<u>Problem 1:</u> Simplex Algorithm and the Two-Phase Method [23 Points]

Consider the following LP:

$$
\begin{array}{llrcl}
\text{(P)} & \max & (1,3,1,2)x & & \\
& \text{s.t.} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{pmatrix} x & = & \begin{pmatrix} 20 \\ -10 \end{pmatrix} \\
& & x & \geq & \mathbb{0}.
\end{array}
$$

(a) [4 Points] Suppose we would like to solve (P) using the Two-Phase Method. Write down the auxiliary problem (AuxP) for (P). (Note: you are <u>not</u> asked to solve (AuxP).)

(b) [4 Points] State a feasible basis for (AuxP), and write (AuxP) in the canonical form for this basis.

The LP (P) has been reproduced here for your convenience:

$$\begin{array}{llrcl} \text{(P)} & \max & (1,3,1,2)x & & \\ & \text{s.t.} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{pmatrix} x & = & \begin{pmatrix} 20 \\ -10 \end{pmatrix} \\ & & x & \geq & \mathbb{0}. \end{array}$$

(c) [15 Points] The following are the canonical forms of (P) in all possible bases.

$$\begin{array}{ll} \max & 34 + (0,0,-4,-5)x \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} x = \begin{pmatrix} -8 \\ 14 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

$$\begin{array}{ll} \max & 2 + (-4,0,0,3)x \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} -2 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

$$\begin{array}{ll} \max & 6 + (0,2,0,1)x \\ \text{s.t.} & \begin{pmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} x = \begin{pmatrix} -1 \\ 7 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

$$\begin{array}{ll} \max & 14 + (-\frac{5}{2},0,-\frac{3}{2},0)x \\ \text{s.t.} & \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

$$\begin{array}{ll} \max & \frac{32}{3} + (0,\frac{5}{3},-\frac{2}{3},0)x \\ \text{s.t.} & \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} x = \begin{pmatrix} \frac{4}{3} \\ \frac{14}{3} \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

$$\begin{array}{ll} \max & 8 + (2,3,0,0)x \\ \text{s.t.} & \begin{pmatrix} 3 & 2 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

Now answer the following questions:

(i) Suppose we solve (P) using the Simplex Algorithm starting with the basis $B = \{3, 4\}$. List all possible entering and leaving variables for the first iteration.

(ii) Does (P) have an optimal solution? Either provide one and state its objective value, or explain why it does not have an optimal solution.

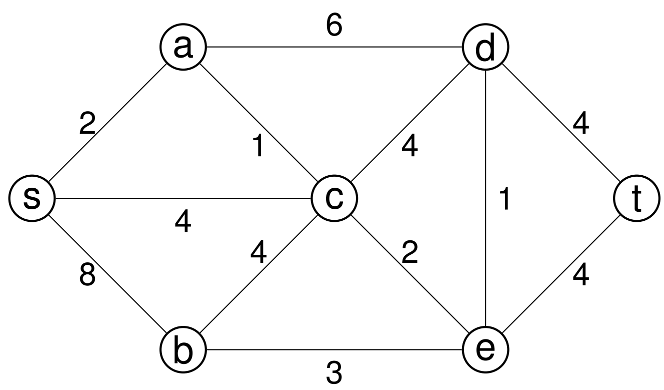
(iii) Write down all of the extreme points (if any) of the feasible region of (P). Briefly explain your answer.

(iv) Recall (AuxP), the auxiliary problem for (P) you provided in (a). Does (AuxP) have an optimal solution? Either provide one and state its objective value, or explain why it does not have an optimal solution.

(v) Is there a feasible solution in (P) with objective value 10? Either provide one such solution, or briefly explain why none exists.

Problem 2: Shortest Path Algorithm [17 Points]

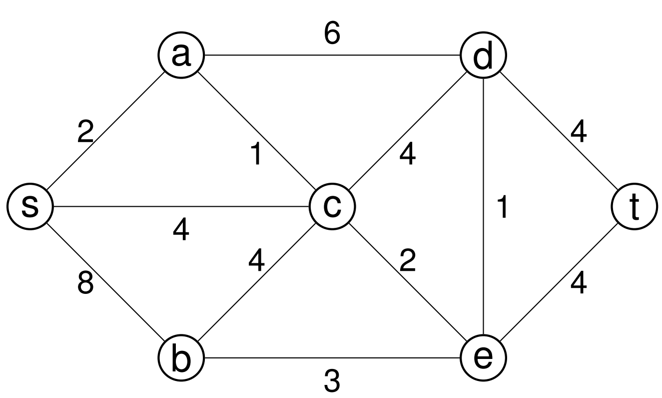
(a) [9 Points] The figure below shows a graph $G = (V, E)$, with each edge labelled by its cost c_e.

