CO 250 – Exercises on LP Duality and Related Topics

(supplement to Chapters 3 & 4 of the textbook)

Please read Chapter 3.1 and Chapter 4, and then attempt as many of the following exercises as possible. Brief outlines of some of the solutions will be posted on Learn.

Problem 1: LP Duality

Write down the dual for each of the linear programs given below using the methods from the course. Some parts have two equivalent LPs (written side-by-side in a box); write down the dual of each of the two LPs and convince yourself that these two dual LPs are equivalent.

(a)

$$\begin{array}{ll} \max & (1, \ 2)\, x \\ \text{subject to} & \\ & \begin{pmatrix} 2 & 3 \\ 5 & 7 \\ 11 & 23 \end{pmatrix} x \begin{array}{c} \le \\ = \\ \ge \end{array} \begin{pmatrix} 100 \\ 200 \\ 300 \end{pmatrix} \\ & x_1 \le \mathbf{0}, \quad x_2 \ge \mathbf{0} \end{array}$$

(b)

$$\begin{array}{ll} \min & (10, \ 20, \ 30)\, x \\ \text{subject to} & \\ & \begin{pmatrix} 4 & 2 & 3 \\ 8 & 1 & 2 \end{pmatrix} x \begin{array}{c} \le \\ \ge \end{array} \begin{pmatrix} 100 \\ -200 \end{pmatrix} \\ & x_1 \le \mathbf{0}, \quad x_2 \ge \mathbf{0} \end{array}$$

(c)

$$\begin{array}{ll} \min & (-1, \ 2, \ -3, \ 4)\, x \\ \text{subject to} & \\ & \begin{pmatrix} -1 & 2 & 1 & 7 \\ 2 & 0 & -3 & 1 \\ 1 & 3 & 2 & -1 \end{pmatrix} x \begin{array}{c} = \\ \ge \\ \le \end{array} \begin{pmatrix} 100 \\ -200 \\ 300 \end{pmatrix} \\ & x_1 \ge \mathbf{0}, \quad x_2 \le \mathbf{0}, \quad x_4 \le \mathbf{0} \end{array}$$

(d)

$$\begin{array}{ll} \max & a^T x \\ \text{subject to} & \\ & \begin{array}{rcl} A\,x & = & b \\ B\,x & \le & c \end{array} \\ & x \quad \text{free}, \end{array}$$

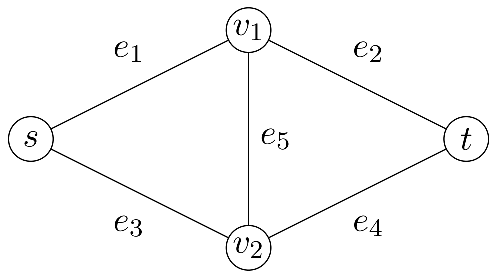
where A, B are matrices and a, b, c are vectors of appropriate dimensions.

(e)

$$
\begin{array}{ll}
\min & a^T x + b^T z \\
\text{subject to} & \\
& \begin{array}{rcrcl} A\,x & + & B\,z & = & c \\ C\,x & - & D\,z & \leq & d \end{array} \\
& x \quad \text{free}\,, \qquad z \geq \mathbf{0}
\end{array}
$$

where A, B, C, D are matrices and a, b, c, d are vectors of appropriate dimensions.

(f) Consider the following graph G. Assume that each edge e_i has weight $c_i \geq 0$.



(i)

$$
\begin{array}{lrll}
\text{(LP-(i))} \quad \min & \sum_{i=1}^{5} c_i x_i & & \\
\text{subject to} & x_1 + x_3 & \geq 1 & (\text{cut } \delta(s) = \{e_1, e_3\}) \\
& x_2 + x_3 + x_5 & \geq 1 & (\text{cut } \delta(\{s, v_1\}) = \{e_2, e_3, e_5\}) \\
& x_1 + x_4 + x_5 & \geq 1 & (\text{cut } \delta(\{s, v_2\}) = \{e_1, e_4, e_5\}) \\
& x_2 + x_4 & \geq 1 & (\text{cut } \delta(\{s, v_1, v_2\}) = \{e_2, e_4\}) \\
& x_i & \geq 0 & (i = 1, \ldots, 5)
\end{array}
$$

(ii)

$$
\begin{array}{lrll}
\text{(LP-(ii))} \quad \min & \sum_{i=1}^{5} c_i x_i & & \\
\text{subject to} & x_1 + x_3 & = 1 & \delta(s) = \{e_1, e_3\} \\
& x_1 + x_2 + x_5 & = 1 & \delta(v_1) = \{e_1, e_2, e_5\} \\
& x_3 + x_4 + x_5 & = 1 & \delta(v_2) = \{e_3, e_4, e_5\} \\
& x_2 + x_4 & = 1 & \delta(t) = \{e_2, e_4\} \\
& x_i & \geq 0 & (i = 1, \ldots, 5)
\end{array}
$$

