Earth 460 Classnotes (Set 2)

I.) Time Series Analysis – Continuous Signals

C.) Fourier Transforms

1.) Let $g(t)$ be a real-valued time signal defined on $-\infty < t < +\infty$ such that

a.) $\int_{-\infty}^{+\infty} |g(t)| dt < +\infty$ (This condition implies that $g(t) \to 0$ as $t \to \pm\infty$)

b.) Any discontinuities in $g(t)$ are finite.

2.) Then there exist a unique Fourier transform pair $g(t) \Leftrightarrow G(f)$ where

a.) $G(f) = \int_{-\infty}^{+\infty} g(t) e^{-2\pi i f t} dt$ (the Fourier transform of $g(t)$)

b.) $g(t) = \int_{-\infty}^{+\infty} G(f) e^{+2\pi i f t} df$ (the inverse Fourier transform of $G(f)$)

Note 1: The condition $\int_{-\infty}^{+\infty} |g(t)| dt < +\infty$ also implies that $G(f) \to 0$ as $f \to \pm\infty$

Note 2: There are variations in the manner that the Fourier transform is defined in different fields, such as the sign convention for phase polarity; the use of angular versus cyclic frequency. It is necessary to be aware of these variants when comparing results from different sources.

Differences in phase polarity: $G(f) = \int_{-\infty}^{+\infty} g(t) e^{+2\pi i f t} dt$ and $g(t) = \int_{-\infty}^{+\infty} G(f) e^{-2\pi i f t} df$

Use of angular frequency: $G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt$ and

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(\omega) e^{+i\omega t} d\omega$$

3.) Notation used to indicate the Fourier transform pair:

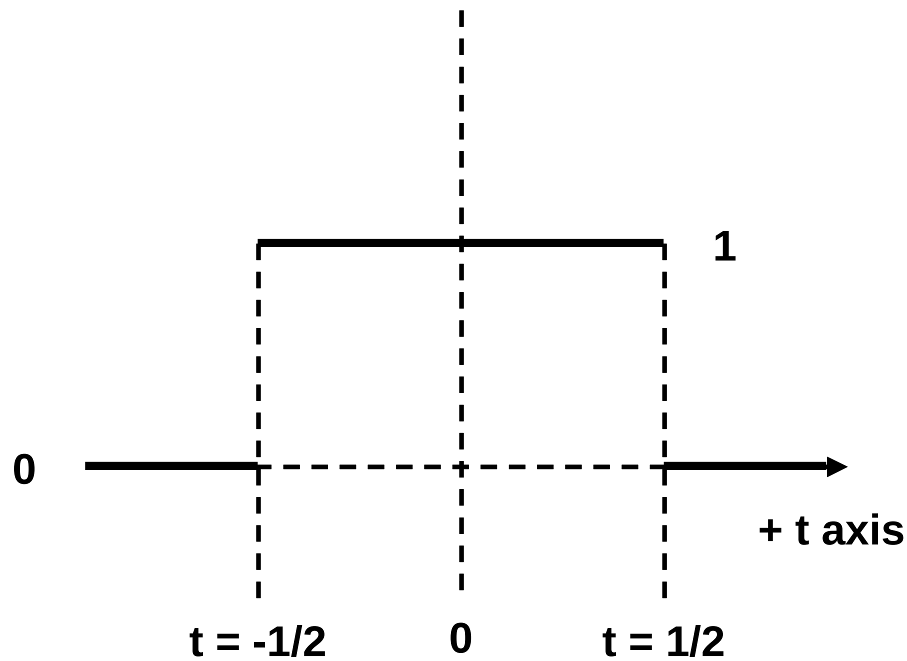
$$g(t) \Leftrightarrow G(f),\ G(f) = \mathbf{F}\left[g(t)\right],\ g(t) = \mathbf{F}^{-1}\left[G(f)\right]$$

4.) In general, $G(f)$ is a complex-valued function:

$G(f) = \mathrm{Re}\left[G(f)\right] + i\,\mathrm{Im}\left[G(f)\right]$.

For a real-value time signal $g(t)$, $\mathrm{Re}\left[G(f)\right]$ is an even function (i.e. $\mathrm{Re}\left[G(f)\right] = \mathrm{Re}\left[G(-f)\right]$) and $\mathrm{Im}\left[G(f)\right]$ is an odd function (i.e. $-\mathrm{Im}\left[G(f)\right] = \mathrm{Im}\left[G(-f)\right]$).

Example: $g(t) = \Pi(t) = \begin{cases} 0, & |t| > 1/2 \\ 1, & |t| \le 1/2 \end{cases}$ (Rectangular or Boxcar Function)

