CO 250: Introduction to Optimization

Module 2: Linear Programs (Finding a Feasible Solution)

The Problem

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}.$$

To run Simplex, we need a feasible basis.

Question

How do we find a feasible basis?

Is there an easier question to answer?

Question

How do we find a feasible solution?

These two questions are equivalent.

Exercise

There is an algorithm that, given a feasible solution, finds a feasible basis.

$\Rightarrow$ We will focus on the second question.

The Key Idea

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbf{0} \right\}$$

Algorithm 1

INPUT: A, b, c, and a feasible solution

OUTPUT: Optimal solution/detect LP unbounded.

OK
Simplex +
exercise

Algorithm 2

INPUT: A and b

OUTPUT: Feasible solution/detect there is none

HOW?

We will show that...

Proposition

We can use Algorithm 1 to get Algorithm 2.

A First Example

Problem: Find a feasible solution/detect none exist for

$$
\begin{array}{ll}
\max & (1,2,-1,3)x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

Remark

It does not depend on the objective function.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ -2 & -9 & 0 & 3 \end{pmatrix} x = \begin{pmatrix} 7 \\ -13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative.

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

min $x_5 + x_6$

s.t.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$x \geq \mathbf{0}$

x_5, x_6 are auxiliary variables

Remark

This auxiliary problem is

- feasible, since $(0, 0, 0, 0, 7, 13)^\top$ is a solution, and
- bounded, as 0 is the lower bound.

➡ Therefore, this auxiliary problem has an optimal solution.

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 5 & 2 & 1 \\ 2 & 9 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad \text{and} \quad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\begin{aligned} \min \quad & x_5 + x_6 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 5 & 2 & 1 & 1 & 0 \\ 2 & 9 & 0 & -3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

x_5, x_6 are auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(2, 1, 0, 0, 0, 0)^\top$ is an optimal solution to the auxiliary problem,

since $x_5 = x_6 = 0$.

$\Rightarrow$ Therefore, $(2, 1, 0, 0)^\top$ is a feasible solution to $(\star)$.

A Second Example

Problem: Find a feasible solution/detect none exist for

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\boxed{\begin{array}{l} \min \quad z = x_4 + x_5 \\ \text{s.t.} \\ \qquad \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \end{array}}$$

x_4, x_5 are auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(0, 0, 1, 0, 3)^\top$ is the optimal solution to the auxiliary problem.

However, $(0, 0, 1)^\top$ is NOT a solution to $(\star)$.

$$\begin{pmatrix} 5 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\begin{array}{ll} \min & z = x_4 + x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 5 & 1 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

auxiliary problem

optimal solution $(0, 0, 1, 0, 3)^\top$

optimal value $= 0 + 3 = 3$.

Claim

$(\star)$ does not have a solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution x'_1, x'_2, x'_3.

Then, $(x'_1, x'_2, x'_3, 0, 0)^\top$ is a feasible solution to the auxiliary problem,

but that solution has of value 0. This is a contradiction.

Formalize

<u>Problem:</u> Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \tag{$\star$}$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct an auxiliary problem (A $m \times n$ matrix).

$$\boxed{\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\begin{array}{c|c} A & I \end{array} \right) x = b \\ & x \geq \mathbf{0} \end{array}}$$

$x_{n+1}, \ldots, x_{n+m}$ are auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ is the optimal solution to the auxiliary problem.

Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to $(\star)$.

<u>Problem:</u> Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \tag{$\star$}$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct an auxiliary problem (A $m \times n$ matrix).

$$\boxed{\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \mid & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}}$$

$x_{n+1}, \ldots, x_{n+m}$ are auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ is the optimal solution to the auxiliary problem.

Proposition

If $z = 0$, then $(x_1, \ldots, x_n)^\top$ is a solution to ($\star$).

Proof

When $z = 0$, we must have $x_{n+1} = \ldots = x_{n+m} = 0$.

<u>Problem:</u> Find a feasible solution/detect none exist for

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that b is non-negative.

Step 2. Construct an auxiliary problem (A $m \times n$ matrix).

$$\boxed{\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \left(\begin{array}{c|c} A & I \end{array}\right) x = b \\ & x \geq \mathbf{0} \end{array}}$$

$x_{n+1}, \ldots, x_{n+m}$ are auxiliary variables

Step 3. Solve the auxiliary problem using Algorithm 1.

$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ is the optimal solution to the auxiliary problem.

Proposition

When $z > 0$, then $(\star)$ has no solution.

$$Ax = b \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

$$\boxed{\begin{array}{ll} \min & z = x_{n+1} + \ldots + x_{n+m} \\ \text{s.t.} & \\ & \begin{pmatrix} A & \mid & I \end{pmatrix} x = b \\ & x \geq \mathbf{0} \end{array}}$$

auxiliary problem

optimal solution
$(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$

Proposition

When $z > 0$, then $(\star)$ has no solution.

Proof

Suppose, for a contradiction, $(\star)$ has a solution $x'_1, \ldots, x'_n$.

Then $(x'_1, \ldots, x'_n, 0, \ldots, 0)^\top$ is feasible solution to auxiliary problem,

but that solution has of value 0. This is a contradiction.