(g)

$$
\begin{array}{ll}
(P^{initial}) \quad \max \quad z = & (1,1)x \\
\text{subject to} & \\
& \begin{pmatrix} -1 & 0 \end{pmatrix} x \ \geq \ \begin{pmatrix} 10 \end{pmatrix} \\
& x \geq \ \mathbf{0}
\end{array}
\qquad
\begin{array}{ll}
(P) \quad \max \quad z = & (1,1,0)x \\
\text{subject to} & \\
& \begin{pmatrix} -1 & 0 & -1 \end{pmatrix} x \ = \ \begin{pmatrix} 10 \end{pmatrix} \\
& x \geq \ \mathbf{0}
\end{array}
$$

(h)

$$(P^{initial}) \quad \max \quad z = (2,1)x$$

subject to

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x \begin{matrix} = \\ \leq \end{matrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

$$(P) \quad \max \quad z = (2,1,0)x$$

subject to

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

Solution: The duals for parts (a)–(e) are stated below:

(a)

$$\begin{array}{ll} \min & (100,\ 200,\ 300)y \\ \text{subject to} & \\ & \begin{pmatrix} 2 & 5 & 11 \\ 3 & 7 & 23 \end{pmatrix} y \begin{matrix} \leq \\ \geq \end{matrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & y_1 \geq \mathbf{0}\ ,\ \ y_2 \text{ free},\ \ y_3 \leq \mathbf{0} \end{array}$$

(b)

$$\begin{array}{ll} \max & (100, -200)y \\ \text{subject to} & \\ & \begin{pmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 2 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix} \\ & y_1 \leq \mathbf{0}\ ,\ \ y_2 \geq \mathbf{0} \end{array}$$

(c)

$$\begin{array}{ll} \max & (100,\ -200,\ 300)x \\ \text{subject to} & \\ & \begin{pmatrix} -1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & -3 & 2 \\ 7 & 1 & -1 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \\ \geq \end{matrix} \begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix} \\ & y_1 \text{ free},\ \ y_2 \geq \mathbf{0}\ ,\ \ y_3 \leq \mathbf{0} \end{array}$$

(d)

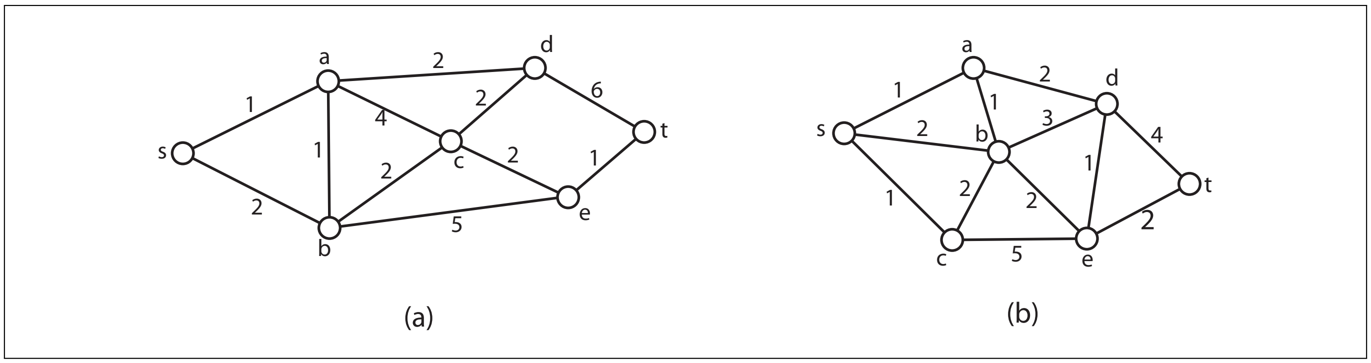
$$\begin{array}{ll} \min & b^T y + c^T w \\ \text{subject to} & \\ & A^T y + B^T w \ = \ a \\ & y \text{ free},\ \ w \geq 0 \end{array}$$