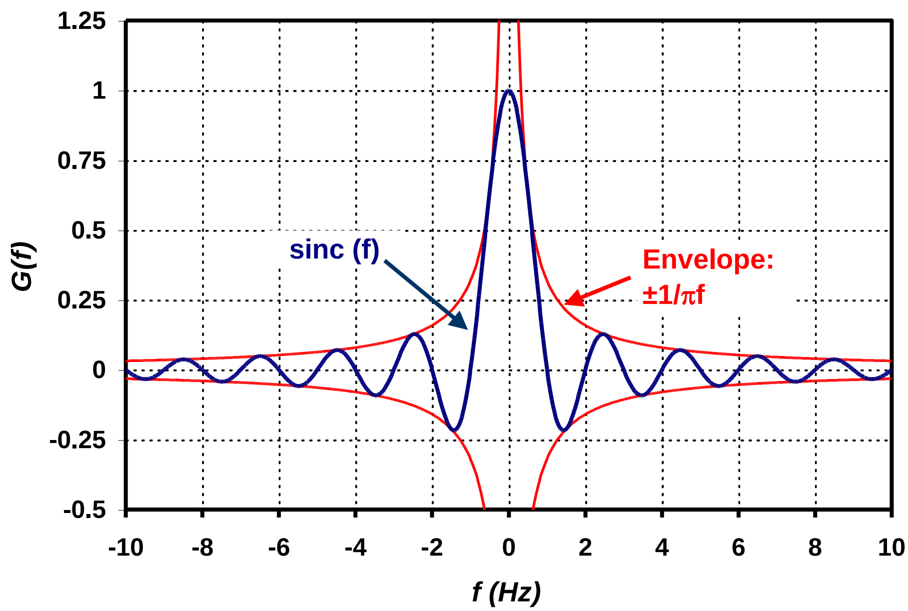


$$G(f) = \int_{-\infty}^{+\infty} g(t) e^{-2\pi i f t}\, dt = \int_{-\infty}^{+\infty} \Pi(t) e^{-2\pi i f t}\, dt = \int_{-\infty}^{-1/2} 0 e^{-2\pi i f t}\, dt + \int_{-1/2}^{+1/2} 1 e^{-2\pi i f t}\, dt + \int_{+1/2}^{+\infty} 0 e^{-2\pi i f t}\, dt$$

$$= \int_{-1/2}^{+1/2} 1 e^{-2\pi i f t}\, dt \,.$$

Using $\int e^{at}\,dt = e^{at}/a$ where $a = -2\pi i f$, $G(f) = \left.\dfrac{e^{-2\pi i f t}}{-2\pi i f}\right|_{-1/2}^{+1/2} = \dfrac{e^{+\pi i f} - e^{-\pi i f}}{2\pi i f}$

Using $\sin(\pi f) = \left(e^{+\pi i f} - e^{-\pi i f}\right)/2i$, $G(f) = \dfrac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$. $\therefore \quad \prod(t) \Leftrightarrow \text{sinc}(f)$.



Correspondingly, it can be shown that $\text{sinc}(t) \Leftrightarrow \prod(f)$ using the inverse Fourier transform.

5.) Amplitude and Phase Spectra

a.) Complex-valued $G(f)$ can be expressed in polar form $G(f) = A(f)e^{i\theta(f)}$.

b.) $A(f) = \sqrt{\left\{\text{Re}\left[G(f)\right]\right\}^2 + \left\{\text{Im}\left[G(f)\right]\right\}^2}$ is the amplitude spectrum of $g(t)$.

c.) $\theta(f) = \tan^{-1}\left\{\text{Im}\left[G(f)\right]/\text{Re}\left[G(f)\right]\right\}$ is the phase spectrum of $g(t)$.

d.) For real-valued signals $g(t)$, $A(f)$ is an even, non-negative function and $\theta(f)$ is odd function. (Note: $\theta(f=0)=0$)

1.) If $g(t)$ is an even function (i.e., $g(t) = g(-t)$), then $\theta(f) = 0$ (i.e., zero phase signal). (Examples: $\text{sinc}(t)$ & $\prod(t)$)

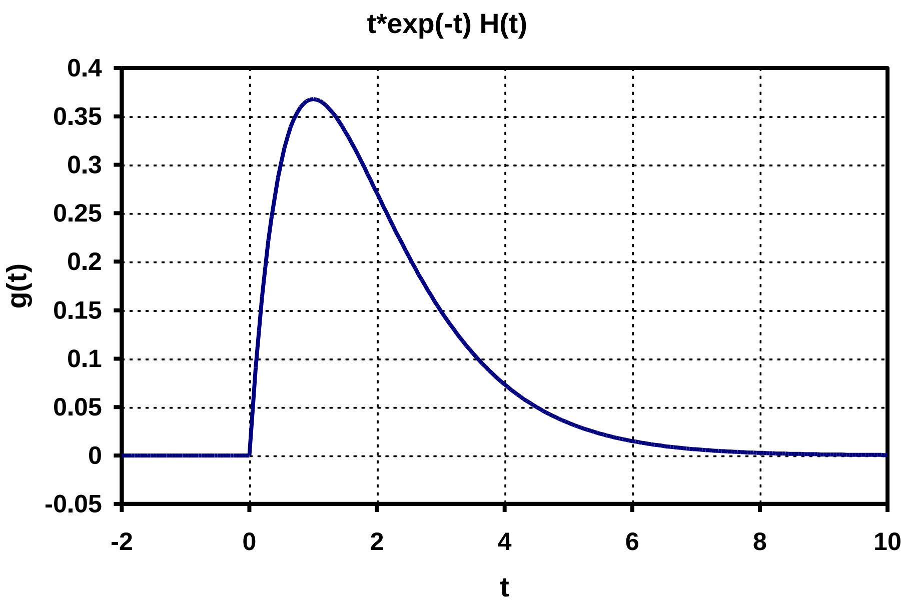
2.) If $g(t)$ is an odd function (i.e., $-g(t) = g(-t)$), then

$\theta(f) = \pm(\pi/2)\operatorname{sgn}(f)$ (± sign is determined by polarity of $\operatorname{Im}[G(f)]$).

