#1. [15 marks] (**Simplex method**)
Consider the following linear programming problem (P) in SEF. Note that (P) is in canonical form for the feasible basis $B = \{3, 4\}$.

$$
(P) \qquad \begin{array}{llrcrcrcrcr}
\text{maximize} & z = & 3x_1 & + & 4x_2 & & & & & & \\
\text{subject to} & & & & & & & & & & \\
& & x_1 & + & x_2 & + & x_3 & & & = & 40 \\
& & -x_1 & + & x_2 & & & + & x_4 & = & 60 \\
& & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}
$$

[10] (a) Solve (P) using the Simplex method, starting with $B = \{3, 4\}$. Choose the entering variable (respectively, leaving variable) using the smallest subscript rule, that is, pick the variable with the smallest index if there is a choice. **Stop after 3 iterations**, even if your solution is incomplete. You may use the following formula for the inverse of a 2×2 matrix:

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
$$

Show all of your work. (You may use either canonical form LPs (Chapter 2.5), or tableaus (Chapter 2.7).)

[5] (b) If (P) has an optimal solution, then write down an optimal basic feasible solution, and write down a certificate of optimality (briefly explain the validity of your certificate).
Otherwise, if (P) is unbounded, then write down a certificate of unboundedness (briefly explain the validity of your certificate).

#2. [20 marks] (**Duality**)

(a) Consider the following linear programming problem (P)

$$\min \begin{pmatrix} 1 & \alpha & 10 & \frac{25}{2} \end{pmatrix} x$$

$$\begin{pmatrix} 1 & 0 & 5 & 6 \\ 2 & 6 & 0 & 1 \end{pmatrix} x \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ free },\ x_4 \geq 0$$

See next page for part (b).

[7] (1) Write down the dual (D) of (P) and the complementary slackness conditions.

[6] (2) For what values of α is the solution $\bar{x}^T = (1\ 0\ 0\ 0)$ optimal for (P)? Justify your answer.

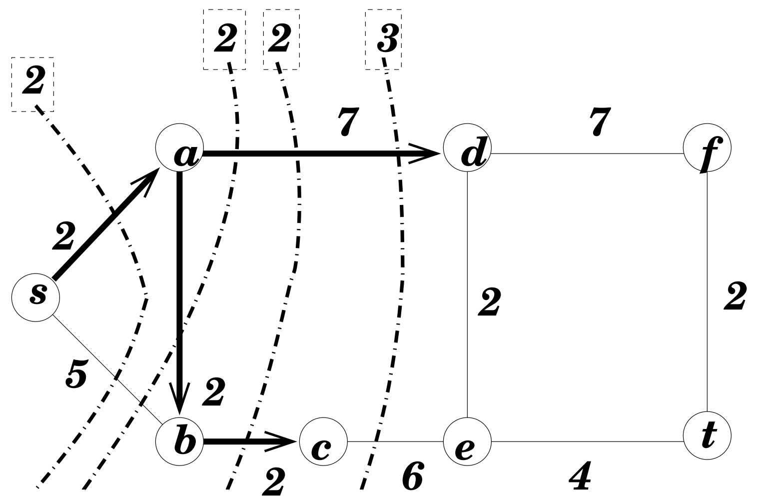
(#2, continued)

[7] (b) Consider the following linear programming problem:

$$\text{(P)} \qquad \min\{c^T x : Ax \geq b, x \leq 0\}.$$

Write the dual (D) of (P).
<u>Prove</u>: the objective value $b^T \bar{y}$ of every feasible solution $\bar{y}$ of (D) gives a lower bound on the objective value $c^T \bar{x}$ of any feasible solution $\bar{x}$ to (P).

