# Mathematics & Music (AMath390)

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## 1 Introduction

**What is Music?**

From a Google search...

- vocal or instrumental sounds (or both) combined in such a way as to produce beauty of form, harmony, and expression of emotion "couples were dancing to the music"?
- the art or science of composing or performing music¿'he devoted his life to music"?
- a sound perceived as pleasingly harmonious."the background music of softly lapping water"

Determining what and what is not music is difficult and is partly determined by culture. Wikepedia has a nice article on the subject "Definition of Music". My favorite definition is that it is "organized sound" ; a phrase coined by the composer Edgard Varg èse.

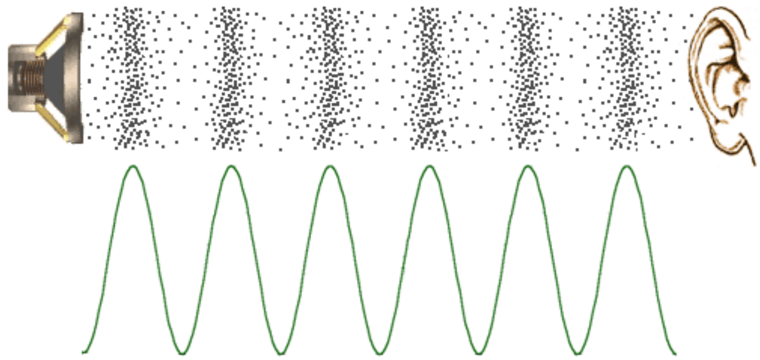
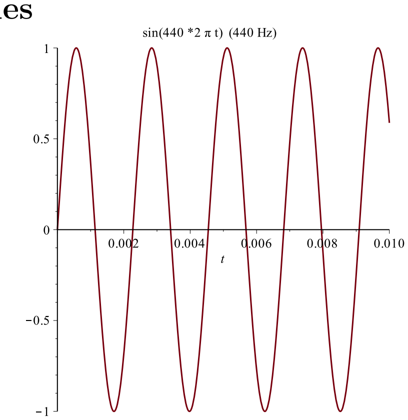
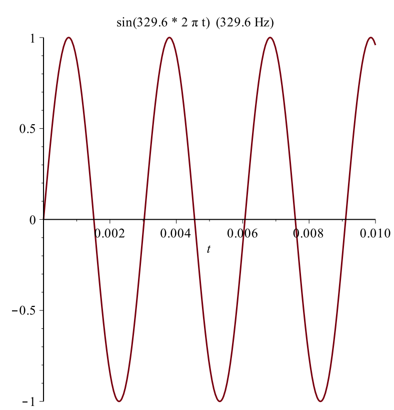


Figure 1: Sound is pressure waves.

**Different frequencies**



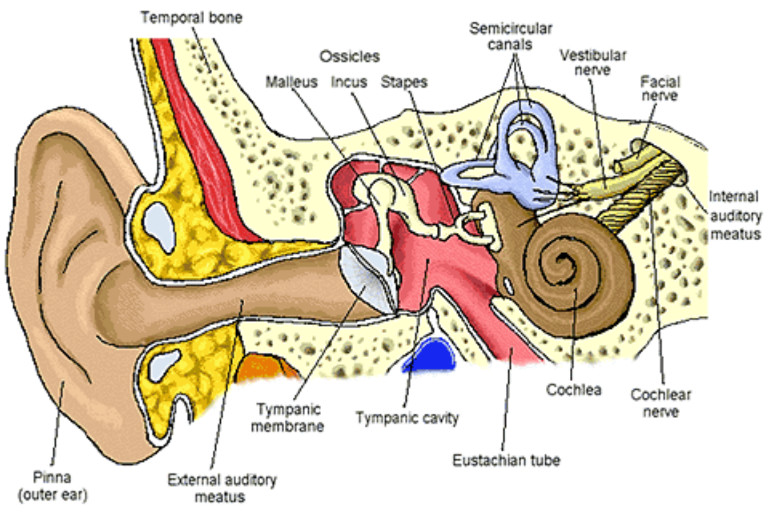

$440Hz\ for\ 0.01s$




$329.6Hz\ for\ 0.5s$

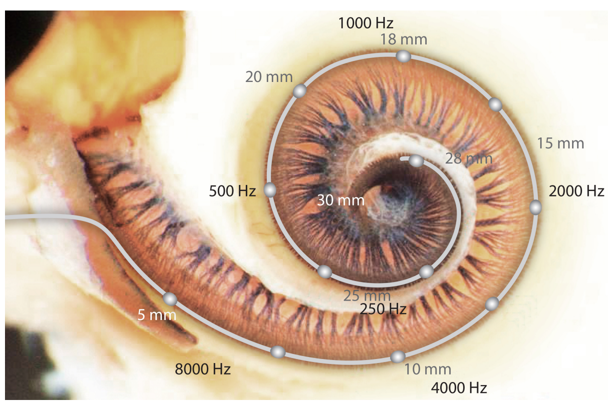
The 440Hz wave is perceived as "higher" in pitch than the 329.6Hz wave. In modern North American music, 440Hz is the note A in the treble clef (A4) and 329.6Hz is the frequency of the note E below it. However, in Europe the same note A4 is played slightly higher, 443Hz. In earlier times, pitch varied widely from place to place. In Baroque times A4 was played a lot lower, around 415Hz.

**Hearing**




**Cochlea (slice)**

**Cochlea Response**




- vibrations on membrane travel through cochlea
- different cilia respond to different frequencies
- cilia nearest ear respond to highest frequencies

**Frequency of Sound**

- the frequency of a sound wave is perceived by our brain
- frequency corresponds to pitch in music
- range of human hearing: 20-20,000Hz
- most music contains multiple frequencies

**Pyschoacoustics**

- much of sound perception is due to processing in our brain
- sounds at the frequency limit are perceived as less loud
- just noticeable difference is perceptible difference in sequential notes
- limit of discrimination is perceptible difference in simultaneous notes
- both just noticeable difference and limit discrimination vary with volume and frequency of the sound
- limit of discrimination is much smaller than just noticeable difference
- if the lowest frequency is absent we often perceive it as being present

**Music is not Mathematics**

> While we're discussing mathematical aspects of music, we should not lose sight of the evocative power of music as a medium of expression for moods and emotions. About the numerous interesting questions this raises, mathematics has little to say.

(Benson, pg. xii)

> Why do rhythms and melodies, which are composed of sound, resemble the feelings, while this is not the case for tastes, colours or or smells? Can it be because they are motions, as actions are also motions?

(Aristotle, quoted in Benson)

## 2 Harmonic Motion

**Newton's Law**

For a body with mass $m$ and an applied force $F$ the resulting acceleration $a$ is

$$ma = F.$$

Letting deflection be $y$,

$$m\frac{d^2y}{dt^2} = F.$$

For many systems the restoring force is (approximately) proportional to deflection. For example, in a spring the restoring force is $-ky$ where $k$ is known as the spring constant. Letting $k > 0$ indicate a proportionality constant,

$$m\frac{d^2y}{dt^2} = -ky.$$

Defining $\omega = \sqrt{\frac{k}{m}}$, rewrite as

$$\ddot{y}(t) + \omega^2 y(t) = 0.$$

$\ddot{y}(t) + \omega^2 y(t) = 0$

The general solution to this equation is

$$y(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1}$$

where $A$, $B$ are determined by initial conditions. To see (1) work out the derivatives of $y$ :

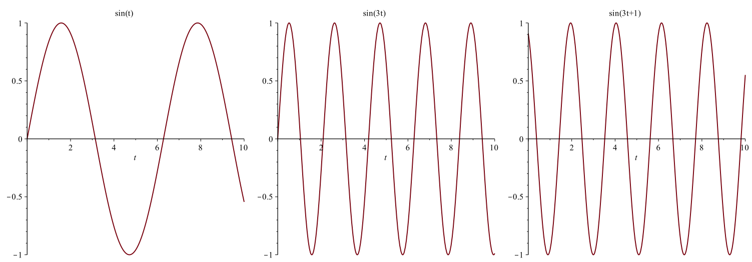
$$\begin{aligned}\dot{y}(t) &= -A\omega\sin(\omega t) + B\omega\cos(\omega)t \\ \ddot{y}(t) &= -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) = -\omega^2 y(t).\end{aligned}$$

Letting $\phi$ be such that

$$\sin(\phi) = \frac{A}{\sqrt{A^2+B^2}}, \quad \cos(\phi) = \frac{B}{\sqrt{A^2+B^2}},$$

and defining $c = \sqrt{A^2+B^2}$,

$$y(t) = c\sin(\phi)\cos(\omega t) + c\cos(\phi)\sin(\omega t).$$



(a) $\sin(t)$ (b) $\sin(3t)$ (c) $\sin(3t+1)$

Figure 2: Harmonic Motion

Using the sum formula

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{2}$$

yields

$$y(t) = c\sin(\omega t + \phi). \tag{3}$$

$c$, $\phi$ are determined by initial conditions.

This is known as harmonic motion.

The advantage of the second representation (3) is that it is clear that the solution is periodic with frequency $\omega$. The amplitude $c$ and phase $\phi$ are determined by initial conditions.

The frequency of the wave corresponds to pitch of an audible sound; amplitude of the wave corresponds to loudness. A difference in phase of two waves is not perceptible unless they occur at the same time.

### Frequency and Pitch

- The value of $\omega$ in $\sin(\omega t + \phi)$ yields the frequency of the solution.
- Frequency often given as cycle/s (Hz)
- $\sin(\omega t)$ has frequency $\omega/2\pi$ Hz
- sounds with higher frequencies are said to have a higher pitch .

### Some Music Notation

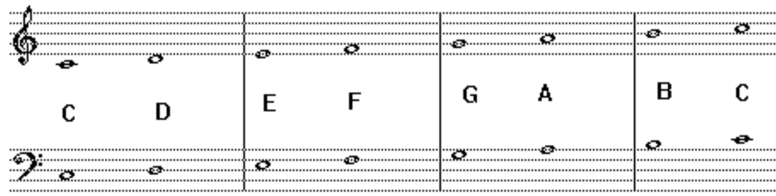



Figure 3: Notes on treble clef and bass clefs. The middle C typically has a frequency of 261.6Hz in North America. The higher C has twice the frequency of middle C; the lowest C has half the frequency of middle C.

**Damping**

Actual systems do not oscillate forever; there are dissipative forces.
A more realistic model includes dissipation

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = 0, \quad 0 < \xi < 1$$

which has solution

$$\begin{aligned} y(t) &= e^{-\xi\omega t}\left(A\cos(\sqrt{1-\xi^2}\omega t) + B\sin(\sqrt{1-\xi^2}\omega t)\right) \\ &= Ce^{-\xi\omega t}\sin((\sqrt{1-\xi^2})\omega t + \phi). \end{aligned}$$

**Damped vs Undamped Oscillations**

- decaying amplitude
- frequency slightly lower

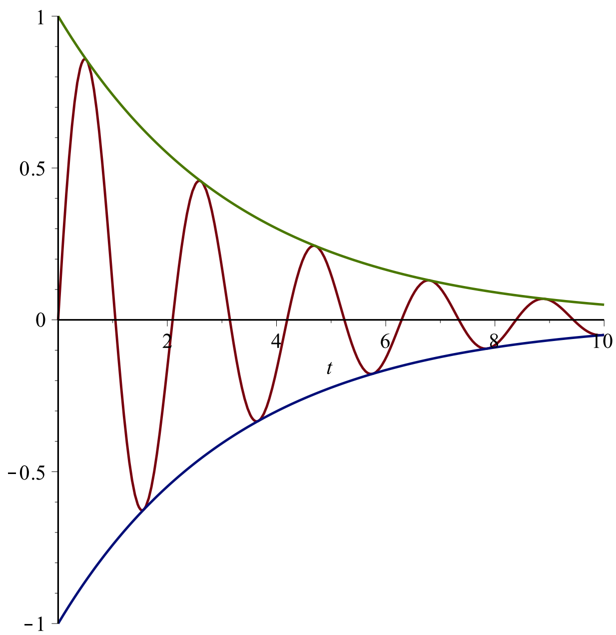
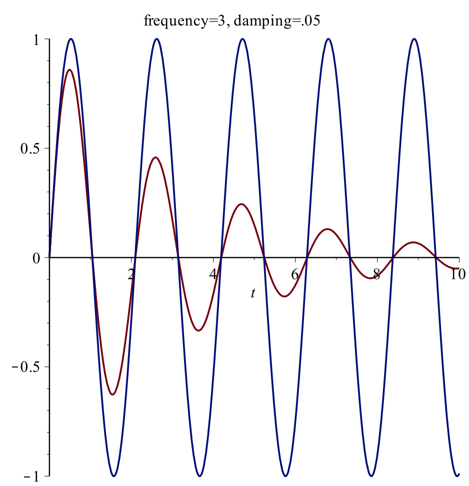
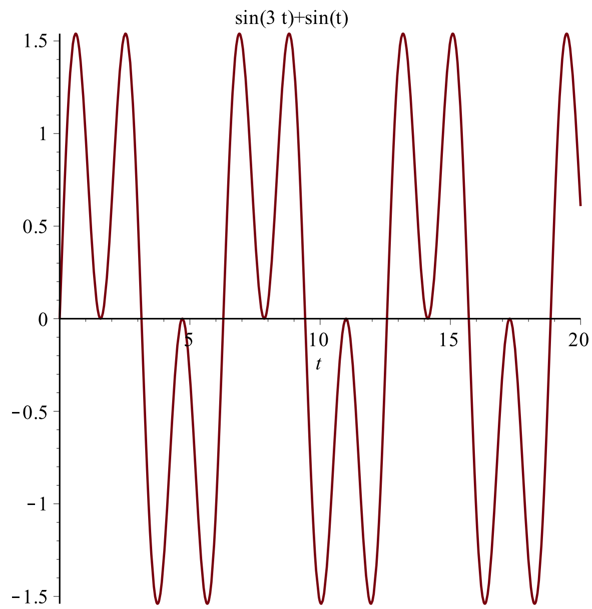



Figure 4: Damped oscillations with frequency $\omega = 3$, damping $\xi = .05$

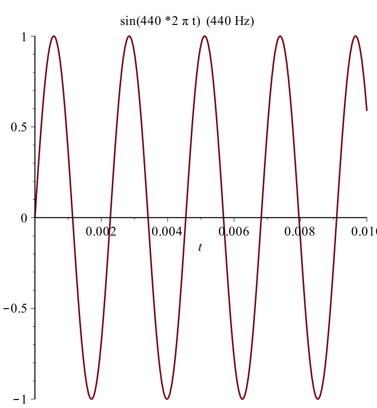



With $\xi = 0.05$, frequency is 99.8% of the undamped frequency. If $\xi = 0.2$, it's 98%. Since we are typically concerned only with frequency, and damping only slightly affects frequency, we will not generally include damping in the analysis.

