CO 250: Introduction to Optimization

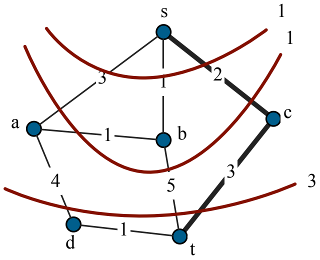
Module 3: Duality Through Examples (Shortest Path Algorithm)

Recap: Feasible Widths via Duality

The figure on the right shows another simple instance of the shortest s,t-path problem.

By inspection, we see the shortest s,t-path (bold edges) has length 5.

We know this because there is a feasible width assignment, of value 5, proving optimality!



Shortest path LP:

$$\begin{aligned} \min \quad & \sum (x_e \,:\, e \in E) \\ \text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(S)) \geq 1 \\ & \qquad\qquad (\delta(S)\ s,t\text{-cut}) \\ & x \geq \mathbb{0} \end{aligned}$$

Shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \,:\, \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \leq c_e \\ & \qquad\qquad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

Recap: Feasible Widths via Duality

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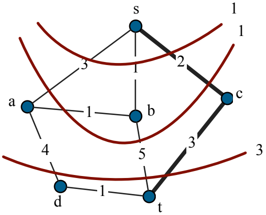

$$x_e = \begin{cases} 1 & e \textbf{ bold} \text{ in figure} \\ 0 & \text{otherwise} \end{cases}$$

for all $e \in E$ is feasible for a shortest path LP.

$$y_{\{s\}} = y_{\{s,b\}} = 1,\ y_{\{s,a,b,c\}} = 3,$$

and $y_S = 0$ for all other s,t-cuts, $\delta(S)$ yields a feasible dual solution of value 5!

Recap: Feasible Widths via Duality

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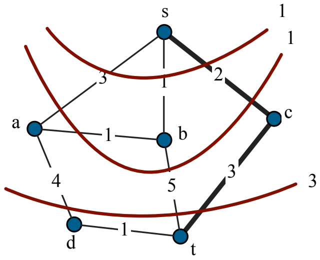
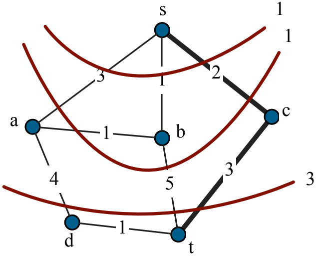


Theorem

[Weak Duality] If $\bar{x}$ is feasible for a shortest path LP and $\bar{y}$ is feasible for its dual, then $b^T\bar{y} \leq c^T\bar{x}$.

$\longrightarrow$ The **bold** path in the figure is a shortest s,t-path!

Recap: Feasible Widths via Duality



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Today:

1. How did we find the bold path?
2. How did we find the dual solution?
3. Is there always a dual solution whose values matches the length of a shortest s,t-path?

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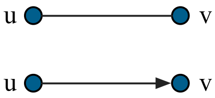
An Algorithm for the Shortest s, t-Path Problem

Arcs and Directed Paths

So far, we know that edges of a graph $G = (V, E)$ are unordered pairs of vertices.

Now we'll introduce arcs – ordered pairs of vertices. We denote an arc from u to v as $\overrightarrow{uv}$, and draw it as an arrow from u to v.



A directed path is then a sequence of arcs

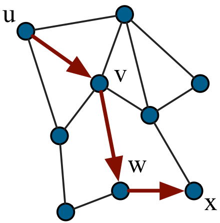
$$\overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \ldots, \overrightarrow{v_{k-1} v_k},$$

where $\overrightarrow{v_i v_{i+1}}$ is an arc in the given graph, and $v_i \neq v_j$ for all $i \neq j$.

Example:

$$\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{wx}$$

is a u, x-dipath.



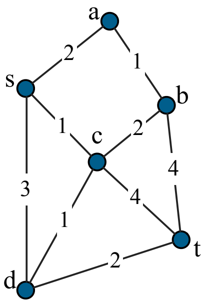
Shortest Paths: Algorithmic Ideas

Idea: Find an s, t-path P and a feasible dual y, s.t. $c(P) = \mathbb{1}^T y$. How do we do this?

