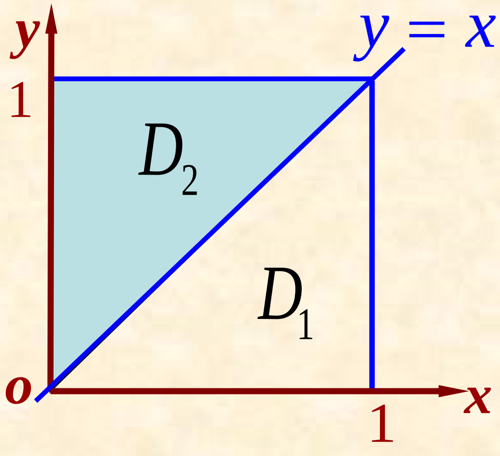
Double Integrals

习题选讲

1. 计算二重积分$\iint\limits_D e^{\max\{x^2,y^2\}}dxdy$, 其中$D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 1\}$.

分析：利用重积分的可加性 令$x^2=y^2 \Rightarrow y=x$

将D分为两个区域$\begin{cases} D_1=\{(x,y)\mid 0\le x\le 1, 0\le y\le x\} \\ D_2=\{(x,y)\mid 0\le x\le 1, x\le y\le 1\} \end{cases}$



$$I=\iint\limits_{D_1} e^{\max\{x^2,y^2\}}dxdy+\iint\limits_{D_2} e^{\max\{x^2,y^2\}}dxdy$$

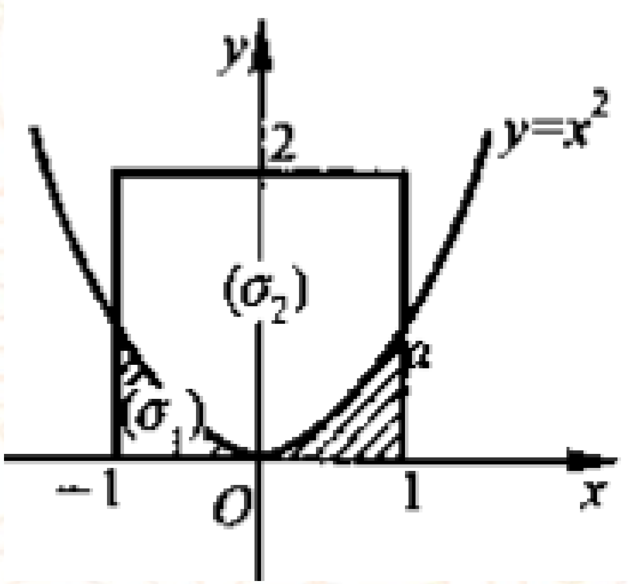
$$=\iint\limits_{D_1} e^{x^2}dxdy+\iint\limits_{D_2} e^{y^2}dxdy$$

$$=\int_0^1 dx\int_0^x e^{x^2}dy+\int_0^1 dy\int_0^y e^{y^2}dx$$

$$=e-1$$

P161 (B) **1.(1)计算二重积分** $\iint\limits_{(\sigma)}\sqrt{|y-x^2|}d\sigma,(\sigma)=\{(x,y)\,|\,|x|\leqslant 1,0\leqslant y\leqslant 2\}$

解:如图,将 (σ) **分为两个区域** (σ_1) **及** (σ_2)**,则**



$$\iint\limits_{(\sigma)}\sqrt{|y-x^2|}d\sigma$$

$$=\iint\limits_{(\sigma_1)}\sqrt{x^2-y}\,\mathrm{d}\sigma+\iint\limits_{(\sigma_2)}\sqrt{y-x^2}\,\mathrm{d}\sigma$$

$$=2\int_0^1\mathrm{d}x\int_0^{x^2}\sqrt{x^2-y}\,\mathrm{d}y+2\int_0^1\mathrm{d}x\int_{x^2}^{2}\sqrt{y-x^2}\,\mathrm{d}y$$

$$=\frac{5}{3}+\frac{\pi}{2}$$

$f(x,y)$带绝对值的积分

形如 $a.\iint\limits_D |f(x,y)|d\sigma$;　$b.\iint\limits_D \max\{f(x,y),g(x,y)\}d\sigma$;

$c.\iint\limits_D [f(x,y)]d\sigma$　$d.\iint\limits_D \min\{f(x,y),g(x,y)\}d\sigma$;

等的被积函数均应看作分区域函数来对待,利用积分的可加性分区域积分求和.