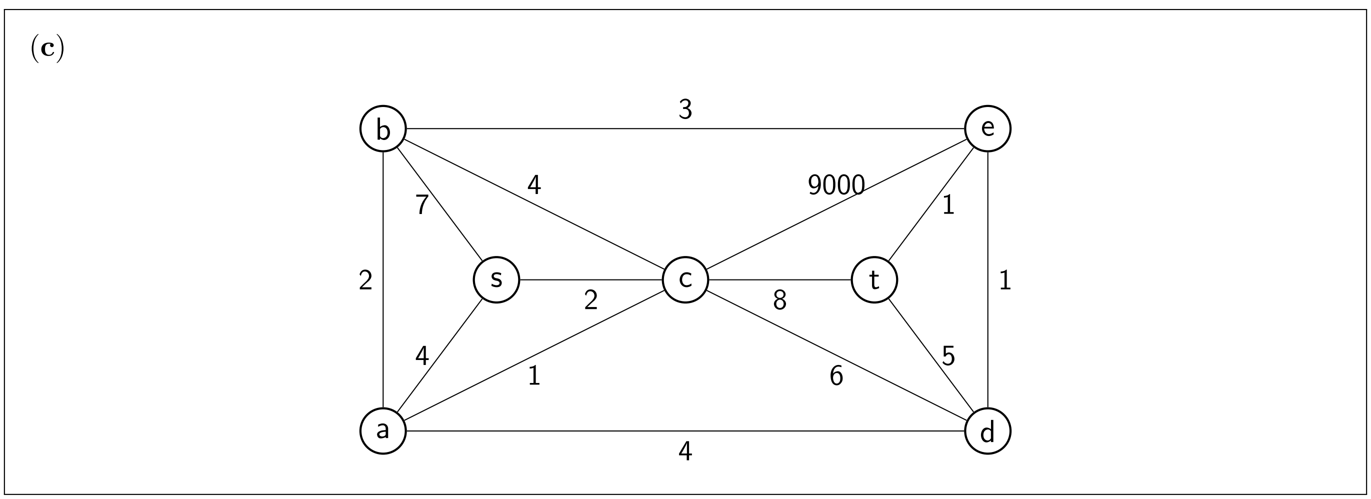
(e)

$$
\begin{array}{ll}
\max & c^T y + d^T w \\
\text{subject to} & \\
& \begin{array}{rcl} A^T y + C^T w & = & a \\ B^T y - D^T w & \leq & b \end{array} \\
& y \text{ free}, \;\; w \leq 0
\end{array}
$$

Problem 2: Shortest Path Algorithm

Several examples of the shortest st-path problem are given in the figures below. Each of these figures has a graph $G = (V, E)$ and each edge $e \in E$ is labeled by its length c_e, and each vertex is labeled by its name. For each of these examples, find a shortest st-path by running the algorithm presented in Chapter 3.1 of the textbook. Show all your work.





Solution: The solution for part (c) is given below. The solutions for the other parts are similar, and are skipped.

(c) We find a shortest st-path using the Shortest Path Algorithm. The details of the algorithm are as follows:

Iteration 1: $y = 0, U = \{s\}, \delta(U) = \{sa, sb, sc\}$. Comparing the slack of edges in $\delta(U)$:

$$\begin{aligned}
\text{slack}_y(sa) &= c_{sa} = 4, \\
\text{slack}_y(sb) &= c_{sb} = 7, \\
\text{slack}_y(sc) &= c_{sc} = 2. \quad \longleftarrow \text{ minimum}
\end{aligned}$$

The edge that has the minimum slack is sc. Thus, we set $y_{\{s\}} = 2$, change edge sc to arc $\overrightarrow{sc}$, and add vertex c to U.

Iteration 2: $U = \{s, c\}, \delta(U) = \{sa, sb, ca, cb, cd, ce, ct\}$. Comparing slacks:

$$\begin{aligned}
\text{slack}_y(sa) &= c_{sa} - y_{\{s\}} = 4 - 2 = 2, \\
\text{slack}_y(sb) &= c_{sb} - y_{\{s\}} = 7 - 2 = 5, \\
\text{slack}_y(ca) &= c_{ca} = 1, \quad \longleftarrow \text{ minimum} \\
\text{slack}_y(cb) &= c_{cb} = 4, \\
\text{slack}_y(cd) &= c_{cd} = 6, \\
\text{slack}_y(ce) &= c_{ce} = 9000, \\
\text{slack}_y(ct) &= c_{ct} = 8.
\end{aligned}$$

The edge ca gives us the smallest slack, so we set $y_{\{s,c\}} = 1$, change edge ca to arc $\overrightarrow{ca}$, and add vertex a to U.

Iteration 3: $U = \{s, c, a\}, \delta(U) = \{sb, cb, cd, ce, ct, ab, ad\}$. The slacks of these edges are:

$$\begin{aligned}
\text{slack}_y(sb) &= c_{sb} - y_{\{s\}} - y_{\{s,c\}} = 7 - 2 - 1 = 4, \\
\text{slack}_y(cb) &= c_{cb} - y_{\{s,c\}} = 4 - 1 = 3, \\
\text{slack}_y(cd) &= c_{cd} - y_{\{s,c\}} = 6 - 1 = 5, \\
\text{slack}_y(ce) &= c_{ce} - y_{\{s,c\}} = 9000 - 1 = 8999, \\
\text{slack}_y(ct) &= c_{ct} - y_{\{s,c\}} = 8 - 1 = 7, \\
\text{slack}_y(ab) &= c_{ab} = 2, \quad \longleftarrow \text{ minimum} \\
\text{slack}_y(ad) &= c_{ad} = 4.
\end{aligned}$$

Now we set $y_{\{s,c,a\}} = 2$, change edge ab to arc $\overrightarrow{ab}$, and add vertex b to U.