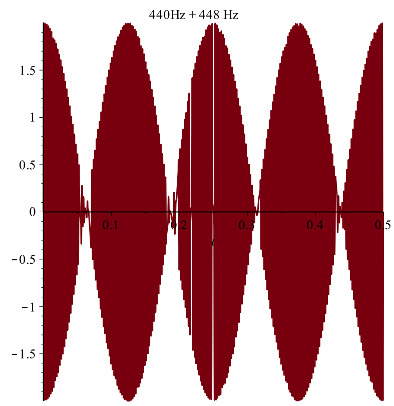
**Superposition**




## Beats




$$440Hz\ for\ 0.01s$$




$$440Hz + 448Hz \; for \; 0.5s$$

**Why is there a lower frequency envelope when the two frequencies are close?**

For simplicity, consider two waves with same phase and amplitude, but different frequencies $\omega_2 > \omega_1$:

$$y(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$$

Defining

$$\bar{\omega} = \frac{1}{2}(\omega_2 + \omega_1), \quad \Delta = \frac{1}{2}(\omega_2 - \omega_1),$$

and using the sum formula (2),

$$\begin{aligned} y(t) &= \sin(\bar{\omega}t - \Delta t) + \sin(\bar{\omega}t + \Delta t) \\ &= 2\cos(\Delta t)\sin(\bar{\omega}t). \end{aligned}$$

If $\Delta$ is small, this looks like a sine wave with frequency $\bar{\omega}/2\pi$ Hz and amplitude a slow cosine wave.

**Forced Motion**

$$\ddot{y}(t) + 2\xi w\dot{y}(t) + \omega^2 y(t) = f(t). \tag{4}$$

$$y(0) = y_0, \quad \dot{y}(0) = y_1.$$

Suppose $y_p(t)$ is found that solves (4), but maybe not the initial conditions. For any $A$, $B$,

$$y_u(t) = e^{-\xi\omega t}\left(A\cos(\sqrt{1-\xi^2}\omega t) + B\sin(\sqrt{1-\xi^2}\omega t)\right)$$

solves

$$\ddot{y}(t) + 2\xi w\dot{y}(t) + \omega^2 y(t) = 0$$

and so $y_u + y_p$ solves (4).

$$\boxed{\text{Choose A, B to satisfy the initial conditions.}}$$

**Periodic forcing**

In this course we are interested in periodic forcing; that is equations of the form

$$\ddot{y}(t) + 2\xi\omega\dot{y}(t) + \omega^2 y(t) = F\sin(\alpha t). \tag{5}$$

What is the solution of this equation?

Since repeated derivatives of $\sin(\alpha t)$, $\cos(\alpha t)$ are also $\sin(\alpha t)$, $\cos(\alpha t)$, try

$$y_p(t) = a\sin(\alpha t) + b\cos(\alpha t).$$

Substituting into the left-hand-side of (5) yields

$$\underbrace{(-a\alpha^2 - 2\xi\omega\alpha b + \omega^2 a)}_{\mathbf{F}}\sin(\alpha t) + \underbrace{(-b\alpha^2 - 2\xi\omega\alpha a + \omega^2 b)}_{\mathbf{0}}\cos(\alpha t).$$

For this to equal the right-hand-side of (5),

$$\begin{aligned}(\omega^2 - \alpha^2)a + (-2\xi\omega\alpha)b &= F \\ (2\xi\omega\alpha)a + (\omega^2 - \alpha^2)b &= 0.\end{aligned}$$

This is 2 linear equations for the unknown parameters $a, b$: If $\alpha \neq \omega$ or $\xi \neq 0$

$$a = \frac{F(\omega^2 - \alpha^2)}{(\omega^2 - \alpha^2)^2 + (2\xi\omega\alpha)^2}, \quad b = \frac{-F(2\xi\omega\alpha)}{(\omega^2 - \alpha^2)^2 + (2\xi\omega\alpha)^2}.$$

**Vibration of forced system**

Writing $\omega_0 = \sqrt{1-\xi^2}\,\omega$,

$$y(t) = e^{-\xi\omega t}(A\sin(\omega_0 t) + B\cos(\omega_0 t)) + a\sin(\alpha t) + b\cos(\alpha t).$$

Using the sum formula (2)

$$y(t) = e^{-\xi\omega_0 t}C\sin(\omega_0 t + \phi) + M\sin(\alpha t + \phi_f).$$

where $M$ and $\phi_f$ are determined by $a$ and $b$ (or the forcing function parameters $F$ and $\alpha$ ) and $C$ and $\phi$ are determined by $A$ and $B$ (the initial conditions). The response is the sum of two waves

- decaying wave at natural frequency $\omega_0$
- persistent wave at forced frequency $\alpha$

**Resonance**

In steady-state, once the effect of the initial conditions has dissipated,

$$y(t) = M\sin(\alpha t + \phi_f).$$

where

$$\begin{aligned} M &= \sqrt{a^2+b^2} \\ &= \frac{F}{\sqrt{(\omega^2-\alpha^2)^2 + (2\xi\omega\alpha)^2}}. \end{aligned}$$

The value of $M$ is the magnitude of the steady-state oscillations. The magnitude increases as the forcing frequency $\alpha$ approaches the natural frequency $\omega$, and the peak is larger for lightly damped systems. A vibrating system that is forced at a frequency close to the natural frequency is said to be in *resonance.*

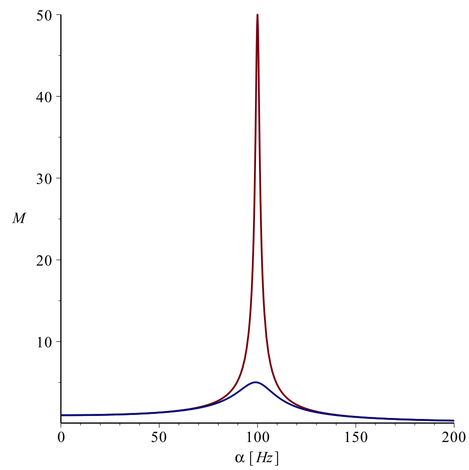



Figure 5: Magnitude $M$ of the steady-state response of a forced oscillator with $\omega = 100Hz$, $\xi = 0.1$, (blue) $\xi = 0.01$ (red) . The horizontal axis $\alpha$ is the frequency of the forcing term. The magnitude increases as the forcing frequency $\alpha$ approaches the natural frequency $\omega$.

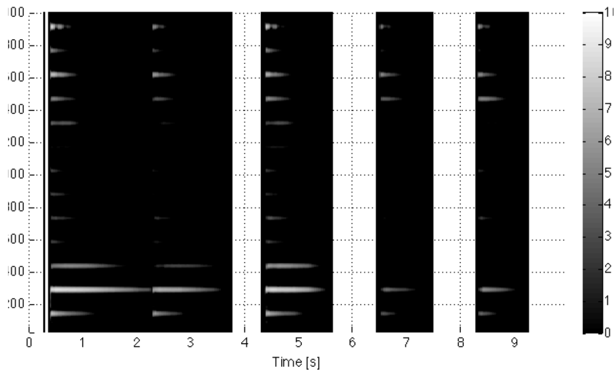



Figure 6: Guitar- Spectrogram of D string, strummed

# 3 Dynamics of Stretched String

The sound in many musical instruments, for instance guitars and violins, is produced by vibrating strings. A string of length $\ell$ is stretched and fixed at each end. The sound is produced by plucking, strumming etc. the string.

**Stringed Instruments**

- How does string thickness affect pitch?
- How does string length affect pitch?
- How does string tension affect pitch?
- Does where the guitar is strummed affect pitch?
- Does where the guitar is strummed affect the sound?

**Mathematical Model of Vibrating String**

Assume constant tension $T$, density $\rho$, uniform cross-sectional area $A$ and small deflections $u(x,t)$. Set the deflection $u = 0$ when the string is not stretched by strumming, striking etc. Consider a small section of string of length $\Delta x$. It has mass $m = \rho A \Delta x$ and acceleration $a = \frac{\partial^2 u(x,t)}{\partial t^2}$. The force on the stretched string is due to tension in the string and its vertical component is, letting $\theta$ be the angle the string makes with its unstretched position,

$$F = -TA\sin(\theta(x)) + TA\sin(\theta(x+\Delta x)).$$

In the above expression, $T$ is force per unit area so if $\tau$ is the applied force (in Newtons) and $A$ is the cross-sectional area (in $m^2$) then $T = \frac{\tau}{A}$.

Substituting these expressions for $m$, $a$ and $F$ into Newton's second law,

$$ma = F$$

and dividing through by $\Delta x$ yields

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = T \frac{1}{\Delta x}(\sin(\theta(x+\Delta x) - \sin(\theta(x))).$$

Take the limit as $\Delta x \to 0$ and define $c^2 = \frac{T}{\rho}$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial}{\partial x} \sin(\theta(x)). \tag{6}$$

(Extra information on partial derivatives such as $\frac{\partial}{\partial x}$ is in [Ben06, App. P] .) For small deflections, that is small $\theta$,

$$\sin(\theta) \approx \tan(\theta) = \frac{\partial u}{\partial x}.$$

Substitution into (6) yields the *wave equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{7}$$

Since the end of each string is fixed, the deflections $u$ at each end are zero:

$$u(0,t) = 0, \quad u(\ell,t) = 0. \tag{8}$$

This equation is to be solved with the boundary conditions (8) and appropriate initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{9}$$

where $f$ and $g$ describe the initial deflection and velocity respectively of the stretched string.

**Separation of variables** Partial differential equations are in general very difficult to solve. Try looking for solutions of the form

$$u(x,t) = M(x)N(t).$$

Substitution into (7) yields, using $'$ to indicate differentiation,

$$MN'' = c^2 M'' N.$$

Rearranging,

$$\frac{N''}{c^2 N} = \frac{M''}{M}.$$

Since the left-side depends only on time $t$ and the right-side depends only on space $x$, each side must be a constant. Call this constant $-\lambda$. This yields two *ordinary* differential equations

$$M''(x) + \lambda M(x) = 0, \tag{10}$$

$$N'' = c^2 \lambda N. \tag{11}$$

The spatial function $M$ should satisfy (10) and the boundary conditions also (8). Clearly $M = 0$ is a solution, but this is not interesting. The general solution to (10) is

$$M(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

From the boundary condition at $x = 0$, $A = 0$. It is also required that

$$B\sin(\sqrt{\lambda}\ell) = 0.$$

The equation (10) will have non-trivial solutions that satisfy the boundary conditions only if

$$\lambda = (\frac{\pi k}{\ell})^2, k = 1, 2, \dots$$

so

$$M_k(x) = \sin(\frac{\pi k}{\ell}x).$$

The functions $M_k$ are known as the *eigenfunctions* of $\frac{\partial^2}{\partial x^2}$ with the boundary conditions (8). They will be indicated by

$$\phi_k(x) = \sin(\frac{\pi k}{\ell}x).$$

**Solution to Wave Equation**

The differential equation for $N$ (11) then has solutions

$$N(t) = A_k \cos(\frac{\pi kc}{\ell}t) + B_k \sin(\frac{\pi kc}{\ell}t)$$

for constants $A_k$, $B_k$.

By linearity, any linear combination of

$$u_k(x,t) = \left[A_k \cos(\frac{\pi kc}{\ell}t) + B_k \sin(\frac{\pi kc}{\ell}t)\right] \sin(\frac{\pi k}{\ell}x)$$

also satisfy (7) and (8):

$$u(x,t) = \sum_{k=1}^{\infty}[A_k \cos(\frac{\pi kc}{\ell}t) + B_k \sin(\frac{\pi kc}{\ell}t)] \sin(\frac{\pi k}{\ell}x)$$

This approach to solving a partial differential equation is known as the *Method of Separation of Variables.*

But in order for $u$ to be a solution, constants $A_k, B_k$ are needed so that the initial conditions (9) are satisfied.

**Initial Conditions**

For any choice of constants $A_k$, $B_k$,

$$u(x,t) = \sum_{k=1}^{\infty}[A_k \cos(\frac{\pi kc}{\ell}t) + B_k \sin(\frac{\pi kc}{\ell}t)] \sin(\frac{\pi k}{\ell}x) \tag{12}$$

solves the wave equation and satisfies the boundary conditions $u(0,t) = 0$, $u(\ell,t) = 0$. The individual terms $u_k$ are called the *modes of vibration* of the response.

The constants $A_k$ and $B_k$ in (12) need to be chosen so that the initial conditions (9) are satisfied:

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} A_k \sin(\frac{\pi k}{\ell}x),$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{k=1}^{\infty} \frac{c\pi k}{\ell} B_k \sin(\frac{\pi k}{\ell}x).$$

For initial conditions that are a finite linear combination of functions of the form $\sin(\frac{\pi k}{\ell}x)$ this is straightforward. But to allow for more general initial conditions, arbitrary initial conditions, such as that shown in Figure 7 need to be written as a sum of sine functions.

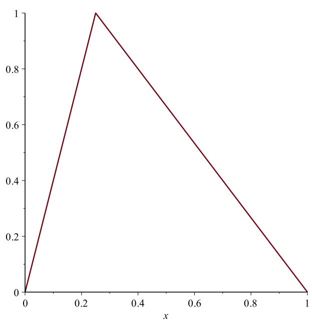



Figure 7: Can an arbitrary function be written as a Fourier sine series $\sum_{k=1}^{\infty} f_k \sin(\frac{\pi k}{\ell} x)$ for some choice of $\{f_k\}$?

**Calculation of coefficients**

The eigenfunctions $\phi_k(x) = \sin(\frac{\pi k}{\ell} x)$ are *orthogonal*:

$$\int_0^\ell \phi_j(x)\phi_k(x)\,dx = \left\{ \begin{array}{ll} \frac{\ell}{2} & j = k \\ 0 & j \neq k \end{array} \right. .$$

Thus multiply each side of

$$f(x) = \sum_{k=1}^{\infty} A_k \sin(\frac{\pi k}{\ell} x)$$

by $\phi_j$ and integrate over $[0, \ell]$ to obtain

$$\int_0^\ell f(x)\sin(k\pi\frac{x}{\ell})dx = A_k\frac{\ell}{2}$$

and so

$$A_j = \frac{2}{\ell}\int_0^\ell f(x)\sin(j\pi\frac{x}{\ell})dx. \tag{13}$$

The series

$$\sum_{k=1}^{\infty} A_k \sin(\frac{\pi k}{\ell} x) \tag{14}$$

is the *Fourier sine series* for $f$.

