Dither in Nonlinear Systems

G. ZAMES, MEMBER, IEEE, AND N. A. SHNEYDOR

Abstract—**A dither is a high-frequency signal introduced into a nonlinear system with the object of augmenting stability. In this paper,[1] it is shown that the effects of dither depend on its amplitude distribution function. The stability of a dithered system is related to that of an equivalent smoothed system, whose nonlinear element is the convolution of the dither distribution and the original nonlinearity. The ability of dithers to stabilize large classes of nonlinear systems is explained in terms of an effective narrowing of the nonlinear sector. A feature of the approach taken here is that a deterministic (i.e., strong) concept of stability is established under probabilistic (i.e., weak) assumptions on the dither.**

I Introduction

A DITHER is a high-frequency signal introduced into a system with the object of modifying its nonlinear characteristics. By sweeping back and forth quickly across the domain of a nonlinear element, a dither has the effect of averaging the nonlinearity making it smoother and in some sense less nonlinear. Usually dithers are periodic or stationary-random functions whose frequencies lie above the system cut-off frequency and which are therefore filtered out before reaching the output.

By dithering a system it is possible to augment stability, quench undesirable limit-cycles, and reduce nonlinear distortion under a surprisingly wide range of conditions. In servomechanism design dithering is frequently used for these purposes; for example, in systems with relays (MacColl [2]), dry friction (Besekerski [3]), and many others. From the point of view of theory, dither phenomena are interesting because they produce a self-linearizing tendency which appears to be a factor in a variety of nonlinear problems.

There have been many studies of dither by empirical methods of the describing function and statistical-linearization type (see for example, Pervozvanskii [4]), and there is a good understanding of the subject at an intuitive level. However, there seem to have been no rigorous general[2] analyses, especially in the area of feedback stability, and the precise manner in which dither augments stability has remained unclear.

In this paper, an analysis will be given of a simple but representative class of dithered systems, illustrated in Fig. 1. Here H is linear and time invariant, Φ is a memoryless nonlinearity, and w' is a dither signal.

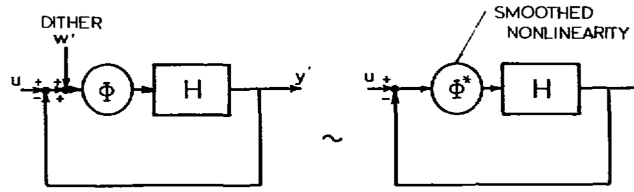

Fig. 1. A dithered feedback system.

It will be shown that the behavior of the dithered system depends on the amplitude-distribution function $F_w(\cdot)$ of w', and $F_w(\cdot)$ will be assumed to be repetitive in time. It is emphasized that $F_w(\cdot)$ will not generate a random process, but will be used in a purely deterministic setting. The behavior of the dithered system will be shown to be comparable to that of a smoothed system (see Fig. 1) in which the smoothed nonlinearity $\Phi^*(\cdot)$ is the convolution of $\Phi(\cdot)$ and $dF_w(\cdot)$. A comparison test will be derived, relating the stability of the smoothed system to that of the dithered original whenever the repetition frequency is high enough.

For example, if the dither w is the (aperiodic) sawtooth function of Fig. 2(a), its distribution function F_w repeats at intervals of ΔT, and is uniformly distributed on $[-A,A]$. If Φ has the saturating characteristic of Fig. 2(c), the resulting Φ^* is shown in the same figure. Observe that Φ^* lies in a narrower sector.

It is well known [4] that the stability of a feedback system depends on the sector of its nonlinear element. We shall show that dither in effect narrows the sector and that this narrowing underlies the ability of dithers to stabilize large classes of systems.

A feature of our analysis will be that a deterministic (i.e., strong) concept of stability will be obtained under a probabilistic (i.e., weak) description of the dither. The approaches taken here to averaging the nonlinearity and reducing the feedback equations to a pair of simultaneous inequalities appear to be novel.

II. Formulation of the Problem

A. Preliminaries

We adopt the framework of input-output stability, along the lines of [5]. Good overviews are [6] and [7].

Let $R(R^+)$ denote the (nonnegative) reals. L_{2e} is the linear space of R-valued locally square-integrable functions on R^+, equipped with a one-parameter family of projection operators $P_T: L_{2e} \to L_{2e}$, $T \geq 0$, with the property that for $x \in L_{2e}$, $(P_T x)(t) = x(t)$ for $t \leq T$, or 0 for $t > T$. L_2 is the normed linear subspace of L_{2e} consisting of square-integrable functions and carrying the norm

Manuscript received July 11, 1975; revised May 18, 1976. Paper recommended by G. L. Blankenship, Chairman of the IEEE S-CS Stability, Nonlinear, and Distributed Systems Committee.

G. Zames is with the Department of Electrical Engineering, McGill University, Montreal, P.Q., Canada.
N. A. Shneydor is with the Department of Electrical Engineering, Technion—Israel Institute of Technology, Haifa, Israel.

[1] This paper is based on the Ph.D. dissertation of N. A. Shneydor [1].
[2] A special class of rectangular dithers has been studied by Steinberg and Kadushin [11] using a method based on a theorem of Warga.

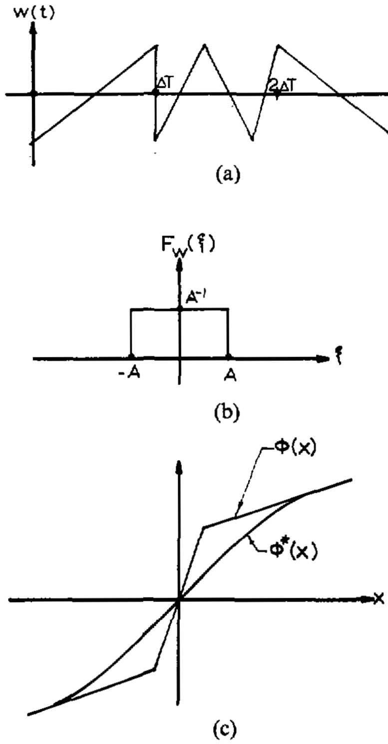

Fig. 2. An F-repetitive dither, its ADF, and the smoothed nonlinearity.