Iteration 4: $U = \{s, c, a, b\}, \delta(U) = \{cd, ce, ct, ad, be\}$. The slacks of these edges are:

$$\begin{aligned}
\text{slack}_y(cd) &= c_{cd} - y_{\{s,c\}} - y_{\{s,c,a\}} = 6 - 1 - 2 = 3, \\
\text{slack}_y(ce) &= c_{ce} - y_{\{s,c\}} - y_{\{s,c,a\}} = 9000 - 1 - 2 = 8997, \\
\text{slack}_y(ct) &= c_{ct} - y_{\{s,c\}} - y_{\{s,c,a\}} = 8 - 1 - 2 = 5, \\
\text{slack}_y(ad) &= c_{ad} - y_{\{s,c,a\}} = 4 - 2 = 2, \quad \longleftarrow \text{ minimum} \\
\text{slack}_y(be) &= c_{be} = 3.
\end{aligned}$$

Thus, we set $y_{\{s,c,a,b\}} = 2$, change edge ad to arc $\overrightarrow{ad}$, and add vertex d to U.

Iteration 5: $U = \{s, c, a, b, d\}, \delta(U) = \{ce, ct, be, de, dt\}$. The slacks of these edges are:

$$\begin{aligned}
\text{slack}_y(ce) &= c_{ce} - y_{\{s,c\}} - y_{\{s,c,a\}} - y_{\{s,c,a,b\}} = 9000 - 1 - 2 - 2 = 8995, \\
\text{slack}_y(ct) &= c_{ct} - y_{\{s,c\}} - y_{\{s,c,a\}} - y_{\{s,c,a,b\}} = 8 - 1 - 2 - 2 = 3, \\
\text{slack}_y(be) &= c_{be} - y_{\{s,c,a,b\}} = 3 - 2 = 1, \quad \longleftarrow \text{ minimum} \\
\text{slack}_y(de) &= c_{de} = 1, \quad \longleftarrow \text{ minimum} \\
\text{slack}_y dt &= c_{dt} = 5.
\end{aligned}$$

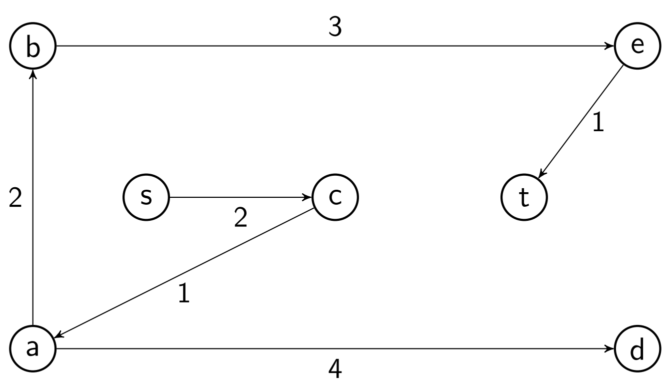
We have a tie! Fortunately, it is fine to choose any of edges that attains the minimum slack, so let's pick be. Then we set $y_{\{s,b,a,c,d,e\}} = 1$, change edge be to arc $\overrightarrow{be}$, and add vertex e to U.

Iteration 6: $U = \{s, c, a, b, d, e\}, \delta(U) = \{ct, dt, et\}$. The slacks of these edges are:

$$\begin{aligned}
\text{slack}_y(ct) &= c_{ct} - y_{\{s,c\}} - y_{\{s,c,a\}} - y_{\{s,c,a,b\}} - y_{\{s,c,a,b,d\}} = 8 - 1 - 2 - 2 - 1 = 2, \\
\text{slack}_y(dt) &= c_{dt} - y_{\{s,c,a,b,d\}} = 5 - 1 = 4, \\
\text{slack}_y(et) &= c_{et} = 1. \quad \longleftarrow \text{ minimum}
\end{aligned}$$

Thus, we set $y_{\{s,c,a,b,d,e\}} = 1$, change edge et to arc $\overrightarrow{et}$, and add vertex t to U.

Finally, now $t \in U$, and we may terminate. The arcs we have collected are:



Thus, an st-path is

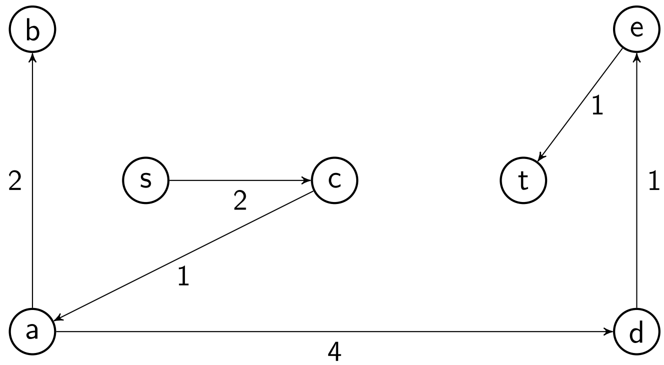
$$P = \{sc, ca, ab, be, et\},$$

with length $2 + 1 + 2 + 3 + 1 = 9$. The final cut widths we have are

$$\begin{aligned}
&y_{\{s\}} = 2, y_{\{s,c\}} = 1, y_{\{s,c,a\}} = 2, \\
&y_{\{s,c,a,b\}} = 2, y_{\{s,c,a,b,d\}} = 1, y_{\{s,c,a,b,d,e\}} = 1,
\end{aligned}$$

and $y_U = 0$ for all other st-cuts $\delta(U)$. These widths sum up to 9, which verifies that the above st-path is indeed (among) the shortest.

