Dither Signals and Their Effect on Quantization Noise

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Abstract—In this paper, the conditions a dither signal must meet so that the quantizer noise can be considered independent of the signal are derived for a quantizer having a finite number of levels. An infinite class of dither signals which satisfy these conditions is given. It is seen that the most useful member of this class is one whose probability density function is uniformly distributed over a quantizing interval. It is also shown that a necessary and sufficient condition for the noise n(t) to be independent of the signal x(t) is that a measurable quantity, called the "*D*" factor, be zero.

I. Introduction

Quantizing is a nonlinear operation which takes an input signal $x(t)$ of arbitrary amplitude distribution and produces an output $y(t)$ whose possible amplitude is, at most, a countably infinite number of values. Figure 1 is a symbolic representation of the quantizer. In general, $y(t)$ can be considered as a sum of two terms: the information signal $x(t)$ and a noise $n(t,x)$ which is a function of the information signal. It has been shown by Widrow[1] that the minimum loss of statistical data due to the quantizer operation occurs when $n(t,x)$ can be made independent of the signal $x(t)$.

$$n(t,x) = n(t) \tag{1}$$

The usefulness in satisfying (1) is not limited to the recovery of the signal statistics. The results of Roberts,[2] in the field of TV communication, and Ishikawa,[3] in the field of control systems, imply that the satisfaction of (1) leads to a minimum loss of the signal waveform accuracy.

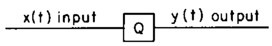

Fig. 1. Block diagram of quantizer.

Since the quantizer is a nonlinear device, one would expect different distortion (noise) effects to be caused by the quantizing operation for different signal inputs. Widrow[1] has pointed out that there are even some signals which produce distortion effects which can be treated from a statistical point of view as if they were independent of the signal. His derived condition for the one-dimensional case, assuming an infinite-level quantizer, is given by (2). An analogous "band-limited" result holds for the n-dimensional case.

$$W_x(u) = 0 \qquad \text{for all } u > \frac{2\pi}{q} \tag{2}$$

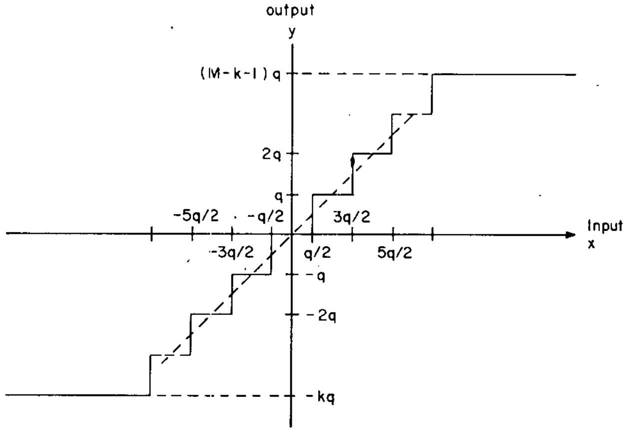

Fig. 2. M-Level quantizer input-output characteristic.

where $W_x(u)$ is the characteristic function of $x(t)$, q is the quantizer interval. The quantizer is assumed to have uniform jumps similar to those described in Fig. 2.

Signals that satisfy (2) have infinite amplitude ranges so that there are relatively few signals of practical interest which have this property. However, there are Gaussian input signals with zero means which will approximately satisfy (2). Widrow has described, in detail, the quantizer output statistics when the input is a zero-mean Gaussian signal.

Of course, not all random signals can be described as zero-mean Gaussian signals. It has been shown[4] that the assumption of (1) can lead to major errors when the signal is distributed uniformly, or as described in (3).

$$p(x) = \frac{a}{2} e^{-a|x|} \tag{3}$$

However, the condition described by (1) can always be met if the proper dither signal is used. A dither signal is a second signal which is added to the input of the quantizer and then subtracted after the quantizing operation. In a communication system, this subtraction operation must take place in the receiver and implies the use of a known dither waveform and the requirement of synchronous information being transmitted to the receiver or the removal of the dither signal by filtering.

The properties of a sine wave dither signal have been studied by Jaffee,[5] and a comparison of the sine wave dither signal and a uniformly distributed dither signal has

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[1] Widrow, B., Statistical analysis of amplitude-quantized sample-data systems, *Trans. AIEE (Applications and Industry)*, vol 79, 1960 (Jan '61 section), pp 555–568.

[2] Roberts, L. G., Picture coding using pseudo-random noise, *IRE Trans. on Information Theory*, vol 8, Feb 1962, pp 145–154.