练习：计算二重积分$\iint\limits_D |x^2+y^2-1|\mathrm{d}\sigma$,

其中$D=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 1\}$.

答案：$\dfrac{\pi}{4}-\dfrac{1}{3}$

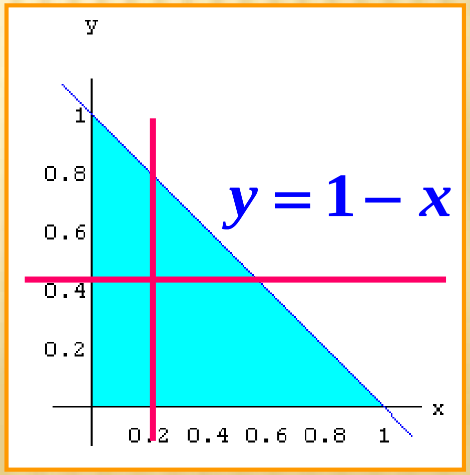
3. 改变积分 $\int_0^1 dx \int_0^{1-x} f(x,y)dy$ 的次序.

☑ (A) $\int_0^1 dy \int_0^{1-y} f(x,y)dx$

(B) $\int_0^{1-x} dy \int_0^1 f(x,y)dx$

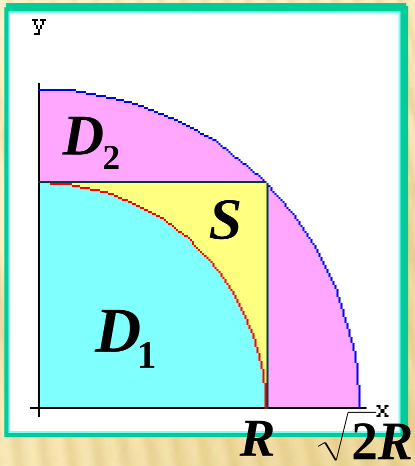
(C) $\int_0^1 dy \int_0^{1-x} f(x,y)dx$

解 积分区域如图



原式 $= \int_0^1 dy \int_0^{1-y} f(x,y)dx.$

4. 求广义积分 $\int_0^{+\infty} e^{-x^2} \quad d$.



解 $D_1 = \{(x,y) \mid x^2 + y^2 \le R^2\}$

$D_2 = \{(x,y) \mid x^2 + y^2 \le 2R^2\}$

$S = \{(x,y) \mid 0 \le x \le R, 0 \le y \le R\}$

$\{x \ge 0, y \ge 0\}$ 显然有 $D_1 \subset S \subset D_2$

$\because \quad e^{-x^2-y^2} > 0,$

$$\therefore \quad \iint_{D_1} e^{-x^2-y^2} dxdy \le \iint_{S} e^{-x^2-y^2} dxdy \le \iint_{D_2} e^{-x^2-y^2} dxdy.$$

$$I=\iint\limits_{S} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y=\int_0^R e^{-x^2}dx\int_0^R e^{-y^2}dy=\left(\int_0^R e^{-x^2}dx\right)^2;$$

$$I_1=\iint\limits_{D_1} e^{-x^2-y^2}dxdy=\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^R e^{-\rho^2}\rho\mathrm{d}\rho=\frac{\pi}{4}(1-e^{-R^2});$$

