AMath390

Math & Music **Fall 2016**

Assignment #1 **SOLUTIONS**

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1. Consider the differential equation

$$\ddot{y}(t) + \omega^2 y(t) = 0. \tag{1}$$

(a) Show that for any constants c, ϕ,

$$y(t) = c\sin(\omega t + \phi) \tag{2}$$

is a solution to the differential equation.

Solution: Differentiate (2) twice:

$$\begin{aligned} \dot{y}(t) &= \omega c\cos(\omega t + \phi) \\ \ddot{y}(t) &= -\omega^2 c\sin(\omega t + \phi) \\ &= -\omega^2 y(t). \end{aligned}$$

(b) Consider a spring with mass on the end of $20.g$, and stiffness $8.0N/m$. It is stretched so the initial deflection is $30.cm$ with 0 initial velocity. What is $y(t)$? What is the frequency of the vibrations? (You may neglect damping.)

Solution: Neglecting damping, the governing equation is

$$m\ddot{y}(t) + ky(t) = 0$$

or (1) with $\omega^2 = \frac{k}{m}$. Here $\omega = \sqrt{\frac{8}{.02}} = 20$. The general solution to the governing equation is

$$y(t) = c\sin(20t + \phi).$$

The initial condition on the velocity yields that

$$c20\cos(\phi) = 0.$$

The value $c = 0$ would yield $y(t) \equiv 0$ so $\phi = \frac{\pi}{2}$. Using the value for the initial position yields

$$c\sin(\frac{\pi}{2}) = 0.30,$$

and so $c = 0.30$. The solution is

$$y(t) = 0.30\sin(20t + \frac{\pi}{2}).$$

(This can also be written $y(t) = 0.30\cos(20t)$.)

2. Express

$$f(t) = 3\cos(2t) + 5\sin(2t)$$

(where f has units cm) in the form $c\sin(\omega t + \phi)$. State the frequency (in Hz), amplitude, and phase shift.

> **Solution:** Define $c = \sqrt{9+25}$ and ϕ so
>
> $$\sin(\phi) = \frac{3}{c} = .54\,\text{rad}.$$
>
> Checking,
>
> $$\cos(\phi) = .86 = \frac{5}{c}.$$
>
> Thus,
>
> $$f(t) = 5.8\sin(2t + .54).$$
>
> The frequency is $\frac{2}{2\pi}$ Hz $=$ .32Hz, the amplitude is 5.8cm and the phase shift is 0.54 radians.

3. Consider two sine waves of different frequencies $\omega_1 < \omega_2$ and the same amplitude.

   (a) What is the frequency of the beats in the amplitude envelope?

   > **Solution:** As in the course notes, define
   >
   > $$\bar{\omega} = \frac{1}{2}(\omega_2 + \omega_1), \quad \Delta = \frac{1}{2}(\omega_2 - \omega_1),$$
   >
   > write the sum of the two functions is
   >
   > $$y(t) = A\sin(\bar{\omega}t - \Delta t) + A\sin(\bar{\omega}t + \Delta t).$$
   >
   > Use the sum formula
   >
   > $$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
   >
   > to rewrite y as
   >
   > $$\begin{aligned} y(t) &= A\sin(\bar{\omega}t - \Delta t) + A\sin(\bar{\omega}t + \Delta t) \\ &= 2A\cos(\Delta t)\sin(\bar{\omega}t). \end{aligned}$$
   >
   > Thus, the frequency of the beats is $\frac{1}{2}(\omega_2 - \omega_1)$ rad/s or $\frac{\omega_2 - \omega_1}{4\pi}$ Hz.

   (b) Listen to the 5 sound clips posted on Learn. One has only a single frequency, $200Hz$. The others have a second sound at different frequency: $220, 240, 280$ and 300 Hz. The filename indicates the particular recording. Can you hear beats on all the clips? What about the frequency of the beats? Does the pitch change? Relate your answer to the expression for $y(t)$ in part (a).

Solution: Mathematically, all the signals can be written

$$y(t) = 2A\cos(\Delta t)\sin(\bar{\omega} t)$$

as in part (a). For the functions where the frequencies are close together, we are unable to distinguish the individual frequencies. As they become farther apart, the variation in the amplitude becomes much faster so it sounds "rough". For well-separated frequencies we can distinguish the individual frequencies as a blended sound.

4. The solution in the notes for periodic forcing of a system is correct if the system is damped and the forcing frequency is not equal to the natural frequency ω. Consider the remaining case of an undamped oscillator forced at its natural frequency:

$$\ddot{y}(t) + 4y(t) = \sin(2t). \tag{3}$$

(a) Show that for some value of b the particular solution is $bt\cos(2t)$. What is b?

