CO 250 - Spring 2018

Assignment 4

Due date : Friday, June 1, 2018, by 4pm on Crowdmark

Submission Guidelines:

- Please submit your solutions to Crowdmark. Late assignments will not be accepted, and will receive a mark of zero. It is the responsibility of the students to make sure that the file they submit is clearly readable. Illegible submissions will receive a mark of zero.
- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- In some questions you are asked to *formulate* the problem. You are *not* asked to actually solve the formulation, e.g., compute optimal solutions. Your formulations should be easy to modify if we change the data and constants defining the problems. Clearly define all your variables (including their units) and any other new notation you use in all your answers. Your solutions must also contain a brief justification of all the constraints (explain the relation between each of the constraints and the requirements stated in the problem) and the objective function.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. Copying or paraphrasing a solution from some fellow student or old solutions from previous offerings qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. Students found to be cheating will be given a mark of 0 on the assignment. In addition, all academic offenses will be reported to the Math Academic Integrity Officer (which may lead to further penalties) and recorded in the student's file.

Re-marking policies: If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that, if you see any marking error, then write a letter detailing clearly the marking errors, and submit this to one of the TAs within one week from the date the graded assignment is returned. If you still have concerns after the final decision of the TA, then please contact your instructor communicating all the correspondence with the TA and the original petition.

[1] It is an academic offense to post this assignment or solutions to any web page.

Question 1. Certificate of optimality **(20 marks)**

(a) Consider the following linear program.

$$\begin{array}{ll} \max & (-2, 10, 1, -3, -2)x \\ \text{subject to} & \\ & \begin{pmatrix} 2 & 1 & 0 & -3 & 1 \\ 4 & -2 & -1 & -1 & 2 \\ 0 & 3 & 1 & 4 & -1 \end{pmatrix} x = \begin{pmatrix} -4 \\ 3 \\ 15 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

You are given that $\overline{x} = (2, 1, 0, 3, 0)^\top$ is a feasible solution to this LP. Prove that $\overline{x}$ is an optimal solution by obtaining a suitable certificate of optimality and justifying why this implies that $\overline{x}$ is an optimal solution to the LP. (15 marks)

(**Hint:** There is a certificate y with $y_3 = 1$.)

(b) Consider the LP (P)

$$\begin{array}{rrcl} \max & c^\top x + c_0 & & \\ \text{subject to} & & & \\ & Ax & \leq & b \\ & x & \leq & \mathbb{0} \end{array}$$

over the variable $x \in \mathbb{R}^n$, where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Suppose we have a vector $e \in \mathbb{R}^m$ such that $e \geq \mathbb{0}$, $e^\top A \leq c^\top$, and $e^\top b = c^\top d$ for some feasible solution d of the LP (P). Prove that d is an optimal solution of (P). (5 marks)

Question 2. Standard Equality Form **(20 marks)**

For some matrices $B, C, D, E, F, G \in \mathbb{R}^{m \times n}$ and vectors $d, e \in \mathbb{R}^n$ and $f, g \in \mathbb{R}^m$, consider the following linear program in variables $y, z, w \in \mathbb{R}^n$

$$\begin{array}{rl} \min & d^\top(y - z) + e^\top(z + w) \\ \text{subject to} & \\ & \begin{array}{rcl} By & \leq & Dw + f \\ Cz & = & Ey \\ Fw & \geq & Gz - g \end{array} \\ & y \text{ free},\ z \leq \mathbb{0},\ w \geq \mathbb{0} \end{array}$$

Write the LP in Standard Equality Form with variables $x \in \mathbb{R}^k$, where k is a carefully chosen constant (depending on m and n). In particular, give an explicit description of c, A, b as per the definition of LPs in SEF on page 50 in the textbook for the course.

Question 3. Basic solutions I **(15 marks)**

Let

$$A = \begin{pmatrix} 2 & 6 & -7 & 5 & 1 & -2 & -1 \\ 9 & 11 & 0 & 7 & 4 & 7 & 11 \\ 13 & 15 & 1 & 10 & 5 & 11 & 16 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 9 \\ 13 \end{pmatrix}$$

define the feasible region of the LP $\{c^\top x : Ax = b,\ x \geq \mathbb{0}\}$. For each of the vectors below, indicate whether the vector is a basic solution and whether it is a basic feasible solution. If it is a basic feasible solution find all corresponding bases. Fully justify your claims.

(a) $\begin{pmatrix} 0 & \frac{1}{6} & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^\top$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix}^\top$

(c) $\begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 \end{pmatrix}^\top$ (d) $\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}^\top$

Question 4. Basic solutions II **(20 marks)**

For a vector $a \in \mathbb{R}^n$, we refer to the set of indices j for which $a_j \neq 0$ as the support of a.

Consider the feasible solutions $d, e \in \mathbb{R}^n$ to the linear program

$$\max\{c^\top x : Ax = b, x \geq \mathbb{0}\} \tag{P}$$

where $b \in \mathbb{R}^m$, and the rows of matrix $A \in \mathbb{R}^{m \times n}$ are linearly independent. Let

$$f := \frac{1}{250}\, d + \frac{249}{250}\, e \ \ .$$

(a) Show that the support of f equals the union of the supports of d and e.

(b) Prove that f is feasible for (P).

(c) Suppose that the vectors d, e are distinct, prove that f is not a basic solution.

(d) If d, e are distinct optimal solutions of (P), prove that (P) admits a non-basic optimal solution.