CO 250 Online- Spring 2018

Assignment 3

Due date : Saturday, May 26th, 2018, by 4:00pm on Crowdmark

Submission Guidelines:

- Please submit your solutions to Crowdmark. Late assignments will not be accepted, and will receive a mark of zero. It is the responsibility of the students to make sure that the file they submit is clearly readable. Illegible submissions will receive a mark of zero.
- You answers **need to be fully justified**, unless specified otherwise. Always remember the WHAT-WHY-HOW rule, namely explain in full detail what you are doing, why are you doing it, and how are you doing it. Dry yes/no or numerical answers will get 0 marks.
- In some questions you are asked to *formulate* the problem. You are *not* asked to actually solve the formulation, e.g., compute optimal solutions. Your formulations should be easy to modify if we change the data and constants defining the problems. Clearly define all your variables (including their units) and any other new notation you use in all your answers. Your solutions must also contain a brief justification of all the constraints (explain the relation between each of the constraints and the requirements stated in the problem) and the objective function.

Assignment policies: While it is acceptable to discuss the course material and the assignments, you are expected to do the assignments on your own. Copying or paraphrasing a solution from some fellow student or old solutions from previous offerings qualifies as cheating and we will instruct the TAs to actively look for suspicious similarities and evidence of academic offenses when grading. Students found to be cheating will be given a mark of 0 on the assignment. In addition, all academic offenses will be reported to the Math Academic Integrity Officer (which may lead to further penalties) and recorded in the student's file.

Re-marking policies: If you have any complaints about the marking of assignments, then you should first check your solutions against the posted solutions. After that, if you see any marking error, then write a letter detailing clearly the marking errors, and submit this to one of the TAs within one week from the date the graded assignment is returned. If you still have concerns after the final decision of the TA, then please contact your instructor communicating all the correspondence with the TA and the original petition.

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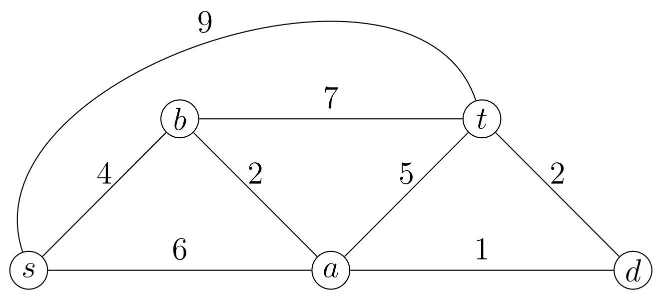
[1] It is an academic offense to post this assignment or solutions to any web page.

Question 1. Shortest Path **(20 marks)**

The figure below shows a graph $G = (V, E)$. Each edge $ij \in E$ in the figure is labeled by its length c_{ij}, and each vertex by its name.



Write down the IP formulation for finding a shortest s, t-path in the graph G, both in compact notation (see Section 1.5.2 of the textbook) and in explicit notation (where each constraint of the IP and the objective function is written in full, and no abbreviations are allowed; you may use either matrix-vector notation or scalar notation).

Question 2. NLP Modeling **(12 marks)**

John would like to rake his lawn, which measures 30 feet by 20 feet. Having waited until the end of the fall to rake his lawn, John is now presented with the following task. There are N sites in the lawn where leaves have fallen; site i is located at (p^i, q^i) with $0 \le p^i \le 30$, $0 \le q^i \le 20$, and there is a total weight of w^i in the leaves accumulated at site i.

John's raking strategy is to first collect all the leaves at some K locations in the lawn, which we will call centers, and then bag the leaves collected at these K centers. (Note that the location of a center could coincide with that of a site.) All leaves at a site i are moved to exactly one center. The cost incurred in raking leaves from site i to a center (x, y) is proportional to the squared distance between site i's location and the center, and is given by $w_i\big((p^i - x)^2 + (q^i - y)^2\big)$. John also wants to ensure that for every site i, there is some center (x, y) at distance $\sqrt{(p^i - x)^2 + (q^i - y)^2}$ at most D from it, where D is a given constant. The cost incurred in bagging leaves at a center depends on the total weight of leaves that have been accumulated at that center: if W is the total weight of leaves collected (from various sites) at a center, then the bagging cost is given by $\lambda \cdot W$, where $\lambda \ge 0$ is some given constant.

Provide a non-linear programming (NLP) formulation to help John rake his lawn (while satisfying the above constraints) so as to minimize the total cost incurred.

The objective function and constraints in your formulation should only involve polynomials of degree at most 3; some examples are: $x_1^3 + 2x_2x_3^2 + x_2 \le 5$, $x_1x_2 + x_2^2 - x_3 = 5$, $2x_1 + 3x_2 \ge 7$. Note that affine and quadratic functions are also polynomials of degree at most 3. Also, note that we can encode a binary variable $x_i \in \{0, 1\}$ via the quadratic constraint $x_i(1 - x_i) = 0$.

Question 3. Certifying outcomes of LPs **(15 marks)**

(a) Consider the following linear program.

$$\begin{array}{ll} \max & (4,1,0,-2,1)x \\ \text{subject to} & \\ & \begin{pmatrix} 3 & 1 & 2 & 1 & 2 \\ 1 & 2 & -4 & 4 & -2 \\ 2 & 1 & -3 & 5 & -1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} \\ & x \geq \mathbb{0}. \end{array}$$

Show that this LP is infeasible by exhibiting a certificate of infeasibility and justifying why this implies that the LP is infeasible. (6 marks)

(b) Consider the following linear program.

$$\begin{array}{ll} \max & (4,3,1,-1,-1)x \\ \text{subject to} & \\ & \begin{pmatrix} 3 & 2 & 5 & 3 & 1 \\ 1 & 1 & 3 & 2 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ & x_1 \leq 0,\ x_4, x_5 \geq 0. \end{array}$$

You are given that $\overline{x} = (0,-2,1,0,0)^T$ is a feasible solution to this LP. Show that this LP is unbounded. Justify your answer. (9 marks)

(**Hint:** Obtain a suitable vector r and proceed similarly to the example in slides "b. Certificates" for week 4. Note that $x_1 \leq 0$ and x_2, x_3 are free; how does this change the requirements on r stated in the from slide 12? There is a suitable r with $r_4 = 0$.)

Question 4. Outcomes of LPs **(10 marks)**

Let $x = (x_1, x_2, \ldots, x_n)^T$.

(a) Consider the following linear program:

$$\min \{c^T x : Ax \leq \mathbb{0},\ x \geq \mathbb{0}\}. \tag{P}$$

Prove that (P) is either unbounded, or has an optimal value of 0. (5 marks)

(b) Consider the following linear program:

$$\max \{c^T x : Ax = b\}. \tag{Q}$$

Suppose that (Q) has an optimal solution. Prove that *every* feasible solution to (Q) is an optimal solution. (5 marks)