Mathematics & Music (AMath390)

K.A. Morris
Dept. of Applied Mathematics, University of Waterloo

1 Introduction

What is Music?

From a Google search...

- vocal or instrumental sounds (or both) combined in such a way as to produce beauty of form, harmony, and expression of emotion "couples were dancing to the music"?
- the art or science of composing or performing music¿'he devoted his life to music"?
- a sound perceived as pleasingly harmonious."the background music of softly lapping water"

Determining what and what is not music is difficult and is partly determined by culture. Wikepedia has a nice article on the subject "Definition of Music". My favorite definition is that it is "organized sound" ; a phrase coined by the composer Edgard Varg èse.

Different frequencies

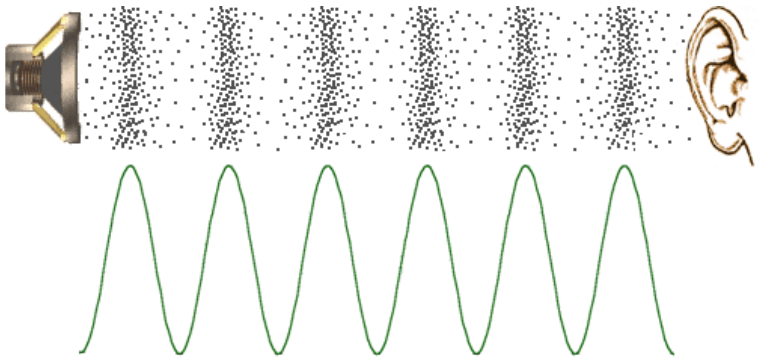
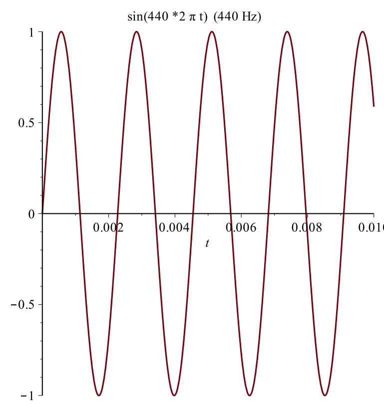
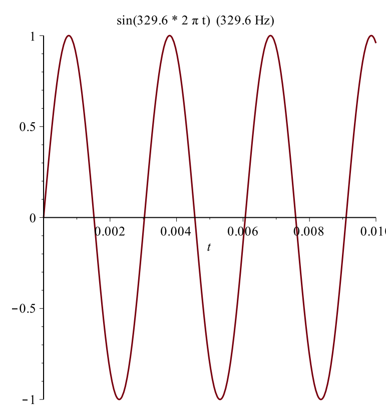


Figure 1: Sound is pressure waves.



