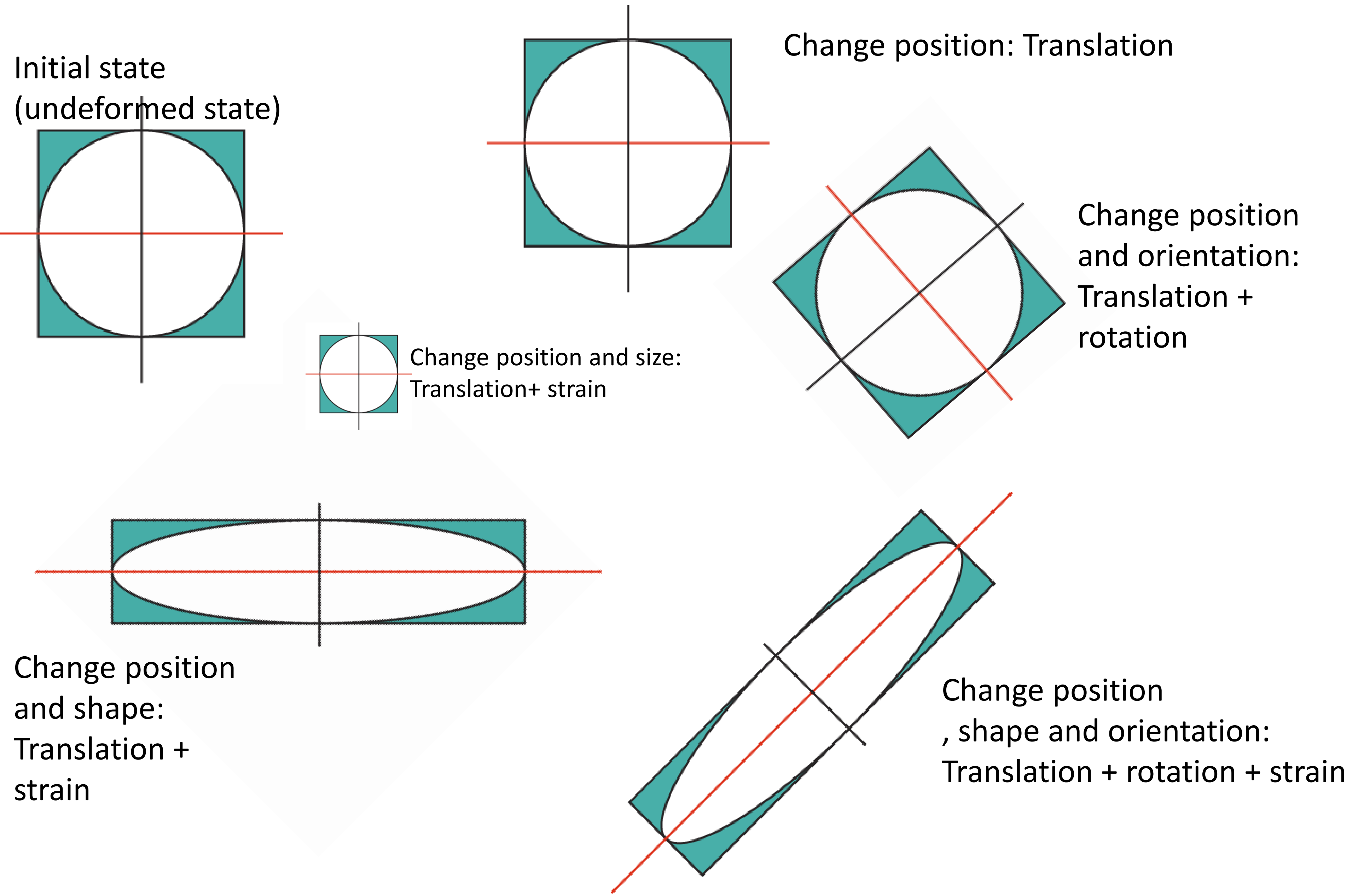
Deformation of rocks (Continuum Mechanics)

Outline

- Part I Review: some concepts of deformation
  - Strain/Rigid body rotation/Translation
  - Longitudinal strain/shear strain
  - Strain ellipse/ellipsoid/Flinn diagram
  - Deformation path
- Part II
  - Study deformation mathematically
  - Deformation gradient tensor
  - Polar decomposition
  - Velocity gradient tensor
  - ………….

Part I

Some concepts of deformation



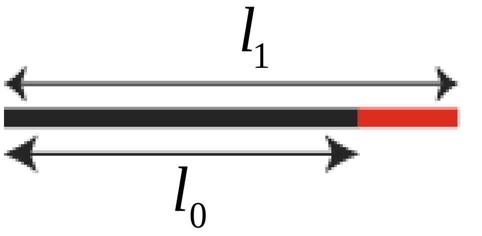
Initial state
(undeformed state)
Change position: Translation
Change position and size:
Translation+ strain
Change position
and orientation:
Translation +
rotation
Change position
and shape:
Translation +
strain
Change position
, shape and orientation:
Translation + rotation + strain

Deformation

- Deformation： change in position, shape/size, and orientation of a body.
  - may or may not involve shape and/or volume change. If neither shape nor volume changes, the body is rigid; otherwise it is deformable.
  - Strain is a measurement of the “distortion part” of a deformation: change in shape and/or volume.

Longitudinal strain

<u>Length change</u> (longitudinal strain)



– Elongation (e)

– Stretch (S)

Quadratic stretch S^2

– Natural strain

Natural strain

For a deformation reaching a large longitudinal strain, the total strain is accumulated by many many strain increments. Each infinitesimal increment is:

$$de = \frac{dl}{l}$$

and the total accumulated strain is naturally:

$$\varepsilon = \int_0^{\varepsilon} de = \int_{l_0}^{l_1} \frac{dl}{l} = \ln\left(\frac{l_1}{l_0}\right)$$

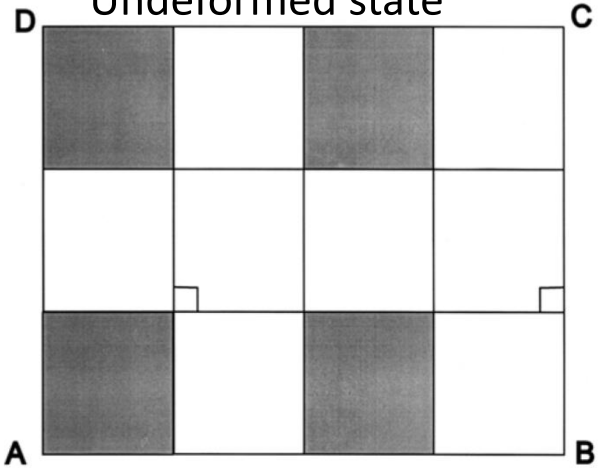
Field examples

Extended dike: S>1, e>0

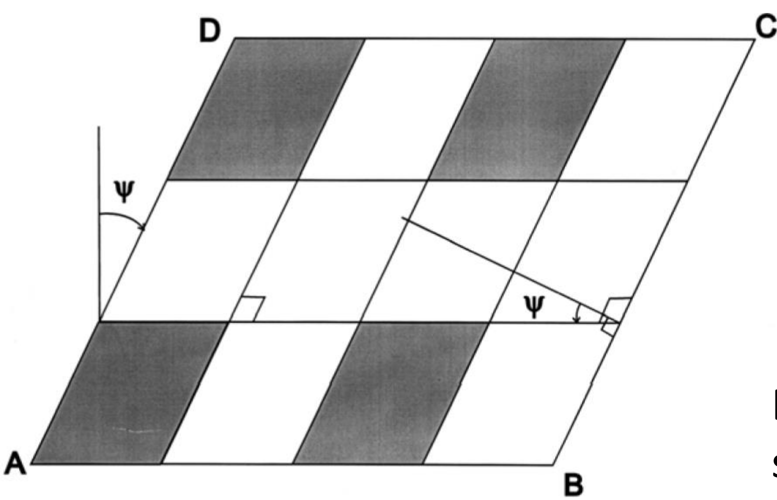
Shortened layers: S<1, e<0

Shear Strain

Undeformed state



- Shear strain: angle change



$$\gamma_{AB} = \tan(\psi)$$

ψ, the shear angle for lines parallel to AB, is clockwise and is positive

The shear angle for lines parallel to BC is ccw and is negative

$$\gamma_{BC} = \tan(-\psi) = -\tan(\psi) = -\gamma_{AB}$$

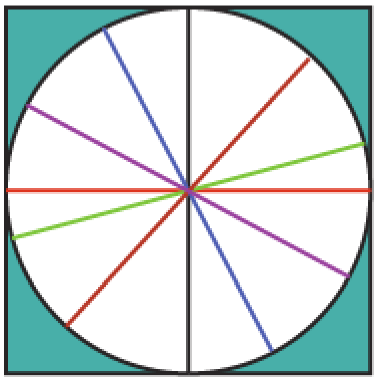
Every line has a shear angle and hence shear strain

Deformed state

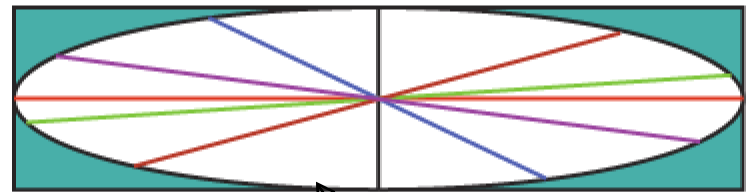
How about strain of a continuous body which is 3 D?

Strain Ellipse (Ellipsoid)

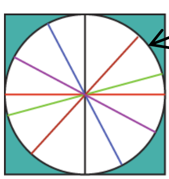
Initial state

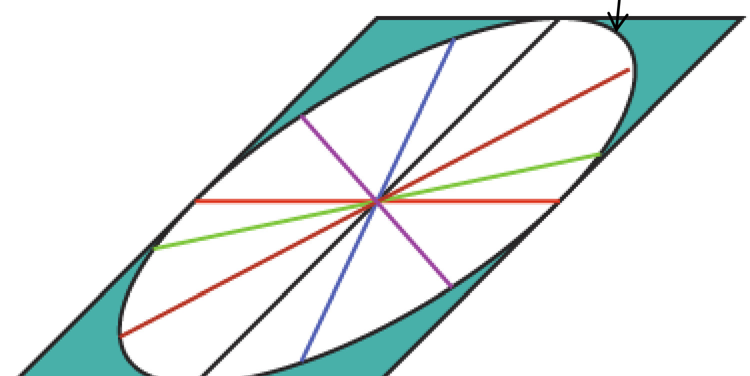


Unit circle/sphere



Strain ellipse/ellipsoid



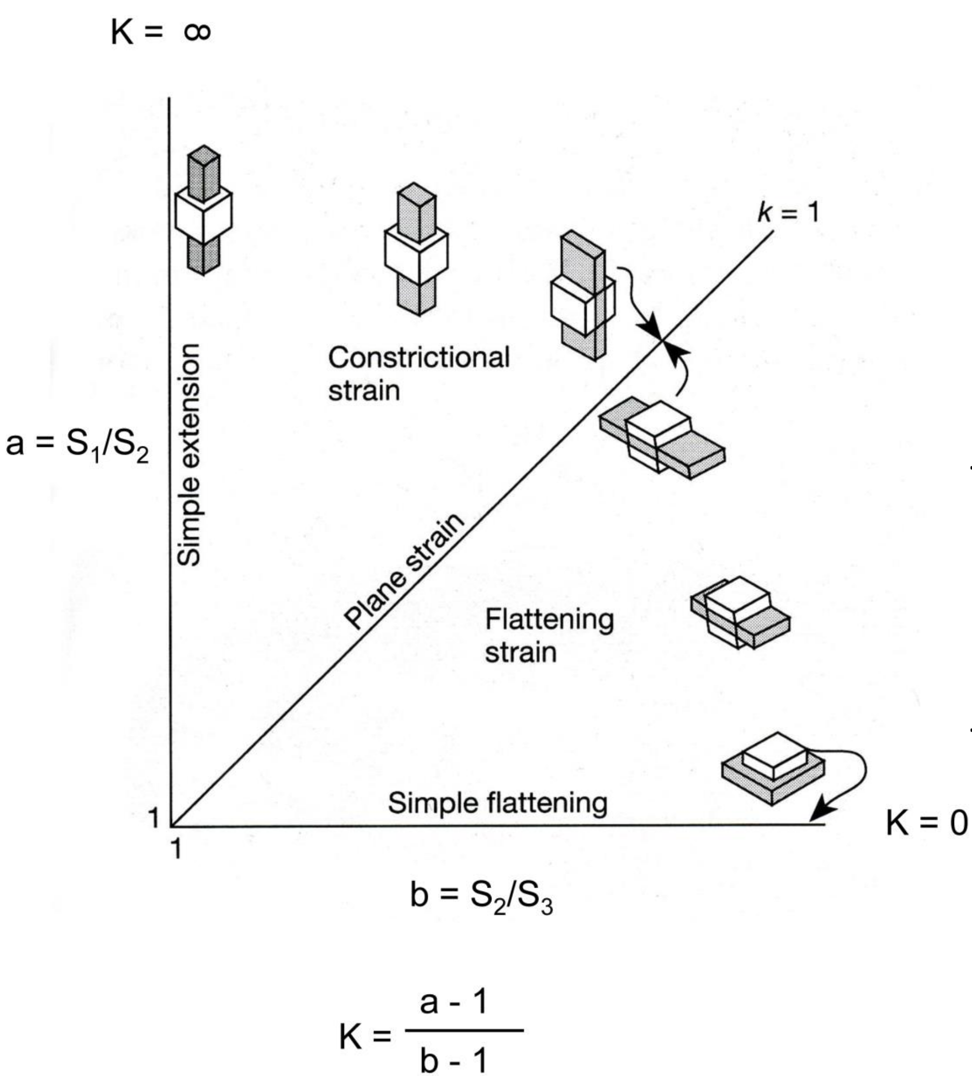


$$\text{volume stretch} = \frac{V_1}{V_0} = \frac{\text{final volume}}{\text{initial volume}}$$

Strain Ellipse/Ellipsoid

- Lines parallel to the long axis of the ellipse have the maximum extension and stretch (e_1, S_1)
- Lines parallel to the short axis have the minimum extension and stretch (e_3, S_3)
- If the initial circle has radius of 1, the final lengths of the two semi-axes of the ellipse are respectively S_1 and S_3
- All above statements are applicable in 3D cases. In a strain ellips*oid, three semi-axes are S_1, S_2, S_3.*

Shape of Strain Ellipsoid –the Flinn Diagram



Plane-strain: If the deformation is restricted in 2D, i.e., in the 3rd dimension, there is no strain (stretch =1, elongation=0). Such a deformation is said to be of plane strain.

Uniaxial extension: extension along one direction $S_1>S_2=S_3$.
–What does the strain ellipsoid look like?

cigarette

Pure flattening: $S_1=S_2>S_3$
–What does the strain ellipsoid look like?

