CO 250: Introduction to Optimization

Module 5: Integer Programs (Convex Hulls)

LP versus IP

LINEAR PROGRAMMING	INTEGER PROGRAMMING
Can solve very large instances	Some small instances cannot be solved
Algorithms exist that are guaranteed to be fast	No fast algorithm exists
Short certificate of infeasibility (Farka's Lemma)	Does not always exist
Short certificate of optimality (Strong Duality)	Does not always exist
The only possible outcomes are infeasible, unbounded, or optimal	Can have other outcomes

Let us look at an example...

A Bad Example

Proposition

The following IP,

$$\boxed{\begin{array}{l} \max \quad x_1 - \sqrt{2}x_2 \\ \text{s.t.} \\ \qquad\qquad \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ \qquad\quad x_1, x_2 \text{ integer} \end{array}}$$

is feasible, bounded, and has no optimal solution.

- It is feasible. ✓
- 0 is an upper bound. ✓
- It has no optimal solution. ???

$$
\boxed{\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{lcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}}
$$

NO OPTIMAL SOLUTION

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim. x_1', x_2' are feasible

Proof

$x_1' = 2x_1 + 2x_2 \geq 1$ and $x_2' = x_1 + 2x_2 \geq 1$ ✓

$$
\begin{array}{lll}
x_1' & \overset{?}{\leq} \sqrt{2}x_2' & \iff \\
2x_1 + 2x_2 & \overset{?}{\leq} \sqrt{2}\,(x_1 + 2x_2) = \sqrt{2}x_1 + 2\sqrt{2}x_2 & \iff \\
x_1(2-\sqrt{2}) & \overset{?}{\leq} (2\sqrt{2}-2)x_2 & \iff \\
x_1 & \overset{?}{\leq} \frac{2\sqrt{2}-2}{2-\sqrt{2}}x_2 = \sqrt{2}x_2 \quad ✓ &
\end{array}
$$

$$\begin{array}{ll} \max & x_1 - \sqrt{2}x_2 \\ \text{s.t.} & \\ & \begin{array}{rcl} x_1 & \leq & \sqrt{2}x_2 \\ x_1, x_2 & \geq & 1 \end{array} \\ & x_1, x_2 \text{ integer} \end{array}$$

No optimal solution

Suppose, for a contradiction, there exists optimal x_1, x_2. Let

$$x_1' = 2x_1 + 2x_2 \qquad x_2' = x_1 + 2x_2$$

Claim. x_1', x_2' are feasible

Claim. $x_1' - \sqrt{2}x_2' > x_1 - \sqrt{2}x_2$

Proof

$$(2x_1 + 2x_2) - \sqrt{2}(x_1 + 2x_2)\sqrt{2} \stackrel{?}{>} x_1 - \sqrt{2}x_2$$

Simplifying, we obtain

$$\sqrt{2}x_2 \stackrel{?}{>} x_1$$

- $\geq$ since x_1, x_2 are feasible for (IP)
- $>$ otherwise $\sqrt{2} = \frac{x_1}{x_2}$ but $\sqrt{2}$ is not a rational number

Bad News/Good News

Bad News:

- IPs are hard to solve.
- IPs' theoretical results are more difficult than LPs'.

Good News:

- IPs can solve a huge number of useful problems.
- LP theory can sometimes be extended to IPs.

This lecture will show:

Integer Programming can, in principle, be reduced to Linear Programming.

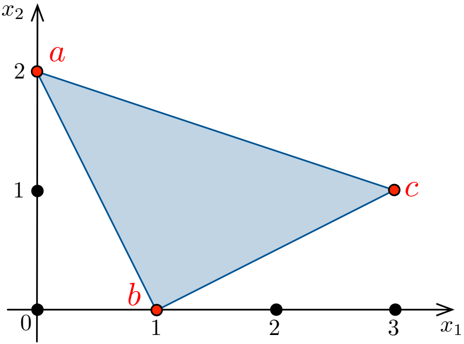
Remark

This will NOT give us a practical procedure to solve IPs,

but it will suggest a strategy.

Definition

Let C be a subset of $\Re^n$.
The convex hull of C is the *smallest convex set* that contains C.



$C = \{a, b, c\}$

Convex hull of C

Definition

Let C be a subset of $\Re^n$.
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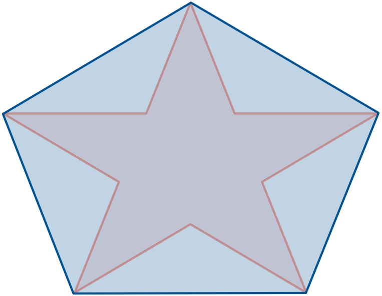


Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

➡ The notion of a convex hull is well defined.