Remark: Observe that in iteration 5 above, we had a tie and could choose between edges be and de. Had we chosen de, instead, we would have obtained the following set of arcs at the termination of the algorithm:



Then we obtain the st-path $\{sc, ca, ad, de, et\}$, which also has length 9, and thus is optimal.

<u>Problem 3:</u> Outcome of Dual LP

Consider the following linear program:

$$
\begin{array}{lll}
\textbf{(P)} & \min & (c_1, \; c_2)x \\
& \text{subject to} & \\
& & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x \;\geq\; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
& & x \;\geq\; 0
\end{array}
$$

(a) Show that (P) is infeasible.
(**Hint**: use a certificate, try $(1,1)^T$.)

(b) Write the dual LP (D) of (P).

(c) Either find values for c_1, c_2 such that the dual (D) is infeasible, or explain (briefly) why this is not possible.

(d) Either find values for c_1, c_2 such that the dual (D) is unbounded, or explain (briefly) why this is not possible.

(e) Either find values for c_1, c_2 such that the dual (D) has an optimal solution, or explain (briefly) why this is not possible.

Solution:

(a) Suppose $\bar{x}$ is a feasible solution for (P), that is, $A\bar{x} \geq b$. Let $y = (1,1)$. Since $y \geq \mathbf{0}$, we have: $y^T A\bar{x} \geq y^T b$. However, $y^T A = \mathbf{0}$, and $y^T b = 2$, which implies that $0 \geq 2$, which is a contradiction. We conclude that (P) is infeasible.

(b) The dual problem (D), of the LP (P), is as follows:

$$
\begin{array}{lll}
\textbf{(D)} & \max & (1, \; 1)y \\
& \text{subject to} & \\
& & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} y \;\leq\; \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
& & y \;\geq\; 0
\end{array}
$$

(c) For $c_1 = c_2 = -1$, we observe that the LP (D) is infeasible. To see why, consider the certificate $f = (1,1)^T$. If $\bar{y}$ is a feasible solution for (D), then $A^T\bar{y} \leq c$, which implies $f^TA^T\bar{y} \leq f^Tc$. Observe that $f^TA^T = \mathbf{0}$ and $f^Tc = c_1 + c_2 = -2$, and thus we have: $0 \leq -2$, which is a contradiction. Hence, if $c_1 = c_2 = -1$, then the dual (D) is infeasible.

(d) For $c_1 = c_2 = 0$, we observe that $(0,0)^T$ is a feasible solution for the dual (D). Recall that the LP (P) is infeasible, by part (a). Hence, by the corollary of the *Weak Duality Theorem* (see Table 4.2 in the textbook), it follows that the dual (D) is either infeasible or unbounded. Since $(0,0)^T$ is a feasible solution for (D), we conclude that (D) is unbounded.

(e) We show that such c_1, c_2 do <u>not</u> exist. Suppose for a contradiction that such c_1 and c_2 do exist. Then by *Strong Duality Theorem* (see Theorem 4.3 in the textbook), since (D) has an optimal solution, its dual (P) must have an optimal solution as well. However, in part (a), we showed that (P) is infeasible, and thus we have a contradiction. We conclude that such c_1, c_2 do <u>not</u> exist.

<u>Problem 4:</u> Duality and CS conditions

Let $a_1, \ldots, a_k$ be a set of <u>distinct</u> numbers. Consider the following linear program (P):

$$\begin{array}{ll} (P) & \max \quad z = x_1 \\ \text{subject to} & \\ & x_1 \leq a_i \quad (i = 1, \ldots, k) \end{array}$$

. (a) Show that the optimal value is $\min\{a_1, \ldots, a_k\}$.

(b) Write the dual (D) of (P).

(c) Write the CS (Complementary Slackness) conditions for (P) and (D).

(d) Using the CS theorem (Theorem 4.5) prove that $\bar{y}$ is an optimal solution to (D) if and only if for all $i = 1, \ldots, k$,

$$\bar{y}_i = \begin{cases} 1 & \text{if } a_i = \min\{a_1, \ldots, a_k\} \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

(a) Clearly the optimal value z cannot be larger than $\min_i a_i$ because of the constraints (namely, $x_1 \leq a_i \quad (i = 1, \ldots, k)$) and $x_1 = min_i a_i$ is feasible. Therefore, it must be optimal.

