CO 250: Introduction to Optimization

Module 2: Linear Programs (Half-Spaces and Convexity)

Definition

For an optimization problem, the

feasible region $=$ set of all feasible solutions.

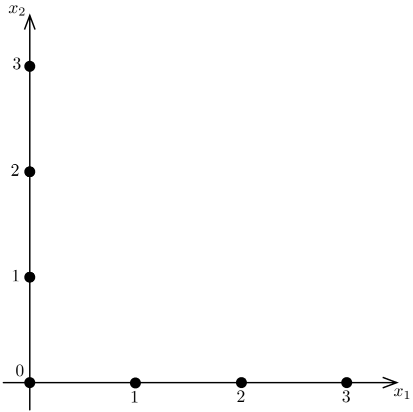
$$\max \quad (1,2)x$$

s.t.

$$\begin{pmatrix}1 & 1\\1 & 0\\0 & 1\end{pmatrix} x \leq \begin{pmatrix}2\\2\\3\end{pmatrix} \qquad \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

$$x \geq \mathbf{0}$$

FEASIBLE REGION $\subseteq \Re^2$



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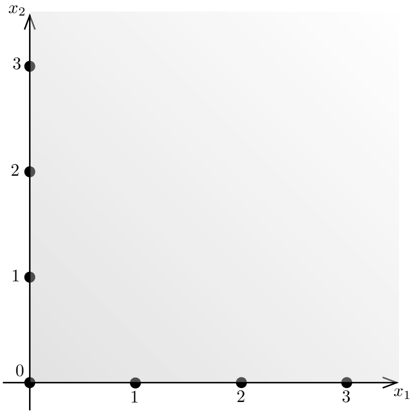
NON-NEGATIVITY CONSTRAINTS



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$$
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\max & (1,2)x \\
\text{s.t.} & \\
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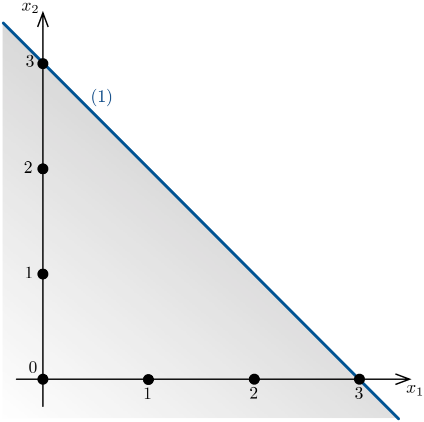
& x \geq \mathbf{0}
\end{array}
$$



CONSTRAINT (1)

Definition

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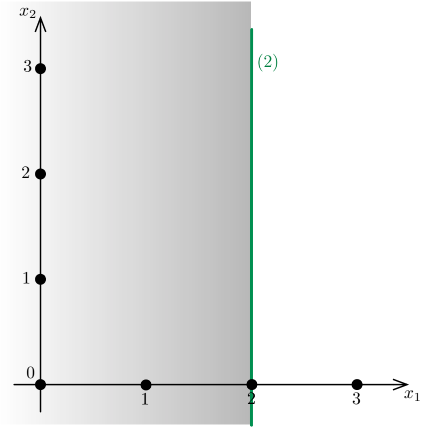
CONSTRAINT (2)



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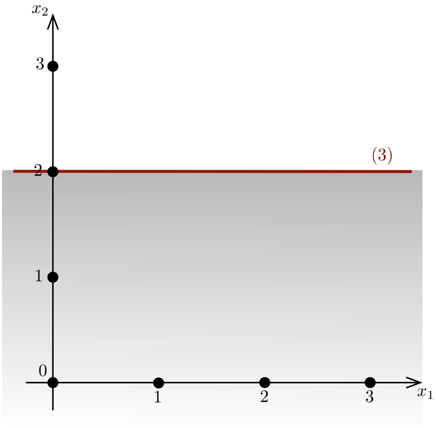
CONSTRAINT (3)



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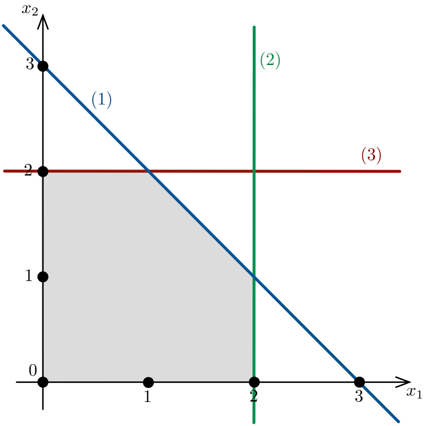
FEASIBLE REGION



Definition

$P \subseteq \Re^n$ is a polyhedron if there exists a matrix A and a vector b such that

$$P = \{x : Ax \leq b\}.$$

Proposition

The feasible region of an LP is a polyhedron.