Example:

Amplitude and phase spectra of $g(t) = t\,e^{-t}\,\mathrm{H}(t) = \begin{cases} t\,e^{-t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

Note: $\mathrm{H}(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$ is the Heaviside step function.



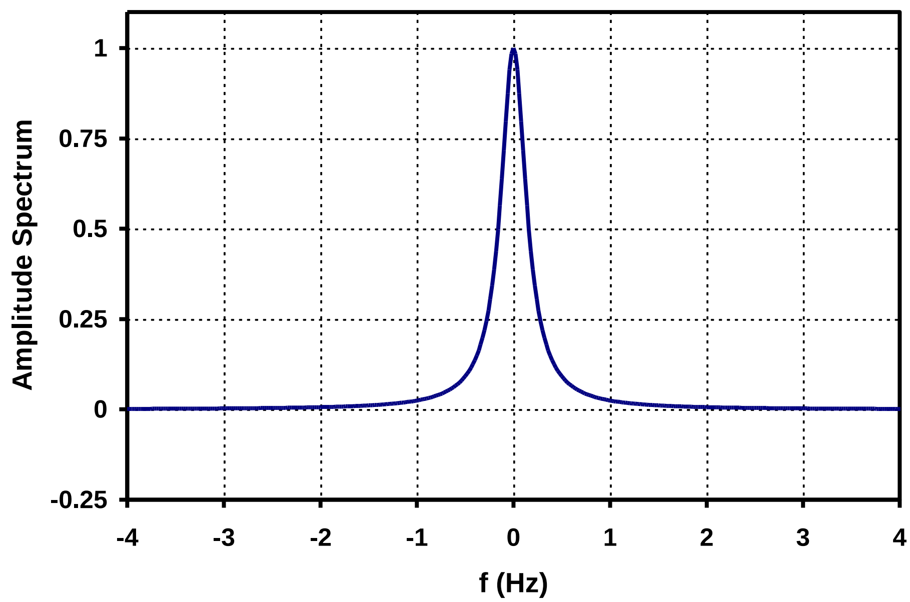
Using the Fourier transform above, it can be shown that

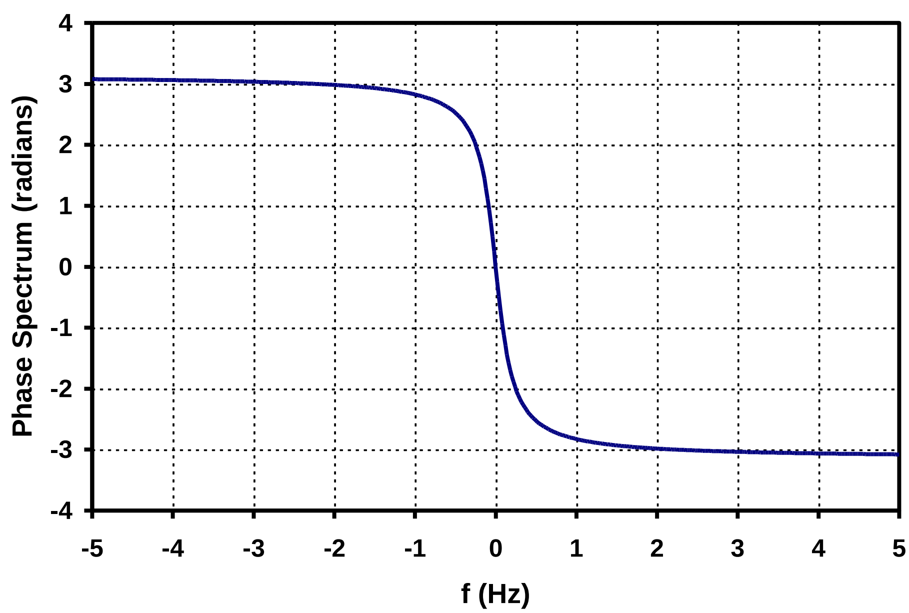
$g(t) = t\,e^{-t}\,\mathrm{H}(t) \Leftrightarrow G(f) = \dfrac{\left(1-4\pi^2 f^2\right) - i\,4\pi f}{\left(1+4\pi^2 f^2\right)^2}$. Hence, $\operatorname{Re}[G(f)] = \dfrac{1-4\pi^2 f^2}{\left(1+4\pi^2 f^2\right)^2}$

and $\operatorname{Im}[G(f)] = \dfrac{-4\pi f}{\left(1+4\pi^2 f^2\right)^2}$. Then,

$$A(f) = \sqrt{\left[\frac{1-4\pi^2 f^2}{\left(1+4\pi^2 f^2\right)^2}\right]^2 + \left[\frac{-4\pi f}{\left(1+4\pi^2 f^2\right)^2}\right]^2} = \frac{1+4\pi^2 f^2}{\left(1+4\pi^2 f^2\right)^2} = \frac{1}{1+4\pi^2 f^2}$$ and

$$\theta(f) = \tan^{-1}\left\{\frac{-4\pi f \Big/ \left(1+4\pi^2 f^2\right)^2}{\left(1-4\pi^2 f^2\right)\Big/\left(1+4\pi^2 f^2\right)^2}\right\} = \tan^{-1}\left\{\frac{-4\pi f}{1-4\pi^2 f^2}\right\}.$$





