AMath390

Math & Music **Fall 2016**

Assignment # 5 **SOLUTIONS**

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1. (a) Find the period T, and coefficients in the complex Fourier series for

$$f(t) = 6 + 4\cos(\pi t) - 3\sin(4\pi t).$$

(Hint: You do not need to compute any integrals to find the Fourier series.)

Solution: The period of $\sin(4\pi t)$ is $\frac{2\pi}{4\pi} = \frac{1}{2}$ and the period of $\cos(\pi t)$ is $\frac{2\pi}{\pi} = 2$. Thus, $f(t)$ is periodic with period 2.
The Fourier series with period T has the form

$$c_0 + \sum_{k=-\infty}^{\infty} c_k \exp(\imath k\pi t) = c_0 + \sum_{k=1}^{\infty} \Big(a_k \cos(k\pi t) + b_k \sin(k\pi t) \Big)$$

Comparing with $f(t)$, $c_0 = 6$, $a_1 = 4$, $b_4 = -3$ and all other coefficients are 0. Since, with $n > 0$,

$$c_n = \frac{a_n - \imath b_n}{2}, \quad c_{-n} = \frac{a_n + \imath b_n}{2},$$

$$c_1 = 2,\ c_{-1} = 2,\ c_4 = -\imath\frac{3}{2},\ c_{-4} = \imath\frac{3}{2},$$

and all the other coefficients are 0. The complex Fourier series is

$$6 + 2\exp(\imath\pi t) + 2\exp(-\imath\pi t) + \imath\frac{3}{2}\exp(\imath 4\pi t) - \imath\frac{3}{2}\exp(-\imath 4\pi t).$$

2. (a) Find the Fourier transform $\hat{f}$ of $\sin(\omega_0 t)$ and also of $\cos(\omega_0 t)$. (Recall that the Fourier transform of $e^{2\pi\imath\omega t}$ is $\delta(\nu - \omega)$.)

Solution: Using the definition of the complex exponential (see course notes, page 3),

$$\begin{aligned} \cos(z) &= \frac{1}{2}\Big(\exp(\imath z) + \exp(-\imath z) \Big) \\ \sin(z) &= \frac{1}{2\imath}\Big(\exp(\imath z) - \exp(-\imath z) \Big) \end{aligned}$$

Thus, by linearity of the Fourier transform,

$$\begin{aligned} \widehat{\cos(\omega_0 \cdot)}\nu &= \frac{1}{2}\left(\delta\left(\nu - \frac{\omega_0}{2\pi}\right) + \delta\left(\nu + \frac{\omega_0}{2\pi}\right) \right), \\ \widehat{\sin(\omega_0 \cdot)}\nu &= \frac{1}{2\imath}\left(\delta\left(\nu - \frac{\omega_0}{2\pi}\right) - \delta\left(\nu + \frac{\omega_0}{2\pi}\right) \right). \end{aligned}$$

(b) Suppose $f(t) = \cos(16\pi t)$ is sampled at 10 times per second. Sketch $f(t)$ and the sampled signal f_s on [0,1].

(c) Calculate the Fourier transform $\hat{f}_s$ of the sampled signal in (b). Sketch $\widehat{f}(\nu)$ and $\hat{f}_s(\nu)$ for $|\nu| < 20$.

Solution: The Fourier transform of $\cos(16\pi t)$ is

$$\hat{f}(\nu) = \frac{1}{2}\Big(\delta(\nu - 8) + \delta(\nu + 8)\Big).$$

Since for any function f,

$$\hat{f}_s(\nu) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \widehat{f}\left(\nu - \frac{n}{T}\right),$$

and in this case $\frac{1}{T} = 10$,

$$\hat{f}_s(\nu) = 5\sum_{n=-\infty}^{\infty}\Big(\delta(\nu - 10n - 8) + \delta(\nu - 10n + 8)\Big).$$

In the interval $|\nu| < 20$, the non-zero terms are

$$5\Big(\delta(\nu-8)+\delta(\nu+8)+\delta(\nu-2)+\delta(\nu+2)+\delta(\nu-12)+\delta(\nu+12)+\delta(\nu-18)+\delta(\nu+18)\Big).$$

Note: There are extra components at $|\nu| = 2 < 8$, caused by the sampling.

(d) Repeat part (c), but with a sampled signal obtained with sample rate of 20 samples/second.

Solution: This is exactly, as (c) but with $N = 20$ instead of $N = 10$, which yields

$$\hat{f}_s(\nu) = 10\sum_{n=-\infty}^{\infty}\Big(\delta(\nu - 20n - 8) + \delta(\nu - 20n + 8)\Big).$$

In the interval $|\nu| < 20$, the non-zero terms are

$$10\Big(\delta(\nu - 8) + \delta(\nu + 8) + \delta(\nu - 12) + \delta(\nu + 12)\Big).$$

Note: On the frequency range where the original function is non-zero, say $|\nu| < 10$, the original and the sampled transform now match.

3. Look at the spectra in Figures 1 and 2.

(a) In each figure, what is the fundamental frequency? How many overtones can you identify?

