CO250 Online Spring 2018
Final Exam Information

- The exam is 2.5 hours long.
- There are 150 marks available in 150 minutes.
- Breakdown of question types:
  - Roughly 20% of the exam consists of proof type questions
  - Roughly 60% of the exam consists of formulations, running algorithms, performing computations, etc.
  - Roughly 20% of the exam tests your understanding of the formulations, algorithms, computations, etc.
- All topics from the course may be covered, including but not limited to:
  - Formulation of linear/integer programs.
  - Optimality/unboundedness/infeasibility for LPs, IPs and NLPs.
  - Certificates
  - Solving LPs, SEF, basic solutions and canonical forms.
  - Simplex Algorithm; you are expected to be able to run the algorithm, and to understand what is going on.
  - Geometry: convexity, extreme points; demonstrate understanding of these concepts and their relation to the optimization problems we consider in this course.
  - Shortest Path Problem: formulation, algorithm, certification. Make sure you understand all of the concepts well!
  - Duality: find the dual of a given LP; weak/strong duality; complementary slackness, etc.
  - Integer Programs: demonstrate understanding of the difficulties involved; cutting planes, etc.
  - Non-Linear Optimization: KKT, etc.
- Sample final exam questions have been posted on LEARN.
- A few practice problems are included below, and additional practice problems can be found in the textbook.

Question 1. LP Modeling I

The company **C & O** manufactures four main pasta products: Alfabet (A), Bowtie (B), Cappelletti (C) and Ditalini (D). The company uses machines 1,2, and 3 to make these products. It has 200 hours of machine time available each month for each of the machines. The table below shows how many hours on each machine is required to produce one ton of a product and the profit (in dollars) the company will pocket for each ton of a product made.

Product	Pasta (A)	Pasta (B)	Pasta (C)	Pasta (D)
Number of hours of machine 1 required per ton	1	3	2	4
Number of hours of machine 2 required per ton	4	1	2	1
Number of hours of machine 3 required per ton	1	2	2	1
Profit ($ per ton)	$350	$190	$400	$ 300

The company would like to make a production plan for one month (decide how much of each pasta product to make) to maximize its total profit.

Formulate this production planning problem as a Linear Programming problem.

Question 2. LP Modeling II

A company wishes to plan its production of two items with seasonal demands over a 12-month period, starting in January. The monthly demand of item 1 is 100000 units during the months of October, November, and December; 10000 units during the months of January, February, March, and April; and 30000 units during the remaining months. The demand of item 2 is 50000 during the months of October through February and 15000 during the remaining months. Suppose that the unit production cost of items 1 and 2 is $ 5.00 and $ 8.00 respectively, provided that these were manufactured until the end of June. After June, the unit costs are reduced to $ 4.50 and $ 7.00 because of the installation of an improved manufacturing system. The total units of items 1 and 2 that can be manufactured during any particular month cannot exceed 120000. Furthermore, each unit of item 1 occupies 2 cubic feet and each unit of item 2 occupies 4 cubic feet of inventory. Suppose that the maximum inventory space allocated to these items is 150000 cubic feet and that the cost of storing inventory in any month is $0.10 per cubic foot. At the end of the 12-month period, the company wants zero inventory.

Formulate the production scheduling problem so that the total production and inventory costs are minimized.

Question 3. IP Modeling

Zihao would like to organize a party. At the moment, he is thinking about the guest list for his party. There are only five people who volunteered to bring cake to the party: Aarav, Boris, Cedric, Dalilah and Eva. Aarav, Boris, Cedric, Dalilah and Eva will bring 4, 6, 4.5, 5 and 3 pounds of cake respectively, if they come to the party.

The relationships between Aarav, Boris, Cedric, Dalilah and Eva are not simple, so Zihao has to respect following constraints.

(i) If Boris is at the party, then Cedric won't be at the party unless Eva is at the party too.

(ii) Dalilah will be at the party only if Cedric is at the party.

(iii) Aarav will be at the party if at least one of Boris, Cedric and Dalilah comes to the party.

(iv) Eva will be at the party if and only if Aarav is not at the party.

Formulate an Integer Programming (IP) problem so that Zihao has a guest list maximizing the total amount of cake brought to his party.

Solution: Let us introduce variables z_A, z_B, z_C, z_D and z_E, which should indicate whether Aarav, Boris, Cedric, Dalilah and Eva, respectively, are at the party. The objective function to maximize the total amount of cake is:

$$\max 4z_A + 6z_B + 4.5z_C + 5z_D + 3z_E\,.$$

First of all z_A, z_B, z_C, z_D and z_E are binary variables: z_A, z_B, z_C, z_D and $z_E \in \{0, 1\}$.

