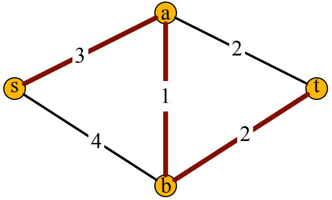
CO 250: Introduction to Optimization

Module 1: Formulations (Shortest Paths)

Recap: Shortest Paths

Input:

- Graph $G = (V, E)$
- Non-negative edge lengths c_e for all $e \in E$
- Vertices $s, t \in V$



Goal: Compute an s,t-path of smallest total length.

Recall: P is an s,t-path if it is of the form

$$v_1v_2,\ v_2v_3,\ \ldots,\ v_{k-1}v_k$$

and

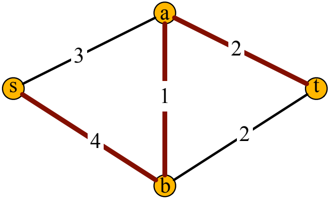
1. $v_iv_{i+1} \in E$ for all $i \in \{1, \ldots, k-1\}$,
2. $v_i \neq v_j$ for all $i \neq j$, and
3. $v_1 = s$ and $v_k = t$.

E.g., $P = sa,\ ab,\ bt$

Recap: Shortest Paths

Shortest Path Problem: Given $G = (V, E)$, $c_e \geq 0$ for all $e \in E$, and $s, t \in V$, compute an s,t-path of smallest total length.

Now: Formulate the problem as an IP!



Useful Observation: Let $C \subseteq E$ be a set of edges whose removal disconnects s and t.

$\longrightarrow$ Every s,t-path P must have at least one edge in C.

Definition

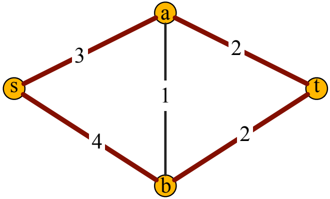
For $S \subseteq V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S.

$$\delta(S) = \{uv \in E \,:\, u \in S, v \notin S\}$$

Cuts

Examples:

1. $S = \{s\} \rightarrow \delta(S) = \{sa, sb\}$
2. $S = \{s, a\} \rightarrow \delta(S) = \{ab, at, sb\}$
3. $S = \{a, b\} \rightarrow \delta(S) = \{sa, sb, at, bt\}$



Definition

$\delta(S)$ is an s, t-cut if $s \in S$ and $t \notin S$.

E.g., 1 and 2 are s, t-cuts, 3 is not.

Definition

For $S \subseteq V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S.

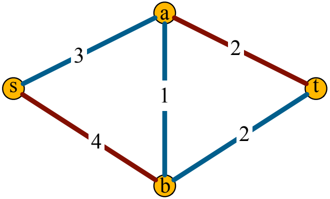
$$\delta(S) = \{uv \in E \,:\, u \in S, v \notin S\}$$

Cuts

Definition

$\delta(S)$ is an s, t-cut if $s \in S$ and $t \notin S$.

E.g., $\delta(\{s, a\}) = \{sb, ab, at\}$ is an s, t-cut.



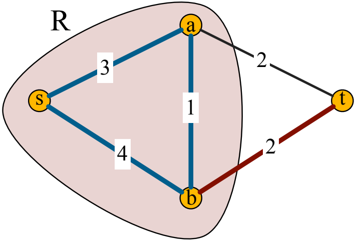
Remark

If P is an s, t-path and $\delta(S)$ is an s, t-cut, then P must have an edge from $\delta(S)$.

E.g., $P = sa,\ ab,\ bt$.

Remark

If $S \subseteq E$ contains at least one edge from every s,t-cut, then S contains an s,t-path.



Proof: (by contradiction)

- Suppose S has an edge from every s,t-cut, but S has no s,t-path.
- Let R be the set of vertices reachable from s in S:

  $R = \{u \in V \ : \ S$ has an s,u–path$\}$.

- $\delta(R)$ is an s,t-cut since $s \in R$ and $t \notin R$.

- Note: There cannot be an edge $uv \in E$ with $u \in R$ and $v \notin R$. Otherwise: v should have been in R!

$\longrightarrow \delta(R) \cap S = \emptyset$.
Contradiction!

