# AMath390

**Math & Music** **Fall 2014**

**Assignment # 4** **SOLUTIONS**

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1. Consider a drum consisting of a membrane stretched over a square frame, of length $\ell$ on each side. Let the tension be $T$ (N/m) and the density $\rho$ (kg/m2) .

   (a) Write down the partial differential equation and boundary conditions describing the motion.

**Solution:** Choose coordinates so the drum is located at $0 \le x \le \ell$ , $0 \le y \le \ell$. Defining $c^2 = \frac{T}{\rho}$, the vibrations $u$ are modelled by

$$\frac{\partial^2 u}{\partial t^2} = c^2 (\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial y^2}), \quad 0 \le x \le \ell,\ 0 \le y \le \ell \tag{1}$$

with boundary conditions

$$u(0,y,t) = 0,\ u(\ell,y,t) = 0,\ 0 \le y \le \ell, \quad u(x,0,t) = 0,\ u(x,\ell,t) = 0, 0 \le x \le \ell$$

and initial conditions, for appropriate functions $f(x,y)$, $g(x,y)$,

$$u(x,y,0) = f(x,y),\ \frac{\partial u}{\partial t}(x,y,0) = g(x,y), \qquad 0 \le x \le \ell,\ 0 \le y \le \ell.$$

   (b) Solve the equation for arbitrary initial conditions.

**Solution:** Using separation of variables, look for solutions of the form

$$u(x,y,t) = X(x)Y(y)T(t).$$

Substituting this into the wave equation (1), and rearranging,

$$\frac{T''}{T} = c^2(\frac{X''}{X} + \frac{Y''}{Y}). \tag{2}$$

The left hand side depends only on time, the left hand side only on spatial variables $x, y$ so each side must equal a constant, call it $\omega^2$. Then

$$T'' + \omega^2 T = 0$$

and so for arbitrary constants $A$, $B$,

$$T(t) = A\sin(\omega t) + B\cos(\omega t).$$

Then

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{\omega^2}{c^2} = -\mu^2$$

where $-\mu^2$ is some constant, since the left hand side depends only on $x$ and the right hand side only on $y$. This implies

$$X'' + \mu^2 X = 0$$

which has general solution

$$X(x) = A_1 \sin(\mu x) + B_1 \cos(\mu(x).$$

From the boundary conditions

$$X(0) = 0, \quad X(\ell) = 0.$$

There are only non-trivial solutions if

$$\mu = \frac{m\pi}{\ell}$$

and the solution is for an arbitrary constant $A_1$,

$$X(x) = A_1 \sin(\mu x).$$

Returning to equation (2), and defining

$$\alpha^2 = \frac{\omega^2}{c^2} - \mu^2,$$

$$Y'' + \alpha^2 Y = 0.$$

The boundary conditions are

$$Y(0) = 0, \quad Y(\ell) = 0.$$

There are only non-trivial solutions if

$$\alpha = \frac{n\pi}{\ell}$$

and the solution is for an arbitrary constant $A_2$,

$$Y(y) = A_2 \sin(\alpha y).$$

Thus, since $\alpha^2 = \frac{\omega^2}{c^2} - \mu^2$, only particular values of $\omega$ are allowed, and

$$\omega_{m,n} = \frac{\pi c\sqrt{m^2+n^2}}{\ell}.$$

Defining

$$u_{m,n}(x,y,t) = (A_{m,n}\sin(\omega_{m,n}t) + B_{m,n}\cos(\omega_{m,n}t))\sin(m\pi\frac{x}{\ell})\sin(n\pi\frac{y}{\ell})$$

the solution is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{m,n}(x,y,t)$$

where the coefficients $A_{m,n}$ , $B_{m,n}$ are chosen so that the initial conditions are satisfied.

The solution can also be written

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M_{m,n}\sin(\omega_{m,n}t + \phi_{m,n})\sin(m\pi\frac{x}{\ell})\sin(n\pi\frac{y}{\ell})$$

where $M_{m,n}$, $\phi_{m,n}$ are chosen so that the initial conditions are satisfied.

(c) In terms of the physical parameters $T$ and $\rho$ what are the natural frequencies of vibration?

> **Solution:** The natural frequencies are
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> $$\omega_{m,n} = \frac{\pi\sqrt{T}\sqrt{m^2+n^2}}{\sqrt{\rho}\ell}.$$

(d) Sketch the nodal lines of the first 2 modes of vibration.

