AMath390

Math & Music **Fall 2016**

Assignment # 6 **SOLUTIONS**

Note: Give all answers to 3 significant digits.

1. Label the note with pitch 440 Hz as A. Find the frequency of the sixth note in the scale, (usuallly called $F^{\#}$) in Pythagorean, just intonation, and equal temperament.

> **Solution:** In Pythagorean, the scale pattern is tone,tone,semi-tone, tone, tone. tone, semi-tone where a tone is $\frac{9}{8}$ and a semi-tone $\frac{256}{243}$. Thus the sixth note has frequency $440(\frac{9}{8})^4(\frac{256}{243})$, or 743Hz.
>
> In just intonation, the sixth note has ratio 5/3 , so the frequency is 733 Hz.
>
> In equal-temperament, the scale pattern is the same as in Pythagorean, except that a tone is $2^{\frac{2}{12}}$ and a semi-tone exactly half a tone, $2^{\frac{1}{2}}$. So the sixth note has frequency $440(2^{\frac{1}{6}})^4)(2^{\frac{1}{2}})$, or 740Hz.

2. In equal temperament, what factor do you multiply the root note (tonic) by to obtain the 7^{th} note in the (major) scale?

> **Solution:**
>
> In equal temperament, there are 12 notes in a scale, equally spaced. A semi-tone is $2^{\frac{1}{2}}$, a tone is $2^{\frac{2}{12}}$ in a major scale. The seventh note is just a semi-tone below the octave, so the factor is $2^{\frac{11}{12}}$.

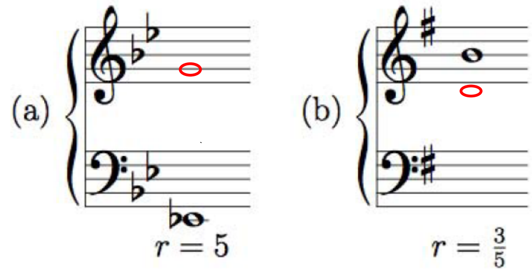
3. (a) How much does a equally tempered fifth differ from a just fifth in cents?

> **Solution:** A just fifth is $\frac{1200}{\ln 2}\ln(\frac{3}{2}) = 702 cents$. An equally tempered fifth is up 7th semitones in the 12 note equally tempered scale so it is 700 cents. The difference is 2 cents.

   (b) Repeat (a) for thirds.

> **Solution:** A just third is $\frac{1200}{\ln 2}\ln(\frac{5}{4}) = 386 cents$. An equally tempered third is the 4th note in the 12 note equally tempered scale so it is 400 cents. The difference is 14 cents.

4. Write on a staff the note that best approximates the frequency having the given interval ratio r from the given note. Use the standard equal temperament scale. Explain your calculations.



Solution:

(a) The note is $E_\flat$. In cents, 5 is 2786. This is two octaves plus 386 cents, so the note should be up 2 octaves plus 4 semitones. The $E_\flat$ 2 octaves above the given note is the bottom line of treble clef. Counting up 4 semi-tones yields G as shown.

(b) The note is B. In cents, $\frac{3}{5}$ is -884 cents, which is 9 semitones below the given note. Alternatively, go down an octave (12 semitones) and then up 3 semitones. This is the note D shown.

5. Consider an equally tempered system with 19 notes.

(a) What note best approximates the ratio $\frac{3}{2}$? Give the error in cents.

Solution: Solve

$$2^{\frac{n}{19}} = \frac{3}{2}$$

or

$$n = 19\log_2(\frac{3}{2}) = 11.1\,.$$

Thus, the best approximation is the $11th$ note. The frequency ratio is $2^{\frac{11}{19}}$ which in cents is $1200\frac{11}{19} = 695$. Since a just fifth is 702 cents (see #3a) the error is 7 cents.

(b) In the usual equally tempered system with 12 notes, which note best approximates the ratio $\frac{3}{2}$? Which system, 12-tet or 19-tet, yields a smaller error?

Solution: In the usual equally tempered 12-tet scale, the closest note to $\frac{3}{2}$ is the 7th note, which has an error of 2 cents (see #3a). The error is smaller for an equally tempered 12-tet scale.