**Definition 1.** *A function is* piecewise smooth *if it is bounded on* $[0, \ell]$ *and both* $f$ *and its derivative are continuous on* $[0, \ell]$ *except at a finite number of points.*

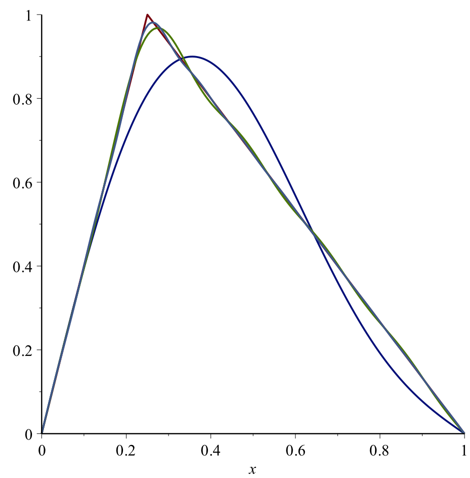



Figure 8: Partial sums of the Fourier sine series for the hat function shown in Figure 7.

Define the partial sums of the Fourier series of $f$ :

$$\tilde{f}_N(t) = \sum_{n=-N}^{N} A_k \sin(\frac{\pi k}{\ell} x)$$

where $A_k$ are determined by (13).

**Theorem 2.** *If $f$ is piecewise smooth on $[0, \ell]$ then at all points $x \in (0, \ell)$ where $f$ is continuous*

$$\lim_{N \to \infty} \tilde{f}_N(x) = f(x).$$

*If $f$ is not continuous at a point $x_0$ then $\tilde{f}_N(x_0) \to \frac{f(x_{0-}+f(x_{0+})}{2}$. Also,*

$$\lim_{N \to \infty} \int_0^\ell |f(x) - \tilde{f}_N(x)|^2 dx = 0.$$

### Harmonics

$$u(x,t) = \sum_{k=1}^{\infty} \left[ A_k \cos(\frac{\pi k c}{\ell} t) + B_k \sin(\frac{\pi k c}{\ell} t) \right] \sin(\frac{\pi k}{\ell} x).$$

The individual frequencies in the response are all integer multiples of the lowest frequency. These are called *harmonics* . The lowest frequency is called the *fundamental frequency.* The harmonics above the the fundamental frequency are *overtones* .

### Location of initial deflection

Consider a hat function such as shown in Figure 7:

$$f(x) = \begin{cases} \frac{x}{x_0} & 0 \leq x < x_0 \\ \frac{\ell - x}{\ell - x_o} & x_0 \leq x \leq \ell \end{cases} \tag{15}$$

The coefficients $A_k$ in its Fourier sine series

$$f(x) = \sum_{k=1}^{\infty} A_k \sin(k\pi \frac{x}{\ell})$$

are, using the formula (13)

$$\begin{aligned} A_k &= \frac{2}{\ell} \int_0^\ell f(x) \sin(k\pi \frac{x}{\ell}) dx \\ &= \frac{2\ell^2}{\pi^2 x_0 (\ell - x_0)} \frac{\sin(k\pi \frac{x_0}{\ell})}{k^2}. \end{aligned} \tag{16}$$

If $u(x,0) = f(x)$ and $\dot{u}(x,0) = 0$, then

$$u(x,t) = \sum_{k=1}^{\infty} A_k \cos(k\pi ct) \sin(k\pi \frac{x}{\ell})$$

where $A_k$ are defined in (16).

If $x_0 = \frac{\ell}{2}$, all the even harmonics are missing; similarly for other values of $x_0$ different harmonics may be weak or missing entirely. This is reflected in the different sound of a guitar when it is plucked at different points.

### Harpsichord

"Clavecin flamand" by Ratigan (instrument et photo)

- Harpsichord mechanics : `http://youtu.be/71x4MSlpGUk`
- Minuet in Gmajor on Harpsichord `http://www.youtube.com/watch?v=2TobXjDXF0s`
- Piano mechanics: `http://www.youtube.com/watch?v=xr21z1CZ54I`
- Minuet in Gmajor on Piano `http://www.youtube.com/watch?v=yIKKDXCP2_M`

**Harmonics of Harpsichord**

The sound on a harpsichord is produced by plucking the individual strings with a quill. This is a non-zero initial position, zero initial velocity. The initial position on a harpsichord is similar to the hat function (15) above and so the individual modes will be

$$\cos(k\pi ct)\sin(k\pi\frac{x}{\ell})$$

with coefficients

$$f_k = \frac{2\ell^2}{\pi^2 x_0(\ell - x_0)}\frac{\sin(k\pi\frac{x_0}{\ell})}{k^2}$$

$$\boxed{\text{The coefficients of the harpsichord harmonics decay as } \frac{1}{k^2}.}$$

**Harmonics of Piano**

The sound on a piano is produced by a hammer striking a string. This corresponds to a zero initial position, non-zero initial velocity. Thus, all the coefficients

$$A_k = 0.$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{k=1}^{\infty} \frac{\pi kc}{\ell}(-A_k \sin(\frac{\pi kct}{\ell}) + B_k \cos(\frac{\pi kct}{\ell})) \sin(k\pi \frac{x}{\ell})$$

and so letting $g(x)$ indicate the initial velocity,

$$g(x) = \dot{u}(x,0) = \sum_{k=1}^{\infty} \frac{\pi kc}{\ell} B_k \cos(\frac{\pi kct}{\ell}) \sin(k\pi \frac{x}{\ell}).$$

Solving for $B_k$, using again orthogonality of the $\{\sin(k\pi \frac{x}{\ell}\}$,

$$B_k = \frac{2}{\pi kc} \int_0^{\ell} g(x) \sin(k\pi \frac{x}{\ell}) dx.$$

The initial velocity is similar to the hat function (15) above. With this assumption,

$$B_k = f_k \frac{\ell}{\pi kc} = \frac{2\ell^3}{\pi^3 c x_0(\ell - x_0)} \frac{\sin(k\pi \frac{x_0}{\ell})}{k^3}$$

$$\boxed{\text{The coefficients } B_k \text{ decay as } \frac{1}{k^3}.}$$

**Sound of Piano vs Harpsichord**

Because a piano's sound is produced by a non-zero initial velocity, while the initial condition for a harpsichord's sound is produced by a non-zero initial position, the higher modes have smaller amplitudes. The sound quality, or timbre, is quite different.

This helps to explain the presence of of more high frequency sound in the harpsichord.

# 4 Wind Instruments

**Wind Instruments**

**Acoustic Plane Waves**

The motion of air in a long thin tube such as a clarinet or flute can be considered one-dimensional and only depending on distance $x$ along the tube.

Consider particles at $x$ when undisturbed and denote displacement from "usual" location $x$ by $u(x,t)$. (Think of a slinky.) Denote similarly pressure $P(x,t)$, density $\rho(x,t)$. Let $P_0 = 0$ be the pressure of the undisturbed air and $\rho_0$ the density. Assume that only motion in the $x$-direction is present; then from Newton's Law on a section $[x, x+\Delta x]$, letting cross-sectional area be $A$,

$$\begin{aligned} ma &= F \\ \rho_0 A(x)\Delta x \frac{\partial^2 u}{\partial t^2} &= A(x)P(x,t) - A(x+\Delta x)P(x+\Delta x,t) \end{aligned}$$

Assume cross-sectional area $A$ is constant and divide through by $A\Delta x$

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{P(x+\Delta x,t) - P(x,t)}{\Delta x}.$$

Taking the limit as $\Delta x \to 0$ yields

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial P(x,t)}{\partial x}. \tag{17}$$

An equation in only one variable is needed.

Write $P'(\rho) = \frac{\partial P}{\partial \rho}$. Then the linear approximation to $P$ as a function of $\rho$ is, recalling that $P(\rho_0) = P_0 = 0$,

$$P(\rho) \approx P'(\rho_0)(\rho - \rho_0). \tag{18}$$

Also, since $\rho = \frac{Mass}{Volume}$,

$$\begin{aligned} \rho(x,t) &= \frac{\rho_0 A \Delta x}{A(x + \Delta x + u(x+\Delta x, t) - (x + u(x,t)))} \\ &= \frac{\rho_0}{1 + \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}}. \end{aligned}$$

Taking the limit as $\Delta x \to 0$, $\rho(x,t) = \rho_0(1 + \frac{\partial u}{\partial x})^{-1} \approx \rho_0(1 - \frac{\partial u}{\partial x})$. Substituting into (18) yields

$$P(x,t) \approx -P'(\rho_0)\rho_0 \frac{\partial u}{\partial x}. \tag{19}$$

Substitute (19 ) into (17) to obtain, after dividing by $\rho_0$, and defining $c^2 = P'(\rho_0)$,

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{20}$$

$\boxed{\text{Same equation as for a stretched string!}}$

The constant $c$ in equation (20) is the speed of sound in the given medium, in this case air. It increases strongly with temperature.

Note: On page 99 of Benson in the coursewares, the bulk modulus $B = P'(\rho_0)\rho_0$. This is equivalent to this derivation, just the constants are relabelled.

**Flute**

A flute is essentially a long tube with constant cross-sectional area, so equation (20)

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}.$$

for the dynamics is valid. Both ends are open, so the pressure $P(x,t) = P_0 = 0$ at the ends. Using (19) this yields the boundary conditions, for a flute of length $\ell$,

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(\ell,t) = 0. \tag{21}$$

There will also be initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

**Flute (cont.)**

Since (20) is the same equation as studied previously for a vibrating string, the same solution procedure can be used. (See Poulin's notes for details.) Separation of variables means substituting $u(x,t) = M(x)N(t)$ into ( 20) and rearranging to obtain

$$\frac{N''(t)}{c^2 N(t)} = \frac{M''(x)}{M(x)} = -\lambda.$$

Since the left-hand-side depends only on $t$, the right-hand-side only on $x$, each side must be a constant, called $-\lambda$. This yields differential equations, for $M$ and for $N$.

**Spatial part of solution,** $M(x)$

$$M''(x) + \lambda M(x) = 0$$

and so, with arbitrary constants, $c_1, c_2$,

$$M(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x).$$

But to satisfy the boundary conditions (21),

$$M'(0) = 0, \quad M'(\ell) = 0$$

and so $c_2 = 0$, $\sqrt{\lambda}\ell = k\pi$ for $k = 0, 2, \ldots$.

$$M(x) = c_1 \cos(\frac{k\pi}{\ell}x).$$

**Temporal part of solution,** $N(t)$

$$N''(t) + \lambda c^2 N(t) = 0$$

Since $\lambda = \frac{k\pi}{\ell}$, define $\omega_k = \frac{k\pi c}{\ell}$, and rewrite this equation as

$$N''(t) + \omega_k^2 N(t) = 0$$

which has general solution

$$N(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t).$$

**Solution of Wave Equation, Open Ends**

$$u_k(x,t) = A_0 + (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \cos(\frac{\omega_k}{c}x)$$

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$$

where $A_k, B_k$ are chosen so initial conditions are satisfied. Since the constant $A_0$ does not affect the sound, it is usually neglected in this context.

This solution applies to situations of tubes with constant cross-section and open ends where only one spatial variable is significant. A flute is a particular example of this.

**Predictions of Model: Fundamental Frequency**

The fundamental frequency is the lowest frequency present:

$$\frac{\pi c}{\ell} rad/s, \quad \frac{c}{2\ell} Hz.$$

The parameter $c$ depends on temperature, weakly on humidity. For dry air, $c = 342$m/s($20°$ C), $c = 345$m/s($25°$ C). Using $c = 344$m/s,

| | length (m) | theo. pitch (Hz) | actual pitch (Hz) |
|---|---|---|---|
| flute | 0.66 | 260 | 262 (C4) |
| short red tube | 0.3 | 573 | 524 (C5) |
| long red tube | 0.63 | 273 | 262 (C4) |

Length clearly corresponds to pitch. The errors, which are more significant for shorter tubes, are due primarily to *end effects*: the pressure is not 0 exactly at the ends.

At lower temperatures, $c$ is smaller, and the theory predicts that an instrument's pitch is lower. This is the case in practice.

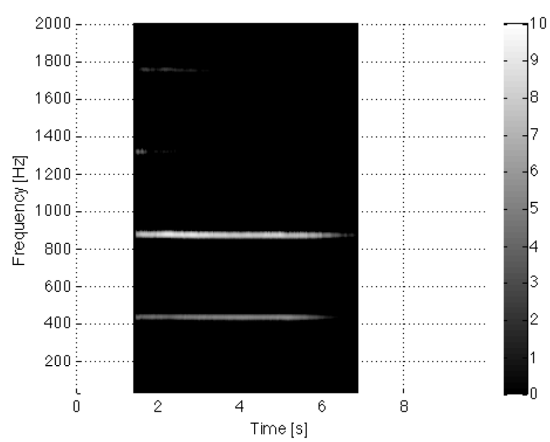
**Predictions of Model: Overtones**

The analysis predicts that the frequencies $\omega_k = \frac{k\pi c}{\ell}$ are present. Writing the fundamental at $\omega_1$, the overtones are integer multiples of the fundamental:

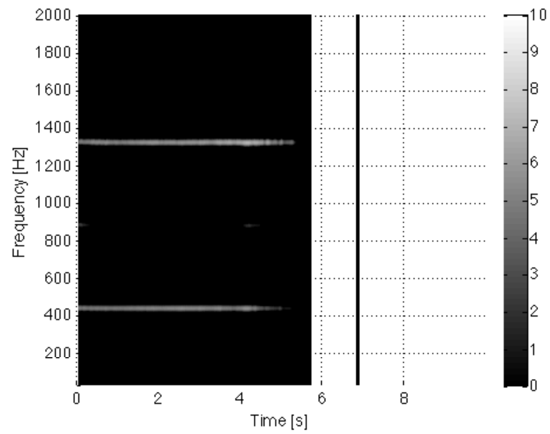
$$\omega_2 = 2\omega_1,\ \omega_3 = 3\omega_1, \ldots$$

Overtones that occur as integer multiples of the fundamental are also called *harmonics.*

**Spectrograms**




Flute




Clarinet

**Clarinet- one open end, one closed end**

The wave equation also applies to sound waves in a clarinet. A clarinet is

- about the same length as a flute
- also has a cross-section that is approximately constant
- one end ($x = \ell$) is open

However, instead of blowing into an open end, sound is produced by vibration of the reed against the mouthpiece. The opening $x = 0$ is very small, and because of the reed vibration, it is often sealed. The pressure at $x = 0$ is not 0. In order for there to be no vacuum at $x = 0$, the air velocity at $x = 0$ is typically 0. Hence, an appropriate set of boundary conditions is

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\ell, t) = 0. \tag{22}$$

**Solution of wave equation - one open end, one closed end**

Separation of variables again is used to solve the wave equation, but now the spatial function $M$ must satisfy

$$M''(x) + \lambda M(x) = 0,$$

$$M(0) = 0, \quad M'(\ell) = 0.$$

Solving yields,

$$M(x) = c_2 \sin(\sqrt{\lambda_k} x)$$

where $c_2$ is arbitrary and $\sqrt{\lambda_k} = (k - \frac{1}{2})\frac{\pi}{\ell} = \frac{(2k-1)\pi}{2\ell}$, $k = 1, 2, \ldots$ so that $M'(\ell) = 0$.

