CO 250: Introduction to Optimization

Module 2: Linear Programs (Formalizing the Simplex)

Finding an Optimal Solution

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Consider $B = \{1, 4\}$.

- A_B is square and non-singular $\Longrightarrow$ B is a basis
- $A_B = I$ and $c_B = \mathbf{0}$ $\Longrightarrow$ LP is in canonical form for B
- $\bar{x} = (2, 0, 0, 5)^\top$ is a basic solution
- $\bar{x} \geq \mathbf{0}$ $\Longrightarrow$ $\bar{x}$ is feasible, i.e., B is feasible

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Question

How do we find a better feasible solution?

$$
\begin{aligned}
\max \quad & \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x \\
\text{s.t.} \quad & \\
& \underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

$B = \{1, 4\}$ is a feasible basis

Canonical form for B

$(2, 0, 0, 5)^\top$ is a basic solution

Idea

Pick $k \notin B$ such that $c_k > 0$.

Set $x_k = t \geq 0$ as large as possible.

Keep all other non-basic variables at 0.

Pick $k = 2$. Set $x_2 = t \geq 0$.

Keep $x_3 = 0$.

$$\max \quad \underbrace{(0 \quad 1 \quad 3 \quad 0)}_{c}\, x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

Idea

Choose basic variables such that $Ax = b$ holds.

$$\begin{aligned}
\begin{pmatrix} 2 \\ 5 \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}$$

$$\boxed{\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}}$$

$$\max \quad \underbrace{\begin{pmatrix} 0 & 1 & 3 & 0 \end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{1, 4\}$ is a basis

$x_2 = t \geq 0$, $x_3 = 0$

$$\underbrace{\begin{pmatrix} x_1 \\ x_4 \end{pmatrix}}_{x_B} = \underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{b} - t \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{A_2}$$

Choose $t \geq 0$ as large as possible.

Basic variables must remain non-negative.

$$x_1 = 2 - t \geq 0 \quad \Longrightarrow \quad t \leq \frac{2}{1}$$

$$x_4 = 5 - t \geq 0 \quad \Longrightarrow \quad t \leq \frac{5}{1}$$

Thus, the largest possible $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\}$.

The new feasible solution is $x = (0, 2, 0, 3)^\top$. It has value $2 > 0$.

$$
\begin{aligned}
\max \quad & (0 \quad 1 \quad 3 \quad 0)x \\
\text{s.t.} \quad & \\
& \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Remark

The new feasible solution $x = (0, 2, 0, 3)^\top$ is a basic solution.

Question

For what basis B is $x = (0, 2, 0, 3)^\top$ a basic solution?

$$
\begin{aligned}
x_2 \neq 0 \quad & \Longrightarrow \quad 2 \in B \\
x_4 \neq 0 \quad & \Longrightarrow \quad 4 \in B
\end{aligned}
$$

As $|B| = 2$, $B = \{2, 4\}$.

$$\begin{aligned}\max \quad & \begin{pmatrix}0 & 1 & 3 & 0\end{pmatrix} x \\ \text{s.t.} \quad & \begin{pmatrix}1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 5\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

$\Downarrow$

$$\begin{aligned}\max \quad & \begin{pmatrix}-1 & 0 & 1 & 0\end{pmatrix} x + 2 \\ \text{s.t.} \quad & \begin{pmatrix}1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 3\end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0\end{aligned}$$

NEW

$\{2, 4\}$ is a feasible basis

Canonical form for $\{2, 4\}$

Remark

We only need to know how to go from the OLD basis to a NEW basis!

- 2 <u>entered</u> the basis.
- 1 <u>left</u> the basis.

WHY?

$$
\begin{array}{ll}
\max & (0 \quad 1 \quad 3 \quad 0)x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

OLD

$\{1, 4\}$ is a feasible basis

Canonical form for $\{1, 4\}$

Pick $2 \notin B$ and set $x_2 = t \geq 0$.

$\Rightarrow$ 2 enters the basis

Set $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} - t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $t = \min\left\{\frac{2}{1}, \frac{5}{1}\right\} = 2$.

$\Rightarrow$ $x_1 = 0$ and 1 leaves the basis

The NEW basis is $\{2, 4\}$.

Example – Continued

$$\max \quad \underbrace{\begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}}_{c} x + 2$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{2, 4\}$ is a feasible basis

Canonical form for B

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$t = \min\left\{\frac{2}{2}, -\right\} = 2$ thus $x_2 = 0$ ➡ 2 leaves the basis

The NEW basis is $B = \{3, 4\}$.

$$\max \quad \underbrace{(-1.5 \quad -0.5 \quad 0 \quad 0)}_{c} x + 3$$

s.t.

$$\underbrace{\begin{pmatrix} 0.5 & 0.5 & 1 & 0 \\ -0.5 & 0.5 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$B = \{3, 4\}$ is a feasible basis

Canonical form for B

$(0, 0, 1, 4)^\top$ is a basic solution

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$: ???

Claim

$(0, 0, 1, 4)^\top$ has value 3. It is optimal because 3 is an upper bound.