同理可得 $I_2=\iint\limits_{D_2} e^{-x^2-y^2}dxdy=\frac{\pi}{4}(1-e^{-2R^2});$

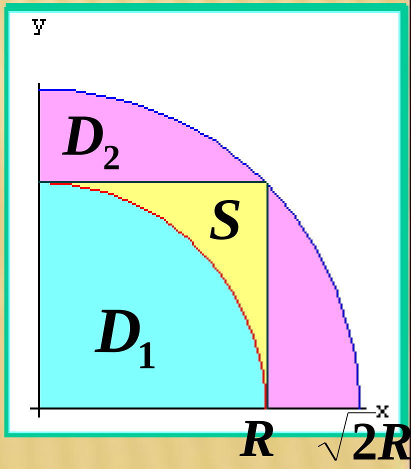
因为 $I_1<I<I_2$,

即 $\frac{\pi}{4}\left(1-e^{-R^2}\right)<\left(\int_0^R e^{-x^2}dx\right)^2<\frac{\pi}{4}\left(1-e^{-2R^2}\right);$

当$R\to\infty$时, $I_1\to\frac{\pi}{4}$, $I_2\to\frac{\pi}{4}$,

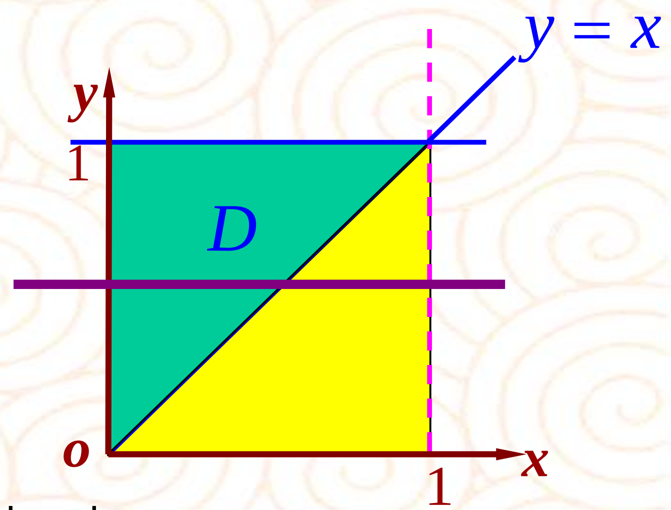
因此当$R\to\infty$时, $I\to\frac{\pi}{4}$, 即$\left(\int_0^{\infty} e^{-x^2}dx\right)^2=\frac{\pi}{4}$,

所求广义积分 $\int_0^{\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.



5. 设 $f(x)$ 在 $[0,1]$ 上连续，并设 $\int_0^1 f(x)dx = A$，

求 $\int_0^1 dx \int_x^1 f(x)f(y)dy$.



解1 令 $I = \int_0^1 dx \int_x^1 f(x)f(y)dy$，

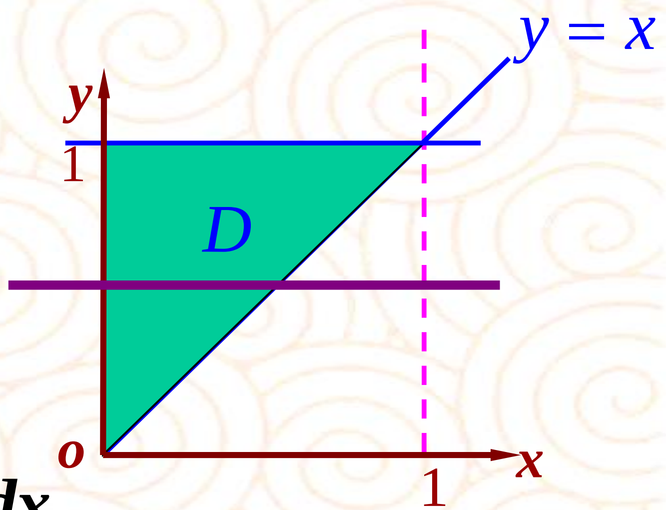
作变换 $\begin{cases} x = y \\ y = x \end{cases}$ 则 $|J| = \left|\dfrac{\partial(x,y)}{\partial(x,y)}\right| = \left\|\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right\| = |-1| = 1$,

$$I = \int_0^1 dy \int_y^1 f(y)f(x)dx$$

故 $2I = \int_0^1 \mathrm{d}x \int_x^1 f(x)f(y)\mathrm{d}y + \int_0^1 \mathrm{d}y \int_y^1 f(x)f(y)\mathrm{d}x$

$$= \int_0^1 dx \int_0^1 f(x)f(y)dy = A^2. \qquad \therefore I = \frac{A^2}{2}$$

5. 设 $f(x)$ 在 $[0,1]$ 上连续，并设 $\int_0^1 f(x)dx = A$，求 $\int_0^1 dx \int_x^1 f(x)f(y)dy$.