Defining $\omega_k = \sqrt{\lambda_k} c = \frac{(2k-1)\pi c}{2\ell}$

$$u_k(x,t) = (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \sin(\frac{\omega_k}{c} x)$$

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$$

**Predictions of Model: Fundamental Frequency**

$$\omega_1 = \sqrt{\lambda_1} c = \frac{\pi c}{2\ell} = \frac{c}{4\ell}\text{Hz}$$

| | length (m) | theo. pitch (Hz) | actual pitch (Hz) |
|---|---|---|---|
| clarinet | 0.6 | 143 | 147 (D3 ) |
| flute | 0.66 | 260 | 262 (C4) |
| closed short red tube | 0.3 | 286 | 262 (C4) |
| open short red tube | 0.3 | 573 | 524 (C5) |
| open long red tube | 0.63 | 273 | 262 (C4) |

- fundamental of the tube with one end closed is half that of the open tube, as predicted by theory.
- clarinet is about the same length as a flute but the fundamental frequency is nearly half that of a flute.

**Predictions of Model: Overtones**

The analysis predicts that the frequencies $\omega_k = \frac{(2k-1)c}{4\ell}$ are present. Writing the fundamental at $\omega_1$, the overtones are integer multiples of the fundamental $\omega_1 = \frac{c}{4\ell}$:

$$\omega_2 = 3\omega_1,\ \omega_3 = 5\omega_1, \ldots$$

Although these overtones are harmonics, only the odd harmonics are present.

**Producing different notes**

Different notes can be produced by opening and covering various wholes, thus changing the effective length of the instrument. Examples: oboe, clarinet, saxophone, flute

On the flute, and many other instruments, different notes are also produced, without changing the fingering, by exciting various overtones, or resonant frequencies of the instrument. There is a nice demonstration of this on various instruments here:

https://www.youtube.com/watch?v=7Yh8OukwwtQ&list=PL10tQt143TnC4GUlaUTuPAQ6xPlbu
index=16

# 5 Modes of Vibration

There is a particular shape of vibration associated with each frequency. These are called modes of vibration.

**Modes of vibration with fixed ends**

For fixed ends, each mode is

$$u_k(x,t) = (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) \sin(\frac{\omega_k}{c} x)$$

Defining

$$M_k = \sqrt{A_k^2 + B_k^2}, \quad \sin\phi_k = \frac{A_k}{\sqrt{A_k^2 + B_k^2}}, \quad \cos\phi_k = \frac{B_k}{\sqrt{A_k^2 + B_k^2}},$$

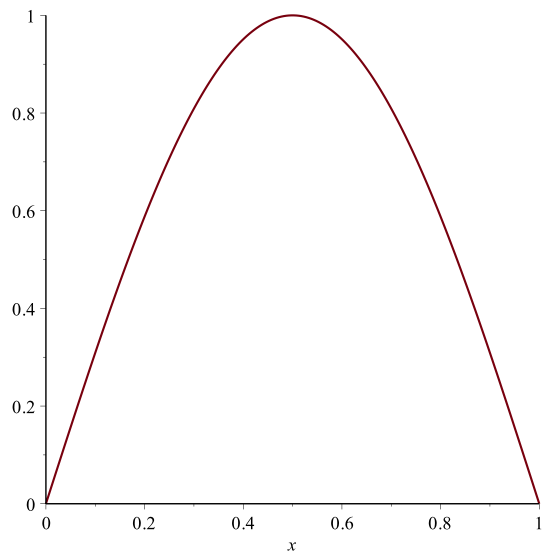
$$u_k(x,t) = M_k \sin(\omega_k t + \phi_k) \sin(\frac{\omega_k}{c} x)$$

The maximum amplitude of each mode is constant with time.

This is the shape of the modes for a vibrating string where deflection $u = 0$ at each end. Also, since pressure $P \approx \frac{\partial u}{\partial x}$, this is also the shape of the pressure in an instrument, such as a flute, that has both ends open.

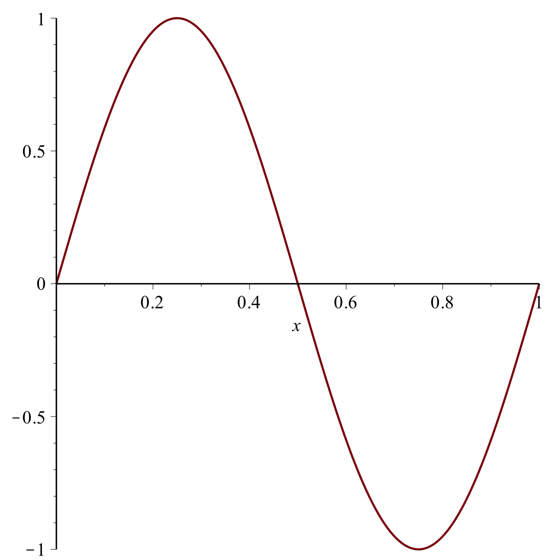
**Mode 1**

$$u_1(x,t) = M_1 \sin(\omega_1 t + \phi_1) \sin(\frac{\omega_1}{c} x)$$



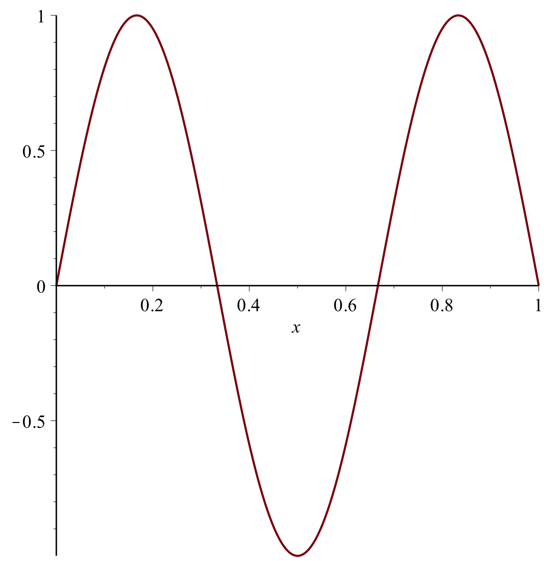

**Mode 2**

$$u_2(x,t) = M_2 \sin(2\omega_1 t + \phi_2) \sin(\frac{2\omega_1}{c} x)$$



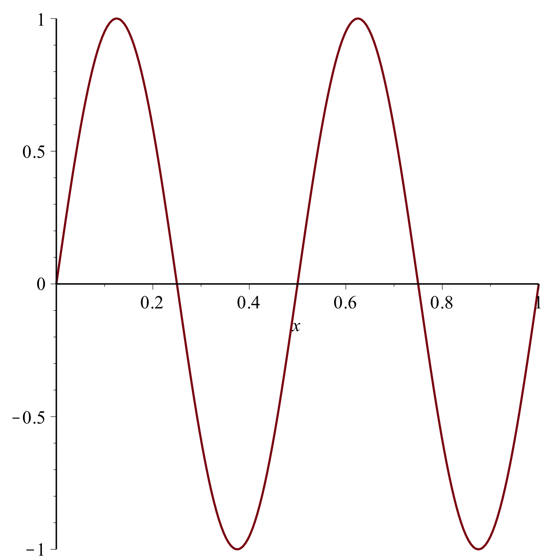

**Mode 3**

$$u_3(x,t) = M_3 \sin(3\omega_1 t + \phi_3) \sin(\frac{3\omega_1}{c} x)$$




**Mode 4**

$$u_4(x,t) = M_4 \sin(4\omega_1 t + \phi_4) \sin(\frac{4\omega_1}{c}x)$$



**Nodes and Anti-nodes**

- in a vibrating string, for each mode, point(s) where deflection is always 0
- in a tube, points where $u$, or similarly $P = \frac{\partial u}{\partial x}$ is 0

  The following link is to a video showing pressures nodes in a long tube. Note the strong response when the forced frequency is at one of the natural frequencies of the tube.

  `https://www.youtube.com/watch?v=ezZ1fZYJjYM`

**Some definitions**

**fundamental (frequency)**
lowest frequency of a note/sound. Generally the perceived pitch

**overtones**
frequencies above the fundamental. The first component above the fundamental is the first overtone.

**partials**
The $m$th partial is the $\hat{m}$th frequency component present.

**harmonics**
frequencies at integer multiples of the fundamental.

**Examples of using terms**
For a clarinet,

- fundamental $\frac{c}{4\ell}$ ; also first harmonic or first partial
- second harmonic $\frac{2c}{4\ell}$ is missing
- $\frac{3c}{4\ell}$ is second partial or third harmonic or first overtone

**Characteristics of Sound**

**pitch**
frequency of vibration of a sound (20Hz-20,000Hz is range audible to humans)

**timbre**
corresponds to frequency response; in particular overtones and their relative strength

**amplitude**
magnitude of vibration; corresponds roughly to loudness (¡mm)

**duration**
length of time note sounds

**Varied Cross-section**

Simplest model, for oboes, saxophones and brass instruments, is a tube of varying cross-section. The wave equation becomes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{c^2}{A(x)} \frac{\partial}{\partial x} A(x) \frac{\partial u(x,t)}{\partial x}.$$

- Can be solved theoretically using separation of variables technique but solution more complicated.
- The precise profile $A(x)$ affects the fundamental and the overtones.
- For example, soprano saxophone and clarinet are about the same length, and look similar but soprano saxophone has a conical profile. The fundamental of a clarinet is almost an octave lower, and the timbre is quite different.
- See material from J. Resch (on Learn) on model for the trumpet that considers not only the varying cross-section, but compressibility and nonlinearities. Computational methods are required for analysis.

# 6 Drums

Drums are the most major example of a class of instruments known as *membranophones*, where the sound is produced by the vibration of a stretched membrane. The sound can be produced by striking, plucked or rubbing the membrane. (Kazoos are also considered membranophones. The sound of a kazoo is produced by singing into a membrane.)

**Mathematical Model for Drum Vibrations**

Consider a simple drum consisting of a membrane stretched over a frame and fixed at the frame. Assume gravity is negligible compared to the tension. Then Newton's Law applied to a small region is, letting $u$ indicate the deflection of the membrane

$$\rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2} = Force\ of\ Tension$$

where $\rho$ is mass/unit area ($kg/m^2$).

Apply the same assumptions as for a vibrating string to a stretched frame:

- constant density of the membrane
- perfectly flexible membrane
- no resistance, friction or other forces
- small deflections so nonlinearities neglected

The same technique as in the derivation of the governing equation for a string is then used to derive the governing equation except vector calculus is needed. This leads to , defining $c^2 = \frac{T}{\rho}$ where T is tension (force/unit length or N/m ),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial y^2} \right).$$

$\boxed{\text{Wave equation in 2 dimensions}}$

Let the drum be the region $(x, y) \in \Omega$.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial y^2} \right), \quad (x, y) \in \Omega$$

$$u(x, y, t) = 0, \quad (x, y) \in bdy\Omega$$

Typically $\Omega$ is a disk, so polar coordinates are natural .

**Solution of Wave equation for Drum**

For a circular drum, polar co-ordinates are natural. Use $x = r\cos\theta$, $y = r\sin\theta$ where $r$ is measured from the centre of the drum, and $\theta$ is measured from some convenient point.

The wave equation in polar co-ordinates is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

In polar co-ordinates, the boundary conditions are

$$u(a,\theta,t) = 0, \quad u(0,\theta,t) < \infty,$$

and since $\theta = 0,\ \theta = 2\pi$ yield the same point,

$$u(r,0,t) = u(r,2\pi,t), \quad \frac{\partial u}{\partial \theta}(r,0,t) = \frac{\partial u}{\partial \theta}(r,2\pi,t).$$

Separation of variables has worked well so far. Since there are 3 variables, now try:

$$u(r,\theta,t) = R(r)H(\theta)T(t).$$

Substituting $u(r,\theta,t) = R(r)H(\theta)T(t)$ into the wave equation, and rearranging, yields

$$\frac{T''}{c^2 T} = \left[\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{H''}{H}\right] = -\lambda$$

since $t, r, \theta$ are independent. The differential equation for $T$ is

$$T'' + c^2\lambda T = 0$$

which is the same equation for $T$ previously obtained; the oscillator equation.

$$T(t) = A_n\cos(c\sqrt{\lambda}t) + B_n\sin(c\sqrt{\lambda}t)$$

$$\left[\frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{H''}{H}\right] = -\lambda$$

Rearranging,

$$\frac{r}{R}(rR')' + \lambda r^2 = -\frac{H''}{H} = \mu^2$$

since the left side depends only on $r$ and the right side depends only on $\theta$. Thus,

$$H'' + \mu^2 H = 0$$
$$(rR')' + \lambda r R - \frac{\mu^2}{r} R = 0.$$

$$H'' + \mu^2 H = 0$$

The general solution is

$$H = A\cos(\mu\theta) + B\sin(\mu\theta).$$

Also,

$$H(0) = H(2\pi),\ H'(0) = H'(2\pi).$$

Clearly $\mu = n$, $n = 0, 1, 2, \ldots$ works. (It can be shown that these are the only values of $\mu$ for which non-trivial solutions for $A, B$ exist.)

The equation for $R$ is

$$(rR')' + \lambda r R - \frac{n^2}{r} R = 0$$

The general solution is, for arbitrary constants $D_n$ and $E_n$,

$$R(r) = D_n J_n(\sqrt{\lambda} r) + E_n Y_n(\sqrt{\lambda} r)$$

where $J_n$, $Y_n$ are called the $n^{th}$-order *Bessel functions* of the first and second kind respectively.

The behaviour of $J_n$ will be discussed shortly. However, $Y_n$ is unbounded at $r = 0$, and so

$$R(r) = D_n J_n(\sqrt{\lambda} r).$$

Also

$$R(a) = 0 = J_n(\sqrt{\lambda} a).$$

For each $n$, $J_n$ has an infinite number of zeros. They yield the values of $\lambda_{n,m}$, $m = 1, 2, \ldots$

**Solution to Wave equation for Drum**

With zero initial velocity, the vibrations are

$$u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_{nm}\cos(c\sqrt{\lambda}_{n,m}t)\sin(n\theta+\phi_{n,m})J_n(\sqrt{\lambda}_{n,m}r) \tag{23}$$

where $\lambda_{n,m}$ are such that $J_n(\sqrt{\lambda}_{n,m}a)=0$.