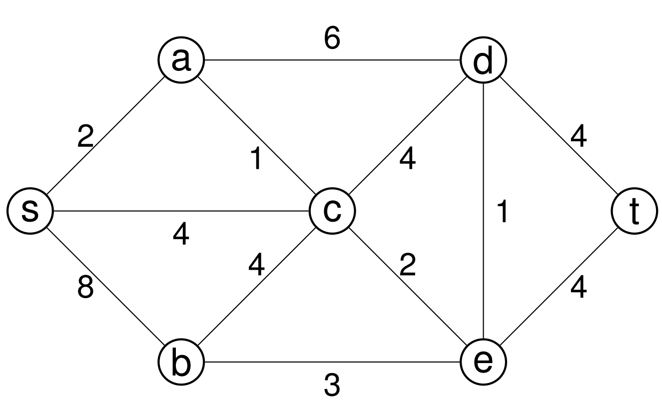


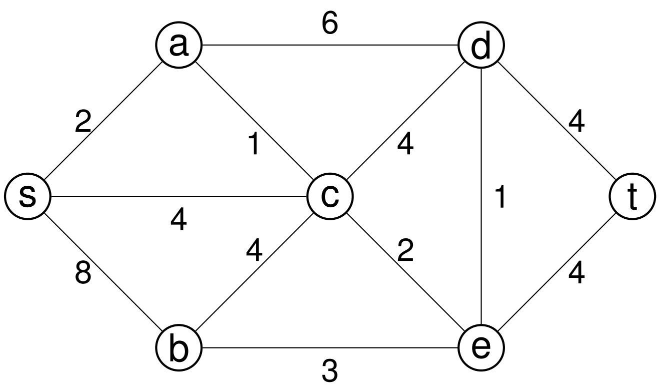
The following table shows the data obtained from running the first 3 iterations of the Shortest Path Algorithm on this graph.

Iteration i	U_i	y_{U_i}	new arc
1	$\{s\}$	2	$\overrightarrow{sa}$
2	$\{s, a\}$	1	$\overrightarrow{ac}$
3	$\{s, a, c\}$	2	$\overrightarrow{ce}$

Using the head start provided above, find a shortest st-path in G using the Shortest Path Algorithm starting at iteration 4. In your solution, you must list all arcs generated throughout the algorithm, and provide a set of feasible cut widths whose total equals the length of the shortest st-path you provide. Copies of the given graph are provided below and on the next page for your convenience (you may not need them all).



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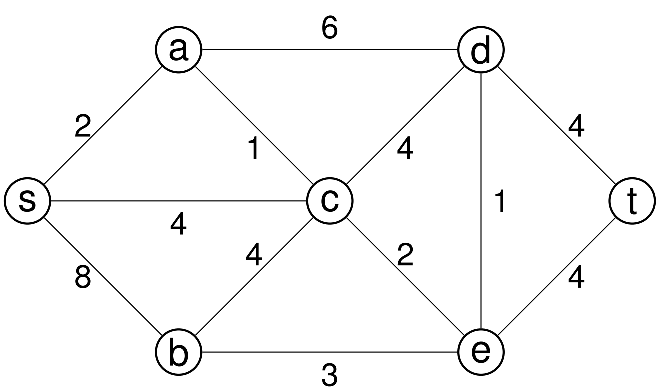
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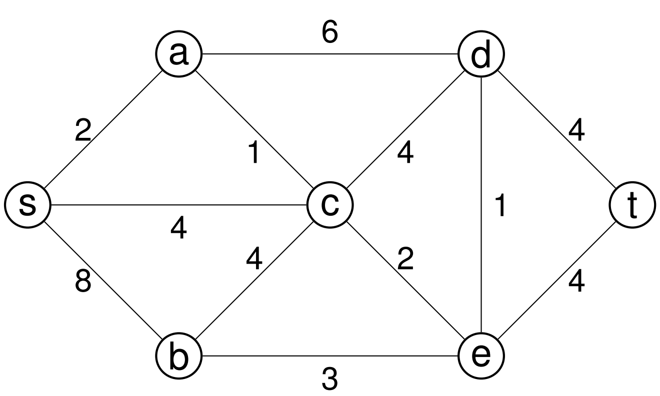
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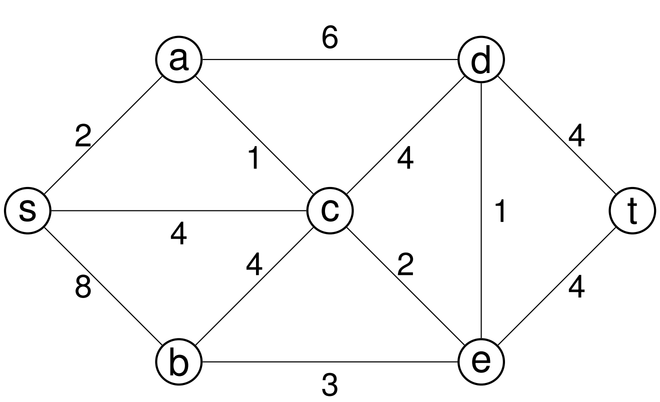
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(b) [4 Points] Given a general graph $G = (V, E)$, two designated vertices $s, t \in V$ and nonnegative edge costs c_e for every edge $e \in E$, state an integer programming formulation for the shortest st-path problem.

