CO 250 – LP Formulations

Example: WatColor and WatSchool

WatSchool needs to buy paint for classes. All the paint is going to be bought from the WatColor store, which sells paint sets. Each of the paint sets at the WatColor store has 3 colors (red, blue, yellow) and each such paint set consists of 7 bottles of paint: 4 bottles of one color, 2 bottles of second color, and 1 bottle of third (e.g., 4 bottles of red, 2 bottles of yellow and 1 bottle of blue). Thus, the WatColor store offers $6 = 3!$ different types of paint sets. Each paint set costs 1 dollar independently of its type. WatSchool would like to have at least 27 bottles of red paint, 32 bottles of blue paint and 10 bottles of yellow paint.

Formulate an LP to buy the required amount of paint at minimal cost.

Formulation:

We are going to introduce variables not for each color, but for each possible paint set. Namely, we introduce a variable $x_{(i,j,k)}$ to denote the number of bought paint sets with i bottles of red color, j bottles of blue color and k bottles of yellow color, where $\{i, j, k\} = \{1, 2, 4\}$.

Since each paint set costs 1 dollar independently of its type, the objective function is as follows

$$\min \qquad x_{(1,2,4)} + x_{(1,4,2)} + x_{(2,1,4)} + x_{(2,4,1)} + x_{(4,1,2)} + x_{(4,2,1)}\,.$$

We now need to ensure that the amount of red paint required by WatSchool is bought. This can be done by imposing the following inequality.

$$x_{(1,2,4)} + x_{(1,4,2)} + 2x_{(2,1,4)} + 2x_{(2,4,1)} + 4x_{(4,1,2)} + 4x_{(4,2,1)} \geq 27\,.$$

The corresponding linear constraints for blue and yellow color are as follows

$$2x_{(1,2,4)} + 4x_{(1,4,2)} + x_{(2,1,4)} + 4x_{(2,4,1)} + x_{(4,1,2)} + 2x_{(4,2,1)} \geq 32\,.$$

and

$$4x_{(1,2,4)} + 2x_{(1,4,2)} + 4x_{(2,1,4)} + x_{(2,4,1)} + 2x_{(4,1,2)} + x_{(4,2,1)} \geq 10\,.$$

Thus, the overall LP problem for WatSchool is going to have the following form

$$
\begin{array}{ll}
\min & x_{(1,2,4)} + x_{(1,4,2)} + x_{(2,1,4)} + x_{(2,4,1)} + x_{(4,1,2)} + x_{(4,2,1)} \\
\text{subject to} & \\
& x_{(1,2,4)} + x_{(1,4,2)} + 2x_{(2,1,4)} + 2x_{(2,4,1)} + 4x_{(4,1,2)} + 4x_{(4,2,1)} \geq 27 \\
& 2x_{(1,2,4)} + 4x_{(1,4,2)} + x_{(2,1,4)} + 4x_{(2,4,1)} + x_{(4,1,2)} + 2x_{(4,2,1)} \geq 32 \\
& 4x_{(1,2,4)} + 2x_{(1,4,2)} + 4x_{(2,1,4)} + x_{(2,4,1)} + 2x_{(4,1,2)} + x_{(4,2,1)} \geq 10 \\
& x \geq 0.
\end{array}
$$

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Example: Cheese Factory

Cheese Factory has a production plan for next three months: January, February and March. To fulfil this production plan Cheese Factory needs the following amount of milk in each of these months:

Month	January	February	March
Demand (liters)	3000	5000	4000

Cheese Factory purchases milk at the beginning of each month and the price for milk is known in advance:

Month	January	February	March
Price (dollars per liter)	0.50	0.60	0.70

Cheese Factory has a storage tank of total capacity 3000 litres, which is filled with 1000 liters at the beginning of January. All milk not used in the current month can be stored in the storage tank for the next month.

Find an LP formulation to minimize the cost of buying milk.

Formulation:

Let us introduce variables p_1, p_2 and p_3 for the amount of milk bought at the beginning of January, February and March, respectively. Variables t_1, t_2 and t_3 denote the amount of milk in the storage tank at the beginning of January, February and March, respectively.

The total cost of purchased milk equals:

$$0.50p_1 + 0.60p_2 + 0.70p_3\,.$$

First, we know the initial amount of milk in the tank, i.e. the amount of milk at the beginning of January:

$$t_1 = 1000\,.$$

The total amount of milk that factory has at the beginning of January consists of t_1 liters in the tank and p_1 liters of milk purchased in January. This, amount should satisfy the demand of 3000 liters and the rest should be stored in the tank for February. Hence:

$$t_1 + p_1 = 3000 + t_2\,.$$

Similarly, for February we get:

$$t_2 + p_2 = 5000 + t_3 \,.$$

Finally, the amount of milk available in March equals $t_3 + p_3$. This amount should be enough to cover the demand in March, which equals 4000 liters:

$$t_3 + p_3 \geq 4000 \,.$$

The storage tank has a limited capacity, thus at the beginning of each month the amount of milk in the tank is bounded by 3000 liters:

$$t_1 \leq 3000, t_2 \leq 3000, t_3 \leq 3000$$

Putting all above inequalities together and imposing non-negativity constraints, we get the following LP formulation:

$$\begin{aligned} \min \quad & 0.50p_1 + 0.60p_2 + 0.70p_3 \\ \text{subject to} \quad & \\ & t_1 = 1000 \\ & t_1 + p_1 = 3000 + t_2 \\ & t_2 + p_2 = 5000 + t_3 \\ & t_3 + p_3 \geq 4000 \\ & t_1 \leq 3000, t_2 \leq 3000, t_3 \leq 3000 \\ & p_1 \geq 0, p_2 \geq 0, p_3 \geq 0 \\ & t_1 \geq 0, t_2 \geq 0, t_3 \geq 0 \,. \end{aligned}$$