Proof

Let x be a feasible solution. Then

$$\underbrace{(-1.5,\ 0.5,\ 0,\ 0)}_{\leq \mathbf{0}} \underbrace{x}_{\geq \mathbf{0}} + 3 \leq 3.$$

Another Example

$$\max \quad (0 \quad -4 \quad 3 \quad 0 \quad 0)x$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 0 \\ 0 & 4 & -2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$\{1, 4, 5\}$ is a feasible basis

Canonical form for $\{1, 4, 5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_3 = t$ ➡ 3 enters the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - t \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$$

$t = \min\left\{\frac{1}{1}, -, -\right\} = 1$ thus $x_1 = 0$ ➡ 1 leaves the basis

The NEW basis is $B = \{3, 4, 5\}$.

$$\begin{array}{ll} \max & (-3 \quad 2 \quad 0 \quad 0 \quad 0)x + 3 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$\{3,4,5\}$ is a feasible basis

Canonical form for $\{3,4,5\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

$x_2 = t \quad \Longrightarrow \quad$ 2 <u>enters</u> the basis

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Choose $t = ???$

Claim

The linear program is unbounded.

$$\max \quad z = \begin{pmatrix} -3 & 2 & 0 & 0 & 0 \end{pmatrix} x + 3$$

s.t.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_2 = t$$

$$x_1 = 0$$

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} - t \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

Claim

The linear program is unbounded.

Proof

$$x(t) = \begin{pmatrix} 0 \\ t \\ 1+2t \\ 4+t \\ 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 4 \end{pmatrix}}_{=\bar{x}} + t \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{=r}$$

- $x(t)$ is feasible for all $t \geq 0$.
- $z \to \infty$ when $t \to \infty$.

($\bar{x}, r$: certificate of unboundedness.)

The Simplex Algorithm

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbf{0} \end{array}}$$

<u>INPUT:</u> a feasible basis B.

<u>OUTPUT:</u> an optimal solution OR it detects that the LP is unbounded.

Step 1. Rewrite in canonical form for the basis B.

Step 2. Find a better basis B or get required outcome.

Trying to Find a Better Basis

$$\begin{array}{|ll|}\hline \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \\ \hline \end{array}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ is a basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$.

Pick $x_B = b - tA_k$.

If $A_k \leq \mathbf{0}$, then STOP. The LP is unbounded.

Choose $t = \min\left\{ \frac{b_i}{A_{ik}} : \text{for all } i \text{ such that } A_{ik} > 0 \right\}$.

Let x_r be a basic variable forced to 0.

The new basis is obtained by having k enter and r leave.

$$\begin{array}{ll} \max & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ basic solution

If $c_N \leq \mathbf{0}$, then STOP. The basic solution $\bar{x}$ is optimal.

Proof

$\bar{x}_B = b$, $\bar{x}_N = \mathbf{0}$.

$\bar{x}$ has value $z = c_N^\top \bar{x}_N + \bar{z} = \bar{z}$.

Let x be a feasible solution.

$$z = \underbrace{c_N^\top}_{\leq \mathbf{0}} \underbrace{x_N}_{\geq \mathbf{0}} + \bar{z} \leq \bar{z}.$$

$$\begin{aligned} \max \quad & z = c_N^\top x_N + \bar{z} \\ \text{s.t.} \quad & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{aligned}$$

B is a feasible basis, $N = \{j \notin B\}$

Canonical form for B

$\bar{x}$ basic solution

If $A_k \leq \mathbf{0}$, then STOP. The LP is unbounded.

Proof

x is feasible for all $t \geq 0$:

$x_k = t \geq 0$, all other non-basic variables have value zero.

$$x_B = b - tA_k = \underbrace{b}_{\geq \mathbf{0}} - \underbrace{t}_{\geq 0} \underbrace{A_k}_{\leq \mathbf{0}} \geq \mathbf{0}$$

$z \to \infty$ when $t \to \infty$:

$$z = \sum_{j \in N} c_j x_j + \bar{z} = c_k x_k + \bar{z} = \underbrace{c_k}_{>0} t + \bar{x}.$$

Proposition

Simplex tells the truth:

- If it claims the LP is unbounded, it is unbounded.
- If it claims the solution is optimal, it is optimal.

Question

Is the Simplex a correct algorithm?

NOT AS STATED! IT MAY NOT TERMINATE!

Potential problem: Start with a feasible basis B_1,

$$\underbrace{B_1 \rightsquigarrow B_2 \rightsquigarrow B_3 \rightsquigarrow \dots \rightsquigarrow B_{k-1} \rightsquigarrow B_k = B_1}_{\text{Cycling}}$$

Theorem

If we use Bland's Rule, then the Simplex algorithm always terminates.

Bland's rule

Theorem

If we use Bland's Rule, then the Simplex algorithm always terminates.

Definition

Bland's rule is as follows:

- If we have a choice for the element entering the basis, pick the smallest one.
- If we have a choice for the element leaving the basis, pick the smallest one.

Let us see an example...

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 2 & 3 \end{pmatrix} x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 4 & 6 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$\{1, 2\}$ is a feasible basis

Canonical form for $\{1, 2\}$

Pick $k \notin B$ such that $c_k > 0$ and set $x_k = t$:

Choices $k = 3$ OR $k = 4$.

Bland's rule says pick $k = 3$ (entering element).

Pick $x_B = b - tA_k$:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} - t \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{and} \quad t = \min\left\{\tfrac{6}{2}, \tfrac{12}{4}\right\} = 3$$

Pick $r \in B$ such that $x_r = 0$:

Choices $r = 1$ OR $r = 2$.

Bland's rule says pick $r = 1$ (leaving element).

The NEW basis is $B = \{3, 4\}$.

Recap

- We have seen a formal description of the Simplex algorithm.
- We showed that if the algorithm terminates, then it is correct.
- We defined Bland's rule and asserted, without proof, that Simplex terminates as long as we are using Bland's rule.
- To get started, we need to get a feasible basis.

  <u>To do:</u> Find a procedure to find a feasible basis.