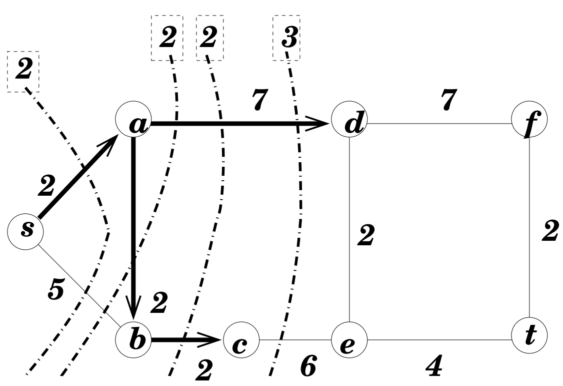
#3. [15 marks] (**Shortest paths**)
The figure on the left below shows a graph $G = (V, E)$. Each of the vertices is labelled by its name, and each edge $e \in E$ has its length c_e next to it. The figure also shows a feasible dual solution that resulted from executing the shortest path algorithm for four iterations. Each of the st-cuts computed by the algorithm is shown in the figure by a dashed curve; the value of the corresponding dual variable is placed next to its st-cut (in a box); the arcs (directed edges) computed by the algorithm are shown as thick lines with arrows. The table on the right below repeats this information for clarity.

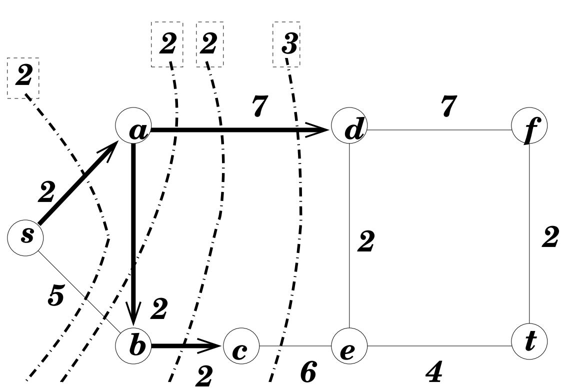


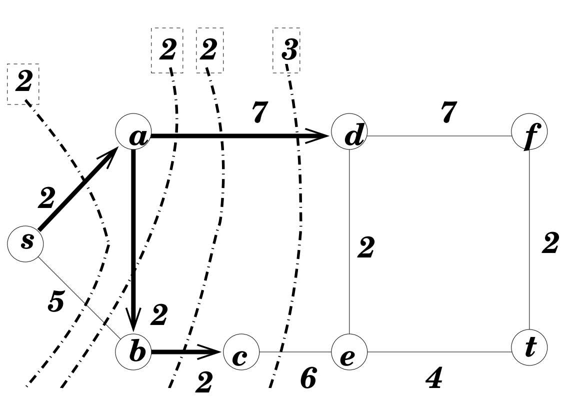
i	U_i	y_{U_i}	new arc
1	$\{s\}$	2	$\vec{sa}$
2	$\{s, a\}$	2	$\vec{ab}$
3	$\{s, a, b\}$	2	$\vec{bc}$
4	$\{s, a, b, c\}$	3	$\vec{ad}$

[8] (a) Starting from the given state, run the shortest path algorithm till a shortest st-path is found. Write down the resulting path (optimal solution) and certify its optimality.
Show all of your work; in particular, in each iteration, show the relevant st-cut, all edges in that st-cut together with their slacks, and show the arc (directed edge) added in that iteration.
Extra copies of the figure are reproduced below, and you may use these to show your work.
STOP after 3 iterations, even if you cannot compute a shortest st-path.

See next page for part (b).







(#3, continued)

[7] (b) (This part is not directly related to the example in the previous part.)

(1) Write down an integer programming formulation (IP) of the Shortest Paths Problem. In more detail, given any graph $G = (V, E)$, two vertices s, t of V, and a positive edge length c_e for each edge $e \in E$, (IP) should formulate the problem of finding an st-path of minimum length.

(2) Let (P) denote the linear programming relaxation of (IP). Write down the dual (D) of (P).

#4. [15 marks] (**Integer Programming**)
Consider the following integer programming problem (IP):

$$
\begin{array}{llcccccccl}
\text{(IP)} & \max & -2x_1 & & -x_3 & & -1/2\,x_5 & & +11 \\
\text{s.t.} & & 1/2\,x_1 & +x_2 & -1/3\,x_3 & & +x_5 & & = 8/3 \\
& & 2x_1 & & -1/4\,x_3 & +x_4 & +1/5\,x_5 & & = 2 \\
& & 5/2\,x_1 & & +2/7\,x_3 & & -5/3\,x_5 & +x_6 & = 7/2 \\
& & \multicolumn{7}{l}{x \geq 0 \text{ and integer}}
\end{array}
$$

[1] (a) Write down an optimal solution x^* of the LP relaxation of (IP). (No explanations needed.)