解2　设 $F'(x) = f(x)$，

则 $\int_0^1 f(x)dx = F(1) - F(0) = A$

$$\int_0^1 dx \int_x^1 f(x)f(y)dy = \int_0^1 f(x)\left[F(1) - F(x)\right]dx$$

$$= AF(1) - \frac{1}{2}F^2(x)\Big|_0^1$$

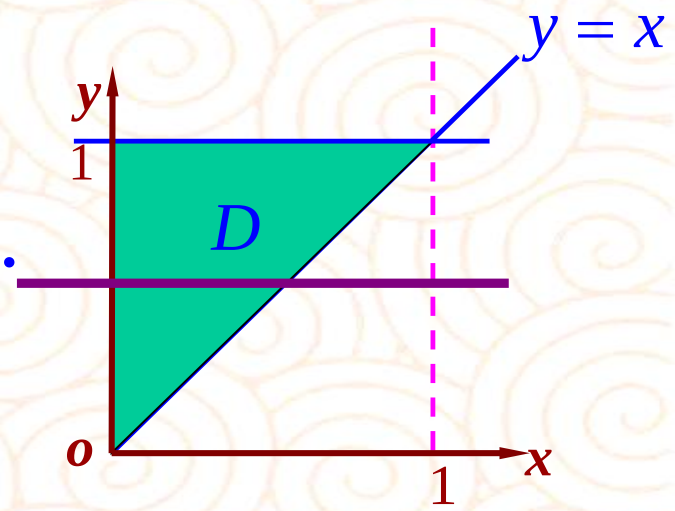
$$= AF(1) - \frac{1}{2}\left[F(1) - F(0)\right]\left[F(1) + F(0)\right]$$

$$= \frac{A}{2}\left[2F(1) - F(1) - F(0)\right]$$

$$= \frac{A}{2}\left[F(1) - F(0)\right] = \frac{A^2}{2}$$

5. 设 $f(x)$ 在 $[0,1]$ 上连续，并设 $\int_0^1 f(x)dx = A$，

求 $\int_0^1 dx \int_x^1 f(x)f(y)dy$.



解3 $\int_x^1 f(y)dy$ 不能直接积出,需改积分次序.

令 $I = \int_0^1 dx \int_x^1 f(x)f(y)dy$,

则原式 $I = \int_0^1 dy \int_0^y f(x)f(y)dx = \int_0^1 dx \int_0^x f(x)f(y)dy$

故 $2I = \int_0^1 dx \int_0^x f(x)f(y)dy + \int_0^1 dx \int_x^1 f(x)f(y)dy$

$$= \int_0^1 dx \int_0^1 f(x)f(y)dy = \int_0^1 f(x)dx \int_0^1 f(y)dy = A^2.$$

$$\therefore I = \frac{A^2}{2}$$

6. 设$a>0, b>0$为常数, $f(t)$是连续函数,且$f(t)\neq 0$, 证明:

$$\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1} \frac{(b+1)f(\frac{x}{a})+(a-1)f(\frac{y}{b})}{f(\frac{x}{a})+f(\frac{y}{b})}dxdy=\frac{\pi}{2}ab(a+b)$$

8. 设函数$f(x)$连续, 平面有界闭区域 D 由$|y|\le|x|\le 1$确定.