(i) This constraint can be modelled as

$$z_C + z_B - z_E \leq 1\,.$$

Indeed, the only assignment of values 0, 1 to the variables z_C, z_B, z_E, which violates the above constraint is as follows $z_C = 1$, $z_B = 1$, $z_E = 0$. The assignment $z_C = 1$, $z_B = 1$, $z_E = 0$ corresponds to the only impossible case for the constraint (i): Cedric and Boris are at the party and Eva is not at the party.

(ii) This constraint can be modelled as

$$z_D \leq z_C\,.$$

The above linear constraint is equivalent to the constraint: z_D equals 1 only if z_C equals 1; which is equivalent to the constraint (ii).

(iii) This constraint can be modelled as

$$\begin{aligned} z_A &\geq z_B \\ z_A &\geq z_C \\ z_A &\geq z_D\,. \end{aligned}$$

The above linear constraints are equivalent to the constraint: z_A equals 1 if at least one of z_B, z_C and z_D equals 1; which is equivalent to the constraint (iii).

(iv) This constraint can be modelled as

$$z_A + z_E = 1\,.$$

The above linear constraint is equivalent to the constraint: z_A equals 1 only if z_E equals 0;

As the result, we get the following (IP) problem:

$$\max 4z_A + 6z_B + 4.5z_C + 5z_D + 3z_E$$

subject to:

$$z_C + z_B - z_E \leq 1$$

$$z_D \leq z_C$$

$$z_A \geq z_B$$

$$z_A \geq z_C$$

$$z_A \geq z_D$$

$$z_A + z_E = 1$$

$$z_A, z_B, z_C, z_D, z_E \in \{0, 1\}.$$

Question 4. Outcomes of LPs

Let $x = (x_1, x_2, \dots, x_n)^T$.

(a) Consider the following linear program:

$$\max \{c^T x : Ax = \mathbb{0},\ x \geq \mathbb{0}\}. \tag{P}$$

Prove that (P) is either unbounded, or has an optimal value of 0. If (P) is unbounded, what is a certificate of unboundedness?

Solution: First, note that $x = \mathbb{0}$ is a feasible solution to (P). So (P) is feasible, and if it is not unbounded, then its optimal value is at least 0. We claim that if (P) has a feasible solution with positive objective value, then it is unbounded; otherwise, its optimal value is 0. To see this, clearly if there is no feasible solution with positive objective value, (P) is not unbounded and has optimal value at most 0; as $x = \mathbb{0}$ is a feasible solution, this means that the optimal value is exactly 0. On the other hand, suppose d is a feasible solution with $c^T d > 0$. Then, d is a certificate of unboundedness for (P): we have $Ad = \mathbb{0}$, $d \geq \mathbb{0}$ (since d is feasible), and $c^T d > 0$ (by assumption).

Note that the above argument holds for all LPs of the form (P).

(b) Let $c, \bar{c} \in \mathbb{R}^n$ be such that $c_i, \bar{c}_i > 0$ for all $i = 1, \dots, n$. Let (P') be the LP with the same constraints as (P) but with objective function $\max \bar{c}^T x$. Prove that (P) and (P') have the same outcome: that is, (P) is unbounded iff (P') is unbounded, and (P) has an optimal value of 0 iff (P') has an optimal value of 0.

Solution: Suppose the system $Ax = \mathbb{0},\ x \geq \mathbb{0}$ has a feasible solution $d \neq \mathbb{0}$. Then, we have $d_i > 0$ for some $i \in \{1, \dots, n\}$, and since $c_i, \bar{c}_i > 0$, we have $c^T d > 0$ and $\bar{c}^T d > 0$, and so d is a certificate of unboundedness for both (P) and (P') showing that both LPs are unbounded. Otherwise, if $x = \mathbb{0}$ is the only feasible solution to the above system, then clearly, both (P) and (P') have optimal value 0.

Question 5. Standard Equality Form

Convert the following LPs into standard equality form.

