CO 250: Introduction to Optimization

Module 1: Formulations (Nonlinear Models)

So Far...

- Linear programs
- Integer linear programs

Both have linear/affine constraints.

Now we begin our study on nonlinear generalization!

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq \mathbb{0} \\ & x \text{ integer} \end{aligned}$$

Nonlinear Programs

A nonlinear program (NLP) is of the form

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_1(x) \le 0 \\
& g_2(x) \le 0 \\
& \dots \\
& g_m(x) \le 0,
\end{aligned}
$$

where

- $x \in \mathbb{R}^n$,
- $f : \mathbb{R}^n \to \mathbb{R}$, and
- $g_i : \mathbb{R}^n \to \mathbb{R}$.

Note: Linear programs are NLPs!

Example 1: Finding Close Points in an LP

Finding Close Points in an LP

Problem: We are given an LP (P), and an infeasible point $\bar{x}$.

Goal: Find a point $x \in P$ that is as close as possible to $\bar{x}$.

I.e., find a point $x \in P$ that minimizes the Euclidean distance to $\bar{x}$:

$$\|x - \bar{x}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - \bar{x}_i)^2}$$

Remark: $\|p\|_2$ is called the L^2-norm of p.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in P \end{array}$$

$$P = \{x \,:\, Ax \leq b\}$$

$$\begin{array}{ll} \min & \|x - \bar{x}\|_2 \\ \text{s.t.} & x \in P \end{array}$$

Example 2: Binary IP via NLP

NLPs and Binary IPs

Suppose we are given a binary IP (i.e., an integer program all of whose variables are binary).

Recall: (Binary) IPs are generally hard to solve!

Now: We can write any binary IP as an NLP!

Ideas?

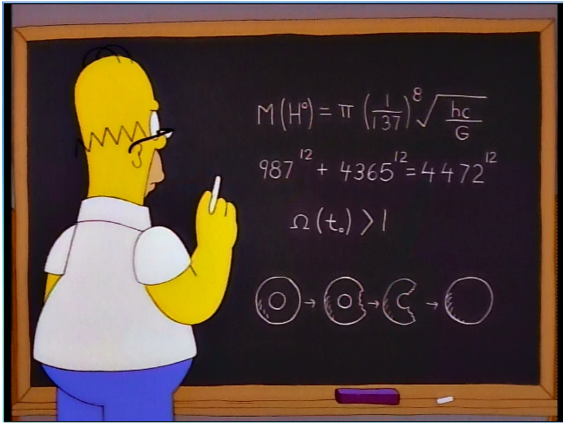
$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq \mathbb{0} \\ & x_j \in \{0,1\} \quad (j \in \{1,\ldots,n\}) \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq \mathbb{0} \\ & x_j(1 - x_j) = 0 \quad (j \in [n]) \qquad (\star) \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq \mathbb{0} \\ & \sin(\pi\, x_j) = 0 \quad (j \in [n]) \qquad (*) \end{aligned}$$

Example 3: Fermat's Last Theorem

This is false...



Swartzwelder, J. (Writer), & Kirkland, M. (Director). (1998). The Wizard of Evergreen Terrace [Television series episode]. In J. L. Brooks, M. Groening, & S. Simon (Executive Producers), The Simpsons. New York, NY: Twentieth Century Fox.

...d'oh!

17th century [Online Image]. Pierre de Fermat. Wikimedia Commons.

Conjecture [Fermat, 1637]

There are no integers $x, y, z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n.$$

Arithmeticorum Liber II. 61

IN QVÆSTIONEM VII.

QVÆSTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

1670 [Online Image]. Diophantus-II-8-Fermat. Wikimedia

In the margins of a copy of a 1670 article of Diophantus Arithmetica he wrote that he had a proof that was a bit too large to fit.

Some 358 years later, Sir Andrew Wiles gave the first accepted proof of the theorem. The proof is over 150 pages long!

NLP for Fermat's Last Theorem

$$\begin{aligned} \min \quad & (x_1^{x_4} + x_2^{x_4} - x_3^{x_4})^2 \\ & + (\sin \pi\, x_1)^2 + (\sin \pi\, x_2)^2 + (\sin \pi\, x_3)^2 + (\sin \pi\, x_4)^2 \\ \text{s.t.} \quad & x_i \geq 1 \quad (i = 1 \ldots 3) \\ & x_4 \geq 3 \end{aligned}$$

- The NLP is trivially feasible, and the value of any feasible solution is non-negative as its objective is the sum of squares.
- In fact, the value of a solution (x_1, x_2, x_3, x_4) is 0 iff
  - $x_1^{x_4} + x_2^{x_4} = x_3^{x_4}$, and
  - $\sin \pi\, x_i = 0$, for all $i = 1 \ldots 3$.

NLP for Fermat's Last Theorem

$$\begin{aligned} \min \quad & (x_1^{x_4} + x_2^{x_4} - x_3^{x_4})^2 \\ & + (\sin \pi\, x_1)^2 + (\sin \pi\, x_2)^2 + (\sin \pi\, x_3)^2 + (\sin \pi\, x_4)^2 \end{aligned}$$

$$\begin{aligned} \text{s.t.} \quad & x_i \geq 1 \quad (i = 1 \ldots 3) \\ & x_4 \geq 3 \end{aligned}$$

Remark

Fermat's Last Theorem is true iff the NLP has optimal value greater than 0.

Note: It is well known that there is an infinite sequence of feasible solutions whose objective value converges to 0!

Proving Fermat's Last Theorem amounts to showing that the value 0 can not be attained!

Nonlinear Models

Recap

- Nonlinear programs are of the form

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \dots \\ & g_m(x) \leq 0, \end{aligned}$$

  where $f, g_1, \dots, g_m$ are nonlinear functions.

- Nonlinear programs are strictly more general than integer programs, and thus likely difficult to solve.

- Some famous questions in math can easily be reduced to solving certain NLPs