CO 250: Introduction to Optimization

Module 4: Duality Theory (Strong Duality)

Recap: Weak Duality

$(\mathrm{P}_{\max})$		$(\mathrm{P}_{\min})$	
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \;?\; b$	≥ 0 variable	$\geq$ constraint	$A^\top y \;?\; c$
$x \;?\; \mathbb{0}$	free variable	$=$ constraint	$y \;?\; \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Last lecture: we described a method to construct the dual of a general linear program.

E.g.: consider the primal LP, (P), on the right – a max LP that falls in the first two columns of the table.

$\longrightarrow$ The dual of (P) is a min LP.

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

Recap: Weak Duality

(P_{max})			(P_{min})	
max $c^\top x$ subject to $Ax \,?\, b$, $x \,?\, \mathbb{0}$	$\leq$ constraint	≥ 0 variable		min $b^\top y$ subject to $A^\top y \,?\, c$, $y \,?\, \mathbb{0}$
	$=$ constraint	free variable		
	$\geq$ constraint	≤ 0 variable		
	≥ 0 variable	$\geq$ constraint		
	free variable	$=$ constraint		
	≤ 0 variable	$\leq$ constraint		

$$\max\ (2,-1,3)x \qquad \text{(P)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x \begin{matrix} \leq \\ = \\ \geq \end{matrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$

$$\text{s.t.}\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Recap: Weak Duality

$$\max\ (2,-1,3)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix}1 & 0 & -1\\ 0 & -2 & 1\\ 1 & 1 & 0\end{pmatrix} x \begin{matrix}\leq\\ =\\ \geq\end{matrix} \begin{pmatrix}2\\ 1\\ -2\end{pmatrix}$$
$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ free}$$

$$\min\ (2,1,-2)y \qquad \text{(D)}$$
$$\text{s.t.}\ \begin{pmatrix}1 & 0 & 1\\ 0 & -2 & 1\\ -1 & 1 & 0\end{pmatrix} y \begin{matrix}\geq\\ \leq\\ =\end{matrix} \begin{pmatrix}2\\ -1\\ 3\end{pmatrix}$$
$$y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0$$

Weak Duality Theorem

if $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D),

$$\implies \quad c^T\bar{x} \leq b^T\bar{y}$$

If $c^T\bar{x} = b^T\bar{y}$, then both $\bar{x}$ and $\bar{y}$ are optimal.

This Lecture: Strong Duality

$(\mathrm{P}_{\max})$			$(\mathrm{P}_{\min})$
	$\leq$ constraint	≥ 0 variable	
max $\quad c^\top x$	$=$ constraint	free variable	min $\quad b^\top y$
subject to	$\geq$ constraint	≤ 0 variable	subject to
$Ax \, ? \, b$	≥ 0 variable	$\geq$ constraint	$A^\top y \, ? \, c$
$x \, ? \, \mathbb{0}$	free variable	$=$ constraint	$y \, ? \, \mathbb{0}$
	≤ 0 variable	$\leq$ constraint	

Question

Can we always find feasible solutions $\bar{x}$ and $\bar{y}$ to a primal-dual pair, $(\mathrm{P}_{\max})$, $(\mathrm{P}_{\min})$, such that $c^T \bar{x} = b^T \bar{y}$?

Strong Duality Theorem

If $(\mathrm{P}_{\max})$ has an optimal solution $\bar{x}$, then $(\mathrm{P}_{\min})$ has an optimal solution $\bar{y}$ such that $c^T \bar{x} = b^T \bar{y}$.

Strong Duality – for LPs in SEF

Let us prove the Strong Duality Theorem in the special case where (P) is in SEF.

$$\begin{aligned} \max\ & c^T x && \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

Let's assume (P) has an optimal solution.