Solution: For arbitrary A, B, setting $y_u(t) = A\sin(2t) + B\cos(2t)$, yields

$$\ddot{y}_u(t) + 4y_u(t) = 0.$$

Setting $y_p(t) = bt\cos 2t$,

$$\ddot{y}_p(t) + 4y_p(t) = -4b\sin 2t.$$

Defining $b = \frac{-1}{4}$ yields that $y_p(t)$ solves (3). Thus, (4) solves (3) for this value of b and any values of A, B.

(b) Show then that for certain values of A, B

$$y(t) = A\sin(2t) + B\cos(2t) + bt\cos(2t) \tag{4}$$

solves (3) with initial conditions $y(0) = 1$, $\dot{y}(0) = -1$.

Solution: The solution from (a) and the initial conditions yield the equations

$$y(0) = B = 1, \quad \dot{y}(0) = 2A - \frac{1}{4} = -1.$$

Solving for A and B,

$$y(t) = \frac{-3}{8}\sin(2t) + \cos(2t) - \frac{1}{4}t\sin(2t).$$

5. This part of the assignment should be done in a group of 3-4 people. Ideally, there should be a male and female student in each group. Each group should submit a common report. Each of you should bring a copy of your report to class on September 23. We will be discussing this part of the assignment in class.

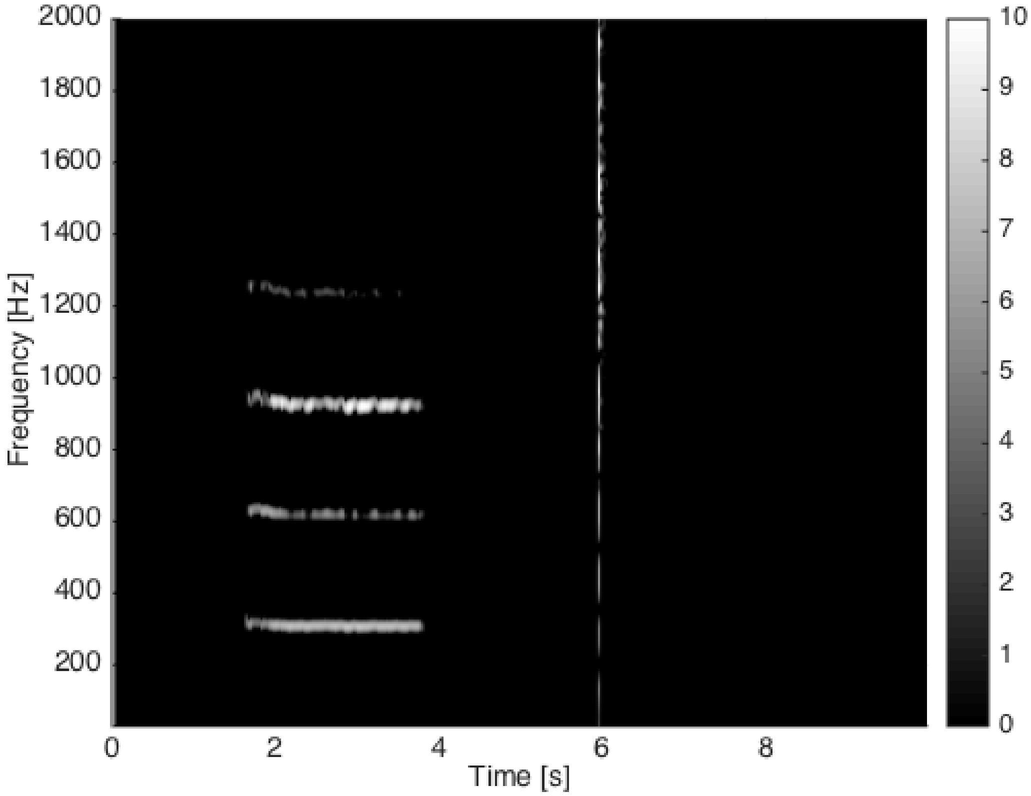

Figure 1: Spectrogram for question 5 b

A spectrogram is a plot showing the frequency content against time of a signal (in our case, sound). An example is shown here. Download the m-file *make_spectrogram.m* from Learn to a directory on your computer. Make sure the file is your matlab path. When you type "make_spectrogram", your voice (or other sound) will be recorded through the microphone on your computer for 10 seconds. A spectrogram will be produced.

(a) Each person in your group should record a spectrogram of a vowel sequence: "a e i o u". The spectrogram will look best if there is a short gap between each vowel. Make sure you speak loud enough and close to the microphone on your computer. Each person should try and keep their pitch roughly the same for each vowel. You may want to make multiple spectrograms, but attach one spectrogram for each person in your group to your report.

(b) How do the spectrograms vary between each person? Suggest reasons.

(c) How do the spectrograms of each vowel different from each other? Suggest reasons.

(d) The first 5 seconds of the spectrogram shown here is someone making a single vowel sound. What vowel they were making? Was the recording made by a man or a woman? Explain your answers.

(e) What is the sound at 6 seconds in the spectrogram?