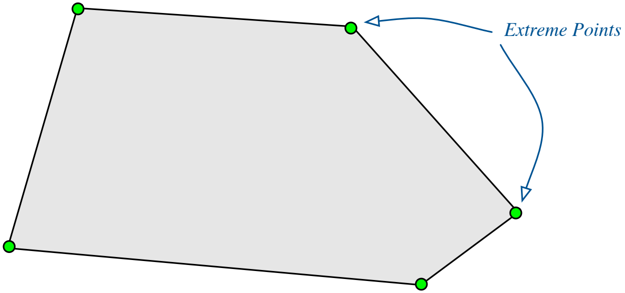
CO 250: Introduction to Optimization

Module 2: Linear Programs (Extreme Points)

Extreme Points

Consider the following convex set:



Question

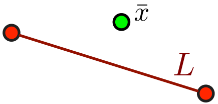
How might we formally describe the “extreme points”?

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
- x is distinct from the endpoints of L.



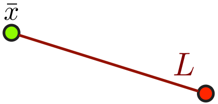
$\bar{x}$ is not contained in L.

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
- x is distinct from the endpoints of L.



$\bar{x}$ is contained in L,

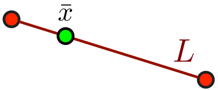
but NOT properly.

Towards a Definition of Extreme Points

Definition

Point $x \in \Re^n$ is properly contained in the line segment L if

- $x \in L$ and
- x is distinct from the endpoints of L.



$\bar{x}$ is PROPERLY contained in L.

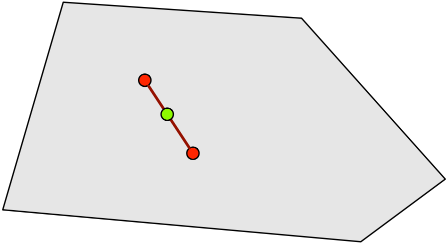
Definition

Let S be a convex set and $\bar{x} \in S$.

It follows that $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.

Definition

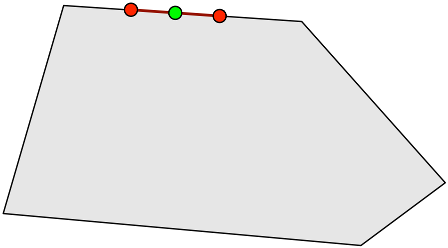
Let S be a convex set and $\bar{x} \in S$. It follows that $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.



Not an extreme point

Definition

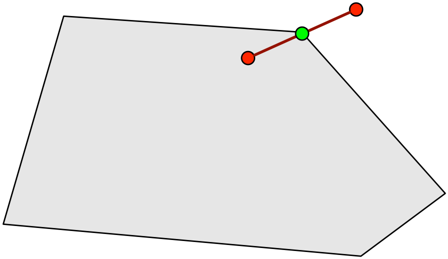
Let S be a convex set and $\bar{x} \in S$. It follows that $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.



Not an extreme point

Definition

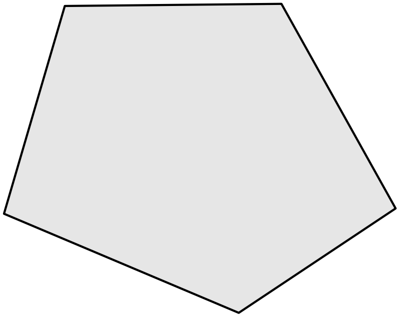
Let S be a convex set and $\bar{x} \in S$. It follows that $\bar{x}$ is NOT an extreme point if there exists a line segment $L \subseteq S$ where L properly contains $\bar{x}$.



An extreme point

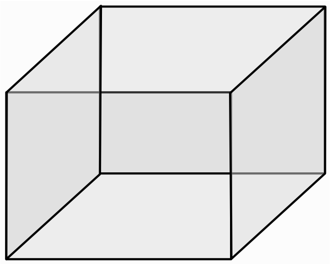
Question

What are the extreme points in the following figure?



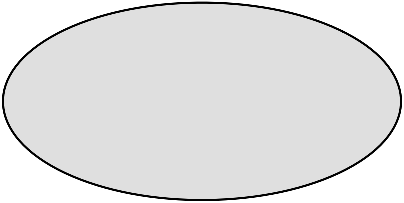
Question

What are the extreme points in the following figure?



Question

What are the extreme points in the following figure?