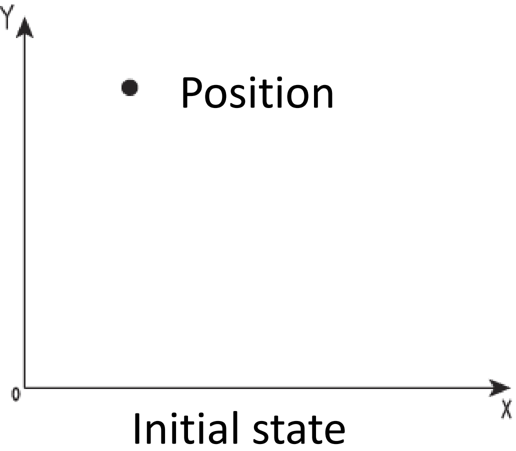
Pancake

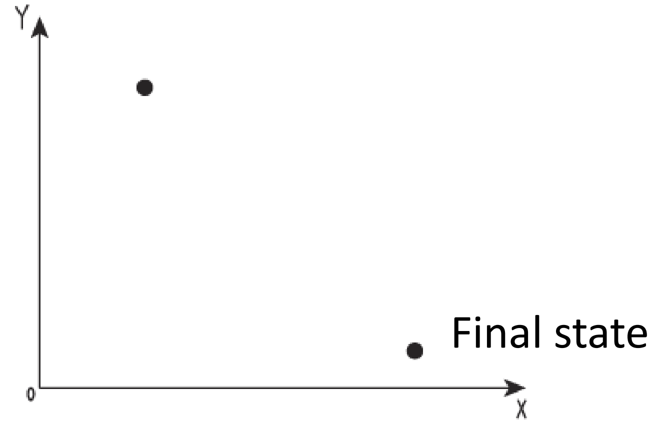
Deformation Path/history

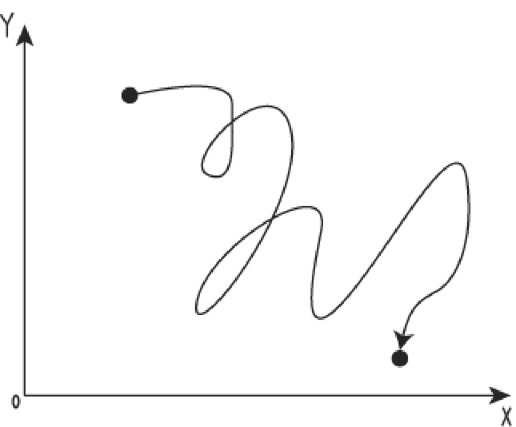
- Strain only considers two states –the initial state and the final state.
- Deformation path refers to the evolution from one state to another
- Deformation history refers to dated deformation path, i.e., path plus the time at which a body was in each intermediate state.

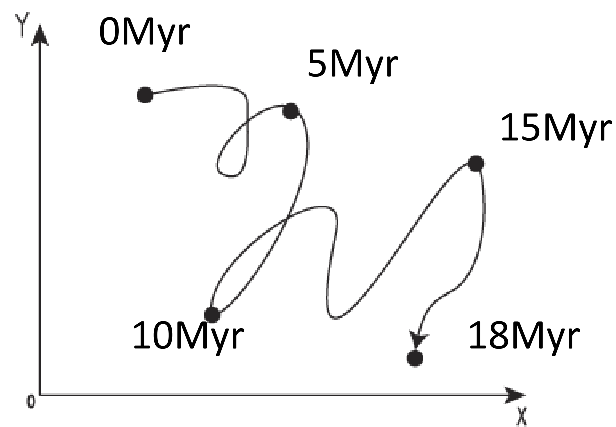
Deformation state/path/history

Use a particle as example

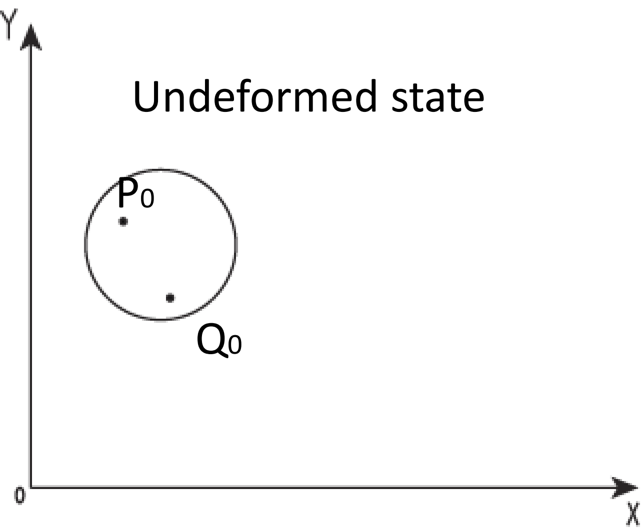


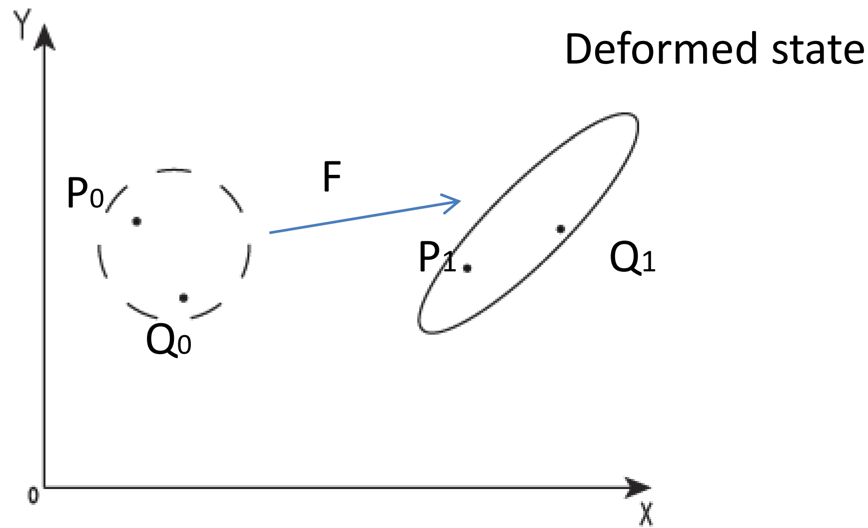






Part II: Mathematical description/study





Changed position & orientation & shape (strain)

$P_0(x0, y0)$

$P_1(x_1, y_1)$

$x_1 = ax0 + by0 + c$
$y_1 = dx0 + ey0 + f$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x0 \\ y0 \end{pmatrix} + \begin{pmatrix} c \\ f \end{pmatrix}$$

Translation

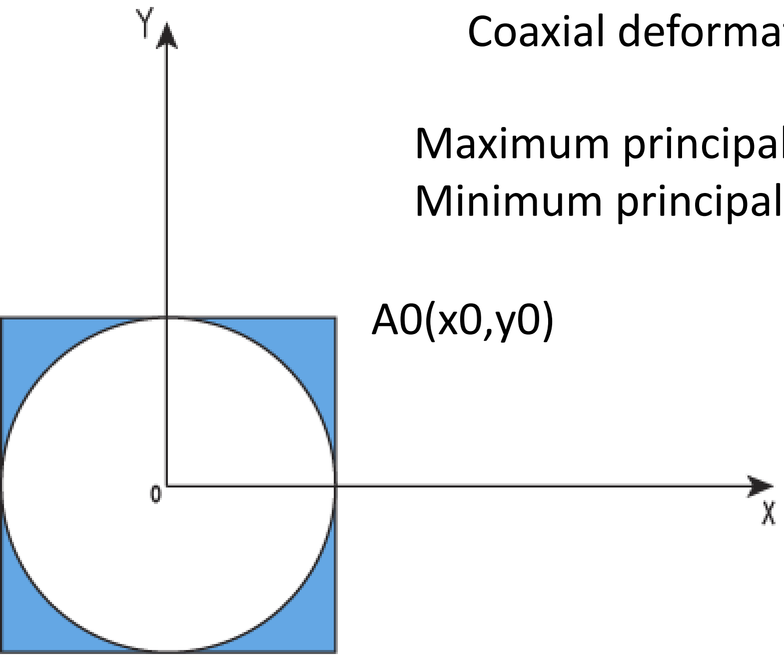
Deformation (position) gradient tensor

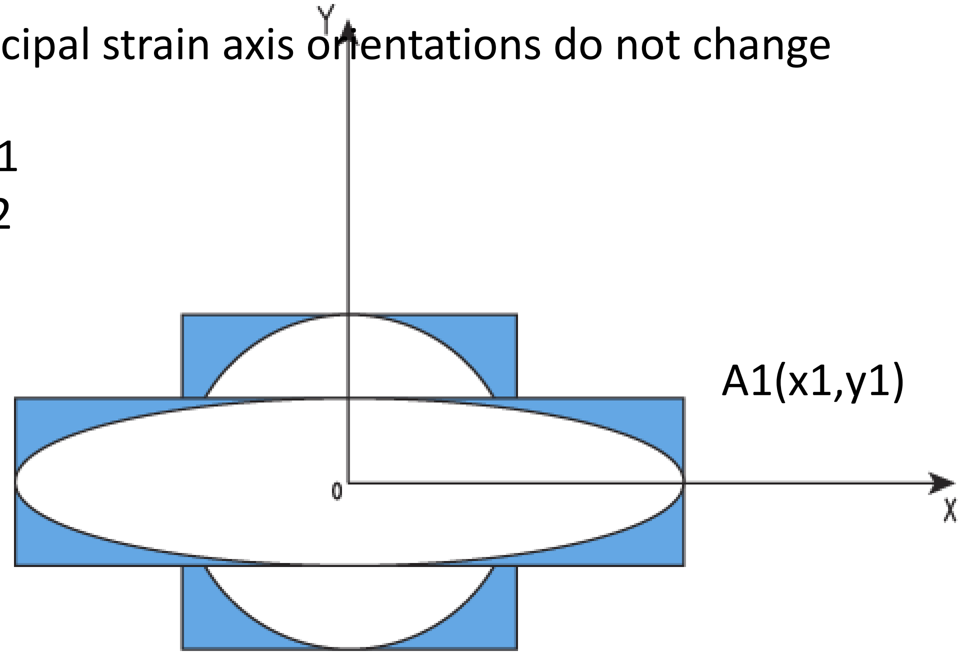
$$\mathbf{F} = \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

Pure shearing

Coaxial deformation: Principal strain axis orientations do not change

Maximum principal strain: S1
Minimum principal strain: S2





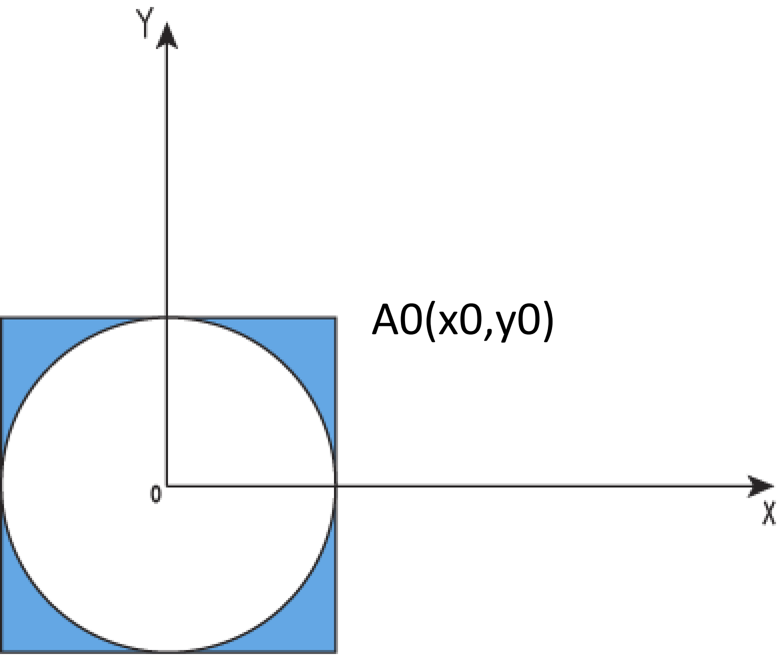
$$\begin{pmatrix} x1 \\ y1 \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} x0 \\ y0 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

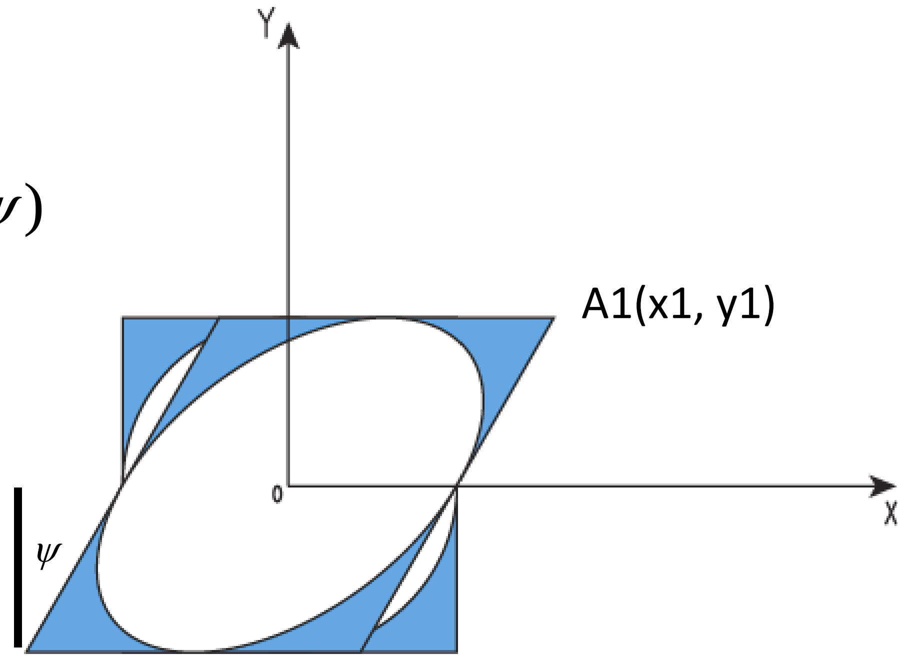
Symmetric tensor

Simple shearing

Non-coaxial deformation: Principal strain axis orientations keep changing



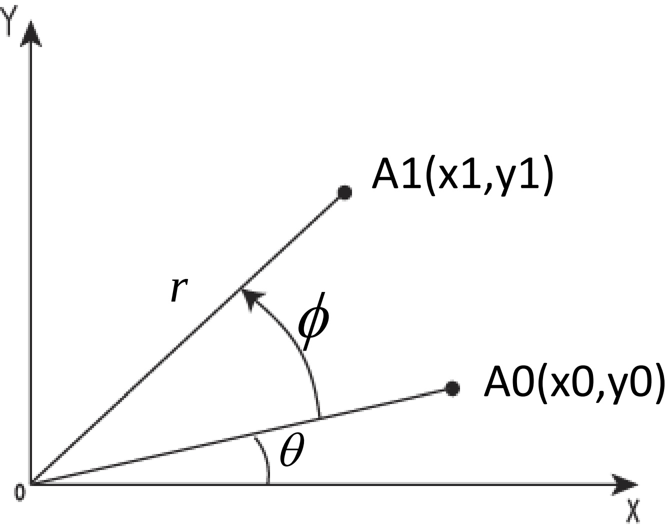
$$\gamma = \tan(\psi)$$



$$\begin{pmatrix} x1 \\ y1 \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x0 \\ y0 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

Rotation



$$\begin{pmatrix} x0 \\ y0 \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} x1 \\ y1 \end{pmatrix} = \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\sin\theta\cos\phi + r\cos\theta\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} x1 \\ y1 \end{pmatrix} = \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x0 \\ y0 \end{pmatrix}$$

Reference frame dependence

Rigid body motion depends on reference frame

Rotation makes no sense unless referred to a reference frame!

$$\mathbf{F} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Orthogonal tensor

Strain is independent of reference frames

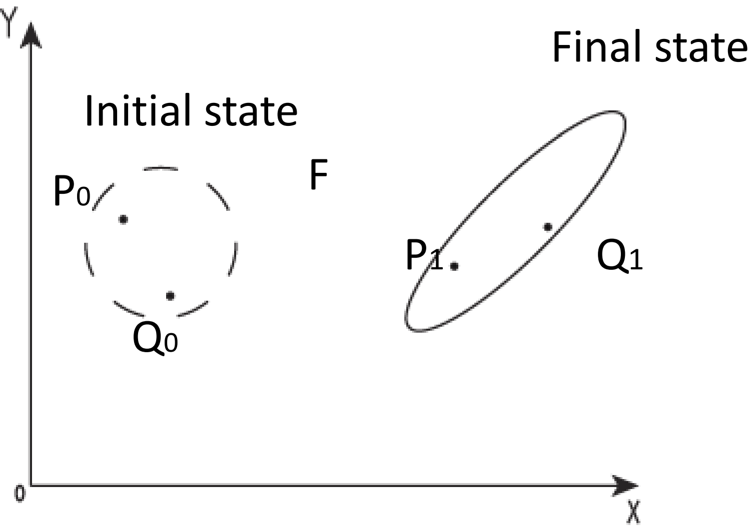
$$\mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{I}$$