证明:$\iint\limits_D f\left(\sqrt{x^2+y^2}\right)dxdy=\pi\int_0^1 xf(x)dx+\int_1^{\sqrt{2}}(\pi-4\arccos\frac{1}{x})xf(x)dx$

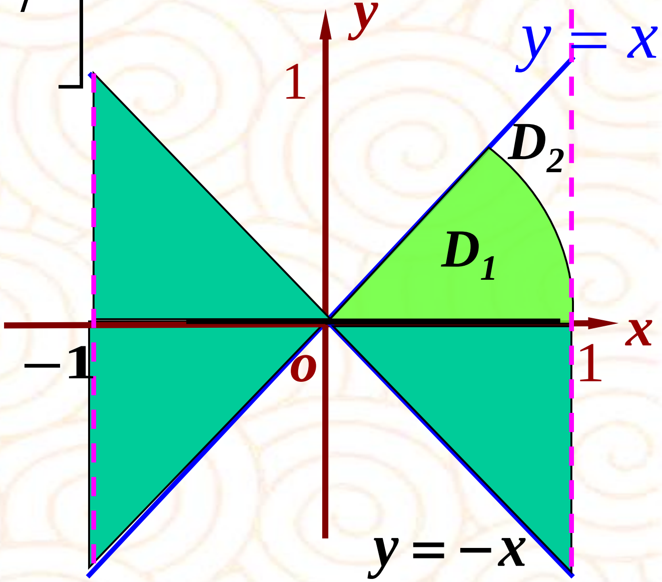
证: $x>0,|y|\le x,|x|\le 1\Rightarrow x>0,-x\le y\le x,0\le x\le 1$

$x<0,|y|\le -x,|x|\le 1\Rightarrow x<0,x\le y\le -x,-1\le x\le 0$

$$左=4\iint\limits_{D_1+D_2} f\left(\sqrt{x^2+y^2}\right)dxdy=4\iint\limits_{D_1} f\left(\sqrt{x^2+y^2}\right)dxdy+4\iint\limits_{D_2} f\left(\sqrt{x^2+y^2}\right)dxdy$$

$$=4\left[\int_0^1 d\rho\int_0^{\frac{\pi}{4}}\rho f(\rho)d\varphi+\int_1^{\sqrt{2}}d\rho\int_{\arccos\frac{1}{\rho}}^{\frac{\pi}{4}}\rho f(\rho)d\varphi\right]$$

$$=\pi\int_0^1 xf(x)dx+\int_1^{\sqrt{2}}(\pi-4\arccos\frac{1}{x})xf(x)dx$$



D关于***x***轴、***y***轴对称;
被积函数 ***f*** 关于***x, y***均为偶函数

左 $=4\iint\limits_{D_1+D_2} f\left(\sqrt{x^2+y^2}\right)dxdy$

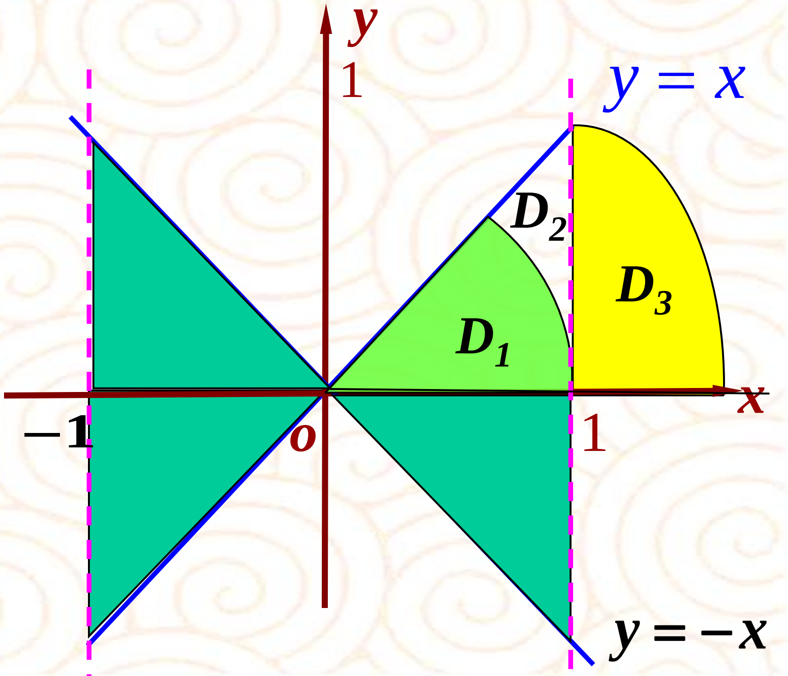
$$=4\int_0^{\frac{\pi}{4}}d\varphi\int_0^1\rho f(\rho)d\rho+4\int_0^{\frac{\pi}{4}}d\varphi\int_1^{\sqrt{2}}\rho f(\rho)d\rho-4\int_0^{\frac{\pi}{4}}d\varphi\int_{\frac{1}{\cos\varphi}}^{\sqrt{2}}\rho f(\rho)d\rho$$

