reality.

Finally, the paper performs sensitivity analysis of parameters. The result shows that the shape and the volume of the tub, different hobbies of people will influence the strategies significantly. Meanwhile, combine with the conclusion of the paper, we provide a one-page non-technical explanation for users of the bathtub.

Fall in love with your bathtub

Abstract

It's pleasant to go home to take a bath with the evenly maintained temperature of hot water throughout the bathtub. This beautiful idea, however, can not be always realized by the constantly falling water temperature. Therefore, people should continually add hot water to keep the temperature even and as close as possible to the initial temperature without wasting too much water. This paper proposes a partial differential equation of the heat conduction of the bath water temperature, and an object programming model. Based on the Analytic Hierarchy Process (AHP) and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), this paper illustrates the best strategy the person in the bathtub can adopt to satisfy his desires.

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Third, the paper conducts the analysis of the diverse parameters in the model to determine the best strategy, respectively, by controlling the other parameters constantly, and adjusting the parameters of the volume, shape of the bathtub and the shape, volume, temperature and the motions and other parameters of the person in turns. All results indicate that the differential model and the evaluation model developed in this paper depends upon the parameters therein. When considering the usage of a bubble bath additive, it is equal to be the obstruction between water and air. Our results show that this strategy can reduce the dropping rate of the temperature effectively, and require fewer number of switching.

The surface area and heat transfer coefficient can be increased because of the motions of the person in the bathtub. Therefore, the deterministic model can be improved as a stochastic one. With the above evaluation model, this paper present the stochastic optimization model to determine the best strategy. Taking the disparity from the initial temperature as the suboptimum objectives, the result of the model reveals that it is very difficult to keep the temperature constant even wasting plentiful hot

water in reality.

Finally, the paper performs sensitivity analysis of parameters. The result shows that the shape and the volume of the tub, different hobbies of people will influence the strategies significantly. Meanwhile, combine with the conclusion of the paper, we provide a one-page non-technical explanation for users of the bathtub.

Keywords: Heat conduction equation; Partial Differential Equation model (PDE Model); Objective programming; Strategy; Analytical Hierarchy Process (AHP)

Problem Statement

A person fills a bathtub with hot water and settles into the bathtub to clean and relax. However, the bathtub is not a spa-style tub with a secondary hearing system, as time goes by, the temperature of water will drop. In that conditions,

we need to solve several problems:(1) Develop a spatiotemporal model of the temperature of the bathtub water to determine the best strategy to keep the temperature even throughout the bathtub and as close as possible to the initial temperature without wasting too much water;(2) Determine the extent to which your strategy depends on the shape and volume of the tub, the shape/volume/temperature of the person in the bathtub, and the motions made by the person in the bathtub.(3)The influence of using bubble to model's results.(4)Give a one-page non-technical explanation for users that describes your strategy

General Assumptions

1.Considering the safety factors as far as possible to save water, the upper temperature limit is set to 45 ℃;

2.Considering the pleasant of taking a bath, the lower temperature limit is set to 33℃;

3. The initial temperature of the bathtub is 40℃.

Table 1

Model Inputs and Symbols

Symbols	Definition	Unit
T_0	Initial temperature of the Bath water	℃
T_∞	Outer circumstance temperature	℃
T	Water temperature of the bathtub at the every moment	℃
t	Time	h
x	X coordinates of an arbitrary point	m
y	Y coordinates of an arbitrary point	m
z	Z coordinates of an arbitrary point	m
α	Total heat transfer coefficient of the system	$W/(m^2 \cdot K)$

S_1	The surrounding-surface area of the bathtub	m^2
S_2	The above-surface area of water	m^2
H_1	Bathtub's thermal conductivity	$W/(m\cdot K)$
D	The thickness of the bathtub wall	m
H_2	Convection coefficient of water	$W/(m^2\cdot K)$
a	Length of the bathtub	m
b	Width of the bathtub	m
h	Height of the bathtub	m
V	The volume of the bathtub water	m^3
c	Specific heat capacity of water	$J/(kg\cdot ℃)$
ρ	Density of water	kg/m^3
$v(t)$	Flooding rate of hot water	m^3/s
T_r	The temperature of hot water	℃

Temperature Model

Basic Model

A spatio-temporal temperature model of the bathtub water is proposed in this paper. It is a four dimensional partial differential equation with the generation and loss of heat. Therefore the model can be described as the Thermal Equation.

The three-dimension coordinate system is established on a corner of the bottom of the bathtub as the original point. The length of the tub is set as the positive direction along the x axis, the width is set as the positive direction along the y axis, while the height is set as the positive direction along the z axis, as shown in figure 1.

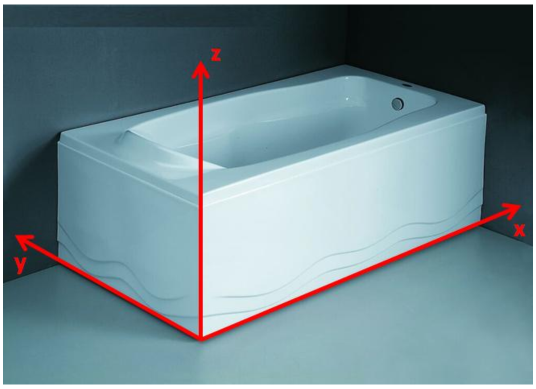

Figure 1. The three-dimension coordinate system

Temperature variation of each point in space includes three aspects: one is the natural heat dissipation of each point in space; the second is the addition of exogenous thermal energy; and the third is the loss of thermal energy. In this way, we build the Partial Differential Equation model as follows:

$$\frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{f_1(x,y,z,t)-f_2(x,y,z,t)}{c\rho V} \tag{1}$$

Where

- t **refers to time;**
- T **is the temperature of any point in the space;**
- f_1 **is the addition of exogenous thermal energy;**
- f_2 **is the loss of thermal energy.**

According to the requirements of the subject, as well as the preferences of people, the article proposes these following optimization objective functions. A precedence level exists among these objectives, while keeping the temperature even throughout the bathtub must be ensured.