Remarks:

It is true, that for an optimal solution the constraint for March is going to be tight, i.e.

$$t_3 + p_3 = 4000 \,,$$

since it is not optimal to buy milk, which is not going to be used later. However, since we are interested in all feasible solutions, we write the constraint for March as a linear inequality.

Another remark concerns the amount of milk left after March. The model implicitly assumes that all remaining milk is stored in the storage tank. For example, this is assumed in the constraint $t_1 + p_1 = 3000 + t_2$. However, it is not clear what happens with the milk after March. If one assumes that the milk not used in March should be stored in the storage tank, then we have to impose the constraint

$$t_3 + p_3 - 4000 \leq 3000 \,,$$

so that the storage tank is not overfilled.

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Example: Tom's Jam

Tom makes two types of jam: jam A and jam B. To make one unit of jam Tom needs strawberries, cherries and sugar. The following table shows how many kilograms (kg) of strawberries, cherries and sugar Tom needs to make one unit of each jam.

	Jam A	Jam B
Strawberries	1	3
Cherries	4	1
Sugar	2	3

Tom has 210 kg of strawberries, 280 kg of cherries and 240 kg of sugar. Each unit of jam A brings Tom 1 dollar of profit, while each unit of jam B brings Tom 2 dollars of profit.

Find an LP formulation to maximize the profit of Tom.

Formulation:

Let x_A denote the quantity of jam A made by Tom, and let x_B denote the quantity of jam B made by Tom.

Then the total profit of Tom is given by the following linear function:

$$x_A + 2x_B\,.$$

Let us write linear constraints. The total amount of strawberries used to produce x_A units of jam A and x_B units of jam B:

$$x_A + 3x_B\,.$$

However, only 210 kg of strawberries are available. Thus:

$$x_A + 3x_B \leq 210\,.$$

In the same way, for cherries we get:

$$4x_A + x_B \leq 280$$

and for sugar we get:

$$2x_A + 3x_B \leq 240\,.$$

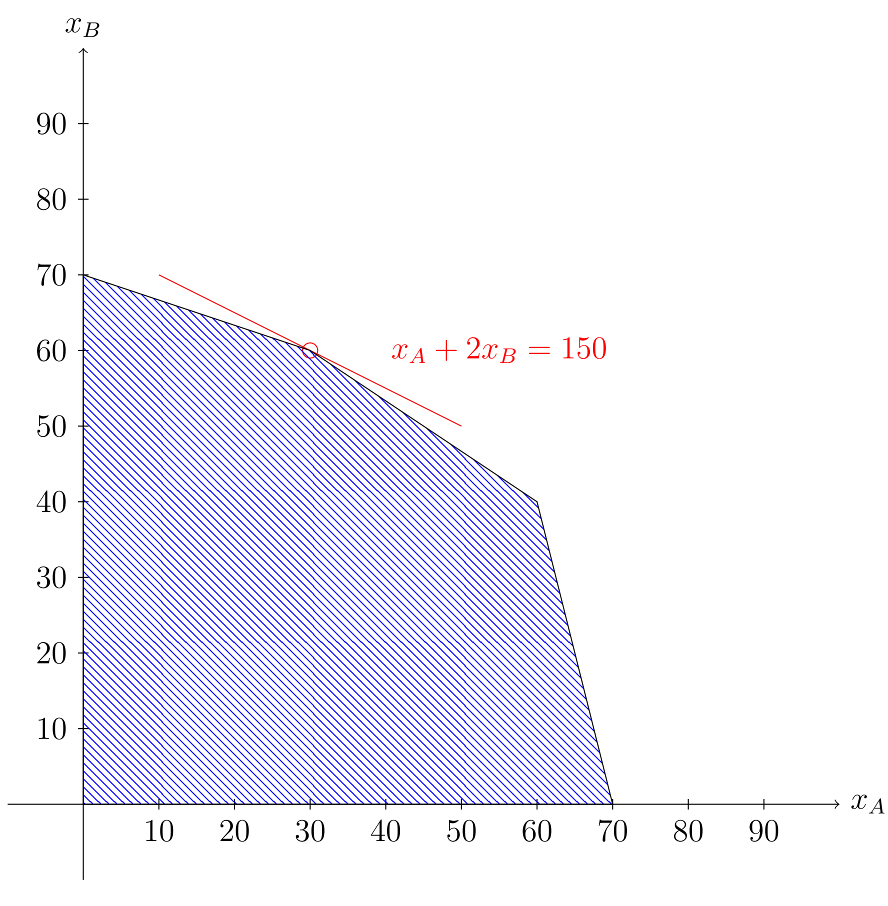
Putting all above inequalities together and imposing non-negativity constraints, we get the following LP formulation

$$
\begin{aligned}
\max \quad & x_A + 2x_B \\
\text{subject to} \quad & \\
& x_A + 3x_B \leq 210 \\
& 4x_A + x_B \leq 280 \\
& 2x_A + 3x_B \leq 240 \\
& x_A \geq 0, x_B \geq 0.
\end{aligned}
$$