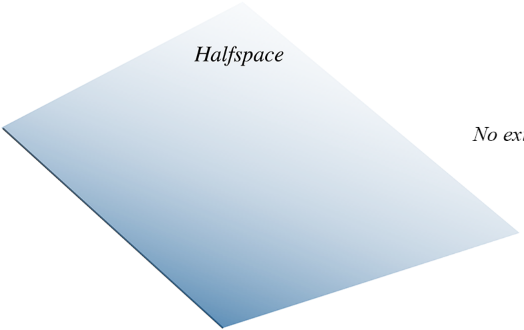


Remark

A convex set may have an infinite number of extreme points.

Question

What are the extreme points in the following figure?



No extreme points!

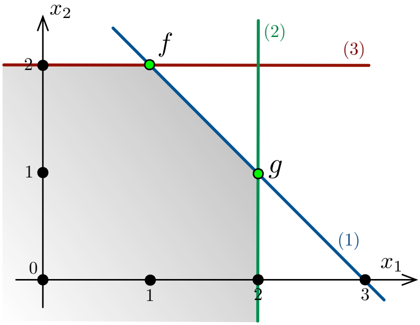
Remark

A convex set may have NO extreme points.

This Lecture

Goals:

1. Characterize the extreme points in a polyhedra.
2. Characterize an extreme point for LP in Standard Equality Form.
3. Gain a geometric understanding of the Simplex algorithm.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \right\}$$

Question

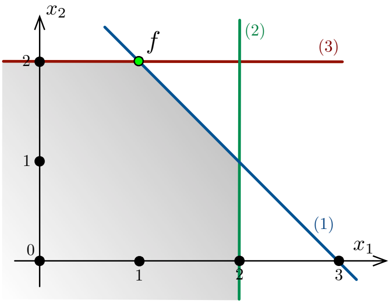
What do the extreme points $f = (1,2)^\top$ and $g = (2,1)^\top$ have in common?

Each satisfy $n = 2$ “independent” constraints with equality!

Definition

Let $P = \{x : Ax \leq b\}$ be a polyhedron and let $x \in P$.

- A constraint is tight for x if it is satisfied with equality, and
- the set of all tight constraints is denoted $\bar{A}x \leq \bar{b}$.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

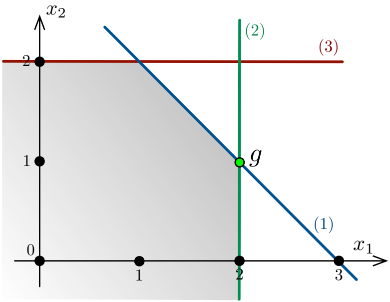
Consider f:

It follows that (1) and (3) are tight. $\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\bar{A}} x \leq \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\bar{b}}$.

Definition

Let $P = \{x : Ax \leq b\}$ be a polyhedron and let $x \in P$.

- A constraint is tight for x if it is satisfied with equality, and
- the set of all tight constraints is denoted $\bar{A}x \leq \bar{b}$.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

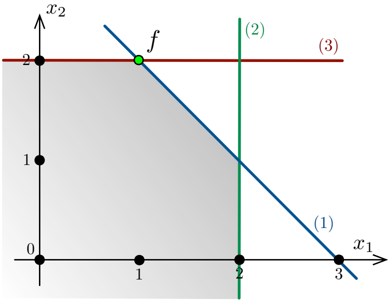
Consider g:

It follows that (1) and (2) are tight. $\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\bar{A}} x \leq \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\bar{b}}$.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $rank(\bar{A}) = n$, then $\bar{x}$ is an extreme point.
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

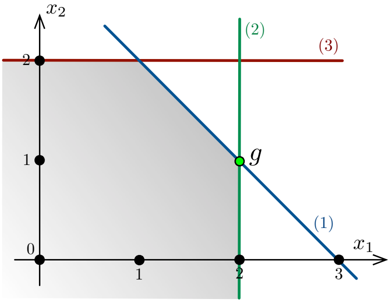
Consider f:

$\bar{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so, since $rank(\bar{A}) = 2$, f is an extreme point.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $rank(\bar{A}) = n$, then $\bar{x}$ is an extreme point.
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

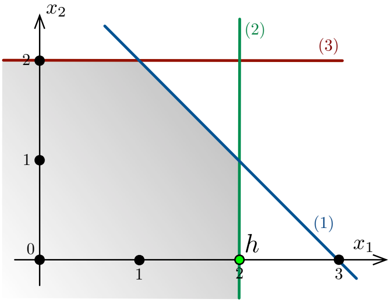
$\boxed{\text{Consider } g\text{:}}$

$\bar{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ so, since $rank(\bar{A}) = 2$, g is an extreme point.