Objective 1($O.1$): keep the temperature even throughout the bathtub;

$$F_1=\min\int_0^t t^2\left[\iiint_V T(x,y,z,t)dxdydz\right]dt-\left[\int_0^t t\left(\iiint_V T(x,y,z,t)dxdydz\right)dt\right]^2 \tag{2}$$

Objective 2($O.2$): keep the temperature as close as possible to the initial temperature;

$$F_2=\min\int_0^t\left(\iiint_V [T(x,y,z,t)-T_0]^2\,dxdydz\right)dt \tag{3}$$

Objective 3($O.3$): do not waste too much water;

$$F_3=\min\int_0^t v(t)\cdot dt \tag{4}$$

Objective 4($O.4$): fewer times of switching.

$$F_4=\min n \tag{5}$$

Since the $O.1$ is the most crucial, we should give priority to this objective. Therefore, the highest priority strategy is given here, which is homogenization of temperature.

Strategy 0 – Homogenization of Temperature

The following three reasons are provided to prove the importance of this strategy.

Reason 1-Simulation

In this case, we use grid algorithm to make discretization of the formula (1), and simulate the distribution of water temperature.

(1) Without manual intervention, the distribution of water temperature as shown in

figure 2. And the variance of the temperature is 0.4962 .

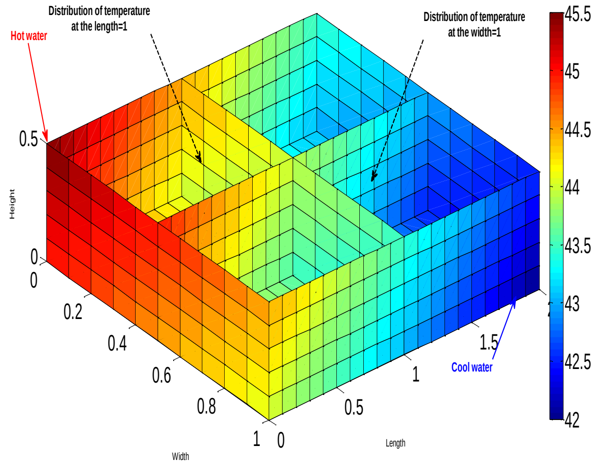

Figure 2. Temperature profiles in three-dimension space without manual intervention

(2) Adding manual intervention, the distribution of water temperature as shown in figure 3. And the variance of the temperature is 0.005.

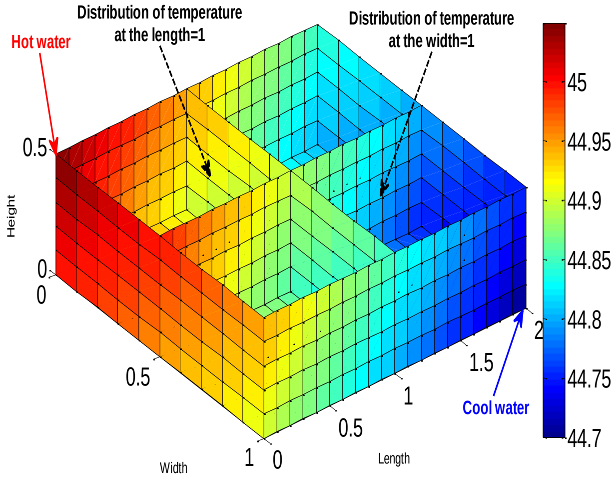

Figure 3. Temperature profiles in three-dimension space with manual intervention

Comparing figure 2 with figure 3, it is significant that the temperature of water will be homogeneous if we add some manual intervention. Therefore, we can assumed that $\alpha(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}) \neq 0$ in formula (1).

Reason 2-Estimation

If the temperature of any point in the space is different, then

$$\alpha(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}) \neq 0$$

Thus, we find two points (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) with:

$$T(x_1, y_1, z_1, t_1) \neq T(x_2, y_2, z_2, t_2)$$

Therefore, the objective function F_1 could be estimated as follows:

$$\int_0^t t^2 \left[\iiint_V T(x, y, z, t) dxdydz \right] dt - \left[\int_0^t t \left(\iiint_V T(x, y, z, t) dxdydz \right) dt \right]^2 \tag{6}$$
$$\geq \left[T(x_0, y_0, z_0, t_0) - T(x_1, y_1, z_1, t_1) \right]^2 > 0$$

The formula (6) implies that some motion should be taken to make sure that the temperature can be homogeneous quickly in general and $F_1 = 0$. So we can assumed that: $\alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) \neq 0$.

Reason 3-Analytical analysis

It is supposed that the temperature varies only on x axis but not on the y-z plane. Then a simplified model is proposed as follows:

$$\begin{cases} T_t = a^2 T_{xx} + A\sin\frac{\pi x}{l} & \left(0 \leq x \leq l\right)\left(0 \leq t\right) \\ T\left(0, t\right) = 0, T\left(l, t\right) = 0 & \left(0 \leq t\right) \\ T\left(x, 0\right) = 0 & \left(0 \leq x \leq l\right) \end{cases} \tag{7}$$

Then we use two ways, Fourier transformation and Laplace transformation, in solving one-dimensional heat equation [Qiming Jin 2012]. Accordingly, we get the solution:

$$T\left(x, t\right) = \frac{Al^2}{\pi^2 a^2}\left(1 - e^{-\pi^2 a^2 t / l^2}\right)\sin\frac{\pi x}{l} \tag{8}$$

Where $x \in \left(0, 2\right)$, $t > 0$, $T|_{x=0} = f_1\left(t\right)$ (assumed as a constant), $T|_{t=0} = T_0$.

Without general assumptions, we choose three specific value of t, and gain a picture containing distribution change of temperature in one-dimension space at different time.