(a)

$$
\begin{array}{ll}
\text{min} & (-1,2,-1,2)x \\
\text{subject to} & \\
& \begin{pmatrix} 2 & 3 & -1 & 4 \\ -1 & -1 & 3 & 8 \\ 14 & 1 & -2 & 8 \\ 5 & -2 & 7 & 12 \end{pmatrix} x \begin{array}{c} \geq \\ \leq \\ \geq \\ = \end{array} \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix} \\
& x \geq \mathbb{0}.
\end{array}
$$

(b)

$$
\begin{array}{ll}
\text{max} & (-2,3,-4,1)x \\
\text{subject to} & \\
& \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & -2 & 16 & 0 \\ 8 & 2 & -3 & 1 \end{pmatrix} x \begin{array}{c} \leq \\ \leq \\ = \end{array} \begin{pmatrix} 10 \\ 14 \\ -2 \end{pmatrix} \\
& x_1, x_4 \geq 0,\ x_3 \leq 0,\ x_2 \text{ free.}
\end{array}
$$

(c)

$$
\begin{array}{ll}
\text{min} & (0,5,0,-1)x \\
\text{subject to} & \\
& \begin{pmatrix} -1 & -2 & 0 & 1 \\ 1 & 2 & 11 & 0 \\ 8 & 0 & -3 & 1 \end{pmatrix} x \begin{array}{c} = \\ \leq \\ = \end{array} \begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix} \\
& x_1, x_2 \geq 0,\ x_3, x_4 \text{ free.}
\end{array}
$$

Question 6. Simplex Iteration

Consider the following LP problem (P):

$$\begin{array}{ll} \max & (0,0,0,\alpha,-3,-1)x \\ \text{subject to} & \\ & \begin{pmatrix} 1 & 0 & 0 & \beta & 3 & 1 \\ 0 & 1 & 0 & -\beta & 0 & 10 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{pmatrix} x = \begin{pmatrix} \beta+2 \\ 0 \\ \beta \end{pmatrix} \\ & x \geq \mathbb{0}, \end{array}$$

where α and β are parameters (these parameters should be treated as constants).

(a) Determine **ALL** values of parameters α and β for which $\bar{x} = (\beta+2, 0, \beta, 0, 0, 0)^T$ is an optimal solution for (P).

(b) Determine **ALL** values of parameters α and β for which $\bar{x} = (\beta+2, 0, \beta, 0, 0, 0)^T$ is a feasible but not an optimal solution for (P).

Question 7. Basic Solutions

(a) Consider the following system:

$$\begin{pmatrix} 1 & 1 & 0 & 3 & 1 & -1 \\ 0 & 2 & -1 & -1 & -2 & 2 \\ 0 & 0 & 1 & 6 & 3 & -3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

For each of the sets B below, indicate if it is a basis or not. If B is a basis, indicate whether the corresponding vector $\bar{x}$ is the basic solution corresponding to B. **Justify your answers inbrief.**

(i) $B = \{1, 2, 3\}$ and $\bar{x} = (0, 2, 3, 0, 1, 1)^T$

(ii) $B = \{1, 2, 3\}$ and $\bar{x} = (0, 2, 3, 0, 0, 0)^T$

(iii) $B = \{4, 5, 6\}$ and $\bar{x} = (0, 0, 0, 1, 0, 1)^T$

(iv) $B = \{1, 4, 6\}$ and $\bar{x} = (0, 0, 0, 1, 0, 1)^T$

(v) $B = \{3, 4, 6\}$ and $\bar{x} = (0, 0, 0, 1, 0, 1)^T$

(vi) $B = \{4, 6\}$ and $\bar{x} = (0, 0, 0, 1, 0, 1)^T$

(vii) $B = \{3, 4, 5\}$ and $\bar{x} = (0, 0, 0, 1, -1, 0)^T$.

(b) Consider the following system:

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 1 & -1 \\ 0 & 1 & 4 & 2 & -2 & 5 \\ 4 & 0 & 5 & -3 & 3 & 9 \end{pmatrix} x = \begin{pmatrix} -1 \\ 5 \\ 9 \end{pmatrix}.$$

For each of the following vectors, indicate if it is a basic solution or not. If it is a basic solution for some basis, state such a basis B, and indicate if it is a basic feasible solution or not. **Justify your answers inbrief.**

(i) $\bar{x} = (1/2, 1/2, 1/2, 0, 0, 1/2)^T$

(ii) $\bar{x} = (0, 0, 0, -1, -1, 1)^T$

(iii) $\bar{x} = (1, 1, 1, 0, 0, 0)^T$

(iv) $\bar{x} = (0, 1, 1, 1, 0, 0)^T$

(v) $\bar{x} = (0, 0, 0, 0, 0, 1)^T$.

Question 8. Simplex method

Solve each of the following LPs using the simplex method. Each LP below is in canonical form for some feasible basis B (which you should be able to identify by inspection); start the simplex algorithm from this basis. Use Bland's rule: break any ties in the choice of the entering and leaving variable by *picking the one with the smallest index*. Show all your steps.

