18. Show that when the cross-sectional area of the bar in Figure 1.16 varies with position, equation (56) is replaced by

$$\frac{\partial^2 y}{\partial t^2} = \frac{c^2}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial y}{\partial x}\right) + \frac{F(x,t)}{\rho}, \qquad c^2 = \frac{E}{\rho},$$

provided expression (53) still gives forces across cross sections of the bar.

19. A bar of unstrained length L is clamped at end $x = 0$. For time $t < 0$, it is at rest, subjected to a force with x-component F distributed uniformly over the other end. If the force is removed at time $t = 0$, formulate the initial boundary value problem for subsequent displacements in the bar.

20. In this exercise we derive the PDE for small vibrations of a suspended heavy cable. Consider a heavy cable of uniform density ρ (mass/length) and length L suspended vertically from one end. Take the origin of coordinates at the position of equilibrium of the lower end of the cable and the positive x-axis along the cable. Denote by $y(x,t)$ small horizontal deflections of points in the cable from equilibrium.

(a) Apply Newton's second law to a segment of the cable to obtain the PDE for small deflections

$$\frac{\partial^2 y}{\partial t^2} = -g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) + \frac{F}{\rho},$$

where $g < 0$ is the acceleration due to gravity and F is the y-component of all external horizontal forces per unit length in the x-direction.

(b) What boundary condition must $y(x,t)$ satisfy?

1.4 Transverse Vibrations of Membranes

In this section we study vibrations of perfectly flexible membranes stretched over regions of the xy-plane (Figure 1.19). When the membrane is very taut and displacements are small, the horizontal components of these displacements are negligible compared with vertical components; that is, displacements may be taken as purely transverse, representable in the form $z(x, y, t)$.

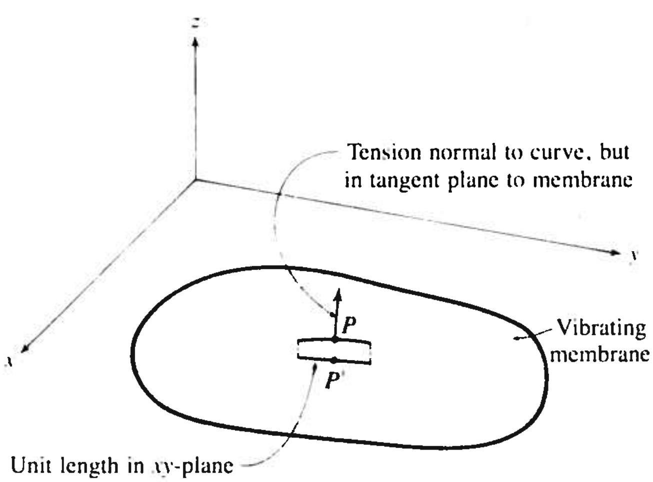

Figure 1.19

In discussing transverse vibrations of strings, tension played an integral role. No less important is the tension in a membrane. Suppose a line of unit length is drawn in any direction at a point P' in the xy-plane and projected onto a curve on the membrane (Figure 1.19). The material on one side of the curve exerts a force on the material on the other side, the force acting normal to the curve and in the tangent plane of the surface at P. This force is called the tension τ of the membrane.

