<u>Problem 1:</u> Extreme points of polyhedra **(14 marks)**

(a) Explain why $\bar{x} = (0, 0, 2, 0, 0)^\top$ is an extreme point of the following polyhedron.

$$\left\{ x \in \mathbb{R}^5 : \begin{pmatrix} 1 & 0 & -2 & 0 & 3 \\ 1 & 2 & 1 & -3 & 0 \end{pmatrix} x = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad x \geq \mathbb{0} \right\}.$$

(4 marks)

(b) Consider the polyhedron Q defined by the following constraints:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 1 \\ 4 \\ 10 \end{pmatrix}.$$

Find an extreme point $\bar{x}$ of Q and explain why $\bar{x}$ is an extreme point. (5 marks)

(c) Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Consider the polyhedron P defined as

$$P = \{x \in \mathbb{R}^n : Ax \leq b, \quad 0 \leq x_i \leq 3 \text{ for all } i = 1, \ldots, n\}.$$

Prove that every point $\bar{x} \in P$, where each entry of $\bar{x}$ is either 0 or 3 (i.e., $\bar{x}_i \in \{0, 3\}$ for all $i = 1, \ldots, n$) is an extreme point of P. (5 marks)

<u>Problem 2:</u> Two-phase simplex method **(16 marks)**

(a) Use the simplex method to find a basic feasible solution to the following LP, or determine that the LP is infeasible. *Use Bland's rule to break ties in the choice of entering and leaving variables.* (You are **not** asked to find an optimal solution to the LP, but only to find a feasible solution if one exists.)

$$\begin{aligned} \max \quad & (5, -1, 4, 1, 0)x \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ -1 & 2 & -3 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{aligned}$$

(8 marks)

If you find it helpful, the inverse of a non-singular 2×2 matrix is given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(b) The following LPs *cannot appear* (as canonical forms) in any of the steps of Phase 1 of the simplex method. Explain why this is the case.

Each part below corresponds to a different LP. In all parts below, the auxilliary problem constructed in Phase 1 has objective function $\max w(x) = -x_5 - x_6$, where x_5 and x_6 are auxilliary variables.

(i)

$$
\begin{array}{llrrrrrrrrrrrrrl}
\max & w(x) = & - & x_1 & + & x_2 & - & 2x_4 & & & - & x_5 & & & + & 1 \\
\text{s.t.} & & & 2x_1 & - & x_2 & + & x_3 & - & x_4 & + & 3x_5 & & & = & 2 \\
& & - & x_1 & + & x_2 & & & + & x_4 & - & x_5 & + & x_6 & = & 3 \\
& & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0.
\end{array}
$$

(2 marks)

(ii)

$$
\begin{array}{llrrrrrrrrrrrrrl}
\max & w(x) = & & x_1 & & & - & x_3 & - & 2x_4 & & & - & 4x_6 & - & 1 \\
\text{s.t.} & & - & x_1 & + & x_2 & + & 3x_3 & + & x_4 & & & + & 2x_6 & = & 4 \\
& & - & 3x_1 & & & - & x_3 & - & x_4 & + & x_5 & - & x_6 & = & 1 \\
& & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0.
\end{array}
$$

(3 marks)

(iii)

$$
\begin{array}{llrrrrrrrrrrrrrl}
\max & w(x) = & - & x_1 & & & & & & & - & x_5 & - & x_6 & + & 0 \\
\text{s.t.} & & & 2x_1 & + & x_2 & - & x_3 & & & - & x_5 & + & 2x_6 & = & 1 \\
& & & x_1 & & & + & x_3 & + & x_4 & & & - & x_6 & = & 2 \\
& & & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0.
\end{array}
$$

(3 marks)

Problem 3: Duality **(20 marks)**

(a) Consider the following linear program.

$$\begin{array}{llcrcrcl}
\max & 3x_1 & - & 3x_2 & + & x_3 & & \\
\text{s.t.} & 3x_1 & - & x_2 & - & x_3 & \leq & 15 \\
 & x_1 & - & x_2 & + & x_3 & = & 2 \\
 & x_1 & - & x_2 & + & 2x_3 & \geq & 1 \\
 & x_1, & x_2 & \geq 0, & & x_3 & \leq & 0.
\end{array} \tag{P}$$

Write down the dual (D) of (P) and the complementary slackness conditions for (P) and (D).
(6 marks)

(b) Use part (a) to show that $x^* = (3, 0, -1)^\top$ is an optimal solution to (P). Is this the unique optimal solution to (P)? Justify your answer. (7 marks)