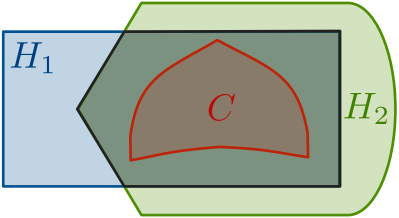
Question

Given $C \subseteq \Re^n$, is there a unique smallest convex set containing C? YES

WHY?

Suppose, for a contradiction, there exists:

- H_1 smallest convex set containing C
- H_2 smallest convex set containing C
- $H_1 \neq H_2$



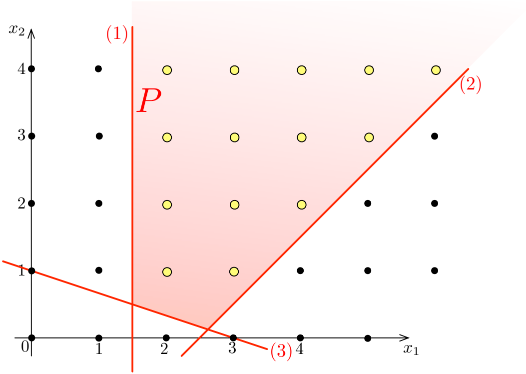
$\Rightarrow$

- $C \subseteq H_1 \cap H_2$,
- $H_1 \cap H_2$ is convex

However, $H_1 \cap H_2$ is smaller than both H_1 and H_2. This is a contradiction.

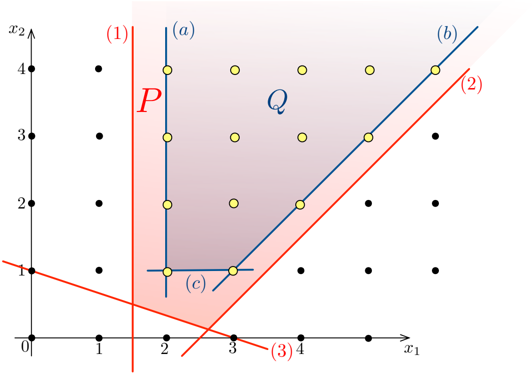
Convex Hulls and Integer Programs

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Integer points in P

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -3/2 \\ 5/2 \\ -3 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \right\}.$$



Q convex hull of integer points in P

$$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \quad \begin{matrix} (a) \\ (b) \\ (c) \end{matrix} \right\}.$$

Polyhedron

Meyer's Theorem

Consider $P = \{x : Ax \leq b\}$ where A, b are rational.
Then, the convex hull of all integer points in P is a polyhedron.

(We'll omit the proof)

Remark

The condition that all entries of A and b are rational numbers cannot be excluded from the hypothesis.

Example

Consider

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \leq \sqrt{2}x_2,\ x_1, x_2 \geq 1 \right\}.$$

The convex hull of all integer points in P is NOT a polyhedron.

<u>Goal</u>: Use Meyer's theorem to reduce the problem of solving *Integer Programs*, to the problem of solving *Linear Program*.

Let A, b be rational.

$$\max\{c^\top x : Ax \le b, x \text{ integer}\}. \tag{IP}$$

The convex hull of all feasible solutions of (IP) is a polyhedron $\{x : A'x \le b\}$.

$$\max\{c^\top x : A'x \le b', x \text{ integer}\} \tag{LP}$$

Theorem

- (IP) is infeasible if and only if (LP) is infeasible,
- (IP) is unbounded if and only if (LP) is unbounded,
- an optimal solution to (IP) is an optimal solution to (LP), and
- an extreme optimal solution to (LP) is an optimal solution to (IP).

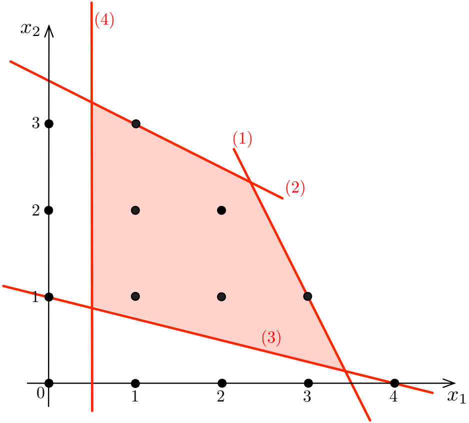
(We'll omit the proofs)

Conceptual way of solving (IP):

Step 1. Compute A', b'.