Proposition

The feasible region of an LP is a polyhedron.

Example:

$$
\begin{array}{ll}
\max & (1,3,2)x \\
\text{s.t.} & \\
& \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \end{pmatrix} \\
& x_1, x_2, x_3 \geq 0
\end{array}
$$

Let's rewrite the constraints:

$$
\begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 3 & 4 & 5 \\ -3 & -4 & -5 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ -2 \\ 12 \\ -12 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Proposition

The feasible region of an LP is a polyhedron.

Exercise

Generalize the previous example and prove the proposition.

GOAL: Understand the geometry of a polyhedra.

Definition

Let $a \neq \mathbf{0}$ be a vector and β a real number.

1. $\{x : a^\top x = \beta\}$ is a hyperplane.
2. $\{x : a^\top x \leq \beta\}$ is a halfspace.

$\Longrightarrow$

A hyperplane is the set of solutions to a single linear equation.

A halfspace is the set of solutions to a single linear inequality.

Remark

A polyhedron is the intersection of a finite set of halfspaces.

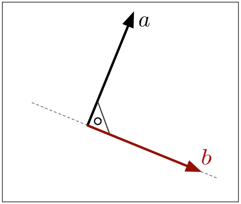
Thus, understanding the geometry of a polyhedra is satisfied by understanding the geometry of halfspaces.

Remark

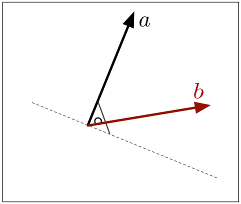
Let a, b be vectors. Then

$$a^\top b = \|a\| \|b\| \cos(\theta)$$

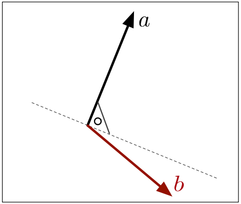
where $\|..\|$ is the norm and θ the angle between a and b.



$a^\top b = 0$



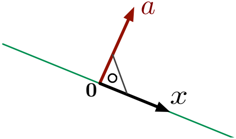
$a^\top b > 0$



$a^\top b < 0$

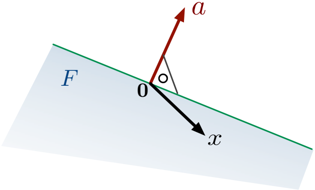
Vector $a \neq \mathbf{0}$, $\beta = 0$

Hyperplane $H = \{x : a^\top x = \beta\}$



Vector $a \neq \mathbf{0}$, $\beta = 0$

Halfspace $F = \{x : a^\top x \leq \beta\}$



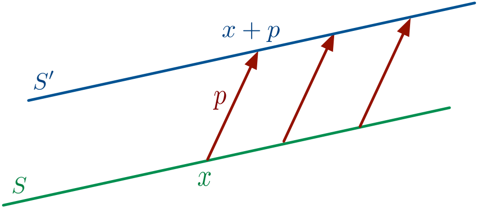
Remark

1. $H =$ the set of vectors orthogonal to a.
2. $F =$ the set of vectors on side of H not containing a.

Definition

Let $S, S' \subseteq \Re^n$. Then S' is a translate of S if there exists $p \in \Re^n$ and

$$S' = \{s + p : s \in S\}.$$



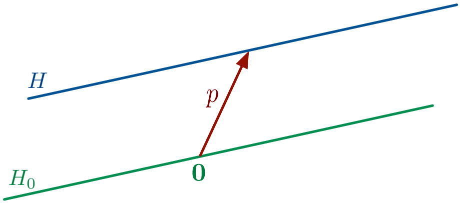
Hyperplanes

Proposition

Let $a \neq \mathbf{0}$ be a vector and β a real number and let

$$H := \{x : a^\top x = \beta\} \qquad \text{and} \qquad H_0 := \{x : a^\top x = 0\}.$$

It follows that H is a translate of H_0.



