# CO 250: Introduction to Optimization

## Module 2: Linear Programs (Basis)

# Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Notation

Let $B$ be a subset of column indices.
Then $A_B$ is a column sub-matrix of $A$ indexed by set $B$.

$$B = \{1, 2, 3\}$$

$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Notation

Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Notation

Let $B$ be a subset of column indices.
Then $A_B$ is a column sub-matrix of $A$ indexed by set $B$.

$$B = \{1, 3, 4\}$$

$$A_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

# Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Notation

Let $B$ be a subset of column indices.
Then $A_B$ is a column sub-matrix of $A$ indexed by set $B$.

$$B = \{5\} \qquad\qquad A_{\{5\}} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

## Notation

$A_j$ denotes column $j$ of $A$.

# Notation

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Notation

Let $B$ be a subset of column indices.
Then $A_B$ is a column sub-matrix of $A$ indexed by set $B$.

$$B = \{5\} \qquad A_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

## Notation

$A_j$ denotes column $j$ of $A$.

# Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

**Definition**

Let $B$ be a subset of column indices. $B$ is a basis if

(1) $A_B$ is a square matrix,

(2) $A_B$ is non-singular (columns are independent).

Is $B = \{1, 2, 3\}$ a basis?

$$A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

YES

# Basis

Consider

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Definition

Let $B$ be a subset of column indices. $B$ is a basis if

(1) $A_B$ is a square matrix,

(2) $A_B$ is non-singular (columns are independent).

Is $B = \{1, 5\}$ a basis?

$$A_B = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}$$

NO

$A_B$ is not square

# Basis

Consider

$$A = \begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

## Definition

Let $B$ be a subset of column indices. $B$ is a basis if

(1) $A_B$ is a square matrix,

(2) $A_B$ is non-singular (columns are independent).

Is $B = \{2,3,4\}$ a basis?

$$A_B = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

NO

columns of $A_B$ are dependent

## Question

Does every matrix have a basis? NO

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 3 & 0 & 1 & 1 \\ -3 & -2 & -1 & -3 & -1 \end{pmatrix}$$

The rows of $A$ are dependent!
There are no $3$ independent columns.

## Theorem

Max number of independent columns $=$
Max number of independent rows.

## Remark

Let $A$ be a matrix with independent rows. Then $B$ is a basis if and only if $B$ is a maximal set of independent columns of $A$.

# Basic Solutions

$$\begin{array}{c}\begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array}\\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A}\end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

## Definition

$x$ is a basic solution for basis $B$ if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$: basic sol'n for $B = \{1, 2, 3\}$

(1) $Ax = b$ ✓

(2) $x_4 = x_5 = 0$ ✓

# Basic Solutions

$$\begin{array}{c} \begin{array}{ccccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{array} \\ \underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

## Definition

$x$ is a basic solution for basis $B$ if

(1) $Ax = b$, and

(2) $x_j = 0$ whenever $j \notin B$.

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$ : basic sol'n for $B = \{1, 3, 4\}$

(1) $Ax = b$ ✓

(2) $x_2 = x_5 = 0$ ✓

$$\underbrace{\begin{array}{cccc} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} \\ \left(\begin{matrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{matrix}\right) \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

## Problem

What is a basic solution $x$ for basis $B = \{1, 4\}$?

$$\begin{aligned} \begin{pmatrix} 2 \\ 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\ &= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_2}_{=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_3}_{=0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \quad \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Thus, the basic solution is $x = (4,\ 0,\ 0,\ 2)^\top$.

## Question

Did we have a choice for a basic solution $x$ given $B = \{1, 4\}$? NO!

# Basic Solutions – Uniqueness

## Proposition

Consider $Ax = b$ and a basis $B$ of $A$.
Then there exists a unique basic solution $x$ for $B$.

Before we proceed with the proof, let's look at some conventions.

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 1 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} & \left.\begin{matrix} -1 \\ -1 \\ -1 \end{matrix}\right) \end{array}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

For $B = \{1, 2, 4\}$,

$$A_B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \text{(basic variables)}$$

columns of $A_B$ and elements of $x_B$ are ordered by $B$!