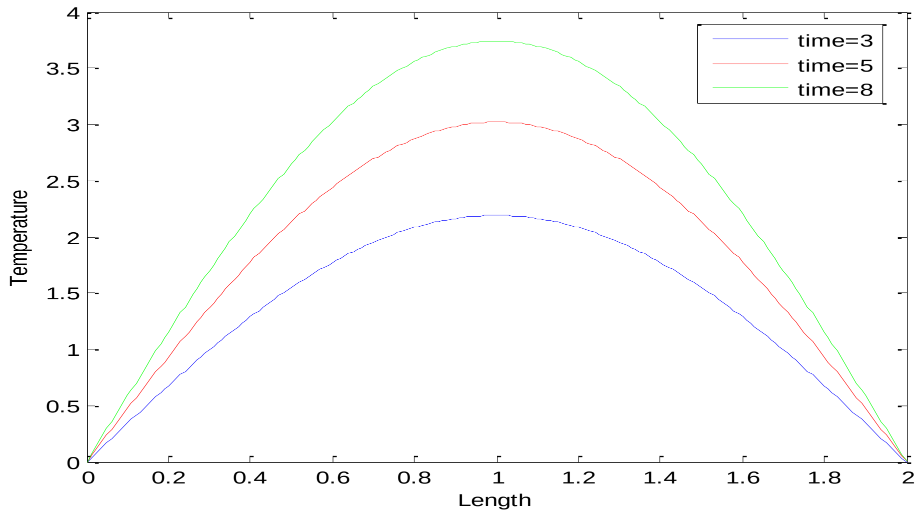

Figure 4. Distribution change of temperature in one-dimension space at different time

Table 2.

Variance of temperature at different time

t	3	5	8
variance	0.4640	0.8821	1.3541

It is noticeable in Figure 4 that temperature varies sharply in one-dimensional space. Furthermore, it seems that temperature will vary more sharply in three-dimension space. Thus it is so difficult to keep temperature throughout the bathtub that we have to take some strategies.

Based on the above discussion, we simplify the four dimensional partial differential equation to an ordinary differential equation. Thus, we take the first strategy that make some motion to meet the requirement of homogenization of temperature, that is $F_1 = 0$.

Results

Therefore, in order to meet the objective function, water temperature at any point in the bathtub needs to be same as far as possible. We can resort to some strategies to make the temperature of bathtub water homogenized, which is $\forall(x, y, z) \in \forall$. That is,

$$T(x, y, z, t) = T(t)$$

Given these conditions, we improve the basic model as temperature does not change with space.

$$\frac{dT}{dt} = \left[\left(\frac{H_1 S_1}{D} + H_2 S_2 + \mu_1\right)(T_\infty - T) + H_3 S_3 (T_p - T) + c\rho v (T_r - T) \right] / \rho c (V_1 - V_2) \quad (9)$$

Where

- μ_1 **is the intensity of people's movement;**
- H_3 **is convection between water and people;**

- **S_3 is contact area between water and people;**
- **T_p is body surface temperature;**
- **V_1 is the volume of the bathtub;**
- **V_2 is the volume of people.**

Where the μ refers to the intensity of people's movement. It is a constant. However, it is a random variable in reality, which will be taken into consideration in the following.

Model Testing

We use the oval-shaped bathtub to test our model. According to the actual situation, we give initial values as follows:

$$\lambda = 0.19,\ D = 0.03,\ H_2 = 0.54,\ T_\infty = 25,\ T_0 = 40$$

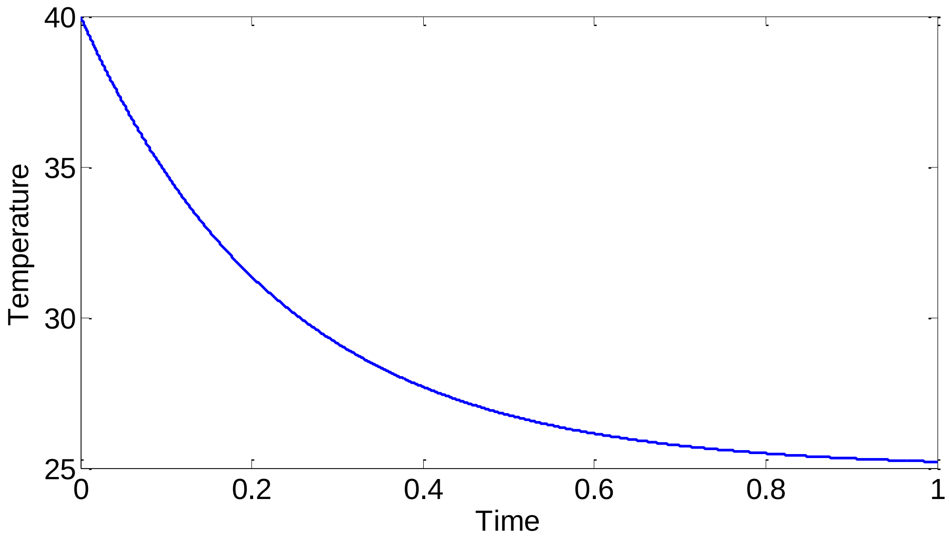

Figure 5. Basic model

The Figure 5 shows that the temperature decreases monotonously with time. And some signs of a slowing down in the rate of decrease are evident in the picture. Reaching about two hours, the water temperature does not change basically and be closely to the room temperature. Obviously, it is in line with the actual situation, indicating the rationality of this model.

Conclusion

Our model is robust under reasonable conditions, as can be seen from the testing above. In order to keep the temperature even throughout the bathtub, we should take some strategies like stirring constantly while adding hot water to the tub. Most important of all, this is the necessary premise of the following question.

Strategy 1 – Fully adapted to the hot water in the tub

Influence of body surface temperature

We select a set of parameters to simulate two kinds of situation separately.

The first situation is that do not involve the factor of human

$$\frac{dT}{dt}=\left[(\frac{H_1S_1}{D}+H_2S_2)(T_\infty-T)\right]/\rho cV \tag{10}$$

The second situation is that involves the factor of human

$$\frac{dT}{dt}=\left[(\frac{H_1S_1}{D}+H_2S_2+\mu_1)(T_\infty-T)+H_3S_3(T_p-T)\right]/\rho c(V_1-V_2) \tag{11}$$

According to the actual situation, we give specific values as follows, and draw a graph of temperature of two functions.