Definition

Let y be a feasible dual solution. The slack of an edge $e \in E$ is defined as

$$\text{slack}_y(e) = c_e - \sum (y_U \,:\, \delta(U)\ s,t\text{-cut},\ e \in \delta(U))$$



Recall the shortest path dual:

$$\begin{aligned} \max \quad & \sum (y_S \,:\, \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S \,:\, e \in \delta(S)) \le c_e \quad (e \in E) \\ & y \ge \mathbb{0} \end{aligned}$$

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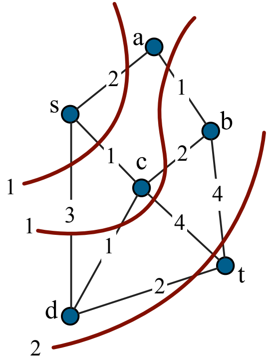


$$\begin{aligned} \max \quad & \sum(y_S \ : \ \delta(S) \ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum(y_S \ : \ e \in \delta(S)) \le c_e \\ & \qquad (e \in E) \\ & y \ge \mathbb{0} \end{aligned}$$

Examples: for the dual y given on the right,

- $\text{slack}_y(sa) = 2 - 1 = 1$
- $\text{slack}_y(sd) = 3 - 1 - 1 = 1$
- $\text{slack}_y(ct) = 4 - 1 - 2 = 1$

We start with the trivial dual $y = \mathbb{0}$.

The simplest s,t-cut is $\delta(\{s\})$.
$\longrightarrow$ Increase $y_{\{s\}}$ as much as we can while still maintaining feasibility
$\longrightarrow$ $y_{\{s\}} = 1$

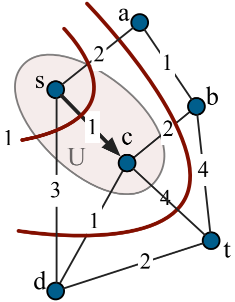
Note: This decreases the slack of sc to 0!
$\longrightarrow$ Replace sc by $\overrightarrow{sc}$

Next we look at all vertices that are reachable from s via directed paths:

$$U = \{s, c\}$$

and consider increasing y_U.

Q: By how much can we increase y_U?



$$\max \quad \sum (y_S \,:\, \delta(S)\ s,t\text{-cut})$$

$$\text{s.t.} \quad \sum (y_S \,:\, e \in \delta(S)) \leq c_e \quad (e \in E)$$

$$y \geq \mathbb{0}$$

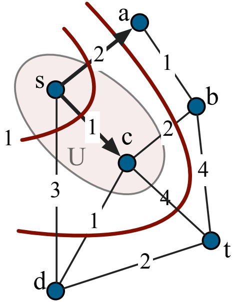
Shortest Paths: Building Duals Incrementally

Q: By how much can we increase y_U?

The maximum increase possible for $y_{\{s,c\}}$ is determined by the slack of edges in $\delta(\{s,c\})$!

$$\begin{aligned} \text{slack}_y(sa) &= 2-1=1 \\ \text{slack}_y(cb) &= 2 \\ \text{slack}_y(ct) &= 4 \\ \text{slack}_y(cd) &= 1 \\ \text{slack}_y(sd) &= 3-1=2 \end{aligned}$$

Edges cd and sa minimize slack. If we pick one arbitrarily, sa for example, we can then set $y_U = \text{slack}_y(sa) = 1$ and convert sa into arc $\overrightarrow{sa}$.



$$\begin{aligned} \max \quad & \sum (y_S : \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_S : e \in \delta(S)) \leq c_e \\ & (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$

Shortest Paths: Building Duals Incrementally

Q: Which vertices are reachable from s via directed paths?

$$U = \{s, a, c\}$$

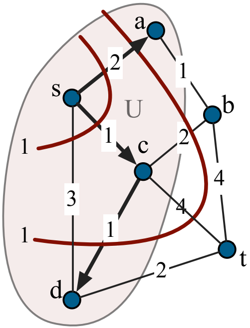
Natural idea: Increase $y_{\{s,a,c\}}$ by as much as we can. How much is this?
$\longrightarrow$ the slack of cd is 0, and hence

$$y_{\{s,a,c\}} = 0$$

Also: we can change cd into $\overrightarrow{cd}$ and let

$$U = \{s, a, c, d\}$$

be the reachable vertices from s.



$$\begin{array}{ll} \max & \sum (y_S : \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} & \sum (y_S : e \in \delta(S)) \le c_e \\ & \qquad (e \in E) \\ & y \ge \mathbb{0} \end{array}$$