# Basic Solutions – Uniqueness

## Proposition

Consider $Ax = b$ and a basis $B$ of $A$.
Then there exists a unique basic solution $x$ for $B$.

## Proof

$$\begin{aligned} b =& Ax = \sum_j A_j x_j \\ &= \sum_{j \in B} A_j x_j + \sum_{j \notin B} A_j \underbrace{x_j}_{=0} \\ &= \sum_{j \in B} A_j x_j = A_B x_B \end{aligned}$$

Since $B$ is a basis, it implies $A_B$ is non-singular, i.e., $A_B^{-1}$ exists. Hence, $x_B = A_B^{-1} b$.

# When Is a Vector Basic?

## Definition

Consider $Ax = b$ with independent rows.
Vector $x$ is a basic solution if it is a basic solution for some basis $B$.

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \end{array}}_{}$$

$$\underbrace{\left(\begin{array}{ccccc} 3 & 2 & 1 & 4 & 1 \\ -1 & 1 & 0 & 2 & 1 \end{array}\right)}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}_{b}$$

## Question

Is $x = (0, 0, 3, 0, 3)^\top$ basic? YES!

Is $x$ basic for $B = \{3, 5\}$?

(1) $Ax = b$ ✓

(2) $x_1 = x_2 = x_4 = 0$ ✓

$$\underbrace{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \end{array}}_{}$$

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \underbrace{\begin{pmatrix} 3 & 2 & 1 & 4 & 1 \\ -1 & 1 & 0 & 2 & 1 \end{pmatrix}}_{A} \end{array} x = \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}_{b}$$

## Question

Is $x = (0, 1, 0, 1, 0)^\top$ basic? NO!

## Proof

By contradiction. Suppose $x$ is basic for basis $B$.

- $x_2 = 1 \neq 0$ implies $2 \in B$.
- $x_4 = 1 \neq 0$ implies $4 \in B$.

Thus,

$$A_{\{2,4\}} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

is a column submatrix of $A_B$. But the columns of $A_{\{2,4\}}$ are dependent, so $A_B$ is singular and $B$ is not a basis, a contradiction.

# Multiple Bases for a Basic Solution

$$\underbrace{\begin{array}{c} \begin{matrix} \color{red}{1} & \color{red}{2} & \color{red}{3} & \color{red}{4} & \color{red}{5} \end{matrix} \\ \begin{pmatrix} 3 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b}$$

Note: $x = (0, 0, 0, 0, 0)^\top$ is a basic solution for

- basis $B = \{1, 2\}$,
- basis $B' = \{1, 3\}$,
- basis $B'' = \{2, 4\}$, ....

**Remark**

A basic solution can be the basic solution for more than one basis.

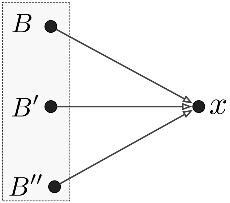
$$\underbrace{\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 3 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix} \end{array}}_{A} x = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b}$$

Note: $x = (0,0,0,0,0)^\top$ is a basic solution for

- basis $B = \{1,2\}$,
- basis $B' = \{1,3\}$,
- basis $B'' = \{2,4\}$, ....

## Remark

A basic solution can be the basic solution for more than one basis.



# Relation to LPs

Problem in SEF:

$$\max\{c^\top x : Ax = b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

**Remark**

If the rows of $A$ are dependent, then either

- there is no solution to $Ax = b$; (P) is infeasible, OR
- a constraint of $Ax = b$ can be removed without changing the solutions.

**Remark**

We may assume, when trying to solve (P), that rows of $A$ are independent.

**Definition**

A basic solution $x$ of $Ax = b$ is feasible if $x \geq 0$, i.e., if it is feasible for (P).

Consider the system

$$Ax = b$$

where the rows of $A$ are independent.

## Recap

(1) $B$ is a basis if $A_B$ is a square, non-singular matrix.

(2) $x$, a solution to $Ax = b$, is a basic solution for $B$ where $x_j = 0$ when $j \notin B$.

(3) $x$ is basic if it is basic for some basis $B$.

(4) Each basis has a unique associated basic solution.

(5) Several bases can have the same basic solution.

(6) A basic solution is feasible if it is non-negative.