$$T_p=33,\ T_0=40$$

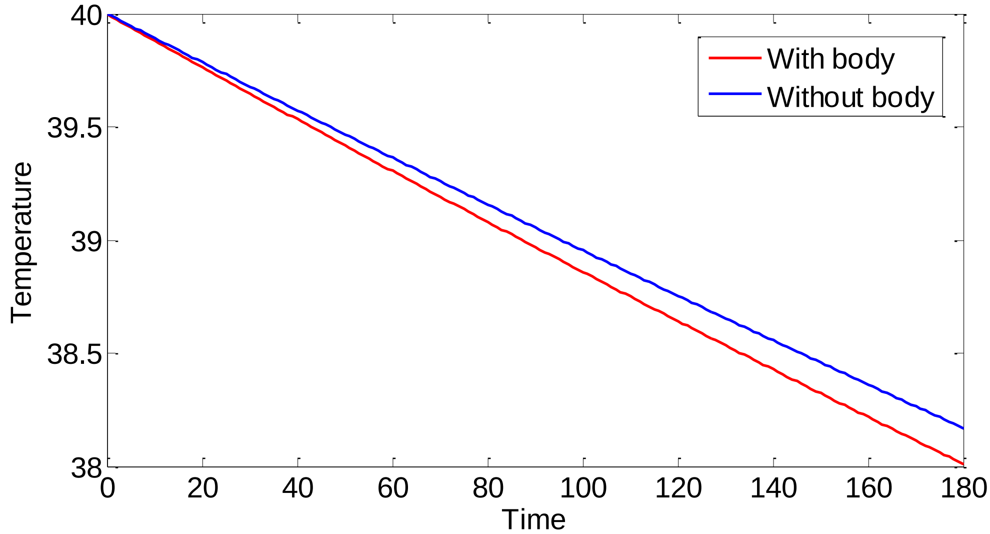

Figure 6a. Influence of body surface temperature

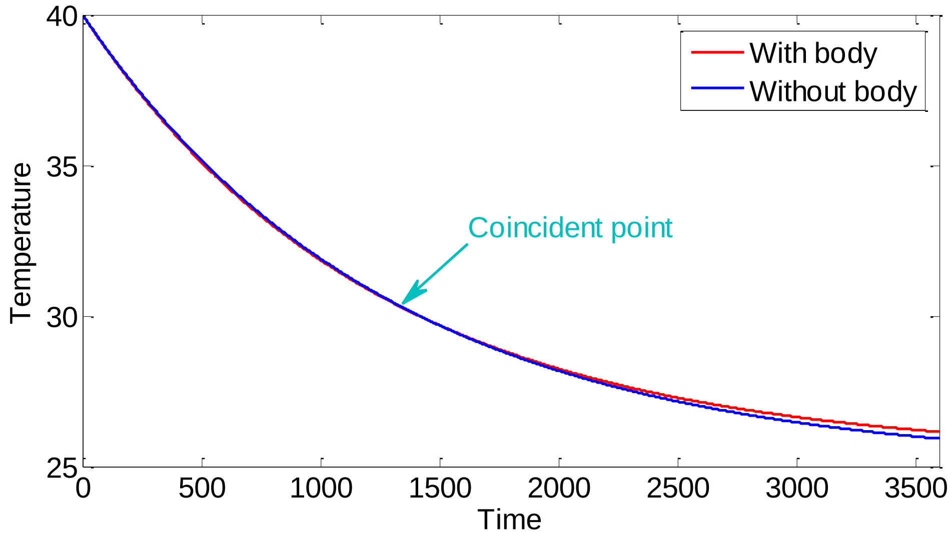

Figure 6b. Influence of body surface temperature

The figure 6 shows the difference between two kinds of situation in the early time (**before the coincident point**), while the figure 7 implies that the influence of body surface temperature reduces as time goes by. Combing with the degree of comfort of bath and the factor of health, we propose the second optimization strategy: Fully adapted to the hot water after getting into the bathtub.

Strategy 2 –Adding water intermittently

Influence of adding methods of water

There are two kinds of adding methods of water. One is the continuous; the other is the intermittent. We can use both different methods to add hot water.

$$\frac{dT}{dt}=\left[(\frac{H_1S_1}{D}+H_2S_2+\mu_1)(T_\infty-T)+c\rho v(T_r-T)\right]/\rho c(V_1-V_2) \tag{12}$$

Where T_r is the temperature of the hot water.

To meet $O.3$, we calculated the minimum water consumption by changing the flow rate of hot water. And we compared the minimum water consumptions of the continuous with the intermittent to determine which method is better.

A. Adding water continuously

According to the actual situation, we give specific values as follows and draw a picture of the change of temperature.

$$T_0=40,\quad T_d=37,\quad T_r=45$$

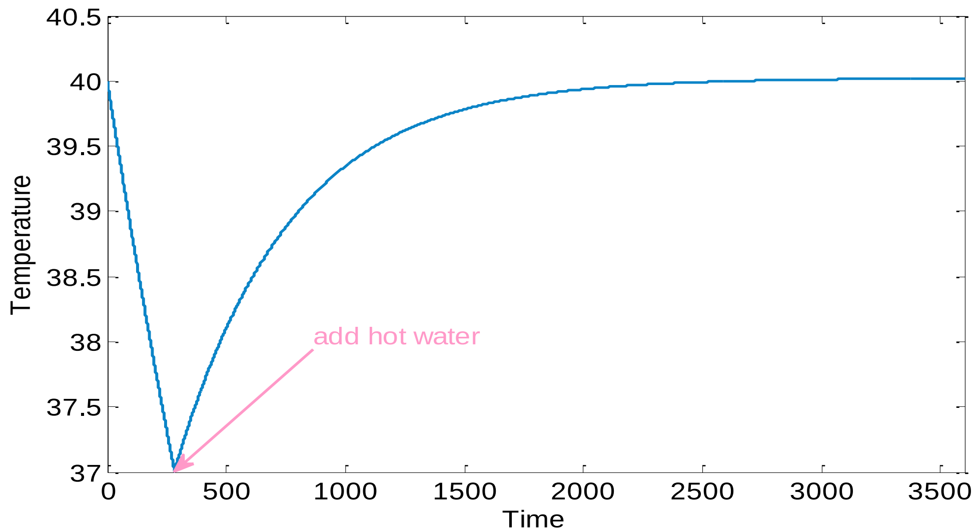

Figure 7. Adding water continuously

In most cases, people are used to have a bath in an hour. Thus we consumed that deadline of the bath: $t_{final}=3600$. Then we can find the best strategy in Figure 5 which is listed in Table 2.

Table 3

Strategy of adding water continuously

t_{start}	t_{final}	Δt	v	T_r	variance	Water flow
4 min	1 hour	56 min	$7.4\times10^{-5}\ m^3/s$	45℃	1.84×10^3	0.2455 m^3

B. Adding water intermittently

Maintain the values of T_0, T_d, T_r, v, we change the form of adding water, and get another graph.