(c) [4 Points] Let $G = (V, E)$ be a general graph with nonnegative edge costs c_e for each edge $e \in E$. Suppose P is an st-path in G, and y is a set of feasible st-cut widths such that

$$\sum \{c_e : e \in P\} = \sum \{y_U : \delta(U) \text{ is an } st\text{-cut}\} + 1.$$

Prove that $\text{slack}_y(e) \leq 1$ for every edge $e \in P$.

<u>Problem 3:</u> Duality Theory I [14 Points]

(a) [8 Points] Write the dual of the following linear program, and state all complementary slackness conditions.

(P$_1$)

$$\begin{aligned} \max \quad & (7, -4, 3)(x_1, x_2, x_3)^T \\ \text{s.t.} \quad & \begin{pmatrix} -1 & 2 & 1 \\ 7 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ & x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0. \end{aligned}$$

(b) [6 Points] The above linear program (P$_1$) can be rewritten as

(P$_2$)

$$\begin{aligned} \max \quad & (7, -4, 3)(x_1, x_2, x_3)^T \\ \text{s.t.} \quad & \begin{pmatrix} 1 & -2 & -1 \\ 7 & -2 & 4 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Consider the feasible solution $\bar{x} = (0, 1, 1)^T$ to (P$_2$). Write the cone of tight constraints at $\bar{x}$, and use it to determine whether $\bar{x}$ is optimal.

<u>Problem 4:</u> Duality Theory II [8 Points]

(a) [4 Points] Consider the following pair of primal and dual linear programs, where α and β are real-valued parameters. (Thus, α and β are unknown items of data, and *not* variables.)

$$
(\mathrm{P}_1)\quad \max\ (-3, \alpha, 13)x \quad \text{s.t.}\ \begin{pmatrix} 4 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 3 & 3 \end{pmatrix} x \begin{matrix} \leq \\ \leq \\ = \end{matrix} \begin{pmatrix} \beta \\ 5 \\ 5 \end{pmatrix}, \quad x \geq \mathbb{0}.
$$

$$
(\mathrm{D}_1)\quad \min\ (\beta, 5, 5)y \quad \text{s.t.}\ \begin{pmatrix} 4 & 1 & -1 \\ 2 & 2 & 3 \\ 2 & -1 & 3 \end{pmatrix} y \geq \begin{pmatrix} -3 \\ \alpha \\ 13 \end{pmatrix}, \quad y_1 \geq 0, y_2 \geq 0, y_3 \text{ free.}
$$

Do there exist values for the parameters α and β such that $\bar{x} = (1, 2, 0)^T$ is feasible and optimal for (P_1), and $\bar{y} = (0, 2, 5)^T$ is feasible and optimal for (D_1)? If *yes*, then write down <u>one</u> setting of such values for α and β, and justify your answer; if *no*, then explain why such values do not exist.

