CO 250: Introduction to Optimization

Module 1: Formulations (LP Models)

Constrained Optimization

- In this course, we consider optimization problems of the following form:
$$\min\{f(x) \,:\, g_i(x) \leq b_i,\, (1 \leq i \leq m), x \in \mathbb{R}^n\},$$
where
  - $n, m \in \mathbb{N}$,
  - $b_1, \ldots, b_m \in \mathbb{R}$, and
  - f, g_1, ..., g_m are functions with from $\mathbb{R}^n$ to $\mathbb{R}$.
- Problems like the above are very hard to solve in general
$\implies$ we focus on special cases.
- This class: All functions are affine.

Modeling: Linear Programs

Affine Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = a^T x + \beta$ for $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}$.
It is linear if, in addition, $\beta = 0$.

Example

(i) $f(x) = 2x_1 + 3x_2 - x_3 + 7$ (affine, but not linear)

(ii) $f(x) = -3x_1 + 5x_3$ (linear)

(iii) $f(x) = 5x - 3\cos(x) + \sqrt{x}$ (not affine and not linear)

Definition

The optimization problem

$$\min\{f(x) \ : g_i(x) \leq b_i, \ \ \forall \ 1 \leq i \leq m, \ x \in \mathbb{R}^n\} \tag{P}$$

is called a linear program if f is affine and $g_1, \ldots, g_m$ is finite number of linear functions.

Comments:

- Instead of set notation, we often write LPs more verbosely.
- Often give non-negativity constraints separately
- May use $\max$ instead of $\min$
- Sometimes replace subject to by s.t.

$$\begin{array}{rl} \max & -2x_1 + 7x_3 \\ \text{subject to} & x_1 + 7x_2 \leq 3 \\ & 3x_2 + 4x_3 \leq 2 \\ & x_1, x_3 \geq 0 \end{array}$$

- We often write $x \geq \mathbb{0}$ as a short form for all variables are non-negative.

$$
\begin{aligned}
\min \quad & -x_1 - 2x_2 - x_3 \\
\text{s.t.} \quad & 2x_1 + x_3 \geq 3 \\
& x_1 + 2x_2 = 2 \\
& x \geq \mathbb{0}
\end{aligned}
$$

- The second mathematical program is not an LP. There are three reasons:
  - Dividing by variables is not allowed.
  - Cannot have strict inequalities.
  - Must have finite number of constraints.

$$
\begin{aligned}
\max \quad & -1/x_1 - x_3 \\
\text{subject to} \quad & 2x_1 + x_3 < 3 \\
& x_1 + \alpha\, x_2 = 2 \quad \forall \alpha \in \mathbb{R}
\end{aligned}
$$

Production Revisited

$$
\begin{array}{llrcr}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u & & & \\
 & 11x_1 + 7x_2 + 6x_3 + 5x_4 & \leq & 700 \\
 & 4x_1 + 6x_2 + 5x_3 + 4x_4 & \leq & 500 \\
 & 8x_1 + 5x_2 + 5x_3 + 6x_4 & \leq & y_s \\
\text{s.t.} & 7x_1 + 8x_2 + 7x_3 + 4x_4 & \leq & y_u \\
 & y_s & \leq & 600 \\
 & y_u & \leq & 650 \\
 & x_1, x_2, x_3, x_4, y_u, y_s & \geq & 0.
\end{array}
$$

The mathematical program for WaterTech example from last class is in fact an LP!

Multiperiod Models

A main feature of the WaterTech production model:
Decisions about production levels have to be made once and for all.

In practice, we often have to make a series of decisions that influence each other.

One such example is multiperiod models:

- Time is split into periods.
- We have to make a decision in each period.
- All decisions influence the final outcome.

KW Oil

KW Oil is local supplier of heating oil.

It needs to decide on how much oil to purchase in order to satisfy demand of its customers.

Years of experience give the following demand forecast for the next 4 months:

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

The projected price of oil fluctuates from month to month:

Month	1	2	3	4
Price ($\$/\ell$)	0.75	0.72	0.92	0.90

Question: When should we purchase how much oil?

KW Oil

Question: When should we purchase how much oil when the goal is to minimize overall total cost?

Additional Complication: The company has a storage tank that

- has a capacity of 4000 litres of oil, and
- currently (beginning of month 1) contains 2000 litres of oil.

Assumption: Oil is delivered at beginning of the month, and consumption occurs in the middle of the month.

KW Oil Model – Variables

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

Month	1	2	3	4
Price ($\$/\ell$)	0.75	0.72	0.92	0.90

(i) Need to decide how many litres of oil to purchase in each month i.
$\longrightarrow$ variable p_i for $i \in [4]$

(ii) How much oil is stored in the tank at beginning of month i?
$\longrightarrow$ variable t_i for $i \in [4]$

Objective Function

Minimize cost of oil procurement.

$$\min \quad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

Constraints: When do

$$t_1, \ldots, t_4, p_1, \ldots, p_4$$

correspond to a feasible purchasing scheme?