[3] Ishikawa, T., Linearization of contactor control systems by external dither signals, Tech Rept 2103-2, Stanford Electronics Lab., Stanford, Calif., Oct 1960.

[4] Schuchman, L., Linearization of the quantizer and its relevance to improved system performance, Engineer's Thesis, Stanford University, Stanford, Calif., Feb 1964.

[5] Jaffee, R. C., Causal and statistical analysis of dithered systems containing three-level quantizer, M.S. thesis, M.I.T., Cambridge, Mass.

been made by Furman.[6] Furman came to the conclusion that the uniform dither signal reduces the quantization distortion effects more than a sine wave dither signal.

In this paper, the conditions a dither signal must meet so that the quantizer noise can be considered independent of the signal are derived for a finite-level quantizer. A class of functions is given that satisfy these conditions. As an important by-product of this result, it should be noted that the uniform dither signal is a member of this class of functions while the sine wave dither signal is not. This classification is in agreement with the conclusion of Furman.[6]

The determination of independence is extremely difficult. It is shown in Section III that a necessary and sufficient condition for the noise $n(t)$ to be independent of the signal $x(t)$ (independence in the sense that $p(n/x) = p(n)$) is that a quantity called the "D" factor be zero. It suffices for now to say that the "D" factor is a measurable statistic which was introduced by L. G. Roberts.

II. The Use of Dither Signals to Reduce the Effects of Amplitude Quantization Noise

Figure 3 illustrates the simple model that is used in this section. Our goal is to derive the conditions the probability density function of the dither signal $[p(d)]$ must meet so that the noise and signal are independent, i.e., the conditional probability density function of the noise given the signal $(p(n/x))$ is equal to the unconditional probability density distribution of the noise $p(n)$.

We first assume Q to be an infinite-level quantizer, and then the results are extended to the finite-level quantizer.

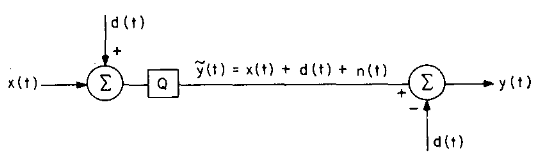

Fig. 3. Block diagram of quantizer when dither signal is used.

A. *Q is an Infinite Uniform-Level Quantizer*

When the signal is known, the noise becomes a function only of the probability density function of the dither signal. That is,

$$p(n|x = X) = f[p(d)]$$

where $p(d)$ is the probability density function of the dither signal. The noise is described by Fig. 4 and is given by

$$\begin{aligned} n &= n(x + d) \\ &= -(x + d) + bq \end{aligned}$$

where

$$bq - \frac{q}{2} \le x + d \le bq + \frac{q}{2}$$

$$b = b(x + d) = \text{an integer}$$

[6] Furman, G. G., Improving the quantization of random signals by dithering, RAND Corp., Memorandum RM-3504-PR, May 1963.

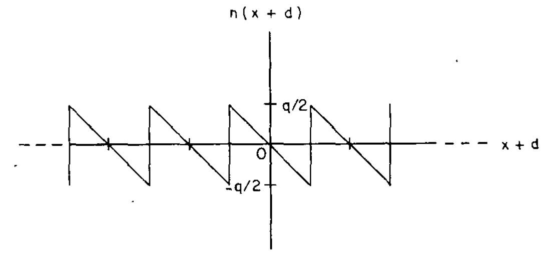

Fig. 4. Quantizer noise as a function of the input.

On the other hand, each value of d is uniquely specified by

$$d = -x - n + bq$$

Hence, we obtain

$$p(n\,|x = X) = \sum_b P_d\,(bq - n - X) \tag{4}$$

where $p_d(\alpha)$ is the probability density function of the dither signal with the variable parameter α.

Since an infinite-level quantizer was assumed with uniformly spaced quantizing levels q distance apart, the following must be true

$$p(n\,|x = X) = 0 \qquad \text{for } |n| > \frac{q}{2}$$

We now find the Fourier transform of (4) and derive conditions on the characteristic function of the dither signal so that

$$p(n|\, x = X) = p(n)$$

The Fourier transform can be written in the following manner

$$\begin{aligned} W_{n/x}(u) &= \int_{-q/2}^{q/2} \sum_b p_d(bq - X - n)\, e^{-j\omega n}\, dn \\ &= \int_{-\infty}^{\infty} v\left(n + \frac{q}{2}\right) - v\left(n - \frac{q}{2}\right) \\ &\qquad \left(\sum_b p_d(bq - X - n)\, e^{-j\omega n}\, dn\right) \end{aligned}$$

where the unit step function is defined as

$$v(x) = \begin{cases} 1 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Using the convolution theorem, we obtain

$$\begin{aligned} W_{n/x}(u) = \int_{-\infty}^{\infty} \left[v\left(n + \frac{q}{2}\right) - v\left(n - \frac{q}{2}\right)\right] e^{-j\omega n}\, dn * \\ \int_{-\infty}^{\infty} \sum_b p_d(bq - X - n)\, e^{-j\omega n}\, dn \end{aligned}$$