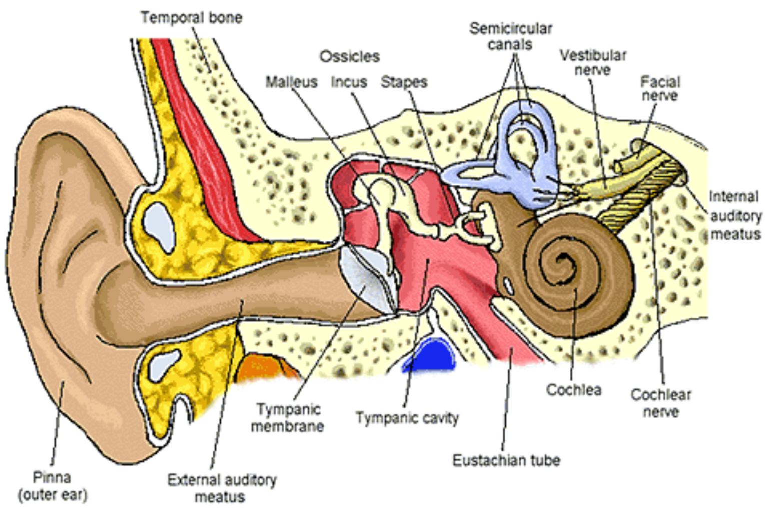
$440Hz\ for\ 0.01s$



$329.6Hz\ for\ 0.5s$

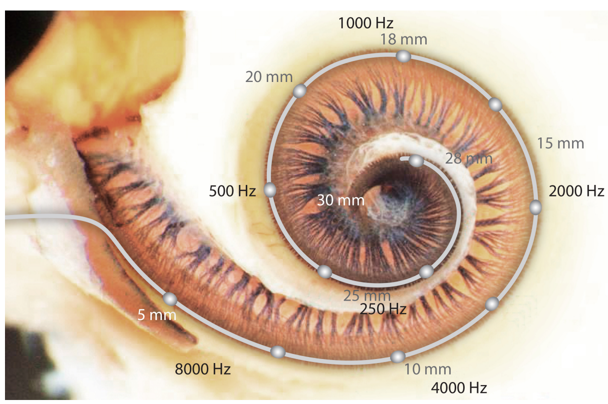
The 440Hz wave is perceived as "higher" in pitch than the 329.6Hz wave. In modern North American music, 440Hz is the note A in the treble clef (A4) and 329.6Hz is the frequency of the note E below it. However, in Europe the same note A4 is played slightly higher, 443Hz. In earlier times, pitch varied widely from place to place. In Baroque times A4 was played a lot lower, around 415Hz.

Hearing



Cochlea (slice)

Cochlea Response



- vibrations on membrane travel through cochlea
- different cilia respond to different frequencies
- cilia nearest ear respond to highest frequencies

Frequency of Sound

- the frequency of a sound wave is perceived by our brain
- frequency corresponds to pitch in music
- range of human hearing: 20-20,000Hz
- most music contains multiple frequencies

Pyschoacoustics

- much of sound perception is due to processing in our brain
- sounds at the frequency limit are perceived as less loud
- just noticeable difference is perceptible difference in sequential notes
- limit of discrimination is perceptible difference in simultaneous notes
- both just noticeable difference and limit discrimination vary with volume and frequency of the sound
- limit of discrimination is much smaller than just noticeable difference
- if the lowest frequency is absent we often perceive it as being present

Music is not Mathematics

> While we're discussing mathematical aspects of music, we should not lose sight of the evocative power of music as a medium of expression for moods and emotions. About the numerous interesting questions this raises, mathematics has little to say.

(Benson, pg. xii)

> Why do rhythms and melodies, which are composed of sound, resemble the feelings, while this is not the case for tastes, colours or or smells? Can it be because they are motions, as actions are also motions?

(Aristotle, quoted in Benson)

2 Harmonic Motion

Newton's Law

$$ma = F$$

Letting deflection be y,

$$m\frac{d^2y}{dt^2} = F$$

For many systems the restoring force is (approximately) proportional to deflection and so, letting $k > 0$ indicate a proportionality constant,

$$m\frac{d^2y}{dt^2} = -ky.$$

Defining $\omega = \sqrt{\frac{k}{m}}$, rewrite as

$$\ddot{y}(t) + \omega^2 y(t) = 0.$$

$\ddot{y}(t) + \omega^2 y(t) = 0$

The general solution to this equation is

$$y(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1}$$

where A, B are determined by initial conditions. To see (1) work out the derivatives of y :

$$\begin{aligned}
\dot{y}(t) &= -A\omega\sin(\omega t) + B\omega\cos(\omega)t \\
\ddot{y}(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) = -\omega^2 y(t).
\end{aligned}$$

Letting ϕ be such that

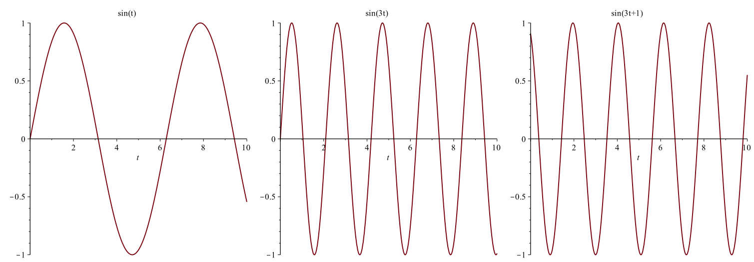
$$\sin(\phi) = \frac{A}{\sqrt{A^2+B^2}}, \quad \cos(\phi) = \frac{B}{\sqrt{A^2+B^2}},$$

and defining $c = \sqrt{A^2+B^2}$,

$$y(t) = c\sin(\phi)\cos(\omega t) + c\cos(\phi)\sin(\omega t).$$

Using the sum formula

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{2}$$



(a) $\sin(t)$ (b) $\sin(3t)$ (c) $\sin(3t+1)$

Figure 2: Harmonic Motion

yields

$$y(t) = c\sin(\omega t + \phi). \tag{3}$$

c, ϕ are determined by initial conditions.

$\boxed{\text{This is known as harmonic motion.}}$

The advantage of the second representation (3) is that it is clear that the solution is periodic with frequency ω. The amplitude c and phase ϕ are determined by initial conditions.

