CO 250: Introduction to Optimization

Module 4: Duality Theory (Geometric Optimality)

Recap: Strong Duality

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

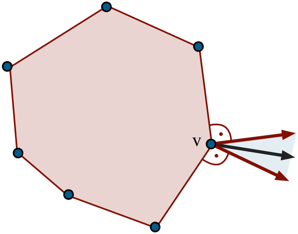
Strong Duality Theorem

For the above primal-dual pair of LPs, (P) and (D), if (P) has an optimal solution, then (D) has one and their objective values equal.

Recap: The Geometry of an LP

In Module 2, we saw that

- The feasible region of an LP is a polyhedron.
- Basic solutions correspond to extreme points of this polyhedron.



Question

When is an extreme point optimal?

Module 2 and strong duality told us that Simplex computes

- a basic solution (if it exists), and
- a certificate of optimality.

Today we will investigate these certificates using geometry.

Revisiting Weak Duality

We can rewrite (P) using slack variables s:

$$\begin{aligned} \max\ & c^T x && \text{(P')} \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned}$$

Note:

- (x, s) feasible for (P') $\longrightarrow$ x feasible for (P)
- x feasible for (P) $\longrightarrow (x, b - Ax)$ feasible for (P')

$$\begin{aligned} \max\ & c^T x && \text{(P)} \\ \text{s.t. } & Ax \leq b \end{aligned}$$

$$\begin{aligned} \min\ & b^T y && \text{(D)} \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned}$$

Revisiting Weak Duality

Suppose $\bar{x}$ is feasible for (P), and $\bar{y}$ is feasible for (D)

$$\max\ c^T x \qquad \text{(P)}$$
$$\text{s.t. } Ax \le b$$

$$\longrightarrow \quad (\bar{x}, \underbrace{b - A\bar{x}}_{\bar{s}} b - A\bar{x}) \text{ feasible for (P')}$$

$$\max\ c^T x \qquad \text{(P')}$$
$$\text{s.t. } Ax + s = b$$
$$s \ge \mathbb{0}$$

Recall the Weak Duality proof:

$$\begin{aligned} \bar{y}^T b &= \bar{y}^T (A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s} \end{aligned}$$

$$\min\ b^T y \qquad \text{(D)}$$
$$\text{s.t. } A^T y = c$$
$$y \ge \mathbb{0}$$

Strong Duality tells us that:

$$\begin{aligned} \bar{x},\ \bar{y} \text{ both optimal} \quad &\iff \quad c^T \bar{x} = \bar{y}^T b \\ &\iff \quad \bar{y}^T \bar{s} = 0 \end{aligned}$$

Revisiting Weak Duality

Recall the Weak Duality proof:

$$\begin{aligned}\bar{y}^T b &= \bar{y}^T(A\bar{x} + \bar{s}) \\ &= (\bar{y}^T A)\bar{x} + \bar{y}^T \bar{s} \\ &= c^T \bar{x} + \bar{y}^T \bar{s}\end{aligned}$$

$$0 = \bar{y}^T \bar{s} = \sum_{i=1}^{m} \bar{y}_i \bar{s}_i \qquad (\star)$$

By feasibility, $\bar{x} \geq \mathbb{0}$ and $\bar{s} \geq \mathbb{0}$ and hence

$(\star)$ holds if and only if $\bar{y}_i = 0$ or $\bar{s}_i = 0$,

for every $1 \leq i \leq m$.

$$\begin{aligned}\max\ & c^T x \\ \text{s.t. } & Ax \leq b\end{aligned} \qquad \text{(P)}$$

$$\begin{aligned}\max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0}\end{aligned} \qquad \text{(P')}$$

$$\begin{aligned}\min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0}\end{aligned} \qquad \text{(D)}$$

Revisiting Weak Duality – Recap

Given: $\bar{x}$ and $\bar{y}$ feasible solutions for (P) and (D)

Define: $\bar{s} = b - A\bar{x}$

Then:

$$\bar{x} \text{ and } \bar{y} \text{ optimal} \quad \Longleftrightarrow \quad \underbrace{\bar{y}_i = 0 \text{ or } \bar{s}_i = 0}_{(\star)}$$

for all $1 \le i \le m$. We can rephrase $(\star)$ equivalently as

$\bar{y}_i = 0$ or ith primal constraint is tight.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \ge \mathbb{0} \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \ge \mathbb{0} \end{aligned} \qquad \text{(D)}$$

Complementary Slackness – Special Case

Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D).

