Mathematics & Music (AMath390) Lecture Notes on Tuning & Temperament

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1 Definitions

Some definitions

fundamental (frequency)
lowest frequency of a note/sound. Generally the perceived pitch

overtones
frequencies above the fundamental. The first component above the fundamental is the first overtone.

partials
The mth partial is the mth frequency component present.

harmonics
frequencies at integer multiples of the fundamental.

Examples of using terms

For a clarinet,

- fundamental $\frac{c}{4\ell}$; also first harmonic
- second harmonic $\frac{2c}{4\ell}$ is missing
- $\frac{3c}{4\ell}$ is second partial or third harmonic or first overtone

Characteristics of Sound

pitch
frequency of vibration of a sound (20Hz-20,000Hz is range audible to humans)

timbre
corresponds to frequency response; in particular overtones and their relative strength

amplitude
magnitude of vibration; corresponds roughly to loudness (¡mm)

duration
length of time note sounds

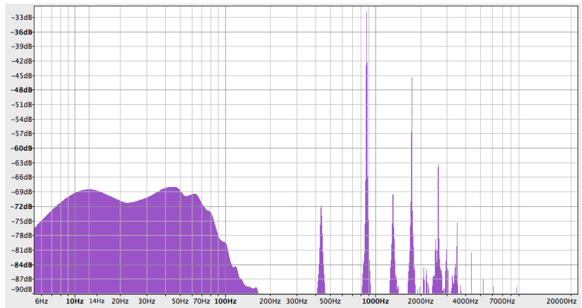
2 Harmonic Series

Home Note

- frequency associated with a given note is a matter of convention
- A4 is now standard at 440Hz in North America, but in the Baroque era, it was almost half a tone lower, and varied significantly
- how notes relate to other notes is critical
- almost all music has a note about which the piece revolves

Harmonics

- For string and wind instruments, partials occur in integer multiples
- amplitude is heard on a logarithmic scale
- frequency also perceived logarithmic



Flute recording (with background noise)

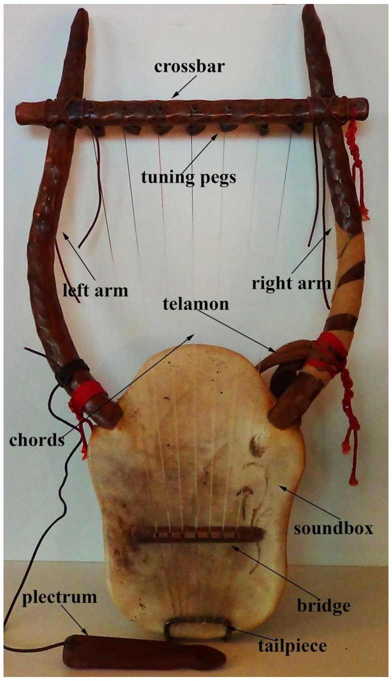
Partials for Flute

partial	kf_1 (Hz) (actual freq)	wavelength	note name
1	434	1	A4
2	869	1/2	A5
3	1302	1/3	E6
4	1738 (1736)	1/4	A6
5	2170 (2139)	1/5	C7
6	2604 (2609)	1/6	E7
7	3038 (?)	1/7	-
8	3472 (3503)	1/8	A7

Constructing a Scale

String with fundamental frequency f

- string with twice the length sound good together
- $2f$ frequency ; 2nd partial
- Greeks noticed that strings in ratio of 3/2 also sounded good together: $3/2f$
- all the harmonics of $2f$ are contained in the harmonics of f; half of the harmonics of $3/2f$ are contained in those of f
- called a *fifth*



3 Pythagorean scale

Pythagorean scale

Fill in notes by iterating on $\frac{3}{2}f$; dropping by an octave whenever the note is above $2f$. Letting $f = 1$, this yields

$\frac{3}{2}$, $\frac{3^2}{2^2}\frac{1}{2}$, $\frac{3^3}{2^4}$, $\frac{3^4}{2^5}\frac{1}{2}$, $\frac{3^5}{2^7}$

do	re	mi	fa	so	la	ti	do
1	$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3}{2}$	$\frac{3^3}{2^4}$	$\frac{3^5}{2^7}$	2
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

