AMath390

Math & Music **Fall 2014**

Assignment # 2 **<u>SOLUTIONS</u>**

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1. Draw graphs of the functions

$$\sin(220\pi t) + \frac{1}{2}\sin(440\pi t)$$

and

$$\sin(220\pi t) + \frac{1}{2}\cos(440\pi t)$$

Explain why these sound the same, even though the graphs look quite different.

> **Solution:** Phase affects transients, but is unimportant to the perception of steady sounds. The only difference between these functions is that the second function has had its phase changed by $\frac{\pi}{2}$, so in steady-state, these will sound identical.

2. Piano wire is manufactured from steel of density $7.86 \times 10^3 kg/m^3$. The diameter of a string is $0.750mm$ and it is $50.1cm$ long.

   (a) What should the tension be so that string sounds at middle C on the piano; that is 262 Hz? Give the answer in Newtons (kgm/s^2).

   (b) How long should a string be to obtain the lowest C on the piano, which has a frequency of about 33Hz? Use the same tension as in (a).

   (c) The length obtained in part (b) is too long for most rooms, so thicker string is needed to obtain the low frequency notes. What should the diameter be to obtain the lowest C, if the length and tension are the same as in part (a).

> **Solution:**
>
> (a) The fundamental frequency is $\frac{\pi c}{\ell} rad/s$ or since $c = \sqrt{\frac{T}{\rho}}$,
>
> $$\frac{\sqrt{T}}{\sqrt{\rho}2\ell} = 262Hz.$$
>
> Solving for T yields
>
> $$T = (262 \cdot 2 \cdot 0.501)^2(7.86)\frac{kg}{ms^2} = 5.42 \times 10^8 \frac{N}{m^2}.$$
>
> Since the cross-sectional area of the wire is $\frac{\pi}{4}(0.750 \times 10^-3)^2 = 4.42 \times 10^{-7} m^2$ this is $239N$.

(Tension is given in terms of $Force/Area$ and density as $Mass/Volume$ in the derivation in the notes, so the diameter of the wire is implicit in the tension. In Benson, tension is in F and density $Mass/Length$, so the diameter of the wire is implicit in the density. Either set of definitions yield the same relationship of the physical variables to frequency.)

(b) The question is to find the length of a string (or wire) so its fundamental frequency is $33Hz$. All the other variables are the same as in part (a) so since fundamental frequency is inversely proportional to length, the length should be $\frac{262}{33}\ell$ where $\ell = 50.1$cm, or $397cm$.

(c) Here the cross-sectional area needs to be increased to lower the frequency. Letting A indicate the new area and A_o the area of the original string,

$$\frac{\sqrt{A}}{\sqrt{A_o}} = \frac{262}{33},$$

and so letting d be the new diameter, since $A = \frac{\pi}{4}d^2$,

$$d = 0.75\frac{262}{33}mm = 5.95mm.$$

3. For a vibrating string of length ℓ with fixed ends, each mode of vibration can be written as

$$u_k(x,t) = M_k \sin(\omega_k t + \phi_k)\sin(\frac{\omega_k}{c}x)$$

where $\omega_k = \frac{k\pi c}{\ell}$ and M_k, ϕ_k are determined by initial conditions. For all $k > 1$, $\sin(\frac{\omega_k}{c}x)$ has zeros in $[0, \ell]$, at say points $x = z$, so $u_k(z,t) = 0$ for all t. Points where the deflection u of a mode is zero ($u(z,t) = 0$) are called nodes.

(a) What is the distance between nodes for each mode $k > 1$?

(b) Where should you place your finger so that it is at a node of the second mode? This will prevent any mode that doesn't have a node at this point, such as the first mode, from vibrating. Express your answer in terms of fraction of the distance from the bridge, so that an answer 1 would be an open string, 0 would be no string at all.

(c) Where should you place your finger so that it is at a node of the third mode? This will prevent any mode that doesn't have a node at this point, such as the first and second modes, from vibrating. Again give your distance in terms of a fraction of the distance from the bridge.