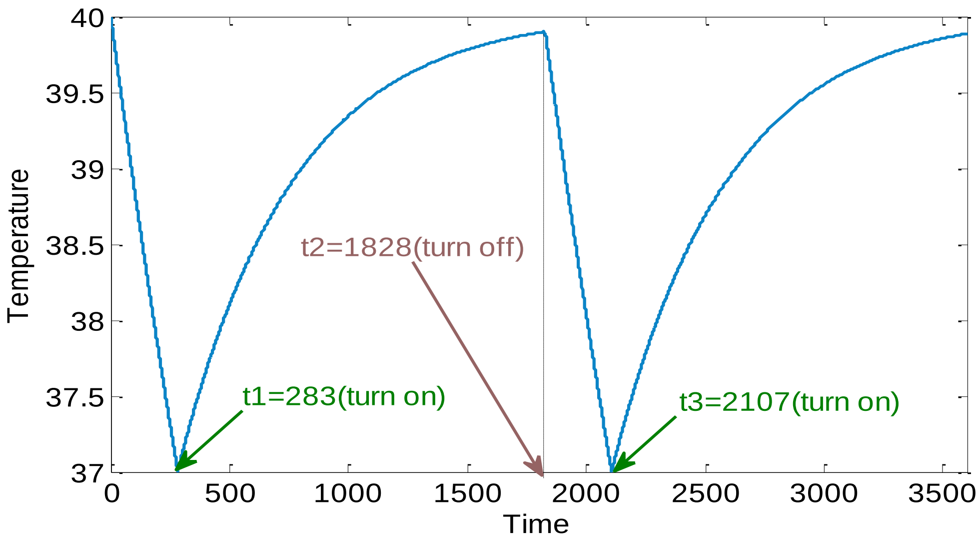

Figure 8. Adding water intermittently

Table 4.

Strategy of adding water intermittently

$t_1(on)$	$t_2(off)$	$t_3(on)$	v	T_r	variance	Water flow
5 min	30 min	35min	$7.4\times10^{-5}\ m^3/s$	45℃	3.6×10^3	0.2248 m^3

Conclusion

Different methods of adding water can influence the variance, water flow and the times of switching. Therefore, we give heights to evaluate comprehensively the methods of adding hot water on the basis of different hobbies of people. Then we build the following model:

$$\begin{cases} F_1 = \int_0^{3600} \left(T(t) - T_0\right)^2 dt \\ F_2 = \sum_{i=1}^{n} \int_{t_{2i-1}}^{t_{2i}} v(t)\, dt \\ F_3 = n \end{cases} \tag{13}$$

$$F = \min\left(w_1F_1 + w_2F_2 + w_3F_3\right) \tag{14}$$

$$s.t. \begin{cases} t_1 > 3\min \\ 5 \le t_{2i+1} - t_{2i} \le 10\min \end{cases}$$

Evaluation on Strategies

For example: Given a set of parameters, we choose different values of v and T_d, and gain the results as follows.

Method 1- *AHP*

Step 1: Establish hierarchy model

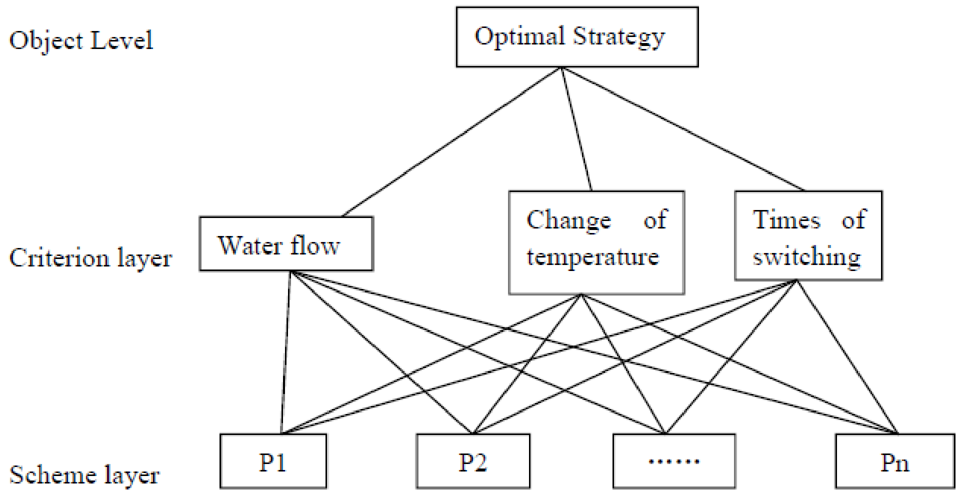

Figure 9. Establish hierarchy model

Step 2: Structure judgment matrix

$$A=\begin{bmatrix} 1 & 5 & 3 \\ 1/5 & 1 & 3 \\ 1/3 & 1/3 & 1 \end{bmatrix}$$

Step 3: Assign weight

w_1	w_2	w_3
0.65	0.22	0.13

Method 2-Topsis

Step1 ： Create an evaluation matrix consisting of m alternatives and n criteria, with the intersection of each alternative and criteria given as x_{ij} we therefore have a matrix

Step2：The matrix $(x_{ij})_{m\times n}$ is then normalised to form the matrix $R=(r_{ij})_{m\times n}$, using the normalisation method

$$r_{ij}=\frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}} \quad , \quad i=1,2,\cdots\cdots,n;\ \ j=1,2,\cdots\cdots,m$$

Step3: Calculate the weighted normalised decision matrix

$$T = \left(t_{ij}\right)_{m\times n} = (w_j r_{ij})_{m\times n} \ , \ i = 1,2,\cdots,m$$

where $w_j = W_j \Big/ \sum_{j=1}^{n} W_j$, $j = 1,2,\cdots,n$

so that $\sum_{j=1}^{n} w_j = 1$, and w_j is the original weight given to the indicator v_j , $j = 1,2,\cdots,n$.