$\|x(\cdot)\|=(\int_0^\infty x^2\,dt)^{1/2}$. The equivalent symbols P_Tx and x_T will be used interchangeably, and similarly for $\|P_Tx\|$ and $\|x\|_T$.

An operator is any map of L_{2e} into itself which maps 0 into 0, and has the causality property $P_TH=P_THP_T$ for all $T\geqslant 0$. If H is an operator mapping a normed subspace A of L_{2e} into another normed subspace B, then H is AB-bounded if $\sup\left\{\frac{\|Hx\|_B}{\|x\|_A}\Big/ x\in A, x\neq 0\right\}<\infty$, and the supremum is the AB-induced norm of H, denoted by $g_{AB}(H)$. H is AB-Lipschitz if $\sup\left\{\frac{\|Hx-Hy\|_B}{\|x-y\|_A}\Big/ x,y\in A, x\neq y\right\}<\infty$. The supremum is the AB-induced Lipschitz norm of H, denoted by $\mathring{g}_{AB}(H)$. Whenever $A=B$, only one symbol will be shown, for example $g_A(H)\triangleq g_{AA}(H)$. In the special case $A=B=L_2$, all norm subscripts will be omitted, i.e., $\|x\|\triangleq\|x\|_{L_2}$.

A function $\Psi:R\rightarrow R$ lies in the *incremental sector* $\{\alpha,\beta\}$, where $-\infty<\alpha\leqslant\beta<\infty$, if

$$\beta=\inf\{\beta'\in R/\Psi(x)-\Psi(y)\leqslant\beta'(x-y),\quad \forall x,y\in R, x\geqslant y\} \tag{1}$$

$$\alpha=\sup\{\alpha'\in R/\Psi(x)-\Psi(y)\geqslant\alpha'(x-y),\quad \forall x,y\in R, x\geqslant y\}. \tag{2}$$

Ψ lies in the *sector* $\{\alpha,\beta\}$ if (1) and (2) are valid with the condition $\forall y\in R$ replaced by $y=0$. If Ψ lies in the (incremental) sector $\{\alpha,\beta\}$, then the center $\theta\triangleq 1/2(\beta+\alpha)$ and radius $\rho\triangleq 1/2(\beta-\alpha)$ of the sector are constants determined by α and β. Ψ is called *Lipschitz* if it lies in an incremental sector $\{\alpha,\beta\}$, and the *Lipschitz constant* of Ψ is $\gamma\triangleq\max(|\alpha|,|\beta|)$.

B. The Dithered Feedback Equation

The system of Fig. 1(a) satisfies the *feedback equation*

$$y'=H\Phi(u-y'+w') \tag{3}$$

in which

1) u, y', and w' are elements of L_{2e} representing input, output, and dither, respectively;

2) H is a convolution operator in L_{2e}, $(Hx)(t)=\int_0^\infty h(t-\tau)x(\tau)d\tau$, $\forall x\in L_{2e}$, $\forall t\geqslant 0$, with $h(\cdot)\in L_1[0,\infty)$ and Fourier transform $\hat{H}(j\omega)=\int_0^\infty h(t)\exp(-j\omega t)dt$;

3) Φ is a (memoryless) operator in L_{2e}, determined by $(\Phi x)(t)=\varphi(x(t))$, where $\varphi(\cdot):R\rightarrow R$ is a given function satisfying the conditions $\varphi(0)=0$, and whose Lipschitz constant is γ.

It will be assumed that for each u and w' in L_{2e}, (3) has a unique solution for y' in L_{2e}.

C. The Unbiased Equation

The rest solution of (3), y'_0, is obtained when $u=0$, and is $y'_0=H\Phi(-y'_0+w')$. In general, $y'_0\neq 0$ and (3) will therefore be called *biased*. The rest solution can be viewed as a residual "ripple" attributable to the dither.

We are interested in the deviations $(y'-y'_0)$ of y' from rest, and therefore introduce the new variables

$$y\triangleq y'-y'_0 \quad \text{(the output deviation)}$$

$$w\triangleq w'-y'_0 \quad \text{(the effective dither)}$$

and the *unbiased nonlinearity* $\Phi_0:L_{2e}\rightarrow L_{2e}$, $\Phi_0(x)\triangleq\Phi(x+w)-\Phi(w)$ for all $x\in L_{2e}$. The new variables satisfy the following *unbiased dithered feedback equation*,

$$y=H\Phi_0(u-y)=H\Phi(u-y+w)-H\Phi(w). \tag{4}$$

Equations (3) and (4) are equivalent, i.e., if w' and y' in L_{2e} satisfy the former equation, then w and y in L_{2e} satisfy the latter, and vice-versa. The sector, center, radius, and Lipschitz constants of φ_0 are identical to those of φ, but φ_0 is time varying.

An explanation of how the new variables are to be interpreted is perhaps in order. There is a class of engineering problems, to which our results are most readily applicable, in which the dither is at the designer's disposal to be shaped for efficient stabilization. In such problems a desirable w is selected first and w' (the signal actually to be generated) can be found explicitly from the equation $w'=w+H\Phi w$. There is a more difficult class of problems, however, in which w' is given, and $w=(I+H\Phi)^{-1}w'$. Calculation of the inverse amounts to the solution of an integral equation. In the latter case, the question of whether solutions for y depend continuously on w' assumes a practical importance, but is deferred to a sequel.

Henceforth, attention will be confined to the unbiased equation (4), and so results will be obtained on the behavior of (3) relative to its rest solution. The term "dither" will mean w.

Let $G_0 : L_{2e} \to L_{2e}$ be the *closed-loop operator* which maps each $u \in L_{2e}$ into the corresponding solution $y \in L_{2e}$ of (4). The object of this paper can now be stated: *to find conditions on the dither w which ensure that G_0 is L_2-bounded.*

D. Amplitude Distribution Functions

Let $\mu(\cdot)$ denote the length of a Lebesgue-measurable subset of R. Let $v(\cdot) \in L_{2e}$, and (t_1, t_2) be any subinterval of R^+.

