CO 250: Introduction to Optimization

Module 1: Formulations (Overview)

Outline

Introducing Optimization

Three Case Studies

A Modeling Example

Optimization - Abstract Perspective

- Abstractly, we will focus on problems of the following form:
  - Given: a set $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}$
  - Goal: find $x \in A$ that minimizes/maximizes f
- This is a very general problem that is enormously useful in virtually every branch of industry.
- Bad News: The above problem is notoriously hard to solve (and may not even be well-defined).

Optimization - Important Special Cases

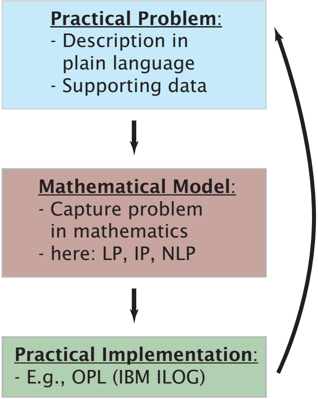
- *Abstract optimization problem (P):*
  - Given: a set $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}$
  - Goal: find $x \in A$ that minimizes/maximizes f

- Will look at *three* special cases of (P) in this course:
  (A) Linear Programming (LP). A is implicitly given by *linear* constraints, and f is a *linear* function.
  (B) Integer Programming (IP). Same as before, but now we want to max/min over the *integer* points in A.
  (C) Nonlinear Programming (NLP). A is given by *non-linear* constraints, and f is a *non-linear* function.

Optimization - Typical Workflow

Typical development process has three stages.

- The starting point is an english language description of practical problem
- We will develop a mathematical model for the problem.
- Finally, we feed the model and data into a solver.
- Iterate!



Optimization in Practice

Optimization is everywhere! Some examples include:

- booking hotel rooms or airline tickets,
- setting the market price of a kwh of electricity,
- determining an "optimal" portfolio of stocks,
- computing energy efficient circuits in chip design,
- and many more!

CSX Rail

- One of the largest transport suppliers in the United States
- Operates 21000 miles of rail network
- 11 Billion in annual revenue
- Serves 23 states, Ontario and Quebec
- Operates 1200 trains per day

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- Has a fleet of 3800 locomotives, and more than 100000 freight cars
- Transports 7.4 million car loads per year

Optimization @ CSX Rail

- [Acharya, Sellers, Gorman '10] use mathematical programming to optimally allocate and reposition empty railcars dynamically.

Angela Cable/iStock/Thinkstock

Implementing system yields the following estimated benefits for CSX:

- Annual savings: $51 million
- Avoided rail car capital investment: $1.4 billion

Optimization in Disease Control

- [Lee et al. '13] Use mathematical programming to prepare for disease outbreak and medical catastrophes.
- Where should we place medical dispensing facilities, and how should we staff these in order to disseminate medication as quickly as possible to the population?
- How should dispensing be scheduled?

Senator Leahy Anthrax Letter, Envelope. Source: FBI,

http://www.fbi.gov/about-us/history/famous-cases/anthrax-

amerithrax/4a.jpg/view

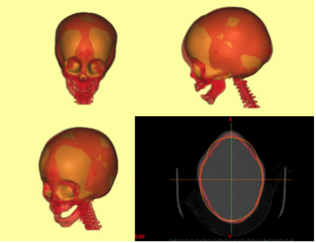
Optimization in Disease Control

- In collaboration with the Center for Disease Control, [Lee et al. '13] developed a decision support suite called RealOpt.
- The suite is being used by ≈ 6500 public health and emergency directors in the USA to design, place, and staff medical dispensing centres.
- In tests, throughput in medical dispensing centres increases by several orders of magnitude.

Senator Leahy Anthrax Letter, Envelope. Source: FBI, http://www.fbi.gov/about-us/history/famous-cases/anthrax-amerithrax/4a.jpg/view

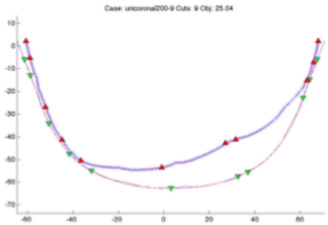
Optimization @ Sick Kids

- Current project with SickKids, Toronto.
- Cranofacial deformations in infants sometimes need to be corrected via an operation.
- Surgeons cut the forehead in 8-11 places and bend the forehead back to reach an "ideal" shape.



Optimization @ Sick Kids

- Skull pieces between incisions are rigid $\rightarrow$ number and position of cuts determines how close the skull maybe be bent to approximate the ideal template.
- Problem: Find a fixed, small number of cut positions that allow for the best approximation of the ideal skull form.
- The cuts given by our model are quite different from those usually performed by surgeons.



- We model this as a (non-linear) IP.

WaterTech Production

- WaterTech produces 4 products, $\mathcal{P} = \{1, 2, 3, 4\}$, from the following resources:
  - Time on two machines, and
  - Skilled and unskilled labour
- The following table gives precise requirements:

Product	Machine 1	Machine 2	Skilled Labor	Unskilled Labor	Unit Sale Price
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
4	5	4	6	4	180

E.g., producing a unit of product 3 requires 6h on machine 1, 5h on machine 2, 5h of skilled, and 7h of unskilled labour. It can be sold at $220 per unit.