(c) Let (P') be the LP: $\max\ \{c^\top x : Ax = b,\ x \geq \mathbb{0}\}$, and let (D') denote the dual of (P'). Let B be a basis, and let the canonical form of (P') for B be given by the following LP:

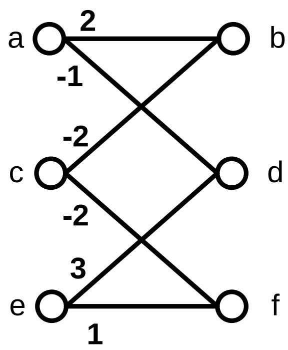
$$\max \quad \bar{c}_N^\top x_N + \bar{z} \qquad \text{s.t.} \qquad x_B + \bar{A}_N x_N = \bar{b}, \quad x \geq \mathbb{0}. \tag{P''}$$

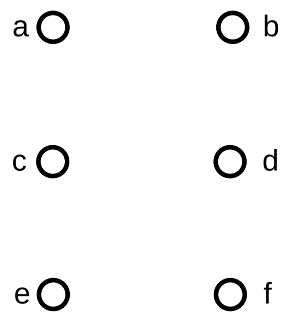
Every basic variable x_i appears in exactly one constraint of (P''), so we index the rows of $\bar{A}$, and the coordinates of $\bar{b}$ by B. Suppose that $\bar{c}_N^\top \leq \mathbb{0}$, and there is some $i \in B$ such $\bar{A}_{ij} \geq 0$ for all $j \in N$ and $\bar{b}_i < 0$. Prove that (D') is unbounded. (7 marks)

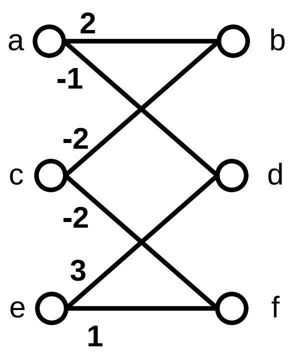
(**Hint:** Use the formulae giving the canonical form for B to obtain a solution to (D'). Deduce something about the outcome of (P''), and hence (P'), and use duality.)

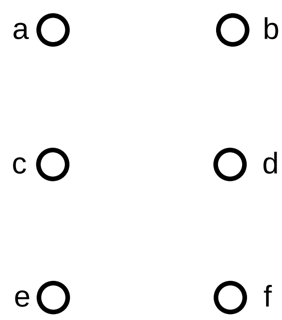
Problem 4: Minimum-cost perfect matching **(17 marks)**

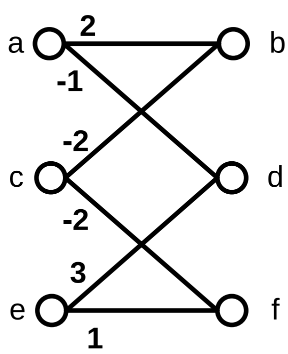
(a) Consider the bipartite graph $G = (V, E)$ shown below. The numbers labeling the edges indicate the edge costs $\{c_e\}_{e \in E}$. Apply the minimum-cost perfect-matching algorithm to find a matching with minimum cost or a deficient set in G. With every iteration, specify the current dual solution and how it is updated. Use the copies of G and its node-set V given below to show your work. You do **not** need to explain why your answer is correct. (8 marks)

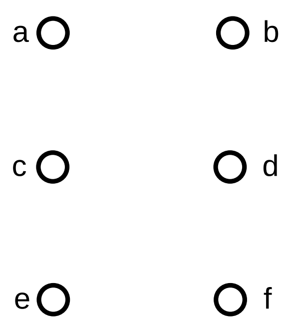


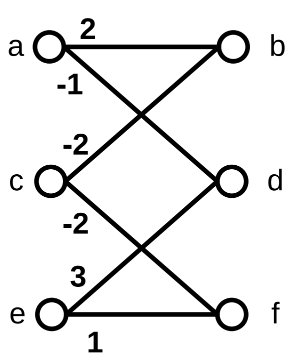


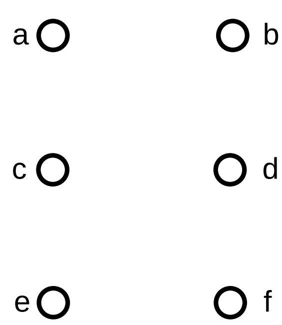












(b) Recall that the LP-relaxation of the IP formulation of the minimum-cost perfect-matching problem for a graph $G = (V, E)$ is as follows.