Definition: The *amplitude distribution function* (ADF) of $v(\cdot)$ on (t_1, t_2) is the function $F_v : R \to [0,1]$,

$$F_v(\xi) = \frac{\mu\{t / t \in (t_1, t_2), v(t) \leqslant \xi\}}{(t_2 - t_1)}.$$

The properties of ADF's are discussed in books on probability (see Doob [7]). $F_v(\xi)$ is monotone nondecreasing in ξ, continuous on the right, $\lim_{\xi \to -\infty} F_v(\xi) = 0$, and $\lim_{\xi \to \infty} F_v(\xi) = 1$. Any function having these properties will be called an ADF.

Definition: v is *F-repetitive*, or F_v is *repetitive*, if there is a sequence $\{t_i\}$, $0 = t_0 < t_1 < \cdots$, unbounded from above, such that for $i = 1, 2, \cdots$, the ADF of $v(\cdot)$ on (t_{i-1}, t_i) equals the ADF of $v(\cdot)$ on (t_0, t_1). The *maximal repetition interval*, is the supremum $\sup\{t_i - t_{i-1} / \ i = 1, 2, \cdots\}$. The sequence $\{t_i\}$ will be called the *F_v-partition.*

Assumption: Henceforth, the dither w of (4) will be fixed, F-repetitive function in L_{2e}, with maximal repetition interval ΔT, and F_w-partition $\{t_i\}$.

The assumption that the dither is F-repetitive implies that the ADF of the dither repeats at a high enough (but not necessarily periodic) rate. The dither itself need not repeat or be periodic. Of course, periodic dithers are F-repetitive.

E. The Smoothed Equation

Definition: For any $\Psi : R \to R$, the *smoothed image* $\Psi^* : R \to R$ is

$$\Psi^*(\eta) = \int_{-\infty}^{\infty} \Psi(\xi)\, d_\xi F_w(\xi - \eta), \tag{5}$$

provided the Lebesgue–Stjeltjes integral (5) exists for almost all η. Moreover, $\Psi_0^* : R \to R$ is defined by the equation $\Psi_0^*(\eta) = \Psi^*(\eta) - \Psi^*(0)$.

Definition: The *smoothed images* of Φ and Φ_0 are the operators Φ^* and Φ_0^* in L_{2e} satisfying the identities $(\Phi^* x)(t) = \varphi^*(x(t))$ and $(\Phi^* x)(t) = \varphi_0^*(x(t))$ for all $x \in L_{2e}$ and $t \in R^+$.

The *smoothed feedback equation* is

$$y^* = H\Phi_0^*(u - y^*) = H\Phi^*(u - y^*) - H\Phi^*(0), \tag{6}$$

u and y^* being in L_{2e}.

It is shown in Appendix A that whenever φ lies in a (incremental) sector $\{a, b\}$, φ^* lies in a (incremental) sector $\{a^*, b^*\}$ and $a \leqslant a^* \leqslant b^* \leqslant b$. In other words *the smoothed nonlinearity φ^* lies in a (incremental) sector not greater than that of the original φ.* By a suitable choice of dither dithribution, the smoothed sector can often be made much smaller than the original (see Fig. 2).

Let γ^* denote the Lipschitz constant of φ^*.

F. High-Frequency Attenuation and the Sobolev Space S_{2p}

The high-frequency attenuation properties of H will be expressed by stipulating that its range be a Sobolev space (see Yosida [9]).

Definition: For any constant $p > 0$, the *Sobolev space* S_{2p} is the linear subspace of L_2 containing every function $x(\cdot)$ which is absolutely continuous on $[0, \infty)$, satisfies $x(0) = 0$, and whose derivative $\dot{x}(\cdot)$ exists, a.e. and is in L_2, and equipped[3] with the norm $\|x\|_{s_2} = (\|x\|^2 + \|p^{-1}\dot{x}\|^2)^{1/2}$.

If $x \in L_2$, a sufficient condition for $(Hx) \in S_{2p}$ is that $\sup_{\omega \in R} |\omega \hat{H}(j\omega)| < \infty$, then (Hx) has a derivative in L_2 by Parseval's theorem, and $(Hx) \in L_2$ because H is L_2-bounded.

The abbreviation g_{LS} will be used for $g_{L_2 S_2}$.

G. Constant Terms and a Formula for p_0

Our results will employ a constant $p_0 > 0$ representing a lower bound to the allowable dither repetition frequency. We shall now give a sequence of formulas which explicitly determine p_0, as well as certain other terms for use in the proofs. The formulas are iterative, i.e., each defines a new constant or function in terms of preceding ones. Let

$$\kappa = g_{s_2}(I - G_0^*)$$

$$k = \sup_{\omega \in R} |H(j\omega)|$$

$$k_D = \sup_{\omega \in R} |\omega H(j\omega)|$$

$$k_{LS} = \sup_{\omega \in R} |(1 + jp^{-1}\omega) H(j\omega)|$$

$$c = \kappa k_{LS} \gamma$$

$$p_1 = \kappa k_{LS}(\gamma + \gamma^*) p\, \Delta T / \sqrt{8}$$

$$p_2 = k_D \gamma / \sqrt{8}$$

$$(\Delta T)_0 = \sup\left\{ \Delta T / \ p_2 \Delta T \leqslant 1,\ p_1 \Delta T + \frac{2cp_2\Delta T}{1 - p_2 \Delta T} \leqslant 1 \right\}$$

$$p_0 = (\Delta T)_0^{-1}.$$

III. Principal Results

Consider the feedback equation (4) and the associated smoothed equation (6). Recall that the dither w is F-repetitive with maximal interval ΔT, and p_0 is a positive constant.

[3]The subscript-free symbol $\|\cdot\|$ will be reserved for the L_2 norm.

Theorem 1 (A comparison test): If

1) $\sup_{\omega\in R}|\omega H(j\omega)|<\infty$, and

2) G_0^* (the smoothed image of G_0) is a causal operator, then, sufficient conditions for G_0 to be L_2-bounded are that G_0^* be S_{2_p}-bounded and $p_0\Delta T<1$.

Corollary 1: If $\sup_{\omega\in R}|\omega H(j\omega)|<\infty$ and $p_0\Delta T<1$, then a sufficient condition for G_0 to be L_2-bounded is that the smoothed system satisfy the circle criterion, i.e., for some constants a and b, $0<a\leqslant b<\infty$,

$$a\leqslant\frac{\varphi^*(\xi)}{\xi}\leqslant b,\qquad \forall\xi\in R,\quad \xi\neq 0,\tag{7}$$

$$|\hat{H}(j\omega)+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)|\leqslant\frac{1}{2}\left(\frac{1}{a}-\frac{1}{b}\right),\qquad \forall\omega\in R,\tag{8}$$

and $\hat{H}(j\omega)$ does not encircle the $-1/2(a^{-1}+b^{-1})$ point.