Intervals:

$$\left| \ \frac{9}{8} \ \right| \ \frac{9}{8} \ \left| \ \frac{256}{243} \ \right| \ \frac{9}{8} \ \left| \ \frac{9}{8} \ \right| \ \frac{9}{8} \ \left| \ \frac{256}{243} \ \right|$$

$$big \ - \ big - small - (big) - big - big - small$$

$$\frac{9}{8} = 1.125 \qquad (\frac{256}{243})^2 = 1.1098 \approx \frac{9}{8}$$

Music played with Pythagorean tuning

Ancient Greek music: `http://www.youtube.com/watch?v=elERNFoEf3Y`

Transposition

Often it is desired to have a musical piece based a different note as the tonic (home note). In order to be able to do this, a new scale is needed with the new tonic. To do this, consider first moving up from the home note "C" to "G" and use "G" as the new home. note (tonic).

The scale can be created either using the recursion formula ($\frac{3}{2}$, mod 2) or intervals. A Pythagorean scale has the pattern of intervals between notes

$$\mid \tfrac{9}{8} \mid \tfrac{9}{8} \mid \tfrac{256}{243} \mid \tfrac{9}{8} \mid \tfrac{9}{8} \mid \tfrac{9}{8} \mid \tfrac{256}{243} \mid .$$

Consider first creating the scale using intervals. If the tonic has frequency $\frac{3}{2}$, then the notes in the new scale are

$$\begin{aligned} & \tfrac{3}{2} \\ \frac{3}{2}\frac{9}{8} &= \tfrac{3^3}{2^4} \\ \frac{3^3}{2^4}\frac{9}{8} &= \tfrac{3^5}{2^7} \\ \frac{3^5}{2^7}\frac{256}{243} &= 2 \\ 2\frac{9}{8} &= \tfrac{3^2}{2^2} \\ \frac{3^2}{2^2}\frac{9}{8} &= \tfrac{3^4}{2^5} \\ \frac{3^4}{2^5}\frac{9}{8} &= \tfrac{3^6}{2^8} \\ \frac{3^6}{2^8}\frac{256}{243} &= 3. \end{aligned} \tag{1}$$

Mapping the notes down an octave by dividing by 2 so all the frequencies are between 1 and 2 yields

$$1,\ \frac{3^2}{2^3},\ \frac{3^4}{2^6},\ \frac{3^6}{2^9},\ \frac{3}{2},\ \frac{3^3}{2^4},\ \frac{3^5}{2^7},\ 2 \tag{2}$$

These are all notes as the original scale, except for the new note $\frac{3^6}{2^9}$ (or $\frac{3^6}{2^8}$) which lies between F and G. This note replaces F in this scale starting on G.

Alternatively, the scale can be created by staring with $\frac{3}{2}$ and multiplying by $\frac{3}{2}$ each time. Dividing so the notes are between $\frac{3}{2}$ and 3 yields the column (1) above; dividing by 2 so they are between 1 and 2 yields the second description (2). (In fact, dividing so the notes are between 1 and 2 yields the original scale until $\frac{3^5}{2^7}$; an additional step yields the new note.)

Similarly, a new scale can be created by moving down from 2 to 4/3 "F" and starting with "F". A new note is created with each new scale.

These two scales are shown in Table 1. This process can be repeated. Each time a scale is created, starting a new note, either a fifth up, or a fifth down, from the previous scale, a new note is created. This is illustrated in Table 1 where the scales created after 6 steps in either direction are shown.

Since there are 7 notes in the scale, a further permutation in either direction should bring us back to the original home note. Moving to the fifth note in the P^6 leads to the new tonic $\frac{3^7}{2^11} = 1.07$. Similarly, moving down a fifth (or up a fourth) from the P_6 scale takes us to the new tonic $\frac{2^8}{3^6} = 1.05$. Neither note is the same as the original tonic, 1, and they are not equal to each other. Further permutations can continue indefinitely; each time a new note is introduced.

Particularly for keyboard instruments, where the tuning cannot be adjusted by the musician, it is desirable for successive key shifts to return to the original note (modulo an octave).

Mathematically, integers n, m are needed so that

$$(\frac{3}{2})^m = 2^n$$

$$3^m = 2^{n+m} \quad \text{NOT POSSIBLE}$$

$$2^7 \approx (\frac{3}{2})^{12}.$$

Going up 12 5^{th}s then down 7 octaves takes you back to almost where you started.

