CO 250: Introduction to Optimization

Module 2: Linear Programs (Canonical Forms)

Consider

$$\max\left\{c^\top x : Ax = b, x \geq \mathbb{0}\right\} \qquad \text{(P)}$$

Definition

Let B be a basis of A. Then (P) is in canonical form for B if

(P1) $A_B = I$, and

(P2) $c_j = 0$ for all $j \in B$.

$$\begin{array}{ll} \max & (0 \quad 0 \quad 2 \quad 4)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Canonical form for $B = \{1, 2\}$

Consider

$$\max \left\{ c^\top x : Ax = b, x \geq \mathbb{0} \right\} \tag{P}$$

Definition

Let B be a basis of A. Then (P) is in canonical form for B if

(P1) $A_B = I$, and

(P2) $c_j = 0$ for all $j \in B$.

$$\begin{array}{ll} \max & \begin{pmatrix} -2 & 0 & 0 & 6 \end{pmatrix} x + 2 \\ \text{s.t.} & \\ & \begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Canonical form for $B = \{2, 3\}$

Consider

$$\max \{c^\top x : Ax = b, x \geq \mathbb{0}\} \qquad \text{(P)}$$

Idea

For any basis B we can "rewrite" (P) so that it is in canonical form for a basis B and such that the resulting LP behaves the same as (P).

More formally, we will show the following:

Proposition

For any basis B, there exists (P') in canonical form for B such that

(1) (P) and (P') have the same feasible region, and

(2) feasible solutions have the same objective value for (P) and (P').

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(2) feasible solutions have the same objective value for (P) and (P').

max $\quad (0 \quad 0 \quad 2 \quad 4)x$ s.t. $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $x_1, x_2, x_3, x_4 \geq 0$	max $\quad (-2 \quad 0 \quad 0 \quad 6)x + 2$ s.t. $\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $x_1, x_2, x_3, x_4 \geq 0$

Example

(1) $\bar{x} = (1, 2, 0, 0)^\top$ is feasible for both LPs.

(2)
$$\boxed{\begin{array}{lcr} (0 \quad 0 \quad 2 \quad 4)\bar{x} & = & 2 \times 0 + 4 \times 0 = 0 \\ (-2 \quad 0 \quad 0 \quad 6)\bar{x} \quad +2 & = & -2 \times 1 + 6 \times 0 + 2 = 0 \end{array}}$$

Illustration with an Example

$$\max \quad \underbrace{\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix}}_{c} x$$

s.t.

$$\underbrace{\begin{pmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 1 & 2\end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix}1\\ 2\end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(P)

Question

How do we rewrite (P) in canonical form for basis $B = \{2, 3\}$?

(P1) Replace $Ax = b$ by $A'x = b'$ with $A'_B = I$.

(P2) Replace $c^\top x$ by $\bar{c}^\top x + \bar{z}$ with $\bar{c}_B = \mathbb{0}$ ($\bar{z}$ constant).

Rewriting Constraints – Example

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

(P1) Replace $Ax = b$ by $A'x = b'$ with $A'_B = I$ for $B = \{2, 3\}$.

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix} -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Rewriting Constraints – General

Consider the system $Ax = b$ with basis B of A.

(P1) Replace $Ax = b$ by $A'x = b'$ with $A'_B = I$ for some basis B.

$$Ax = b$$

$$\Longleftrightarrow$$

$$\underbrace{A_B^{-1} A}_{A'} x = \underbrace{A_B^{-1} b}_{b'}$$

Remarks

- $A'_B = I$.
- $Ax = b$ and $A'x = b'$ have the same set of solutions.

Rewriting the Objective Function – Example

$$\max \quad z = \underbrace{\begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix}}_{c^\top} x$$

s.t.

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{b}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(P2) Replace $c^\top x$ by $\bar{c}^\top x + \bar{z}$ with $\bar{c}_B = \mathbb{0}$ ($\bar{z}$ constant) for $B = \{2, 3\}$.

Step 1. Construct a new objective function by

- multiplying constraint 1 by y_1,
- multiplying constraint 2 by y_2, and
- adding the resulting constraints to the objective function.

Step 2. Choose y_1, y_2 to get $\bar{c}_B = 0$.

$$\begin{array}{ll} \max & z = \begin{pmatrix} 0 & 0 & 2 & 4 \end{pmatrix} x \end{array}$$

s.t.