$$\begin{array}{lll} \min & \sum(c_e x_e : e \in E) & \\ \text{s.t.} & \sum(x_e : e \in \delta(v)) = 1 & \forall v \in V \\ & x_e \geq 0 & \forall e \in E. \end{array} \tag{P}$$

Write down the dual LP (D) and the complementary slackness conditions. (5 marks)

(c) Consider the primal-dual pair of LPs from part (b) for the min-cost perfect-matching instance of part (a). Using the results of part (a), give an optimal solution $\bar{x}$ for (P) and an optimal solution $\bar{y}$ for (D). Briefly justify why your solutions are optimal. (4 marks)

<u>Problem 5:</u> Cutting planes **(15 marks)**

(a) Consider the following integer program.

$$
\begin{array}{lrrrrrrrrl}
\max & & & - & x_3 & & & - & 2x_5 & + \ 16 \\
\text{s.t.} & x_1 & & + & \frac{1}{3}x_3 & & & + & \frac{1}{4}x_5 & = \ \frac{9}{4} \\
 & & x_2 & + & \frac{1}{2}x_3 & & & - & \frac{1}{2}x_5 & = \ \frac{5}{3} \\
 & & & & \frac{2}{3}x_3 & + & x_4 & + & \frac{1}{2}x_5 & = \ 2 \\
 & & & & x \geq \mathbb{0}, & & x \text{ integer.} & & &
\end{array} \tag{IP}
$$

Find an optimal solution x^* to the LP-relaxation of (IP). Write down all the cutting planes for x^* that can be obtained from the above equality constraints of (IP). No justification is needed. (3 marks)

(b) Explain why $\frac{1}{2}x_3 + \frac{1}{2}x_5 \geq \frac{2}{3}$ is a cutting plane for x^*. Using this inequality (or otherwise), justify why the inequality $x_3 + x_5 \geq 2$ is a cutting plane for x^*. (4 marks)

(c) Answer True or False: Let (IP’) be an integer program and (LP’) be its LP-relaxation. Suppose that the feasible region of (LP’) is the convex hull of the feasible region of (IP’). One can find an inequality that is valid for (IP’), but is not valid for (LP’) (i.e., is violated by some feasible solution to (LP’)). *Justify your answer.* (Answers with no justification will not receive any credit.) (3 marks)

(d) Let (IP’) be a maximization integer program and (LP’) be its LP-relaxation. Show that if the optimal value of (LP’) is strictly larger than the optimal value of (IP’), then one can find an inequality that is valid for (IP’) but is not valid for (LP’). (5 marks)

<u>Problem 6:</u> Convexity and convex NLPs **(18 marks)**

(a) Consider the following NLP, which you assume is a convex NLP for all $c_1 \geq 0$, $c_2, c_3 \in \mathbb{R}$.

$$\begin{array}{lrcl} \min & c_1 x_1^2 + c_2 x_1 + c_3 x_2 + 2 & & \\ \text{s.t.} & 2x_1^2 + x_2^2 - x_1 x_2 - 4 & \leq & 0 \\ & x_1 - 2 & \leq & 0 \\ & 3x_1 - x_2 - 1 & \leq & 0 \end{array} \tag{NLP}$$

Find values of $c_1 \geq 0$, c_2, and c_3, other than $(c_1, c_2, c_3) = (0, 0, 0)$ (one specific vector (c_1, c_2, c_3) is enough) such that the point $(x_1, x_2) = (1, 2)$ is optimal for (NLP). (7 marks)

(b) Consider the NLP of part (a) for $c = (1, 1, 1)$, which is listed below for convenience.

$$\begin{array}{lrcl} \min & x_1^2 + x_1 + x_2 + 2 & & \\ \text{s.t.} & 2x_1^2 + x_2^2 - x_1 x_2 - 4 & \leq & 0 \\ & x_1 - 2 & \leq & 0 \\ & 3x_1 - x_2 - 1 & \leq & 0 \end{array} \tag{NLP'}$$

By using the KKT theorem, explain why $(x_1, x_2) = (0, -1)$ is **not** optimal for (NLP'). (6 marks)

(c) Let functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, be convex functions. Prove that the set

$$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0,\ i = 1, \ldots, m\}$$

is a convex set.

(You may use without proof the fact that if $C_1, \ldots, C_m$ are convex sets, then $C = \bigcap_{i=1}^{m} C_i$ is also a convex set.)

(5 marks)