- Each vibrational mode corresponds to a particular Bessel function $J_n$ and a particular zero of $J_n$.
- Let the $m^{th}$ zero of $J_n$ be $j_{n,m}$,

$$\sqrt{\lambda}_{n,m}a=j_{n,m}$$

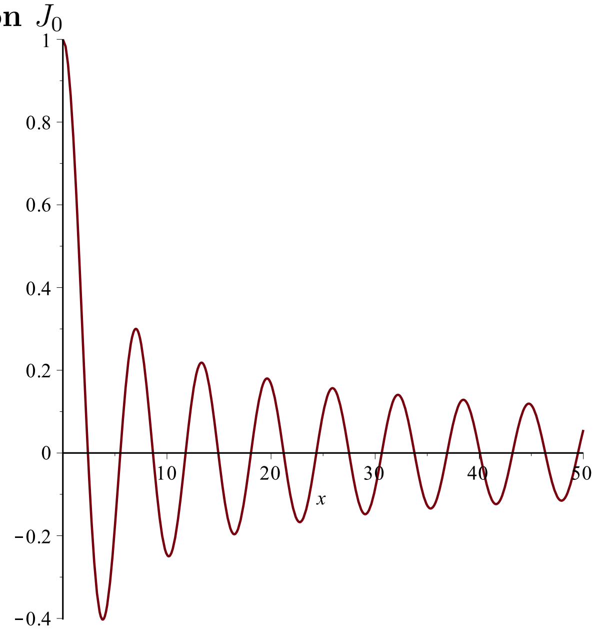
- the natural frequencies are

$$\frac{cj_{n,m}}{a}(rad/s), \quad \text{or}\frac{cj_{n,m}}{2\pi a}(rad/s).$$

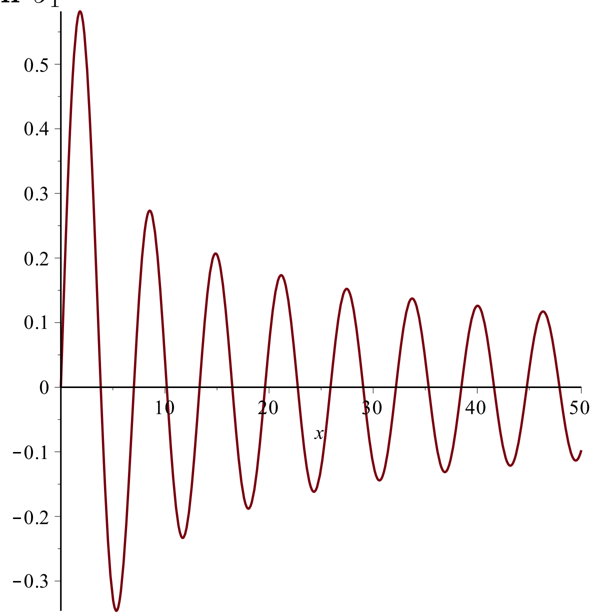
Defining $R_{n,m}(r)=J_n(\sqrt{\lambda}_{n,m}r)=J_n(\frac{j_{n,m}}{a}r)$, (23) can be written

$$u(r,\theta,t)=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}A_{nm}\cos(\frac{cj_{n,m}}{a}t)\sin(n\theta+\phi_{n,m})R_{n,m}(r).$$

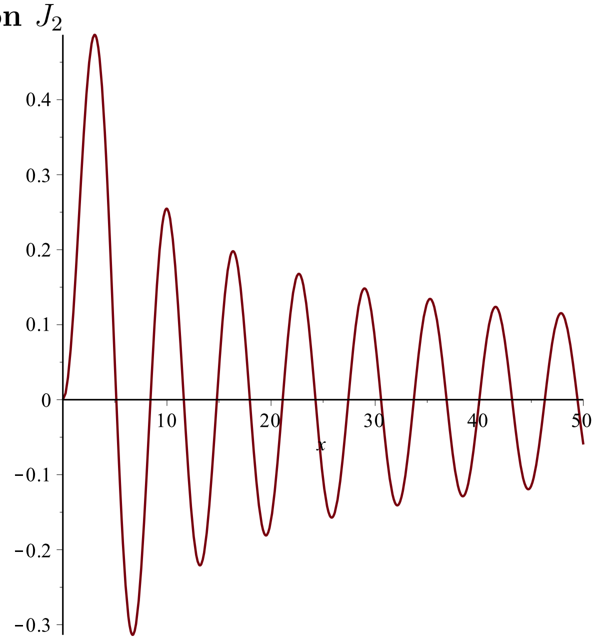
**Bessel function $J_0$**



**Bessel function $J_1$**



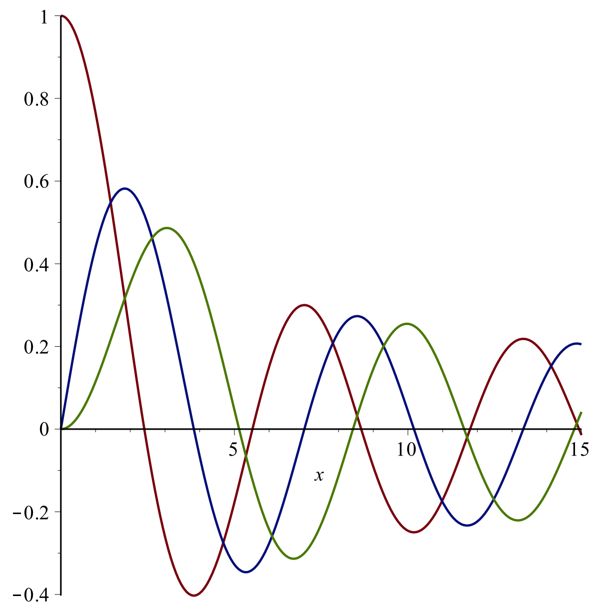

**Bessel function $J_2$**




**Zeros of** $J_0,\ J_1,\ J_2$

| m | $J_0$ | $J_1$ | $J_2$ | $J_3$ |
|---|---|---|---|---|
| 1 | 2.40 | 3.83 | 5.14 | 6.38 |
| 2 | 5.52 | 7.01 | 8.42 | 9.76 |
| 3 | 8.65 | 10.2 | 11.6 | 13.0 |

Table 1: Zeros $j_{n,m}$ of Bessel's function of the first kind $J_n$ [Ben06, pg. 114]




**Zeros of Bessel function $J_n$**

- Each $J_n$ has an infinite number of zeros, approaching infinity
- Except for $r = 0$, $J_n$ and $J_m$, $n \neq m$ have no zeros in common.
- The zeros are not evenly spaced.

**Zeros of Bessel functions $J_n$**

The overtones are not integer multiples of the fundamental frequency and are not harmonic.

The natural frequencies are $c\sqrt{\lambda_{n,m}}$ where $\sqrt{\lambda_{n,m}}a$ are zeros of $J_n$.

$\boxed{\text{Drums have overtones, but they are not harmonics.}}$

| m | $\frac{j_{0,m}}{j_{0,1}}$ | $\frac{j_{1,m}}{j_{0,1}}$ | $\frac{j_{2,m}}{j_{0,1}}$ | $\frac{j_{3,m}}{j_{0,1}}$ |
|---|---|---|---|---|
| 1 | 1 | 1.59 | 2.14 | 2.65 |
| 2 | 2.30 | 2.92 | 3.51 | 4.07 |
| 3 | 3.6 | 4.25 | 4.83 | 5.42 |

Table 2: Ratio of zeros of Bessel's function of the first kind $J_n$ [Ben06, pg. 114]

**Overtones of Drums**

The frequencies of vibration are $c\lambda_{n,m}$ where $\lambda_{n,m}$ solves ($\ell = 1$)

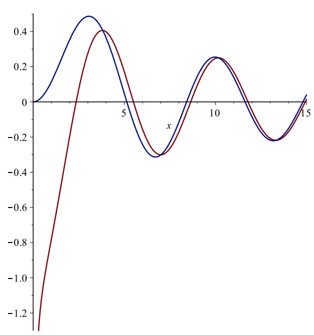
$$J_n(\lambda_m) = 0, \quad m = 1, 2, \ldots; \quad n = 0, 1, 2, \ldots$$

- infinite number of zeros, $\lim_{m\to\infty} \lambda_{n,m} = \infty$
- Since for large $z$

$$J_n(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z - \frac{n\pi}{2} - \frac{\pi}{4})$$

  asymptotically

$$\lambda_{n,m} \approx \pi(m + \frac{n}{2} + \frac{\pi}{4})$$




$J_2$ and approximation

**Overtones of Drums (cont.)**

| 1st Root of $J_n$ | approx. zero |
|---|---|
| 2.405 | 2.356 |
| 3.832 | 3.927 |
| 5.136 | 5.498 |
| 6.381 | 7.068 |

*The overtones are not close to harmonic.*

### Modes of Vibration

Assume zero initial velocity:

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta))J_n(\sqrt{\lambda}_{n,m}r)\cos(c\sqrt{\lambda}_{n,m}t)$$

Spatial part of the response for the mode $(n,m)$ is of the form

$$\cos(n\theta+\phi_n)J_n(\sqrt{\lambda}_{n,m}r)$$

Some modes depend only on $\theta$, some only on $r$, most on both.

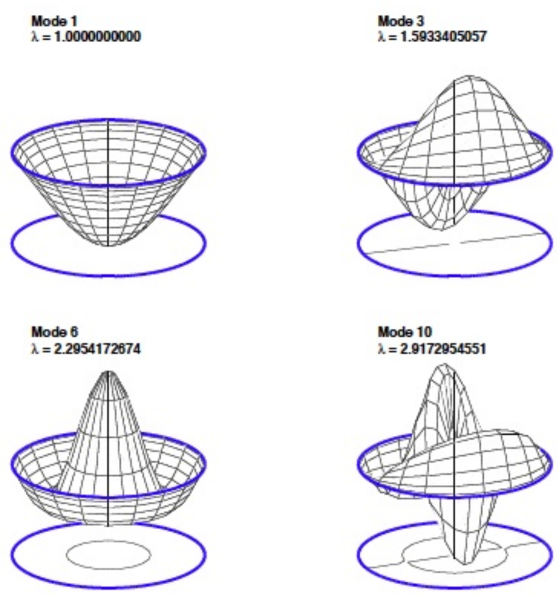
See Figures 3.20, 3.21 in Benson and there are many videos on the web. For example:

`https://www.youtube.com/watch?annotation_id=annotation_964432&feature=iv&src_vid=wvJAgrUBF4w&v=1yaqUI4b974`

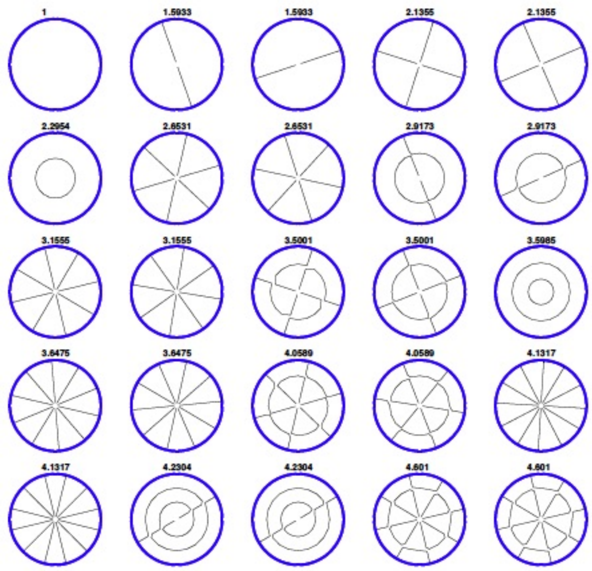
or my personal favorite:

`https://www.youtube.com/watch?v=Bs3uPbhIZxc`

### Membrane Modes from Spectral Methods in Matlab



**Nodes of the first 20 modes from Spectral Methods in Matlab**



**Xylophones, Mbira etc (Idiophones)**

In these instruments, the sound is produced by striking a bar. This sets up transverse vibrations in the bar, which are different from the longitudal vibrations that occur in a tube or string.

`https://www.youtube.com/watch?v=ExGGG1R0liM`

The force is due to a moment on each element of the bar. Use of some assumptions (such as no twisting, linearity, constant physical parameters) leads to the differential equation

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) + EI \frac{\partial^4 u}{\partial x^4}(x,t) = 0, \quad 0 < x < \ell \tag{24}$$

where $E$, $I$, $\rho$ are physical parameters. Since they are constant, define $c^2 = \frac{EI}{\rho}$ ($m^4/s^2$) which yields

$$\frac{\partial^2 u}{\partial t^2}(x,t) + c^2 \frac{\partial^4 u}{\partial x^4}(x,t) = 0. \quad \boxed{\text{beam equation}}$$

Various boundary conditions are possible, depending on how the bars are fastened.

**Solution of the beam equation**

Use separation of variables (!) and assume a solution of the form

$$u(x,t) = M(x)T(t).$$

Substituting into the beam equation and rearranging yields

$$\frac{T''}{T} = -c^2 \frac{M^{IV}}{M} = -\omega^2$$

where $M^{IV}$ indicates the $4^{th}$ derivative. Each side must be a constant, call it $-\omega^2$ (as in sect. 3.9 of Benson).

The equation for $T$ is the familiar oscillator equation

$$T'' + \omega^2 T = 0.$$

**Solution for spatial dependence** $M$

$$M^{IV} = \frac{\omega^2}{c^2} M.$$

Since this is $4^{th}$ order, the general solution involves 4 functions. Defining $\kappa = (\frac{\omega^2}{c^2})^{\frac{1}{4}} = (\frac{\omega}{c})^{\frac{1}{2}} E$

$$M(x) = A \sin \kappa x + B \cos \kappa x + C \sinh \kappa x + D \cosh \kappa x.$$

Now the boundary conditions need to be considered.

