CO 250: Introduction to Optimization

Module 3: Duality Through Examples (Correctness)

Recap: Shortest Path Algorithm

Previous lecture: we showed an algorithm for the shortest path problem that computes

- An s,t-path P whose characteristic vector, x^P, is feasible for the shortest path LP, and
- a feasible solution, y, for the dual of the shortest path LP.

Important: $c^T x = \mathbb{1}^T y \to P$ is a shortest path!

We will start this lecture with another example!

Shortest path LP:

$$
\begin{aligned}
\min \quad & \sum (c_e x_e \,:\, e \in E) \\
\text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(S)) \geq 1 \\
& \qquad\qquad (\delta(S) \ s,t\text{-cut}) \\
& x \geq \mathbb{0}
\end{aligned}
$$

Shortest path dual:

$$
\begin{aligned}
\max \quad & \sum (y_S \,:\, \delta(S) \ s,t\text{-cut}) \\
\text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\
& \qquad\qquad (e \in E) \\
& y \geq \mathbb{0}
\end{aligned}
$$

Recall the algorithm we developed previously:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

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y_W := 0 for all st-cuts δ(W). Set U := {s}
while t ∉ U do
    Let ab be an edge in δ(U) of smallest slack for y where a ∈ U, b ∉ U
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    U := U ∪ {b}
    change edge ab into an arc →ab
end while
return A directed st-path P.
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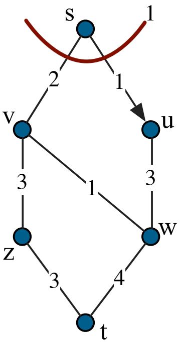

$\longrightarrow$ Run this on the example instance on the right.

Initially: $y = \mathbb{0}$ and $U = \{s\}$

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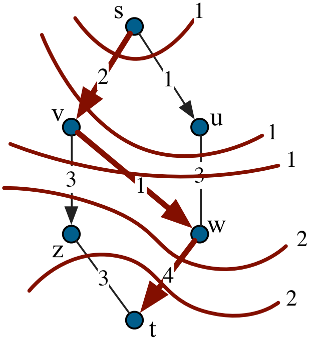
Step 1 su edge with smallest slack in $\delta(U)$
$\longrightarrow$ increase y_U by 1

Step 2 Now: $U = \{s, u\}$
Slack-minimal edge is sv
$\longrightarrow$ increase y_U by 1

Step 3 $U = \{s, v, u\}$
Slack minimizer is vw
$\longrightarrow$ increase y_U by 1

Step 4 $U = \{s, v, u, w\}$
Slack minimizer is vz
$\longrightarrow$ increase y_U by 2

Step 5 $U = \{s, v, u, w, z\}$
Slack minimizer is wt
$\longrightarrow$ increase y_U by 2



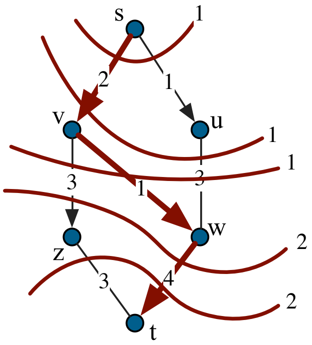
Now: We have a directed s, t-path P of length 7, and a dual feasible solution of the same value!

$\longrightarrow$ P is a shortest path!

Question

Will the algorithm always terminate? Will it always find an s, t-path P whose length is equal to the value of a feasible dual solution?

This lecture: We will show the answers to the above are yes & yes!



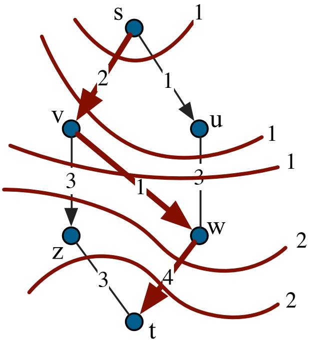
Revisited: Shortest Path Optimality Conditions

Recall: the slack of an edge $uv \in E$ for a feasible dual solution y is

$$c_{uv} - \sum (y_U \,:\, e \in \delta(U))$$

We call an edge $uv \in E$ an equality edge if its slack is 0.

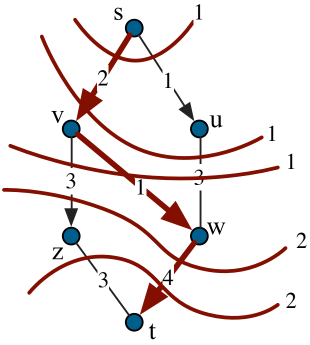
Example: edge vz is an equality edge, and zt is not!



Revisited: Shortest Path Optimality Conditions

We will also call a cut $\delta(U)$ active for a dual solution y if $y_U > 0$.

Example: $\delta(\{s, v, u\})$ is active, while $\delta(\{s, v\})$ is not!



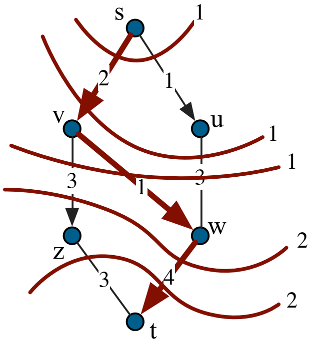
Proposition

Let y be a feasible dual solution, and P and s,t-path. P is a shortest path if

(i) all edges on P are equality edges, and

(ii) every active cut $\delta(U)$ has exactly one edge of P.

Note: Both conditions are satisfied in the example on the right.