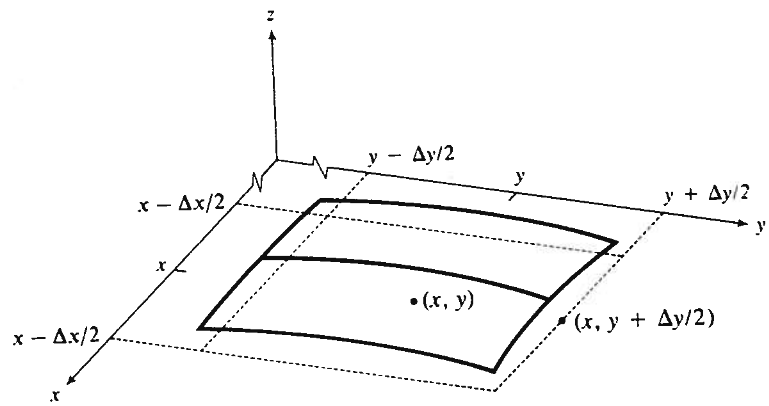

Figure 1.20

To obtain a PDE for displacements $z(x, y, t)$ of the membrane, we examine forces acting on an element of the membrane that projects onto a small rectangle in the xy-plane (Figure 1.20). The vertical component of the tension force on the element is obtained by taking vertical components of the tensions on the boundaries. The tension at the point on the membrane corresponding to the point $(x, y + \Delta y/2)$ in the xy-plane acts in the tangential direction of the curve

$$x = x \quad \text{(fixed)}, \qquad y = y, \qquad z = z(x, y, t),$$

namely,

$$\left(0, 1, \frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \tag{65}$$

A unit vector in this direction is

$$\frac{(0, 1, \partial z/\partial y)}{\sqrt{1 + (\partial z/\partial y)^2}}_{|(x, y+\Delta y/2, t)} \tag{66}$$

When vibrations of the membrane are such that $\partial z/\partial y$ is very much less than unity (and we consider only this case), the denominator in (66) may be approximated by 1, and (66) is replaced by (65). The vertical component of the tension force acting along that part of the boundary containing the point $(x, y + \Delta y/2, z)$ may therefore be approximated by

$$\tau_{|(x, y+\Delta y/2, t)}\left(0, 1, \frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \Delta x \cdot \hat{\mathbf{k}} = \left(\tau\frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \Delta x. \tag{67}$$

A similar analysis may be made on the remaining three boundaries, resulting in a total vertical force on the element (due to tension) of approximately

$$\left[\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}\right]\Delta x + \left[\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}\right]\Delta y. \tag{68}$$

When Newton's second law (force equals time rate of change of momentum) is applied to this element of the membrane, the result is

$$\left[\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}\right]\Delta x + \left[\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}\right]\Delta y + F\,\Delta x\,\Delta y = \frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\Delta x\,\Delta y\right), \tag{69}$$

where ρ is the density of the membrane (mass per unit area) and F is the sum of all vertical external forces on the membrane per unit area in the xy-plane. If we divide both sides of this equation by $\Delta x\,\Delta y$ and take limits as Δx and Δy approach zero,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right) = \lim_{\Delta y\to 0}\frac{\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}}{\Delta y} + \lim_{\Delta x\to 0}\frac{\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}}{\Delta x} + F$$

or

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right) + \frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right) + F. \tag{70}$$

For most applications, both the density of and the tension in the membrane may be taken as constant, in which case (70) reduces to

$$\boxed{\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F}{\rho}, \qquad c^2 = \frac{\tau}{\rho}.} \tag{71}$$

This is the PDE for transverse vibrations of the membrane, called the *two-dimensional wave equation.*

For an external force due only to gravity,

$$F = \rho g, \qquad g < 0; \tag{72}$$

for a damping force proportional to velocity,

$$F = -\beta \frac{\partial z}{\partial t}, \qquad \beta > 0; \tag{73}$$

and for a restoring force proportional to displacement,

$$F = -kz, \qquad k > 0. \tag{74}$$

Initial conditions that accompany (71) describe the displacement and velocity of the membrane at some initial time (usually $t = 0$):

$$z(x, y, 0) = f(x, y), \qquad (x, y) \text{ in } R, \tag{75a}$$

$$\frac{\partial z(x, y, 0)}{\partial t} = g(x, y), \qquad (x, y) \text{ in } R, \tag{75b}$$

where R is the region in the xy-plane onto which the membrane projects. A Dirichlet boundary condition for (71) prescribes the value of $z(x, y, t)$ on the boundary $\beta(R)$ of R,

$$z(x, y, t) = f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \tag{76}$$

$f(x, y, t)$ some given function.