Step 4: Determine the worst alternative $\left(A_w\right)$ and the best alternative $\left(A_b\right)$

$$A_w = \left\{\left\langle \max\left(t_{ij} \,|i = 1,2,\cdots,m\right) | \, j \in J_-\right\rangle, \left\langle \min\left(t_{ij} \,|i = 1,2,\cdots,m\right) | \, j \in J_+\right\rangle\right\} = \left\{t_{wj} \,|\, j = 1,2,\cdots,\mathrm{n}\right\},$$
$$A_b = \left\{\left\langle \min\left(t_{ij} \,|i = 1,2,\cdots,m\right) | \, j \in J_-\right\rangle, \left\langle \max\left(t_{ij} \,|i = 1,2,\cdots,m\right) | \, j \in J_+\right\rangle\right\} = \left\{t_{bj} \,|\, j = 1,2,\cdots,\mathrm{n}\right\},$$

where, $J_+ = \left\{ j = 1,2,\cdots,n \middle| j \right\}$ associated with the criteria having a positive impact,

and $J_- = \left\{ j = 1,2,\cdots,n \middle| j \right\}$ associated with the criteria having a negative impact.

Step 5: Calculate the L2-distance between the target alternative i and the worst condition A_w

$$d_{iw} = \sqrt{\sum_{j=1}^{n}\left(t_{ij} - t_{wj}\right)^2} \ , \ i = 1,2,\cdots,\mathrm{m}$$

and the distance between the alternative i and the best condition A_b

$$d_{ib} = \sqrt{\sum_{j=1}^{n}\left(t_{ij} - t_{bj}\right)^2} \ , \ i = 1,2,\cdots,\mathrm{m}$$

where d_{iw} and d_{ib} are L2-norm distances from the target alternative i to the worst and best conditions, respectively.

Step 6 : Calculate the similarity to the worst condition

Step 7 : Rank the alternatives according to $s_{iw}\left(i = 1,2,\cdots,m\right)$

Step 8 : Assign weight

w_1	w_2	w_3
0.55	0.17	0.23

Conclusion

AHP gives height subjectively while TOPSIS gives height objectively. And the heights are decided by the hobbies of people. However, different people has different hobbies, we choose AHP to solve the following situations.

Impact of parameters

Different customers have their own hobbies. Some customers prefer enjoying in the bath, so the $O.2$ is more important . While other customers prefer saving water, the $O.3$ is more important. Therefore, we can solve the problem on basis of APH .

1. Customers who prefer enjoying: $w_2 = 0.83$, $w_3 = 0.17$

According to the actual situation, we give initial values as follows:

$$S_1 = 3,\ V_1 = 1,\ S_2 = 1.4631,\ V_2 = 0.05,\ T_p = 33,\ \mu_1 = 10$$

Ensure other parameters unchanged, then change the values of these parameters including $S_1, V_1, S_2, V_2, T_d, \mu_1$. So we can obtain the optimal strategies under different conditions in Table 4.

Table 5.

Optimal strategies under different conditions

Transform	Best strategy						Results	
	$v(t)$	T_d	t_1 (on)	t_2 (off)	t_3 (on)	t_4 (off)	F_2	F_3
Initial settings	7.4×10^{-5}	38	181	3218	3401	3600	0.12306	0.94049
change S_1 to 2.5	7.5×10^{-5}	38	182	1785	1969	3570	0.15597	0.94463
change V_1 to 0.7	7.4×10^{-5}	37	193	2403	2599	3600	0.24299	0.91887
change S_2 to 0.963	7.4×10^{-5}	38	182	3219	3402	3600	0.12297	0.94046
change V_2 to 0.07	7.4×10^{-5}	37	276	3440	3600	3600	0.15184	0.91686
change T_p to 37	7.5×10^{-5}	38	184	1786	1970	3571	0.15540	0.94462
change μ_1 to 50	7.5×10^{-5}	37	277	3600	3600	3600	0.14522	0.94154

2.Customers who prefer saving: $w_2 = 0.17$, $w_3 = 0.83$

Just as the former, we give the initial values of these parameters including S_1 , V_1 , S_2 , V_2 , T_d , μ_1 , then change these values in turn with other parameters unchanged. So we can obtain the optimal strategies as well in these conditions.

Table 6.

Optimal strategies under different conditions

Transform	Best strategy						Results	
	$v(t)$	T_d	t_1 (on)	t_2 (off)	t_3 (on)	t_4 (off)	F_2	F_3
Initial settings	1.15×10^{-4}	33	798	1392	2199	2790	1	0.50290
change S_1 to 2.5	1.15×10^{-4}	33	802	1394	2205	2797	0.99836	0.50119
change V_1 to 0.7	8.0×10^{-5}	33	546	1545	2097	3096	0.97688	0.60188
change S_2 to 0.963	1.15×10^{-4}	33	802	1396	2203	2797	1	0.50298
change V_2 to 0.07	1.11×10^{-4}	33	781	1401	2191	2811	1	0.50773
change T_p to 37	1.15×10^{-4}	33	812	1406	2213	2807	0.99501	0.50313
change μ_1 to 50	1.13×10^{-4}	33	783	1405	2196	2818	0.99803	0.94154

Influence of bubble

Using the bubble bath additives is equivalent to forming a barrier between the bath water and air, thereby slowing the falling velocity of water temperature. According to the reality, we give the values of some parameters and gain the results as follows:

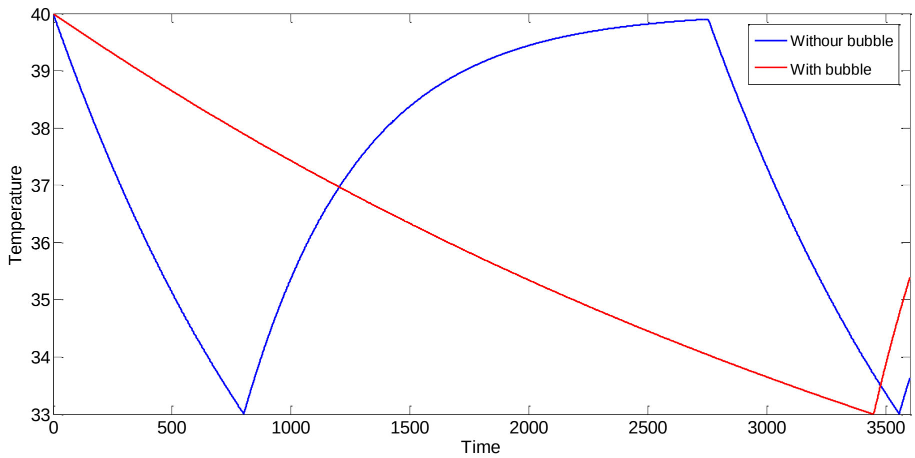

Figure 10. Influence of bubble

Table 7.