[5] (b) For which of the equality constraints of the LP relaxation of (IP) can you derive a cutting plane for x^*? Explain your answer in brief.

[5] (c) For each of the constraints you listed in part (b), write down a cutting plane that can be derived from it. (No explanations needed.)

[4] (d) Explain why the inequality $x_1 + x_2 + x_5 \geq 3$ is a valid cutting plane for x^*.
(**Hint**: Start from the first constraint, and use arguments similar to the arguments used to derive the cutting planes in the course notes.)

#5. [15 marks] (**Nonlinear optimization**)
Consider the following convex non-linear programming problem (NLP) (you may assume that the relevant functions are convex):

$$\begin{array}{lll} \text{(NLP)} & \min & -3x_1 + 4x_2 \\ \text{s.t.} & & x_1^2 + 2x_2 - 9 \leq 0 \\ & & x_1 - x_2 - 3 \leq 0 \end{array}$$

[3] (a) Write down a Slater point x^* for (NLP). Briefly explain why x^* is a Slater point.

[7] (b) Find all possible vectors $x' \in \mathbb{R}^2$ such that both constraints hold as equations (that is, find all solutions to $x_1^2 + 2x_2 - 9 = 0$ and $x_1 - x_2 - 3 = 0$).
<u>Prove</u> that one of these vectors is an optimal solution for (NLP).
(**Hint**: Use the KKT conditions.)
See next page for part (c).

(#5, continued)

The convex non-linear programming problem (NLP) is reproduced below for convenience.

$$\begin{array}{lll} \text{(NLP)} & \min & -3x_1 + 4x_2 \\ \text{s.t.} & & x_1^2 + 2x_2 - 9 \leq 0 \\ & & x_1 - x_2 - 3 \leq 0 \end{array}$$

[5] (c) Does (NLP) have any other optimal solution, <u>different</u> from the ones you listed in the previous part? Justify your answer.
(**Hint**: Use the KKT conditions. Let $\hat{x}$ be another optimal solution. Then, either none of the constraints is tight for $\hat{x}$, or exactly one of constraints is tight for $\hat{x}$.)

#6. [20 marks] (**Theory**)

[7] (a) Consider the integer programming problem

$$\text{(IP)} \quad \max\{c^T x : Ax = b, x \geq 0, x \text{ integer } \}$$

Let $(P) := \max\{c^T x : Ax = b, x \geq 0\}$ be the linear programming relaxation of (IP), and let $(D) := \min\{b^T y : A^T y \geq c\}$ be the dual of (P).

For each of the following statements, answer whether it is **True** or **False**, and explain your answer in brief.

(1) Let $\bar{x}$ be a *feasible* solution for (IP) and let $\hat{x}$ be a *feasible* solution for (P) such that $c^T\bar{x} = c^T\hat{x}$. Then $\bar{x}$ is optimal for (IP).

(2) Let $\bar{x}$ be a *feasible* solution for (IP) and let $\bar{y}$ be a *feasible* solution for (D) such that $c^T\bar{x} = b^T\bar{y}$. Then $\bar{x}$ is optimal for (IP).

See next page for parts (b,c,d).

(#6, continued)

[6] (b) Answer whether the following statement is **True** or **False**, and explain your answer in brief. (If True, give a proof, and if False, give a counter-example or complete explanation.)
Let $C_1 \subseteq \mathbb{R}^n$ and $C_2 \subseteq \mathbb{R}^n$ be convex sets. Then $C_1 \cap C_2$ is a convex set.

[2] (c) Let S be a subset of $\mathbb{R}^n$. Define what is meant by the *convex hull* of S.

[5] (d) Let S be the set of feasible solutions of a given integer programming problem. Let the convex hull of S be $\text{conv}(S) := \{x \in \mathbb{R}^n : A'x \leq b'\}$.
Prove: if $\bar{x}$ is an extreme point of $\text{conv}(S)$, then $\bar{x}$ is integer.
Note: You may use the following result without proving it:
($\star$) Given any fractional (i.e., non-integer) extreme point $\hat{x}$ of a polyhedron that contains S, there exists a cutting plane $c^T x \leq \beta$ for $\hat{x}$.