Proposition

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$$H := \{x : a^\top x = \beta\} \qquad \text{and} \qquad H_0 := \{x : a^\top x = 0\}.$$

It follows that H is a translate of H_0.

Proof

Choose $p \in H$. To show: $x \in H_0 \Leftrightarrow x + p \in H$.

$$\begin{aligned}
& x \in H_0 && \Leftrightarrow \\
& a^\top x = 0 && \Leftrightarrow \\
& a^\top x + a^\top p = 0 + a^\top p && \Leftrightarrow \\
& a^\top (x + p) = \beta && \Leftrightarrow \\
& x + p \in H
\end{aligned}$$

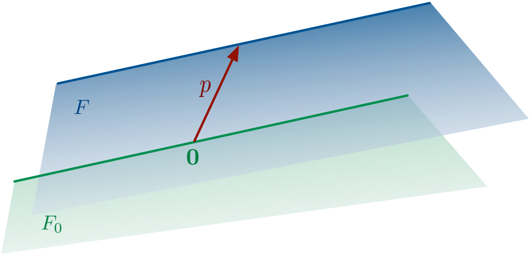
Halfspaces

Proposition

Let $a \neq \mathbf{0}$ be a vector and β a real number and let

$$F := \{x : a^\top x \leq \beta\} \qquad \text{and} \qquad F_0 := \{x : a^\top x \leq 0\}.$$

It follows that F is a translate of F_0.



Exercise

Prove the previous proposition.

Question

What is the dimension of a hyperplane in $\Re^n$?

Let $a \in \Re^n$, $a \neq \mathbf{0}$ and let $\beta \in \Re$. Define

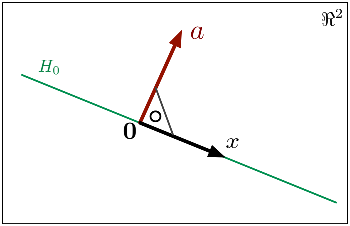
$$H = \{x : a^\top x = \beta\} \qquad \text{and} \qquad H_0 = \{x : a^\top x = 0\}.$$

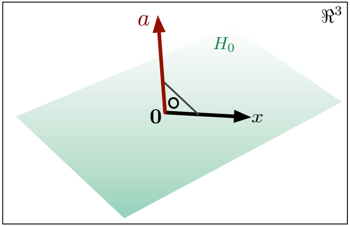
- We define the dimension of H to be the dimension of H_0.
- H_0 is a vector space and its dimension can be computed as follows:

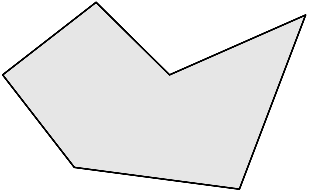
$$dim(H_0) = n - rank(a) = n - 1.$$

Proposition

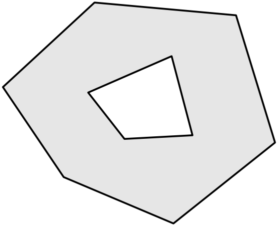
The dimension of a hyperplane in $\Re^n$ is $n - 1$.







Is this shaded figure a polyhedron?
NO



Is this shaded figure a polyhedron?
NO

Remark

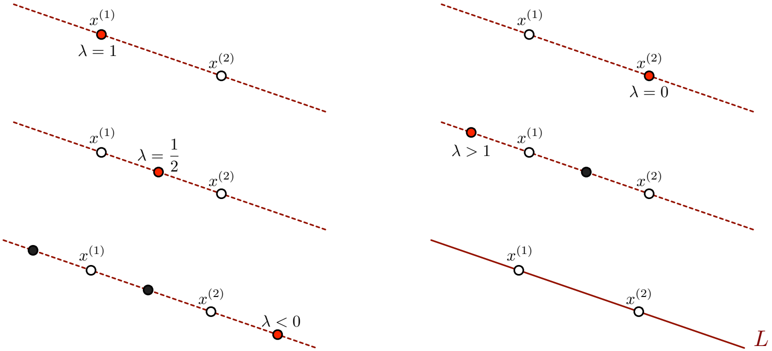
A polyhedron has no "dents" and no "holes".