$$=\pi\int_0^1\rho f(\rho)d\rho+\pi\int_1^{\sqrt{2}}\rho f(\rho)d\rho-4\int_1^{\sqrt{2}}\rho f(\rho)d\rho\int_0^{\arccos\frac{1}{\rho}}d\varphi$$

$$=\pi\int_0^1 xf(x)dx+\int_1^{\sqrt{2}}(\pi-4\arccos\frac{1}{x})xf(x)dx$$

也可补一块黄色部位造出扇形，如右图,则:

$$D_1+D_2=D_1+(D_2+D_3)-D_3$$



P162, 13题　设函数$f(t)$在$[0,+\infty)$上连续, 且满足方程

$$f(t)=e^{4\pi t^2}+\iint\limits_{x^2+y^2\le 4t^2} f\left(\frac{1}{2}\sqrt{x^2+y^2}\right)dxdy,\text{求}f(t)$$

例 设有闭区域 $D: x^2+y^2 \leqslant y, x \geqslant 0$. $f(x, y)$ 是 D 上的连续函数，

且 $f(x,y) = \sqrt{1-x^2-y^2} - \dfrac{8}{\pi}\iint\limits_D f(u,v)\,\mathrm{d}u\mathrm{d}v$, 求 $f(x,y)$.

8. **(本题 10 分)** 设 $f(x,y)$ 在区域 $D=\{(x,y): x^2+y^2\le 4\}$ 上连续，且满足

$$\pi f(x,y)=\sqrt{4-x^2-y^2}+\iint\limits_D f(x,y)\mathrm{d}x\mathrm{d}y,$$

求 $f(x,y)$.

二重积分的定义和计算

二重积分在直角坐标下的计算公式

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \int_a^b \mathrm{d}x\int_{y_1(x)}^{y_2(x)} f(x,y)\mathrm{d}y. \quad [X\text{—型}]$$