WaterTech Production

Restrictions:

- WaterTech has 700h on machine 1 and 500h on machine 2 available.
- It can purchase 600h of skilled labour at $8 per hour and at most 650h of unskilled labour at $6 per hour.

Question: How much of each product should WaterTech produce in order to maximize profit?

Formulate this as a mathematical program!

Ingredients of a Math Model

- Decision Variables: Capture unknown information
- Constraints: Describe which assignments to variables are feasible.
- Objective function: A function of the variables that we would like to maximize/minimize.

WaterTech Model – Variables

- WaterTech needs to decide how many units of each product to produce
  $\Longrightarrow$ introduce variable x_i for number of units of product i to produce
- For convenience, we also introduce:
  y_s, y_u : number of hours of skilled/unskilled labour to purchase.

WaterTech Model – Constraints

- What makes an assignment to $\{x_i\}_{i \in \mathcal{P}}$, y_s, y_u a feasible?
- E.g., a production plan described by an assignment may not use more than 700h of time on machine 1.

t	Machine 1	Machine 2	Skil
	11	4	
	7	6	
	6	5	
	5	4	

$$\implies \quad 11x_1 + 7x_2 + 6x_3 + 5x_4 \le 700$$

Similarly, we may not use more than 500h of machine 2 time.

$$\implies \quad 4x_1 + 6x_2 + 5x_3 + 4x_4 \le 500$$

WaterTech Model – Constraints

- Producing x_i units of product $i \in \mathcal{P}$ requires

  $$8x_1 + 5x_2 + 5x_3 + 6x_4$$

  units of skilled labour, and this must not exceed y_s.

  $$\implies 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s$$

Skilled Labor	Unskilled Labor	Un
8	7	
5	8	
5	7	
6	4	

Similarly for unskilled labour:

$$\implies \quad 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u$$

... and $y_s \leq 600$ as well as $y_u \leq 650$ as only limited amounts of labour can be purchased.

WaterTech Model – Objective Function

- Revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

- Cost of labour:

$$8y_s + 6y_u$$

bor	Unit Sale Price
	300
	260
	220
	180

- Objective function:

$$\text{maximize} \quad 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u$$

WaterTech – Entire Model

$$
\begin{aligned}
\max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t.} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
& 7x_1 + 8x_2 + 7x_3 + 4x_4 \leq y_u \\
& y_s \leq 600 \\
& y_u \leq 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s \geq 0.
\end{aligned}
$$

Solution (via CPLEX): $x = (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T$, $y_s = 583 + \frac{1}{3}$, $y_u = 650$ of profit $\$15433 + \frac{1}{3}$.

Correctness of Model

- Is our model correct? What does this mean?
- Some terminology:
  - (i) Word description of problem
  - (ii) Formulation
- A solution to the formulation is an assignment to all of its variables.
- This is feasible if all constraints are satisfied, and optimal if no better feasible solution exists.

To clarify these ideas let us consider a simple example. Suppose WaterTech manufactures four products, requiring time on two machines and two types (skilled and unskilled) of labour. The amount of machine time and labor (in hours) needed to produce a unit of each product and the sales prices in dollars per unit of each product are given in the following table:

Product	Machine 1	Machine 2	Skilled Labor	Unskilled Labor	Unit Sale Price
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
		4			180

$$
\begin{array}{lrcl}
\max & 300x_1+260x_2+220x_3+180x_4-8y_s-6y_u & & \\
\text{subject to} & & & \\
& 11x_1+7x_2+6x_3+5x_4 & \le & 700 \\
& 4x_1+6x_2+5x_3+4x_4 & \le & 500 \\
& 8x_1+5x_2+5x_3+6x_4 & \le & y_s \\
& 7x_1+8x_2+7x_3+4x_4 & \le & y_u \\
& y_s & \le & 600 \\
& y_u & \le & 650 \\
& x_1,x_2,x_3,x_4,y_u,y_s & \ge & 0.
\end{array}
$$

- Similar: A solution to the word description is an assignment to the unknowns.

Correctness of Model

- One way of showing correctness: define a mapping between feasible solutions to the word description, and feasible solutions to the model, and vice versa.
- E.g., feasible solution to WaterTech word description is given by
  - (i) Producing 10 units of product 1, 50 units of product 2, 0 units of product 3, and 20 units of product 4, and by
  - (ii) Purchasing 600 units of skilled and unskilled labour.
- It is easily checked that

$$x_1 = 10, x_2 = 50, x_3 = 0, x_4 = 20, y_s = y_u = 600$$

is feasible for the mathematical program we wrote.

Fomulations (Overview)

- Your map should preserve cost.
  In the example, the profit from the solution to the word description should correspond to the objective value of its image (under map), and vice versa. Check this!
- In the example, the map was simply the identity. This need not necessarily be the case in general!