By evaluating the first integral and interchanging the order of summation and integration, the expansion becomes

$$\begin{aligned} W_{n/x}(u) &= q \operatorname{sinc} q\, u * \sum_b \int_{-\infty}^{\infty} p_d(bq - X - n)\, e^{-j\omega n}\, dn \\ &= q \operatorname{sinc} q\, u * \sum_b \int_{-\infty}^{\infty} p_d(v)\, e^{j\omega(v + X - bq)}\, dv \end{aligned}$$

We have changed our variable in integration so that $v = bq - X - n$. The above integral becomes

$$W_{n/x}(u) = q \operatorname{sinc} q\, u * \sum_b e^{-j\omega(bq-X)} \int_{-\infty}^{\infty} pd(v)\, e^{-j\omega v}\, dv$$

$$= q \operatorname{sinc} q\, u * \frac{1}{q} \sum_b \delta\left(u - \frac{b}{q}\right) e^{j2\pi u X}\, W_d(u)$$

$$W_{n/x}(u) = \sum_b W_d\left(\frac{b}{q}\right) e^{-j\pi\left(\frac{b}{q}\right) X} \operatorname{sinc} q\left(u - \frac{b}{q}\right) \quad (5)$$

Therefore, if the transform is zero at the points

$$W_d\left(\frac{b}{q}\right) = 0 \qquad \text{for } b = \pm 1, \pm 2, \ldots \quad (6)$$

then, the characteristic function $W_{n/x}(u)$ is independent of x and $p(n/x) = p(n)$. The result shows the sufficiency of (6) but not its necessity.

If $W_d(b/q) \neq 0$ for some value of b, then, from (5), $W_{n/x}(u)$ must be a periodic function of x. If $W_{n/x}(u)$ is a periodic function of x, then its transform is a function of x. Therefore, $p(n/x)$ is not independent of x and $p(n/x) \neq p(n)$. Thus, (6) is a necessary and sufficient condition for $p(n/x) = p(n)$.

We now note the following:

1) All density functions that satisfy (2) can also satisfy (6),

2) An infinite class of probability density functions that are not band-limited and that satisfy (6) are

$$p_d(t) = \frac{1}{kq}\left[v\left(t + \frac{kq}{2}\right) - v\left(t - \frac{kq}{2}\right)\right] * f\alpha(t) \quad (7)$$

where $f\alpha(t)$ can be any arbitrary density distribution, k is any positive integer $(1, 2, 3, \ldots)$.

B. Q is a Finite M-Level Quantizer (as shown in Fig. 2)

It is easily shown[4] that there is no way to make the noise independent of the signal while restricting the possibility of having infinite amplitude quantizing noise impulses if saturation (end effects) can occur. Therefore, no dither signal satisfying (2) can make the noise independent of the signal when a finite-level quantizer is used since band-limited characteristic functions require infinite amplitude probability density functions. However, dither signals that satisfy (7) under the following constraint

$$x_{max} + d_{max} \leq (M - k - 1)q + \frac{q}{2}$$

$$x_{min} + d_{min} \geq -kq - \frac{q}{2}$$

will make the quantizer noise independent of the signal.

It is interesting to note that, in a sampled-data system, an optimum dither signal would be one that has independent samples, each of which is uniformly distributed with k in (7) equal to one.

III. The "D" Factor as a Measure of the Independence of the Quantizing Noise

In this section, we show that it is sufficient to measure a second-order statistic (the "D" factor) in order to determine whether the noise is independent of the signal.

Roberts,[2] in a very interesting paper, describes the use of pseudo-random noise in the transmission of quantized TV information. The metric used as a measure of performance when using dither signals is what Roberts calls the "D" factor. The "D" factor is defined as follows:

$$D = \int (x - \bar{y}_x)^2\, p(x)\, dx \quad (8)$$

where

$$\bar{y}_x \triangleq \int p(y/x)\, y\, dy$$

and

$$y = n + x + d$$

Roberts has shown, experimentally, that, with the use of dither signals which reduce the "D" factor, there is a corresponding improvement in TV picture performance.

