CO 250: Introduction to Optimization

Module 3: Duality through Examples (Weak Duality)

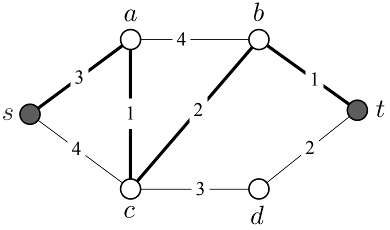
Suppose we are given an instance of the shortest path problem...

- a graph $G = (V, E)$,
- a non-negative length c_e for each edge $e \in E$, and
- a pair of vertices s and t in V.

A width-assignment is of the form

$$\{y_U \; : \; \delta(U) \; s,t\text{-cut}\}$$

and this is feasible if the total width of cuts containing edge e is no more than c_e, for all $e \in E$.



Proposition

If y is a feasible width assignment, then any s,t-path must have length at least

$$\sum (y_U \; : \; U \; s,t\text{-cut}).$$

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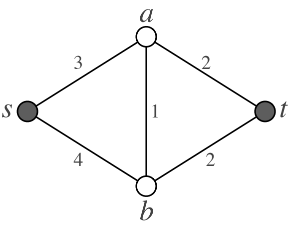


Seemingly, we used an adhoc argument, taylormade for shortest paths. . .

but, as we will now see, there is a constructive and quite mechanical way to derive the Proposition via linear programming!

An Instructive Example LP

The LP on the right is feasible...

E.g., $x^1 = (8, 16)^\top$ and $x^2 = (5, 13)^\top$ are feasible.

$$
\begin{array}{ll}
\min & (2,3)x \\
\text{s.t.} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\
& x \geq \mathbb{0}
\end{array}
$$

Question

Can you find an optimal solution?

x^1 has an objective of value 64 and x^2 has a value of 49 $\longrightarrow$ x^1 is definitely not optimal, but is x^2?

Feasible widths provide a lower-bound on the length of a shortest s, t-path...

Question

Can we find a good lower-bound on the objective value of the above LP?

Deriving Valid Inequalities

Let's suppose that x is feasible for the LP on the right.

It follows that x satisfies

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

and it also satisfies

$$\begin{array}{r} (2,1)x \geq 20 \\ +\ (1,1)x \geq 18 \\ +\ (-1,1)x \geq 8 \\ \hline =\ (2,3)x \geq 46 \end{array}$$

$$\begin{array}{ll} \min & (2,3)x \\ \text{s.t.} & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0} \end{array}$$

Additionally, it satisfies

$$\begin{array}{r} y_1 \cdot (2,1)x \geq y_1 \cdot 20 \\ +\ y_2 \cdot (1,1)x \geq y_2 \cdot 18 \\ +\ y_3 \cdot (-1,1)x \geq y_3 \cdot 8 \\ \hline =\ (2y_1 + y_2 - y_3, y_1 + y_2 + y_3)x \\ \geq 20y_1 + 18y_2 + 8y_3 \end{array}$$

for $y_1, y_2, y_3 \geq 0$.

So, if x is feasible for the LP on the right, it also satisfies

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

for any $y_1, y_2, y_3 \geq 0$.

E.g., for $y = (0, 2, 1)^\top$, we obtain

$$(1, 3)x \geq 44$$

or

$$0 \geq 44 - (1, 3)x \qquad (\star)$$

$$\begin{aligned} \min \quad & (2, 3)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Therefore,

$$\begin{aligned} z(x) &= (2, 3)x \\ &\geq (2, 3)x + 44 - (1, 3)x \\ &= 44 + (1, 0)x \end{aligned}$$

Since $x \geq \mathbb{0}$, it follows that

$$z(x) \geq 44$$

for every feasible solution x!

State of Affairs

We now know that

(i) $x^2 = (5, 13)^\top$ is a solution to the LP of value 49 and

(ii) $z(x) \geq 44$ for every feasible solution to the LP.

$\longrightarrow$ The optimal value of the LP is in the interval $[44, 49]$.

Can we find a better lowerbound on $z(x)$ for a feasible x?

$$\begin{aligned} \min \quad & (2,3)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Lowerbounding $z(x)$ Systematically!

We know that a feasible x satisfies

$$0 \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x$$

for any $y_1, y_2, y_3 \geq 0$. Therefore,

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} + \left((2, 3) - (y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \right) x \quad (\star)$$

We want the second term to be non-negative. Since $x \geq \mathbb{0}$, this amounts to choosing y such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2, 3)$$

With such a y we then have from $(\star)$:

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix}$$

Lowerbounding $z(x)$ Systematically!

So, we choose $y \geq \mathbb{0}$ such that

$$(y_1, y_2, y_2) \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \leq (2, 3) \quad (\star)$$

yields

$$z(x) \geq (y_1, y_2, y_3) \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \qquad (\Diamond)$$

Idea

Find the best possible lower-bound on z. I.e., find $y \geq \mathbb{0}$ such that $(\star)$ holds, and the right-hand side of $(\Diamond)$ is maximized!