(b) The primal (P) has k constraints and one variable. Thus, the dual (D) has k variables and one constraint:

$$\begin{array}{ll} (D) \quad \min & \displaystyle\sum_{j=1}^{k} a_j y_j \\ \text{subject to} & \\ & \displaystyle\sum_{j=1}^{k} y_j = 1 \\ & y \geq 0 \end{array}$$

(c) The CS conditions for a feasible solution $\bar{x}$ of (P) and a feasible solution $\bar{y}$ of (D) state that

$$\bar{y}_i > 0 \implies \bar{x}_1 = a_i \quad (i = 1, \ldots, k)$$

(d) Let i^* be the index such that $a_{i^*} = \min_{j=1,\ldots,k}\{a_j\}$. Then $\bar{x}_1 = a_{i^*}$ is optimal for the primal (P). By the CS conditions, we see that $\bar{y}_j = 0$, $\forall j \in \{1, \ldots, k\} - \{i^*\}$, because $\bar{x}_1 \neq a_j \quad (j \neq i^*$ and $j \in \{1, \ldots, k\})$, since the numbers a_j are distinct.

We must have $\sum_{j=1}^{k} a_j \bar{y}_j = \bar{x}_1 = a_{i^*}$, and this implies that $\bar{y}_{i^*} = 1$ (observe that $\bar{y}_j = 0$ for all other indices j).

<u>Problem 5:</u> Duality and CS conditions

Consider the following linear programming problem (P), where α denotes a real number.

$$(P) \qquad \begin{array}{lrcrcrcrcl} \text{maximize} & z = 5x_1 & + & x_2 & + & 8x_3 & + & \alpha x_4 & & \\ \text{subject to} & & & & & & & & & \\ & x_1 & - & x_2 & + & x_3 & + & 3x_4 & \leq & 1 \\ & & & x_2 & & & + & x_4 & \leq & 3 \\ & x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 5 \\ & x_1 \geq 0 & , & x_2 \geq 0 & , & x_3 \text{ free} & , & x_4 \leq 0 & & \end{array}$$

(a) Write down the dual (D) of (P) and the complementary slackness conditions.

(b) Suppose that $\alpha = 0$. Use the complementary slackness conditions to determine whether or not the following vectors are optimal solutions for (P), and justify your answers:

(i) $\hat{x} = [3, 3, 1, 0]^T$

(ii) $\bar{x} = [3, 2, 0, 0]^T$

(c) For what values of α is the solution $\bar{x} = (3, \;\; 2, \;\; 0, \;\; 0)^T$ optimal for (P)? Justify your answer.

Solution:

(a)

$$\min \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} y$$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} y \begin{matrix} \geq \\ \geq \\ = \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 1 \\ 8 \\ \alpha \end{pmatrix}$$

$$y_1 \geq 0, \; y_2 \geq 0, \; y_3 \text{ free}$$

CS conditions:
(CS1) $x_1 - x_2 + x_3 + 3x_4 < 1 \implies y_1 = 0$
(CS2) $x_2 + x_4 < 3 \implies y_2 = 0$
(CS3) $x_1 > 0 \implies y_1 + y_3 = 5$
(CS4) $x_2 > 0 \implies -y_1 + y_2 + y_3 = 1$
(CS5) $x_4 < 0 \implies 3y_1 + y_2 - y_3 = \alpha$

(b) If $\bar{x}$ is optimal, then there exists a $\bar{y}$ feasible for (D) such that $(\bar{x}, \bar{y})$ satisfies the complementary slackness conditions.

(i) $\hat{x}$ is not feasible, since the third constraint of (P) is violated.

(ii) $\bar{x}$ is feasible but is not optimal for $\alpha = 0$. From (CS2), $y_2 = 0$, and then from (CS3), (CS4), we get a 2×2 system of equations with solution $y_1 = 2, \quad y_3 = 3$. Then, check whether $y = [2, 0, 3]^T$ satisfies all of the dual constraints. Constraints (1)-(3) of the dual hold, but constraint (4) is violated. Hence, $\bar{x}$ is not optimal for (P) for $\alpha = 0$.

(c) Continuing from the solution to the previous part, we plug $y = [2, 0, 3]^T$ into constraint (4) of the dual to get $6 + 0 - 3 \leq \alpha$, thus $\alpha \geq 3$ to ensure that $\bar{x}$ is optimal for (P).

Problem 6: Duality and CS conditions

Consider the following linear-programming problem (P), where $\alpha \in \mathbb{R}$ is a parameter (α is NOT a variable).

$$(P) \quad \min \quad z = \begin{pmatrix} 1 & \alpha & 10 & \frac{25}{2} \end{pmatrix} x$$

subject to

$$\begin{pmatrix} 1 & 0 & 5 & 6 \\ 2 & 6 & 0 & 1 \end{pmatrix} x \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x_1 \geq 0, \; x_2 \leq 0, \; x_3 \text{ free }, \; x_4 \geq 0$$

(a) Write down the dual (D) of (P) and the CS (Complementary Slackness) conditions.

(b) Suppose that $\alpha = 0$. Then which of the three possible outcomes (infeasible, unbounded, optimal) applies to the dual (D)? Justify your answer.

(c) For what values of α is the solution $\bar{x}^T = (1\ 0\ 0\ 0)$ optimal for (P)? Justify your answer.