Strategies (influence of bubble)

Situation	Dropping rate of temperature (the larger the number, the slower)	Disparity to the initial temperature	Water flow	Times of switching
Without bubble	802	1.4419	0.1477	4
With bubble	3449	9.8553	0.0112	2

The Figure 10 and the Table 7 indicates that adding bubble can slow down the dropping rate of temperature effectively. It can decrease the disparity to the initial temperature and times of switching, as well as the water flow.

Improved Model

In reality, human's motivation in the bathtub is flexible, which means that the parameter μ_1 is a changeable measure. Therefore, the parameter can be regarded as a random variable, written as $\mu_1(t) = random[10,50]$. Meanwhile, the surface of water will come into being ripples when people moves in the tub, which will influence the parameters like S_1 and S_2. So, combining with reality, we give the range of values as follows:

$$\begin{cases} S_1(t) = random[S_1, 1.1S_1] \\ S_2(t) = random[S_2, 1.1S_2] \end{cases}$$

Combined with the above model, the improved model is given here:

$$\begin{cases} \frac{dT}{dt}=\left[(\frac{H_1S_1}{D}+H_2S_2+\mu_1)(T_\infty - T)+c\rho v(T_a - T) \right] / \rho c(V_1 - V_2) \\ \mu_1(t)=random[10,50] \quad S_1(t)=random[S_1,1.1S_1] \quad S_2(t)=random[S_2,1.1S_2] \end{cases} \tag{15}$$

Given the values, we can get simulation diagram:

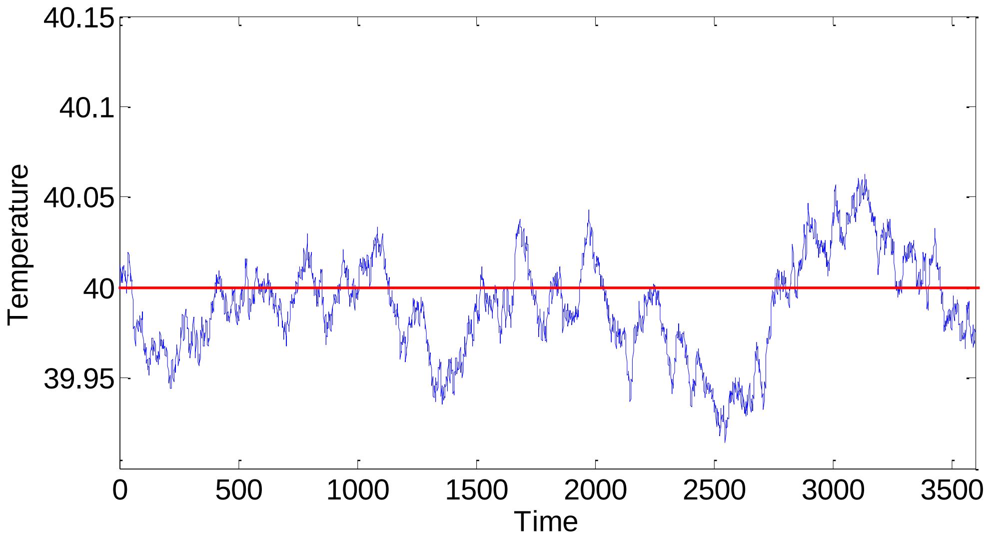

Figure 11. Improved model

The figure shows that the variance is small while the water flow is large, especially the variance do not equals to zero. This indicates that keeping the temperature of water is difficult though we regard $O.2$ as the secondary objective.

Sensitivity Analysis

Some parameters have a fixed value throughout our work. By varying their values, we can see their impacts.

Impact of the shape of the tub

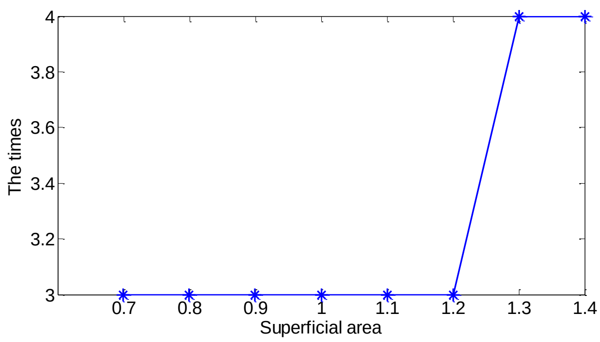

Figure 12a. Times of switching

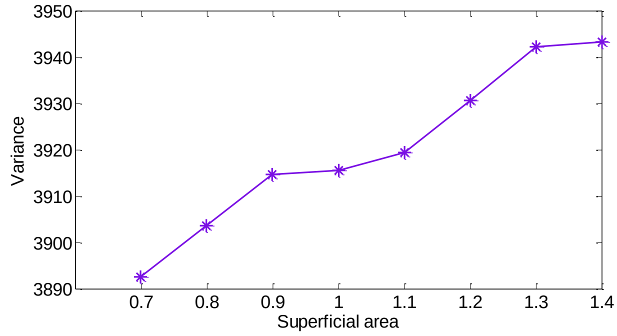

Figure 12b. Variance of temperature

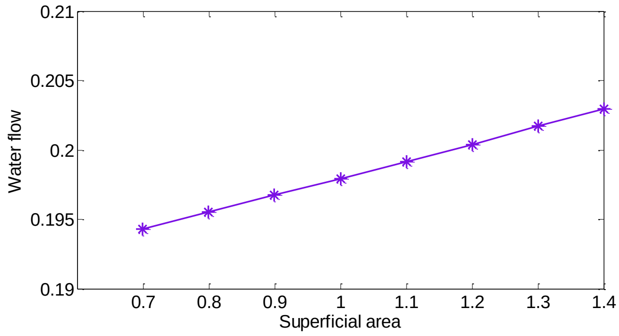

Figure 12c. Water flow

By varying the value of some parameters, we can get the relationships between the shape of tub and the times of switching, variance of temperature, and water flow et.