Deformation (position) gradient tensor

$\mathbf{F}$

- F determines the rotation and stretch of all material lines. Therefore it has all the information for a homogeneous deformation!
- volume stretch = det (F) Determinant of F: $|\mathbf{F}|$
- F symmetric: irrotational deformation: $\mathbf{F}=\mathbf{F}^{\mathrm{T}}$
- F asymmetric: rotational deformation

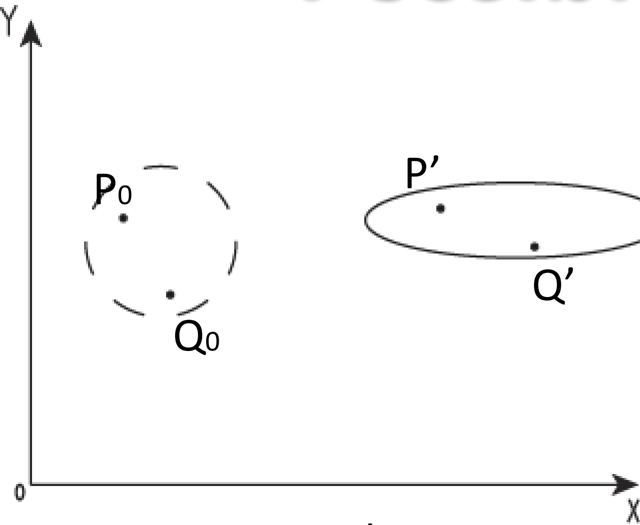
Deformation gradient tensor



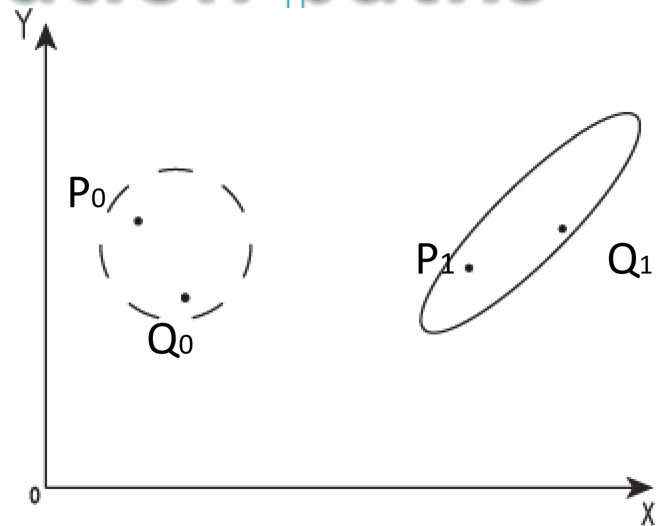
F

Gives information about the changes of two states; No information about deformation path

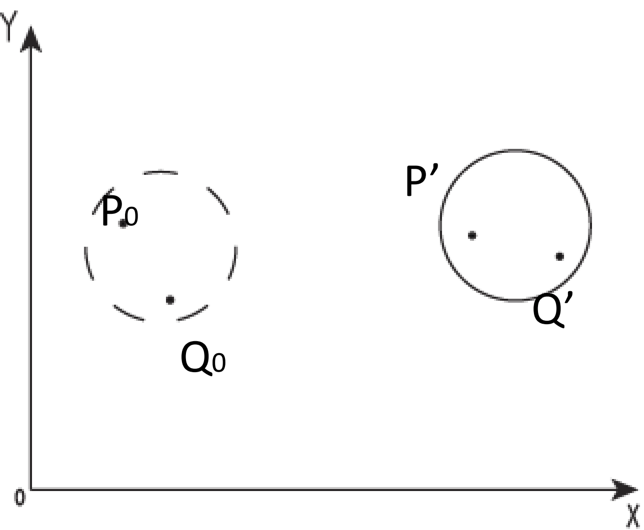
Possible deformation paths



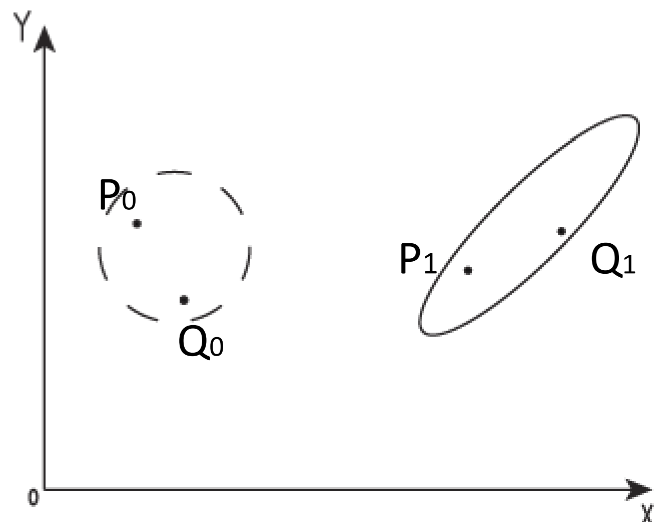
Step 1: Translation + Strain



Step 2: Rotation

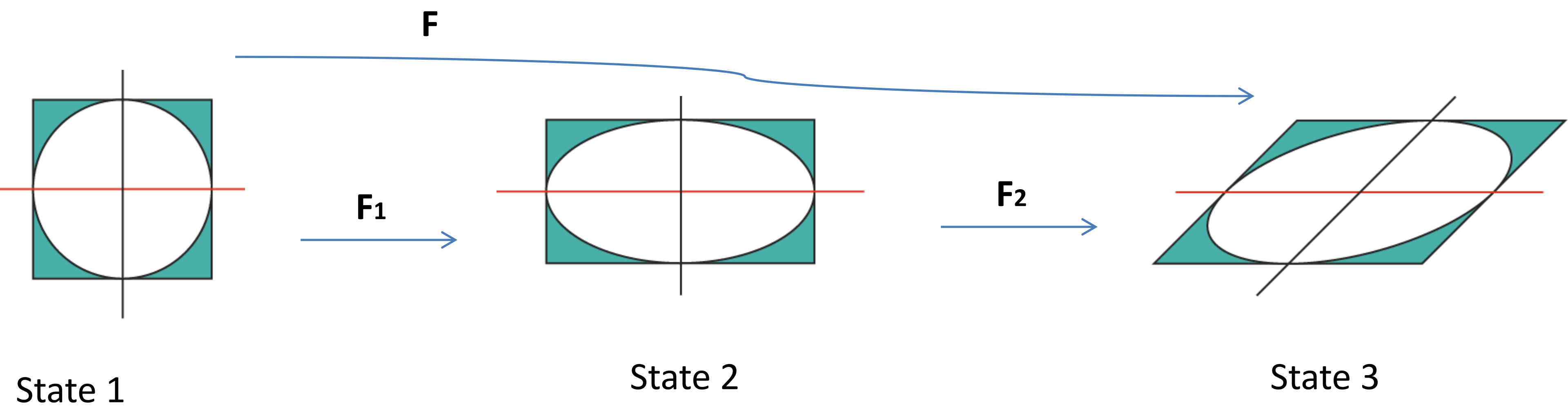


Step 1: Translation + Rotation



Step 2: Strain

Overprinting/superposition of finite deformations



$F=F_2F_1$

Determine Principal stretches (and strain axes) from F

$$\mathbf{F} = \begin{pmatrix} \sqrt{2} & -\dfrac{\sqrt{2}}{4} \\ \sqrt{2} & \dfrac{\sqrt{2}}{4} \end{pmatrix}$$

Polar decomposition

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}$$

$\mathbf{R}$ Orthogonal tensor

$\mathbf{U}, \mathbf{V}$ Symmetrical tensor

Green's deformation tensor or the left/right Cauchy-Green deformation tensor

$$\mathbf{B} = \mathbf{FF}^{\mathrm{T}} = (\mathbf{VR})(\mathbf{VR})^{\mathrm{T}} = \mathbf{VRR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} = \mathbf{VV}^{\mathrm{T}}$$
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = (\mathbf{RU})^{\mathrm{T}}(\mathbf{RU}) = \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{RU} = \mathbf{U}^{\mathrm{T}}\mathbf{U}$$

B: left Cauchy-Green tensor
C: right Cauchy-Green tensor

Eigenvalues of B are 3 quadric principal stretches: S_1^2, S_2^2, S_3^2.

Eigenvectors of B are principal strain axes in the deformed state

Eigenvectors of C are principal strain axes in the undeformed state.

$$\mathbf{F} = \mathbf{RU}$$

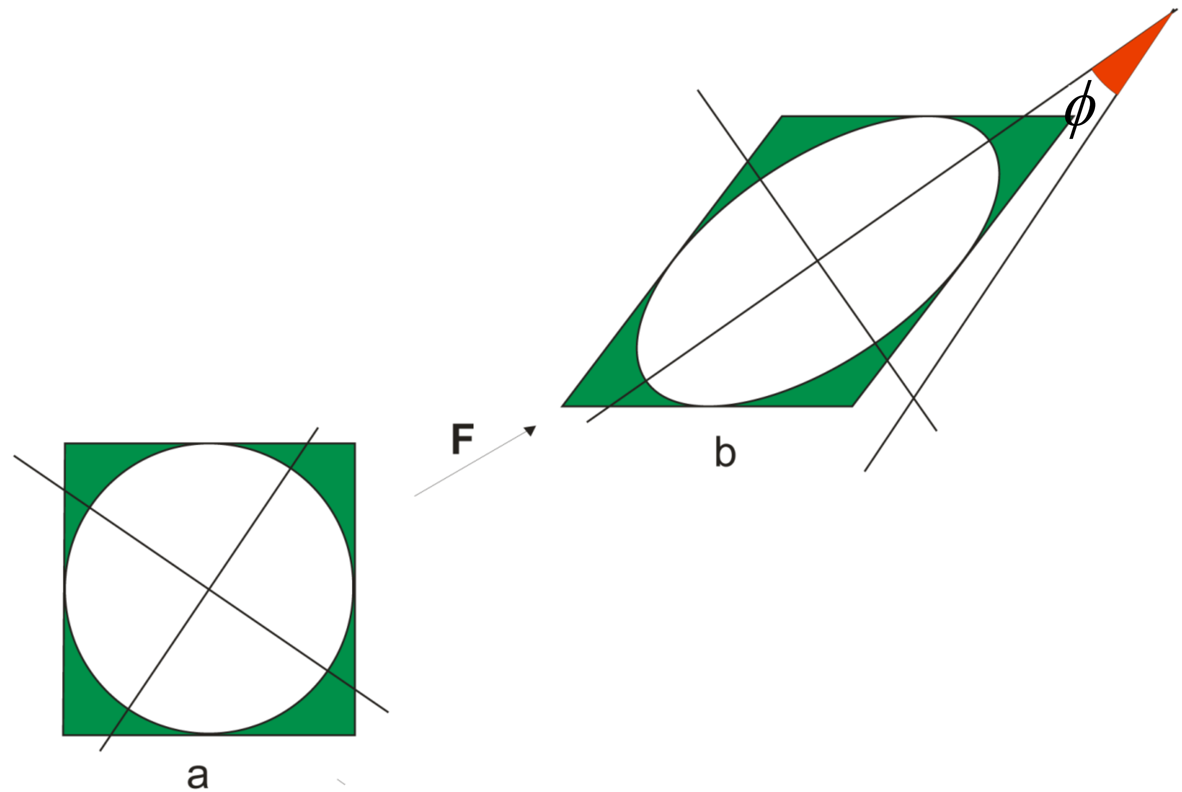
$$\mathbf{F} = \mathbf{VR}$$

a: initial state

b: final state

RU: a—c—b

VR: a—d—b



Polar decomposition

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$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = (\mathbf{VR})(\mathbf{VR})^{\mathrm{T}} = \mathbf{V}\mathbf{R}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} = \mathbf{V}\mathbf{V}^{\mathrm{T}}$$

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = (\mathbf{RU})^{\mathrm{T}}(\mathbf{RU}) = \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{U}$$

B: left Cauchy-Green tensor

C: right Cauchy-Green tensor

Eigenvalues of B are 3 quadric principal stretches.

Eigenvectors of B are principal strain axes in the deformed state

Eigenvectors of C are principal strain axes in the undeformed state.

Exercise

Determine Principal stretches (and strain axes) from F

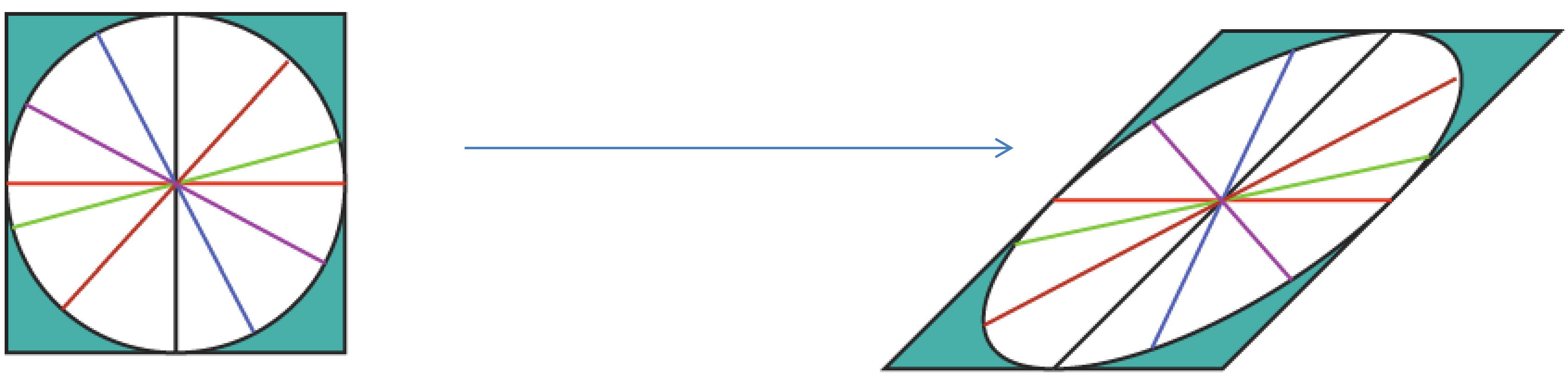
$$\mathbf{F} = \begin{pmatrix} \sqrt{2} & -\dfrac{\sqrt{2}}{4} \\ \sqrt{2} & \dfrac{\sqrt{2}}{4} \end{pmatrix}$$

Outline

- Deformation path:
- Fabric evolution

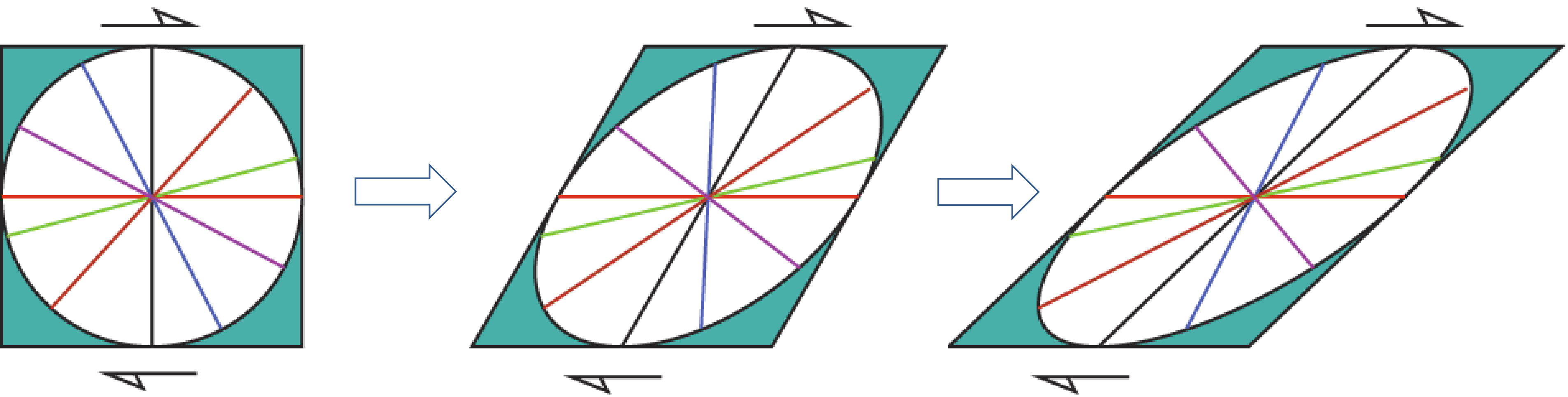
Material lines (passive deformation): Rotation and stretch

Homogenous deformation: Simple shear, Pure shear, or General shear



Do all material lines rotate?
Do all material lines change their length (stretched)?