6.) Operational Properties of the Fourier Transform a.) Linearity: The Fourier transform is a linear process. Consider the Fourier transform pairs $g(t) \Leftrightarrow G(f)$ and $h(t) \Leftrightarrow H(f)$, then for any two complex valued constants a and b:

$$a \cdot g(t) + b \cdot h(t) \Leftrightarrow a \cdot G(f) + b \cdot H(f)$$

Example: $\prod(t) \Leftrightarrow \text{sinc}(f)$ and $e^{-|t|} \Leftrightarrow 2 \big/ \left(1 + 4\pi^2 f^2\right)$, then $3\prod(t) - 2e^{-|t|} \Leftrightarrow 3\,\text{sinc}(f) - \left[4 \big/ \left(1 + 4\pi^2 f^2\right)\right]$.

b.) Shift Relationships: If $g(t) \Leftrightarrow G(f)$, then

1.) $g(t-a) \Leftrightarrow G(f)e^{-2\pi aif}$

2.) $g(t)e^{+2\pi ait} \Leftrightarrow G(f-a)$

Proof of relationship $g(t-a) \Leftrightarrow G(f)e^{-2\pi aif}$

$g(t-a) \Leftrightarrow H(f) = \int_{t=-\infty}^{t=+\infty} g(t-a)e^{-2\pi ift}\,dt$ (Using $s = t-a \rightarrow t = s+a$ and $dt = ds$)

$$= \int_{s=-\infty}^{s=+\infty} g(s)e^{-2\pi if(s+a)}\,ds = e^{-2\pi ifa}\int_{s=-\infty}^{s=+\infty} g(s)e^{-2\pi ifs}\,ds = e^{-2\pi ifa}G(f)$$

3.) Shifting a time signal only affects the phase spectrum, the amplitude spectrum is unchanged.

$g(t) \Leftrightarrow G(f) = A(f)e^{i\theta(f)}$, then

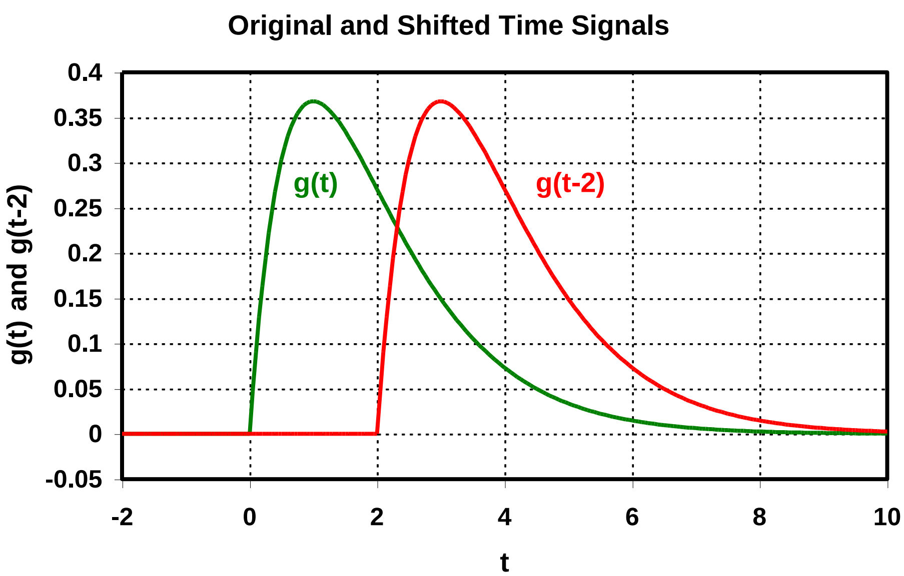
$g(t-a) \Leftrightarrow e^{-2\pi iaf}G(f) = A(f)e^{i\theta(f)}e^{-2\pi iaf} = A(f)e^{i\left[\theta(f)-2\pi af\right]} = A(f)e^{i\theta'(f)}$ where $\theta'(f) = \theta(f) - 2\pi af$ (the phase spectrum of the shifted time signal has an additional linear term).

Examples:

1.) Starting with $t\,e^{-t}\,\mathrm{H}(t) \Leftrightarrow \dfrac{\left(1-4\pi^2 f^2\right)-i\,4\pi f}{\left(1+4\pi^2 f^2\right)^2}$, consider $(t-2)e^{-(t-2)}\,\mathrm{H}(t-2)$

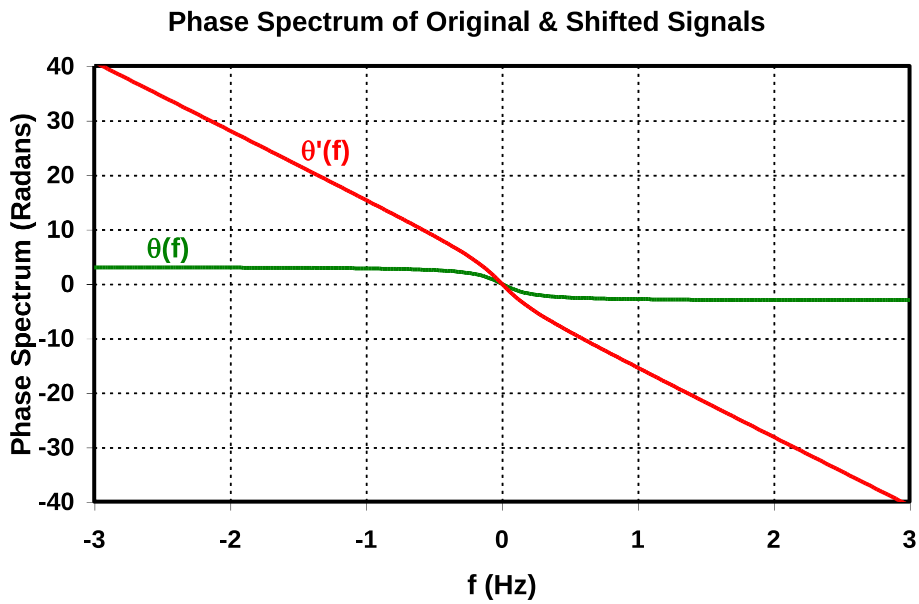
(i.e., $g(t-a)$ with $a=2$).

