CO 250 - Spring 2018

Assignment 5

Due date : Friday, June 8, 2018, by 12 noon (sharp)

Submission Guidelines:

- Please submit your solutions to Crowdmark. Late assignments will not be accepted, and will receive a mark of zero. It is the responsibility of the students to make sure that the pdf file they submit is clearly readable. Illegible submissions will receive a mark of zero, and hard to read submissions may receive deductions.
- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- In some questions you are asked to *formulate* the problem. You are *not* asked to actually solve the formulation, e.g., compute optimal solutions. Your formulations should be easy to modify if we change the data and constants defining the problems. Clearly define all your variables (including their units) and any other new notation you use in all your answers. Your solutions must also contain a brief justification of all the constraints (explain the relation between each of the constraints and the requirements stated in the problem) and the objective function.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. Copying or paraphrasing a solution from some fellow student or old solutions from previous offerings qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. Students found to be cheating will be given a mark of 0 on the assignment. In addition, all academic offenses will be reported to the Math Academic Integrity Officer (which may lead to further penalties) and recorded in the student's file.
Re-marking policies: If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that, if you see any marking error, then write a letter detailing clearly the marking errors, and submit this to one of the head TAs Julian Barbosa (MC 5121, Jaromero@uwaterloo.ca) or Jiyoung Im (MC 5466, j5im@uwaterloo.ca) within one week from the date the graded assignment is returned. If you still have concerns after the final decision of the TA, then please contact your instructor communicating all the correspondence with the TA and the original petition.

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[1] It is an academic offense to post this assignment or solutions to any web page.

Question 1. Basic Solutions **(20 marks)**

Consider a system $Ax = b,\ x \geq \mathbb{0}$ where A is an $m \times n$ matrix whose rows are linearly independent. Let $\bar{x}$ be a solution to $Ax = b$. Let J be the support of $\bar{x}$; so J consists exactly of the column indices j of A for which $\bar{x}_j \neq 0$.

(a) Show that if the columns of A_J are linearly dependent, then $\bar{x}$ is not a basic solution. (3 marks)

(b) Conversely, show that if the columns of A_J are linearly independent then $\bar{x}$ is a basic solution. (5 marks)

(**Hint:** Recall the following fact from linear algebra: let S be a set of vectors and let t be the rank of S, i.e., the maximum number of linearly independent vectors in S. Then, for any $R \subseteq S$ where the vectors in R are linearly independent, there is a subset R' of S with $R' \supseteq R$, $|R'| = t$ such that the vectors in R' are linearly independent.)

(c) Let $\bar{x} \geq \mathbb{0}$, and suppose that $\bar{x}$ is not a basic solution. Show that there exists a vector d such that $d_j = 0$ for all $j \notin J$, $Ad = \mathbb{0}$, and $d_j < 0$ for some $j \in J$. Deduce that there exists an $\epsilon > 0$ such that $x' = \bar{x} + \epsilon d$ is also as a feasible solution to $Ax = b,\ x \geq \mathbb{0}$, and $|\{j : x'_j > 0\}| < |J|$. (7 marks)

(**Hint:** To obtain d, use part (b), possibly replacing d by $-d$.)

(d) Suppose that the system $Ax = b,\ x \geq \mathbb{0}$ has a feasible solution $\bar{x}$. Explain how to obtain a basic feasible solution to that system. You may use the d vector from part c) with respect to any feasible solution, without further justification. (4 marks)

Question 2. Canonical Forms I **(15 marks)**

Consider the following linear program (P):

$$\begin{array}{ll} \max & (1,-1,3,2,1)x \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 1 & -1 & -1 & 2 \\ 0 & 2 & 1 & 1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

(a) For each of the following sets B_i of column indices, determine whether they form a basis. If B_i does form a basis, convert (P) to canonical form with respect to that basis; then determine the corresponding basic solution, and state whether it is feasible.

(i) $B_1 = \{1, 3\}$

(ii) $B_2 = \{1, 5\}$

(iii) $B_3 = \{2, 5\}$

(9 marks)

(b) Determine an optimal solution to (P), and provide a certificate of optimality.

(3 marks)

(c) Prove that there is a unique optimal solution to (P).

(3 marks)

Question 3. Canonical Forms II **(15 marks)**

(a) Consider the following LP (P):

$$\begin{array}{ll} \max & (0,-3,0,-1,0)x + 4 \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 10 & 0 & 2 & 0 \\ 0 & 6 & 1 & 2 & 0 \\ 0 & -3 & 0 & -1 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 5 \\ -2 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

(P) is in canonical form for the basis $\{1,3,5\}$. Find an equivalent LP that is in canonical form for the basis $B = \{1,3,4\}$. Are the basic solutions corresponding to the bases $\{1,3,5\}$ and $\{1,3,4\}$ feasible? (7 marks)

(Hint: There is a better method than computing A_B^{-1}. Instead, use row reduction to convert to canonical form.)

(b) What is the basic solution corresponding to $\{2,3,4\}$? Using your work from the previous part, explain why it is an optimal feasible solution. (4 marks)

(c) Show that there are infinitely many optimal feasible solutions to (P). (4 marks)

(Hint: if α, β are two optimal solutions, show that for any $t \in [0,1]$, the solution $t\alpha + (1-t)\beta$ is also an optimal solution.)

Question 4. Simplex Method **(20 marks)**

Solve each of the following LPs using the simplex method. Each LP below is in canonical form for some feasible basis B (which you should be able to identify by inspection); start the simplex algorithm from this basis. Use Bland's rule: break any ties in the choice of the entering and leaving variable by *picking the one with the smallest index.* Show all your steps.

(When running the simplex method, as noted earlier, to obtain the canonical form for a basis, it is typically more efficient to apply row operations to the canonical form for the previous basis, rather than compute A_B^{-1} and apply the formula in the text.)

(a)

$$\begin{array}{ll} \max & (0,0,1,9,-4)x + 4 \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & -3 & 1 & 5 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

(8 marks)

(b)

$$\begin{array}{ll} \max & (0,0,0,4,1)x - 3 \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0.5 & -0.5 \\ 0 & 0 & 1 & 0.5 & 1.5 \end{pmatrix} x = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

(12 marks)