Solution:

(a) The frequency of the function $\sin(\frac{\omega_k}{c}x)$ is $\frac{\omega_k}{2\pi c}$, or cycles/m so the period is $\frac{2\pi c}{\omega_k}$ m/cycle. The distance between adjacent nodes is half this, or $\frac{\pi c}{\omega_k}$. Substituting in the value for ω_k yields $\frac{\ell}{k}$.

(b) The second mode has a node in the middle of the string, so your finger should be halfway along. (On a guitar this is near the 12th fret.) The first mode has a peak there, so this will stop the first mode.

(c) The third mode has nodes at $1/3\ell$ and $2/3\ell$. Thus, your finger should go 1/3 of the way along. (On a guitar, this is about the 7th fret.)

4. Suppose one stringed instrument has a non-zero initial position with $u_p(x,0) = F(x)$, zero initial velocity, while another initially has a zero position and non-zero velocity, $\dot{u}_v(x,0) = F(x)$.The length of the string, tension, cross-sectional area and density are all the same for both instruments. Compare the coefficients of each mode against frequency k. For which instrument are the high frequency modes more audible?

Solution:

Define

$$f_k = \int_0^\ell F(x)\sin(\pi k \frac{x}{\ell})dx.$$

At time $t = 0$, the solution to the wave equation

$$u(x,0) = \sum_{k=1}^{\infty} A_{p,k}\sin(\frac{\pi k}{\ell}x)$$

where the coefficients $A_{p,k}$ are determined by the initial condition $u(x,0) = F(x)$. Since the eigenfunctions ϕ_j satisfy

$$\int_0^\ell \phi_j(x)\phi_k(x)\,dx = \begin{cases} \frac{2}{L} & j = k \\ 0 & j \neq k \end{cases} \tag{1}$$

Multiply each side of

$$F(x) = \sum_{k=1}^{\infty} A_{p,k}\sin(\frac{\pi k}{\ell}x)$$

by ϕ_j and integrate over $[0,\ell]$ to obtain

$$\int_0^\ell f(x)\sin(k\pi\frac{x}{\ell})dx = A_{p,k}\frac{\ell}{2}$$

and so for an instrument with non-zero initial position, the coefficients of each mode are

$$A_{p,k} = \sqrt{\frac{2}{\ell}}f_k, \quad B_{p,k} = 0.$$

Letting $c^2 = \frac{\tau}{\rho A}$, where τ, ρ and A are the tension, density and cross-sectional area respectively of the string, the vibrations are

$$u_p(x,t) = \sqrt{\frac{2}{\ell}}\sum_{k=1}^{\infty} f_k\cos(k\pi ct)\sin(k\pi\frac{x}{\ell}).$$

For the instrument with non-zero initial velocity,

$$\dot{u}(x,0) = \sum_{k=1}^{\infty} \frac{\pi k c}{\ell} A_{v,k} \cos(\frac{\pi k c t}{\ell}) \sin(k\pi \frac{x}{\ell})$$

and coefficients $A_{v,k}$ are needed so that

$$F(x) = \dot{u}(x,0) = \sum_{k=1}^{\infty} \frac{\pi k c}{\ell} A_{v,k} \cos(\frac{\pi k c t}{\ell}) \sin(k\pi \frac{x}{\ell}).$$

The coefficient of each mode is

$$A_{v,k} = 0, \quad B_{v,k} = \sqrt{\frac{2\ell\rho A}{\tau}} \frac{1}{\pi k} f_k.$$

If the initial velocity is non-zero,

$$u_v(x,t) = \sqrt{\frac{2\ell\rho A}{\tau\pi^2}} \sum_{k=1}^{\infty} \frac{1}{k} f_k \cos(k\pi c t) \sin(k\pi \frac{x}{\ell}).$$

The amplitudes of the higher frequencies is larger for the instrument with non-zero initial position.

5. Listen to the cover of Cheap Thrills by Kina Grannis & Kurt Schneider made using a bicycle for the instruments. (It's on youtube.)

   List all the different ways a sound was made using the bicycle. For which sounds, was the pitch adjustable and how was the pitch adjusted?

Solution: The musicians used the sound of

- gears, when the pedals are moved
- rung bell
- struck spokes (pitch adjusted by holding spoke partway down)
- drumsticks on bicycle wheel