CO 250: Introduction to Optimization

Module 5: Integer Programs (Cutting Planes)

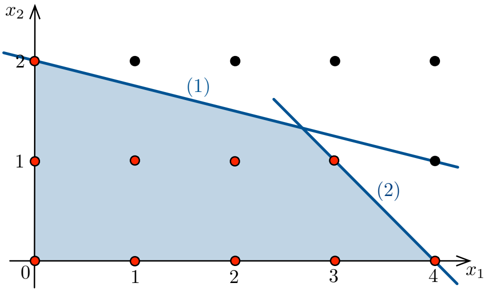
Overview

In this lecture, we will:

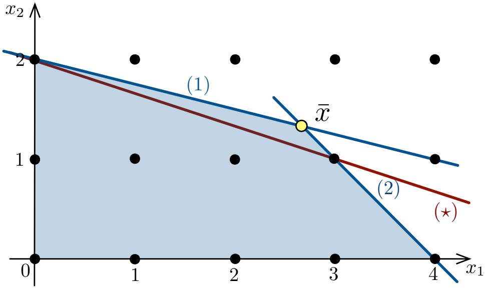
1. investigate a class of algorithms known as cutting planes,
2. restrict ourselves to pure Integer Programs, and
3. present a highly simplified view.

Our First Integer Program:

$$\begin{aligned} \max\ & (2,5)x \\ \text{s. t.}\ & \\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix} \\ & x \ge \mathbf{0},\ x \text{ integer} \end{aligned}$$



$$\max \ (2,5)x$$
s. t.
$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8 \\ 4 \end{pmatrix} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$
$$x \ge \mathbf{0}$$



Using Simplex, we find that $\bar{x} = \left(\frac{8}{3}, \frac{4}{3}\right)^\top$ is optimal. NOT INTEGER!

We now search for a constraint $\alpha^\top x \le \beta$ that

- is satisfied for all feasible solutions to the IP, and
- is not satisfied for $\bar{x}$.

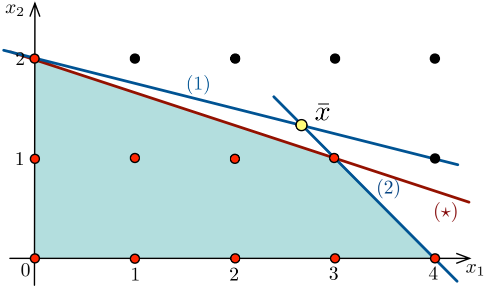
We will call this constraint a cutting plane for $\bar{x}$.

Example:

$$x_1 + 3x_2 \le 6. \qquad (\star)$$

After adding $(\star)$ to our relaxation, we get

$$\begin{aligned} \max\ & (2,5)x \\ \text{s. t.}\ & \\ & \begin{pmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \\ 6 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \\ & x \geq \mathbf{0} \end{aligned}$$



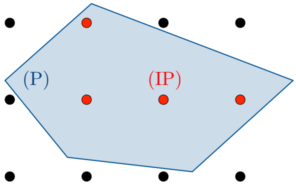
Using Simplex, we get: $x' = (3,1)^\top$ is optimal. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

We have now solved our first IP.

Cutting Plane Scheme

$$\max\left\{c^\top x : Ax \le b, x \text{ integer}\right\} \qquad \text{(IP)}$$



- Let (P) denote $\max\{c^\top x : Ax \le b\}$.
- If (P) is infeasible, then STOP. (IP) is also infeasible.
- Let $\bar{x}$ be the optimal solution to (P).
- If $\bar{x}$ is integral, then STOP. $\bar{x}$ is also optimal for (IP).
- Find a cutting plane $a^\top x \le \beta$ for $\bar{x}$.
- Add a constraint $a^\top x \le \beta$ to the system $Ax \le b$.

Question

How can we find cutting planes?

SIMPLEX DOES THIS FOR US!

Definition

Let $a \in \Re$, then $\lfloor a \rfloor$ denotes the largest integer $\leq a$.