Remarks:

It is not hard to check that $x_A = 30$ and $x_B = 40$ is a feasible solution for the LP formulation above, i.e. it satisfies all linear constraints in the above LP formulation.

The set of feasible solutions defines the pentagon depicted below. It is not hard to check that all feasible solutions satisfy $x_A + 2x_B \leq 150$ and for $x_A = 30$, $x_B = 60$ this inequality is tight. Hence, the optimal solution is $(30, 60)$.



CO 250 – LP Formulations

The Transportation Problem

- Factories $1, \ldots, p$, where factory i produces s_i units per month.
- Shops $1, \ldots, q$, where shop j orders t_j units per month (a 'unit' is a single item, such as a 'car', or a 'Dell laptop' or a 'barrel of oil'.)
- Shipping cost from factory i to shop j is c_{ij}/unit.

Goal:

Find the cheapest 'shipping pattern' (send units from factories to shops) while satisfying 'shop demands' and 'factory supplies'.

Example:

Factories \ Shops	1 $t_1 = 35$	2 $t_2 = 45$	3 $t_3 = 45$
1 $s_1 = 50$	1	2	∞
2 $s_2 = 75$	5	2	10

Shipping costs c_{ij}

Formulation:

Key: Choose decision variables!

Denote by x_{ij} the number of units sent from factory i to shop j. Here is one possible solution:

$$[x_{ij}] = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} = \begin{bmatrix} 5 & 45 & 0 \\ 30 & 0 & 45 \end{bmatrix}$$

Recall: shipping costs $[c_{ij}] = \begin{bmatrix} 1 & 2 & \infty \\ 5 & 2 & 10 \end{bmatrix}$.

The cost of the above solution is

$$\sum_{j=1}^{q} \sum_{i=1}^{p} c_{ij} x_{ij} = 1 \cdot 5 + 2 \cdot 45 + 0 + 5 \cdot 30 + 2 \cdot 0 + 10 \cdot 45 = 695.$$

Goal: find solution $[x_{ij}]$ of cheapest cost such that

- $\forall i$: row i of $[x_{ij}]$ sums to s_i, and
- $\forall j$: column j of $[x_{ij}]$ sums to t_j.

These constraints can be written as linear constraints (of the form $a^T x \overset{\geq}{\underset{\leq}{=}} \beta$) as follows:

- $\sum_{j=1}^{q} x_{ij} = s_i$ (row i sums to s_i)
- $\sum_{i=1}^{p} x_{ij} = t_j$ (column j sums to t_j)

LP model for the Transportation Problem:

$$
\begin{array}{lrl}
\text{minimize} & \displaystyle\sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}x_{ij} & \\
\text{subject to} & \displaystyle\sum_{j=1}^{q} x_{ij} = s_i & \forall\, i \in \{1, ..., p\} \\
& \displaystyle\sum_{i=1}^{p} x_{ij} = t_j & \forall\, j \in \{1, ..., q\} \\
& x_{ij} \geq 0 & \forall\, i \in \{1, ..., p\}\ \forall\, j \in \{1, ..., q\}
\end{array}
$$

Remarks:

In the model above, we assumed that the total supply equals the total demand, i.e. $\sum_{i=1}^{p} s_i = \sum_{j=1}^{q} t_j$. Thus we imposed the constraints:

$$\sum_{j=1}^{q} x_{ij} = s_i \qquad \forall\, i \in \{1, ..., p\}\,,$$

since in this case every factory has to ship all produced units to shops. On the other hand, under the same assumption $\sum_{i=1}^{p} s_i = \sum_{j=1}^{q} t_j$, we can instead impose the constraints:

$$\sum_{j=1}^{q} x_{ij} \leq s_i \qquad \forall\, i \in \{1, ..., p\}\,,$$

obtaining the same set of feasible solutions to the LP formulation. Hence, the following LP model for the Transportation Problem is also correct:

$$
\begin{aligned}
\text{minimize} \qquad & \sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}x_{ij} \\
\text{subject to} \qquad & \sum_{j=1}^{q} x_{ij} \leq s_i \qquad \forall\, i \in \{1,...,p\} \\
& \sum_{i=1}^{p} x_{ij} = t_j \qquad \forall\, j \in \{1,...,q\} \\
& x_{ij} \geq 0 \qquad \forall\, i \in \{1,...,p\}\ \forall\, j \in \{1,...,q\}
\end{aligned}
$$