CO 250: Introduction to Optimization

Module 4: Duality Theory (Weak Duality)

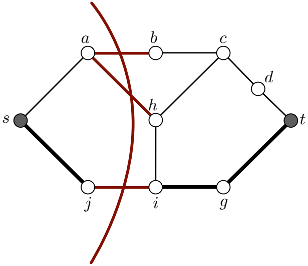
Recap: Shortest Path LP

Solutions to a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, correspond to feasible $0,1$-solutions for the LP

$$\begin{aligned} \min\ & \sum \big(c_e x_e : e \in E\big) \\ \text{s.t.}\ & \sum \big(x_e\ :\ e \in \delta(U)\big) \geq 1 \\ & \qquad\qquad (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbb{0} \end{aligned}$$

This LP is of the form:

$$\min\{c^T x\ :\ Ax \geq b, x \geq \mathbb{0}\}$$



where

- $b = \mathbb{1}$;
- A has a row for every s,t-cut $\delta(U)$, and a column for every edge e; and
- $A_{Ue} = 1$ if $e \in \delta(U)$ and $A_{Ue} = 0$ otherwise.

Recap: Shortest Path Dual

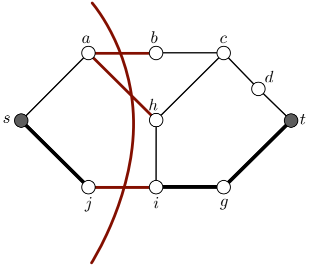
$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \quad \text{(P)}$$

The dual of (P) is given by

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \quad \text{(D)}$$

If (P) is a shortest path LP, then we can rewrite (D) as

$$\begin{aligned} \max \quad & \sum (y_U \,:\, s \in U, t \notin U) \\ \text{s.t.} \quad & \sum (y_U \,:\, e \in \delta(U)) \leq c_e \\ & \qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$



Theorem

If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $b^T \bar{y} \leq c^T \bar{x}$.

Equivalent: y feasible widths and P an s,t-path $\longrightarrow$ $\mathbb{1}^T y \leq c(P)$

This Lecture

Question: Can we find lower-bounds on the optimal value of a general LP?

In the LP on the right,

$$Ax \; ? \; b$$

stands for a system of inequalities whose sign is one of $\leq, =$ or $\geq$, and

$$x \; ? \; \mathbb{0}$$

indicates that variables are either non-negative, non-positive, or free.

$$\begin{aligned} \max \; & c^T x \\ \text{s.t. } & Ax \; ? \; b \\ & x \; ? \; \mathbb{0} \end{aligned}$$

Recall: in the primal-dual pair

$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \qquad \text{(D)}$$

- each non-negative variable, x_e, in (P) corresponds to a '$\leq$'-constraint in (D), and
- each '$\geq$'-constraint in (P) corresponds to a non-negative variable y_U in (D).

Weak Duality in General

Consider the primal LP

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \ ?\ b \\ & x \ ?\ \mathbb{0} \end{aligned}$$

Its dual LP is given by

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \ ?\ c \\ & y \ ?\ \mathbb{0} \end{aligned}$$

Question: What are the question marks?

A: As before:

$$\begin{aligned} \text{primal variables} &\equiv \text{dual constraints} \\ \text{primal constraints} &\equiv \text{dual variables} \end{aligned}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

(P_{max})			(P_{min})	
max $c^\top x$ subject to $Ax \; ? \; b$ $x \; ? \; \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \; ? \; c$ $y \; ? \; \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max \; (1,0,2)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x \begin{matrix} \leq \\ = \end{matrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Its dual LP:

$$\min \; (3,4)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \; ? \; \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y \; ? \; \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \mathbin{?} b$	≥ 0 variable	$\geq$ constraint	$A^\top y \mathbin{?} c$
$x \mathbin{?} \mathbb{0}$	free variable	$=$ constraint	$y \mathbin{?} \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 1:

$$\max\ (1,0,2)x \qquad \text{(P)}$$

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$$x_1, x_2 \geq 0, x_3 \text{ free}$$

Its dual LP:

$$\min\ (3,4)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} y \begin{matrix} \geq \\ \geq \\ = \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
max $c^\top x$	$\leq$ constraint	≥ 0 variable	min $b^\top y$
subject to	$=$ constraint	free variable	subject to
$Ax \; ? \; b$	$\geq$ constraint	≤ 0 variable	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	≥ 0 variable	$\geq$ constraint	$y \; ? \; \mathbb{0}$
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Example 2:

$$\begin{aligned} \min\ & d^T y \\ \text{s.t. }\ & W^T y \geq e \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} \max\ & e^T x \\ \text{s.t. }\ & W x \; ? \; d \\ & x \; ? \; \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

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	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 2:

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$$\begin{aligned} \max\ & e^T x \\ \text{s.t. } & W x \leq d \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Primal-Dual Pairs

Example 2:

$$\min\ d^T y \qquad \text{(P)}$$
$$\text{s.t. } W^T y \geq e$$
$$y \geq \mathbb{0}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\max\ e^T x \qquad \text{(D)}$$
$$\text{s.t. } Wx \leq d$$
$$x \geq \mathbb{0}$$

Primal-Dual Pairs

Example 2:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

Substitute:

- $d \longrightarrow c$
- $e \longrightarrow b$
- $y \longrightarrow x$
- $W^T \longrightarrow A$
- $x \longrightarrow y$

To compute dual LP, check right-hand side of table:

$$\begin{aligned} \max\ & b^T x \\ \text{s.t. } & A^T y \leq c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

This is consistent with the earlier discussion we had!

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \ ?\ b$	≥ 0 variable	$\geq$ constraint	$A^\top y \ ?\ c$
$x \ ?\ \mathbb{0}$	free variable	$=$ constraint	$y \ ?\ \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Example 3:

$$\max \ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min \ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \ ?\ \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y \ ?\ \mathbb{0}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(P_{\max})$		$(P_{\min})$	
max $c^\top x$ subject to $Ax \; ? \; b$, $x \; ? \; \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \; ? \; c$, $y \; ? \; \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

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$$x_1 \geq 0, x_2 \text{ free}, x_3 \geq 0$$

Its dual LP:

$$\min \ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0, y_3 \text{ free}$$

Primal-Dual Pairs

The following table shows how constraints and variables in primal and dual LPs correspond:

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $c^\top x$	$=$ constraint	free variable	min $b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \; ? \; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \; ? \; c$
$x \; ? \; \mathbb{0}$	free variable	$=$ constraint	$y \; ? \; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Theorem

Let $(\mathrm{P}_{\max})$ and $(\mathrm{P}_{\min})$ represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T \bar{x} \leq b^T \bar{y}$$

If $c^T \bar{x} = b^T \bar{y}$, then $\bar{x}$ is optimal for $(\mathrm{P}_{\max})$, and $\bar{y}$ is optimal for $(\mathrm{P}_{\min})$.

Primal-Dual Pairs

Theorem

Let (P_{max}) and (P_{min}) represent the above. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \le b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

Example 3 (continued):

$$\max\ (12, 26, 20)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & 1 \\ 4 & 6 & 5 \\ 2 & -1 & -3 \end{pmatrix} x \begin{matrix} \ge \\ \le \\ = \end{matrix} \begin{pmatrix} -2 \\ 2 \\ 13 \end{pmatrix}$$

$$x_1 \ge 0, x_2 \text{ free}, x_3 \ge 0$$

Its dual LP:

$$\min\ (-2, 2, 13)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 4 & 2 \\ 2 & 6 & -1 \\ 1 & 5 & -3 \end{pmatrix} y \begin{matrix} \ge \\ = \\ \ge \end{matrix} \begin{pmatrix} 12 \\ 26 \\ 20 \end{pmatrix}$$

$$y_1 \le 0, y_2 \ge 0, y_3 \text{ free}$$

Feasible solutions: $\bar{x} = (5, -3, 0)^T$ and $\bar{y} = (0, 4, -2)^T$.
Since $(12, 26, 20)\bar{x} = (-2, 2, 13)\bar{y} = -18 \longrightarrow$ both are optimal!

Proving the General Weak Duality Theorem

(P$_{\max}$)			(P$_{\min}$)
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \;?\; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \;?\; c$
$x \;?\; \mathbb{0}$	free variable	$=$ constraint	$y \;?\; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

General Primal LP:

$$
\begin{aligned}
&\max\ c^T x \\
&\quad \text{s.t. } \mathrm{row}_i(A)x \leq b_i \ (i \in R_1) \\
&\qquad\quad \mathrm{row}_i(A)x \geq b_i \ (i \in R_2) \\
&\qquad\quad \mathrm{row}_i(A)x = b_i \ (i \in R_3) \\
&\qquad\quad x_j \geq 0 \ (j \in C_1) \\
&\qquad\quad x_j \leq 0 \ (j \in C_2) \\
&\qquad\quad x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
&\min\ b^T y \\
&\quad \text{s.t. } \mathrm{col}_j(A)^T y \geq c_j \ (j \in C_1) \\
&\qquad\quad \mathrm{col}_j(A)^T y \leq c_j \ (j \in C_2) \\
&\qquad\quad \mathrm{col}_j(A)^T y = c_j \ (j \in C_3) \\
&\qquad\quad y_i \geq 0 \ (i \in R_1) \\
&\qquad\quad y_i \leq 0 \ (i \in R_2) \\
&\qquad\quad y_i \text{ free } (i \in R_3)
\end{aligned}
$$