Solution: In Figure 1, the fundamental frequency is about 250 Hz. (The note is in fact C4 with 261Hz but on this graph it can't be distinguished from B3, 247 Hz.) Five overtones above this can be seen; the fourth and fifth are quite small.

In Figure 2, the fundamental frequency is just above 500Hz (C5 at 523 Hz). Four overtones above this can be seen although the fourth is very small.

(b) One sound was produced by my blowing into the open end of a tube, the other was produced with the same tube, but with one end closed. Which is which? Explain your answer briefly with a sentence.

Solution: Closing a tube means cuts the fundamental frequency in half. So the sound recorded in the graph in Figure 1 was produced by the tube with one end closed. Also, the frequency response in Figure 1 is missing the odd harmonics. (This extra information indicates that the note in Figure 1 was in fact C4 since this is the note with fundamental frequency half that of C5.)

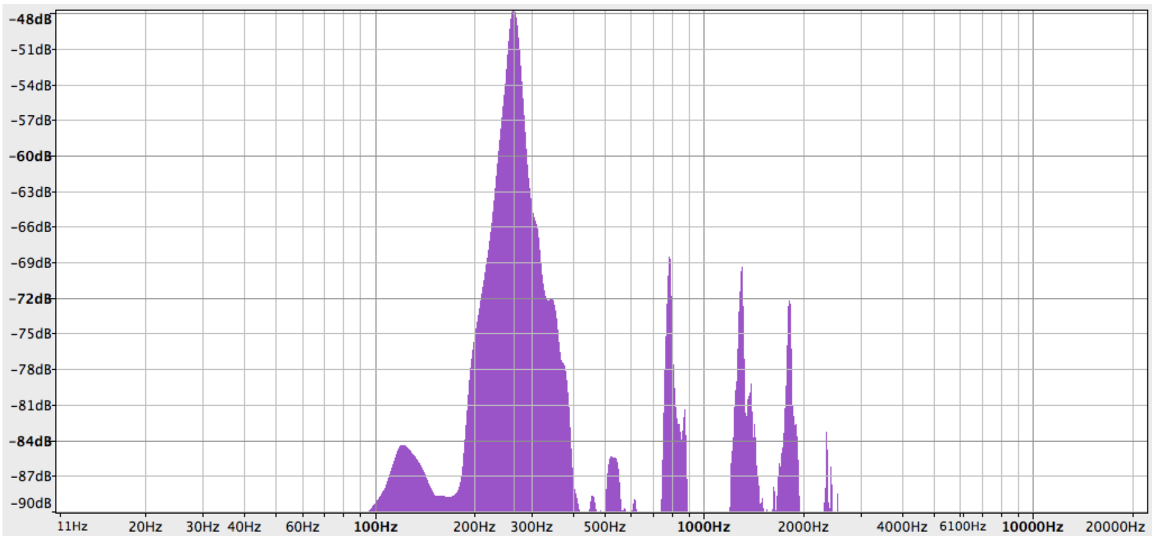

Figure 1: Sound sample (a)

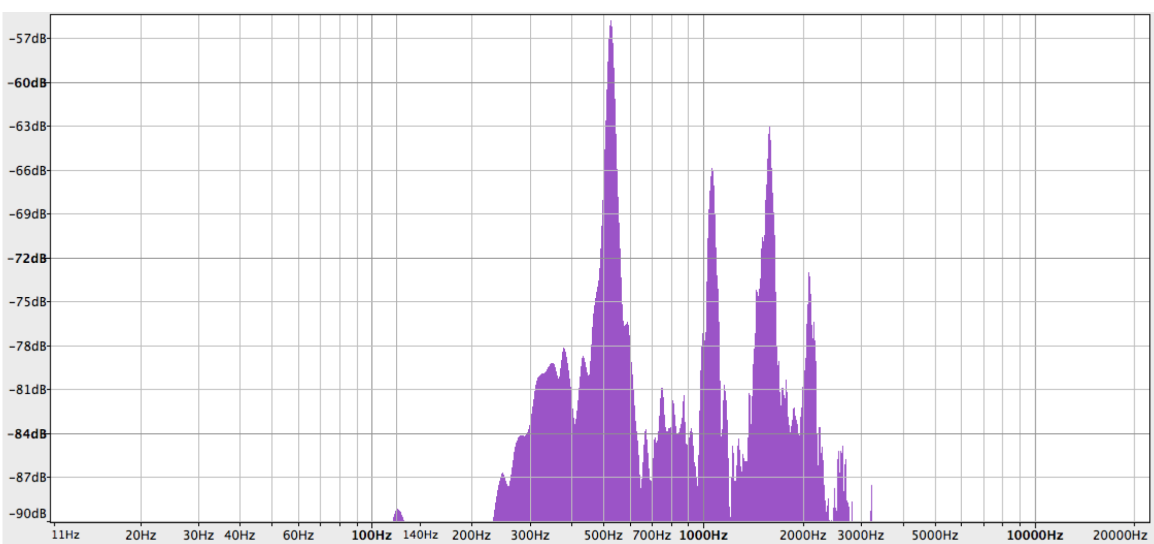

Figure 2: Sound sample (b)