The vertices reachable from s by directed paths are in

$$U = \{s, a, c, d\}$$

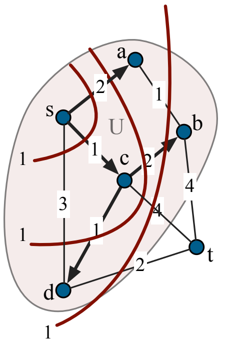
Let us compute the slack of edges in $\delta(U)$.

$$\begin{aligned}
\text{slack}_y(ab) &= 1 \\
\text{slack}_y(cb) &= 2 - 1 = 1 \\
\text{slack}_y(ct) &= 4 - 1 = 3 \\
\text{slack}_y(dt) &= 2
\end{aligned}$$

We let $y_{\{s,a,c,d\}} = 1$, add the equality arc $\overrightarrow{cb}$, and update the set

$$U = \{s, a, b, c, d\}$$

of vertices reachable from s.



$$\begin{aligned}
\max \quad & \sum (y_S : \delta(S) \ s,t\text{-cut}) \\
\text{s.t.} \quad & \sum (y_S : e \in \delta(S)) \leq c_e \quad (e \in E) \\
& y \geq \mathbb{0}
\end{aligned}$$

Shortest Paths: Building Duals Incrementally

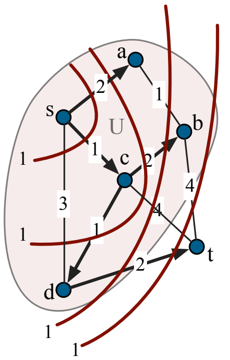
The vertices reachable from s by directed paths are now in

$$U = \{s, a, b, c, d\}$$

Let us compute the slack of edges in $\delta(U)$:

$$\begin{aligned} \text{slack}_y(bt) &= 4 \\ \text{slack}_y(ct) &= 4 - 2 = 2 \\ \text{slack}_y(dt) &= 2 - 1 = 1 \end{aligned}$$

We let $y_{\{s,a,b,c,d\}} = 1$ and add the equality arc $\overrightarrow{dt}$.



$$\begin{aligned} \max \quad & \sum(y_S : \delta(S)\ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum(y_S : e \in \delta(S)) \le c_e \\ & \qquad (e \in E) \\ & y \ge \mathbb{0} \end{aligned}$$

Shortest Paths: Building Duals Incrementally

Note: We now have a directed s, t-path in our graph:

$$P = \overrightarrow{sc}, \overrightarrow{cd}, \overrightarrow{dt},$$

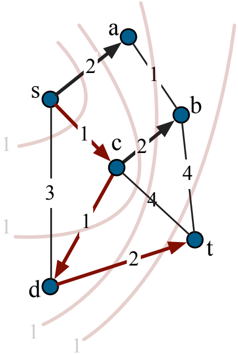
Its length is 4 and its value if 4!

We also have a feasible dual solution:

$$y_{\{s\}} = y_{\{s,c\}} = y_{\{s,a,c,d\}} = y_{\{s,a,b,c,d\}} = 1,$$

and $y_U = 0$ otherwise.

Therefore, we know that path P is a shortest path!



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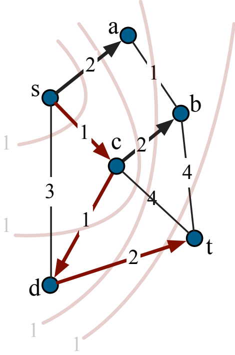
To compute the shortest Path for the instance on the right, we used the following algorithm:

Algorithm 3.2 Shortest path.

Input: Graph $G = (V, E)$, costs $c_e \geq 0$ for all $e \in E$, $s, t \in V$ where $s \neq t$.

Output: A shortest st-path P

```
y_W := 0 for all st-cuts δ(W). Set U := {s}
while t ∉ U do
    Let ab be an edge in δ(U) of smallest slack for y where a ∈ U, b ∉ U
    y_U := slack_y(ab)
    U := U ∪ {b}
    change edge ab into an arc →ab
end while
return A directed st-path P.
```



Recap

- We saw a shortest path algorithm that simultaneously computes
  (a) an s, t-path P, and
  (b) a feasible solution y for the dual of the shortest path LP.
- We will soon show that the length of the output path P, and the value of the dual solution y are the same, thus showing that both P and y are optimal.