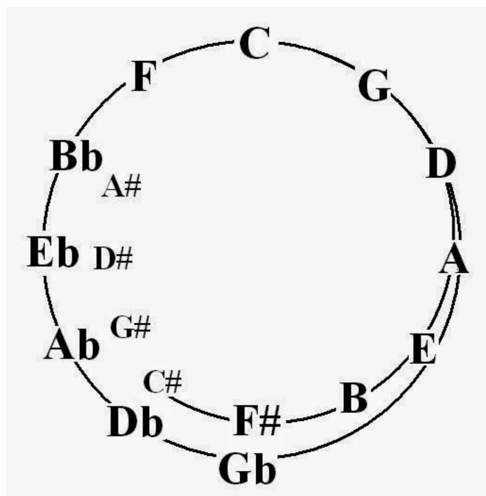


Figure 1: Transposition into a scale with a different tonic creates new notes in a never ending spiral. This is one problem with Pythagorean tuning.

	C		D		E	F		G		A		B	C
P^6		$\frac{3^7}{2^{11}}$		$\frac{3^9}{2^{14}}$		$\frac{3^{11}}{2^{17}}$	$\frac{3^6}{2^9}$		$\frac{3^8}{2^{12}}$		$\frac{3^{10}}{2^{15}}$	$\frac{3^5}{2^7}$	
P^5													
P^4													
P^3													
P^2													
P^1	1		$\frac{3^2}{2^3}$		$\frac{3^4}{2^6}$		$\frac{3^6}{2^9}$	$\frac{3}{2}$		$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$	2
P	1		$\frac{3^2}{2^3}$		$\frac{3^4}{2^6}$	$\frac{2^2}{3}$		$\frac{3}{2}$		$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$	2
P_1	1		$\frac{3^2}{2^3}$		$\frac{3^4}{2^6}$	$\frac{2^2}{3}$		$\frac{3}{2}$		$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$		2
P_2													
P_3													
P_4													
P_5													
P_6		$\frac{2^8}{3^5}$		$\frac{2^5}{3^3}$		$\frac{2^2}{3}$	$\frac{2^{10}}{3^6}$		$\frac{2^7}{3^4}$		$\frac{2^4}{3^2}$	$\frac{2^{12}}{3^7}$	

Table 1: Scales in Pythagorean system. The tonic (home note) is indicated in red.

Intervals and Pythagorean Tuning

- often sounds from the same scale do not sound "harmonious" together
- intervals other than octave, fourth, and fifth can sound discordant
- limits polyphony

Harmonics and Harmony

do	re	mi	fa	so	la	ti	do	
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2	Pythagorean Scale

- All the harmonics of $2f$ are in common with those of f
- $1/2$ the harmonics of $\frac{3}{2}f$ are in common with those of f
- $1/3$ harmonics of $\frac{4}{3}f$ are in common with those of f

- Most of the notes in Pythagorean scale are in ratios of large numbers of the root note.

4 Just Intonation

Create a scale using low ratios:

do	re	mi	fa	so	la	ti	do
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

- interval between the third note and the root (do-mi) is called a third; fourth, fifth and sixth are defined similarly
- ratios for a third, fourth, fifth, sixth are low and notes sound harmonious
- major triad (do-mi-so) very important in Western music
- ratio of $do - mi - so$ is $1 : \frac{5}{4} : \frac{3}{2}$ or $4 : 5 : 6$; on the fourth, fa have $\frac{4}{3} : \frac{5}{3} : 2$ or $4 : 5 : 6$
- Set the remaining intervals to have same ratios in a triad built on the fifth: $\frac{3}{2} : x : y$ or $4 : 5 : 6$ yields $x = \frac{15}{8}$ and $y = \frac{18}{8}$ or $y = \frac{9}{8}$.
- difference between a Pythagorean third and a just third is

$$\frac{\frac{81}{64}}{\frac{5}{4}} = \frac{81}{16 \cdot 5} = \frac{81}{80} = 1.0125$$

  This interval is called a *syntonic comma* or just a *comma*

Scale in Just Intonation

do	re	mi	fa	so	la	ti	do
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

Intervals:

$$\Big| \ \frac{9}{8} \ \Big| \ \frac{10}{9} \ \Big| \ \frac{16}{15} \ \Big| \ \frac{9}{8} \ \Big| \ \frac{10}{9} \ \Big| \ \frac{9}{8} \ \Big| \ \frac{16}{15} \ \Big|$$