Solution:

(a) The dual LP (D) of (P) is stated below; after that, the CS conditions are stated.

$$\max \begin{pmatrix} 1 & 2 \end{pmatrix} y$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 6 \\ 5 & 0 \\ 6 & 1 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \\ \leq \end{matrix} \begin{pmatrix} 1 \\ \alpha \\ 10 \\ \frac{25}{2} \end{pmatrix}$$

$$y_1 \geq 0, \; y_2 \leq 0$$

The CS conditions for a feasible solution $\bar{x}$ of (P) and a feasible solution $\bar{y}$ of (D) are as follows:
(CS1) $x_1 + 5x_3 + 6x_4 > 1 \implies y_1 = 0$
(CS2) $2x_1 + 6x_2 + x_4 < 2 \implies y_2 = 0$
(CS3) $x_1 > 0 \implies y_1 + 2y_2 = 1$
(CS3) $x_2 < 0 \implies 6y_2 = \alpha$
(CS4) $x_4 > 0 \implies 6y_1 + y_2 = \frac{25}{2}$

(b) (D) is infeasible for $\alpha = 0$, since the 2nd constraint of (D) fixes $y_2 = 0$ (note that y_2 is non-positive), and then (after that) 1st constraint gives $y_1 \leq 1$ while 3rd constraint gives $y_1 = 2$.

(c) If $\bar{x}$ is optimal, then there exists a $\bar{y}$ feasible for (D) such that $(\bar{x}, \bar{y})$ satisfies the complementary slackness conditions. Since $\bar{x}_1 > 0$, $\bar{y}$ satisfies $\bar{y}_1 + 2\bar{y}_2 = 1$. Moreover, the third dual constraint implies that $5\bar{y}_1 = 10$, and therefore $\bar{y}^T = (2 \ -1/2)$. Such $\bar{y}$ is feasible whenever $6\bar{y}_2 \geq a$ and therefore $\alpha \leq -3$.

<u>Problem 7:</u> Geometry and CS conditions

Consider the following linear program:

$$
(P) \quad \begin{array}{lrcrcr}
\text{maximize} & x_1 & - & x_2 & & \\
\text{subject to} & & & & & \\
& x_1 & - & 2x_2 & \leq & 0 \\
& x_1 & - & x_2 & \leq & 1 \\
& -x_1 & - & x_2 & \leq & -3 \\
& -x_1 & & & \leq & -1
\end{array}
$$

The feasible region for (P) is indicated in the figure.

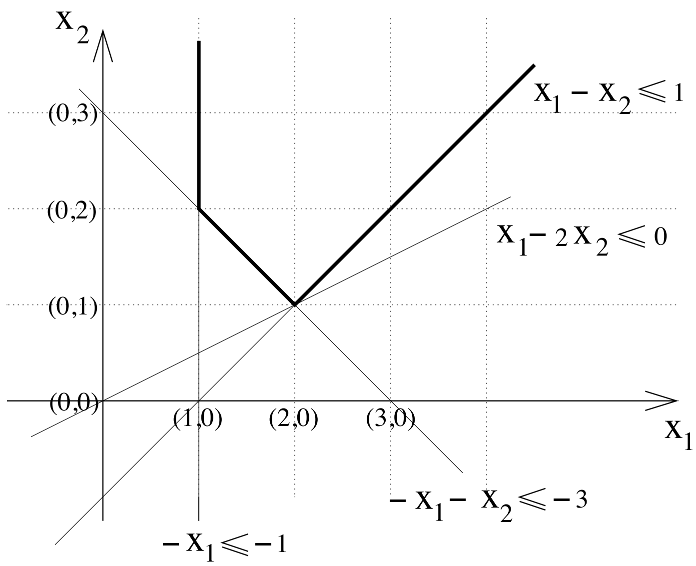


Figure 1: The feasible region of (P).

(a) Indicate what are the cones of tight constraints for each of the following points:

$$[1 \ \ 2]^T \quad [2 \ \ 1]^T \quad [3 \ \ 2]^T.$$

(b) Using geometry (and inspection), find an optimal solution x^* of (P).

(c) Write the dual (D) of (P) and write the complementary slackness conditions.

(d) Using parts (a),(b),(c) and complementary slackness, either construct two distinct optimal solutions for (D), or explain why this is not possible.

Solution:

(a) We introduce some notation. The LP (P) is of the form: max $\{c^T x \;\text{ subject to }\; Ax \le b\}$

where $c^T = [c_1, c_2] = [1, -1]$, and $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -1 & -1 \\ -1 & 0 \end{bmatrix}$, and $b = \begin{bmatrix} 0 \\ 1 \\ -3 \\ -1 \end{bmatrix}$.

By plugging $[1, 2]^T$ into $Ax \le b$, we see that the third and fourth constraints are tight. Thus, the cone of tight constraints for $[1, 2]^T$ is $\mathsf{cone}\{[-1, -1]^T, [-1, 0]^T\}$.

