CO 250 - Spring 2018

Assignment 8

Due date : Friday, July 6th, 2018, by 4:00pm

Submission Guidelines:

- Please submit your solutions to Crowdmark. Late assignments will not be accepted, and will receive a mark of zero. It is the responsibility of the students to make sure that the file they submit is clearly readable. Illegible submissions will receive a mark of zero.
- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- In some questions you are asked to *formulate* the problem. You are *not* asked to actually solve the formulation, e.g., compute optimal solutions. Your formulations should be easy to modify if we change the data and constants defining the problems. Clearly define all your variables (including their units) and any other new notation you use in all your answers. Your solutions must also contain a brief justification of all the constraints (explain the relation between each of the constraints and the requirements stated in the problem) and the objective function.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. Copying or paraphrasing a solution from some fellow student or old solutions from previous offerings qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. Students found to be cheating will be given a mark of 0 on the assignment. In addition, all academic offenses will be reported to the Math Academic Integrity Officer (which may lead to further penalties) and recorded in the student's file.

Re-marking policies: If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that, if you see any marking error, then write a letter detailing clearly the marking errors, and submit this to one of the TAs within one week from the date the graded assignment is returned. If you still have concerns after the final decision of the TA, then please contact your instructor communicating all the correspondence with the TA and the original petition.

[1] It is an academic offense to post this assignment or solutions to any web page.

Question 1. Duality **(20 marks)**

Find the dual of each of the following LPs.

(a) (5 marks)

$$\begin{array}{ll} \max & (2,3,-1)x \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 7 & -2 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix} x \begin{array}{c} \leq \\ = \\ \leq \\ = \end{array} \begin{pmatrix} 1 \\ 2 \\ 6 \\ 2 \end{pmatrix} \\ & x_1, x_2 \geq \mathbb{0},\ x_3 \text{ free} \end{array}$$

(b) (5marks)

$$\begin{array}{ll} \min & (2,4)x \\ \text{subject to} & \\ & \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -3 & 2 \\ -1 & 3 \end{pmatrix} x \begin{array}{c} \geq \\ = \\ \leq \\ = \end{array} \begin{pmatrix} 1 \\ 2 \\ 6 \\ 2 \end{pmatrix} \\ & x_1 \geq \mathbb{0},\ x_2 \text{ free} \end{array}$$

(c) (10 marks)

$$\begin{array}{ll} \min & c^\top x + \mathbb{1}^\top u \\ \text{subject to} & \\ & \begin{array}{rcl} Ax & = & b \\ Dx + Iu & \leq & d \end{array} \\ & x \geq \mathbb{0},\ u \text{ free} \end{array}$$

Where A, D are matrices, I is the identity matrix, b, c, d are vectors, and x, u are vectors of variables.

Question 2. Feasibility and Optimality **(20 marks)**

(a) (10 marks) Suppose you have an algorithm that solves LP problems in SEF starting from a given basic feasible solution. Show how to use this algorithm, in an efficient manner, to solve a system of linear inequalities $\tilde{A}x \leq \tilde{b}$. Solving such a system means, we either produce $\bar{x}$ such that $\tilde{A}\bar{x} \leq \tilde{b}$, or we output "INFEASIBLE" correctly. Prove all your claims.

Clarification/hint: You should call the algorithm solving LPs only once, for an LP problem whose data are defined using $\tilde{A}$ and $\tilde{b}$ so that if the number of variables and inequalities in $\tilde{A}x \leq \tilde{b}$ are n and m respectivley, your LP problem has at most $3(m+n)$ variables and at most $3(m+n)$ constraints.

(b) (10 marks) Suppose you have an algorithm that solves systems of linear inequalities (as explained in part (a)). Show how to use this algorithm, in an efficient manner, to solve a given LP in SEF with data (A, b, c). Solving such an LP means, we either produce an optimal $\bar{x}$ or we output "LP is UNBOUNDED", or "LP is INFEASIBLE" correctly. Prove all your claims.

Clarification/hint: You may want to use the solver for linear inequality systems two or three times with different $(\tilde{A}, \tilde{b})$ that you make up using the data $(A, b, c,)$. If the given LP problem has n variables and m linear equations, your linear systems of inequalities constructed from (A, b, c) should each have at most $3(m+n)$ variables and at most $3(m+n)$ inequalities.

Question 3. Complementary slackness **(20 marks)**

Consider the following LP (P):

$$
\begin{aligned}
\max \quad & (2,3,-3)x \\
\text{subject to} \quad & \\
& \begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 8 \\ 6 \end{pmatrix} \\
& x \geq \mathbb{0}.
\end{aligned}
$$

(a) Write down the dual problem (D). Take the dual directly without converting (P) into SEF. (4 marks)

(b) Write down all the complementary slackness conditions for (P) and (D). (6 marks)

(c) Use complementary slackness to show that $\bar{x} = (2,6,2)^\top$ is an optimal solution to (P). (4 marks)

(d) Use complementary slackness to describe the set of *all* optimal solutions to (P) and (D). Write down explicitly the polyhedron that specifies this set. (6 marks)

Question 4. Duality and consequences **(15 marks)**

You may want to read Section 4.4 of the text before attempting this question.

(a) Prove that exactly one of the following statements holds:

- there exists x such that $Ax \geq b,\ x \leq \mathbb{0}$;
- there exists y such that $A^\top y \geq \mathbb{0},\ y \geq \mathbb{0}$ and $b^\top y > 0$. (7 marks)

(b) Prove that the following statements are equivalent:

- $x = \mathbb{0}$ is the only solution to $Ax = \mathbb{0},\ x \geq \mathbb{0}$;
- there exists y such that $A^\top y \geq \mathbb{0}$, and all entries of $A^\top y$ are positive. (8 marks)

(**Hint:** Construct a primal LP that attempts to find x such that $Ax = \mathbb{0},\ x \geq \mathbb{0}$ and $x \neq 0$, and apply duality.)