The frequency of the wave corresponds to pitch of an audible sound; amplitude of the wave corresponds to loudness. A difference in phase of two waves is not perceptible unless they occur at the same time.

Frequency and Pitch

- The value of ω in $\sin(\omega t + \phi)$ yields the frequency of the solution.
- Frequency often given as cycle/s (Hz)
- $\sin(\omega t)$ has frequency $\omega/2\pi$ Hz
- sounds with higher frequencies are said to have a higher pitch .

Some Music Notation

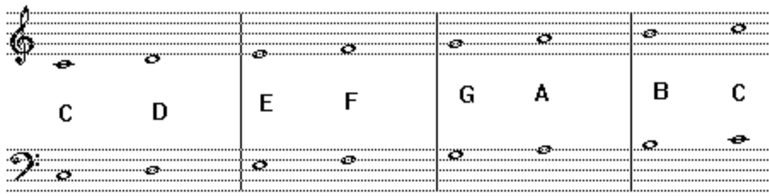


Figure 3: Notes on treble clef and bass clefs. The middle C typically has a frequency of 261.6Hz in North America. The higher C has twice the frequency of middle C; the lowest C has half the frequency of middle C.

Damping

Actual systems do not oscillate forever; there are dissipative forces.
A more realistic model includes dissipation

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = 0, \quad 0 < \xi < 1$$

which has solution

$$\begin{aligned} y(t) &= e^{-\xi\omega t}\left(A\cos(\sqrt{1-\xi^2}\omega t) + B\sin(\sqrt{1-\xi^2}\omega t)\right) \\ &= Ce^{-\xi\omega t}\sin((\sqrt{1-\xi^2})\omega t + \phi). \end{aligned}$$

Damped vs Undamped Oscillations

- decaying amplitude
- frequency slightly lower

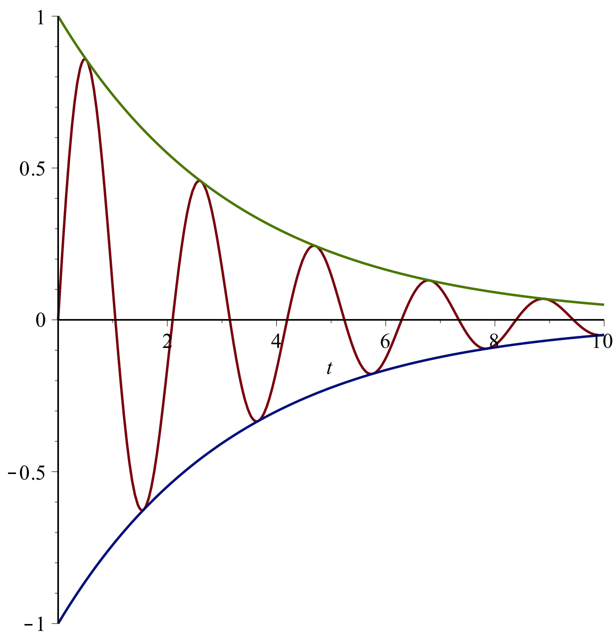
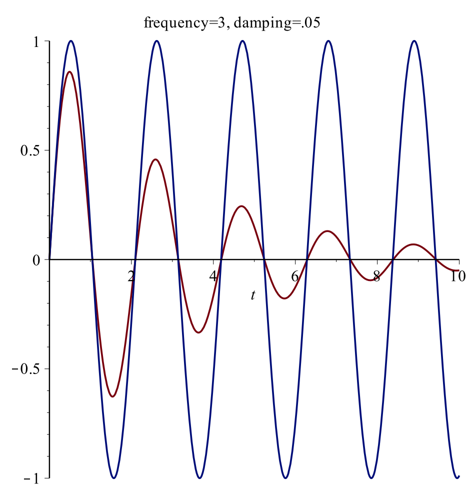
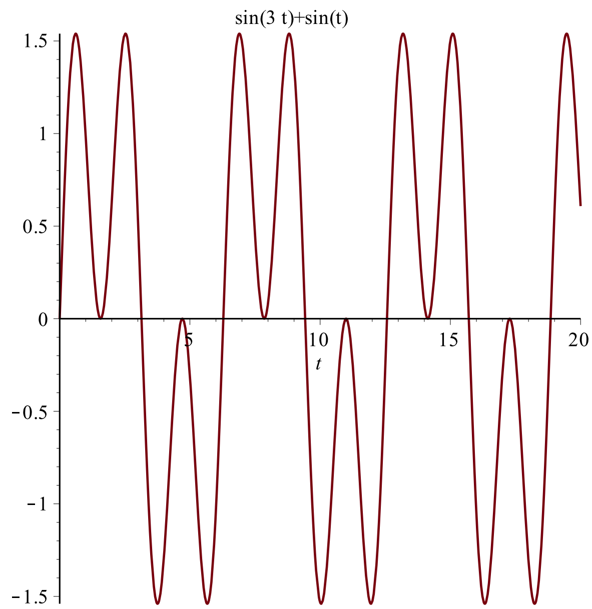