Proving the General Weak Duality Theorem

General Primal LP:

$$
\begin{aligned}
\max\ & c^T x \\
\text{s.t. } & \mathrm{row}_i(A)x \le b_i \ (i \in R_1) \\
& \mathrm{row}_i(A)x \ge b_i \ (i \in R_2) \\
& \mathrm{row}_i(A)x = b_i \ (i \in R_3) \\
& x_j \ge 0 \ (j \in C_1) \\
& x_j \le 0 \ (j \in C_2) \\
& x_j \text{ free } (j \in C_3)
\end{aligned}
$$

Its dual according to the table:

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t. } & \mathrm{col}_j(A)^T y \ge c_j \ (j \in C_1) \\
& \mathrm{col}_j(A)^T y \le c_j \ (j \in C_2) \\
& \mathrm{col}_j(A)^T y = c_j \ (j \in C_3) \\
& y_i \ge 0 \ (i \in R_1) \\
& y_i \le 0 \ (i \in R_2) \\
& y_i \text{ free } (i \in R_3)
\end{aligned}
$$

We can rewrite the above LPs using slack variables!

Proving the General Weak Duality Theorem

General Primal LP:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s_i \geq 0 \ (i \in R_1) \\ & s_i \leq 0 \ (i \in R_2) \\ & s_i = 0 \ (i \in R_3) \\ & x_j \geq 0 \ (j \in C_1) \\ & x_j \leq 0 \ (j \in C_2) \\ & x_j \text{ free } (j \in C_3) \end{aligned}$$

Its dual according to the table:

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y + w = c \qquad (\star) \\ & w_j \leq 0 \ (j \in C_1) \\ & w_j \geq 0 \ (j \in C_2) \\ & w_j = 0 \ (j \in C_3) \\ & y_i \geq 0 \ (i \in R_1) \\ & y_i \leq 0 \ (i \in R_2) \\ & y_i \text{ free } (i \in R_3) \end{aligned}$$

Suppose $\bar{x}$ and $\bar{y}$ are feasible for the original primal and dual LPs
Let $\bar{s} = b - A\bar{x}$ and $\bar{w} = c - A^T\bar{y}$. It follows that

$$\bar{y}^T b = \bar{y}^T(A\bar{x} + \bar{s}) = (\bar{y}^T A)\bar{x} + \bar{y}^T\bar{s} \overset{(\star)}{=} (c - \bar{w})^T\bar{x} + \bar{y}^T\bar{s} = c^T\bar{x} - \bar{w}^T\bar{x} + \bar{y}^T\bar{s}.$$

We can show that $\bar{w}^T\bar{x} \leq 0$ and $\bar{y}^T\bar{s} \geq 0 \longrightarrow \quad \bar{y}^T b \geq c^T\bar{x}$

Consequences of Weak Duality

Theorem

Let (P_{max}) and (P_{min}) represent the above table. If $\bar{x}$ and $\bar{y}$ are feasible for the two LPs, then

$$c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ is optimal for (P_{max}), and $\bar{y}$ is optimal for (P_{min}).

(i) (P_{max}) is unbounded $\longrightarrow$ (P_{min}) infeasible

(ii) (P_{min}) is unbounded $\longrightarrow$ (P_{max}) infeasible

(iii) (P_{max}) and (P_{min}) feasible $\longrightarrow$ both must have optimal solutions

Proof: (i) Suppose, for a contradiction, that $\bar{y}$ is feasible for (P_{min}). By weak duality $\longrightarrow$ $c^T\bar{x} \leq b^T\bar{y}$ for all $\bar{x}$ feasible for (P_{max}), and hence the latter is bounded.

(ii) Similar to (i)

(iii) weak duality $\longrightarrow$ both (P_{max}) and (P_{min}) bounded

Fundamental Theorem of LP $\longrightarrow$ Both LPs must have an optimal solution! $\square$

(P_{max})		(P_{min})	
max $c^\top x$ subject to $Ax \, ? \, b$, $x \, ? \, \mathbb{0}$	$\leq$ constraint	≥ 0 variable	min $b^\top y$ subject to $A^\top y \, ? \, c$, $y \, ? \, \mathbb{0}$
	$=$ constraint	free variable	
	$\geq$ constraint	≤ 0 variable	
	≥ 0 variable	$\geq$ constraint	
	free variable	$=$ constraint	
	≤ 0 variable	$\leq$ constraint	

Recap

- We can use the above table to compute duals of general LPs
- Weak duality theorem: if $\bar{x}$ and $\bar{y}$ are feasible for (P_{max}) and (P_{min}), then:

$$c^T \bar{x} \leq b^T \bar{y}$$

Both are optimal if equality holds!