Then $\bar{x}$ and $\bar{y}$ are optimal if and only if

(i) $\bar{y}_i = 0$, or

(ii) the ith constraint of (P) is tight for $\bar{x}$,

for every row index i.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \tag{P}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax + s = b \\ & s \geq \mathbb{0} \end{aligned} \tag{P'}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \tag{D}$$

Complementary Slackness Conditions – Example

Consider the following LP:

$$\max\ (5,3,5)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

Its dual is:

$$\min\ (2,4,-1)y \qquad \text{(D)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ -1 & 2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$$
$$y \geq \mathbb{0}$$

Claim

$\bar{x} = (1,-1,1)^T$ and $\bar{y} = (0,2,1)^T$ are optimal!

Complementary Slackness

Feasible solutions $\bar{x}$ and $\bar{y}$ for (P) and (D) are optimal if and only if

$\bar{y}_i = 0$ or the ith primal constraint is tight for $\bar{x}$, for all row indices i.

It is easy to check if $\bar{x}$ and $\bar{y}$ are feasible.

(i) $\bar{y}_1 = 0$ or $(1,2,-1)\bar{x} = 2$

(ii) $\bar{y}_2 = 0$ or $(3,1,2)\bar{x} = 4$

(iii) $\bar{y}_3 = 0$ or $(-1,1,1)\bar{x} = -1$

$\longrightarrow$ $\bar{x}$ and $\bar{y}$ are optimal!

General Complementary Slackness

(P_{max})					
max $\quad c^\top x$ subject to $Ax \mathbin{?} b$ $x \mathbin{?} \mathbb{0}$		$\leq$ constraint	≥ 0 variable	min $\quad b^\top y$ subject to $A^\top y \mathbin{?} c$ $y \mathbin{?} \mathbb{0}$	
		$=$ constraint	free variable		
		$\geq$ constraint	≤ 0 variable		
		≥ 0 variable	$\geq$ constraint		
		free variable	$=$ constraint		
		≤ 0 variable	$\leq$ constraint		

Suppose: (P_{max}) and (P_{min}) are a pair of primal and dual LPs according to the above table, with feasible solutions $\bar{x}$, and $\bar{y}$

$\bar{x}$ and $\bar{y}$ satisfy the complementary slackness conditions if ...

for all variables x_j of (P_{max}):

(i) $\bar{x}_j = 0$, or

(ii) jth constraint of (P_{min}) is satisfied with equality for $\bar{y}$

for all variables y_i of (P_{min}):

(i) $\bar{y}_i = 0$, or

(ii) ith constraint of (P_{max}) is satisfied with equality for $\bar{x}$

$\bar{x}$ and $\bar{y}$ satisfy the CS conditions if ...

for all variables x_j of (P_{max}):

(i) $\bar{x}_j = 0$, or

(ii) jth constraint of (P_{min}) is satisfied with equality for $\bar{y}$

for all variables y_i of (P_{min}):

(i) $\bar{y}_i = 0$, or

(ii) ith constraint of (P_{max}) is satisfied with equality for $\bar{x}$

Note: The two or's above are inclusive!

Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

General CS Conditions – Example

(P_{max})					
max $\quad c^\top x$ subject to $Ax \, ? \, b$ $x \, ? \, \mathbb{0}$	$\leq$ constraint	≥ 0 variable		min $\quad b^\top y$ subject to $A^\top y \, ? \, c$ $y \, ? \, \mathbb{0}$	(P_{min})
	$=$ constraint	free variable			
	$\geq$ constraint	≤ 0 variable			
	≥ 0 variable	$\geq$ constraint			
	free variable	$=$ constraint			
	≤ 0 variable	$\leq$ constraint			

Consider the following LP...

$$\max \ (-2, -1, 0)x \qquad \text{(P)}$$

$$\text{s.t.} \ \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$x_1 \leq 0, x_2 \geq 0$$

... and its dual LP:

$$\min \ (5, 7)y \qquad \text{(D)}$$

$$\text{s.t.} \ \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ 2 & 2 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \end{matrix} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$y_1 \leq 0, y_2 \geq 0$$

General CS Conditions – Example

$$\max\ (-2,-1,0)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$
$$x_1 \leq 0, x_2 \geq 0$$

$$\min\ (5,7)y \qquad \text{(D)}$$
$$\text{s.t.}\ \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ 2 & 2 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \end{matrix} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$
$$y_1 \leq 0, y_2 \geq 0$$

Check: $\bar{x} = (-1,0,3)^T$ and $\bar{y} = (-1,1)^T$ are feasible for (P) and (D). Are they also optimal?

Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

General CS Conditions – Example

$$\begin{aligned} \max\ & (-2,-1,0)x \\ \text{s.t. } & \begin{pmatrix} 1 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} \begin{matrix} \geq \\ \leq \end{matrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x_1 \leq 0, x_2 \geq 0 \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & (5,7)y \\ \text{s.t. } & \begin{pmatrix} 1 & -1 \\ 3 & 4 \\ 2 & 2 \end{pmatrix} y \begin{matrix} \leq \\ \geq \\ = \end{matrix} \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \\ & y_1 \leq 0, y_2 \geq 0 \end{aligned} \qquad \text{(D)}$$

Claim

$\bar{x} = (-1,0,3)^T$ and $\bar{y} = (-1,1)^T$ are optimal

Primal conditions:

(i) $\bar{x}_1 = 0$ or the first (D) constraint is tight for $\bar{y}$.

(ii) $\bar{x}_2 = 0$ or the second (D) constraint is tight for $\bar{y}$.

(iii) $\bar{x}_3 = 0$ or the third (D) constraint is tight for $\bar{y}$.

Dual conditions:

(i) $\bar{y}_1 = 0$ or the first (P) constraint is tight for $\bar{x}$.

(ii) $\bar{y}_2 = 0$ or the second (P) constraint is tight for $\bar{x}$.

Complementary Slackness – Geometry

Complementary Slackness Theorem

Let (P) and (D) be an arbitrary primal-dual pair of LPs, and let $\bar{x}$ and $\bar{y}$ be feasible solutions. Then these solutions are optimal if and only if the CS conditions hold.

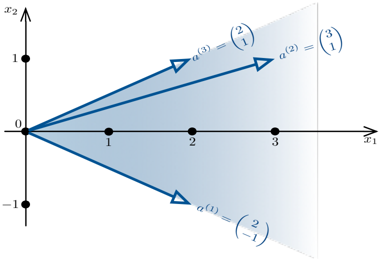
Will now see a geometric interpretation of this theorem!

But some basics first!

Definition

Let $a^{(1)}, \ldots, a^{(k)}$ be vectors in $\mathbb{R}^n$. The cone generated by these vectors is given by

$$C = \{\lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \ldots + \lambda_k a^{(k)} \ : \ \lambda \geq \mathbb{0}\}$$

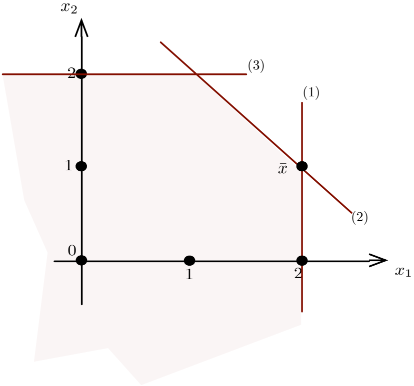


Example: The cone generated by $a^{(1)}, a^{(2)}$ and $a^{(3)}$ is the blue-shaded area.

Geometry – Cone of Tight Constraints

Consider the following polyhedron:

$$P = \{x \in \mathbb{R}^2 \; : \; \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}}_{A} x \leq \underbrace{\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}}_{b} \}$$



Consider: $\bar{x} = (2,1)^T$

(i) $\bar{x} \in P \longrightarrow$ Check!