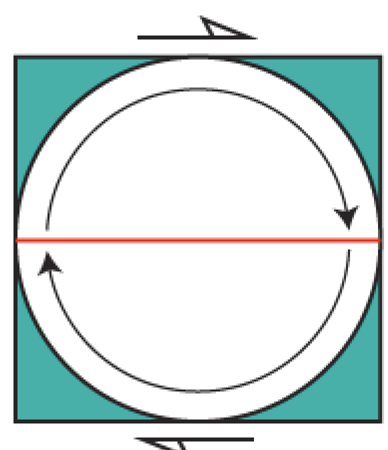
Material lines: Rotation in simple shear



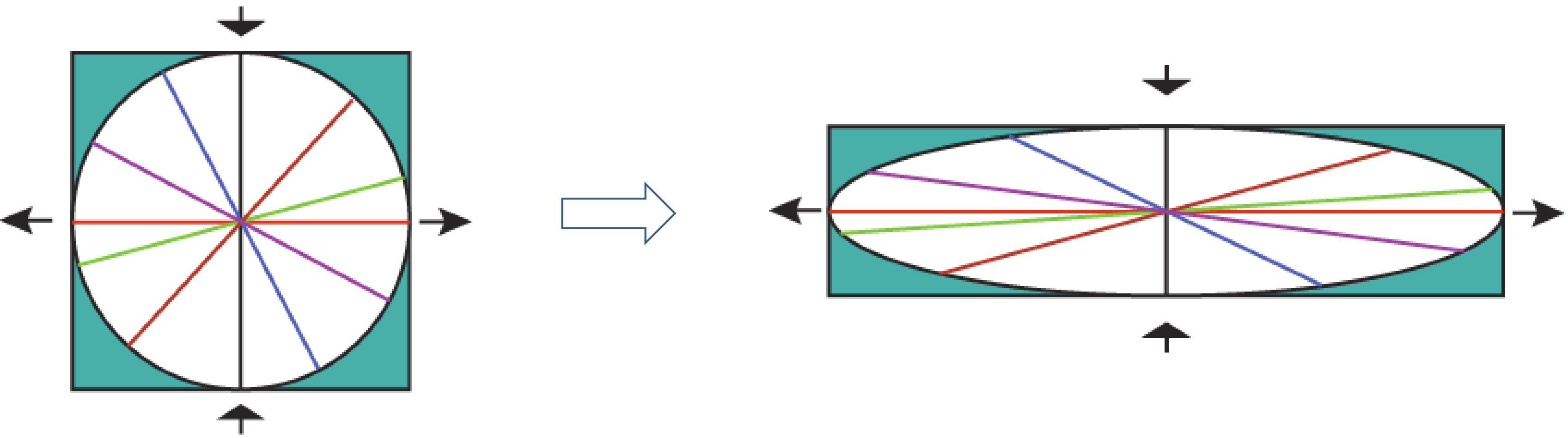
Which material line is non-rotational? (flow apophysis)

Horizontal red line; lines on the shear plane

What is the final orientation of all material lines if strain is high enough?

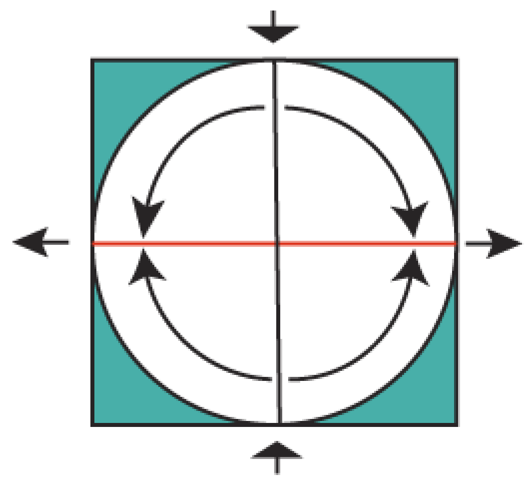


Material lines: Rotation in pure shear



Which material lines are non-rotational?

What are the final orientations of material lines if strain is high enough

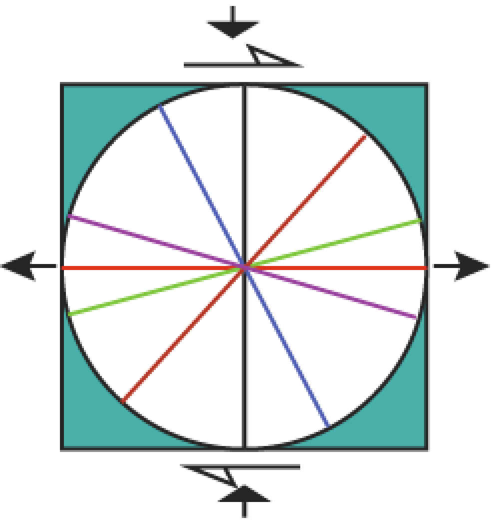


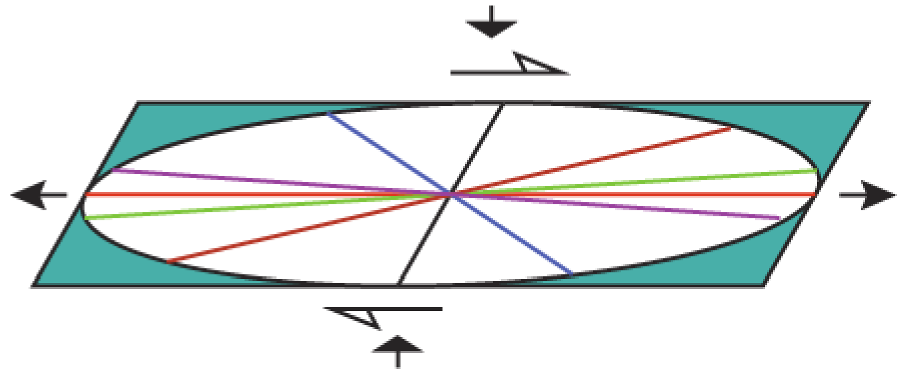
(flow apophyses)

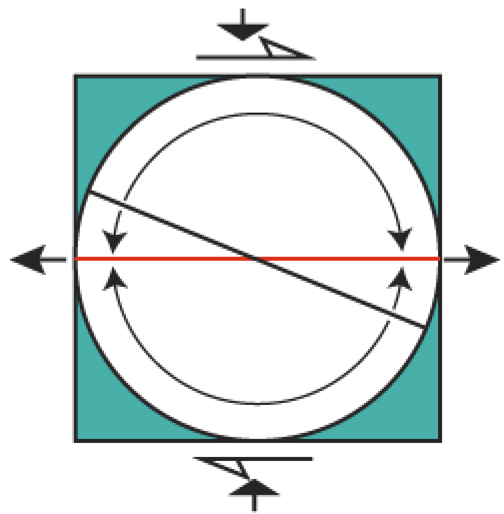
Semistable

Stable

Material lines: Rotation in general shear



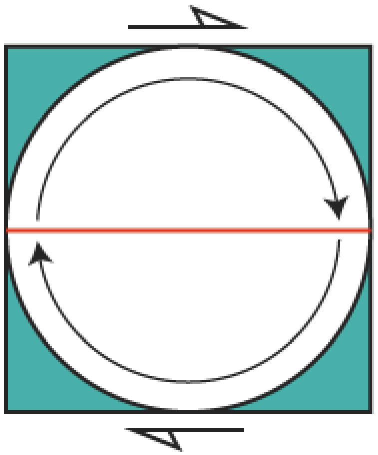




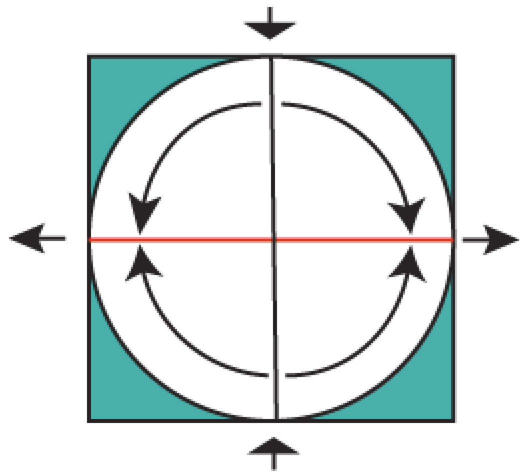
Which material lines are non-rotational?

What are the final orientations of material lines if strain is high enough?

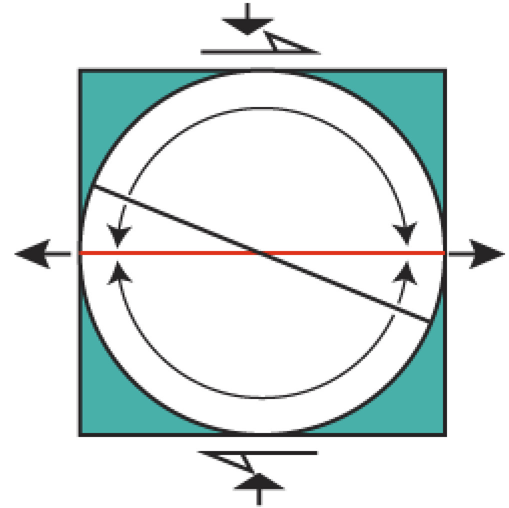
Fabric attractor



Simple shear



Pure shear



General shear

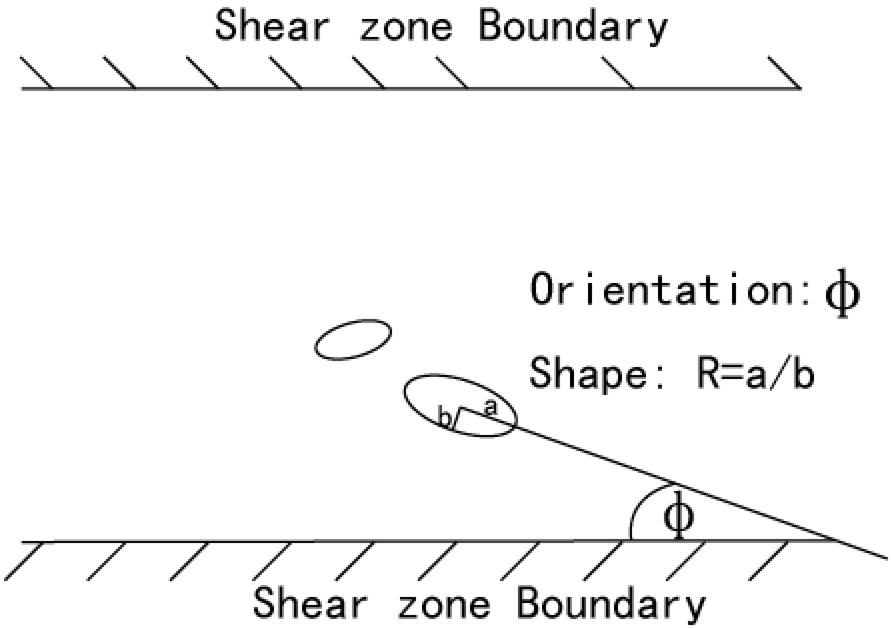
Fabric attractor

Alsop, G. I. and Carreras, J. (2007)

Sheath fold

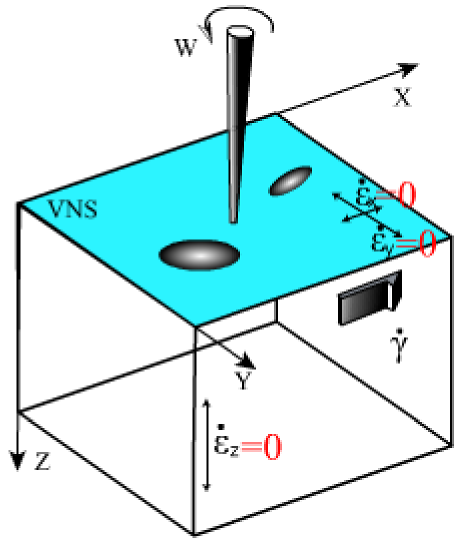
Fabric cannot be regarded as material lines: Rigid porphyroclasts

Competence contrast: Not passively deformed

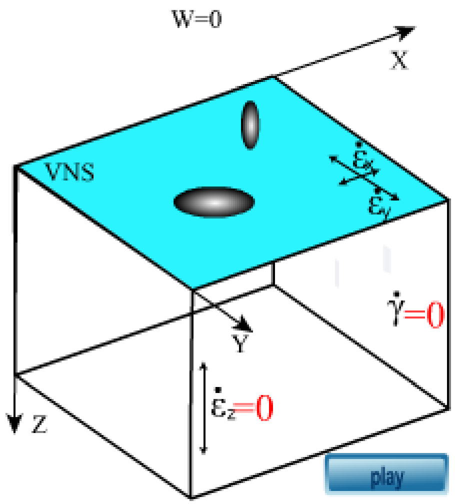


Some geologists use shape preferred orientation of porphyrocalsts to estimate the contribution of the simple shear component in general shears

Rotation of rigid porphyroclasts



Simple shear



Pure shear

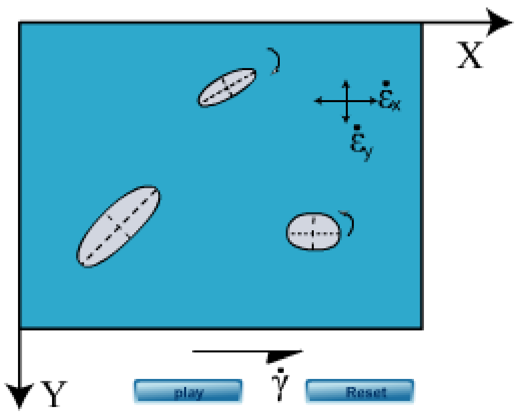
Assumptions:

- Behaves according to Jeffery's theory: Need tested

  Matrix: Newtonian material (linear relationship between stress and strain rate)
- one axis // vorticity vector w: on the shear plane and perpendicular to the shear direction: Almost impossible

Rotation of rigid porphyroclasts

In general shear



Porphyroclast aspect ratio: R=a/b

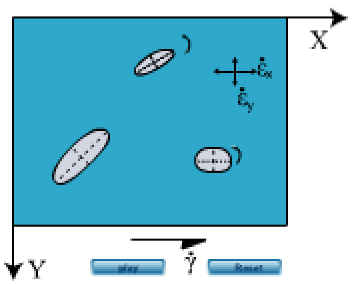
Critical aspect ratio: Rc
Related to the percentage of simple shear component

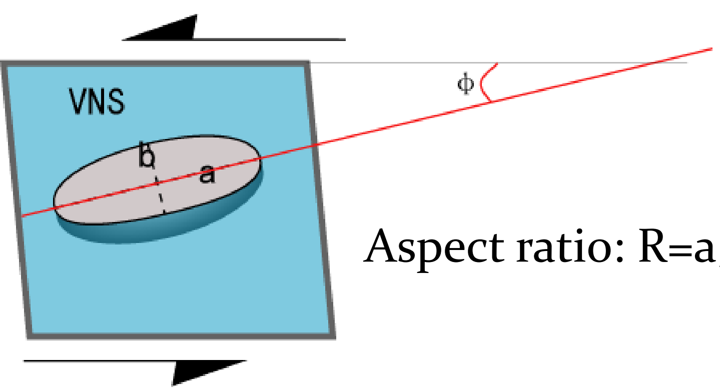
2)R>Rc:
Porphyroclasts rotate to their stable orientations

3)R<Rc:
Porphyroclast rotates permanently with vorticity.