Suppose instead that the edge of the membrane can move vertically and that it is subjected to an external vertical force $f(x, y, t)$ per unit length. The edge is also acted on by the tension in the membrane, and the magnitude of the z-component of the tension acting across a unit length along $\beta(R)$ is $|\tau \partial z/\partial n|$, where n is a coordinate measuring distance in the xy-plane normal to $\beta(R)$ (Figure 1.21). Consequently, if we take the edge of the membrane as massless, Newton's second law for vertical components of forces on an element ds of $\beta(R)$ gives

$$-\left(\tau \frac{\partial z}{\partial n}\right)_{|\beta(R)} ds + f(x, y, t)\, ds = 0 \tag{77a}$$

or

$$\frac{\partial z}{\partial n} = \frac{1}{\tau} f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0. \tag{77b}$$

This is a nonhomogeneous Neumann boundary condition. When the only force acting on the edge of the membrane is that due to tension, $z(x, y, t)$ must satisfy a homogeneous Neumann condition,

$$\frac{\partial z}{\partial n} = 0, \qquad (x, y) \text{ on } \beta(R), \qquad t > 0. \tag{78}$$

Another possibility is to have the edge of the membrane elastically attached to the xy-plane in such a way that the restoring force per unit length along $\beta(R)$ is proportional to displacement. Then, according to (77a),

$$-\left(\tau \frac{\partial z}{\partial n}\right) ds + [-kz + f(x, y, t)]\, ds = 0, \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \tag{79a}$$

where $k > 0$, and $f(x, y, t)$ now represents all external forces acting on $\beta(R)$ other than tension and the restoring force. Equation (79a) can be written in the equivalent form

$$\tau\frac{\partial z}{\partial n} + kz = f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \tag{79b}$$

a Robin condition.

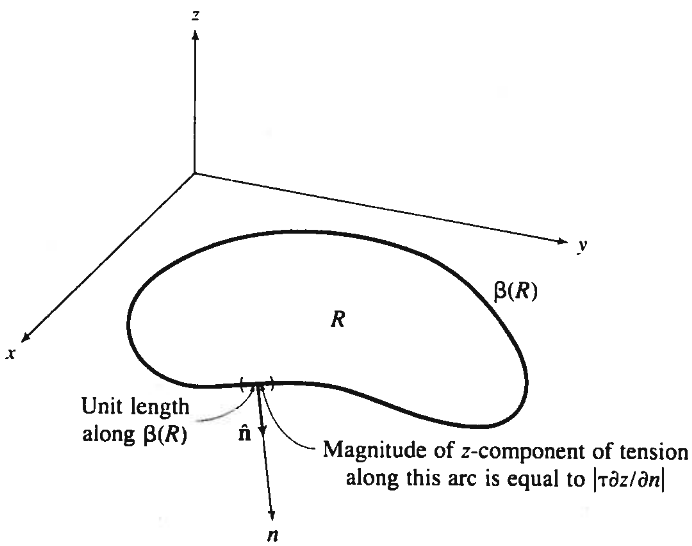

Figure 1.21

The initial boundary value problem for displacements in the membrane is

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F(x, y, t)}{\rho}, \qquad (x, y) \text{ in } R, \qquad t > 0, \tag{80a}$$

$$\text{Boundary conditions,} \tag{80b}$$

$$z(x, y, 0) = f(x, y), \qquad (x, y) \text{ in } R, \tag{80c}$$

$$z_t(x, y, 0) = g(x, y), \qquad (x, y) \text{ in } R. \tag{80d}$$

If boundary conditions (80b) and external force $F(x, y, t)$ are independent of time, there may exist solutions of (80a, b) that are also independent of time. Such solutions, called *static deflections*, satisfy Poisson's equation

$$\nabla^2 z = -\frac{F(x, y)}{\tau} \tag{81a}$$

and

$$\text{Boundary conditions.} \tag{81b}$$

If, in addition, no external forces are present, the PDE reduces to Laplace's equation

$$\nabla^2 z = 0. \tag{82}$$