(ii) Tight constraints:

$$\text{row}_1(A)\bar{x} = b_1 \quad \longrightarrow \quad (1,0)\bar{x} = 2$$
$$\text{row}_2(A)\bar{x} = b_2 \quad \longrightarrow \quad (1,1)\bar{x} = 3$$

Cone of tight constraints:

Cone generated by rows of tight constraints

Cone of tight constraints:

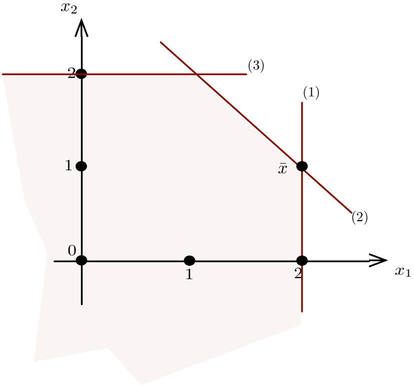
Cone generated by rows of tight constraints

Tight constraints:

$$(1,0)\bar{x} = 2 \quad (1)$$
$$(1,1)\bar{x} = 3 \quad (2)$$

Cone of tight constraints:

$$\{\lambda_1(1,0)^T + \lambda_2(1,1)^T \,:\, \lambda_1, \lambda_2 \geq 0\}$$



Consider an LP of the form

$$\max\{c^T x \,:\, Ax \leq b\}$$

and a feasible solution $\bar{x}$.

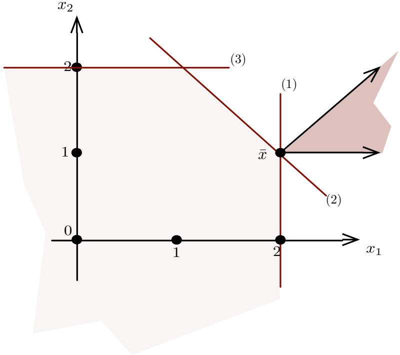
The cone of tight constraints at $\bar{x}$ is defined by the rows of A corresponding to tight constraints at $\bar{x}$.

Theorem

Let $\bar{x}$ be a feasible solution to

$$\max\{c^T x \,:\, Ax \leq b\}$$

Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.



Example: Consider the LP

$$\max\{(3/2, 1/2)x \,:\, x \in P\}$$

$$P = \{x \in \mathbb{R}^2 \,:\, \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}\}$$

Tight constraints at $\bar{x} = (2,1)^T$:

$$(1,0)\bar{x} = 2 \qquad (1)$$
$$(1,1)\bar{x} = 3 \qquad (2)$$

Geometry – Cone of Tight Constraints

Example: Consider the LP

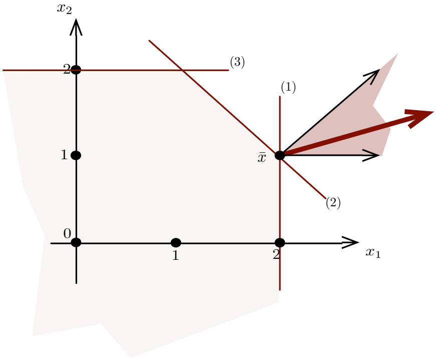
$$\max\{(3/2, 1/2)x \ : \ x \in P\} \qquad (\star)$$

Tight constraints at $\bar{x} = (2,1)^T$:

$$(1,0)\bar{x} = 2 \qquad (1)$$
$$(1,1)\bar{x} = 3 \qquad (2)$$

Note: $(3/2, 1/2)^T$ in cone of tight constraints as

$$\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$$P = \{x \in \mathbb{R}^2 \ : \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \le \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}\}$$

Geometry – Cone of Tight Constraints

Example: Consider the LP

$$\max\{(3/2, 1/2)x \,:\, x \in P\} \qquad (\star)$$

Tight constraints at $\bar{x} = (2,1)^T$:

$$(1,0)\bar{x} = 2 \qquad (1)$$
$$(1,1)\bar{x} = 3 \qquad (2)$$

Note: $(3/2, 1/2)^T$ is in cone of tight constraints as

$$\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore: $\bar{x}$ is an optimal solution!

Theorem

Let $\bar{x}$ be a feasible solution to

$$\max\{c^T x \,:\, Ax \le b\}$$

Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

Proving the "if" direction of the above theorem amounts to

(i) finding a feasible solution $\bar{y}$ to the dual of $(\star)$, and

(ii) showing that $\bar{x}$ and $\bar{y}$ satisfy the CS conditions!

The above theorem follows from CS Theorem!