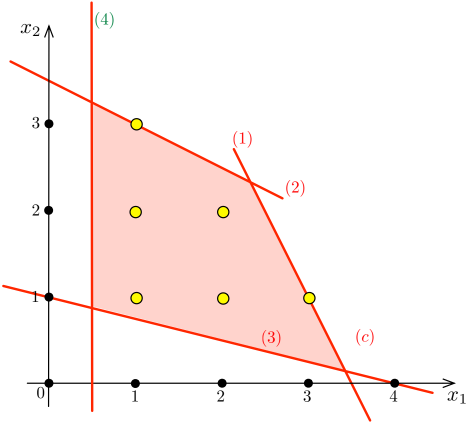
Step 2. Use Simplex to find an extreme optimal solution to (LP).

$$\max\left\{ \begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & -4 \\ -1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 7 \\ 7 \\ -4 \\ -\frac{1}{2} \end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer} \right\} \qquad \text{(IP)}$$



A feasible region for (IP) relaxation

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix} x \leq \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer}\right\} \qquad \text{(IP)}$$



A feasible region for (IP)

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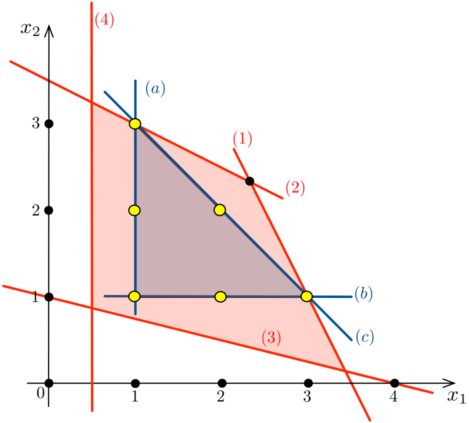
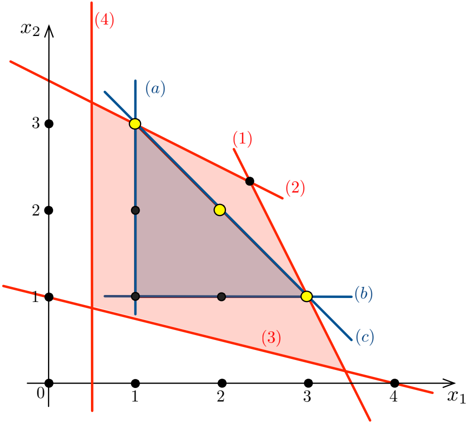


The convex hull of feasible solutions of (IP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

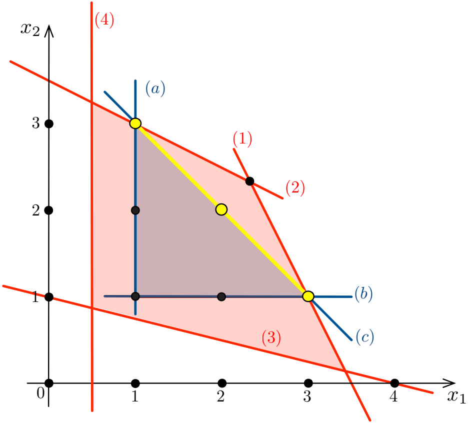
$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix} x \leq \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer}\right\} \qquad \text{(IP)}$$



Optimal solutions of (IP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix} x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix} x \leq \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix} \right\} \qquad \text{(LP)}$$

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}2 & 1\\ 1 & 2\\ -1 & -4\\ -1 & 0\end{pmatrix}x \le \begin{pmatrix}7\\ 7\\ -4\\ -\frac{1}{2}\end{pmatrix} \quad \begin{matrix}(1)\\(2)\\(3)\\(4)\end{matrix} \quad x \text{ integer}\right\} \qquad \text{(IP)}$$



Optimal solutions of (LP)

$$\max\left\{\begin{pmatrix}1 & 1\end{pmatrix}x : \begin{pmatrix}-1 & 0\\ 0 & -1\\ 1 & 1\end{pmatrix}x \le \begin{pmatrix}-1\\ -1\\ 4\end{pmatrix} \quad \begin{matrix}(a)\\(b)\\(c)\end{matrix}\right\} \qquad \text{(LP)}$$

$$\max\{c^\top x : Ax \le b, x \text{ integer}\} \qquad \text{(IP)}$$

The convex hull of the feasible region is a polyhedron $\{x : A'x \le b\}$.

$$\max\{c^\top x : A'x \le b', x \text{ integer}\} \qquad \text{(LP)}$$

Conceptual way of solving (IP):

Step 1. Compute A', b'.

Step 2. Use Simplex to find an extreme optimal solution.

Remark

This is NOT a practical way to solve an LP!

WHY NOT?

- We do not know how to compute A', b', and
- A', b' can be MUCH more complicated than A, b.

Question

How do we fix these problems?

Idea

Construct an approximation of the convex hull of the solutions of (IP).

Recap

- Integer Programs are much harder to solve than Linear Programs.
- Linear Programming theory does not always extend to Integer Programs.
- We defined the notion of convex hulls.
- The convex hull of the integer points in a rational polyhedron is a polyhedron.
- Integer programming reduces to Linear programming, but it is not a practical reduction.