Variables:

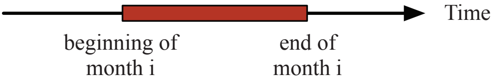
p_i : oil purchase in month i
t_i : tank level in month i

Constraints

Assumptions:

(i) Oil is purchased at beginning of month.

(ii) Oil is consumed afterwards.



Variables:

p_i : oil purchased in month i

t_i : tank level in month i

We need: $p_i + t_i \geq \{\text{demand in month } i\}$

[Balance Equation] $p_i + t_i = \{\text{demand in month } i\} + t_{i+1}$

Constraints

[Balance Equation] $p_i + t_i =$ {demand in month i} $+t_{i+1}$

Tank content in month 1: 2000 litres

Month	1	2	3	4
Demand (ℓ)	5000	8000	9000	6000

- Month 1:
  $p_1 + 2000 = 5000 + t_2$
- Month 2:
  $p_2 + t_2 = 8000 + t_3$
- Month 3:
  $p_3 + t_3 = 9000 + t_4$
- Month 4:
  $p_4 + t_4 \geq 6000$

KW Oil: Entire LP

$$
\begin{array}{ll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
\text{subject to} & \\
& \begin{array}{lcll}
p_1 + t_2 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
\end{array}
$$

Solution: $p = (3000, 12000, 5000, 6000)^T$, and
$t = (2000, 0, 4000, 0)^T$

KW Oil: Add-Ons

Can easily capture additional features. E.g.,...

- Storage comes at a cost: storage cost is \$.15 per litre/month.

  Add $\sum_{i=1}^{4} 0.15t_i$ to objective.

- Minimize the maximum # of litres of oil purchased over all months.
  (i) We will need a new variable M for maximum # of litres purchased.
  (ii) We will have to add constraints

$$
\begin{array}{lll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & \\
\text{subject to} & & \\
& \begin{array}{rcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2,3,4) \\
t_i, p_i & \geq & 0 & (i = 1,2,3,4)
\end{array} &
\end{array}
$$

KW Oil: Add-Ons

(i) Add variable M for maximum $\#$ of litres purchased over all months.

(ii) Add constraints

$$p_i \leq M$$

for all $i \in [4]$.

(iii) Done? No! We need to replace the objective function with

$$\min \quad M$$

$$\begin{array}{lllll}
\min & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 & & & \\
\text{subject to} & & & & \\
& p_1 + t_1 & = & 5000 + t_2 & \\
& p_2 + t_2 & = & 8000 + t_3 & \\
& p_3 + t_3 & = & 9000 + t_4 & \\
& p_4 + t_4 & \geq & 6000 & \\
& t_1 & = & 2000 & \\
& t_i & \leq & 4000 & (i = 2, 3, 4) \\
& t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}$$

Goal: Minimize maximum $\#$ of litres of oil purchased over all months.

Minimizing the Maximum Purchase: LP

$$\min \quad M$$

s.t.

$$
\begin{array}{lcll}
p_1 + t_1 & = & 5000 + t_2 & \\
p_2 + t_2 & = & 8000 + t_3 & \\
p_3 + t_3 & = & 9000 + t_4 & \\
p_4 + t_4 & \geq & 6000 & \\
t_1 & = & 2000 & \\
t_i & \leq & 4000 & (i = 2, 3, 4) \\
p_i & \leq & M & (i = 1, 2, 3, 4) \\
t_i, p_i & \geq & 0 & (i = 1, 2, 3, 4)
\end{array}
$$

KW Oil: Correctness

- Why is this a correct model?
- Suppose that $M, p_1, \ldots, p_4, t_1, \ldots, t_4$ is an optimal solution to the LP.
- Clearly: $M \geq \max_i p_i$
- Since M, p, t is optimal we must have $M = \max_i p_i$. Why?

$$
\begin{array}{llll}
\min & M & & \\
\text{s.t.} & & & \\
& p_1 + t_1 & = & 5000 + t_2 \\
& p_2 + t_2 & = & 8000 + t_3 \\
& p_3 + t_3 & = & 9000 + t_4 \\
& p_4 + t_4 & \geq & 6000 \\
& t_1 & = & 2000 \\
& t_i & \leq & 4000 \qquad (i = 2, 3, 4) \\
& p_i & \leq & M \qquad (i = 1, 2, 3, 4) \\
& t_i, p_i & \geq & 0 \qquad (i = 1, 2, 3, 4)
\end{array}
$$

Otherwise, we could decrease M by a little bit, without violating feasibility. This would contradict optimality because we would get a new feasible solution that has a smaller objective function.