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$(y_1, y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = (y_1, y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{ll} \max & z = (0 \quad 0 \quad 2 \quad 4)x \\ \text{s.t.} & \\ & \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

$$\begin{array}{rcr} 0 & = & -(y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x + (y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ z & = & (0 \quad 0 \quad 2 \quad 4)x \\ \hline z & = & \left[(0 \quad 0 \quad 2 \quad 4) - (y_1 \quad y_2) \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \right] x + (y_1 \quad y_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{array}$$

Remark

For any choice of y_1, y_2 and any feasible solution x,

objective value of x for old objective function $=$

objective value of x for new objective function

$$z = \underbrace{\left[\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix} - \begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 1 & 2\end{pmatrix}\right]}_{\bar{c}^\top} x + \underbrace{\begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix}}_{\bar{z}}$$

Question

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

$$\begin{pmatrix}0 & 0\end{pmatrix} = \bar{c}_B^\top = \begin{pmatrix}0 & 2\end{pmatrix} - \begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 2\end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^\top \begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0\\ 2\end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}2\\ 0\end{pmatrix}$$

$$z = \underbrace{\left[\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix} - \begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2\end{pmatrix}\right]}_{\bar{c}^\top} x + \underbrace{\begin{pmatrix}y_1 & y_2\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix}}_{\bar{z}}$$

Question

How do we choose y_1, y_2 such that $\bar{c}_B = 0$ for $B = \{2, 3\}$?

Choose

$$\begin{pmatrix}y_1 \\ y_2\end{pmatrix} = \begin{pmatrix}2 \\ 0\end{pmatrix}$$

$\Rightarrow$

$$z = \left[\begin{pmatrix}0 & 0 & 2 & 4\end{pmatrix} - \begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2\end{pmatrix}\right] x + \begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix}$$

$\Rightarrow$

$$z = \begin{pmatrix}-2 & 0 & 0 & 6\end{pmatrix} x + 2$$

$$\boxed{\begin{array}{ll} \max & z = c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbb{0} \end{array}}$$

(P2) Replace $c^\top x$ by $\bar{c}^\top x + \bar{z}$ with $\bar{c}_B = \mathbb{0}$ ($\bar{z}$ constant) for some basis B.

$$\begin{array}{rcr} 0 & = & -y^\top Ax + y^\top b \\ z & = & c^\top x \\ \hline z & = & \left[c^\top - y^\top A\right] x + y^\top b \end{array}$$

Remark

For any choice of y_1, y_2 and any feasible solution x,

objective value of x for old objective function $=$
objective value of x for new objective function

$$z = \underbrace{\left[c^\top - y^\top A\right]}_{\bar{c}^\top} x + \underbrace{y^\top b}_{\bar{z}}$$

Question

How do we choose y such that $\bar{c}_B = 0$ for a basis B?

$$\mathbb{0}^\top = \bar{c}_B^\top = c_B^\top - y^\top A_B$$

$\Longleftrightarrow$

$$y^\top A_B = c_B^\top$$

$\Longleftrightarrow$

$$A_B^\top y = c_B$$

$\Longleftrightarrow$

$$y = \left(A_B^\top\right)^{-1} c_B = A_B^{-\top} c_B$$

Remark

For any non-singular matrix M,

$$(M^\top)^{-1} = (M^{-1})^\top =: M^{-\top}$$

Recap

Proposition

Consider A with basis B,

$$\boxed{\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & \\ & Ax = b \\ & x \geq \mathbb{0} \end{array}} \quad \text{(P)}$$

$$\boxed{\begin{array}{ll} \max & \underbrace{\left[c^\top - y^\top A\right]}_{\bar{c}} x + y^\top b \\ \text{s.t.} & \\ & \underbrace{A_B^{-1} A}_{A'} x = A_B^{-1} b \\ & x \geq \mathbb{0} \end{array}} \quad \text{(P')}$$

where $y = A_B^{-\top}$. Then

(1) (P') is in canonical form for basis B, i.e., $\bar{c}_B = 0$ and $A'_B = I$.

(2) (P) and (P') have the same feasible region.

(3) <u>Feasible solutions</u> have the same objective value for (P) and (P').