Example

$$
\begin{aligned}
\lfloor 3.7 \rfloor &= 3 \\
\lfloor 62 \rfloor &= 62 \\
\lfloor -2.1 \rfloor &= -3
\end{aligned}
$$

Example

$$\max \ (2,5)x$$
s. t.
$$\begin{pmatrix}1 & 4\\ 1 & 1\end{pmatrix} x \leq \begin{pmatrix}8\\4\end{pmatrix} \qquad \begin{matrix}(1)\\(2)\end{matrix}$$
$$x \geq \mathbf{0}, \ x \text{ integer}$$

Add a slack variable, $x_3 \geq 0$, and rewrite (1) as $x_1 + 4x_2 + x_3 = 8$.

Add another slack variable, $x_4 \geq 0$, and rewrite (2) as $x_1 + x_2 + x_4 = 4$.

Since x_1 and x_2 are integers, $x_3 = 8 - x_1 - 4x_2$ and $x_4 = 4 - x_1 - x_2$ are integers, too.

Thus, we can rewrite the IP as

$$\max \ (2,5,0,0)x$$
s. t.
$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x \leq \begin{pmatrix}8\\4\end{pmatrix}$$
$$x \geq \mathbf{0}, \ x \text{ integer}$$

Solving the IP

$$\max \ (2,5,0,0)x$$
s. t.
$$\begin{pmatrix}1 & 4 & 1 & 0\\ 1 & 1 & 0 & 1\end{pmatrix} x \leq \begin{pmatrix}8\\4\end{pmatrix}$$
$$x \geq \mathbf{0}, \ x \text{ integer}$$

We will now find a relaxation for the integer program.

Solving the IP

$$\begin{aligned}
&\max \quad (2,5,0,0)x \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}$$

We will use the Simplex algorithm to solve this.

Get an optimal basis $B = \{1,2\}$ and rewrite the basis in canonical form for B:

$$\begin{aligned}
&\max \quad (0,0,-1,-1)x + 12 \\
&\text{s. t.} \\
&\qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x \leq \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
&\qquad x \geq \mathbf{0}
\end{aligned}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$. NOT INTEGER

Let us use the canonical form to get a cutting plane for $\bar{x}$.

$$\begin{array}{|l|}\hline \max\ (0,0,-1,-1)x + 12 \\ \text{s. t.} \\ \qquad \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x \leq \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\ \qquad x \geq \mathbf{0} \\ \hline \end{array}$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the LP relaxation satisfies,

$$x_1 - \frac{1}{3}x_3 + \frac{4}{3}x_4 \leq \frac{8}{3}$$

$$x_1 + \left\lfloor -\frac{1}{3} \right\rfloor x_3 + \left\lfloor \frac{4}{3} \right\rfloor x_4 \leq \frac{8}{3}$$

$$x_1 - x_3 + x_4 \leq \frac{8}{3}$$

For every feasible solution to the IP, $x_1 - x_3 + x_4$ is integer.

Hence, every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq \left\lfloor \frac{8}{3} \right\rfloor = 2$$

$$
\begin{array}{ll}
\max & (0,0,-1,-1)x + 12 \\
\text{s. t.} & \\
& \begin{pmatrix} 1 & 0 & -1/3 & 4/3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} x \leq \begin{pmatrix} 8/3 \\ 4/3 \end{pmatrix} \\
& x \geq \mathbf{0}
\end{array}
$$

The basic solution is $\bar{x} = (8/3, 4/3, 0, 0)^\top$.

Every feasible solution to the IP satisfies

$$x_1 - x_3 + x_4 \leq 2 \qquad (\star)$$

However, $\bar{x}$ does not satisfy $(\star)$ as

$$1 \times \frac{8}{3} - 1 \times 0 + 1 \times 0 > 2$$

$\Rightarrow$ $(\star)$ is a cutting plane for $\bar{x}$.

We can rewrite $(\star)$ as

$$x_1 - x_3 + x_4 + x_5 = 2 \qquad \text{where} \qquad x_5 \geq 0.$$

We now add this to the relaxation.

$$\begin{array}{ll} \max & (0,0,-1,-1,0)x + 12 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x \le \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ & x \ge \mathbf{0} \end{array}$$

Solve this using the Simplex algorithm.