Suppose one end is clamped and the other free (mbira etc )

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(\ell, t) = 0, \quad \frac{\partial^3 u}{\partial x^3}(\ell, t) = 0$$

so that

$$M(0) = 0, \quad M'(0) = 0, \quad , M''(\ell) = 0, \quad M'''(\ell) = 0.$$

Using the other 2 boundary conditions,

$$A(-\sin \kappa\ell - \sinh \kappa\ell) + B(-\cos \kappa\ell - \cosh \kappa\ell) = 0,$$

$$A(-\cos \kappa\ell - \cosh \kappa\ell) + B(\sin \kappa\ell - \sinh \kappa\ell) = 0.$$

This is a linear system in $A, B$ which has non-trivial solutions only if the determinant is 0, so

$$1 + \cos \kappa\ell \cosh \kappa\ell = 0.$$

**Solution to Beam Equation**

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t) M_k(x)$$

where $M_k$ is the solution to $M^{IV} + \kappa^4 M = 0$ and

$$\omega_k = \kappa_k^2 c$$

where $\kappa_k$ solve

$$1 + \cos \kappa_k \ell \cosh \kappa_k \ell = 0.$$

**Overtones**

Since $\kappa^4 = \frac{\omega^2}{c^2}$, the natural frequencies are $\omega_k = \kappa_k^2 c$ where $\kappa_k = \frac{z}{\ell}$ and $z$ solves

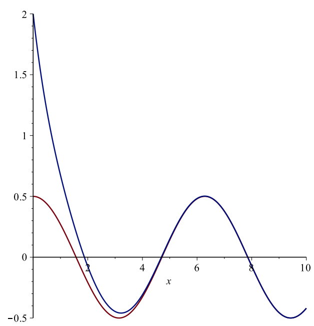
$$1 + \cos z \cosh z = 1 + \frac{1}{2} \cos z (e^z + e^{-z})$$

- There are an infinite number of real zeros approaching $\infty$, $\kappa_k$.
- For large $z$

  $$1 + \cos z \cosh(z) \approx \frac{1}{2} (\cos z) e^z$$

  asymptotically

  $$\kappa_k \approx \frac{(2k-1)\pi}{2}.$$




$e^{-z}(1 + \cos z \cosh z)$ and approximation

**Overtones (cont.)**

| Roots of $1 + \cos z \cosh z$ | $\frac{(2k-1)\pi}{2}$ |
|---|---|
| 1.875 | 1.571 |
| 4.694 | 4.712 |
| 7.854 | 7.854 |
| 11.00 | 11.00 |

*The overtones are approximately harmonic.*

# 7 Frequency Response

Consider the first-order forced differential equation

$$\dot{z}(t) = -az(t) + bu(t), \quad z(0) = z_o$$

where $a > 0$. (This describes a number of situations. For instance $z$ can be the temperature of a well-mixed tank and $u$ is the heat added/removed.) The solution to this differential equation is

$$z(t) = e^{-at}z_o + \int_0^t e^{-a(t-\tau)}bu(\tau)d\tau = e^{-at}z_o + \int_0^t e^{-a\tau}bu(t-\tau)d\tau.$$

As $t \to \infty$ the response is entirely due to the forcing function. This is called the *steady-state response.*

Consider a periodic forcing term $u$. It's easier to integrate an exponential than a sine or cosine, so consider, (letting $\imath = \sqrt{-1}$, and set

$$u(t) = \cos(\omega t) + \imath \sin(\omega t) = e^{\imath\omega t}.$$

$$z(t) = e^{-at}z_o + \int be^{-a(t-\tau)}e^{\imath\omega\tau}d\tau = e^{-at}z_o + \frac{-be^{-at}}{a+\imath\omega} + \frac{be^{\imath\omega t}}{a+\imath\omega}.$$

As $t \to \infty$, the first two terms become insignificant. The steady-state response is

$$z_{ss}(t) = \frac{be^{\imath\omega t}}{a+\imath\omega}. \tag{25}$$

Note

$$|z_{ss}(t)| = \frac{b}{\sqrt{a^2+\omega^2}}.$$

Let Re indicate the real part of a complex number. Since $\cos(\omega t) = \mathrm{Re}\, e^{\imath\omega t}$,

$$\begin{aligned}\int_0^t e^{-a(t-\tau)}b\cos(\omega t)d\tau &= \int_0^t e^{-a(t-\tau)}b\mathrm{Re}\, e^{\imath\omega\tau}d\tau \\ &= \mathrm{Re} \int_0^t e^{-a(t-\tau)}be^{\imath\omega\tau}d\tau.\end{aligned}$$

Thus, the steady-state response to a cosine is the real part of (25). This works out to

$$\frac{b(a\cos(\omega t) + \omega\sin(\omega t))}{a^2+\omega^2}$$

or defining $\phi = \arctan(\frac{\omega}{a})$,

$$\frac{b\cos(\omega t - \phi)}{\sqrt{a^2+\omega^2}}.$$

For small forcing frequencies, $\phi \approx 0$ and the response is about $\frac{1}{a}$. As $\omega$ increases, the phase shift increases to $-90$ degrees and the magnitude decreases to 0.

A plot of the response of a system to a periodic forcing function as the frequency of the forcing function changes is called the *frequency response* . A example of the frequency response of a first order system is shown in Figure 9.

Note that, defining $h(t) = e^{-at}b$, the response to the forcing function $u$ is described by

$$\int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau.$$

*Example* 1. (optional) A system with one forcing term can be modelled by a set of linear ordinary differential equations can be written

$$\dot{z}(t) \;\; = \;\; Az(t) + bu(t), \quad z(0) = z_o\,,$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$. Roughly, $A$ describes the internal dynamics, $B$ the effect of the forced input $u$. The function $u$ is the *input*, and $z$ is called the *state*. Note that $z(t) \in R^n$ so in general it will be a vector. (The number of state variables $n$ is called the *order* of the system. ) The solution can be written

$$z(t) = \exp(At)z_o + \int_0^t \exp(A(t-\tau))bu(\tau)d\tau.$$

Usually only part of the state is measured so letting $y$ be the measurement, and $c$ a vector describing the measurement,

$$y(t) = c^T z(t).$$

The measurement, with zero initial condition is

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$

where

$$h(t) = c^T\exp(At)b.$$

This generalizes to systems of ordinary differential equations and also systems modelled by partial differential equations, such as a vibrating string, or pressure waves. There is a function $h$, called the *impulse response* with $h(t) = 0$ for $t < 0$ so that, once the effect of any initial condition has decayed, the relationship between the measurement and the forced term is described by

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau. \tag{26}$$

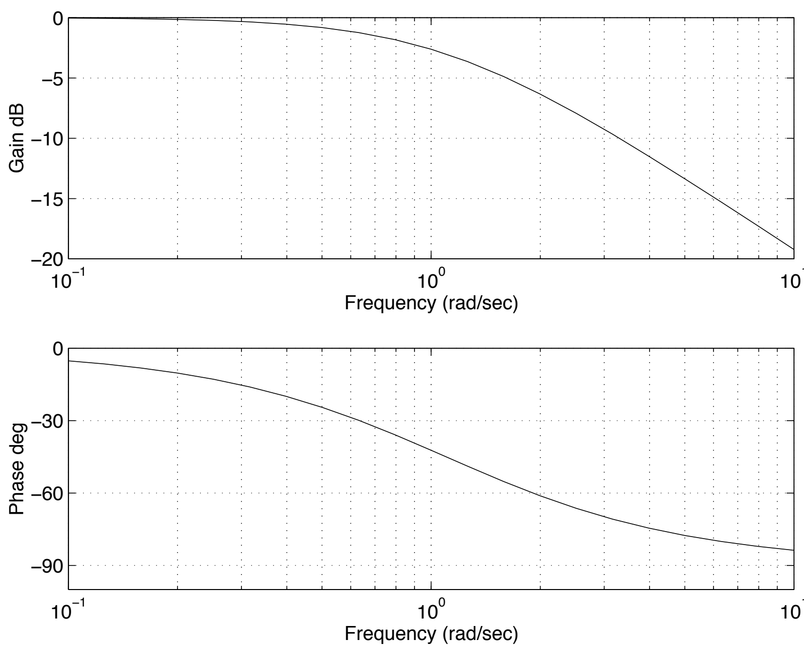



Figure 9: Frequency response of $\frac{1.1}{s+1.1}$ The magnitude and frequency are on a logarithmic scale.

Consider a forced damped oscilator with natural frequency 5 and damping parameter $0 < \xi < 1$:

$$\ddot{w}(t) + 10\xi\dot{w}(t) + 25w(t) = 25u(t).$$

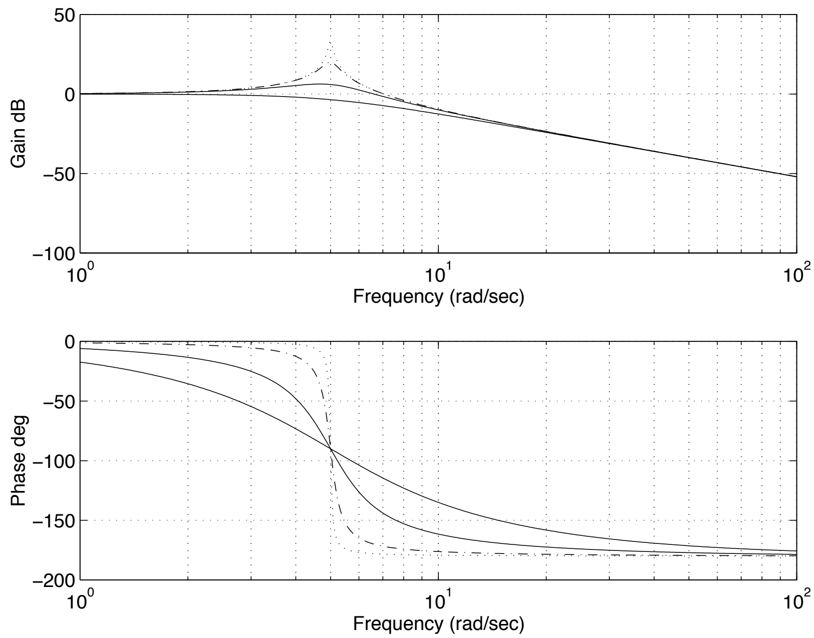



Figure 10: The frequency response of a damped oscillator with natural frequency $\omega_n = 5$ and $\xi = .01,\ .05,\ .25,\ .75\,.$

A plot of the frequency response for different values of the damping $\xi$ is shown in Figure 10. Note that the peak of the magnitude of the frequency response is at the natural frequency. The sharpness of the peak increases as damping decreases.

# 8 Fourier series

For some fundamental frequency $\omega_0$ (rad/s) consider a Fourier series

$$\tilde{f}(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)). \tag{27}$$

Individual terms are called harmonics. The period is $T = \frac{2\pi}{\omega_0}$. The fundamental frequency in $Hz$ is $\frac{\omega_0}{2\pi}$ or $\frac{1}{T}$.

Series of this form occurred (for example) when solving the wave equation for the response of vibrating strings and wind instruments. The function $\tilde{f}$ can be considered the response at a particular value of the spatial variable $x$. Clearly any function of this form is periodic. Any reasonable function with period $\frac{2\pi}{\omega_0}$ can be written in this form.

It is convenient to use the exponential form of (27) to calculate the coefficients. Using the notation

$$\imath = \sqrt{-1}$$

for real $z$

$$\exp(iz) = \cos z + \imath \sin(z).$$

Then

$$\exp(-\imath z) = \cos z - \imath(\sin z).$$

It follows that

$$\begin{aligned} \cos(z) &= \frac{1}{2}(\exp(\imath z) + \exp(-\imath z)) \\ \sin(z) &= \frac{1}{2\imath}(\exp(\imath z) - \exp(-\imath z). \end{aligned}$$

Then, defining

$$c_0 = a_0, \quad c_n = \frac{a_n - \imath b_n}{2}, \quad c_{-n} = \frac{a_n + \imath b_n}{2} \quad (n > 0).$$

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} c_n e^{\imath n\omega_0 t}. \tag{28}$$

Since $f(t)$ is real-valued, $c_{-n} = \bar{c}_n$ where $\bar{c}$ denotes the complex conjugate.

The complex form of the Fourier series is more compact and also it's simpler to calculate the single set of coefficients $c_n$ rather than the two sets $a_n$ and $b_n$. For any $n \neq 0$,

$$\int_0^T \cos(n\omega_0 t)dt = 0, \quad \int_0^T \sin(n\omega_0 t)dt = 0,$$

so

$$\int_0^T e^{\imath n\omega_0 t}dt = 0.$$

This is true when integrating over any interval of the form $[t_0, t_0 + T]$.

Multiply both sides of (28) by $e^{-\imath m\omega_0 t}$ and integrate to obtain

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{f}(t)e^{-\imath m\omega_0 t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{n=-\infty}^{\infty} c_n e^{\imath(n-m)\omega_0 t}dt = Tc_n.$$

Thus,

$$c_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{f}(t)e^{-\imath m\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\imath m\omega_0 t}dt.$$

Define the partial sums of the Fourier series of $f$ :

$$\tilde{f}_N(t) = \sum_{n=-N}^{N} c_n e^{\imath n\omega_0 t}.$$

A similar theorem to the following result, Theorem 2, was given earlier in the course.

**Theorem 3.** *Suppose that $f$ is periodic with period $T$ and piecewise smooth. The value of the Fourier series $\tilde{f}(t)$ (27) or (28) converges to $f(t)$ at all points where $f$ is continuous. That is,*

$$\lim_{N\to\infty} \tilde{f}_N(t) = f(t).$$

*If $f$ is not continuous at a point $t_0$ then $\tilde{f}_N(t_0)$ converges to $\frac{f(t_{0-}+f(t_{0+})}{2}$.*

**Theorem 4.** *Suppose that $f$ is periodic with period $T$ and piecewise smooth.*

$$\lim_{N\to\infty} \int_0^T |f(t) - \tilde{f}_N(t)|^2 dt = 0$$

## 8.1 Functions defined on an interval

Suppose we have a function $f(t)$ defined on $[-\frac{A}{2}, \frac{A}{2}]$ (or any other interval of length $A$.) We can define the Fourier series for $f$ by defining $T = A$ and considering $f$ to be extended periodically outside of the original interval. This yields a periodic function $\tilde{f}$ that equals $f$ on $[-\frac{A}{2}, \frac{A}{2}]$ in the sense described in the above theorems. Outside of $[-\frac{A}{2}, \frac{A}{2}]$ it is possible that the two functions are not equal, and $f$ may even not be defined. This yields a Fourier series (28) with

$$c_n = \frac{1}{A}\int_{\frac{-A}{2}}^{\frac{A}{2}} f(t)e^{-\imath n\omega_0 t}dt; \quad \omega_0 = \frac{2\pi}{A}.$$

By extending $f$ outside of its original interval of definition we can define a Fourier series for $f$ with any period $T > A$. Suppose we set $f(t) = 0$ for $|t| > \frac{A}{2}$. Then we could consider the function to be defined on any interval $[-\frac{T}{2}, \frac{T}{2}]$ where $T \geq A$. The Fourier series is then

$$c_n = \frac{1}{T}\int_{\frac{-A}{2}}^{\frac{A}{2}} f(t)e^{-\imath n\omega_0 t}dt; \quad \omega_0 = \frac{2\pi}{T}. \tag{29}$$

Altenatively, consider a function $f$ defined on an interval $[0, \ell]$. By constructing the odd extension of $f$ to $[-\ell, 0]$ an odd periodic function with period $T = 2\ell$ is obtained. Since cosine is even, this Fourier series contains only sine terms and it equals $f$ on $[0, \ell]$.