Figure 4: Damped oscillations with frequency $\omega = 3$, damping $\xi = .05$

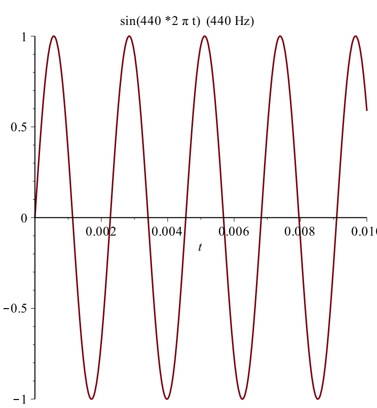


With $\xi = 0.05$, frequency is 99.8% of the undamped frequency. If $\xi = 0.2$, it's 98%. Since we are typically concerned only with frequency, and damping only slightly affects frequency, we will not generally include damping in the analysis.

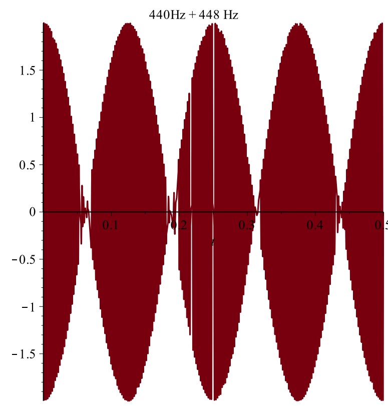
Superposition



Beats



$$440Hz\ for\ 0.01s$$



$440Hz + 448Hz\ for\ 0.5s$

Why is there a lower frequency envelope when the two frequencies are close?

For simplicity, consider two waves with same phase and amplitude, but different frequencies $\omega_2 > \omega_1$:

$$y(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$$

Defining

$$\bar{\omega} = \frac{1}{2}(\omega_2 + \omega_1), \quad \Delta = \frac{1}{2}(\omega_2 - \omega_1),$$

and using the sum formula (2),

$$\begin{aligned} y(t) &= \sin(\bar{\omega}t - \Delta t) + \sin(\bar{\omega}t + \Delta t) \\ &= 2\cos(\Delta t)\sin(\bar{\omega}t). \end{aligned}$$

If Δ is small, this looks like a sine wave with frequency $\bar{\omega}/2\pi$ Hz and amplitude a slow cosine wave.

Forced Motion

$$\ddot{y}(t) + 2\xi w\dot{y}(t) + \omega^2 y(t) = f(t). \tag{4}$$
$$y(0) = y_0, \quad \dot{y}(0) = y_1.$$

Suppose $y_p(t)$ is found that solves (4), but maybe not the initial conditions. For any A, B,

$$y_u(t) = e^{-\xi\omega t}\left(A\cos(\sqrt{1-\xi^2}\omega t) + B\sin(\sqrt{1-\xi^2}\omega t)\right)$$

solves

$$\ddot{y}(t) + 2\xi w\dot{y}(t) + \omega^2 y(t) = 0$$

and so $y_u + y_p$ solves (4).

$$\boxed{\text{Choose A, B to satisfy the initial conditions.}}$$

Periodic forcing

In this course we are interested in periodic forcing; that is equations of the form

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = F\sin(\alpha t). \tag{5}$$

What is the solution of this equation?

