CO 250: Introduction to Optimization

Module 2: Linear Programs (Certificates)

Recap and a Question

Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- It is infeasible.
- It has an optimal solution.
- It is unbounded.

Questions

Consider a linear program.

- If it is infeasible, how can we prove it?
- If we have an optimal solution, how can we prove it is optimal?
- If it is unbounded, how can we prove it?

This can be always be done!

Proving Infeasibility

The following linear program is infeasible:

$$\begin{aligned} \max \quad & (3,4,-1,2)^T x \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Question

How can we prove this problem is, in fact, infeasible?

We cannot try all possible assignments of values to x_1, x_2, x_3, and x_4.

Claim

There is no solution to (1), (2) and $x \geq \mathbb{0}$ where

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof

Construct a new equation:

$$\begin{array}{rrrrrrrrl} & -1 \times (1): & (-3 & 2 & 6 & -7)x & = & -6 & \\ + & 2 \times (2): & (4 & -2 & -4 & 8)x & = & 4 & \\ \hline & & (1 & 0 & 2 & 1)x & = & -2 & (\star) \end{array}$$

Suppose there exists $\bar{x} \geq \mathbb{0}$ satisfying (1), (2). Then $\bar{x}$ satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} \bar{x}}_{\geq 0} = \underbrace{-2}_{<0}.$$

Contradiction!

Repeat using matrix formulations.

Proof

Suppose for a contradiction there is a solution $\bar{x}$ to $x \geq \mathbb{0}$ and

$$\underbrace{\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}_{b} \qquad Ax = b$$

Construct a new equation:

$$\underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \underbrace{\begin{pmatrix} -1 & 2 \end{pmatrix}}_{y^T} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix} x = -2 \qquad (\star) \qquad y^T A x = y^T b$$

Since $\bar{x}$ satisfies the equations it satisfies $(\star)$:

$$\underbrace{\begin{pmatrix} 1 & 0 & 2 & 1 \end{pmatrix}}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{-2}_{<0}. \qquad \underbrace{y^T A}_{\geq \mathbb{0}^\top} \underbrace{\bar{x}}_{\geq \mathbb{0}} = \underbrace{y^T b}_{<0}$$

Contradiction.

Proposition

There is no solution to $Ax = b,\ x \geq \mathbb{0}$, if there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Exercise

Give a proof of this proposition.

Question

If no solution to $Ax = b, x \geq \mathbb{0}$ can we always prove it in that way?

YES!!!!!

Farkas' Lemma

If there is no solution to $Ax = b,\ x \geq \mathbb{0}$, then there exists y where

$$y^T A \geq \mathbb{0}^\top \qquad \text{and} \qquad y^T b < 0.$$

Proving Optimality

$$\max \quad z(x) := (-1\ -4\ \ 0\ \ 0)x + 4$$

s.t.

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$x \geq \mathbb{0}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Question

How can we prove this solution is, in fact, optimal?

We cannot try all possible feasible solutions.

$$\begin{array}{ll} \max & z(x) := (-1 \; -4 \;\; 0 \;\; 0)x + 4 \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Optimal solution:

$$\bar{x}_1 = 0$$
$$\bar{x}_2 = 0$$
$$\bar{x}_3 = 4$$
$$\bar{x}_4 = 5$$

Claim

- $\bar{x}$ is feasible solution of value 4. (easy)
- 4 is an upper bound.

Proof

Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \; -4 \;\; 0 \;\; 0)}_{\leq \mathbb{0}} \underbrace{x'}_{\geq \mathbb{0}} + 4$$

Proving Unboundedness

$$\max \quad z := \begin{pmatrix}-1 & 0 & 0 & 1\end{pmatrix} x$$

s.t.

$$\begin{pmatrix}-1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix}$$

$$x \geq \mathbb{0}$$

Problem
is unbounded

Question

How can we prove that this problem is unbounded?

Idea

Construct a family of feasible solutions $x(t)$ for all $t \geq 0$ and show that as t goes to infinity, the value of the objective function goes to infinity.

$$\boxed{\begin{aligned} &\max \quad z := \begin{pmatrix}-1 & 0 & 0 & 1\end{pmatrix} x \\ &\text{s.t.} \\ &\qquad \begin{pmatrix}-1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 1\end{pmatrix} \\ &\qquad x \geq \mathbb{0}\end{aligned}}$$

$$x(t) := \begin{pmatrix}0 \\ 0 \\ 2 \\ 1\end{pmatrix} + t \begin{pmatrix}1 \\ 0 \\ 1 \\ 2\end{pmatrix}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Claim 2

$z \to \infty$ when $t \to \infty$.

$$\begin{aligned} \max \quad & z := \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix} x \\ \text{s.t.} \quad & \\ & \underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b} \\ & x \geq \mathbb{0} \end{aligned}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 1

$x(t)$ is feasible for all $t \geq 0$.

Proof

$$x(t) = \bar{x} + tr \geq \mathbb{0} \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \bar{x}, r \geq \mathbb{0}.$$

$$Ax(t) = A\left[\bar{x} + tr\right] = \underbrace{A\bar{x}}_{b} + t \underbrace{Ar}_{\mathbb{0}} = b.$$

$$\max \quad z := \underbrace{\begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}}_{c^T} x$$

s.t.

$$\underbrace{\begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b}$$

$$x \geq \mathbb{0}$$

$$x(t) := \underbrace{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\bar{x}} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}_{r}$$

Claim 2

$z \to \infty$ when $t \to \infty$.

Proof

$$z = c^T x(t) = c^T [\bar{x} + tr] = c^T \bar{x} + t \underbrace{c^T r}_{=1>0}.$$

Exercise

Generalize and prove the following proposition.

Proposition

The linear program,

$$\max\{c^T x : Ax = b, x \geq \mathbb{0}\}$$

is unbounded if we can find $\bar{x}$ and r such that

$$\bar{x} \geq \mathbb{0}, \quad r \geq \mathbb{0}, \quad A\bar{x} = b, \quad Ar = \mathbb{0} \quad \text{and} \quad c^T r > 0.$$

Recap

1. For linear programs, exactly one of the following holds. It is
   (A) infeasible,
   (B) unbounded, or
   (C) has an optimal solution.

2. If (A) occurs, there is a short proof of that fact.

3. If (B) occurs, there is a short proof of that fact.

4. For an optimal solution, there is a short proof that it is optimal.

Remark

We have not yet shown you how to find such proofs.