AMath390

Math & Music **Fall 2016**

Extra Questions on Frequency Response and Sampling **SOLUTIONS**

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1. Find the period T, the complex coefficients in the Fourier series and the energy in the interval ($\int_0^T f(t)^2 dt$) of
$$f(t) = 2 + 5\sin 6t + 4\cos 8t.$$
(Hint: You do not to compute any integrals to find the Fourier series.)

Solution: The period of $\sin 6t$ is $\frac{2\pi}{6} = \frac{\pi}{3}$ and the period of $\cos 8t$ is $\frac{2\pi}{8} = \frac{\pi}{4}$. Since
$$3\frac{\pi}{3} = 4\frac{\pi}{4}\pi$$
$f(t)$ is periodic with period π.
The Fourier series is
$$2 + 2\exp(-\imath 8t) + \imath\frac{5}{2}\exp(-\imath 6t) + \imath\frac{5}{2}\exp(\imath 6t) + 2\exp(\imath 8t).$$
The energy is
$$\int_0^{\pi} f(t)^2 dt = 4\pi + 25\int_0^{\pi}\sin^2 6t dt + 16\int_0^{T}\cos^2 8t dt = \frac{49}{2}\pi.$$
(Note this is the sum of the squared magnitudes of the coefficients in the Fourier series, scaled by π. This illustrates the Fourier series version of Parseval's Theoerem.)

2. (a) Find the Fourier series for
$$f(t) = t^2, \quad t \in [-\pi, \pi]$$
where f is extended to all values of t as a 2π periodic function. Calculate both the real and complex forms of the Fourier series.

Solution: Real: All coefficients of sin in the sin/cos form of the Fourier series are 0. The coefficients of $\cos(nt)$, $n \neq 0$, are
$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\cos(nt)dt \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

where the integral can be computed by integrating by parts twice (or use Maple or a table of integrals). Also,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}.$$

Thus, the Fourier series for f, extended to at 2π periodic function is

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nt).$$

Complex:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3} = a_0.$$

For $n \neq 0$, integrating by parts twice yields

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 \exp(\imath nt) dt = \frac{2(-1)^n}{n^2}.$$

The complex form of the Fourier series is

$$c_0 + \sum_{n=-\infty}^{n=\infty} c_n \exp(\imath nt).$$

Using $c_n = \frac{a_n - \imath b_n}{2}$, $c_{-n} = \frac{a_n + \imath b_n}{2}$, reveals that the two representations are identical.

3. Let f be a Fourier-transformable function with Fourier transform $\hat{f}$. Let a be any non-zero constant.
   (a) Show that the Fourier transform of $f(at)$ is $\frac{1}{|a|}\hat{f}(\frac{\nu}{a})$.
   (b) Show that the Fourier transform of $f(t-a)$ is $e^{-\imath 2\pi\nu a}\hat{f}(\nu)$.
   (c) Suppose that f is differentiable and f' is also Fourier transformable. Show that $\widehat{(f')}(\nu) = (\imath 2\pi\nu)\hat{f}(\nu)$.

4. Any Fourier transform $F(\nu)$ of a function $f(t)$ can be decomposed into real and imaginary parts:
$$F(\nu) = R(\nu) + \imath X(\nu)$$
where R, X are real-valued functions.

   Show that if $f(t)$ is a real-valued function then R is an even function and X is an odd function. Then show that
$$F(-\nu) = R(\nu) - \imath X(\nu).$$
   In other words, if f is real-valued, its Fourier transform F satisfies
$$F(-\nu) = \overline{F(\nu)}.$$
   This also implies that $|F(-\nu)| = |F(\nu)|$.

5. A function $f(t) = 0$ outside of the interval $[0, A]$ and its Fourier transform $\hat{f}(\nu)$ is negligible for $|\nu| > \sigma = 10^3$Hz. What is the minimum rate of samples/s at which the signal f must be sampled so it can be reconstructed from its samples? (This is the Nyquist rate.)

Solution: 2000Hz.