> **Solution:** The first mode, associated with the fundamental frequency, has no nodes.
>
> The second two modes (1,2) and (2,1) each have a single line crossing the centre of the drum.

(e) Would you expect to be able to hear the difference between the sound of a square drum and a round drum? Explain your answer briefly.

> **Solution:**
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> The square drum has overtones that are different from those of a round drum with the same fundamental frequency, so it might have a different timbre. They are not harmonic though, so the difference would be subtle. (Listening to some videos verifies this.)

(f) The *adufe* is a traditional tambourine-like Portugese instrument consisting of a square frame covered with a membrane on both sides. How do you think enclosing the drum changes the sound?

> **Solution:** Most drums are enclosed in this way. Likely the second membrane emphasizes the vibrations, since there are two membranes. The resonance of the cavity causes certain overtones to be emphasized, improving the sound.

2. The vibrations in a struck bar of length $\ell$, such as in a xylophone, mbira, glockenspiel are governed by the equation (assuming constant width and small deflections)

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \underbrace{\frac{EI}{\rho}}_{c^2}\frac{\partial^4 u}{\partial x^4}(x,t) = 0, \quad 0 < x < L \tag{3}$$

where $E$, $I$, $\rho$ are positive physical constants. Since they are constant, define $c^2 = \frac{EI}{\rho}$. Various boundary conditions are possible. For a glockenspiel, the ends are fixed, but free to bend so the boundary conditions are

$$u(0,t) = 0,\ \frac{\partial^2 u}{\partial x^2}(0,t) = 0,\ u(\ell,t) = 0,\ \frac{\partial^2 u}{\partial x^2}(\ell,t) = 0.$$

(a) Assume that the solution is separable, $u(x,t) = M(x)N(t)$ and show that, for arbitrary $A$

$$M(x) = A\sin(ax)$$

solves the spatial differential equation and boundary conditions, for particular values of $a$. What are the allowable values of $a$? (Hint: call the separation constant $-a^4$.)

(b) Write out the general solution. (There will be arbitrary constants, since the initial position and velocity is not specified.

(c) What is the fundamental frequency? What are the overtones?

(d) What is the fundamental frequency, if the bar is 1.00 m long? The first overtone? Use $c = \sqrt{\frac{EI}{\rho}} = 222m^2/s$.

(e) Using the values from part (d), find the length of a second bar so that the fundamental frequency is twice that of the first bar.

**Solution:**

(a) Assuming that $u(x,t) = M(x)N(t)$ and substituting into (3) yields (after rearranging)

$$\frac{N''(t)}{c^2 N(t)} = -\frac{M^{IV}(x)}{M(x)}.$$

Since one side depends only on $t$, the other only on $x$ they must be constant; call this constant $-a^4$.

This yields

$$M^{IV}(x) = a^4 M(x).$$

It can be easily verified that $M(x) = A\sin ax$ solves this equation and also yields

$$M(0) = 0, \quad M''(0) = 0.$$

It is also required that

$$M(\ell) = 0, \quad M''(\ell) = 0.$$

This means that

$$\sin(a\ell) = 0$$

and so for a non-trivial solution, $a = \frac{k\pi}{\ell}$.

(b) Continuing from part (a), the time equation is

$$N''(t) = -a_k^4 c^2 N(t)$$

which has general solution, defining

$$\omega_k = a_k^2 c = \frac{k^2\pi^2 c}{\ell^2},$$

$$A_k\cos(\omega_k t) + B_k\sin(\omega_k t).$$

The general solution to (3) with the given boundary conditions is

$$u(x,t) = \sum_{k=1}^{\infty}(A_k\cos(\omega_k t) + B_k\sin(\omega_k t))\sin(\frac{k\pi}{\ell}x).$$

(c) The fundamental frequency is $f_1 = \frac{\omega_1}{2\pi} = \frac{\pi c}{2\ell^2}$. The overtones are $\frac{k^2 \pi c}{2\ell^2}$, $k = 2, 3, \ldots$

(d) For a $1m$ bar, $f_1 = 349$Hz. The first overtone is $f_2 = 4f_1 = 1396$ Hz.

(e) The second bar should have length $\frac{1}{\sqrt{2}}$m, or 0.707m.