Since repeated derivatives of $\sin(\alpha t)$, $\cos(\alpha t)$ are also $\sin(\alpha t)$, $\cos(\alpha t)$, try

$$y_p(t) = a\sin(\alpha t) + b\cos(\alpha t).$$

Substituting into the left-hand-side of (5) yields

$$\underbrace{(-a\alpha^2 - 2\xi\omega\alpha b + \omega^2 a)}_{F}\sin(\alpha t) + \underbrace{(-b\alpha^2 - 2\xi\omega\alpha a + \omega^2 b)}_{0}\cos(\alpha t).$$

For this to equal the right-hand-side of (5), solve linear equations for a, b:

$$\begin{aligned} (\omega^2 - \alpha^2)a + (-2\xi\omega\alpha)b &= F \\ (-2\xi\omega\alpha)a + (\omega^2 - \alpha^2) &= 0. \end{aligned}$$

If $\alpha \neq \omega$ or $\xi \neq 0$

$$a = \frac{F(\omega^2 - \alpha^2)}{\sqrt{(\omega^2 - \alpha^2)^2 + (2\xi\omega\alpha)^2}}, \quad b = \frac{-F(2\xi\omega\alpha)}{\sqrt{(\omega^2 - \alpha^2)^2 + (2\xi\omega\alpha)^2}}.$$

Vibration of forced system

Writing $\omega_0 = \sqrt{1-\xi^2}\,\omega$,

$$y(t) = e^{-\xi\omega t}(A\sin(\omega_0 t) + B\cos(\omega_0 t)) + a\sin(\alpha t) + b\cos(\alpha t).$$

Using the sum formula (2)

$$y(t) = e^{-\xi\omega_0 t}C\sin(\omega_0 t + \phi) + M\sin(\alpha t + \phi_f).$$

where M and ϕ_f are determined by a and b (or the forcing function parameters F and α) and C and ϕ are determined by A and B (the initial conditions). The response is the sum of two waves

- decaying wave at natural frequency ω
- persistent wave at forced frequency α

Resonance

In steady-state, once the effect of the initial conditions has dissipated,

$$y(t) = M\sin(\alpha t + \phi_f).$$

where

$$\begin{aligned} M &= \sqrt{a^2+b^2} \\ &= \frac{F}{\sqrt{(\omega^2-\alpha^2)^2 + (2\xi\omega\alpha)^2}}. \end{aligned}$$

The value of M is the magnitude of the steady-state oscillations. The magnitude increases as the forcing frequency α approaches the natural frequency ω, and the peak is larger for lightly damped systems. A vibrating system that is forced at a frequency close to the natural frequency is said to be in resonance.

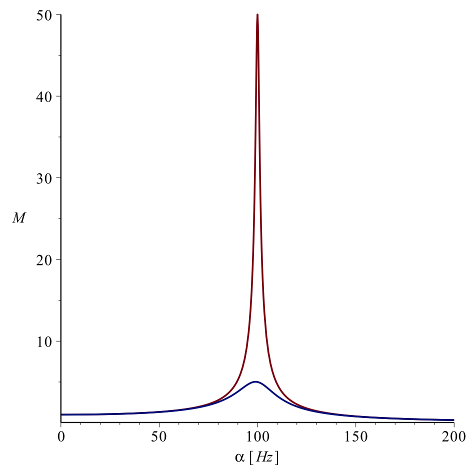


Figure 5: Magnitude M of the steady-state response of a forced oscillator with $\omega = 100Hz$, $\xi = 0.1$, (blue) $\xi = 0.01$ (red) . The horizontal axis α is the frequency of the forcing term. The magnitude increases as the forcing frequency α approaches the natural frequency ω.

3 Dynamics of Stretched String

Mathematical Model of Vibrating String

The sound in many musical instruments, for instance guitars and violins, is produced by vibrating strings. A string of length ℓ is stretched and fixed at each end. The sound is produced by plucking, strumming etc. the string.

Assume constant tension T, density ρ, uniform cross-sectional area A and small deflections $u(x,t)$. Set the deflection $u = 0$ when the string is not stretched by strumming, striking etc. Consider a small section of string of length Δx. It has mass $m = \rho A \Delta x$ and acceleration $a = \frac{\partial^2 u(x,t)}{\partial t^2}$. The force on the stretched string is due to tension in the string and its vertical component is, letting θ be the angle the string makes with its unstretched position,

$$F = -TA\sin(\theta(x)) + TA\sin(\theta(x + \Delta x)).$$

Substituting these expressions for m, a and F into Newton's second law,

$$ma = F$$

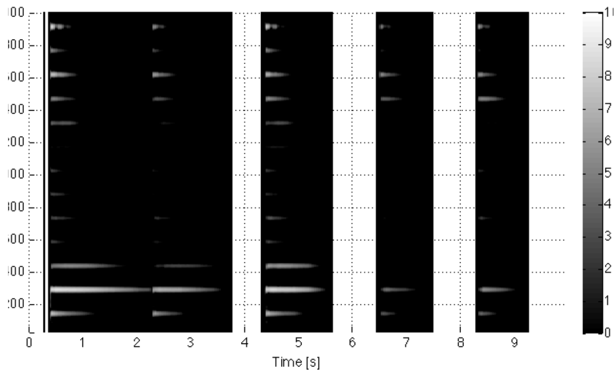


Figure 6: Guitar- Spectrogram of D string, strummed

and dividing through by Δx yields

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = T \frac{1}{\Delta x}(\sin(\theta(x+\Delta x) - \sin(\theta(x))).$$