Geometric Optimality – Towards a Proof

If we write out the LP:

$$\max\ (3/2, 1/2)x \qquad (\star)$$

$$\text{s.t. } \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

We can write the dual of $(\star)$ as:

$$\min\ (2, 3, 2)y \qquad (\Diamond)$$

$$\text{s.t. } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} y = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

$$y \geq \mathbb{0}$$

We know that:

$$\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1/2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence: $\bar{y} = (1, 1/2, 0)^T$ is feasible for $(\Diamond)$.

Also: $\bar{y}_i > 0$ only if the constraint i is tight at $\bar{x}$.
$\longrightarrow$ Dual CS Conditions hold!

How about primal CS conditions?
$\longrightarrow$ they always hold as all constraints in the dual are equality constraints!

CS Theorem $\longrightarrow$ $(\bar{x}, \bar{y})$ optimal!

Suppose $\bar{x}$ is a solution to (P), and let $J(\bar{x})$ be the indices of tight constraints for $\bar{x}$. i.e.,

$$\text{row}_i(A)\bar{x} = b_i$$

for $i \in J(\bar{x})$ and

$$\text{row}_i(A)\bar{x} < b_i$$

for $i \notin J(\bar{x})$.

Suppose c is in the cone of tight constraints at $\bar{x}$, and thus

$$c = \sum_{i \in J(\bar{x})} \lambda_i \text{row}_i(A)^T$$

for some $\lambda \geq \mathbb{0}$.

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \leq b \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t.}\ & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \qquad \text{(D)}$$

(x, y) satisfy CS Conditions if

$$y_i = 0 \quad \text{or} \quad \text{row}_i(A)x = b_i \qquad (\star)$$

Suppose c is in the cone of tight constraints at $\bar{x}$, and thus for some $\lambda \geq \mathbb{0}$:

$$\begin{aligned} c &= \sum_{i \in J(\bar{x})} \lambda_i \text{row}_i(A)^T \\ &= A^T \bar{y} \end{aligned}$$

Where we define:

$$\bar{y}_i = \begin{cases} \lambda_i & : \ i \in J(\bar{x}) \\ 0 & : \ \text{otherwise} \end{cases}$$

Since $\lambda \geq \mathbb{0}$: $\bar{y}$ is feasible for (D)!

Also note: $\bar{y}_i > 0$ only if $\text{row}_i(A)\bar{x} = b_i$
$\longrightarrow$ CS conditions $(\star)$ hold!

$$\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \end{aligned} \tag{P}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y = c \\ & y \geq \mathbb{0} \end{aligned} \tag{D}$$

(x, y) satisfy CS Conditions if

$$y_i = 0 \quad \text{or} \quad \text{row}_i(A)x = b_i \tag{$\star$}$$

Hence: $(\bar{x}, \bar{y})$ are optimal!

Wrapping up...

We almost proved:

Theorem

Let $\bar{x}$ be a feasible solution to

$$\max\{c^T x \,:\, Ax \leq b\}$$

Then $\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

Missing: $\bar{x}$ is optimal $\quad \longrightarrow \quad$ c is in the cone of tight constraints

CS Theorem $\longrightarrow$ there is a feasible dual solution $\bar{y}$ that, together with $\bar{x}$, satisfies CS conditions.

We can use CS conditions and $\bar{y}$ to show that c lies in cone of tight constraints for $\bar{x}$. This is an exercise!

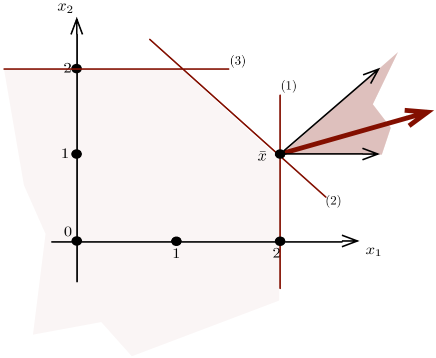
Recap

Given a feasible solution $\bar{x}$ to

$$\max\{c^T x \,:\, Ax \leq b\}$$

$\bar{x}$ is optimal if and only if c is in the cone of tight constraints for $\bar{x}$.

This provides a nice geometric view of optimality certificates



$$\max\ (3/2, 1/2)x \qquad \text{(P)}$$
$$\text{s.t.}\ \begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix} x \leq \begin{pmatrix}2\\3\\2\end{pmatrix}$$