Thus, there are many ways to extend a function defined on an interval to a periodic function defined on the real line. The different extensions lead to different Fourier series. On the original region of definition, all the series equal the original function at points where it is continuous. Outside of the original interval they will be different in general. However once the periodic extension of the function is defined, the coefficients $c_n$ are determined. Conversely, each set of $\{c_n\}$ with

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

determine a unique function.

# 9 Fourier transform

Consider now a function $f$ defined on the whole real line. If $f$ is periodic with period $T$ then a Fourier series can be defined that equals $f$. However, many functions are not periodic. Consider the $T-$periodic extension of the function defined on some interval $[-\frac{T}{2}, \frac{T}{2}]$. The corresponding Fourier series $\tilde{f}_T$ will equal $f$ on that interval, but not generally outside that interval. Defining the integral

$$\hat{f}(\nu) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-\imath 2\pi\nu t} dt, \tag{30}$$

and noting that $\omega_0 = \frac{2\pi}{T}$, the Fourier coefficients can be written

$$c_n = \frac{1}{T}\hat{f}(\frac{n}{T}).$$

Substituting into (28),

$$\tilde{f}_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \hat{f}(\frac{n}{T}) e^{\imath 2\pi \frac{n}{T} t}. \tag{31}$$

The subscript $T$ here indicates that it is the Fourier series for $f$, considered as a function on $[-\frac{T}{2}, \frac{T}{2}]$. Defining $\Delta = \frac{2\pi}{T}$, (31) can be rewritten as

$$\tilde{f}_T(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(\frac{1}{2\pi} n\Delta) e^{-\imath n\Delta t}\Delta.$$

This is a Riemann sum. Taking $\Delta \to 0$, (or, equivalently, $T \to \infty$ ) yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\frac{1}{2\pi}\omega) e^{\imath\omega t} d\omega.$$

Defining $\nu = \frac{1}{2\pi}\omega$, a change of variables leads to

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu) e^{2\pi\imath\nu t} d\nu$$

where $\hat{f}$ is calculated as the integral in (30) as $T \to \infty$.

**Definition 5.** *The Fourier transform of a real (or complex)-valued function $f$ of a real variable $t$ is*

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi\imath\nu t}dt \tag{32}$$

*for any function for which the integral is well-defined.*

(There are many slightly different definitions of the Fourier transform. They vary in their handling of constants, but are fundamentally all equivalent. )

Clearly the Fourier transform of any function that is only non-zero on a bounded interval and integrable on that interval is well-defined. More generally, if

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

the Fourier transform is defined.

The above calculations are a heuristic argument showing the relationship between a function, its Fourier series, the Fourier transform. This can be rigorously justified.

**Theorem 6.** *[Ben06, Thm. 2.13.3] Let $f$ be a piecewise smooth function that is also integrable on $(-\infty, \infty)$. At points where $f$ is continuous,*

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu)e^{2\pi\imath\nu t}d\nu.$$

*At discontinuities, the value of the above integral is the average of the right and left limits of $f$.*

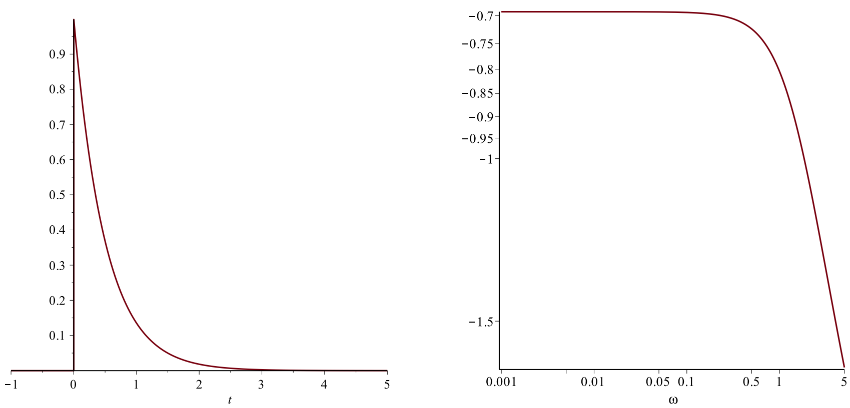
Since the transform is defined via an integral, it is linear: if $a$ is a scalar and $f, g$ have Fourier transforms.

$$(\widehat{af + g})(\nu) = a\widehat{f}(\nu) + \widehat{g}(\nu).$$

Similarly, the inverse Fourier transform defined is also linear.

*Example* 2. For $t \geq 0$, and $a > 0$, define

$$f(t) = e^{-at},$$




(a) $f(t) = \int_{-\infty}^{\infty} \hat{f}(\nu) e^{-\imath 2\pi\nu t} dt$

(b) $\hat{f}(\nu) = \int_{-\infty}^{\infty} \hat{f}(\nu) e^{\imath 2\pi\nu t} dt$

Figure 11: There is a one-to-one correspondence between a function $f(t)$ and its Fourier transform $\hat{f}(\nu)$.

and set $f(t) = 0$ for $t < 0$.

$$\begin{aligned}
\hat{f}(\nu) &= \int_0^\infty e^{-at} e^{-\imath 2\pi\nu t} dt \\
&= \lim_{T\to\infty} \int_0^T e^{-at} e^{-\imath 2\pi\nu t} dt \\
&= \lim_{T\to\infty} \frac{1}{-a - 2\imath\pi\nu} e^{-at - \imath 2\pi\nu t} \Big|_{t=0}^T \\
&= \frac{1}{a + \imath 2\pi\nu}.
\end{aligned}$$

*Example* 3. Consider a function

$$\hat{f}(\nu) = 1 \quad |\nu| < \sigma$$

and 0 outside of this band. Then in the time domain

$$\begin{aligned}
f(t) &= \int_{-\sigma}^{\sigma} e^{2\pi\imath\nu t} d\nu \\
&= \frac{1}{2\imath\pi t}(e^{\imath 2\pi\sigma t} - e^{-\imath 2\pi\sigma t}) \\
&= \frac{1}{\pi t}\sin(2\pi\sigma t).
\end{aligned}$$

Defining the function

$$\text{sinc}(z) = \frac{\sin z}{z},$$

$$f(t) = 2\sigma \text{sinc}(2\pi\sigma t).$$

The following identity holds.

**Theorem 7. Parseval's Theorem.** *For any Fourier-transformable function $f$ with $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$,*

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 d\nu.$$

Since there is a bijection between $f$ and its transform $\hat{f}$, a function is uniquely defined in terms of its Fourier transform. The representation $f(t)$ or $\hat{f}$ can be used to analyze a function.

First, the Fourier transform can be viewed as the frequency response of a system. For a Fourier series, the size of the coefficients reveals the response to various frequencies. The Fourier transform was obtained as a limiting Fourier series as the gap between frequencies becomes small. It thus reveals the response of a system to all frequencies. A perfectly harmonic system, such as an ideal vibrating string will only have a non-zero frequency response at a discrete number of evenly spaced frequencies. However, most systems show a response at all frequencies with a stronger response at certain frequencies.

The frequency viewpoint obtained by the Fourier transform is critical in showing a continuous signal can be reconstructed from samples, provided it is sampled fast enough. It is also important in reconstructing the original signal from its samples.

(Optional) The Fourier transform can also be shown directly to be the steady state response to $e^{\imath\omega t}$ of a system with impulse response $f$. Let $f$ be the impulse response of some system. The impulse response of a causal is system is 0 for negative time. Letting $\imath = \sqrt{-1}$, suppose a steady-state response of the system to the periodic input

$$e^{\imath\omega t} = cos(\omega t) + \imath \sin(\omega t)$$

exists. This means that the effect of the initial condition has disappeared.

For such large $t$, or when the initial condition is zero, For large $t$,

$$
\begin{aligned}
y(t) &= \int_0^t f(\tau) e^{\imath\omega(t-\tau)} d\tau \\
&= \int_0^t f(\tau) e^{-\imath\omega\tau} d\tau e^{\imath\omega t} \\
&= \int_{-\infty}^t f(\tau) e^{-\imath\omega\tau} d\tau e^{\imath\omega t} \\
&\approx \int_{-\infty}^{\infty} f(\tau) e^{-\imath\omega\tau} d\tau e^{\imath\omega t}.
\end{aligned}
$$

With frequency $\omega = 2\pi\nu$ where $\nu$ is the frequency in Hertz, this is

$$y(t) = \hat{f}(\nu) e^{\imath 2\pi\nu t}.$$

If $u(t) = \cos(2\pi\nu t) = \operatorname{Re} e^{\imath 2\pi\nu t}$; then, using the fact that $g(t)$ is real-valued, $y(t) = \operatorname{Re} \hat{f}(\nu) e^{\imath 2\pi\nu t}$.

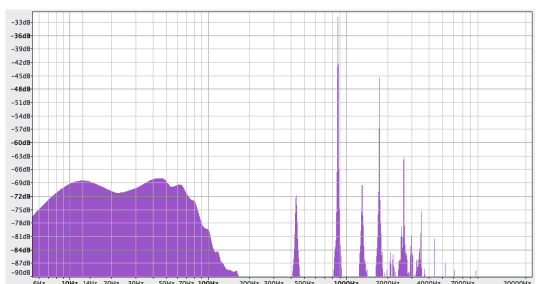



Figure 12: Fourier transform of recording of of flute blowing A4 in room with fan. This shows the frequency response.

# 10 Sampling

A signal can only be measured at a finite number of time intervals. Clearly, it is possible to sample music at a discrete number of time instants, store it and then recover the original sound. But how many samples are needed?

Let $N$ be the number of samples per second. and $T = \frac{1}{N}$ the interval between sample times. (Note: Benson uses the symbol $\Delta t$ for the sample interval.) This yields a sequence of real numbers:

$$f(0),\ f(T),\ f(2T),\ \ldots.$$

There are a couple to ways to approach this mathematically. The approach of Benson and many other authors is used here. Define an *impulse train* or *sampling operation* $\delta_s$ as follows. Let $\delta(t)$ be the impulse distribution that maps a function to its value at $t = 0$:

$$f(0) = \int_{-\infty}^{\infty} \delta(t) f(t) dt.$$

or more generally

$$f(t_0) = \int_{-\infty}^{\infty} \delta(t - t_0) f(t) dt.$$

With this notation, the first sample can be regarded as $f(t)\delta(t)$, the second as $f(t)\delta(t - T)$ and so on. Define *sampling*

$$\delta_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

Then

$$f(t)\delta_s(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$$

represents the sampled signal, $f_s$.

The integral over some time interval $[-R, R]$ of the sampled signal, multiplied by $T$, approximates the integral of the original continuous (analog) signal over the same time interval:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\tau)u(\tau)d\tau &\approx \sum_{n=-\infty}^{\infty} f(nT)u(nT)\,T \\
&= \sum_{n=-\infty}^{\infty} f(nT)\int_{-\infty}^{\infty}\delta(t-nT)u(t)dt\,T \\
&= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} f(nT)\delta(t-nT)u(t)dt\,T \\
&= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)u(t)dt\,T \\
&= \int_{-\infty}^{\infty} f(t)\delta_s(t)u(t)dt\,T
\end{aligned}$$

## 10.1 Fourier transform of $\delta_s$

For any real $r$, the Fourier transform of $\delta(t-r)$ is

$$\begin{aligned}
\widehat{\delta(t-r)}(\nu) &= \int_{-\infty}^{\infty}\delta(t-r)e^{-2\pi\imath\nu t}dt \\
&= e^{-2\pi\imath\nu r}.
\end{aligned}$$

In particular, with $r=0$,

$$\widehat{\delta}(\nu) = 1.$$

The Fourier coefficients of $\delta(t)$ extended to a $T-$ periodic function are, defining $\omega_0 = \frac{2\pi}{T}$, (see formula (29))

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\delta(t)e^{\imath n\omega_0 t}dt = \frac{1}{T}.$$

Thus, the $T$- periodic extension of $\delta$ has Fourier series

$$\begin{aligned}
\tilde{\delta}(t) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{\imath n\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} = 2\pi N \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{\imath n 2\pi N t}.
\end{aligned}$$

But the $T$-periodic extension of an impulse $\delta$ is the sampling operation $\delta_s$ and so $\delta_s$ has Fourier series

$$\delta_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{\imath 2\pi nNt}. \tag{33}$$

Now

$$e^{\imath 2\pi rt} = \int_{-\infty}^{\infty} \delta(\nu - r) e^{2\pi\imath\nu t} d\nu.$$

And so, using the formula for the inverse Fourier transform,

$$\widehat{(e^{2\pi\imath r\cdot})}(\nu) = \delta(\nu - r).$$

Using the Fourier series (33) for the sampling operation , linearity yields that

$$\widehat{\delta_s}(\nu) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(\nu - nN) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{T}). \tag{34}$$

That is, the Fourier transform of an impulse train is an impulse train with period $N$.

## 10.2 Fourier transform of $f_s$

The Fourier transform of the sampled function is periodic with period $N$.

**Theorem 8.** *[Ben06, Thm. 7.5.1]*

$$\hat{f}_s(\nu) = \widehat{(f\delta_s)}(\nu) = \sum_{n=-\infty}^{\infty} f(nT) e^{-2\pi\imath nT\nu}.$$

**Proof:** required Using the definition of the Fourier transform,

$$\begin{aligned}
\hat{f}_s(\nu) = \widehat{(f\delta_s)}(\nu) &= \int_{-\infty}^{\infty} f(t)\delta_s(t) e^{-2\pi\imath\nu t} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) e^{-2\pi\imath\nu t} dt \\
&= \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta(t - nT) e^{-2\pi\imath\nu t} dt \\
&= \sum_{n=-\infty}^{\infty} f(nT) e^{-2\pi\imath\nu nT}. \square
\end{aligned}$$

Another form of the transform of $f_s$ can be obtained.

Define a general operation, *convolution*, on two functions (or distributions) of a scalar variable:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

The result is another function (or distribution).