The scale does not have the same tone-tone-semitone pattern as for Pythagorean

Here is an example of the choice of different temperaments on a violin: `https://www.youtube.com/watch?v=buZOs-czOUg`

Tuning demonstrations

- Several clips of "Innsbruck" by Isaac (c. 1500) in different tuning systems

  Innsbruck-Equal

  Innsbruck-Pythagorean

  Innsbruck-Just

Problems with Just Intonation- Transposition

Just Scale

C	D	E	F	G	A	B	C
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

Suppose instead of starting scale on original note "C" with frequency f want to start on the $5th$, "G", which has frequency $\frac{3}{2}f$. Set $f = 1$ since everything is multiplied by it.

$\frac{3}{2}$	$\frac{27}{16}$	$\frac{15}{8}$	2	$\frac{9}{4}\frac{1}{2}$	$\frac{15}{6}\frac{1}{2}$	$\frac{45}{16}\frac{1}{2}$	3
G	$A\sharp$	B	C	D	E	$F\sharp$	G

- Spiral of fifths again, with more and more notes.
- $\frac{3}{2}^m \neq 2^n$
- Transposition worse than Pythagorean: 2 notes introduced.

Chords

- A chord is several notes played at the same time to produced an effect
- Movement through chords important part of Western music

  `https://www.youtube.com/watch?v=oOlDewpCfZQ`

Triads in Just Intonation

Just Scale

do	re	mi	fa	so	la	ti	do
C	D	E	F	G	A	B	
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

- triads on C, F , G (I , IV, V) are ratios of 6 : 5 : 4: "justly" tuned
- D (II) has fifth $\frac{\frac{5}{3}}{\frac{9}{8}} = \frac{40}{27} \neq \frac{3}{2}$
- E (III) has third $\frac{\frac{3}{2}}{\frac{5}{4}} = \frac{12}{10} \neq \frac{5}{4}$

5 Meantone Scales

Meantone scales

- Mean-tone scales sacrifice fifths slightly in order to improve thirds and progression through different chords
- ensure that thirds on C, F, G (I, IV, V) are "just": $\frac{5}{4}$
- take interval within these thirds to be mean of this: $\frac{\sqrt{5}}{2}$
- to keep a Pythagorean scale pattern, have 2 semitones left:

$$(\frac{\sqrt{5}}{2})^5 s^2 = 2$$

so the semitone is

$$s = \frac{8}{5^{\frac{5}{4}}}$$

A meantone scale

One type of mean-tone scale is as follows.

do	re	mi	fa	so	la	ti	do
C	D	E	F	G	A	B	C
1	$\frac{\sqrt{5}}{2}$	$\frac{5}{4}$	$\frac{2}{5^{\frac{1}{4}}}$	$5^{\frac{1}{4}}$	$\frac{5^{\frac{3}{4}}}{2}$	$\frac{5^{\frac{5}{4}}}{4}$	2

- fifth is $r = 5^{\frac{1}{4}} = 1.49535 \approx \frac{3}{2}$
- transposing 4 fifths is a third (modulo octaves)
- 12 fifths is 3 thirds and

$$(\frac{5}{4})^3 = \frac{125}{64} < 2$$

so don't get cycle of fifths

- usually make a fifth large in a seldom used key - yields a very discordant *wolf fifth*

6 Equal Temperament

Temperaments

- Goal is to preserve consonances of octave, fifth, third and sixths while allowing transposition into different keys and chord progressions.
- Can't be done.
- Compromises must be made.

Tempering

> Since, through the grace of God, music has so progressed and changed, it would be absurd if we had not tried to improve the keyboard, so that well-composed modern pieces should not be ruined, and a howl come out of them .... Some would like to say that one should not compose in every key, such as C sharp, F sharp and G sharp. But I say that if one does not do it, another will ...... And why should I set limits for this person or that, and want to prohibit him from composing in this key? .... The free arts want free geniuses.