Get an optimal basis $B = \{1,2,3\}$ and rewrite the basis in canonical form for B:

$$\begin{array}{ll} \max & (0,0,0,-\frac{1}{2},-\frac{3}{2})x + 11 \\ \text{s. t.} & \\ & \begin{pmatrix} 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{pmatrix} x \le \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ & x \ge \mathbf{0} \end{array}$$

The basic optimal solution is $x' = (3,1,1,0,0)^\top$. INTEGER!

Since x' is optimal for the IP relaxation, x' is also optimal for the IP!

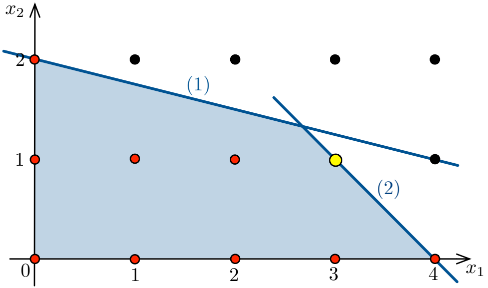
$(3,1,1,0,0)^\top$ is optimal for

$$\begin{array}{ll}\max & (0,0,-1,-1,0)x+12 \\ \text{s. t.} & \begin{pmatrix} 1 & 0 & -1/3 & 4/3 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 \\ 1 & 0 & -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8/3 \\ 4/3 \\ 2 \end{pmatrix} \\ & x \geq \mathbf{0} \;\; x \text{ integer} \end{array}$$

$\Rightarrow$

$(3,1)^\top$ is optimal for

$$\begin{array}{ll}\max & (2,5)x \\ \text{s. t.} & \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 8 \\ 4 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix} \\ & x \geq \mathbf{0}, \;\; x \text{ integer} \end{array}$$



Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}$$

$N = \{j : j \notin B\}$

$\bar{x}$ basic ($\bar{x}_N = \mathbf{0}, \bar{x}_B = b$)

$r(i)$ index of i^{th} basic variable

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

We know that every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j = b_i.$$

Getting Cutting Planes in General

Solve the relaxation and get the LP in a canonical form for B.

$$\begin{array}{|l|}\hline \max \quad \bar{c}^\top x + \bar{z} \\ \text{s. t.} \\ \qquad x_B + A_N x_N = b \\ \qquad x \geq \mathbf{0} \\ \hline \end{array}$$

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$r(i)$ index of i^{th} basic variable.

Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to the LP relaxation satisfies

$$x_{r(i)} + \sum_{j \in N} A_{ij} x_j \leq b_i. \quad \Longrightarrow \quad \underbrace{x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j}_{\text{integer for all } x \text{ integer}} \leq b_i.$$

Hence, every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b_i \rfloor$$

Solve the relaxation and get the LP in a canonical form for B.

$$\boxed{\begin{array}{ll} \max & \bar{c}^\top x + \bar{z} \\ \text{s. t.} & \\ & x_B + A_N x_N = b \\ & x \geq \mathbf{0} \end{array}}$$

$N = \{j : j \notin B\}$

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Suppose $\bar{x}$ is NOT INTEGER. Then, b_i is fractional for some value i.

Every feasible solution to IP satisfies

$$x_{r(i)} + \sum_{j \in N} \lfloor A_{ij} \rfloor x_j \leq \lfloor b \rfloor$$

However, $\bar{x}$ does not satisfy ($\star$) as

$$\underbrace{x_{r(i)}}_{b_i} + \sum_{j \in N} \lfloor A_{ij} \rfloor \underbrace{x_j}_{=0} = b_i > \lfloor b_i \rfloor .$$

$\Rightarrow$ ($\star$) is a cutting plane for $\bar{x}$.

The Good and the Bad

The Good News:

- The Simplex based cutting plane algorithm eventually will terminate.

The Bad News

- If implemented in this way, it will be terribly slow.

Ways we can Improve the Algorithm

- Do not use the 2-phase Simplex algorithm to reoptimize; work with the dual instead.
- Add more than one cutting plane at at time.
- Combine it with a divide and conquer strategy (branch and bound).

Recap

- We solved the LP relaxation of an integer program.
- If we get an integer solution, we know it is optimal for an integer program; otherwise, we add a cutting plane.
- Cutting planes can be obtained from the final canonical form.
- Careful implementation is key to success.