$\longrightarrow$ 2-Phase Simplex terminates with an optimal basis B (Why?)

$$\begin{aligned} \min\ & b^T y && \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

We can rewrite (P) for basis B:

$$\begin{aligned} \max\ & z = \bar{y}^T b + \bar{c}^T x && \text{(P')} \\ \text{s.t. } & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x \geq \mathbb{0} \end{aligned}$$

where:

$$\bar{y} = A_B^{-T} c_B$$

$$\bar{c}^T = c^T - \bar{y}^T A$$

Thus, $\bar{x}_N = \mathbb{0}$ and $\bar{x}_B = A_B^{-1} b$

Strong Duality – for LPs in SEF

We can rewrite (P) for basis B:

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$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned} \qquad \text{(P)}$$

Thus, $\bar{x}_N = \mathbb{0}$ and $\bar{x}_B = A_B^{-1} b$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \end{aligned} \qquad \text{(D)}$$

Recall that (P) and (P') are equivalent!

$\longrightarrow$ $\bar{x}$ has same value in (P) and (P')

$$\begin{aligned} c^T \bar{x} &= \bar{y}^T b + \bar{c}^T \bar{x} \\ &= \bar{y}^T b + \bar{c}_N^T \bar{x}_N \\ &= b^T \bar{y} \end{aligned}$$

where:

$$\bar{y} = A_B^{-T} c_B$$
$$\bar{c}^T = c^T - \bar{y}^T A$$

Goal: Show that $\bar{y}$ is dual feasible.

Strong Duality – for LPs in SEF

We can rewrite (P) for basis B:

$$\begin{aligned} \max\ & z = \bar{y}^T b + \bar{c}^T x && \text{(P')} \\ \text{s.t. } & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x \geq \mathbb{0} \end{aligned}$$

Thus, $\bar{x}_B = A_B^{-1} b$ and $\bar{x}_N = \mathbb{0}$ and $c^T \bar{x} = b^T \bar{y}$.

Note that B is an optimal basis $\longrightarrow$ $\bar{c} \leq \mathbb{0}$

$$\longrightarrow \qquad c^T - \bar{y}^T A \leq \mathbb{0}$$

Equivalently, $A^T \bar{y} \geq c$, meaning $\bar{y}$ is dual feasible!

$$\begin{aligned} \max\ & c^T x && \text{(P)} \\ \text{s.t. } & Ax = b \\ & x \geq \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min\ & b^T y && \text{(D)} \\ \text{s.t. } & A^T y \geq c \end{aligned}$$

where:

$$\bar{y} = A_B^{-T} c_B$$

$$\bar{c}^T = c^T - \bar{y}^T A$$

Strong Duality Theorem

Strong Duality Theorem

Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.

Note: (P) is feasible and (D) is feasible $\longrightarrow$ (P) cannot be unbounded

Fundamental Theorem of LP $\longrightarrow$ (P) has an optimal solution.

Subtly different version via previous results:

Strong Duality Theorem – Feasibility Version

Let (P) and (D) be primal-dual pair of LPs. If both are feasible, then both have optimal solutions of the same objective value.

Possible Outcomes of Primal-Dual Pair (P), (D)

(D)(P)	optimal solution	unbounded	infeasible
optimal solution	possible ①	impossible ②	impossible ③
unbounded	impossible ④	impossible ⑤	possible ⑥
infeasible	impossible ⑦	possible ⑧	possible ⑨

- ①, ⑥, and ⑧ many examples exist
- ② follows directly from Weak Duality as follows:

  Suppose, for a contradiction, that (D) has an optimal solution $\bar{y}$.
  $c^T\bar{x} \leq b^T\bar{y}$ for all feasible primal solutions $\bar{x}$ by Weak Duality $\longrightarrow$ (P) is bounded!
  Similar arguments apply to ④ and ⑤
- ③, ⑦ follow directly from Strong Duality
- I'll leave ⑨ for you to do as an exercise!

$$\max\ c^Tx \qquad \text{(P)}$$
$$\text{s.t. } Ax = b$$
$$x \geq \mathbb{0}$$

$$\min\ b^Ty \qquad \text{(D)}$$
$$\text{s.t. } A^Ty \geq c$$

Recap

Strong Duality Theorem

Let (P) and (D) be a primal-dual pair of LPs. If (P) has an optimal solution, then (D) has one, and their objective values equal.

(D)(P)	optimal solution	unbounded	infeasible
optimal solution	possible ①	impossible ②	impossible ③
unbounded	impossible ④	impossible ⑤	possible ⑥
infeasible	impossible ⑦	possible ⑧	possible ⑨