**Theorem 9.** *[Ben06, Thm. 2.18.1] For any Fourier-transformable functions $f$ and $g$,*

$$\widehat{(f * g)}(\nu) = \widehat{f}(\nu)\widehat{g}(\nu),$$
$$f(t)g(t) = (\widehat{f} * \widehat{g})(\nu).$$

**Proof:** (optional) First prove (i). From the definition of convolution, and the Fourier transform

$$\begin{aligned}
\widehat{(f * g)}(\nu) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t-s)ds e^{-2\pi\imath\nu t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t-s)e^{-2\pi\imath\nu t}dsdt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(u)e^{-2\pi\imath\nu(s+u)}dsdu \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)e^{-2\pi\imath\nu s}g(u)e^{-2\pi\imath\nu u}dsdu \\
&= \int_{-\infty}^{\infty} f(s)e^{-2\pi\imath\nu s}ds \int_{-\infty}^{\infty} g(u)e^{-2\pi\imath\nu u}du \\
&= \widehat{f}(\nu)\widehat{g}(\nu).
\end{aligned}$$

Part (ii) is proven identically. □

Now, consider again the Fourier transform of the sampled signal, $f_s(t)$. Since

$$f_s(t) = f(t)\delta_s(t),$$

Theorem 9 implies, using (34) for the Fourier transform of $\delta_s$,

$$\begin{aligned}
\widehat{f_s}(\nu) &= (\widehat{f} * \widehat{\delta_s})(\nu) \\
&= \int_{-\infty}^{\infty} \widehat{f}(s)\widehat{\delta_s}(\nu - s)ds \\
&= \int_{-\infty}^{\infty} \widehat{f}(s) \sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{T} - s)ds \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(s)\delta(\nu - \frac{n}{T} - s)ds \\
&= \frac{1}{T} \sum_{n=-\infty}^{\infty} \widehat{f}(\nu - \frac{n}{T}).
\end{aligned}$$

Thus, we have shown the following important result.

**Theorem 10.** *[Ben06, Cor. 7.5.4] Suppose a Fourier-transformable signal $f$ is sampled at rate $N$ samples/sec and define $T = \frac{1}{N}$. The sampled signal $f_s$ has Fourier transform*

$$\widehat{f_s}(\nu) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \widehat{f}(\nu - \frac{n}{T}). \tag{35}$$

This result also shows that $\widehat{f_s}(\nu)$ is periodic. However, the Fourier transform of a sampled signal is not only periodic, with period $\frac{1}{T} = N$, but it consists of a sum of delayed copies of the original transform $\hat{f}$.

## 11 Recovering the original signal

The expression (35) for the Fourier transform of the sampled signal can be written

$$\begin{aligned}\widehat{f_s}(\nu) &= N \sum_{n=-\infty}^{\infty} \widehat{f}(\nu - nN) \\ &= N \left(\hat{f}(\nu) + \hat{f}(\nu - N) + \hat{f}(\nu + N) + \hat{f}(\nu - 2N) + \hat{f}(\nu + 2N) + \dots\right)\end{aligned}$$

Suppose that $f(t)$ is a function so

$$\hat{f}(\nu) = 0, \quad |\nu| > \sigma. \tag{36}$$

Such functions are said to be *band-limited*. Then if $N > 2\sigma$

$$\hat{f}_s(\nu) = N\hat{f}(\nu), \quad |\nu| \leq \sigma.$$

If this condition fails, $\hat{f}_s(\nu) \neq N\hat{f}(\nu)$ on this frequency range. There are extra contributions to $\hat{f}_s$ caused by the delayed copies of $\hat{f}$. This is called *aliasing*.

In order to recover the original signal, the sampling rate must be at least twice the frequency range of the original signal.

The remaining questions are: how large a sampling rate is sufficient, and how can the original function be recovered?

To answer these questions, consider

$$\widehat{\Pi}(\nu) = \begin{cases} 1 & |\nu| \leq \sigma, \\ 0 & |\nu| > \sigma. \end{cases}$$

This is an ideal *low-pass filter*. Multiplication of any transform by $\hat{\Pi}$ will yield a function that is 0 for frequencies above $\sigma$. The corresponding time function is

$$\Pi(t) = \frac{\sin(2\pi\sigma t)}{\pi t}.$$

For any function, sampled $N$ times/second, the sampled function $f_s$ is, defining $T = \frac{1}{N}$,

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT).$$

Define $g(t) = (f_s * \Pi)(t)$. By the convolution theorem, the Fourier transform

$$\hat{g}(\nu) = \widehat{f_s * \Pi}(\nu) = \hat{f}_s(\nu)\widehat{\Pi}(\nu).$$

If $f$ is band-limited, that is, the Fourier transform satisfies (36) for some $\sigma > 0$, and also $N > 2\sigma$,

$$\hat{g}(\nu) = N\hat{f}(\nu).$$

Since Fourier transforms are unique, the inverse transform of $g$ equals $f$. We can recover $f$ by calculating $g$.

$$\begin{aligned} g(t) &= (f_s * \Pi)(t) = \int_{-\infty}^{\infty} f_s(s)\Pi(t-s)ds \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(nT)\delta(s-nT)\Pi(t-s)ds \\ &= \sum_{n=-\infty}^{\infty} f(nT) \int_{-\infty}^{\infty} \delta(s-nT)\Pi(t-s)ds \\ &= \sum_{n=-\infty}^{\infty} f(nT)\frac{\sin(2\pi\sigma(t-nT))}{\pi(t-nT)}. \end{aligned}$$

Since $\hat{g}(\nu) = N\hat{f}(\nu)$, $g(t) = Nf(t)$ and this yields the following fundamental result.

**Theorem 11. Nyquist Sampling Theorem.** *Suppose a Fourier-transformable function $f$ has the property that its Fourier transform is band-limited; that is;*

$$\hat{f}(\nu) = 0, \quad |\nu| > \sigma.$$

*Then $f$ can be recovered exactly from its samples, provided that the sampling rate $N > 2\sigma$. In this case, defining $T = \frac{1}{N}$,*

$$f(t) = \sum_{n=-\infty}^{\infty} Tf(nT)\frac{\sin(2\pi\sigma(t-nT))}{\pi(t-nT)}. \tag{37}$$

The actual implementation of this principle is more complicated than indicated here. One important point is as follows. In practice, no signal has a frequency response that drops immediately to 0, and similarly it is impossible to construct an ideal filter $\Pi$; the frequency response of an actual filter will drop off to 0 over a non-zero frequency range. Suppose the frequency range of interest of $f$ is $\sigma$. Choose a filter with a frequency response of the form

$$\widehat{\Pi}(\nu) = \begin{cases} 1 & |\nu| \leq \sigma \\ \hat{h}(\nu) & \sigma < |\nu| \leq \sigma + R \\ 0 & |\nu| > \sigma + R \end{cases}$$

where $\hat{h}$ is some function with magnitude that moves from 1 to 0 over $|\nu| \in [\sigma, \sigma + R]$. To avoid aliasing, choose a sampling rate $N > 2(\sigma + R)$.

In reconstruction of the original signal, only a finite number of terms in (37) can be calculated. However, since $\frac{\sin(z)}{z} \to 0$ quite quickly with increasing $z$, using a relatively small number of terms yields a small error.

# 12 Discrete Fourier Transform (optional)

The formulas above for the sampled signal $\{fnT\}$ involve an infinite series of samples. In practice of course, only a finite number are sampled, say $n = 0 \ldots M-1$. Think of the original function $f$ as being defined on all of the real line, but only non-zero on an interval. If the sampling period is $T$, the Fourier transform of the sampled signal $f_s$ is, using Theorem 8,

$$\widehat{f_s}(\nu) = \sum_{n=0}^{M-1} f(nT) e^{-2\pi n \imath \nu T}.$$

Since the Fourier transform is constructed from $M$ values of $f$, it should be possible to reconstruct $f_s$ from $\hat{f}_s$ using only $M$ values of $\hat{f}_s$.

Note that

$$\begin{aligned} e^{2\pi\imath \frac{k}{M}} \sum_{n=0}^{M-1} e^{-2\pi\imath n \frac{k}{M}} &= \sum_{n=0}^{M-1} e^{-2\pi\imath (n+1) \frac{k}{M}} \\ &= \sum_{n=1}^{M} e^{-2\pi\imath n \frac{k}{M}}. \end{aligned}$$

But

$$e^{-2\pi\imath M \frac{k}{M}} = e^{2\pi\imath k} = 1$$

so

$$e^{2\pi\imath \frac{k}{M}} \sum_{n=0}^{M-1} f(nT) e^{-2\pi\imath n \frac{k}{M}} = \sum_{n=0}^{M-1} f(nT) e^{-2\pi\imath n \frac{k}{M}}. \tag{38}$$

Either $\frac{k}{M}$ is an integer, so that $e^{2\pi\imath \frac{k}{M}} = 1$, or the sum is 0.

**Lemma 12.** *[Ben06, Prop. 7.9.1] If there is an integer $j$ so that $\frac{k}{M} = j$ then*

$$\sum_{n=0}^{M-1} f(nT) e^{-2\pi\imath n \frac{k}{M}} = \begin{cases} M & \textit{if there is an integer } j \textit{so that } \frac{k}{M} = j \\ 0 & \textit{else.} \end{cases}$$

**Proof:** If $k = jM$ for some integer $j$, then each term in the sum is 1 and the total is $M$. If there is no such integer $j$, then $e^{2\pi\imath \frac{k}{M}} \neq 1$ and so from (38), the sum must be 0. $\square$

Define

$$\hat{F}(k) = \widehat{f}_s(\frac{k}{MT}) = \sum_{m=0}^{M-1} f(mT)e^{-2\pi\imath\frac{mk}{M}}. \tag{39}$$

The expression (39) defines the *discrete Fourier transform* of the sequence $F(m) = f(mT)$. The inverse of this transform can be calculated using a similar formula.

**Theorem 13.** *[Ben06, Thm. 7.9.1] If $f(nT) = 0$ except for $0 \le n < M$, then*

$$f(nT) = \frac{1}{M}\sum_{k=0}^{M-1} \hat{F}(k)e^{2\pi\imath n\frac{k}{M}}. \tag{40}$$

**Proof:** Using the definition of $\hat{F}(k)$, and letting $n$ be any integer $0 \le n < M$,

$$\begin{aligned}
\frac{1}{M}\sum_{k=0}^{M-1} \hat{F}(k)e^{2\pi\imath n\frac{k}{M}} &= \frac{1}{M}\sum_{k=0}^{M-1}\left(\sum_{m=0}^{M-1} f(mT)e^{-2\pi\imath\frac{mk}{M}}\right)e^{2\pi\imath n\frac{k}{M}} \\
&= \frac{1}{M}\sum_{m=0}^{M-1} f(mT)\sum_{k=0}^{M-1}\left(e^{-2\pi\imath k\frac{m}{M}}\right)e^{2\pi\imath k\frac{n}{M}} \\
&= \frac{1}{M}\sum_{m=0}^{M-1} f(mT)\sum_{k=0}^{M-1} e^{2\pi\imath k\frac{n-m}{M}}.
\end{aligned}$$

Since $n - m < M$, $\frac{n-m}{M}$ is not an integer unless $m = n$. In this case the sum is $M$. Thus,

$$\frac{1}{M}\sum_{k=0}^{M-1} \hat{F}(k)e^{2\pi\imath n\frac{k}{M}} = f(nT). \qquad \square$$

The expressions (39) and (40) describe a discrete Fourier transform pair between the sequences $F$ and $\hat{F}$ .

Calculation of the discrete Fourier transform (39) (and similarly the inverse discrete transform (40) ) requires operations on the order of $M^2$. Efficient calculation of exponentials is possible.

The number of operations, and the corresponding computation time, can be significantly reduced if the number of samples $M$ is chosen properly. Suppose that $M$ is an even number. Then (39) can be separated into even and odd terms, writing $M_1 = \frac{M}{2} - 1$,

$$\hat{F}(k) = \sum_{n=0}^{M_1-1} f(2nT)e^{-2\pi\imath\frac{2nk}{M}} + \sum_{n=0}^{M_1-1} f((2n+1)T)e^{-2\pi\imath\frac{(2n+1)k}{M}}.$$

This can be written more concisely by defining, for any integer $M$, the root of unity

$$w_M = e^{\imath\frac{2\pi}{M}}.$$

With this notation,

$$\hat{F}(k) = \sum_{n=0}^{M_1-1} f(2nT)w_M^{-2nk} + w_M^{-k}\sum_{n=0}^{M_1-1} f((2n+1)T)w_M^{-2nk}. \qquad (41)$$

Since

$$w_M^M = e^{\imath 2\pi} = 1, \quad w_{2M_1}^{M_1} = e^{\imath\pi} = -1,$$

$$\begin{aligned}
\hat{F}(k + M_1) &= \sum_{n=0}^{M_1-1} f(2nT)w_M^{-2n(k+M_1)} + w_M^{(-k-M_1)}\sum_{n=0}^{M_1-1} f((2n+1)T)w_M^{-2n(k+M_1)} \\
&= \sum_{n=0}^{M_1-1} f(2nT)w_M^{-2nk} - w_M^{-k}\sum_{n=0}^{M_1-1} f((2n+1)T)w_M^{-2nk}. \qquad (42)
\end{aligned}$$

Comparing (41) to (42) it is apparent that only half the terms need to be calculated. If $M_1$ is even, this process can be repeated. The most efficient calculation of the Fourier transform occurs if $M = 2^j$ for some integer $j$. Then the Fourier transform can be calculated in $\mathcal{O}(N \log_2 N)$ operations instead of the $N^2$ required in the original formulation (39). This is known as the *fast Fourier transform.* Since the formula for the inverse discrete Fourier transform is similar, it can also be efficiently calculated when $M$ is a power of 2.

# 13 Quantization (optional)

With digital computation, the samples $\{f(nT)\}$ are not stored as real numbers, but as a binary number with a finite number of bits. This means that only a finite number of levels of volume can be recorded. If the number of bits is low, this leads to serious degradation of the recorded signal, particularly at low volume: any number below the least significant bit will be recorded as 0! The effects of this can be alleviated in two ways. One is to use enough bits that the resolution is good enough that the difference between 2 bits is inaudible. About 24-32 bits is generally regarded as sufficient for this. The other way to reduce the effect of quantiziation is to add noise when quantizing the signal. With dither, the original signal is more apparent. Also, the error sounds more like noise and is less noticeable. This can be mathematically justified using statistics; [LV04] is a good tutorial paper. Most recorded music uses 16 bits with dither. The noise is barely audible.

# References

[Ben06, chap.1,sect. 2.18,3.1-3.3,3.5-3.6,3.8-3.10,7.1,7.5,7.6 ], [Pap80].

Extra material on differential equations can be found in many books; see for instance [Zil01].

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