The proofs of Theorem 1 and its corollary are in Section VI.

Remarks: Assumption 1) of Theorem 1 is a high-frequency attenuation requirement which ensures that the dither is smoothed out by H, provided the dither is fast (i.e., repeats often) enough. The condition $p_0\Delta T<1$ gives an upper bound to the repetition interval ΔT under which smoothing is sufficient. The constant p_0 measures the effective bandwidth of H.

Theorem 1 implies that for fast dithers the stability of the dithered system can be deduced from the S_2-boundedness of the smoothed one. If the smoothed system satisfies the circle criterion it is S_2-bounded, and so the original system must be L_2-bounded.

Stability in the bounded-input-bounded-output sense, or in the asymptotic sense, can be deduced from L_2-boundedness by standard methods [5].

The smoothed nonlinearity lies in a smaller sector than the original, and the stabilizing effect of dither can be viewed as caused by this effective reduction in sector.

Example (see Fig. 2): φ is a saturation, and w is a sawtooth function. Although w is aperiodic, its ADF repeats on intervals of width ΔT. (In general, there is no requirement that the widths of successive intervals should be equal.) The nonlinearity φ and F_w satisfy the equations

$$\begin{aligned}
\varphi(x)&=3x,\ x\in[0,1) & F_w(\xi)&=\frac{1}{8}\xi,\ |\xi|\leqslant 4\\
&=\frac{8}{3}+\frac{1}{3}x,\ x>1 & &=0,\ \xi<-4\\
&=-\varphi(-x),\ x<0 & &=1,\ \xi>4.
\end{aligned}$$

Under these assumptions, the smoothed nonlinearity φ^* satisfies

$$\begin{aligned}
\varphi^*(x)&=x, && \text{for } x\in[0,3),\\
&=-\frac{1}{6}x^2+2x-\frac{3}{2}, && \text{for } x\in(3,5),\\
&=\frac{8}{3}+\frac{x}{3}, && \text{for } x>5,\\
&=-\varphi^*(-x), && \text{for } x<0.
\end{aligned}$$

a b

The (incremental) sectors of φ and φ^* are $\{\frac{1}{3},3\}$ and $\{\frac{1}{3},1\}$, respectively, so dither has effectively reduced the sector-radius by a factor of 4. The region in which $\hat{H}(j\omega)$ may lie without causing instability has correspondingly been increased by the introduction of dither.

The constant p_0 depends on the high-frequency attenuation properties of H, and can be found by means of the sequence of formulas given in Section II-G (but has not been calculated). The repetition interval can be any number smaller than $(\Delta T)^{-1}$.

IV. Averages of φ^* and φ

The distribution of a sum $x+w$ of independent variables is the convolution of their distributions, by a well-known theorem of probability (Doob [8]). A similar result will be obtained here for the special case where w is F-periodic and x is a step-function. It will follow that $\varphi(x+w)$ and $\varphi^*(x)$ have equal averages, and this fact will underlie our method of proof. In later sections, arbitrary elements of a Sobolev space will be approximated by step-functions.

Let (t_m,t_n) be a fixed, finite interval with end-points in the F_w-partition, and $l=t_m-t_n$.

A. Averages and Means

The *average* of any $v\in L_{2e}$ *on* (t_m,t_n), denoted by $\bar{v}$, is

$$\bar{v}\triangleq\frac{1}{l}\int_{t_m}^{t_n}v(t)\,dt.\tag{9}$$

If $\Psi:R\to R$ is Lebesgue–Stjeltjes integrable with respect to F_v, the *mean of* Ψ *with respect to* F_v, denoted by $E_v(\Psi)$, is the Lebesgue–Stjeltjes integral

$$E_v(\Psi)\triangleq\int_{-\infty}^{\infty}\Psi(\eta)\,dF_v(\eta).\tag{10}$$

By a theorem of Van Vleck [10] if either $\overline{\Psi v}$ or $E_v(\Psi)$ exists, they are equal, i.e.,

$$\overline{\Psi v}=E_v(\Psi).\tag{11}$$

B. Step Functions and ADF's of Sums

Definition: A *step-function* is any piecewise-constant function in L_{2e} continuous on the right. It will be assumed that every step-function x is *synchronized* with $w(\cdot)$, i.e., the discontinuity points of $x(\cdot)$ are F_w-partition points.

Lemma 1: If $x(\cdot)$ is any step-function, and all ADF's are measured on (t_m,t_n), then (see Appendix A for proof),

$$F_{x+w}(\xi)=\int_{-\infty}^{\infty}F_w(\xi-\eta)\,dF_x(\eta).\tag{12}$$

Lemma 2: If $\Psi:R\to R$ is Borel-measurable, and x any step-function, then

$$\overline{\Psi(x+w)}=\overline{\Psi^*(x)}\tag{13}$$

whenever either average exists on (t_m,t_n).

In particular, the conclusions of Lemma 2 hold when $\Psi(\cdot)$ is the function $\varphi(\cdot)$.

Proof: If $\overline{\Psi(x+w)}$ exists, then

$$\begin{aligned}\overline{\Psi(x+w)} &= \int_{-\infty}^{\infty} \Psi(\xi)\, dF_{x+w}(\xi), && [\text{by } (10)]\\ &= \int_{-\infty}^{\infty} \Psi(\xi)\, d_\xi \int_{-\infty}^{\infty} F_w(\xi-\eta)\, dF_x(\eta), && [\text{by } (12)]\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi(\xi)\, d_\xi F_w(\xi-\eta)\, dF_x(\eta),\\ &= \int_{-\infty}^{\infty} \Psi^*(\eta)\, dF_x(\eta), && [\text{by } (5)]\\ &= \overline{\Psi^*(x)}\,. && [\text{by } (11)]\end{aligned}$$

Conversely, if $\overline{\Psi^*(x)}$ exists, the same arguments in reversed order prove (13).

As φ is Lipschitz it is Borel-measurable, so it may be substituted for Ψ in Lemma 2. Q.E.D.