Rotation of rigid porphyroclasts

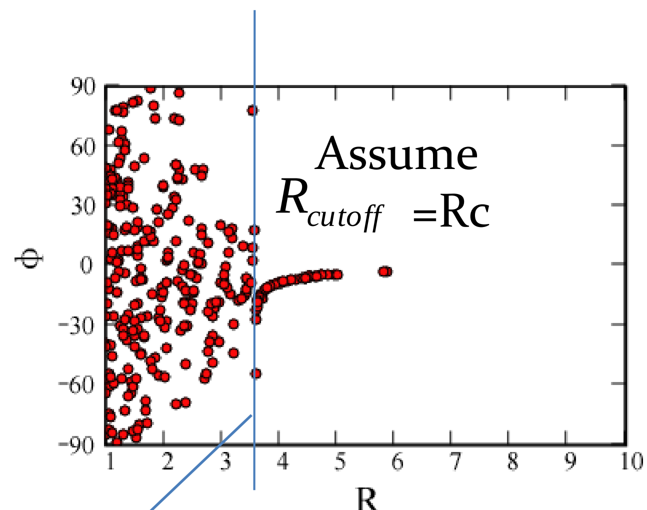
In general shear





Aspect ratio: R=a/b

ϕ Versus R plot

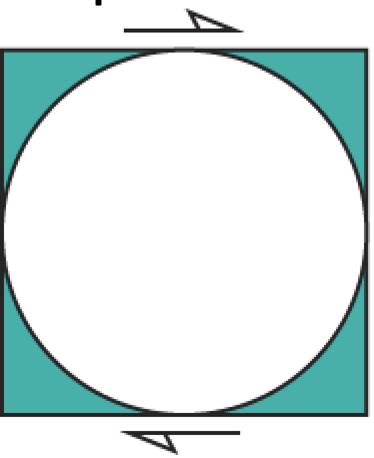


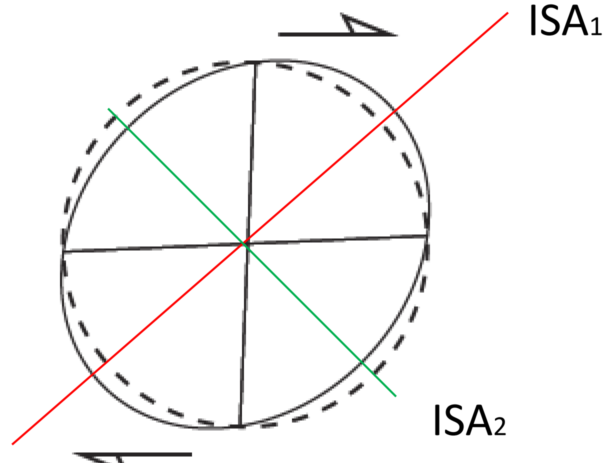
Wallis plot

Rc is used to determine the contribution of simple shear component

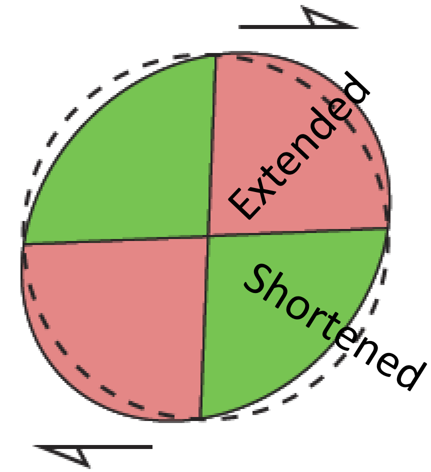
Material lines: Stretch in shearing

Simple shear

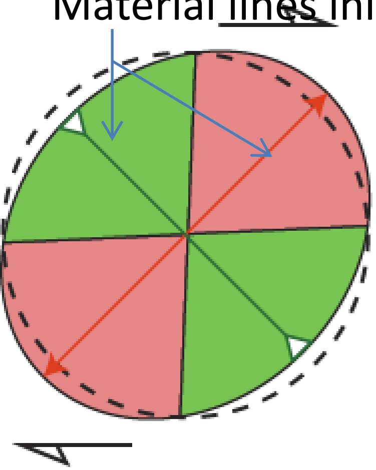




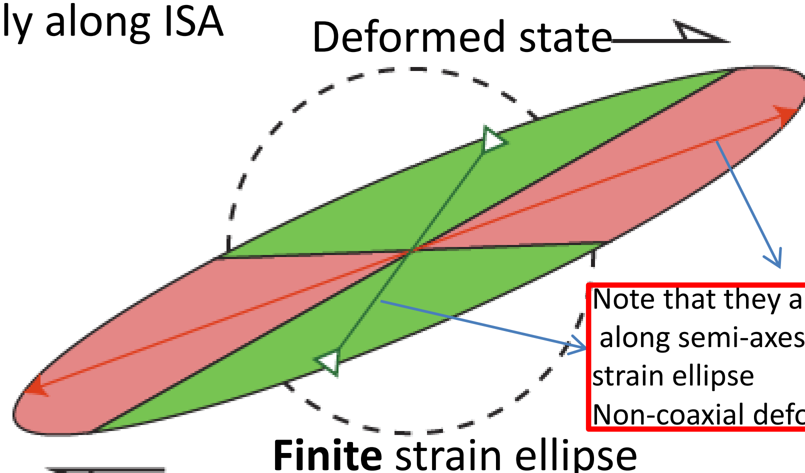
Infinitesimal strain ellipse
ISA: Instantaneous stretching axis



Material lines initially along ISA

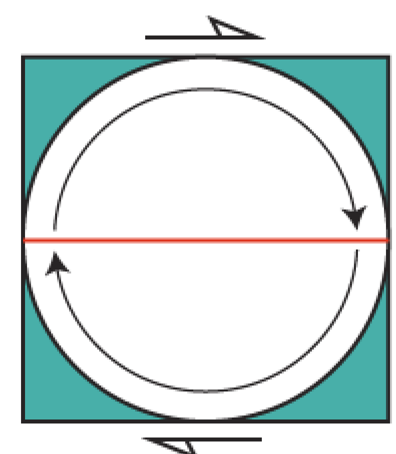


Deformed state

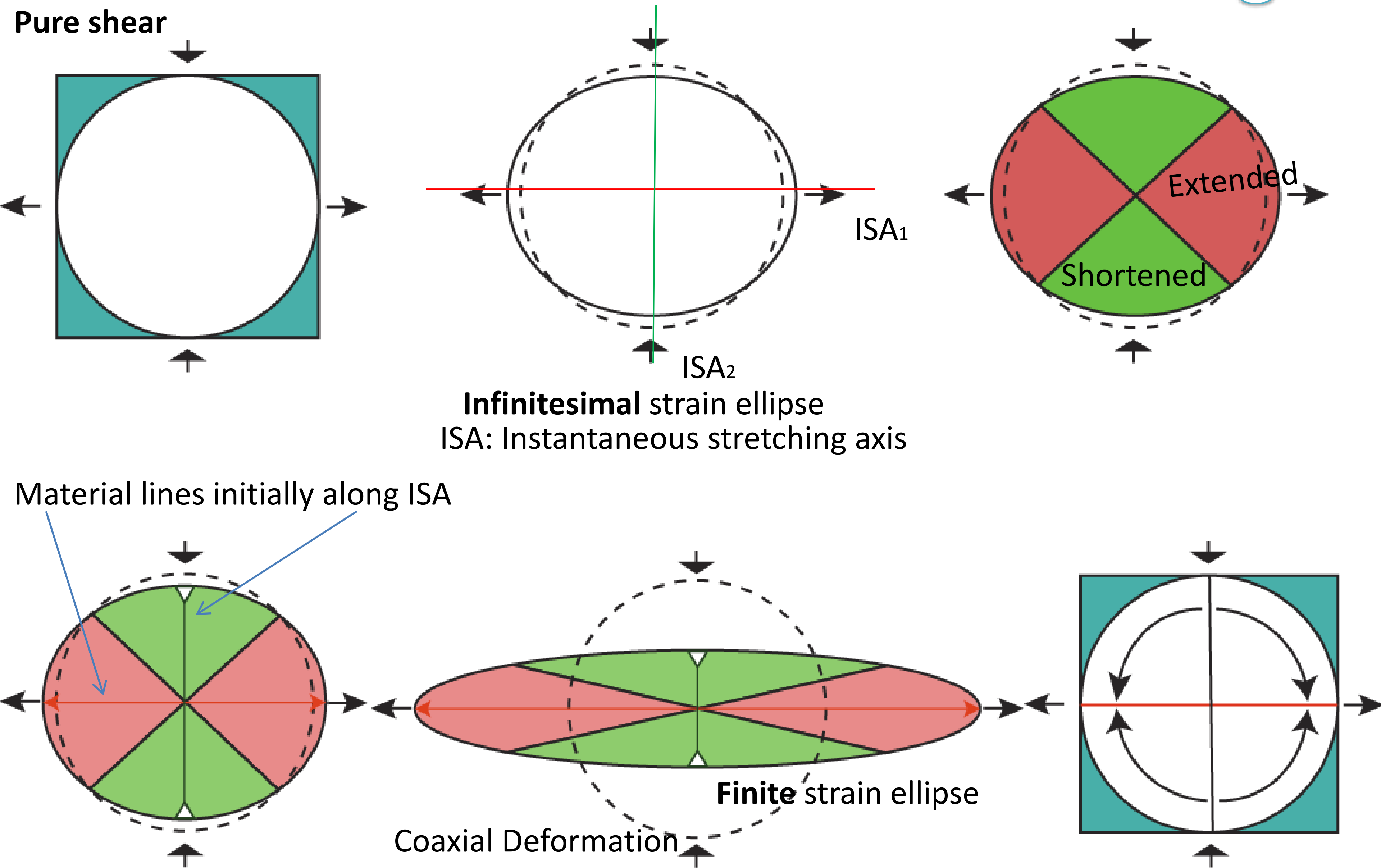


Note that they are NOT along semi-axes of the finite strain ellipse
Non-coaxial deformation !

Finite strain ellipse

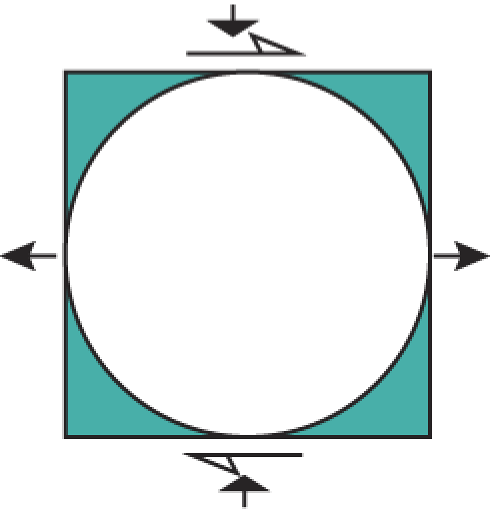


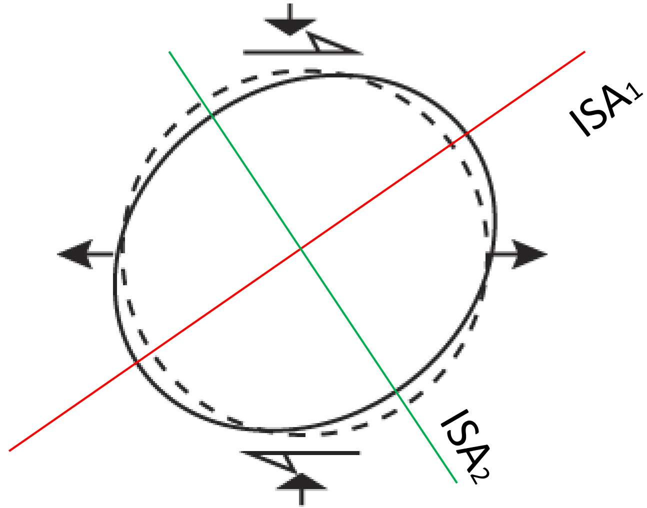
Material lines: Stretch in shearing

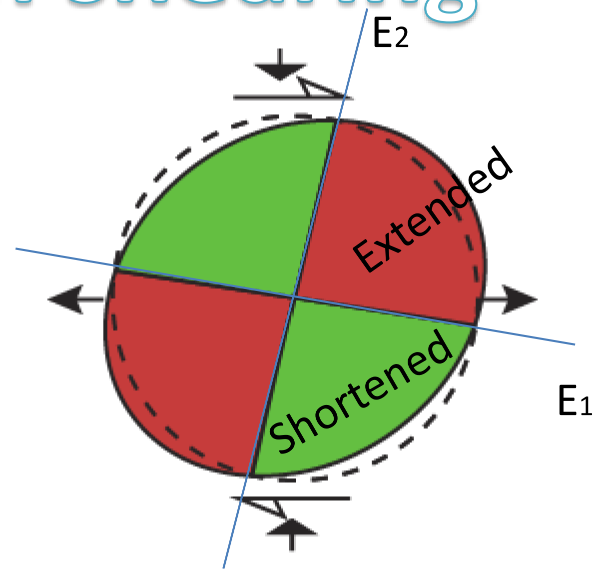


Material lines: Stretch in shearing

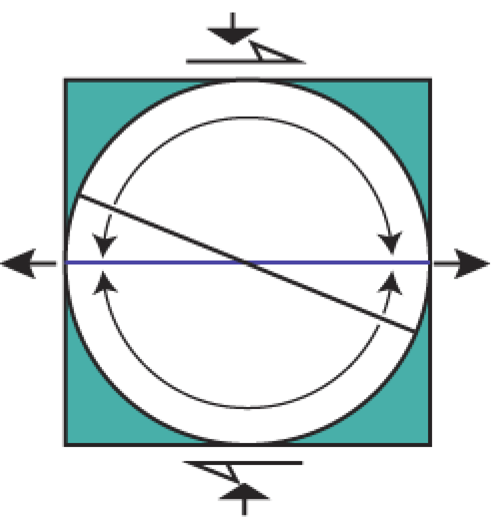
General shear

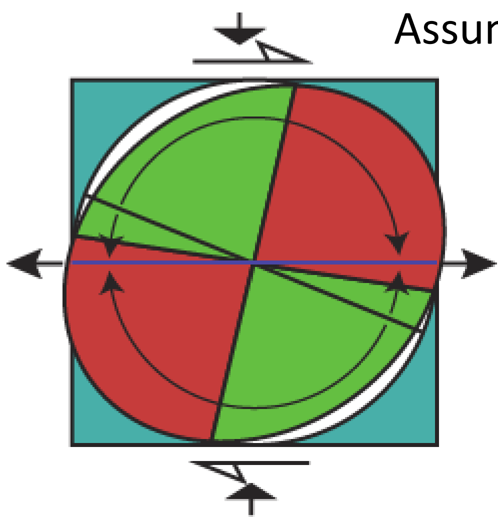




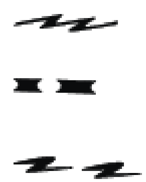


Infinitesimal strain ellipse
ISA: Instantaneous stretching axis





Assume that dikes in shear zones deform passively



Folded dike

Boudinaged dike

Folded-then-boudinaged dike

Flow

- **Deformation:**

Position change (transformation): position gradient tensor F

- **Flow: progressive deformation**

Velocity field: velocity gradient tensor L

Strain rate and angular velocity

$$e = \frac{l_1 - l_0}{l_0} = \frac{\text{change in length}}{\text{initial length}}$$