By plugging $[2, 1]^T$ into $Ax \le b$, we see that the first, second and third constraints are tight. Thus, the cone of tight constraints for $[2, 1]^T$ is $\mathsf{cone}\{[1, -2]^T, [1, -1]^T, [-1, -1]^T\}$.

By plugging $[3, 2]^T$ into $Ax \le b$, we see that the only tight constraint is the second one. Thus, the cone of tight constraints for $[3, 2]^T$ is $\mathsf{cone}\{[1, -1]^T\}$.

(b) From the geometry, it is clear that $[2, 1]^T$ is an optimal solution of (P), and the corresponding optimal value is 1. Another optimal solution is $[3, 2]^T$.

(c) The dual (D) is as follows:

$$
\begin{array}{lll}
 & \text{minimize} & y_2 - 3y_3 - y_4 \\
(D) & \text{subject to} & \\
 & & y_1 + y_2 - y_3 - y_4 = 1 \\
 & & -2y_1 - y_2 - y_3 = -1 \\
 & & y_1, y_2, y_3, y_4 \ge 0
\end{array}
$$

Complementary slackness conditions:

Now, let $\bar{x}$ be a feasible solution to the LP (P) and $\bar{y}$ be a feasible solution to the dual (D). Then $\bar{x}$ is an optimal solution to (P), and $\bar{y}$ is an optimal solution to (D) if and only if

$$\bar{x}_1 - 2\bar{x}_2 = 0 \quad \text{or} \quad \bar{y}_1 = 0 \tag{1}$$
$$\bar{x}_1 - \bar{x}_2 = 1 \quad \text{or} \quad \bar{y}_2 = 0 \tag{2}$$
$$-\bar{x}_1 - \bar{x}_2 = -3 \quad \text{or} \quad \bar{y}_3 = 0 \tag{3}$$
$$-\bar{x}_1 = -1 \quad \text{or} \quad \bar{y}_4 = 0 \tag{4}$$

(d) In part (b), we found an optimal solution $x^* = [3, 2]^T$ to the LP (P). By part (a), the cone of tight constraints for x^* is: $\mathsf{cone}\{[1, -1]^T\}$.

By the proof of Theorem 4.7 in the textbook, y^* is an optimal solution to the dual if and only if y^* is the vector of multipliers for certifying that $c = [1, -1]^T$ is in the cone of tight constraints for x^*. In particular, if $y^* = (y_1^*, y_2^*, y_3^*, y_4^*)^T$, then $y_i^* = 0$ if the i^{th} constraint (row) of the LP (P) is not tight for the solution x^*. It follows that $y^* = [0, 1, 0, 0]^T$ is the unique optimal solution to the dual (D).

Alternative solution: (Note that the proof of Theorem 4.7 invokes complementary slackness conditions. The proof below uses these conditions explicitly.)

For any optimal solution x^* of the primal (P), every optimal solution y^* of the dual must satisfy the complementary slackness conditions with respect to x^*. Now, by part (c), if $x^* = [3, 2]^T$, then it must be the case that $y_1^* = 0$, $y_3^* = 0$ and $y_4^* = 0$. Then, it follows from the constraints of the dual (D) that $y_1^* = 1$. Thus, $y^* = [0, 1, 0, 0]^T$ is the unique optimal solution to the dual (D).

<u>Problem 8:</u> Variants of Farkas' Lemma

(a) Prove that exactly one of the following statements holds:

- there exists x such that $Ax \leq b$, $x \geq \mathbf{0}$;
- there exists y such that $A^\top y \geq \mathbf{0}$, $y \geq \mathbf{0}$ and $b^\top y < 0$.

(b) Prove that exactly one of the following statements holds:

- there exists x such that $Ax \leq b$ and $A'x = b'$;
- there exists y and z such that $y \geq \mathbf{0}$, $y^\top A + z^\top A' = 0$, and $y^\top b + z^\top b' < 0$.

Solution: The proof of part (a) is given below; the proof of part (b) is skipped.

(a) We follow a similar proof to Theorem 4.8 of the textbook. First we prove that both statements cannot be true at the same time. By contradiction, if that happens, as $y \geq \mathbf{0}$, we can left multiply both sides of $Ax \leq b$ by $y^\top$ and keep the inequality; $y^\top Ax \leq y^\top b$. From $A^\top y \geq \mathbf{0}$ and $x \geq \mathbf{0}$ we have $y^\top Ax \geq 0$, which is a contradiction because $b^\top y < 0$.

Now we prove that if the first statement is not true, then the second one is true. Consider the following primal-dual LP problems:

$$\begin{aligned} (P) \quad & \max\{ \mathbf{0}^\top x : Ax \leq b, x \geq \mathbf{0} \}, \\ (D) \quad & \min\{b^\top y : A^\top y \geq \mathbf{0}, y \geq \mathbf{0} \}. \end{aligned}$$

The first statement is not true is equivalent to (P) is infeasible. $y = \mathbf{0}$ is a feasible point for (D), so by Table 4.2 of the textbook, (D) is unbounded. This means that there exists a dual feasible solution with negative objective value, which is exactly the second statement.