This is a Linear Program:

$$\begin{aligned} \max \quad & (20, 18, 8)y \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} y \leq (2, 3) \\ & y \geq \mathbb{0} \end{aligned}$$

Lowerbounding $z(x)$ Systematically!

This is a Linear Program:

$$\begin{aligned} \max \quad & (20, 18, 8)y \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} y \le (2, 3) \\ & y \ge \mathbb{0} \end{aligned}$$

Solving it gives:

$$\begin{aligned} \bar{y}_1 &= 0 \\ \bar{y}_2 &= 5/2 \\ \bar{y}_3 &= 1/2 \end{aligned}$$

and the objective value is 49.

There is no feasible solution x to

$$\begin{aligned} \min \quad & (2, 3)x \\ \text{s.t.} \quad & \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} x \ge \begin{pmatrix} 20 \\ 18 \\ 8 \end{pmatrix} \\ & x \ge \mathbb{0} \end{aligned}$$

which has an objective value smaller than 49.

Since $x^2 = (5, 13)^\top$ is a feasible solution with value 49, it must be optimal!

A General Argument

Suppose now we are given the LP

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax \geq b \\ & x \geq \mathbb{0} \end{array}$$

Any feasible solution x must satisfy

$$y^\top Ax \geq y^\top b,$$

for $y \geq \mathbb{0}$, and hence also

$$0 \geq y^\top b - y^\top Ax$$

Therefore,

$$\begin{array}{rcl} z(x) & = & c^\top x \\ & \geq & c^\top x + y^\top b - y^\top Ax \\ & = & y^\top b + (c^\top - y^\top A)x \end{array}$$

If we also know that

$$A^\top y \leq c$$

then $x \geq \mathbb{0}$ implies that $z(x) \geq y^\top b$.

The best lower-bound on $z(x)$ can be found by the following LP:

$$\begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y \leq c \\ & y \geq \mathbb{0} \end{array}$$

The Dual LP

The linear program is called the dual of primal LP

$$\begin{aligned} \max \quad & b^T y & \text{(D)} \\ \text{s.t.} \quad & A^T y \le c \\ & y \ge \mathbb{0} \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x & \text{(P)} \\ \text{s.t.} \quad & Ax \ge b \\ & x \ge \mathbb{0} \end{aligned}$$

Theorem

[Weak Duality] If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), then $b^T\bar{y} \le c^T\bar{x}$.

Proof:

$$\begin{aligned} b^T\bar{y} &= \bar{y}^T b \\ &\le \bar{y}^T(A\bar{x}) && \text{as } \bar{y} \ge \mathbb{0} \text{ and } b \le A\bar{x}, \\ &= (A^T\bar{y})^T\bar{x} \\ &\le c^T\bar{x} && \text{as } \bar{x} \ge \mathbb{0} \text{ and} A^T\bar{y} \le c. \end{aligned}$$

□

Lowerbounding the Length of s, t-Paths

Recap: Shortest Path LP

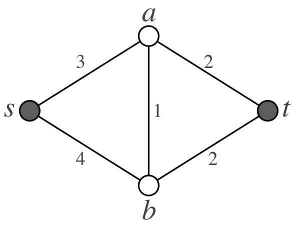
Given a shortest path instance $G = (V, E)$, $s, t \in V$, $c_e \geq 0$ for all $e \in E$, the shortest-path LP is

$$\begin{array}{lll} \min & \sum \left(c_e x_e : e \in E \right) & \\ \text{s.t.} & \sum \left(x_e : e \in \delta(U) \right) \geq 1 & (U \subseteq V, s \in U, t \notin U) \\ & x \geq \mathbb{0}, x \text{ integer} & \end{array}$$

Let's look at an example!

Shortest Path: Example

On the right, we see a sample instance of the shortest-path problem.



Here is the corresponding IP:

$$\min \quad (3,4,1,2,2)x$$

$$\text{s.t.} \quad \begin{array}{c} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} x \geq \mathbb{1}$$

$x \geq \mathbb{0}, x$ integer

Note that if P is an s,t-path, then letting

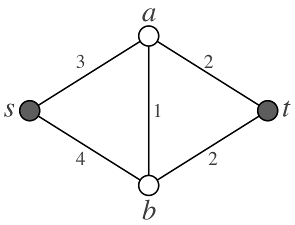
$$\bar{x}_e = \begin{cases} 1 & \text{if } e \text{ is an edge of } Q \\ 0 & \text{otherwise.} \end{cases}$$

for all $e \in E$ yields a feasible IP solution and its objective value is $c(P)$.

$$\min \quad (3,4,1,2,2)x$$

$$\text{s.t.} \quad \begin{array}{c} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} x \geq \mathbb{1}$$

$$x \geq \mathbb{0}, x \text{ integer}$$



Example:

$$P = sa, ab, bt$$

is an s,t-path.

$$x = (1,0,1,0,1)^T$$

is feasible for the IP, and its value is 6.

