CO 250: Introduction to Optimization

Module 2: Linear Programs (Standard Equality Forms)

The Key Definition

Definition

A Linear Program (LP) is in Standard Equality Form (SEF) if

1. it is a maximization problem,
2. for every variable x_j we have the constraint $x_j \geq 0$, and
3. all other constraints are equality constraints.

$$
\begin{array}{ll}
\max & (1,-2,4,-4,0,0)x + 3 \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 5 & 3 & -3 & 0 & -1 \\ 2 & -1 & 2 & -2 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}
$$

Question

Is the following LP in SEF?

$$\boxed{\begin{array}{ll} \max & x_1 + x_2 + 17 \\ \text{s.t.} & \\ & x_1 - x_2 = 0 \\ & x_1 \geq 0 \end{array}}$$

NO! There is no constraint $x_2 \geq 0$. We say x_2 is free.

Remarks

- $x_2 \geq 0$ is implied by the constraints.
- x_2 is still free since $x_2 \geq 0$ is not given explicitly.

Motivation

We will develop an algorithm called the Simplex that can solve any LP

as long as it is in Standard Equality Form (SEF)

Question

What do we do if the LP is not in SEF?

Idea

1. Find an "equivalent" LP in SEF.
2. Solve the "equivalent" LP using Simplex.
3. Use the sol'n of "equivalent" LP to get the sol'n of the original LP.

Question

What do we mean by equivalent?

Equivalent LPs

Idea

A pair of LPs are equivalent if they behave in the same way.

Definition

Linear programs (P) and (Q) are equivalent if

- (P) infeasible $\iff$ (Q) infeasible,
- (P) unbounded $\iff$ (Q) unbounded,
- can construct optimal sol'n of (P) from optimal sol'n of (Q),
- can construct optimal sol'n of (Q) from optimal sol'n of (P).

Theorem

Every LP is equivalent to an LP in SEF.

We will illustrate the proof with a series of examples.

Dealing with Minimization

$$\min \quad (1, 2, -4)(x_1, x_2, x_3)^\top$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{array}{c} = \\ \leq \end{array} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

EQUIVALENT!

$$\max \quad -(1, 2, -4)(x_1, x_2, x_3)^\top$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{array}{c} = \\ \leq \end{array} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$x_1, x_2 \geq 0$$

Replacing an Inequality by an Equality

Suppose an LP has the constraint

$$x_1 - x_2 + x_4 \leq 7.$$

We can replace it by

$$x_1 - x_2 + x_4 + s = 7, \quad \text{where} \quad s \geq 0.$$

Suppose an LP has the constraint

$$x_1 - x_2 + x_4 \geq 7.$$

We can replace it by

$$x_1 - x_2 + x_4 - s = 7, \quad \text{where} \quad s \geq 0.$$

Free Variables

$$\begin{array}{ll} \max & z = (1,2,3)(x_1,x_2,x_3)^\top \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \\ & x_1, x_2 \geq 0,\ x_3 \text{ is free.} \end{array}$$

Find an equivalent LP without the free variable x_3. How?

Idea

Any number is the difference between two non-negative numbers.

$\boxed{\text{Set } x_3 := a - b \text{ where } a, b \geq 0.}$

Free Variables – Rewrite the Objective Function

$\boxed{\text{Set } x_3 := a - b \text{ where } a, b \geq 0.}$

$$
\begin{aligned}
z &= (1,2,3)(x_1, x_2, x_3)^\top \\
&= x_1 + 2x_2 + 3x_3 \\
&= x_1 + 2x_2 + 3(a - b) \\
&= x_1 + x_2 + 3a - 3b \\
&= (1,2,3,-3)(x_1, x_2, a, b)^\top
\end{aligned}
$$

Free Variables – Rewrite the Constraints

$\boxed{\text{Set } x_3 := a - b \text{ where } a, b \geq 0.}$

$$
\begin{aligned}
\begin{pmatrix} 5 \\ 4 \end{pmatrix} &= \begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
&= x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -1 \end{pmatrix} + (a - b) \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
&= x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -1 \end{pmatrix} + a \begin{pmatrix} 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} -3 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 5 & 3 & -3 \\ 2 & -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ a \\ b \end{pmatrix}
\end{aligned}
$$

$$\max \quad z = (1,2,3)(x_1, x_2, x_3)^\top$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$x_1, x_2 \geq 0,\ x_3$ is free.

EQUIVALENT!

$$\max \quad z = (1,2,3,-3)(x_1, x_2, a, b)^\top$$

s.t.

$$\begin{pmatrix} 1 & 5 & 3 & -3 \\ 2 & -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$x_1, x_2, a, b \geq 0$

Generalize and Consequences

We have shown, for an LP, how to

- change $\min$ to $\max$,
- replace inequalities by equalities, and
- get rid of free variables.

Exercise

1. Generalize to arbitrary LPs.
2. Show each step yields an equivalent LP.

$\Downarrow$

Theorem

Every LP is equivalent to an LP in SEF.

Recap

1. We defined what it means for an LP to be in SEF.
2. We defined what it means for two LPs to be equivalent.
3. We showed how to convert any LP into an equivalent LP in SEF.
4. To solve any LP, it suffices to know how to solve LPs in SEF.