Theorem

Let $P = \{x \in \Re^n : Ax \le b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $rank(\bar{A}) = n$, then $\bar{x}$ is an extreme point.
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}$$

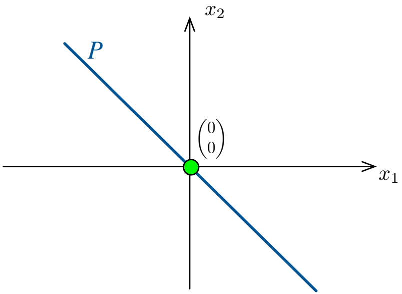
$\boxed{\text{Consider } h\text{:}}$

$\bar{A} = \begin{pmatrix} 1 & 0 \end{pmatrix}$ so, since $rank(\bar{A}) < 2$, h is NOT an extreme point.

Is the following true? NO!

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $\bar{A}$ has n rows then $\bar{x}$ is an extreme point.
2. If $\bar{A}$ has $< n$ rows then $\bar{x}$ is NOT an extreme point.



$$P = \left\{ x : \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$\bar{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ has $n = 2$ rows, but $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is NOT extreme.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. $\boxed{\text{If } rank(\bar{A}) = n\text{, then } \bar{x} \text{ is an extreme point.}}$
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.

Let's prove part (1).

Remark

Let $a, b, c \in \Re$, and suppose

$$a = \frac{1}{2}b + \frac{1}{2}c \qquad \text{and} \qquad b \leq a,\ c \leq a.$$

It follows that $a = b = c$.

Proof

$a = \frac{1}{2}\underbrace{b}_{\leq a} + \frac{1}{2}\underbrace{c}_{\leq a} \leq \frac{1}{2}a + \frac{1}{2}a = a.$

Thus, equality holds throughout $\Rightarrow$ $b = a$ and $c = a$.

Remark

Let $a, b, c \in \Re$, and let λ where $0 < \lambda < 1$. Suppose

$$a = \lambda b + (1 - \lambda)c \qquad \text{and} \qquad b \leq a, \; c \leq a.$$

It follows that $a = b = c$.

Exercise

Prove the previous remark.

Remark

Let $a, b, c \in \Re^n$, and let λ where $0 < \lambda < 1$. Suppose

$$a = \lambda b + (1 - \lambda)c \qquad \text{and} \qquad b \leq a,\ c \leq a.$$

It follows that $a = b = c$.

Exercise

Prove the previous remark.

Theorem

Let $P = \{x \in \Re^n : Ax \le b\}$ be a polyhedron and let $\bar{x} \in P$.

1. $\boxed{\text{If } rank(\bar{A}) = n, \text{ then } \bar{x} \text{ is an extreme point.}}$

Proof

Suppose $\bar{x}$ is not an extreme point.

$\bar{x}$ is properly contained in a line segment with endpoints $x^{(1)}, x^{(2)} \in P$.

$\bar{x} \neq x^{(1)}, x^{(2)} \in P$ and for some λ, $0 < \lambda < 1$, $\bar{x} = \lambda x^{(1)} + (1-\lambda)x^{(2)}$.

$$\bar{b} = \bar{A}\bar{x} = \bar{A}\left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) = \lambda \bar{A}x^{(1)} + (1-\lambda)\bar{A}x^{(2)}.$$

$\bar{A}x^{(1)} \le \bar{b}$ and $\bar{A}x^{(2)} \le \bar{b}$.

Our remark implies that $\bar{b} = \bar{A}x^{(1)} = \bar{A}x^{(2)}$.

However, since $rank(\bar{A}) = n$, $x^{(1)} = x^{(2)}$. This is a contradiction.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

2. $\boxed{\text{If } rank(\bar{A}) < n, \text{ then } \bar{x} \text{ is NOT an extreme point.}}$

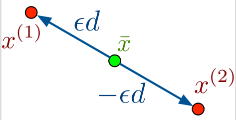
Proof

Since $rank(\bar{A}) < n$, there exists a vector d such that $\bar{A}d = \mathbf{0}$.

