CO 250 Online- Spring 2018

Assignment 6

Due date : Friday, June 22nd, 2018, by 4:00pm (sharp)

Submission Guidelines:

- Please submit your solutions to Crowdmark. Late assignments will not be accepted, and will receive a mark of zero. It is the responsibility of the students to make sure that the file they submit is clearly readable. Illegible submissions will receive a mark of zero.
- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- In some questions you are asked to *formulate* the problem. You are *not* asked to actually solve the formulation, e.g., compute optimal solutions. Your formulations should be easy to modify if we change the data and constants defining the problems. Clearly define all your variables (including their units) and any other new notation you use in all your answers. Your solutions must also contain a brief justification of all the constraints (explain the relation between each of the constraints and the requirements stated in the problem) and the objective function.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. Copying or paraphrasing a solution from some fellow student or old solutions from previous offerings qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. Students found to be cheating will be given a mark of 0 on the assignment. In addition, all academic offenses will be reported to the Math Academic Integrity Officer (which may lead to further penalties) and recorded in the student's file.

Re-marking policies: If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that, if you see any marking error, then write a letter detailing clearly the marking errors, and submit this to one of the TAs within one week from the date the graded assignment is returned. If you still have concerns after the final decision of the TA, then please contact your instructor communicating all the correspondence with the TA and the original petition.

[1] It is an academic offense to post this assignment or solutions to any web page.

Question 1. Canonical forms, simplex method, and certificates **(15 marks)**

In all parts below, (P) refers to an LP of the following form:

$$\max \{c^\top x + \bar{z} : \quad Ax = b, \quad x \geq \mathbb{0}\}$$

where A is an $m \times n$ matrix with full row rank.

(a) Suppose that (P) is in canonical form for a feasible basis B and $c_j < 0$ for all $j \in N$. Let $\bar{x}$ be the basic solution corresponding to basis B. Show that $\bar{x}$ is the *unique* optimal solution to (P). (4 marks)

(b) Let B be an optimal basis for (P) and $\bar{x}$ be the corresponding basic feasible solution. That is, when the simplex method is run on (P), it encounters the canonical form of (P) for the basis B, and terminates at this point declaring that the corresponding basic solution $\bar{x}$ is an optimal solution.

Prove that $y = A_B^{-\top} c_B$ (the vector used to convert (P) to the canonical form for B) is a certificate of optimality for $\bar{x}$. ($A_B^{-\top}$ denotes $(A_B^{-1})^\top$, which is equal to $(A_B^\top)^{-1}$.) (5 marks)

(c) Let $\bar{x}$ be a feasible solution to (P). Let B be a basis of A. Let the following LP be the canonical form of (P) for basis B.

$$\max \quad z(x) := \bar{c}^\top x + \bar{z} \qquad \text{s.t.} \qquad \bar{A}x = \bar{b}, \quad x \geq \mathbb{0}. \qquad (\mathrm{P_{CF}})$$

Let $k \in N$ be such that $\bar{c}_k > 0$, and $\bar{A}_k \leq \mathbb{0}$. Define the vector d by setting $d_k = 1$, $d_B = -\bar{A}_k$, and $d_j = 0$ for all $j \in N \setminus \{k\}$. Show that d (together with $\bar{x}$) is a certificate of unboundedness for the *original LP* (P). (6 marks)

Question 2. Two-phase Simplex method I **(15 marks)**

For each of the following LPs, run the two-phase Simplex method with Bland's tie-breaking rule (i.e., smallest subscript rule) to either prove that the LP is infeasible or that it is unbounded, or find an optimal solution. Clearly separate your work into Phase 1 and Phase 2. Specify clearly the auxiliary problem created in Phase 1. Show all iterations in both phases. Clearly specify the entering and leaving variables in each iteration.

(a)

$$\begin{array}{ll} \max & (2,0,1,-1)x \\ \text{subject to} & \\ & \begin{pmatrix} 2 & 1 & 1 & 3 \\ -1 & 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

(8 marks)

(b)

$$\begin{array}{ll} \max & (4,2,0,1)x \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 1 & -1 & 3 \\ 1 & -1 & -2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

This LP is infeasible. Show that the certificate of optimality obtained for the auxiliary problem from Phase 1 of the 2-phase simplex method gives a certificate of infeasibility for the above LP. (7 marks)

Question 3. Two-phase simplex method II **(15 marks)**

Consider an LP with constraints

$$Ax = b, \quad x \geq \mathbb{0}.$$

where A is an $m \times n$ matrix (with full row rank). Running the two-phase simplex method on such an LP creates the following auxilliary LP (Q) in Phase 1:

$$\begin{array}{ll} \max & (\underbrace{0, \ldots, 0}_{n} \quad \underbrace{-1, \ldots, -1}_{m})x \\ \text{subject to} & \\ & [\ A \ \mid \ I\ ]x = b \\ & x \geq \mathbb{0} \end{array}$$

, where $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})^\top$ with $x_{n+1}, \ldots, x_{n+m}$ being auxillliary variables.

(a) Generalize Q2(b), prove that if the optimal value of (Q) is negative, then every certificate of optimality y for (Q) is a certificate of infeasibility for the system $Ax = b, \quad x \geq \mathbb{0}$. (9 marks)

(b) Suppose that when Phase 1 of the two-phase simplex method terminates, we determine that the optimal value of (Q) is 0, and obtain an optimal solution $(\bar{x}_1, \ldots, \bar{x}_n, \ \underbrace{0, \ldots, 0}_{m})^\top$.

Prove that $\bar{x} := (\bar{x}_1, \ldots, \bar{x}_n)^\top$ is a basic solution to the system $Ax = b, \quad x \geq \mathbb{0}$. (6 marks)

(**Hint:** You may find Q1(a) and Q1(b) of Assignment 5 to be useful.)

Question 4. Convexity and extreme points **(15 marks)**

For $x \in \mathbb{R}^n$, define $\|x\| = \sqrt{x^\top x}$, which corresponds to the length of the vector x. Define $B := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, i.e., B is the set of points at distance at most 1 from the origin, which is often called the n-dimensional unit ball. Intuitively, we can see that B is a convex set.

In both parts below, you may use the following inequality, which is called the *Cauchy-Schwarz* inequality:

$$\text{for any vectors } x, y \in \mathbb{R}^n, \quad x^\top y \leq \|x\| \, \|y\| \quad \text{with equality holding if and only if } x = y.$$

(a) Give an algebraic proof that B is a convex set. (6 marks)

(b) Show that $x \in B$ is an extreme point of B if and only if $\|x\| = 1$. (9 marks)