CO 250: Introduction to Optimization

Module 2: Linear Programs (Simplex – A First Attempt)

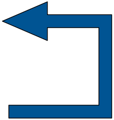
A Naive Strategy for Solving an LP

Step 1. Find a feasible solution, x.

Step 2. If x is optimal, STOP.

Step 3. If LP is unbounded, STOP.

Step 4. Find a "better" feasible solution.



Many details missing!

Questions

- How do we find a feasible solution?
- How do we find a "better" solution?
- Will this ever terminate?

The SIMPLEX algorithm works along these lines.

In this lecture: A first attempt at this algorithm.

First Example

Consider

$$
\begin{array}{ll}
\max & (4,3,0,0)x + 7 \\
\text{s.t.} & \\
& \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
- The objective function is $z = 4x_1 + 3x_2 + 7$.

Question

The feasible solution has objective value: $4 \times 0 + 3 \times 0 + 7 = 7$.

- Can we find a feasible solution with value larger than 7?

YES!

First Example

Consider

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Remarks

- We have a feasible solution: $x_1 = 0, x_2 = 0, x_3 = 2$, and $x_4 = 1$.
- The objective function is $z = 4x_1 + 3x_2 + 7$.

Idea

Increase x_1 as much as possible, and keep x_2 unchanged, i.e.,

$$\begin{aligned} x_1 &= t \qquad \text{for some } t \geq 0 \text{ as large as possible} \\ x_2 &= 0 \end{aligned}$$

$$\boxed{\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}}$$

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= ? \\ x_4 &= ? \end{aligned}$$

Choose $t \geq 0$ as large as possible.

It needs to satisfy

1. the equality constraints, and
2. the non-negativity constraints.

Satisfying the Equality Constraints

$$\begin{array}{ll} \max & (4,3,0,0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = t$$
$$x_2 = 0$$
$$x_3 = ?$$
$$x_4 = ?$$

$$\begin{aligned} \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\ &= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{aligned}$$

$$\boxed{\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

Remark

Equality constraints hold for any choice of t.

Satisfying the Non-Negativity Constraints

$$\begin{array}{ll} \max & (4,3,0,0)x+7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$x_1 = t$$
$$x_2 = 0$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Choose $t \geq 0$ as large as possible.

$x_1 = t \geq 0$ ✓

$x_2 = 0$ ✓

$x_3 = 2 - 3t \geq 0 \Rightarrow t \leq \frac{2}{3}$

$x_4 = 1 - t \geq 0 \Rightarrow t \leq 1$

Thus, the largest possible t is $\min\left\{1, \frac{2}{3}\right\} = \frac{2}{3}$. The new solution is

$$x = (t,\ 0,\ 2-3t,\ 1-t)^\top = \left(\frac{2}{3}, 0, 0, \frac{1}{3}\right)^\top$$

Repeating the Argument?

$$\max \quad (4,3,0,0)x + 7$$

s.t.

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = \frac{2}{3}$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = \frac{1}{3}$$

Question

Is the new solution optimal? NO!

Question

Can we use the same trick to get a better solution? NO!

What made it work the first time around?

Remark

The LP needs to be in "canonical" form.

$$\begin{array}{ll} \max & (4 \quad 3 \quad 0 \quad 0)x + 7 \\ \text{s.t.} & \\ & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$x_1 = 0$
$x_2 = 0$
$x_3 = 2$
$x_4 = 1$

Revised strategy:

Step 1. Find a feasible solution, x.

Step 2. Rewrite LP so that it is in "canonical" form.

Step 3. If x is optimal, STOP.

Step 4. If LP is unbounded, STOP.

Step 5. Find a "better" feasible solution.

From Here to a Complete Algorithm

(1) Define what we mean by "canonical" form.

(2) Prove that we can always rewrite LPs in canonical form.

algorithm known as the SIMPLEX.

First on "To do list":

- Define basis and basic solutions.
- Define canonical forms.