V. S_{2p}-Approximations and Properties

In order to exploit the relation between the original φ and smoothed φ^* expressed by (13), L_2 functions will be approximated by step-functions. The approximations will converge for those functions having the smoothness properties associated with S_{2p}.

The F_w-partition is denoted by $\{t_i\}$.

Definition: The *step-average* $\tilde{x}(\cdot)\in L_{2e}$ of $x(\cdot)\in L_{2e}$ is the function

$$\tilde{x}(t)=(t_i-t_{i-1})^{-1}\int_{t_{i-1}}^{t_i} x(\tau)\,d\tau, \quad \forall t\in[t_{i-1},t_i),\ i=1,2,\cdots.$$

$\tilde{x}(\cdot)$ is a step-function, and the best L_2 approximation to $x(\cdot)$ in the sense that if x_s is any other step-function (with discontinuity points in $\{t_i\}$ only), then

$$\|x_s-x\|\geqslant\|\tilde{x}-x\|. \tag{14}$$

Equation (14) is proved by calculating the difference $\|x_s-x\|^2-\|\tilde{x}-x\|^2$ and showing it to be equal to $\|x_s-\tilde{x}\|^2$ which is nonnegative. Another inequality we shall need is

$$\|\tilde{x}\|\leqslant\|x\| \tag{15}$$

as $\|x\|^2-\|\tilde{x}\|^2=\|x-\tilde{x}\|^2\geqslant 0$. Similarly,

$$\|x-\tilde{x}\|\leqslant\|x\|. \tag{16}$$

$(x-\tilde{x})$ is not necessarily small for small ΔT, but its integral is. In fact, if $D^{-1}:L_2\to S_{2p}$ denotes the integration operator, $(D^{-1}x)(t)=\int_0^t x(\tau)\,d\tau$, the following lemma, proved in Appendix B, is obtained.

Lemma 3: For any $x(\cdot)\in L_2$ and $T\in\{t_i\}$,

$$\|D^{-1}(x-\tilde{x})\|_T\leqslant\frac{\Delta T}{\sqrt{8}}\|x-\tilde{x}\|_T\leqslant\frac{\Delta T}{\sqrt{8}}\|x\|_T.$$

Let $D:S_{2p}\to L_2$ denote the differentiation operator, $Dx=\dot{x}$, $\dot{x}$ being the derivative of x, a.e. Let $Q:S_{2p}\to L_{2e}$ be the mapping $Q=I+p^{-1}D$, I being the identity on S_{2p}. Then, by a simple calculation, for any $x\in S_{2p}$,

$$\|x\|_{S_{2p}}=\|Qx\|_{L_2} \tag{17}$$

$$\|P_Tx\|^2+\|P_T\dot{x}\|^2\leqslant\|P_TQx\|^2 \tag{18}$$

$$\|P_Tx\|\leqslant\|P_TQx\|. \tag{19}$$

Lemma 4: For any $x(\cdot)\in S_{2p}$ and $T\in\{t_i\}$,

$$\|x-\tilde{x}\|_T\leqslant\frac{\Delta T}{\sqrt{8}}\|\dot{x}\|_T\leqslant\frac{p\Delta T}{\sqrt{8}}\|Qx\|_T\leqslant\frac{p\Delta T}{\sqrt{8}}\|x\|_{S_{2p}}.$$

In other words, the step-average approximation to x converges as $\Delta T\to 0$ provided $x\in S_{2p}$. The proof of Lemma 4 is in Appendix B.

A. L_2-S_2 Induced Norms

If $\sup_{\omega\in R}|\omega\hat{H}(j\omega)|<\infty$, the range of H is in S_{2p}, so the composition QH is well defined. Then, by Parseval's theorem and elementary Fourier transform properties, the L_2-S_2 induced norm of H is

$$g_{LS}(H)=g(QH)=\sup_{\omega\in R}|(1+p^{-1}j\omega)\hat{H}(j\omega)|. \tag{20}$$

Let $Q^{-1}:L_2\to S_{2p}$ be the inverse of Q. For any operator K it can be shown that

$$\mathring{g}_{S_2}(K)=\mathring{g}(QKQ^{-1}). \tag{21}$$

VI. Proofs of Theorem 1 and Corollary 1

All norms will be L_2-norms.

Consider (4). The input $u\in L_2$ can be replaced by an equivalent smoothed input u_S, defined to be $u_S\triangleq H\Phi(u-y+w)-H\Phi(-y+w)$, which lies in S_{2p}, as Φ is L_2-Lipschitz and H maps L_2 into S_{2p}. From (4) the new equation

$$y=H\Phi(-y+w)-H\Phi w+u_S=H\Phi_0(-y)+u_S \tag{22}$$

is obtained.

A. Approximation of $H\Phi_0$ by $H\Phi_0^*$

The integrals defining φ_0^* converge, because $w\in L_{2e}$ implies w is locally L_1 so $\int_{-\infty}^{\infty}|\xi|\,dF_w(\xi)<\infty$, and as φ is Lipschitz. Let $\delta:L_{2e}\to L_{2e}$ be the mapping,

$$\begin{aligned}\delta(x)&\triangleq H\Phi_0(x)-H\Phi_0^*(x)\\ &=H[\Phi(x+w)-\Phi w-\Phi^*x+\Phi^*(0)].\end{aligned} \tag{22a}$$

δ gives the error in approximating $H\Phi_0$ by $H\Phi_0^*$. δ can be expressed as a sum of three mappings, $\delta=\delta_1+\delta_2+\delta_3$, defined by the equations

$$\delta_1(x)=H\Phi(x+w)-H\Phi(\tilde{x}+w)$$

$$\delta_2(x)=H[\Phi(\tilde{x}+w)-\Phi(w)-\widetilde{(\Phi(\tilde{x}+w)-\Phi w)}\,]$$

$$\delta_3(x) = H\Phi^*\tilde{x} - H\Phi^*x$$

in which the identities $\widetilde{\widetilde{\Phi w}} = \Phi^*(0)$ and $\widetilde{\widetilde{\Phi(\tilde{x}+w)}} = \Phi^*\tilde{x}$, valid by definition of Φ^*, have been employed. Write $\delta_{13} \triangleq \delta_1 + \delta_3$. Equation (22) yields

$$y = H\Phi_0^*(-y) + \delta_{13}(-y) + \delta_2(-y) + u_S. \quad (23)$$

Bounds on δ_{13} and δ_2. The term $\delta_{13}(x)$ represents errors resulting from the approximation of x by a step function. Let T be any point in $\{t_i\}$. It is shown in Appendix C that for any $x \in L_2$,

$$\|Q\delta_{13}x\|_T \leqslant k_{LS}(\gamma+\gamma^*)\|x-\tilde{x}\|_T \quad (24)$$

and, for any $x \in S_{2p}$

$$\|Q\delta_{13}x\|_T \leqslant k_{LS}(\gamma+\gamma^*)\frac{p\Delta T}{\sqrt{8}}\|Qx\|_T. \quad (25)$$