Elongation

$$\Delta e = \frac{\Delta l}{l}$$

Infinitesimal elongation

$$\lim_{\Delta t \to 0} \frac{\Delta e}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta l}{l\ \Delta t}$$

Strain rate

$$\frac{de}{dt} = \frac{1}{l}\frac{dl}{dt} = \frac{d(\ln l)}{dt} = \frac{d\varepsilon}{dt} = \dot{\varepsilon}$$

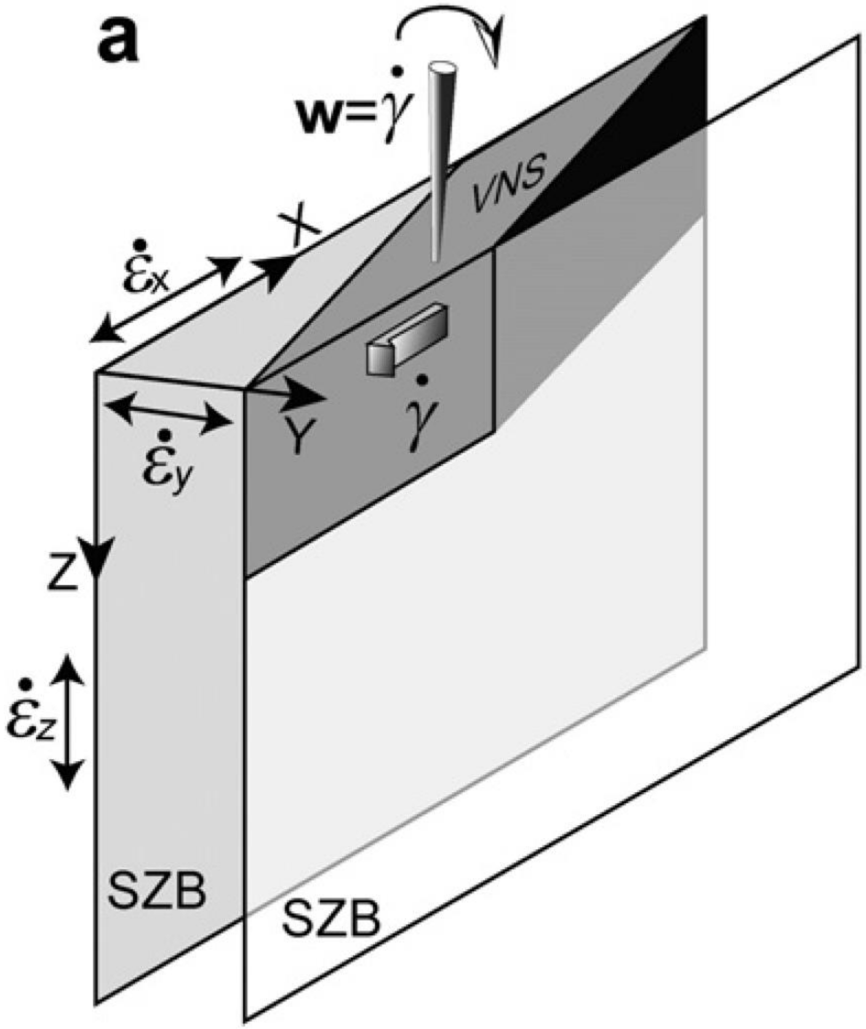
$$\dot{\theta} = \frac{d\theta}{dt}$$

Angular velocity

$$\dot{\gamma}_{ab} = \dot{\theta}_a - \dot{\theta}_b$$

Shear strain rate

Flow field

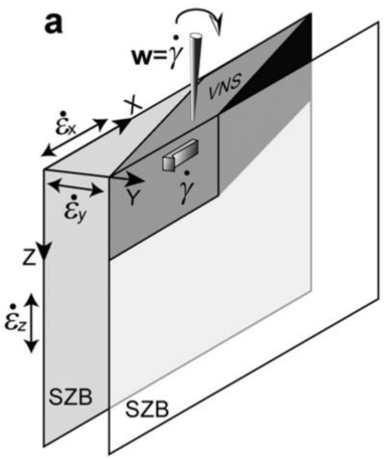


- Describes velocity of any particle in terms of its instant position

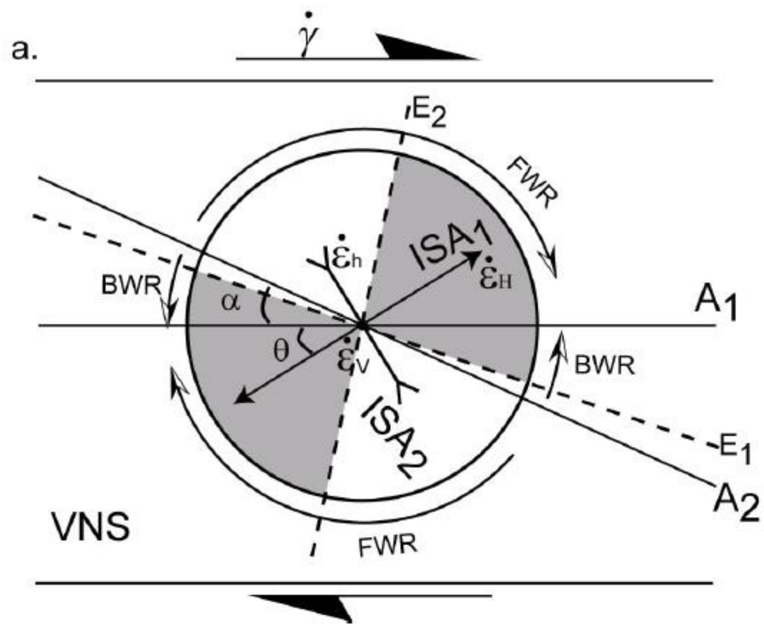
$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{\varepsilon}_x & -\dot{\gamma} & 0 \\ 0 & \dot{\varepsilon}_y & 0 \\ 0 & 0 & \dot{\varepsilon}_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

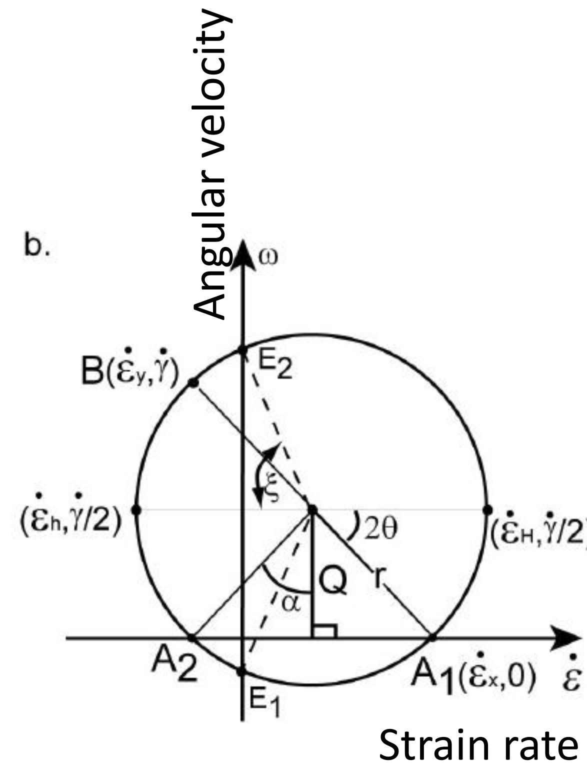
$$\mathbf{L} = \begin{pmatrix} \dot{\varepsilon}_x & -\dot{\gamma} & 0 \\ 0 & \dot{\varepsilon}_y & 0 \\ 0 & 0 & \dot{\varepsilon}_z \end{pmatrix}$$

Mohr circle of velocity gradient tensor

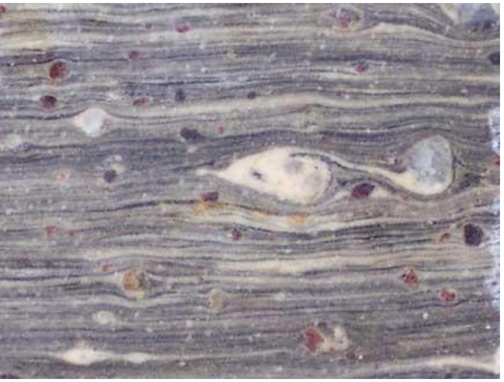


$$\mathbf{L}=\begin{pmatrix} \dot{\varepsilon}_x & -\dot{\gamma} & 0 \\ 0 & \dot{\varepsilon}_y & 0 \\ 0 & 0 & \dot{\varepsilon}_z \end{pmatrix}$$