Pick a small $\epsilon > 0$.

$x^{(1)} = \bar{x} + \epsilon d$

$x^{(2)} = \bar{x} - \epsilon d$



It suffices to prove the following:

(a) $\bar{x}$ is properly contained in the line segment between $x^{(1)}$ and $x^{(2)}$.

(b) $x^{(1)}, x^{(2)} \in P$.

Proof

Since $rank(\bar{A}) < n$, there exists a vector d such that $\bar{A}d = \mathbf{0}$.

Pick a small $\epsilon > 0$. Let $x^{(1)} = \bar{x} + \epsilon d$ and $x^{(2)} = \bar{x} - \epsilon d$.

(a) $\bar{x}$ is properly contained in the line segment between $x^{(1)}$ and $x^{(2)}$.

Why?

$$\frac{1}{2}x^{(1)} + \frac{1}{2}x^{(2)} = \frac{1}{2}(\bar{x} + \epsilon d) + \frac{1}{2}(\bar{x} - \epsilon d) = \bar{x}. \quad \checkmark$$

Proof

Since $rank(\bar{A}) < n$, there exists a vector d such that $\bar{A}d = \mathbf{0}$.

Pick a small $\epsilon > 0$. Let $x^{(1)} = \bar{x} + \epsilon d$ and $x^{(2)} = \bar{x} - \epsilon d$.

(a) $\bar{x}$ is properly contained in the line segment between $x^{(1)}$ and $x^{(2)}$.

(b) $x^{(1)}, x^{(2)} \in P$. (It is sufficient to show this for $x^{(1)}$ only.)

$\boxed{\text{Consider tight constraints } \bar{A}x \le \bar{b}.}$

$$\bar{A}x^{(1)} = \bar{A}(\bar{x} + \epsilon d) = \underbrace{\bar{A}\bar{x}}_{\bar{b}} + \epsilon \underbrace{\bar{A}d}_{\mathbf{0}} = \bar{b}. \qquad \checkmark$$

$\boxed{\text{Consider non-tight constraint } a^\top x \le \beta.}$

$$a^\top x^{(1)} = a^\top(\bar{x} + \epsilon d) = \underbrace{a^\top \bar{x}}_{<\beta} + \epsilon \underbrace{a^\top d}_{=??} < \beta$$

for a small enough ϵ!

Consider

$$P = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \qquad \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \text{ is a basic solution}$$

Question

Is $(2, 4, 0)^\top$ an extreme point?

Let's use our theorem to find an answer.

Theorem

Let $P = \{x \in \Re^n : Ax \leq b\}$ be a polyhedron and let $\bar{x} \in P$.

1. If $rank(\bar{A}) = n$, then $\bar{x}$ is an extreme point.
2. If $rank(\bar{A}) < n$, then $\bar{x}$ is NOT an extreme point.

We need to rewrite the constraints in P so they are all in the form "$\leq$".

Consider

$$P = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \qquad \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \text{ is a basic solution}$$

Question

Is $(2,4,0)^\top$ an extreme point?

We need to rewrite the constraints in P so they are all in the form "$\leq$".

$P = \{x : Ax \leq b\}$, where

$$A = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 3 \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right) \quad \text{and} \quad b = \left(\begin{array}{r} 2 \\ 4 \\ \hline -2 \\ -4 \\ \hline 0 \\ 0 \\ 0 \end{array} \right)$$

For $P = \{x : Ax \leq b\}$, where

$$A = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 3 \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array}\right) \quad , \quad b = \left(\begin{array}{r} 2 \\ 4 \\ \hline -2 \\ -4 \\ \hline 0 \\ 0 \\ 0 \end{array}\right)$$

and $(2, 4, 0)^\top$, we have

$$\bar{A} = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 0 & 1 & 3 \\ \hline -1 & 0 & 1 \\ 0 & -1 & -3 \\ \hline 0 & 0 & -1 \end{array}\right)$$

Since $rank(\bar{A}) = 3$, we know that $(2, 4, 0)^\top$ is an extreme point!

This is no accident...

Theorem

Let $P = \{x \geq \mathbf{0} : Ax = b\}$ where rows of A are independent. The following are equivalent:

1. $\bar{x}$ is an extreme point of P.
2. $\bar{x}$ is a basic feasible solution of P.

Exercise

Prove the previous theorem.