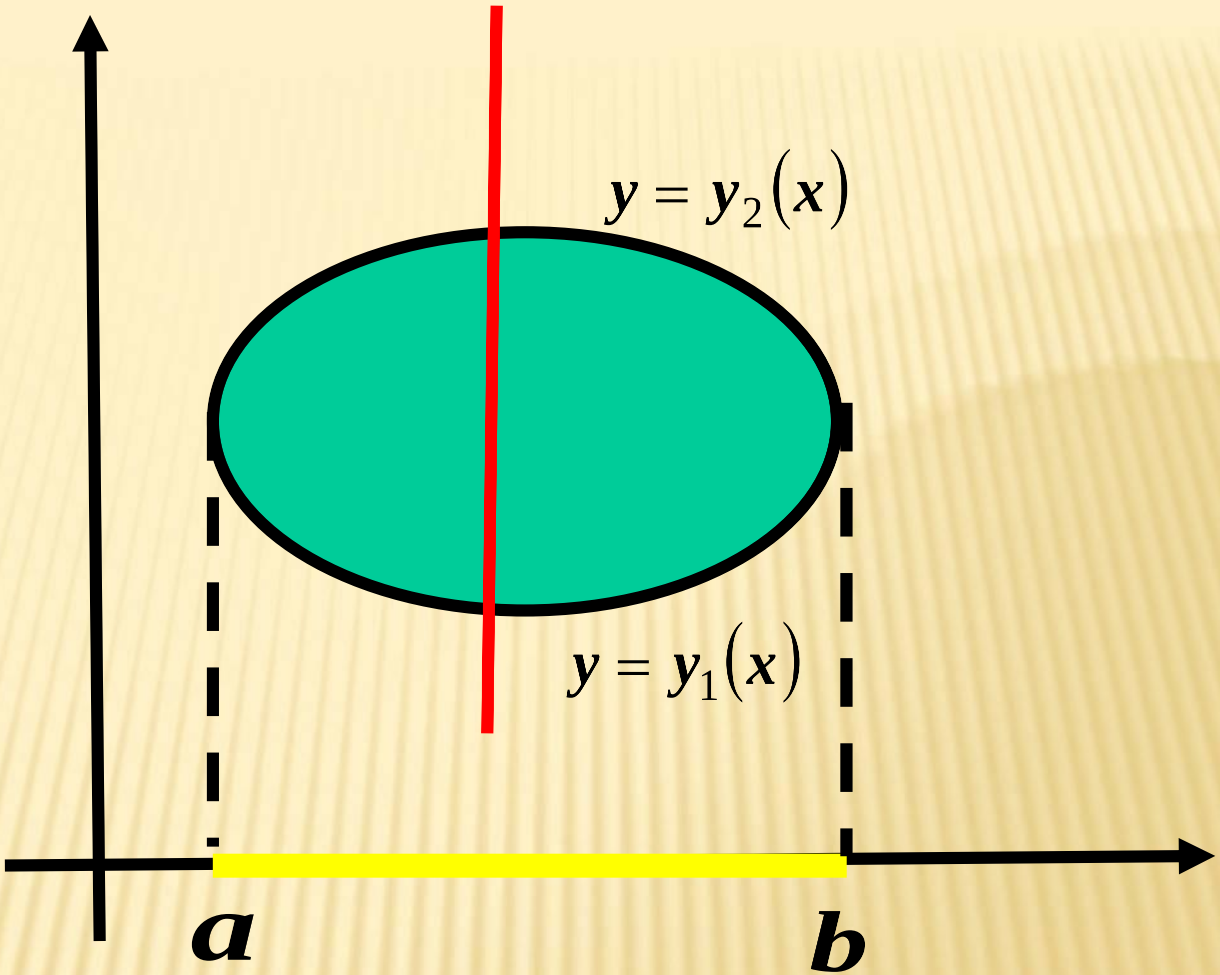
$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \int_c^d \mathrm{d}y\int_{x_1(y)}^{x_2(y)} f(x,y)\mathrm{d}x. \quad [Y\text{—型}]$$

（在积分中要正确选择**积分次序**）

二重积分在极坐标下的计算公式

$$\iint\limits_D f(x,y)\mathrm{d}\sigma = \iint\limits_D f(r\cos\theta,\ r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

二重积分的一般换元法



1画域

2投影

3发射

4定限

$$\iint\limits_D f(x,y)\,\mathrm{d}\sigma = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x,y)\,\mathrm{d}y.$$

练习

交换下列积分顺序

$$I=\int_0^2 \mathrm{d}x\int_0^{\frac{x^2}{2}} f(x,y)\mathrm{d}y+\int_2^{2\sqrt{2}}\mathrm{d}x\int_0^{\sqrt{8-x^2}} f(x,y)\mathrm{d}y$$

空间曲面

球面 $x^2+y^2+z^2=R^2$

双叶双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

双曲抛物面 $-\frac{x^2}{2p}+\frac{y^2}{2q}=z \quad (p\cdot q>0)$

圆锥面 $z^2=a^2(x^2+y^2)$

柱面及其方程 $F(x,y)=0$

旋转抛物面 $\frac{x^2}{2p}+\frac{y^2}{2q}=z$

椭圆柱面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

旋转曲面 $f(\pm\sqrt{x^2+y^2},z)=0$

双曲柱面 $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

单叶双曲面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

抛物柱面 $y^2=2x$