$\delta_2(x)$ passes through the origin at all partition points, and can be viewed as a ripple produced because the dither is imperfectly smoothed. As shown in Appendix C, for any $x \in L_2$,

$$\|\delta_2 x\|_T \leqslant k_D\gamma\frac{\Delta T}{\sqrt{8}}\|x\|_T. \quad (26)$$

Inequalities (25) and (26) show that δ_{13} and δ_2 converge to zero as $\Delta T \to 0$. However, they do so in different norms, δ_{13} in S_{2p} (recall that $\|Qx\| = \|x\|_{S_{2p}}$) and δ_2 in L_2, and they will be treated separately. To separate them, a pair of new variables will be introduced in (23), which will lead to a pair of simultaneous inequalities.

B. A Change of Variables

Define $r \triangleq -\delta_2(-y)$ and $z \triangleq -(y+r)$. The new variables $-r$ and $-z$ represent a ripple term and thederippled output. From (23), the new equation $-z = (H\Phi_0^* + \delta_{13})(z+r) + u_S$ is obtained. Define $J \triangleq H\Phi_0^* + \delta_{13}$. Then, $(I + H\Phi_0^*)z = -\delta_{13}z + J(z) - J(z+r) - u_S$. The assumed existence of G_0^* implies that the inverse $(I+H\Phi_0^*)^{-1}$ exists and equals $E_0^* = I - G_0^*$, so

$$z = E_0^*[J(z) - J(z+r) - \delta_{13}z - u_S]. \quad (27)$$

Lipschitz constant for J. It is shown in Appendix C that for any x and x_1 in L_2 and T in $\{t_i\}$,

$$\|Q(Jx - Jx_1)\|_T \leqslant 2k_{LS}\gamma\|x-x_1\|_T. \quad (28)$$

C. Simultaneous Inequalities for z and r

Let T be any point of the F_w-partition. As all operators in (27) are causal,

$$P_TQz = P_TQE_0^*Q^{-1}[QJ(z_T) - QJ(z_T+r_T) - Q\delta_{13}(z_T) + Qu_S]. \quad (29)$$

Bound on z: As $g(QE_0^*Q^{-1}) = g_{S_2}(E_0^*) = \kappa$, the triangle inequality gives

$$\|Qz\|_T \leqslant \kappa\{\|QJ(z_T) - QJ(z_T+r_T)\|_T + \|Q\delta_{13}(z)\|_T + \|Qu_S\|_T\}. \quad (30)$$

The first two terms on the right-hand side of (30) are bounded using (28) and (25). For (25) to be applicable the condition $x \in S_{2p}$ must be fulfilled. Now $z = \delta_{13}(-y) - y$, so z is in the range of H, and there is a function e in L_{2e} such that $H(e_T)$ is in S_{2p}, and $(\delta_{13}(z))_T = (\delta_{13}(He_T))_T$. Consequently, (25) is applicable.

The last term of (30) is bounded by the inequality

$$\|Qu_S\|_T \leqslant g(QH)\mathring{g}(\Phi)\|u\|_T = k_{LS}\gamma\|u\|_T. \quad (31)$$

Consequently, $\|Qz\|_T \leqslant \kappa[2k_{LS}\gamma\|r\|_T + k_{LS}(\gamma+\gamma^*)(p\Delta T/\sqrt{8})\|Qz\|_T + k_{LS}\gamma\|u\|_T]$ which gives the first of two simultaneous inequalities,

$$(1-p_1\Delta T)\|Qz\|_T \leqslant 2c\|r\|_T + c\|u\|_T. \quad (32)$$

Bound on r: The second simultaneous inequality is obtained as follows. Let $f:(0,1)\to R_1$ be the function $F(x) = x/1-x$. We get

$$\|r\|_T = \|\delta_2(-y)\|_T \leqslant g(\delta_2)\|y\|_T \leqslant k_D\gamma\frac{\Delta T}{\sqrt{8}}(\|z\|_T + \|r\|_T) \quad (33)$$

by (26) and the triangle inequality. But $k_D\gamma\Delta T/\sqrt{8} = p_2\Delta T \leqslant p_0\Delta T < 1$, so (33) can be solved to yield the second of our simultaneous inequalities,

$$\|r\|_T \leqslant f(p_2\Delta T)\|z\|_T \leqslant F(p_2\Delta T)\|Qz\|_T. \quad (34)$$

Solution of the simultaneous inequalities: From (34) applied to (32), an explicit bound on Qz,

$$(1-p_1\Delta T - 2cF(p_2\Delta T))\|Qz\|_T \leqslant c\|u\|_T, \quad (35)$$

is obtained. As $\|y\|_T \leqslant \|r\|_T + \|z\|_T$, $\|y_T\| \leqslant (1 + f(p_2\Delta T))\|Qz\|_T$, [by (34)] so that

$$\|y\|_T \leqslant \frac{c(1+F(p_2\Delta T))}{(1-p_1\Delta T - 2cf(p_2\Delta T))}\|u\|_T \quad (36)$$

where (35) and the fact that $p_1\Delta T + 2cf(p_2\Delta T) < 1$ have been used.

Since (36) holds for all T in the partition, the conclusions of Theorem 1 are true.