Note: $\mathrm{H}(t-2) = \begin{cases} 1, & t-2 \geq 0 \rightarrow t \geq 2 \\ 0, & t-2 < 0 \rightarrow t < 2 \end{cases}$



Then $(t-2)e^{-(t-2)}\,\mathrm{H}(t-2) \Leftrightarrow \dfrac{\left(1-4\pi^2 f^2\right)-i\,4\pi f}{\left(1+4\pi^2 f^2\right)^2}\,e^{-4\pi i f}$ and

$$\theta'(f) = \theta(f) - 2\pi a f = \tan^{-1}\left\{\frac{-4\pi f}{1-4\pi^2 f^2}\right\} - 4\pi f$$



2.) $\Pi(t) \Leftrightarrow \mathrm{sinc}(f)$, then $\Pi(t+3) \Leftrightarrow \mathrm{sinc}(f)e^{+6\pi if}$ (i.e., $a=-3$)

3.) $e^{-|t|} \Leftrightarrow 2/(1+4\pi^2 f^2)$, then $e^{-|t-2|} \Leftrightarrow 2e^{-4\pi if}/(1+4\pi^2 f^2)$ (i.e., $a=2$)

c.) Scaling Relationships: If $g(t) \Leftrightarrow G(f)$, then

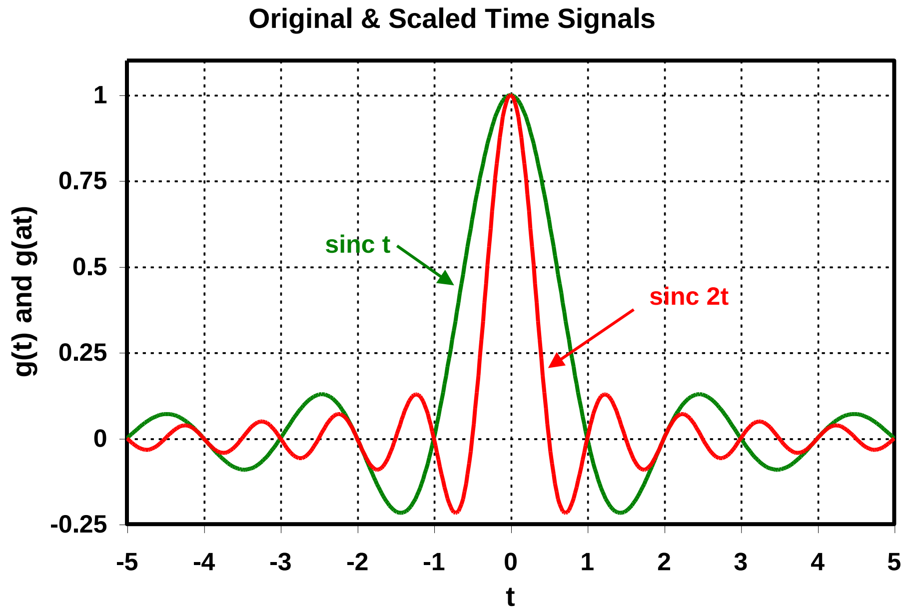
1.) $g(at) \Leftrightarrow \dfrac{1}{|a|}G(f/a)$

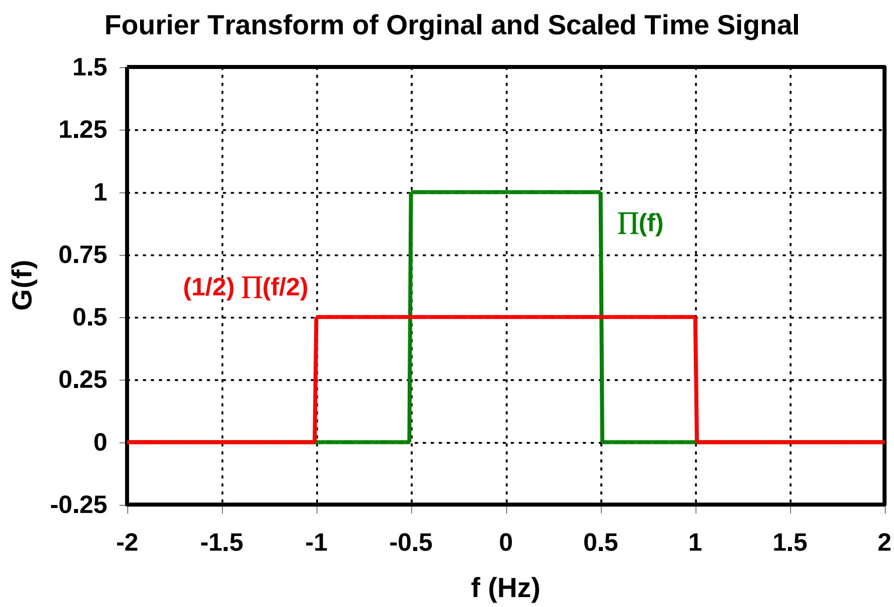
2.) $\dfrac{1}{|a|}g(t/a) \Leftrightarrow G(af)$

Example:

Starting with $\mathrm{sinc}(t) \Leftrightarrow \Pi(f)$, consider $\mathrm{sinc}(2t)$ (i.e., $g(at)$ with $a=2$).