The 2-Phase Method

To solve

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\},$$

we proceed in two phases:

Phase 1. Find a feasible solution/detect none exist.

Phase 2. Given the feasible solution, find an optimal solution/detect LP unbounded.

Problem: Solve the following LP,

$$\boxed{\begin{array}{ll} \max & (1,1,1)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\begin{array}{ll} \min & z = x_4 + x_5 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

NOT in SEF

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\begin{aligned} &\max \quad z = -x_4 - x_5 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

To rewrite $B = \{4, 5\}$ in canonical form, you can

- use the formulae, OR
- notice $A_B = I$ and rewrite the objective function as follows...

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\begin{aligned} &\max \quad z = -x_4 - x_5 \\ &\text{s.t.} \\ &\qquad \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ &\qquad x \geq \mathbf{0} \end{aligned}$$

In SEF

feasible basis $B = \{4, 5\}$

NOT in canonical form

$$\begin{array}{rcl} z &=& (0 \quad 0 \quad 0 \quad -1 \quad -1)x \\ 0 &=& (1 \quad 2 \quad -1 \quad 1 \quad 0)x - 4 \\ 0 &=& (1 \quad -1 \quad 1 \quad 0 \quad 1)x - 4 \\ \hline z &=& (2 \quad 1 \quad 0 \quad 0 \quad 0)x - 8 \end{array}$$

Phase 1. Find a feasible solution/detect none exist for

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \text{and} \qquad x \geq \mathbf{0} \qquad\qquad (\star)$$

Step 1. Multiply the equations such that the RHS is non-negative. OK

Step 2. Construct an auxiliary problem.

$$\begin{aligned} \max \quad & z = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \end{pmatrix} - 8 \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{aligned}$$

In SEF

feasible basis $B = \{4, 5\}$

canonical form for B

Step 3. Solve the auxiliary problem using Simplex, starting from B.

$B = \{1, 4\}$ is an optimal basis with the basic solution $(4, 0, 0, 0, 0)^\top$.

$z = 0$ implies that $(4, 0, 0)^\top$ is a feasible solution for $(\star)$.

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\boxed{\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}}$$

Remark

$(4,0,0)^\top$ is a basic solution.

Exercise

Show that this will <u>always</u> be the case!

Phase 1. $(4,0,0)^\top$ is a feasible solution for

$$\begin{array}{ll} \max & (2,-1,2)x \\ \text{s.t.} & \\ & \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}}_{A} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

Question

For what basis, B, is $x = (4,0,0)^\top$ a basic solution?

$x_1 \neq 0 \Longrightarrow 1 \in B$

Cardinality of maximal set of independent columns of $A =$
Cardinality of maximal set of independent rows of $A = 2$

Thus, for some $i \in \{2,3\}$, columns 1 and i of A are independent.

In this case, we can pick $i = 2$. In particular, $B = \{1,2\}$ is a basis.

Phase 2. Find an optimal solution/detect LP unbounded.

$$\begin{array}{ll} \max & (1,1,1)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ & x \geq \mathbf{0} \end{array}$$

$B = \{1,2\}$ is a feasible basis (from Phase 1).

We can now solve the problem using Simplex, starting from B. ➡

$x = (0,8,12)^\top$ is an optimal solution.

Consequences

Theorem

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

Exactly one of the following holds for the LP:

(A) it is infeasible,

(B) it is unbounded, or

(C) it has an optimal solution that is basic.

Proof

Run 2-Phase method with Simplex using Bland's rule.
(Recall that Bland's rule ensures that Simplex terminates.)

Theorem

$$\max\left\{c^\top x : Ax = b, x \geq \mathbf{0}\right\}$$

Exactly one of the following holds for the LP:

(A) it is infeasible,

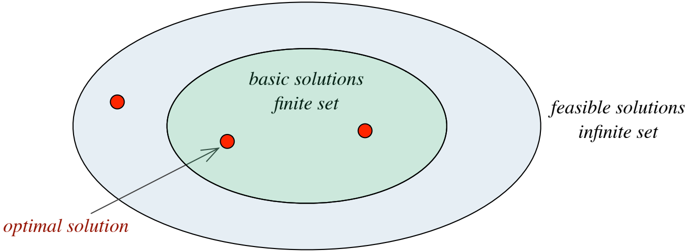
(B) it is unbounded, or

(C) it has an optimal solution that is basic.

Remark

A finite number of basis implies a finite number of basic solutions.

If a LP has at least two feasible solutions and one optimal solution,



Fundamental Theorem of Linear Programming

Consider an arbitrary LP. Exactly one of the following holds:

(A) it is infeasible,

(B) it is unbounded, or

(C) it has an optimal solution.

Proof

Convert the LP into an equivalent LP in SEF.

Apply the previous theorem.

Recap

- Using 2-Phase + the Simplex algorithm, we can solve arbitrary LPs.
- We proved the fundamental theorem of linear programming.

We have given a bare bone version of the Simplex procedure.

Careful implementation is key to having a practical algorithm.

If Simplex were a bike,

Sergio Schnitzler/Hemera/Thinkstock

State of the art implementation

gamegfx/iStock/Thinkstock

Our implementation