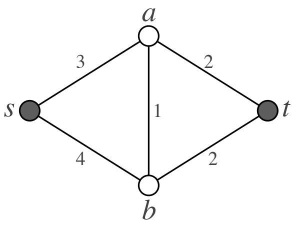
Remark

The optimal value of the shortest path IP is, at most, the length of a shortest s,t-path.

$$\min \quad (3,4,1,2,2)x \qquad \text{(P)}$$

$$\text{s.t.} \quad \begin{array}{c} \\ \{s\} \\ \{s,a\} \\ \{s,b\} \\ \{s,a,b\} \end{array} \begin{array}{c} \begin{array}{ccccc} sa & sb & ab & at & bt \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array} x \geq \mathbb{1}$$

$$x \geq \mathbb{0}, x \ \cancel{\text{integer}}$$



The resulting LP is called the linear programming relaxation of the IP.

Note that dropping the integrality restriction can not increase the optimal value.

Straight from Weak Duality theorem, we have that:

Remark

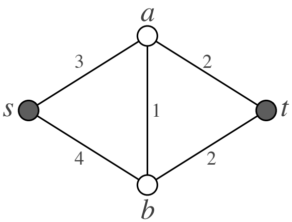
The dual of (P) has optimal value no larger than that of (P)!

The dual of the shortest path LP on the previous slide is given by

$$\max \quad \mathbb{1}^\top y$$

$$\text{s.t.} \quad \begin{array}{c} \\ sa \\ sb \\ ab \\ at \\ bt \end{array} \begin{array}{c} \begin{array}{cccc} \{s\} & \{s,a\} & \{s,b\} & \{s,a,b\} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} y \le \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$y \ge \mathbb{0}$$



Note that dual solutions assign the value $y_U \ge 0$ to every s,t-cut $\delta(U)$!

Focus on the constraint for edge ab:

$$y_{\{s,a\}} + y_{\{s,b\}} \le 1$$

Remark

y is feasible for the above LP if and only if it is a feasible width assignment for the s,t-cuts in the given shortest path instance!

General Shortest Path Instances

Input: $G = (V, E)$, $c_e \geq 0$ for all $e \in E$, $s, t \in V$.

Shortest path LP:

$$\begin{aligned} \min \quad & \sum (c_e x_e \,:\, e \in E) \\ \text{s.t.} \quad & \sum (x_e \,:\, e \in \delta(U)) \geq 1 \\ & \qquad\qquad (\delta(U)\ s, t - \text{cut}) \\ & x \geq \mathbb{0} \end{aligned}$$

The LP is of the form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq \mathbb{1} \\ & x \geq \mathbb{0} \end{aligned} \tag{P}$$

where

(i) A has a column for every edge and a row for every s, t-cut $\delta(U)$.

(ii) $A[U, e] = 1$ if $e \in \delta(U)$ and 0 otherwise.

The LP is of the form

$$\min \quad c^T x \qquad \text{(P)}$$
$$\text{s.t.} \quad Ax \geq \mathbb{1}$$
$$x \geq \mathbb{0}$$

where

(i) A has a column for every edge and a row for every s,t-cut $\delta(U)$.

(ii) $A[U,e] = 1$ if $e \in \delta(U)$, and 0 otherwise.

Its dual is of the form

$$\max \quad \mathbb{1}^T y \qquad \text{(D)}$$
$$\text{s.t.} \quad A^T y \leq c$$
$$y \geq \mathbb{0}$$

Note that the dual has a constraint for every edge $e \in E$. The left-hand side of this constraint is

$$\sum (y_U \ : \ e \in \delta(U))$$

and the right-hand side is c_e.

Remark

Feasible solutions to (D) correspond precisely to feasible width assignments. Weak duality implies that $\sum y_U$ is, at most, the length of a shortest s,t-path!

Recap

- The dual LP of
$$\min\{c^T x \,:\, Ax \geq b, x \geq \mathbb{0}\} \tag{P}$$
is given by
$$\max\{b^T y \,:\, A^T y \leq c, y \geq \mathbb{0}\} \tag{D}$$
- If x is feasible for (P) and y feasible for (D), then $b^T y \leq c^T x$.
- The LP relaxation of an integer program is obtained by dropping the integrality restriction.
- The dual of the shortest path LP is given by

$$\begin{aligned} \max \quad & \sum (y_U \,:\, \delta(U) \ s,t\text{-cut}) \\ \text{s.t.} \quad & \sum (y_U \,:\, e \in \delta(U)) \leq c_e \quad (e \in E) \\ & y \geq \mathbb{0} \end{aligned}$$