D. Proof of Corollary 1

The hypotheses of the Circle Criterion (See [4] and [5]) are fulfilled, therefore, E_0^* is a causal, L_2-bounded operator. But $G_0^* = H\Phi_0^*E_0^*$, so for any $u \in S_{2p}$,

$$\|QG_0^*u\|_T = \|QH\Phi_0^*E_0^*u\|_T \leqslant g_{LS}(H\Phi_0^*)g(E_0^*)\|u\|_T \leqslant k_{LS}\gamma^*g(E_0^*)\|Qu\|_T.$$

Therefore, $g_{S_2}(G_0^*) \leqslant k_{LS}\gamma^*g(E_0^*) < \infty$, G_0^* is S_2-bounded, and the hypotheses of Theorem 1 are fulfilled. The corollary follows. Q.E.D.

Appendix A—ADF Inequalities

Lemma A1: If $\Psi: R\to R$, Ψ^* is defined, the sectors of Ψ and Ψ^* are $\{a,b\}$ and $\{a^*,b^*\}$, and the incremental sectors are $\{\alpha,\beta\}$ and $\{\alpha^*,\beta^*\}$, then $\alpha\leqslant\alpha^*\leqslant\beta^*\leqslant\beta$ and $a\leqslant a^*\leqslant b^*\leqslant b$.

Proof: For any ξ_1 and ξ_2 in R, say $\xi_2\leqslant\xi_1$,

$$\begin{aligned}\Psi^*(\xi_2)-\Psi^*(\xi_1)&=\int_{-\infty}^{\infty}\left[\Psi(\xi_2-\eta)-\Psi(\xi_1-\eta)\right]dF_w(\eta)\\&\leqslant\sup_{\eta\in R}\left[\Psi(\xi_2-\eta)-\Psi(\xi_1-\eta)\right]\\&\quad\cdot\int_{-\infty}^{\infty}dF_w(\eta)\\&\leqslant\beta(\xi_2-\xi_1)\end{aligned}$$

from which $\beta^*\leqslant\beta$. The inequality $\alpha^*\leqslant\alpha$ is proved similarly. The (nonincremental) sector inequalities are obtained by fixing $\xi_1=0$. Q.E.D.

Proof of Lemma 1:

$$\begin{aligned}F_{x+w}(\xi)&=l^{-1}\mu\{t/\ t\in(t_m,t_n),\ (x+w)(t)\leqslant\xi\}\\&=l^{-1}\sum_{i=m+1}^{n}\mu\{t/\ t\in(t_{i-1},t_i),\ (x+w)(t)\leqslant\xi\}\\&=l^{-1}\sum_{i=m+1}^{n}(t_i-t_{i-1})F_w(\xi-x_i)\end{aligned}$$

where $x_i\triangleq x(t_{i-1})$. Therefore,

$$F_{x+w}(\xi)=\sum_{i=m+1}^{n}F_w(\xi-\eta)dF_x(\eta).$$

As $\{x_i\}$ contains all points of increase of F_x, (12) follows. Q.E.D.

Appendix B—Step-Function Approximations

A preliminary lemma is derived first.

Lemma A2: If $x(\cdot)\in L_2[a,b]$ has a derivative $\dot{x}$, a.e., on $[a,b]$, and $x(0)=0$, then $\int_a^b x^2(\sigma)\,d\sigma\leqslant\frac{1}{2}(b-a)^2\int_a^b(\dot{x})^2\,d\mu$.

Proof: By Schwartz's inequality,

$$\begin{aligned}\int_a^b x^2(\sigma)\,d\sigma&=\int_a^b\left(\int_a^\sigma\dot{x}(\mu)\,d\mu\right)^2d\sigma\\&\leqslant\int_a^b(\sigma-a)\int_a^\sigma\dot{x}^2(\mu)\,d\mu\,d\sigma\\&\leqslant\frac{1}{2}(b-a)^2\int_a^b\dot{x}^2\,d\mu.\end{aligned}\qquad\text{Q.E.D.}$$

Proof of Lemma 3: Say $T=t_n$, and let $\tau_i\triangleq\frac{1}{2}(t_{i-1}+t_i)$, $i=1,2\cdots$. Then,

$$\|D^{-1}(x-\tilde{x})\|_T^2=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left[\int_0^t[x(\sigma)-\tilde{x}(\sigma)]\,d\sigma\right]^2dt\tag{A1}$$

$$=\sum_{i=1}^{n}\left\{\int_{t_{i-1}}^{\tau_i}\left[\int_{t_{i-1}}^{t}[x(\sigma)-\tilde{x}(\sigma)]\,d\sigma\right]^2dt+\int_{\tau_i}^{t_i}\left[\int_{t_{i-1}}^{t}[x(\sigma)-\tilde{x}(\sigma)]\,d\sigma\right]^2dt\right\}\tag{A2}$$

where the vanishing of the inner integral of (A1) at t_i (by definition of $\tilde{x}$) has been used. Lemma A2 applied to the integrals in (A2) gives

$$\begin{aligned}\|D^{-1}(x-\tilde{x})\|_T^2&\leqslant\sum_{i=1}^{n}\frac{1}{8}(t_i-t_{i-1})^2\int_{t_{i-1}}^{t_i}[x(\sigma)-\tilde{x}(\sigma)]^2\,d\sigma\\&\leqslant\frac{(\Delta T)^2}{8}\|x-\tilde{x}\|^2\end{aligned}$$

which proves the first inequality of Lemma 3. The second inequality follows from (16). Q.E.D.

Proof of Lemma 4: Let $T=t_n$; $\tau_i=\frac{1}{2}(t_{i-1}-t_i)$, $i=1,2,\cdots,n$; and let x_s be the step function $x_s(t)=x(\tau_i)$ for all $t\in[t_{i-1},t_i)$. An application of (14), followed by an argument resembling the proof of Lemma 3, gives

$$\begin{aligned}\|x-\tilde{x}\|^2&\leqslant\|x-x_s\|^2\\&=\sum_{i=1}^{n}\left\{\int_{t_{i-1}}^{\tau_i}[x(\sigma)-x(\tau_i)]^2\,d\sigma+\int_{\tau_i}^{t_i}[x(\sigma)-x(\tau_i)]^2\,d\sigma\right\}\\&\leqslant\frac{(\Delta T)^2}{8}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\dot{x})^2\,d\mu\end{aligned}$$

which establishes the first inequality of Lemma 4. The other inequalities follow from the definition of Q and the S_{2p} norm. Q.E.D.