The Simplex algorithm moves from extreme points to extreme points.

Simplex - a Geometric Illustration

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Solve using Simplex:

- Basis $B = \{3,4,5\}$, basic solution $(0,0,10,6,4)^\top$
- Basis $B = \{1,4,5\}$, basic solution $(5,0,0,1,9)^\top$
- Basis $B = \{1,2,5\}$, basic solution $(4,2,0,0,6)^\top$
- Basis $B = \{1,2,3\}$, basic solution $(1,5,3,0,0)^\top$: optimal

$\Rightarrow$ Simplex visits extreme points of P_1 in order:

$$\begin{pmatrix} 0 \\ 0 \\ 10 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

However, we cannot draw a picture of this...

$$\boxed{\begin{array}{ll} \max & (2,3,0,0,0)x \\ \text{s.t.} & \\ & x \in P_1 \end{array}} \qquad P_1 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

is obtained by adding slack variables to

$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}} \qquad P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Remark

$(0,0,10,6,4)^\top$	extreme point of P_1	$\Rightarrow$	$(0,0)^\top$	extreme point of P_2,
$(5,0,0,1,9)^\top$	extreme point of P_1	$\Rightarrow$	$(5,0)^\top$	extreme point of P_2,
$(4,2,0,0,6)^\top$	extreme point of P_1	$\Rightarrow$	$(4,2)^\top$	extreme point of P_2,
$(1,5,3,0,0)^\top$	extreme point of P_1	$\Rightarrow$	$(1,5)^\top$	extreme point of P_2.

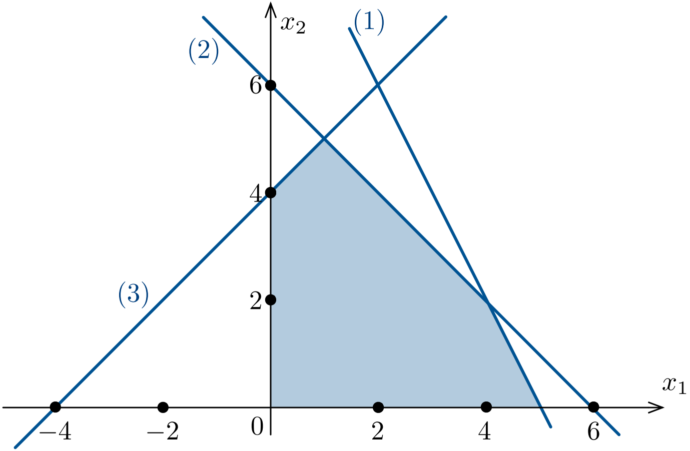
➡ Simplex visits extreme points of P_2 in order:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}}$$

$$P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

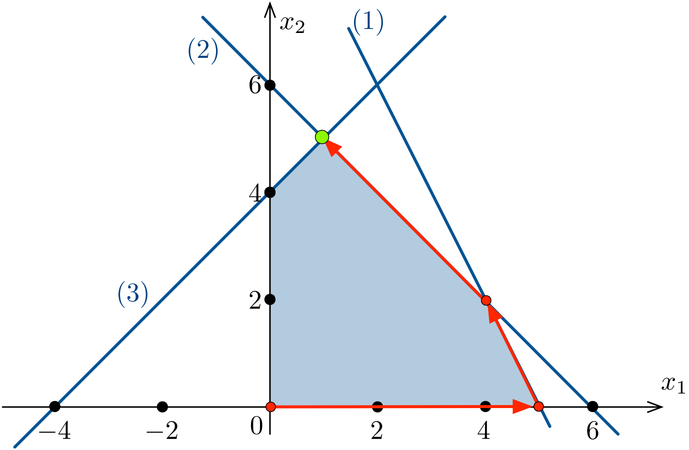
Simplex visits extreme points of P_2 in order: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}$.



$$\boxed{\begin{array}{ll} \max & (2,3)x \\ \text{s.t.} & \\ & x \in P_2 \end{array}}$$

$$P_2 = \left\{ x \geq \mathbf{0} : \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 10 \\ 6 \\ 4 \end{pmatrix} \right\}$$

Simplex visits extreme points of P_2 in order: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$



Recap

- We defined extreme points of convex sets.
- We characterized extreme points in polyhedra.
- We saw that extreme points $=$ basic solutions for problems in SEF.
- We showed that Simplex lets us moves from extreme point to extreme point.