An IP for Shortest Paths

Variables: We have one binary variable x_e for each edge $e \in E$. We want:

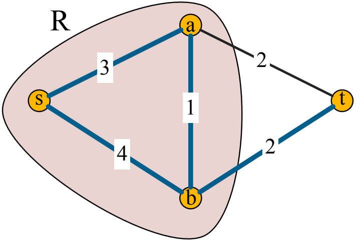
$$x_e = \begin{cases} 1 : e \in P \\ 0 : \text{otherwise} \end{cases}$$

Constraints: We have one constraint for each s,t-cut $\delta(U)$, forcing P to have an edge from $\delta(S)$.

$$\sum(x_e \; : \; e \in \delta(U)) \geq 1 \qquad (1)$$

for all s,t-cuts $\delta(U)$.

Objective: $\sum(c_e x_e \; : \; e \in E)$



Remark

If $S \subseteq E$ contains at least one edge from every s,t-cut, then S contains an s,t-path.

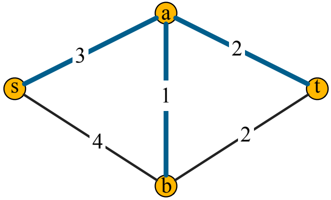
An IP for Shortest Paths

$$
\begin{aligned}
\min \quad & \sum (c_e x_e \;:\; e \in E) \\
\text{s.t.} \quad & \sum (x_e \;:\; e \in \delta(U)) \geq 1 \;\; (U \subseteq V, s \in U, t \notin U) \\
& x_e \geq 0, x_e \text{ integer} \quad (e \in E)
\end{aligned}
$$

$$
\min \quad (3,4,1,2,2)x
$$

$$
\text{s.t.} \quad
\begin{array}{c c}
 & \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\
\begin{array}{c} \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} &
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array} x \geq \mathbb{1}
$$

$$
x \geq \mathbb{0} \quad x \text{ integer}
$$



An IP for Shortest Paths

$$\begin{aligned}
\min\ & \sum (c_e x_e \,:\, e \in E) \\
\text{s.t. } & \sum (x_e \,:\, e \in \delta(U)) \geq 1 \ (U \subseteq V, s \in U, t \notin U) \\
& x_e \geq 0, x_e \text{ integer} \quad (e \in E)
\end{aligned}$$

Suppose: $c_e > 0$ for all $e \in E$

Then: In an optimal solution, $x_e \leq 1$ for all $e \in E$. Why?

Suppose $x_e > 1$.
Then let $x_e = 1$. This is cheaper and maintains feasibility!

For a binary solution x, define

$$S_x = \{e \in E \,:\, x_e = 1\}.$$

An IP for Shortest Paths

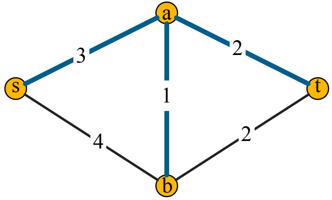
Note: If x is feasible for an IP, then S_x satisfies the remark, but S_x may contain more than just an s,t-path!

E.g., $x_e = 1$ for all blue edges in the figure and $x_e = 0$ otherwise. Then,

$$S_x = \{sa, ab, at\}$$

Note: x cannot be optimal for the IP!

Why?



Remark

If $S \subseteq E$ contains at least one edge from every s,t-cut, then S contains an s,t-path.

An IP for Shortest Paths

$$\begin{aligned} \min\ & \sum (c_e x_e \,:\, e \in E) \\ \text{s.t.}\ & \sum (x_e \,:\, e \in \delta(U)) \geq 1 \ (U \subseteq V, s \in U, t \notin U) \\ & x_e \geq 0, x_e \text{ integer} \quad (e \in E) \end{aligned}$$

Remark

If x is an optimal solution for the above IP and $c_e > 0$ for all $e \in E$, then S_x contains the edges of a shortest s,t-path.

Recap

- Given $G = (V, E)$ and $U \subseteq V$, we define

$$\delta(U) = \{uv \in E \,:\, u \in U, v \notin U\}.$$

- $\delta(U)$ is an s,t-cut if $s \in U$ and $t \notin U$.
- If $S \subseteq E$ intersects every s,t-cut $\delta(U)$, then S contains an s,t-path.
- Feasible solutions to the shortest path LP correspond to edge-sets that intersect every s,t-cut; optimal solutions are minimal in this respect if $c_e > 0$ for all $e \in E$.