It is significant that the three indexes will change as the shape of the tub changes. Therefore the shape of the tub makes an obvious effect on the strategies. It is a sensitive parameter.

Impact of the volume of the tub

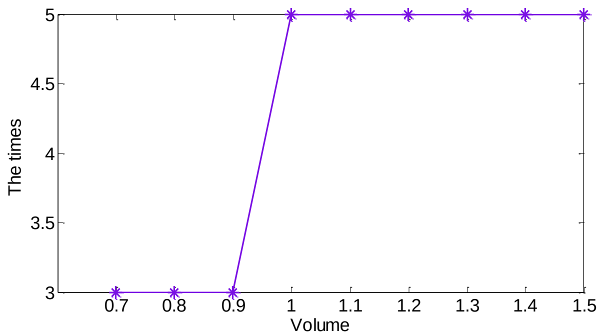

Figure 13a. Times of switching

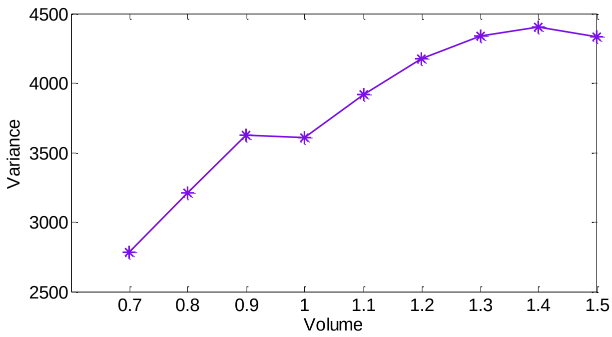

Figure 13b. Variance of temperature

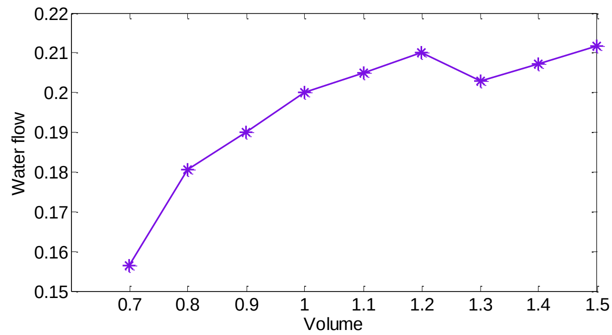

Figure 13c. Water flow

By varying the value of some parameters, we can get the relationships between the volume of tub and the times of switching, variance of temperature, and water flow et.

It is significant that the three indexes will change as the volume of the tub changes. Therefore the shape of the tub makes an obvious effect on the strategies. It is a sensitive parameter.

Impact of different hobbies

Different people have different hobbies, which will influence the height. Then different strategies are given as follows:

Table 8.

Strategies of different hobbies

variance	water flow	T_d	v	t_1 (on)	t_2 (off)	t_3 (on)	t_4 (off)	t_5 (on)
0.023	0.181	35	8.4×10^{-5}	9 min	27 min	35 min	53 min	1 hour
0.026	0.176	35	8.5×10^{-5}	9 min	26 min	35 min	52 min	1 hour
0.556	0.237	36	7.4×10^{-5}	7 min	60 min	60 min	1 hour	1 hour
0.558	0.211	36	7.5×10^{-5}	7 min	41 min	48 min	1 hour	1 hour
0.841	0.230	37	7.6×10^{-5}	5 min	31 min	36 min	1 hour	1 hour
0.842	0.218	37	7.7×10^{-5}	5 min	28 min	33 min	57 min	1 hour
0.959	0.239	38	7.4×10^{-5}	3 min	54 min	57 min	1 hour	1 hour
0.960	0.243	38	7.5×10^{-5}	3 min	32 min	35 min	1 hour	1 hour
0.995	0.253	39	7.4×10^{-5}	1.5 min	47 min	48 min	1 hour	1 hour

We can see in Table that the strategies changes sharply under different heights. Therefore the hobby makes an obvious effect on the strategies. It is a sensitive parameter.

Strengths and Weaknesses

Strengths

1. The model is established on the basis of the physical formula of the thermal conductivity, with a strong theoretical support.
2. The model is close to the life, which can combine with the actual situation to solve the issues.
3. The model is comprehensive. It includes many factors, and discuss the effect to strategy, which makes the model feasible.

Weaknesses

1. The random factor of the model can't be control.
2. To meet the conditions, the person should do motion all time.

Reference

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[2] Warzoha R J, Fleischer A S. Effect of graphene layer thickness and mechanical compliance on interfacial heat flow and thermal conduction in solid-liquid phase change materials.[J]. Acs Applied Materials & Interfaces, 2014, 6(15):12868-12876.
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Non-technical letter

Dear customer：

Do you want to have a bath after a busy day? Recently, people are used to having a bath to relax. However, it is difficult for us to relax well because of the constantly dropping of the water temperature. This instruction will help you control the temperature of water in bath more flexibly. A series of strategies are written as follows:

1. You can choose the initial temperature of water which depends on yourself and your family members.
2. The temperature is higher where close to the faucet but lower on the other side, so you can choose appropriate place to find the most comfortable temperature.
3. If you do not like the fluctuation of water temperature changes too big, you can turn on switch, let hot water turn into bathtub constantly, but this strategy will consume too much water.
4. If you want to keep the fluctuation of water temperature, besides, you want to save water, you can adopt the way of add hot water discontinuously, just turn on switch when you feel cool and turn off switch when you feel hot.
5. Shell temperature in winter is lower than in summer ,at the meantime ,indoor temperature is also lower than in summer, so when you take a bath in winter， the speed of the dropping of temperature will increase, so the frequency of add hot water can higher than in summer.
6. When you wash， you can increase the temperature/time of hot water properly due to too large of motion amplitude; when you relax, you can decrease the temperature/time of hot water appropriately.
7. When you child wash, you could increase the temperature/time of hot water properly because she/he is energetic.

The reason about why the bathtub water temperature is so difficult to keep balance:

1. It takes much time for the thermal energy to transform all around in the bathtub due to the big volume of bathtub.
2. Human motion is random, unfixed when people in bath, while the addition of hot water is fixed, so heat dissipation is not equal to heating, The water temperature is not uniform.