Then $\mathrm{sinc}(2t) \Leftrightarrow \dfrac{1}{2}\Pi(f/2)$.





3.) Scaling in time/frequency domain causes the inverse scaling in the frequency/time domain. In the above example, the scaling produces a more compressed time signal and a corresponding Fourier transform with broader bandwidth.

d.) Derivative Relationships: If $g(t) \Leftrightarrow G(f)$, then

1.) $\dfrac{d^n}{dt^n} g(t) \Leftrightarrow (2\pi i f)^n G(f)$

2.) $(-2\pi i t)^n g(t) \Leftrightarrow \dfrac{d^n}{df^n} G(f)$

Proof of relationship $\dfrac{d}{dt} g(t) \Leftrightarrow (2\pi i f) G(f)$ (the $n = 1$ case)

$$\frac{d}{dt} g(t) \Leftrightarrow H(f) = \int_{-\infty}^{+\infty} \left[\frac{d}{dt} g(t) \right] e^{-2\pi i f t}\, dt$$

(Using integration by parts: $\int_a^b u(t) \cdot \left[\dfrac{d}{dt} v(t) \right] dt = u(t) \cdot v(t) \Big|_a^b - \int_a^b v(t) \cdot \left[\dfrac{d}{dt} u(t) \right] dt$

where $\dfrac{d}{dt} v(t) = \dfrac{d}{dt} g(t) \rightarrow v(t) = g(t)$

and $u(t) = e^{-2\pi i f t} \rightarrow \dfrac{d}{dt} u(t) = (-2\pi i f) e^{-2\pi i f t}$).

$$= g(t) e^{-2\pi i f t} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} g(t)(-2\pi i f) e^{-2\pi i f t}\, dt = (2\pi i f) \int_{-\infty}^{+\infty} g(t) \cdot e^{-2\pi i f t}\, dt = (2\pi i f) G(f)$$

3.) Derivatives of the time signal affect the amplitude and phase spectra in the following manner:

$g(t) \Leftrightarrow G(f) = A(f) e^{i\theta(f)}$, then

$$\frac{d^n}{dt^n} g(t) \Leftrightarrow (2\pi i f)^n G(f) = (2\pi i f)^n A(f) e^{i\theta(f)} = \left[2\pi |f| \right]^n A(f) e^{i\left[\theta(f) + n(\pi/2)\operatorname{sgn}(f)\right]} = A'(f) e^{i\theta'(f)}$$

where $A'(f) = \left[2\pi |f| \right]^n A(f)$ and $\theta'(f) = \theta(f) + n(\pi/2)\operatorname{sgn}(f)$.

Note: $\operatorname{sgn}(f) = \begin{cases} +1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$

Taking the derivative of a time signal enhances the higher frequency components and suppresses the lower frequency components of its Fourier transform.

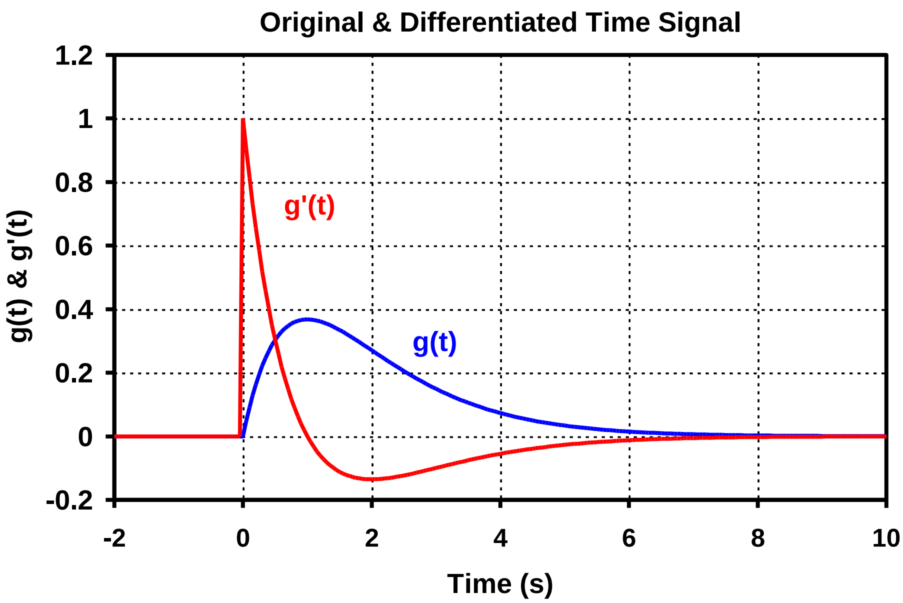
Example:

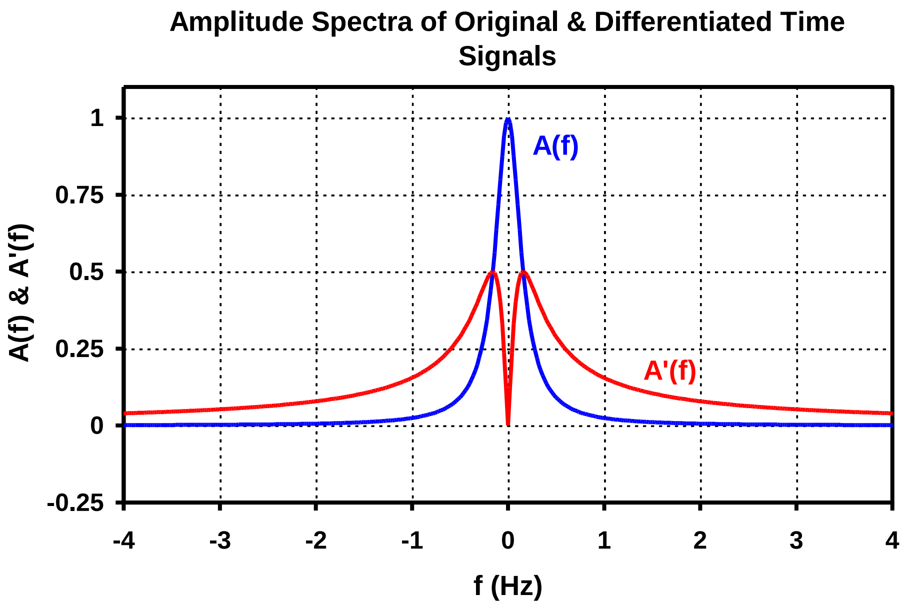
Consider $g(t) = t\,e^{-t}\,\mathrm{H}(t) \Leftrightarrow G(f) = \dfrac{\left(1-4\pi^2 f^2\right) - i\,4\pi f}{\left(1+4\pi^2 f^2\right)^2}$ with $A(f) = \dfrac{1}{1+4\pi^2 f^2}$ and

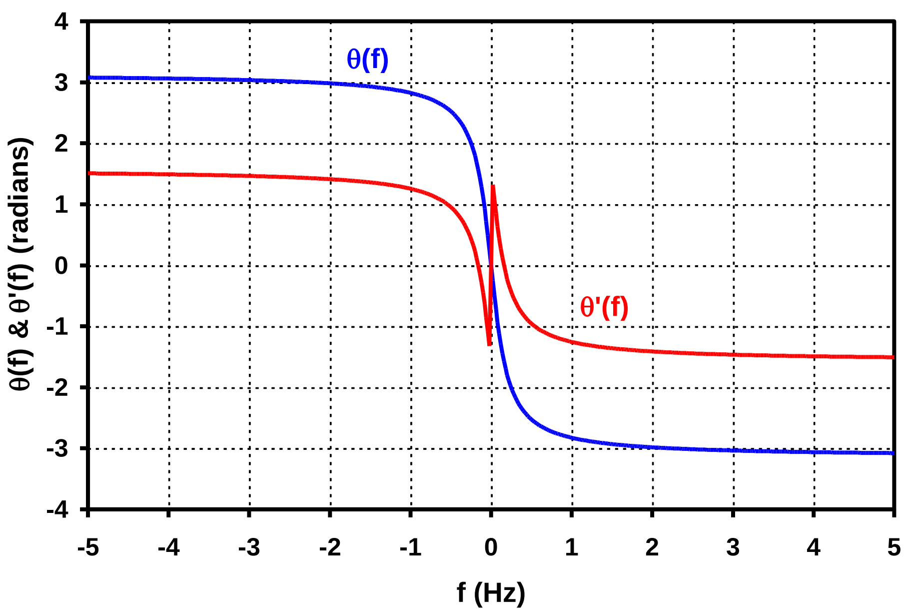
$\theta(f) = \tan^{-1}\left\{\dfrac{-4\pi f}{1-4\pi^2 f^2}\right\}$.

Then $\dfrac{d}{dt}g(t) = e^{-t}(1-t)\mathrm{H}(t) \Leftrightarrow (2\pi i f)G(f) = (2\pi f)\dfrac{4\pi f + i\left(1-4\pi^2 f^2\right)}{\left(1+4\pi^2 f^2\right)^2}$ where

$A'(f) = \dfrac{2\pi|f|}{1+4\pi^2 f^2}$ and $\theta'(f) = \left\{\tan^{-1}\left[\dfrac{-4\pi f}{1-4\pi^2 f^2}\right]\right\} + \dfrac{\pi}{2}\mathrm{sgn}(f)$.







4.) The derivative relationships can be used to construct Fourier transform pairs for time signals involving powers of t.

Example:

Starting with $g(t) = e^{-t}\,\mathrm{H}(t) \Leftrightarrow G(f) = (1 - 2\pi i f)^{-1}$, then we can obtain

$g'(t) = te^{-t}\mathrm{H}(t) \Leftrightarrow G'(f) = (2\pi i)^{-1}\dfrac{d}{df}G(f) = (1-2\pi i f)^{-2}$ and

$g''(t) = t^2 e^{-t}\mathrm{H}(t) \Leftrightarrow G''(f) = (2\pi i)^{-1}\dfrac{d}{df}G'(f) = 2(1-2\pi i f)^{-3}$.

e.) Integral Relationships: If $g(t) \Leftrightarrow G(f)$, then

1.) $\int_{-\infty}^{t} g(s)ds \Leftrightarrow (2\pi i f)^{-1}G(f)$

2.) $(-2\pi i t)^{-1}g(t) \Leftrightarrow \int_{-\infty}^{f} G(s)ds$

3.) Integration of the time signal affect the amplitude and phase spectra in the following manner:

$g(t) \Leftrightarrow G(f) = A(f)e^{i\theta(f)}$, then

$$\int_{-\infty}^{t} g(s)ds \Leftrightarrow (2\pi i f)^{-1}G(f) = -i(2\pi f)^{-1}A(f)e^{i\theta(f)} = \left[2\pi|f|\right]^{-1}A(f)e^{i[\theta(f)-(\pi/2)\mathrm{sgn}(f)]} = A'(f)e^{i\theta'(f)}$$

where $A'(f) = \left[2\pi|f|\right]^{-1}A(f)$ and $\theta'(f) = \theta(f) - (\pi/2)\mathrm{sgn}(f)$.

