CO 250: Introduction to Optimization

Module 2: Linear Programs (Possible Outcomes)

What Does Solving an Optimization Problem Mean?

input

LP or
IP or
NLP

algorithm
(software)

output

solution

$$\begin{array}{ll} \max & 2x_1 - 3x_2 \\ \text{s.t.} & \\ & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

$$x_1 = 1$$
$$x_2 = 0$$

Optimal solution

Remark

Sometimes, the answer is not so straightforward!

Definition

An assignment of values to each of the variables is a feasible solution if all the constraints are satisfied.

Definition

An optimization problem is feasible if it has at least one feasible solution. It is infeasible otherwise.

$$
\begin{array}{ll}
\max & x_1 - x_2 \\
\text{s.t.} & \\
& \begin{array}{rcrcr} 4x_1 & + & x_2 & \geq & 2 \\ 2x_1 & - & 3x_2 & \leq & 4 \end{array} \\
& x_1,\ x_2 \geq 0
\end{array}
$$

$x_1 = 1$

$x_2 = 3$

Feasible solution

⬇

Problem is feasible

Definition

An assignment of values to each of the variables is a feasible solution if all the constraints are satisfied.

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An optimization problem is feasible if it has at least one feasible solution. It is infeasible otherwise.

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& x_1,\ x_2 \geq 0
\end{array}
$$

$x_1 = 3$

$x_2 = 0$

NOT a feasible solution

But, the problem is feasible

Definition

- For a maximization problem, an optimal solution is a feasible solution that maximizes the objective function.
- For a minimization problem, an optimal solution is a feasible solution that minimizes the objective function.

$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \leq 1 \\ & x_2 \geq 1 \end{array}}$$

$x_1 = 1, x_2 = \alpha$ optimal for all $\alpha \geq 1$

Remark

An optimization problem can have several optimal solutions.

Question

Does the following linear program have an optimal solution?

$$\boxed{\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & & & \\ & x_1 & \geq & 2 \\ & x_1 & \leq & 1 \end{array}}$$

Infeasible problem, so no optimal solution

Question

Does every feasible optimization problem have an optimal solution? NO.

$$\boxed{\begin{array}{ll} \max & x_1 \\ \text{s.t.} & \\ & x_1 \geq 1 \end{array}}$$

Feasible ($x_1 = 1$), but still no optimal solution!

Definition

- A maximization problem is unbounded if for every value M, there exists a feasible solution with objective value greater than M.
- A minimization problem is unbounded if for every value M, there exists a feasible solution with objective value smaller than M.

We have seen three possible outcomes for an optimization problem:

- It has an optimal solution.
- It is infeasible.
- It is unbounded.

Question

Can anything else happen? YES.

Consider

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

- Feasible: set $x = 0$.
- Not unbounded: 1 is an upper bound.
- But there is no optimal solution!

Proof

Suppose for a contradiction x is an optimal solution. Let

$$x' := \frac{x+1}{2}.$$

Then $x' < 1$ is feasible. Moreover, $x' > x$.

Thus, x is not optimal, a contradiction.

Question

Is there any other example without strict inequalities? YES.

Consider

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

- Feasible: set $x = 1$.
- Not unbounded: 0 is a lower bound.
- But there is no optimal solution!

Exercise

Check this optimization problem has no optimal solution.

$$\boxed{\begin{array}{ll} \max & x \\ \text{s.t.} & \\ & x < 1 \end{array}}$$

Not a linear program

Strict inequality

$$\boxed{\begin{array}{ll} \min & \dfrac{1}{x} \\ \text{s.t.} & \\ & x \geq 1 \end{array}}$$

Not a linear program
Objective function non-linear

Remark

Linear programs are nicer than general optimization problems.

Fundamental Theorem of Linear Programming

For any linear program, one of the following holds:

- It has an optimal solution.
- It is infeasible.
- It is unbounded.

$$\boxed{\begin{aligned} \min \quad & \frac{1}{x} \\ \text{s.t.} \quad & \\ & x \geq 1 \end{aligned}}$$

Not a linear program
Objective function non-linear

Remark

Linear programs are nicer than general optimization problems.

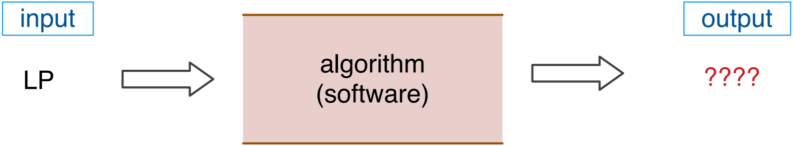
Fundamental Theorem of Linear Programming

For any linear program exactly one of the following holds:

- It has an optimal solution.
- It is infeasible.
- It is unbounded.

We will prove it later in the course.

We can now describe what we mean by solving a linear program.



LP has an optimal solution

Return an optimal solution $\bar{x}$

LP is infeasible

Say the LP is infeasible

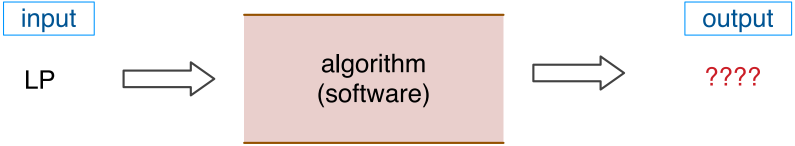
LP is unbounded

Say the LP is unbounded

Remark

Algorithms should justify their answers!

We can now describe what we mean by solving a linear program,



$\boxed{\text{LP has an optimal solution}}$

Return an optimal solution $\bar{x}$ + proof that $\bar{x}$ is optimal

$\boxed{\text{LP is infeasible}}$

Return a proof the LP is infeasible

$\boxed{\text{LP is unbounded}}$

Return a proof the LP is unbounded

Remark

Algorithms always need to justify their answers!

Recap

1. Optimization problems can be
   (A) infeasible,
   (B) unbounded, or
   (C) have an optimal solution.

2. There are optimization problems where none of (A), (B), (C) hold.

3. For LPs, exactly one of (A), (B), (C) holds.

4. By solving an LP, we mean
   - indicating which of (A), (B), (C) hold;
   - if (C) holds, giving an optimal solution; and
   - giving a proof the answer is correct.