Formalizing this remark will lead us to the notion of convexity.

Definition

Let $x^{(1)}, x^{(2)} \in \Re^n$. The line through $x^{(1)}$ and $x^{(2)}$ is defined as

$$L = \left\{ x = \lambda x^{(1)} + (1-\lambda) x^{(2)} : \lambda \in \Re \right\}.$$



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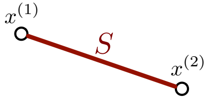
Definition

Let $x^{(1)}, x^{(2)} \in \Re^n$. The line segment between $x^{(1)}$ and $x^{(2)}$ is defined as

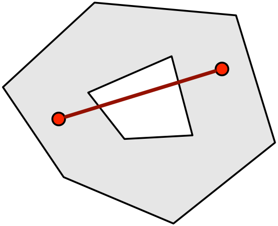
$$S = \left\{ x = \lambda x^{(1)} + (1-\lambda) x^{(2)} : \lambda \in \Re, 0 \le \lambda \le 1 \right\}.$$



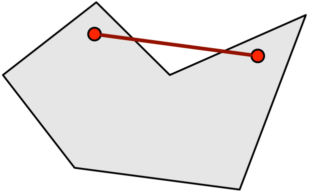
Definition

A set $S \subseteq \Re^n$ is convex if, for any pair of points $x^{(1)}, x^{(2)} \in S$,

the line segment between $x^{(1)}$ and $x^{(2)}$ is in S.



Is this shaded figure convex? NO



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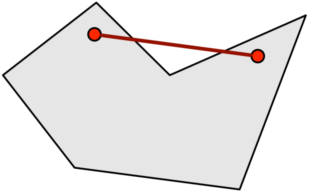
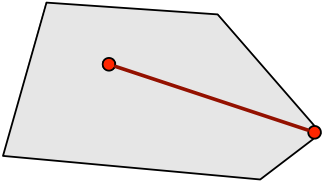


Is this shaded figure convex? NO



Is this shaded figure convex? YES

Let $A, B \subseteq \Re^n$. Suppose A and B are both convex.

Question

Is $A \cup B$ convex?

NO.

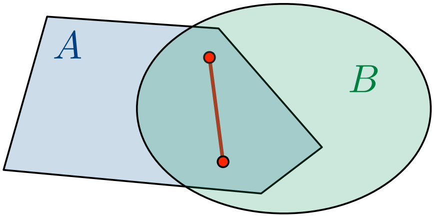
Example: $A = \{0\}$ and $B = \{1\}$.

Question

Is $A \cap B$ convex?

YES.

Example:



Proposition

Let $H = \{x : a^\top x \leq \beta\}$ be a halfspace. It follows that H is convex.

Proof

Pick two arbitrary points $x^{(1)}, x^{(2)} \in H$.

Pick an arbitrary point $\bar{x}$ in the line segment between $x^{(1)}$ and $x^{(2)}$.

We now need to show that $\bar{x} \in H$.

$$
\begin{aligned}
a^\top \bar{x} &= a^\top \left(\lambda x^{(1)} + (1-\lambda)x^{(2)}\right) \\
&= \underbrace{\lambda}_{\geq 0} \underbrace{a^\top x^{(1)}}_{\leq \beta} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{a^\top x^{(2)}}_{\leq \beta} \\
&\leq \lambda\beta + (1-\lambda)\beta \\
&= \beta
\end{aligned}
$$

Proposition

Let $H = \{x : a^\top x \le \beta\}$ be a halfspace. It follows that H is convex.

Corollary

If P is a polyhedron, then P is convex.

Proof

P is the intersection of halfspaces.

Each halfspace is convex.

The intersection of convex sets is convex.

The feasible region of an LP is always a convex set!

Recap

- We defined hyperplanes and halfspaces.
- A polyhedron is the intersection of a finite number of halfspaces.
- We showed that the feasible region of an LP is a polyhedron.
- We defined the line segment between two points.
- We defined convex sets.
- We proved that a polyhedron is always convex.