From Fig. 4, it can be seen that the quantizer output $\tilde{y}$ is related to the system output y by

$$\tilde{y} - d = y$$

Hence, the following relations exist.

$$E(\tilde{y} - d) = E(y)$$

$$E(\tilde{y} - d | x = X) = E(y | x = X)$$

$$E(\tilde{y} | x = X) - E(d | x = X) = E(y | x = X)$$

From this point on, the following assumptions are to be made

1) $p(d | x = X) = p(d)$

2) $E[d] = 0$.

Then,

$$E(\tilde{y} | x = X) = E(y | x = X) \quad (9)$$

Writing out $E(\tilde{y} | x = X)$ in integral form and using (9), we have

$$E(\tilde{y} | x = X) = \int [n(x + d) + x + d]\, p(d)\, dd$$

$$E(y | x = X) = x + \int n(x + d)\, p(d)\, dd \quad (10)$$

Substituting (10) into (8), we obtain

$$D = \int [E(n | x = X)]^2\, p(x)\, dx \quad (11)$$

Since both $p(x)$ and $[E(n|x)]^2$ are non-negative,[7]

$$D = 0$$

if, and only if,

$$E(n | x) = 0$$

[7] D could equal zero when $E[n/x = X] \neq 0$ if $p(x) = 0$ for all x. In order that the results be nontrivial, we assume $p(x) \neq 0$ for all x.

From our previous work, (5), we know that

$$W_{n/x}(u) = \sum_b W_d\left(\frac{b}{q}\right) e^{-j\pi(b/q)X} \operatorname{sinc} q\left(u - \frac{b}{q}\right)$$

Thus, it can be seen that

$$E(n|x=X) = -j \frac{d\,W_{n/x}(u)}{du}\bigg|_{u=0}$$

$$= \sum_b W_d\left(\frac{b}{q}\right) e^{-j\pi(b/q)X} \left(\frac{d\left[\operatorname{sinc} q\left(u - \frac{b}{q}\right)\Big|_{u=0}\right]}{du}\right)$$

$$E(n|x=X) = -\sum_{b\neq 0} W_d\left(\frac{b}{d}\right) e^{-j\pi(b/q)X} \frac{q}{b} \cos \pi b \qquad (12)$$

Therefore,

$$E(n|\,x=X) = 0 \qquad \text{for all } x$$

if, and only if,

$$W_d\left(\frac{b}{q}\right) = 0 \qquad \text{for } b = \pm 1, \pm 2, \ldots,$$

but this is the same criterion that must be met by a dither signal so that

$$p(n|\,x=X) = p(n)$$

We conclude then that

$$D = 0$$

if, and only if,

$$p(n|\,x=X) = p(n)$$

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New Loop-Around Systems

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Abstract—**The system described here provides centralized two-way transmission measurements on two-way dial trunks and on one-way incoming dial trunks from remote offices to a control office.**

I. Purpose of System

MANPOWER ECONOMIES can be obtained in the operation of a telephone system when control offices can perform routine two-way transmission measurements on circuits to and from remote offices, either on a one-man basis or automatically. Such measurements can be performed on two-way dial trunks and outgoing one-way dial trunks if the "Far End" is equipped with milliwatt supplies and conventional "Loop-Around" equipment. Measurements on incoming one-way dial trunks, such as AMA and CAMA trunks, generally require the use of personnel or the installation of special switch trains at the far end.

The purpose of the TTS 20 loop-around system described here is to permit an operator or automatic equipment, located at a control office, to make transmission measurements on incoming one-way dial trunks without requiring personnel or special switch trains at the far-end office. The system requires installation of far-end equipment at each remote office and of a single group of near-end equipment at the control office. The far-end equipment is normally connected to a single subscriber appearance in the far-end switching equipment.

If the transmission measurement shows the circuit to be faulty, it is generally desirable to eliminate it from service promptly. For this reason, the system described here includes provisions for remotely "busying-out" the circuit under test, thereby "blocking" it from accepting further traffic. Similarly, circuits which have been previously blocked can be unblocked from the control office. Incorporation of the remote "blocking" and "unblocking" feature also makes it possible to perform selective control functions at the remote office, such as transferring traffic to a standby carrier or radio channel and others.

In a number of cases, it is necessary to test "direct" circuits between remote offices. Some additional features have therefore been incorporated in the equipment to permit a control office to make transmission measurements on such direct circuits.

Four-wire systems have been put in operation for a number of special purposes. To permit transmission measurements on such circuits, the equipment has been designed to provide measurements on either two-wire circuits or four-wire circuits.

The TTS 20 loop-around system, when equipped with its different options and accessory units, incorporates all

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