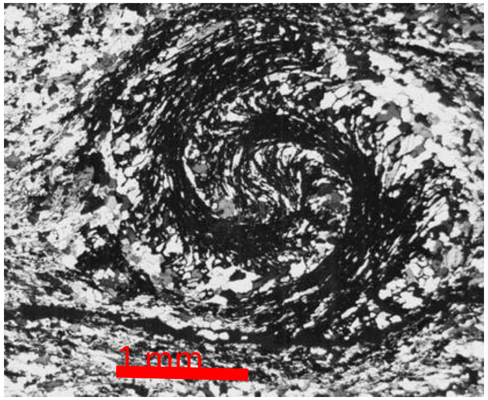




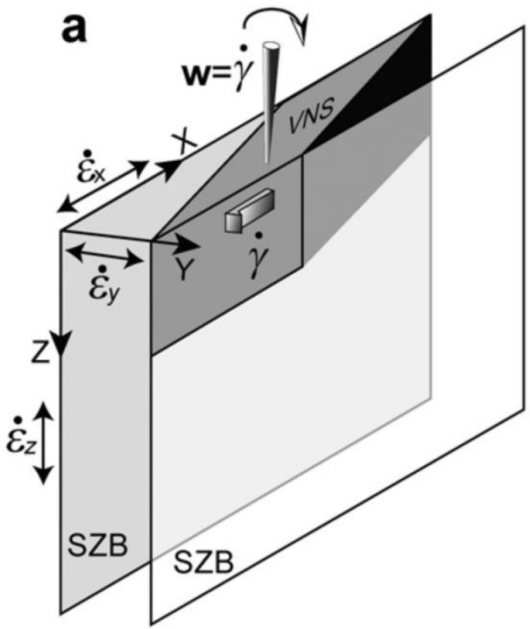
Vorticity: w

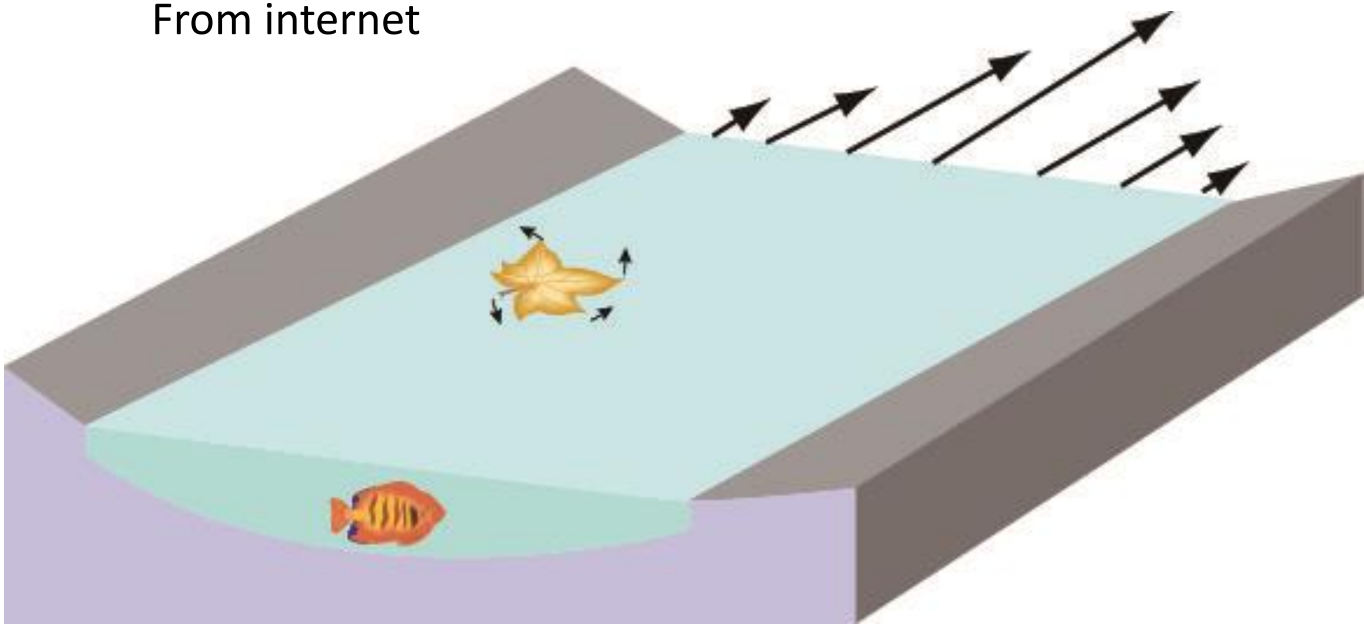


From internet



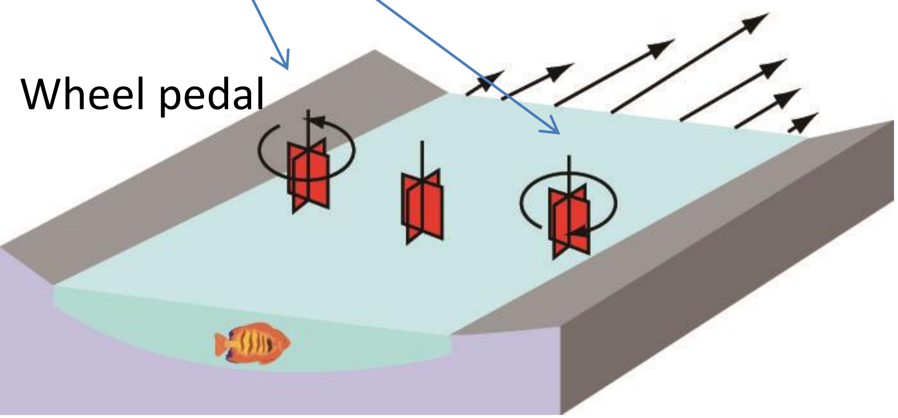
From internet



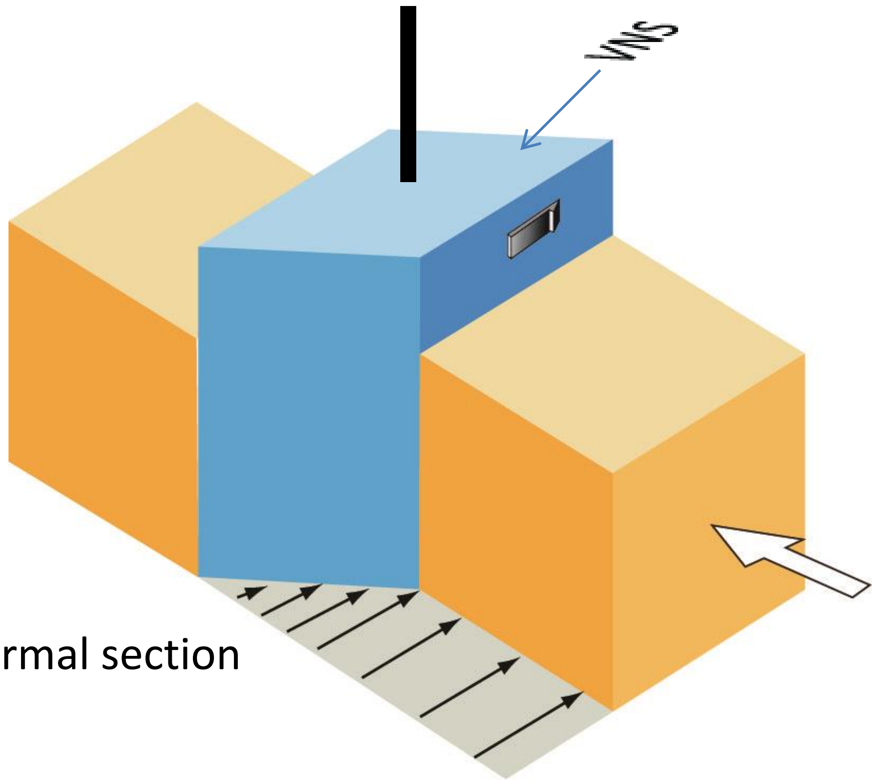


Vorticity: w

Vorticity vector



Vorticity vector



Simple shear: has vorticity

VNS: vorticity normal section

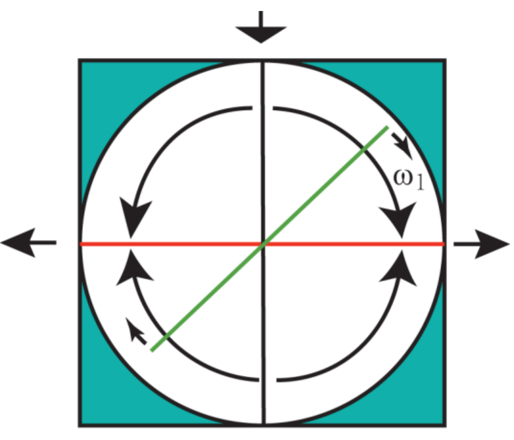
Pure shear
Flatting: no vorticity

General shear

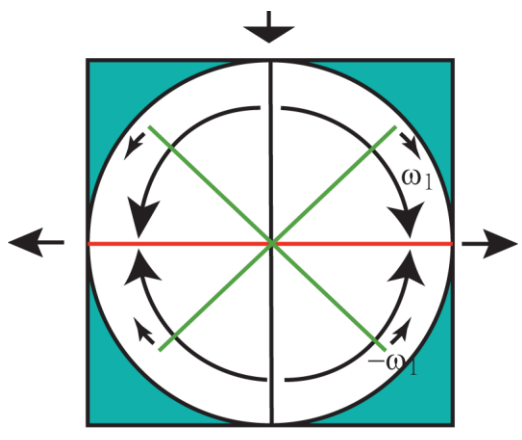
Combination of simple shear and pure shear
Has vorticity

Vorticity: w

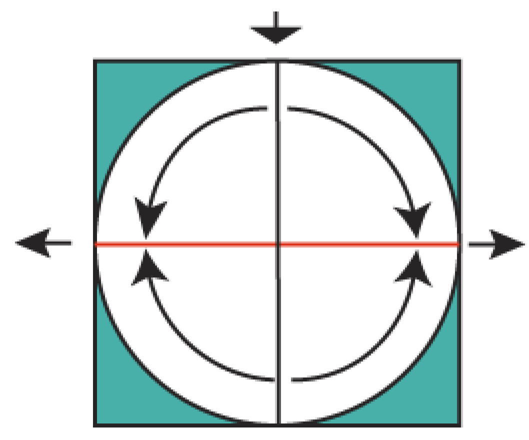
- Material line rotating: has angular velocity
- IwI (vorticity magnitude): Sum of angular velocity of any two orthogonal material lines (on the VNS); Two times of averaged angular velocity of all material lines on the VNS



Pure shear

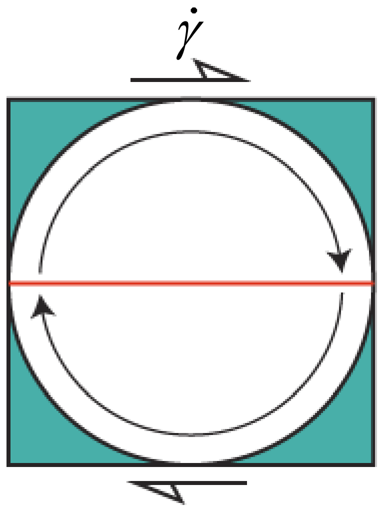


Pure shear



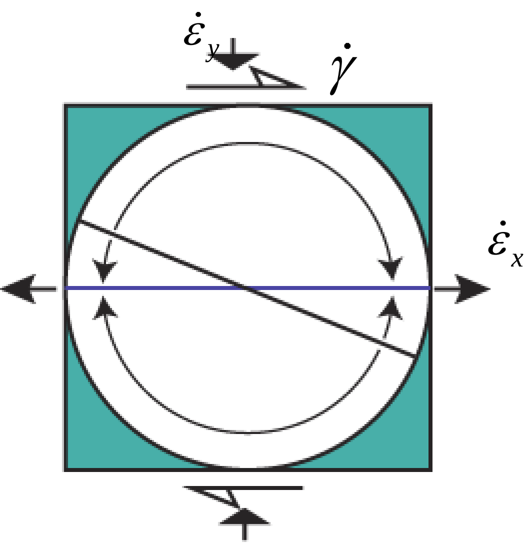
Pure shear; w=0

Vorticity



Simple shear: |w|= $\dot{\gamma}$

(simple shear strain rate)



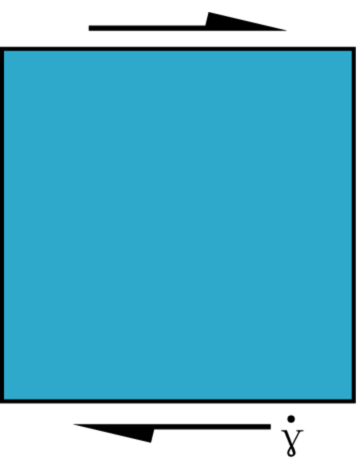
General shear: |w|= $\dot{\gamma}$

kinematic vorticity number (W_k)

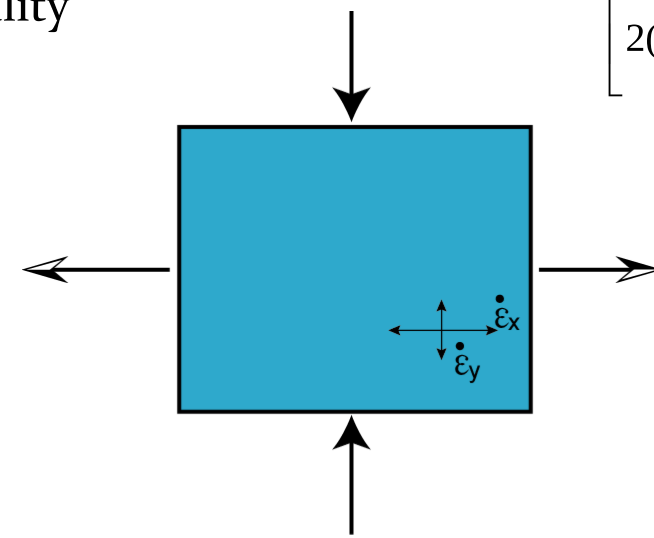
- Kinematic vorticity number (W_k)

$$W_k = \frac{\dot{\gamma}}{\left[2(\dot{\varepsilon}_x^{\,2} + \dot{\varepsilon}_y^{\,2}) + \dot{\gamma}^2\right]^{\frac{1}{2}}}$$

Measure of degree of non-coaxiality

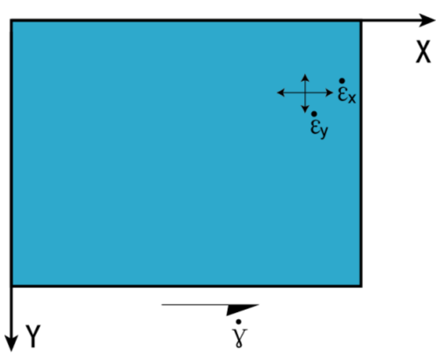


Simple shear
W_k=1



Pure shear
W_k=0

$$\mathbf{L} = \begin{pmatrix} \dot{\varepsilon}_x & \dot{\gamma} \\ 0 & \dot{\varepsilon}_y \end{pmatrix}$$



General shear: $0<W_k<1$

Contributions of simple shear and pure shear

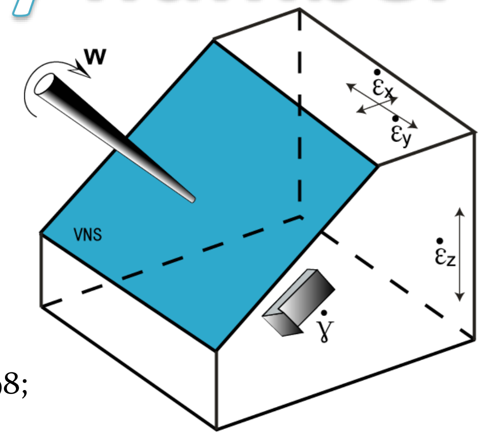
kinematic vorticity number (W_k)

3D general shear

- Triclinic shear
- Monoclinic shear

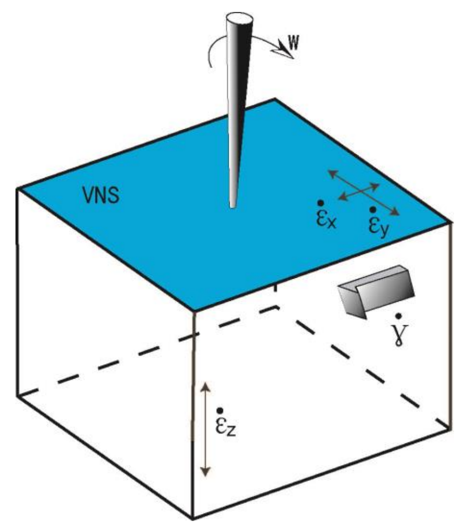
$$W_k = \frac{\dot{\gamma}}{\left[2(\dot{\varepsilon}_x^{\,2}+\dot{\varepsilon}_y^{\,2}+\dot{\varepsilon}_z^{\,2})+\dot{\gamma}^2\right]^{\frac{1}{2}}}$$

(Jiang and Williams, 1998; Lin et al., 1998)



triclinic

W $\not\parallel$ $\dot{\varepsilon}_z$



Monoclinic

W// $\dot{\varepsilon}_z$

W_s: Sectinal kinematic vorticity number (Passchier, 1988, 1991; Robin and Cruden,1994; Jiang and Williams, 1998; Lin et al., 1998)

W_s on VNS:

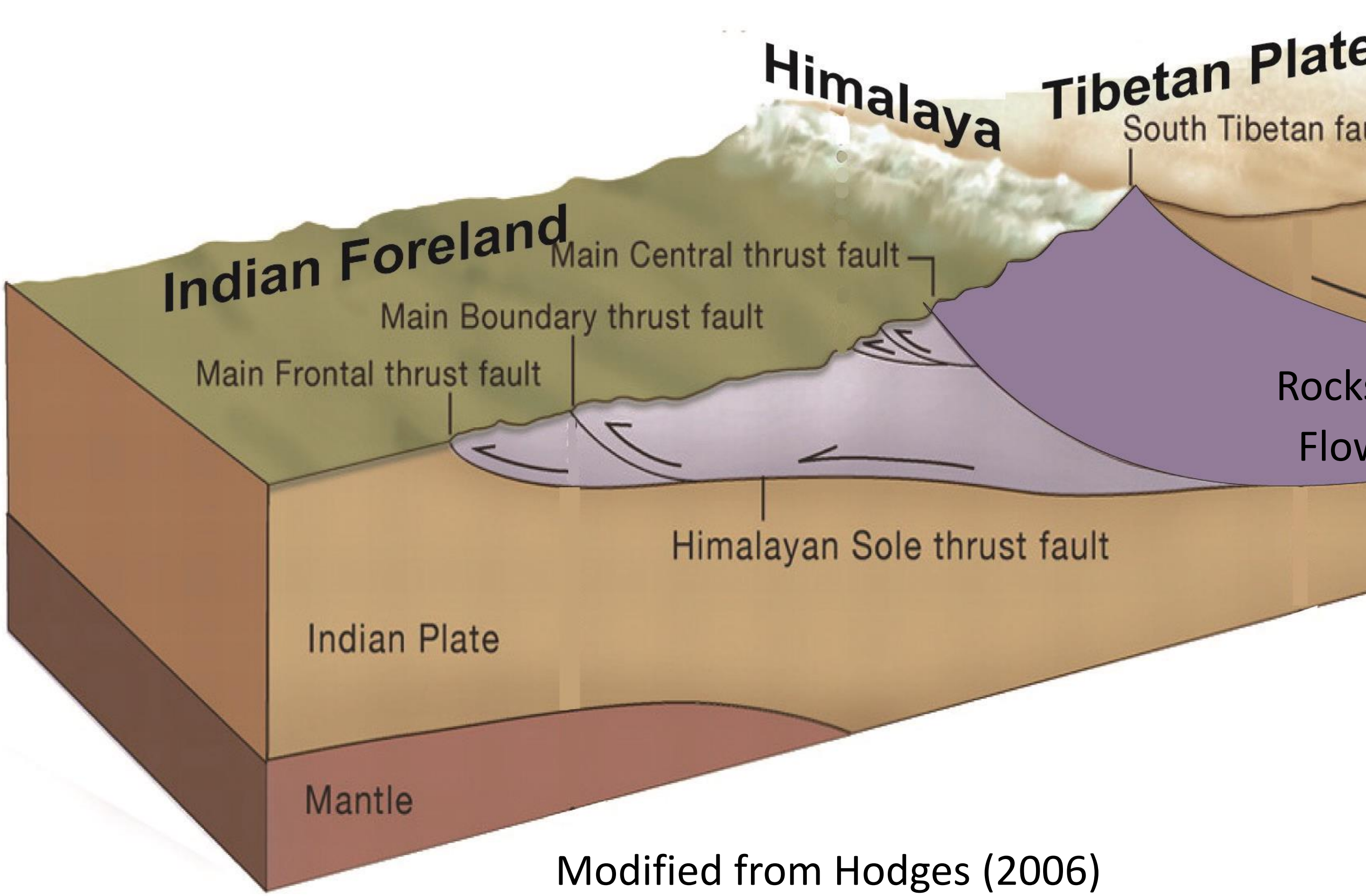
$$W_s = \frac{\dot{\gamma}}{[\dot{\gamma}^2+(\dot{\varepsilon}_x-\dot{\varepsilon}_y)^2]^{1/2}}$$