Revisited: Shortest Path Optimality Conditions

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Proof: Let's suppose that P and y satisfy (i) and (ii) of the proposition. Then,

$$\sum_{e \in P} c_e = \sum_{e \in P} (\sum (y_U \ : \ e \in \delta(U))$$

because every edge on P is an equality edge by (i). The right-hand side equals

$$\sum (y_U \cdot |P \cap \delta(U)| \ : \ \delta(U))$$

But, by (ii), $y_U > 0$ only if $|P \cap \delta(U)| = 1$. Hence:

$$\sum_{e \in P} c_e = \sum_U y_U$$

□

Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G=(V,E)$, costs $c_e \geq 0$ for all $e \in E$, $s,t \in V$ where $s \neq t$.
Output: A shortest st-path P

```
y_W := 0 for all st-cuts δ(W). Set U := {s}
while t ∉ U do
    Let ab be an edge in δ(U) of smallest slack for y where a ∈ U, b ∉ U
    y_U := slack_y(ab)
    U := U ∪ {b}
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end while
return A directed st-path P.
```
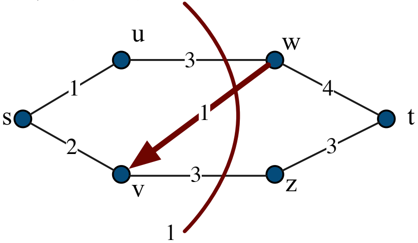

Note: The algorithm terminates since one vertex is added to U in every step and V is finite.



It suffices to show:

Proposition

The Shortest Path Algorithm maintains throughout its execution if:

(I1) y is a feasible dual,

(I2) arcs are equality arcs (i.e., have 0 slack),

(I3) no active cut $\delta(U)$ has an entering arc: an arc wu with $w \notin U$, and $u \in U$,

(I4) for every $u \in U$ there is a directed s,u-path, and

(I5) arcs have both ends in U.

Correctness of the Shortest Path Algorithm

Suppose the invariants hold when the algorithm terminates. Then:

- $t \in U$ and (I4) implies that there is a directed s,t-path P,
- y is feasible by (I1), and
- arcs on P are equality arcs by (I2)

To show this: $\delta(U)$ active $\longrightarrow$ P has exactly one edge in $\delta(U)$.

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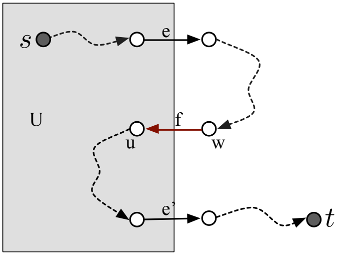
Correctness of the Shortest Path Algorithm

For a contradiction suppose $\delta(U)$ active $\longrightarrow$ P has more than one edge in $\delta(U)$

Let e and e' be the first two edges on P that leave $\delta(U)$.

Then, there must also be an arc f on P that enters U, but this contradicts (I3)!



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Correctness of the Shortest Path Algorithm

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

1: $y_W := 0$ for all st-cuts $\delta(W)$. Set $U := \{s\}$
2: **while** $t \notin U$ **do**
3:    Let ab be an edge in $\delta(U)$ of smallest slack for y where $a \in U$, $b \notin U$
4:    $y_U := \text{slack}_y(ab)$
5:    $U := U \cup \{b\}$
6:    change edge ab into an arc $\overrightarrow{ab}$
7: **end while**
8: **return** A directed st-path P.

Let's now prove the proposition!

Trivial: (I1) – (I5) hold after Step 1.
Suppose (I1) – (I5) hold before Step 3.
We will show that they also hold after Step 6.

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(I5) arcs have both ends in U.

(I1) y is Dual Feasible

Algorithm 3.2 Shortest path.

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Shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \ : \ \delta(S) \ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \ : \ e \in \delta(S)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

Note: In Step 3-6, only y_U for the current U changes.

y_U arises only on the left-hand sides of constraints for edges in $\delta(U)$.

The smallest slack of any of these constraints is precisely the increase in y_U.

$\longrightarrow$ y remains feasible!

Also: The constraint of the newly created arc holds with equality after the increase
$\longrightarrow$ (I2) continues to hold and constraints for arcs have slack 0.

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4: $y_U := \text{slack}_y(ab)$
5: $U := U \cup \{b\}$
6: change edge ab into an arc $\overrightarrow{ab}$
7: **end while**
8: **return** A directed st-path P.

- the only new active cut created is $\delta(U)$
- (I5) $\longrightarrow$ all old arcs have both ends in U
- one new arc has tail in U, and head outside U

$\longrightarrow$ (I3) holds after Step 6

Proposition

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(I4) for every $u \in U$ there is a directed s, u-path, and

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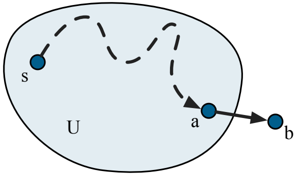
Correctness of the Shortest Path Algorithm

Note: Algorithms adds arc ab in current step, and (I4) implies that there is a directed s, a-path P.



(I5) $\longrightarrow$ arcs different from ab have both ends in U

$\longrightarrow$ since b is outside U, it cannot be on P, and thus, P together with ab is a directed s, b-path

$\longrightarrow$ (I4) holds at the end of loop

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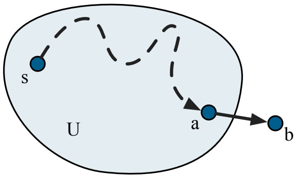
Correctness of the Shortest Path Algorithm



Finally, the only new arc added is ab. As b is added to U, (I5) continues to hold.

We are now done!

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Recap

- We saw that the shortest path algorithm
  (i) always produces an s,t-path P, and
  (ii) a feasible dual solution y.
- Moreover, the length of P equals the objective value of y, and hence, P must be a shortest s,t-path.
- Implicitly, we therefore showed that the shortest path LP always has an optimal integer solution!