(b) [4 Points] In the following let A be an m by n matrix, and let b and c be vectors of appropriate dimensions. Consider the following LP (P_2) and its dual (D_2):

$$
(\mathrm{P}_2)\quad \min\ c^T x \quad \text{s.t.}\ Ax \geq b,\ x \geq \mathbb{0}
$$

$$
(\mathrm{D}_2)\quad \max\ b^T y \quad \text{s.t.}\ A^T y \leq c,\ y \geq \mathbb{0}
$$

Let $\bar{x}$ and $\bar{y}$ be feasible solutions for (P_2) and (D_2), respectively. Suppose that $\bar{x}$ and $\bar{y}$ satisfy the following *relaxed complementary slackness* (RCS) conditions:

[i] For all $j \in \{1, \dots, n\}$ we have $\bar{x}_j = 0$ <u>or</u> $\sum_{i=1}^m A_{ij}\bar{y}_i = c_j$ (or both);

[ii] For all $i \in \{1, \dots, m\}$ we have $\bar{y}_i = 0$ <u>or</u> $\sum_{j=1}^n A_{ij}\bar{x}_j \leq 3b_i$ (or both).

Prove that $c^T\bar{x} \leq 3c^T x^*$, where x^* is an optimal solution for (P_2).
(<u>Hint</u>: Recall the proof of weak duality!)

<u>Problem 5:</u> Integer Programming and Cutting Planes [14 Points]

(a) [6 Points] Let (IP) be an integer program on variables $x_1, \ldots, x_7$. Suppose that we apply the Simplex Algorithm to its LP relaxation, and that the final canonical form is as follows:

$$\text{(P)} \qquad \begin{aligned} \max \;& (-1,0,0,0,-3,0,-2)x \\ \text{s.t.} \;& \begin{pmatrix} 1/3 & 1 & 0 & 0 & 7/2 & 0 & -2/9 \\ -3/5 & 0 & 5/8 & 1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 & -8 & 1 & 19/16 \end{pmatrix} x = \begin{pmatrix} 3 \\ 6/7 \\ 3/2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0. \end{aligned}$$

Determine the basic solution $\bar{x}$ corresponding to the above canonical form. Indicate which constraints lead to cutting planes and for each such constraint, generate a corresponding cutting plane.

(b) [4 Points] Consider (IP), its LP relaxation (P) and basic solution $\bar{x}$ from part (a). Prove that the constraint

$$x_1 + x_3 + x_5 + x_7 \geq 1$$

is a cutting plane for $\bar{x}$.

(c) [4 Points] Does (IP) have an integer feasible solution x^* such that $x_7^* = 0$? Either provide one such solution, or prove that no such solution exists.

Problem 6: Nonlinear Optimization [12 Points]

Consider the following convex nonlinear programming problem (NLP):

$$\begin{array}{llrcl} \text{(NLP)} & \min & & x_3 & \\ & \text{s. t.} & & & \\ & & (x_1-10)^4 + x_2^2 - x_3 - 17 & \le & 0, \\ & & (x_1-7)^2 - x_2 & \le & 0, \\ & & -x_1 & \le & 0, \\ & & -x_2 + 4 & \le & 0, \\ & & -x_3 - 17 & \le & 0. \end{array}$$

Note: you may assume (without proof) that the functions defining the objective function and the constraints are convex and differentiable. Moreover, you may assume that there exists a Slater point.

(a) [3 Points] Let $\bar{x} := (9,4,0)^T$. Determine whether or not $\bar{x}$ is a Slater point. Explain your answer in brief.

(b) [9 Points] Write down the optimality conditions for $\bar{x} = (9,4,0)^T$ for (NLP) as described in the Karush-Kuhn-Tucker (KKT) Theorem. Using these conditions and the theorem, determine whether or not $\bar{x}$ is an optimal solution for (NLP).

<u>Problem 7:</u> True of False [12 Points]

For each of the following statements, answer whether it is **True** or **False**. If true, give a proof; and if false, give one counter-example or a complete explanation.

(a) Let $C_1 \subseteq \mathbb{R}^n$ and $C_2 \subseteq \mathbb{R}^n$ be convex sets. Then $C_1 \cap C_2$ is a convex set.

(b) Let (P) be a linear program that has an optimal solution. Suppose that we obtain a new linear program (P^{new}) from (P) by changing the right-hand side vector b of (P) to b^{new}. (Thus, (P^{new}) is the same as (P), except for the right-hand side vector.) Then (P^{new}) cannot be unbounded.

(c) Consider the linear programs

$$\begin{array}{ll} (P) & \max\left\{c^T x : Ax \leq b\right\}, \\ (P') & \max\left\{c^T x : Ax \leq b, x \geq \mathbb{0}\right\}. \end{array}$$

Suppose that (P) has an optimal solution, while (P') is infeasible. Then the dual of (P') is unbounded.