Generally: $W_k \neq W_s$

VNS: vorticity normal section

Vorticity analysis: W_s not W_k

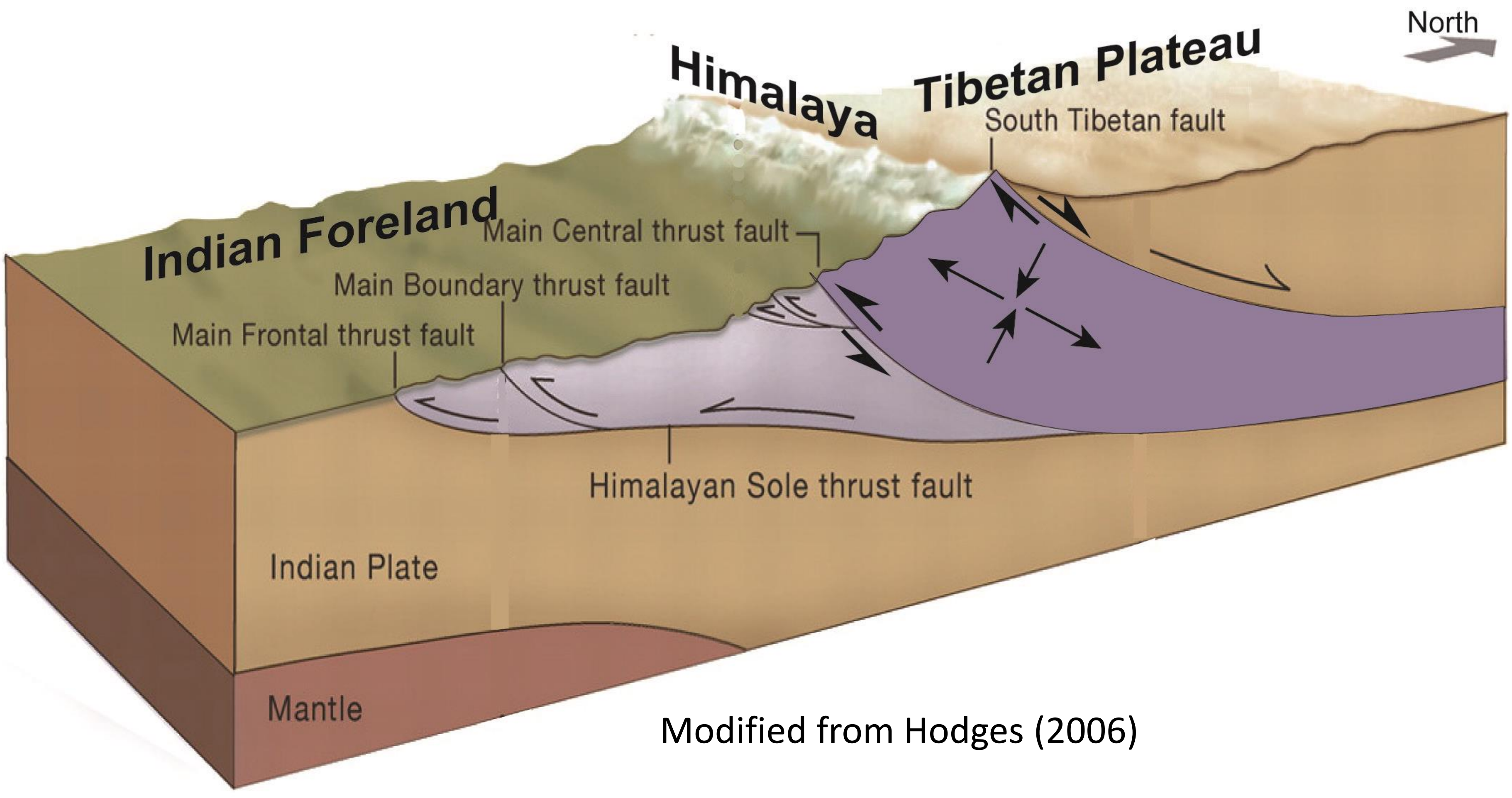
Vorticity analysis implication



Modified from Hodges (2006)

From internet

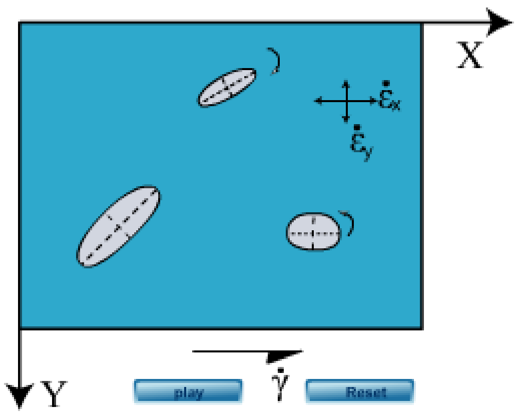
Vorticity analysis implication



Modified from Hodges (2006)

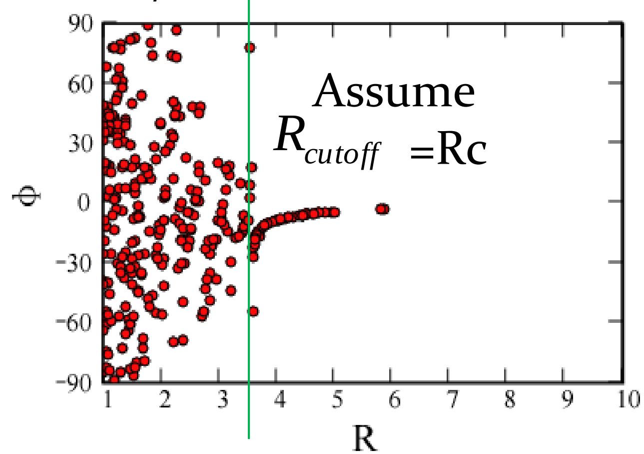
How to estimate Ws?

Sectional kinematic vorticity number



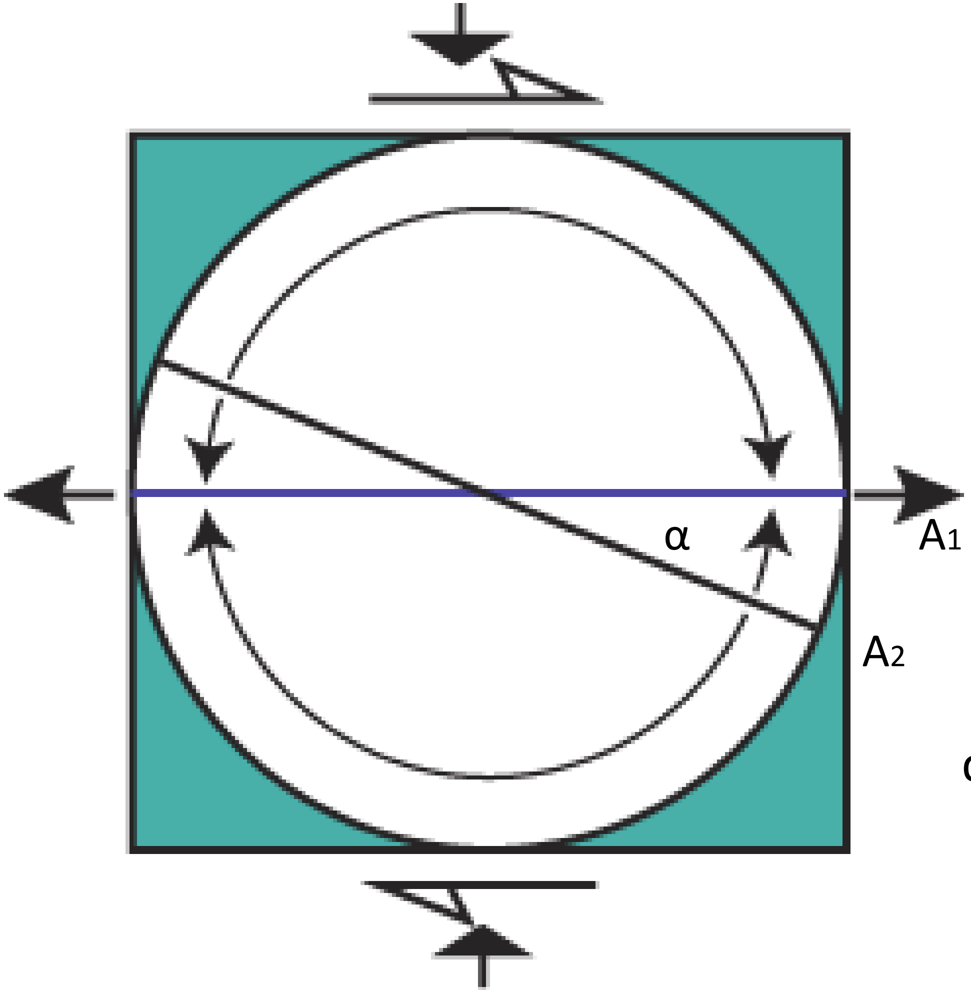
Critical aspect ratio: $$Rc = \sqrt{\frac{1+W_S}{1-W_S}}$$

ϕ Versus R plot



Wallis plot

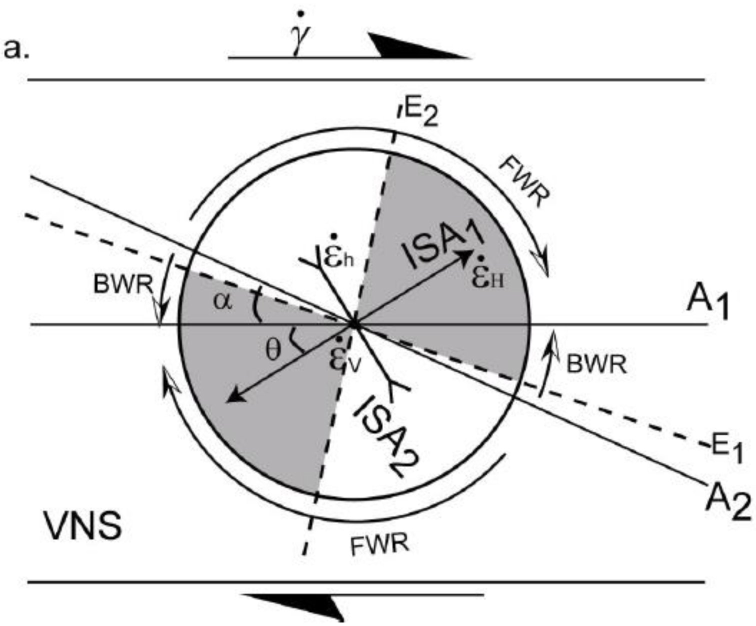
Sectional kinematic vorticity number

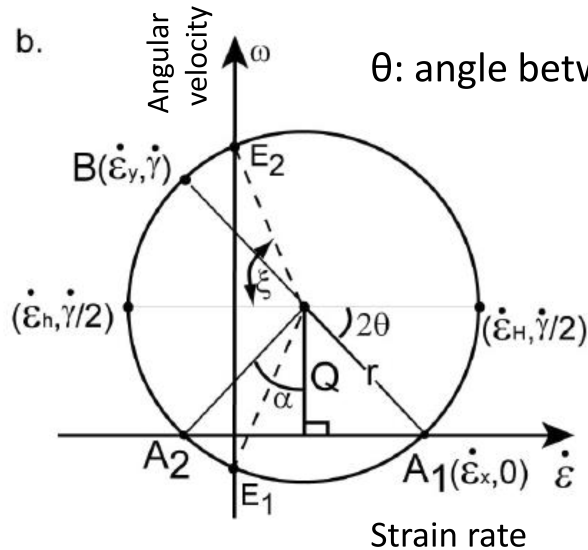


$$W_s = \cos(\alpha)$$

α: angle between two flow apophyses

Sectional kinematic vorticity number

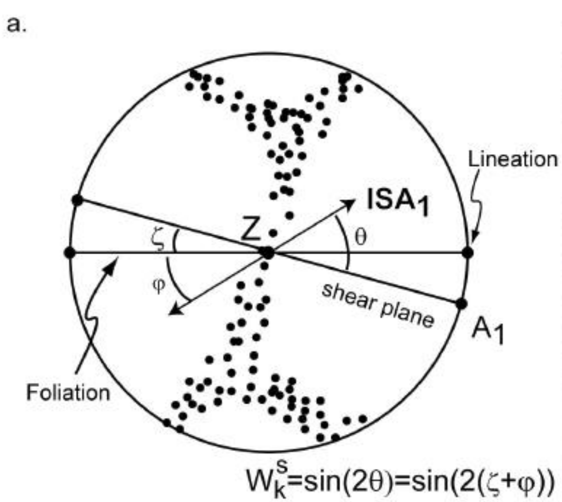


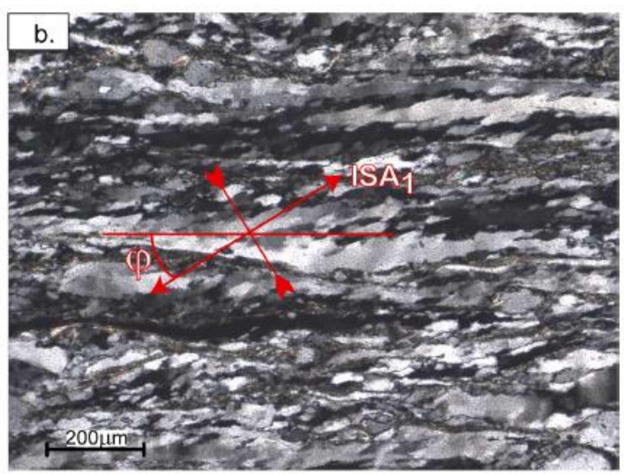


θ: angle between ISA1 and the shear plane

$$\alpha = 90 - 2\theta$$

$$W_s = \sin(2\theta)$$

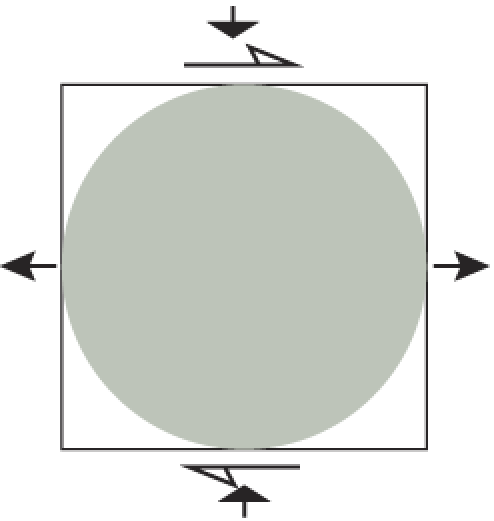


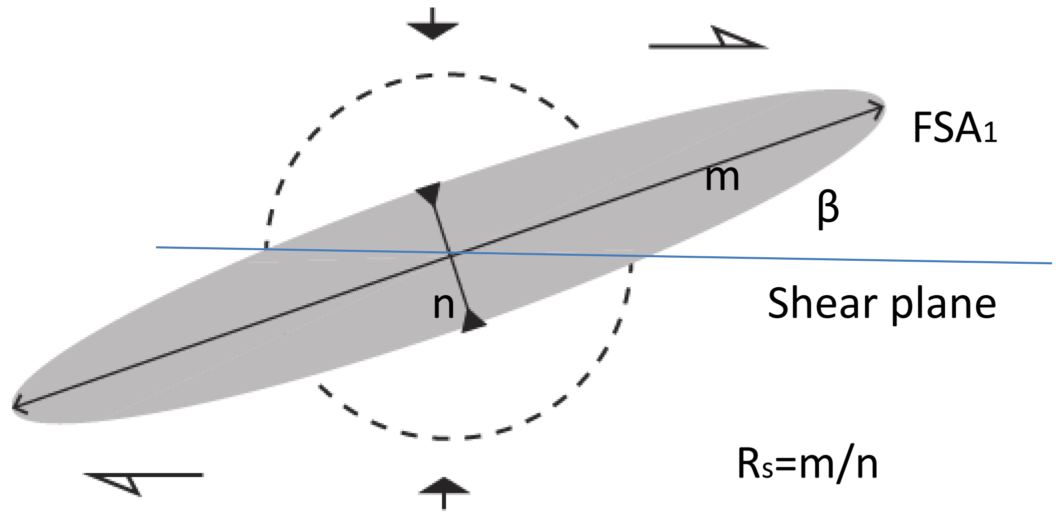


Assuming that oblique grains of recrystallized quartz align along the ISA_1

Can it represent ISA_1?

Sectional kinematic vorticity number





Finite strain ellipse

$$W_s = \sin\{\arctan[\frac{\sin(2\beta)}{\frac{R_s+1}{R_s-1}-\cos(2\beta)}]\}\times\frac{R_s+1}{R_s-1}$$