(Werckmeister, Orgel-Probe, 1698, quoted from Douglas Hollick, http://www.rvrcd.com)

Irregular Temperaments

- possible that a keyboard was tuned differently for different pieces
- various modifications of mean tone to make it easier to play in different keys
- intervals slightly different in different keys so the spiral of fifths becomes a circle
- commonly used for several centuries before 2000
- many different schemes

Equal Temperament

- Original Pythagorean scale has 5 tones and 2 semitones $\rightarrow$ 12 semitones
- Set each semi-tone to $s = 2^{\frac{1}{12}}$, tone is $s^2 = 2^{\frac{1}{6}}$

do	re	mi	fa	so	la	ti	do
C	D	E	F	G	A	B	C
1	$2^{\frac{1}{6}}$	$2^{\frac{1}{3}}$	$2^{\frac{5}{12}}$	$2^{\frac{7}{12}}$	$2^{\frac{3}{4}}$	$2^{\frac{11}{12}}$	2

- shifting through 12 fifths goes back to start

$$(2^{\frac{7}{12}})^{12} = 2^7$$

- cycle of fifths
- all keys sound the same

Equal Temperament (cont.)

do	re	mi	fa	so	la	ti	do
C	D	E	F	G	A	B	C
1	$2^{\frac{1}{6}}$	$2^{\frac{1}{3}}$	$2^{\frac{5}{12}}$	$2^{\frac{7}{12}}$	$2^{\frac{3}{4}}$	$2^{\frac{11}{12}}$	2
cents:	200	400	500	700	900	1100	1200

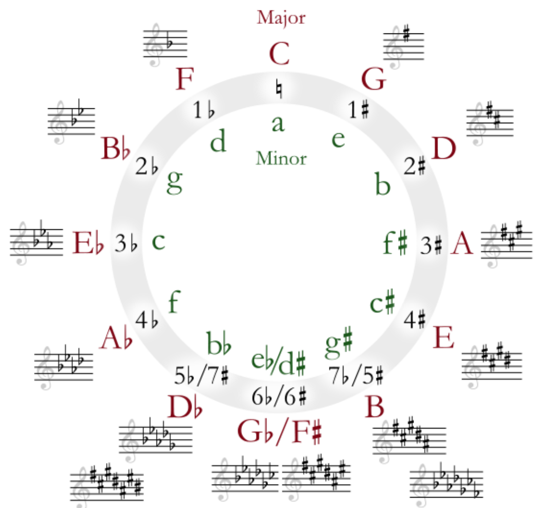
- Converting to cents: 1200 cents =octave . If $r > 1$ is frequency ratio in relation to tonic (home note), in cents

$$1200 \log_2 r = 1200 \ln r / \ln 2$$

- fifth equally tempered fifth: $2^{\frac{7}{12}} = 1.49831 \approx \frac{3}{2}$
- perfect fifth is 702 cents; an equally tempered fifth is 700 cents - very close
- equal tempered third $2^{\frac{1}{3}} = 1.2992$ or 400 cents
- perfect third is 1.25 or 386 cents; an equally tempered third is 400 cents, higher by 14 cents
- perfect sixth is 884 cents; equally tempered sixth is 900 cents, higher by 16 cents

Circle of Fifths

Because a multiple of a "fifth" is a multiple of an octave, spiral of fifths



collapses into a circle

Comparisions

- Prelude & Fugue in F minor by Krebs in meantone and equal temperament `http://www.youtube.com/watch?v=54mE1hxAvyY`[2ex]

  > Key of F minor: Deep depression, funereal lament, groans of misery and longing for the grave

  (Key characteristics from Schubart,1806, reproduced on pg 191 of Benson)

- Pachelbel Canon in just, meantone and equal (11 minutes) `https://www.youtube.com/watch?v=d2I1zNw2w-c`

Equal vs Well tempering

Only an issue for keyboard instruments

Irregular & Meantone	Equal
thirds and sixths harmonic	*almost* harmonic
some keys sound strange	can play in any key
different keys have different character	every key sounds the same

> The equal-tempered scale inherits nearly all the important components of the Pythagorean scale and can also transpose. Now every key sounds as in tune (or out of tune), as every other key, just as we wanted but at the expense of the pure integer ratios, which have been virtually banished. It is somewhat reminiscent of the modern practice where an oak grove is ripped out to build a shopping center and then the shopping center is named Oak Grove. We are left with the impression of the pure intervals but not their reality. We get the advantage of the modern conveniences (transposition) but at the expense of the reason we wanted it. Isn't it interesting that not even music is immune to the inevitable downside of technological advance? The moral: nothing is free:

Loy, *Musimathics*