(When running the simplex method, as noted in Q2(b), to obtain the canonical form for a basis, it is typically more efficient to apply row operations to the canonical form for the previous basis, rather than compute A_B^{-1} and apply the formula in the text.)

(a)

$$\begin{aligned} \max \quad & (0,0,4,-11,-1)x + 17 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1 & 0 & 2 & 7 & -1 \\ 0 & 1 & -4 & -5 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Solution: This LP is in canonical form for basis $B^{(1)} = \{1,2\}$, and the corresponding basic solution is $x^{(1)} = (2,1,0,0,0)^\top$, which is feasible. The objective value of $x^{(1)}$ is $z(x^{(1)}) = 17$. There is only one $j \in N$ with $c_j > 0$, namely, c_3, so $k = 3$ enters the basis. Setting $x_3 = t$ while keeping $x_4 = x_5 = 0$, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

So to ensure that $x_1, x_2 \geq 0$, the maximum possible value we may take for t is $\min\{\frac{2}{2}, -\} = 1$, which corresponds to the basic variable x_1, so $\ell = 1$ leaves the basis.

The new basis is therefore $B^{(2)} = \{2,3\}$. We now need to convert our LP into the canonical form for the new basis (which we do using row operations), which is:

$$\begin{aligned} \max \quad & (-2,0,0,-25,1)x + 21 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 0.5 & 0 & 1 & 3.5 & -0.5 \\ 2 & 1 & 0 & 9 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

$B^{(2)}$ yields the basic feasible solution $x^{(2)} = (0,5,1,0,0)^T$. Repeating the same process, there is only one entry of c_N that is positive, namely, c_5, so $k = 5$ enters the basis. We can increase x_5 to $t = \min\{-, \frac{1}{1}\} = 1$, so $\ell = 2$ leaves the basis.

The new basis is $B^{(3)} = \{3,5\}$. We write the LP in the canonical form for the new basis, which is:

$$\begin{aligned} \max \quad & (-4,-1,0,-34,0)x + 26 \\ \text{subject to} \quad & \\ & \begin{pmatrix} 1.5 & 0.5 & 1 & 8 & 0 \\ 2 & 1 & 0 & 9 & 1 \end{pmatrix} x = \begin{pmatrix} 3.5 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

$B^{(3)}$ yields the basic feasible solution $x^{(3)} = (0,0,3.5,0,5)^T$. We now have $c_N \leq \mathbb{0}$, so $B^{(3)} = \{3,5\}$ is the optimal basis, and $x^{(3)} = (0,0,3.5,0,5)^T$ is an optimal solution (in fact, the unique optimal solution) with objective value 26.

(b)

$$\begin{array}{ll} \max & (5,2,-1,0,0,0)x \\ \text{subject to} & \\ & \begin{pmatrix} 2 & 1 & -4 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 16 \\ 2 \\ 1 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Solution: This LP is in canonical form for basis $B^{(1)} = \{4,5,6\}$. The corresponding basic feasible solution is $x^{(1)} = (0,0,0,16,2,1)^\top$, which has objective value $z(x^{(1)}) = 0$. Since $c_1, c_2 > 0$, both x_1 and x_2 are candidates for the entering variable. Bland's rule dictates that we break ties in favor of x_1 and choose x_1 as the entering variable, so $k = 1$ enters the basis. We can increase x_1 to $t = \min\{\frac{16}{2}, \frac{2}{1}, -\} = 2$ while maintaining feasibility, $\ell = 5$ leaves the basis.

The new basis is $B^{(2)} = \{1,4,6\}$. We use row operations to convert the canonical form for $B^{(1)}$ into the canonical form for $B^{(2)}$. This yields the following LP:

$$\begin{array}{ll} \max & (0,-3,9,0,-5,0)x + 10 \\ \text{subject to} & \\ & \begin{pmatrix} 0 & -1 & 0 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 12 \\ 2 \\ 3 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

The basic feasible solution corresponding to $B^{(2)}$ is $x^{(2)} = (2,0,0,12,0,3)^\top$. Only $c_3 > 0$, so $k = 3$ enters the basis. The column corresponding to x_3 in the above constraint matrix is nonpositive, so this indicates that the LP is unbounded. (Indeed, we can obtain the family $x(t) = x^{(2)} + t(2,0,1,0,0,1)^\top$ of feasible solutions with objective value $(0,-3,9,0,-5,0)x(t) + 10 = 9t + 10$, which tends to ∞ as $t \to \infty$, showing that the above canonical form, and hence the original LP, is unbounded.)