Appendix C—Inequalities for Theorem 1

Proofs of (24) and (25): For any $x\in L_2$ and T in the F_w-partition,

$$\|Q\delta_{13}x\|_T=\|QH\{\Phi(x+w)-\Phi(\tilde{x}+w)+\Phi^*\tilde{x}-\Phi^*x\}\|_T.$$

As $g(QH)=g_{LS}(H)=k_{LS}$, and the Lipschitz constants of Φ and Φ^* are γ and γ^*,

$$\begin{aligned}\|Q\delta_{13}x\|_T&\leqslant k_{LS}\{\|\Phi(x+w)-\Phi(\tilde{x}+w)\|_T\\&\qquad+\|\Phi^*\tilde{x}-\Phi^*x\|_T\}\\&\leqslant k_{LS}(\gamma+\gamma^*)\|x-\tilde{x}\|_T\\&\leqslant k_{LS}(\gamma+\gamma^*)\|x\|_T\end{aligned}\tag{A3}$$

which proves (24). For $x\in S_{2p}$, (A3) and Lemma 4 imply (25). Q.E.D.

Proof of (26):

$$\begin{aligned}\|\delta_2x\|_T&=\|D^{-1}DH\{\Phi(\tilde{x}+w)-\Phi w-\widetilde{\Phi(\tilde{x}+w)}+\widetilde{\Phi w}\}\|_T\\&\leqslant g(DH)\|D^{-1}\{\Phi(\tilde{x}+w)-\Phi w-\widetilde{\Phi(\tilde{x}+w)}+\widetilde{\Phi w}\}\|_T\\&\leqslant k_D\frac{\Delta T}{\sqrt{8}}\|\Phi(\tilde{x}+w)-\Phi w\|_T\qquad\text{(A4)}\\&\leqslant k_D\gamma\frac{\Delta T}{\sqrt{8}}\|x\|_T.\qquad\text{(A5)}\end{aligned}$$

Equation (A4) was obtained using Lemma 3, and (15) was used in deriving (A5). Q.E.D.

Proof of (28): From the definition of J and (22a), $Jx = H\Phi(w+x) - H\Phi w - \delta_2(x)$, so that

$$\begin{aligned}\|Q(Jx - Jx_1)\|_T &= \|QH\{\Phi(w+x) - \Phi(w+x_1) \\ &\quad - \Phi(w+\tilde{x}) + \Phi(w+\tilde{x}_1) \\ &\quad + \widetilde{\Phi(w+\tilde{x})} - \widetilde{\Phi(w+\tilde{x}_1)}\}\|_T \\ &\leqslant g(QH)\{\|\Phi(w+x) - \Phi(w+x_1)\|_T \\ &\quad + \|\Phi(w+\tilde{x}) - \Phi(w+\tilde{x}_1)\|_T\} \qquad \text{(A6)} \\ &\leqslant 2k_{LS}\gamma\|x - x_1\|_T \qquad \text{(A7)}\end{aligned}$$

which implies the desired result (28). The fact that $\|\tilde{z}\| \leqslant \|z\|$ for any $z \in L_2$ was used in deriving (A6) and (A7).

References

[1] N. Schneydor, "Dither in nonlinear systems," Ph.D dissertation, Dept. Elec. Eng., Technion—Israel Inst. Technol., Haifa, Feb. 1974.

[2] L. A. MacColl, *Fundamentals of Servomechanisms.* Princeton, NJ: Van Nostrand, 1945, pp. 78–87.

[3] V. A. Bersekerskii, "Applying vibrators to eliminate nonlinearities in automatic regulators," *Automat. i Telemech.*, no. 8, pp. 411–417, 1947.

[4] A. A. Pervozvanskii, "An approximate method for the investigation of self-oscillating systems with random perturbations," *Izv. Akad. Nauk SSSR*, OTN, no. 3, pp. 14–24, 1958.

[5] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems—Part I: Conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 228–238, Apr. 1966.

——, "On the input-output stability of time-varying nonlinear feedback systems—Part II: Conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 465–476, July 1976.

[6] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Ouput Properties.* New York: Academic, 1975.

[7] J. C. Willems, *The Analysis of Feedback Systems.* Cambridge, MA: M.I.T. Press, 1971.

[8] J. L. Doob, *Stochastic Process.* New York: Wiley, 1953.

[9] K. Yosida, *Functional Analysis.* New York: Springer, 1968.

[10] E. B. Van Vleck, "Haskin's momental theorem and its connection with Stjeltjes's problem of moments," *Trans. Amer. Math. Soc.*, vol. 18, pp. 326–330, 1917.

[11] A. M. Steinberg and I. Kadushin, *Stabilization of Nonlinear Systems with a Dither Control*, Dept. Aeoro. Eng., Technion—Israel Inst. Technol., Haifa, TAE Rep. 150, 1972.

G. Zames (S'57–M'61) received the Sc.D. degree in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 1960.

From 1965 to 1970 he was Senior Scientist at the NASA Electronics Research Center, Cambridge, MA, and has served on the faculties of the Massachusetts Institute of Technology from 1960 to 1965; Harvard University, Cambridge, MA, from 1962 to 1963, and the Technion—Israel Institute of Technology, Haifa, from 1972 to 1974. He is presently Professor of Electrical Engineering at McGill University, Montreal, P.Q., Canada. Among the awards he has received are the Guggenheim, National Academy of Sciences, and Athlone (Imperial College) Fellowships, British Association Medal (McGill University) and AACC Best Paper Prize.

Dr. Zames is a member of the Editorial Board of the *SIAM Journal on Control*, and author of many papers on control and nonlinear systems.

N. A. Shneydor was born in Jerusalem, Israel, on August 28, 1932. He received the B.Sc. and M.Sc. degrees in electrical engineering, and the D.Sc. degree from the Technion—Israel Institute of Technology, Haifa, in 1955, 1961, and 1975, respectively.

From 1954 to 1958, he was in the Israeli Navy where he specialized in fire control systems, in general, and servo equipment, in particular. From 1958 to 1960, while studying for the M.Sc. degree, he was an instructor in electrical engineering at the Technion—Israel Institute of Technology. In 1960 he joined the Scientific Department of the Ministry of Defense. His areas of research there were feedback systems and guidance and control. He joined the Technion—Israel Institute of Technology in 1970 where his teaching and research are mostly in the area of nonlinear systems.