Take the limit as $\Delta x \to 0$ and define $c^2 = \frac{T}{\rho}$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial}{\partial x} \sin(\theta(x)). \tag{6}$$

For small deflections, that is small θ,

$$\sin(\theta) \approx \tan(\theta) = \frac{\partial u}{\partial x}.$$

Substitution into (6) yields the *wave equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{7}$$

Since the end of each string is fixed, the deflections u at each end are zero:

$$u(0,t) = 0, \quad u(\ell,t) = 0. \tag{8}$$

This equation is to be solved with the boundary conditions (8) and appropriate initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{9}$$

where f and g describe the initial deflection and velocity respectively of the stretched string.

Separation of variables

Partial differential equations are in general very difficult to solve. Try looking for solutions of the form

$$u(x,t) = M(x)N(t).$$

Substitution into (7) yields, using $'$ to indicate differentiation,

$$MN'' = c^2 M'' N.$$

Rearranging,

$$\frac{N''}{c^2 N} = \frac{M''}{M}.$$

Since the left-side depends only on time t and the right-side depends only on space x, each side must be a constant. Call this constant $-\lambda$. This yields two *ordinary* differential equations

$$M''(x) + \lambda M(x) = 0, \tag{10}$$
$$N'' = c^2 \lambda N. \tag{11}$$

The spatial function M should satisfy (10) and also the boundary conditions (8). Clearly $M = 0$ is a solution, but this is not interesting. The equation (10) will have non-trivial solutions that satisfy the boundary conditions only if

$$\lambda = (\frac{\pi k}{\ell})^2, k = 1, 2, \ldots$$

so

$$M_k(x) = \sin(\frac{\pi k}{\ell} x).$$

The functions M_k are known as the *eigenfunctions* of $\frac{\partial^2}{\partial x^2}$ with the boundary conditions (8). The other differential equation (11) then has solutions

$$N(t) = A_k \cos(\frac{\pi k c}{\ell} t) + B_k \sin(\frac{\pi k c}{\ell} t)$$

for constants A_k, B_k.

For any integer k, and constants A_k, B_k

$$u_k(x,t) = A_k \cos(\frac{\pi k c}{\ell} t) + B_k \sin(\frac{\pi k c}{\ell} t)] \sin(\frac{\pi k}{\ell} x)$$

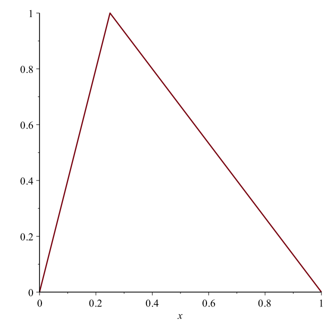


Figure 7: Can a arbitiary function be written as a Fourier sine series $\sum_{k=1}^{\infty} f_k \sin(\frac{\pi k}{\ell} x)$ for some choice of $\{f_k\}$?

solves the wave equation (7) and satisfies the boundary conditions (8). By linearity, any linear combination of the u_k,

$$u(x,t) = \sum_{k=1}^{\infty} [A_k \cos(\frac{\pi k c}{\ell} t) + B_k \sin(\frac{\pi k c}{\ell} t)] \sin(\frac{\pi k}{\ell} x) \tag{12}$$

also satisfy (7) and (8). This approach to solving a partial differential equation is known as the *Method of Separation of Variables* . But in order for u to be a solution, constants A_k, B_k are needed so that the initial conditions (9) are satisfied.

Initial Conditions

The constants A_k and B_k in (12) need to be chosen so that the initial conditions (9) are satisfied:

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} A_k \sin(\frac{\pi k}{\ell} x) =, \quad \frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{k=1}^{\infty} \frac{\pi k}{\ell} B_k \sin(\frac{\pi k}{\ell} x).$$

For initial conditions that are a finite linear combination of functions of the form $\sin(\frac{\pi k}{\ell} x)$ this is straightforward. But to allow for more general initial conditions, arbitrary initial conditions, such as that shown in Figure 7 need to be written as a sum of sine functions.