Integrating a time signal enhances the lower frequency components and suppresses the higher frequency components of its Fourier transform.

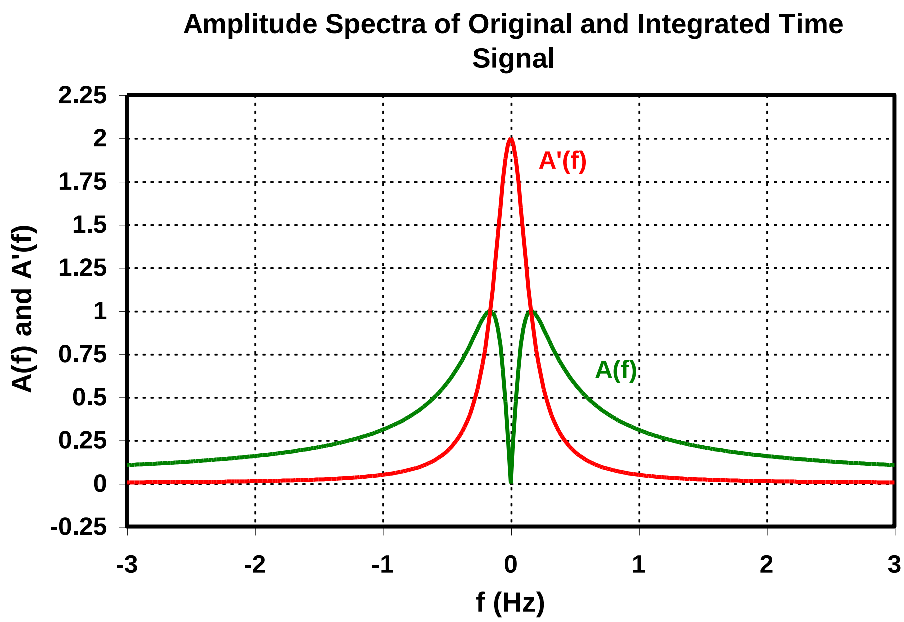
Example:

Starting with $e^{-|t|}\mathrm{sgn}(t) \Leftrightarrow \dfrac{-4\pi i f}{1+(2\pi f)^2}$ with $A(f) = \dfrac{4\pi|f|}{1+(2\pi f)^2}$ and

$\theta(f) = -(\pi/2)\mathrm{sgn}(f)$.

Then $\int_{-\infty}^{t} e^{-|s|}\mathrm{sgn}(s)ds \Leftrightarrow (2\pi i f)^{-1}\dfrac{-4\pi i f}{1+(2\pi f)^2} = \dfrac{-2}{1+(2\pi f)^2}$ with $A'(f) = \dfrac{2}{1+(2\pi f)^2}$

and $\theta'(f) = -\pi\,\mathrm{sgn}(f)$

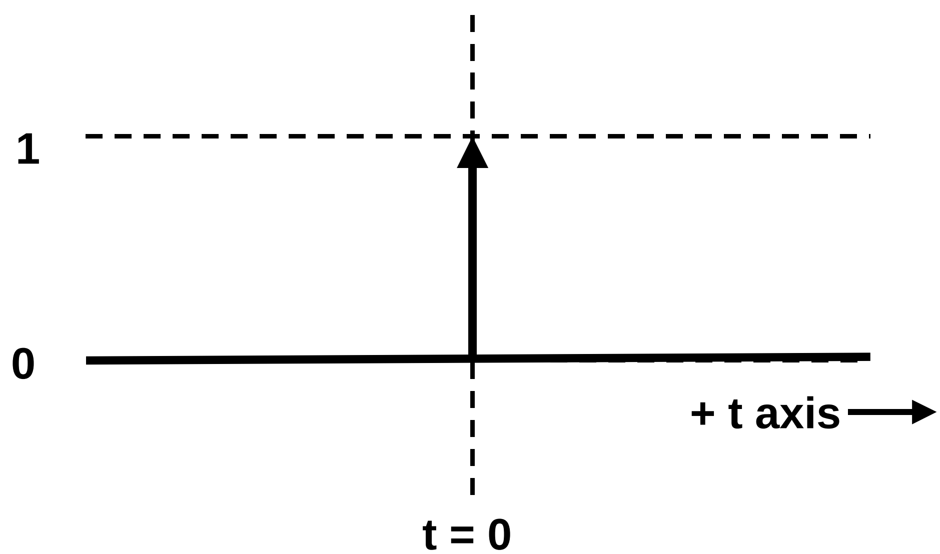


7.) Generalized Functions and Transforms in the Limit

a.) Dirac delta or impulse function $\delta(t)$

1.) This function has a unit area centered at $t = 0$ with infinitesimally short duration. These conditions imply an infinitely large magnitude.

2.) Graphical representation of $\delta(t)$:



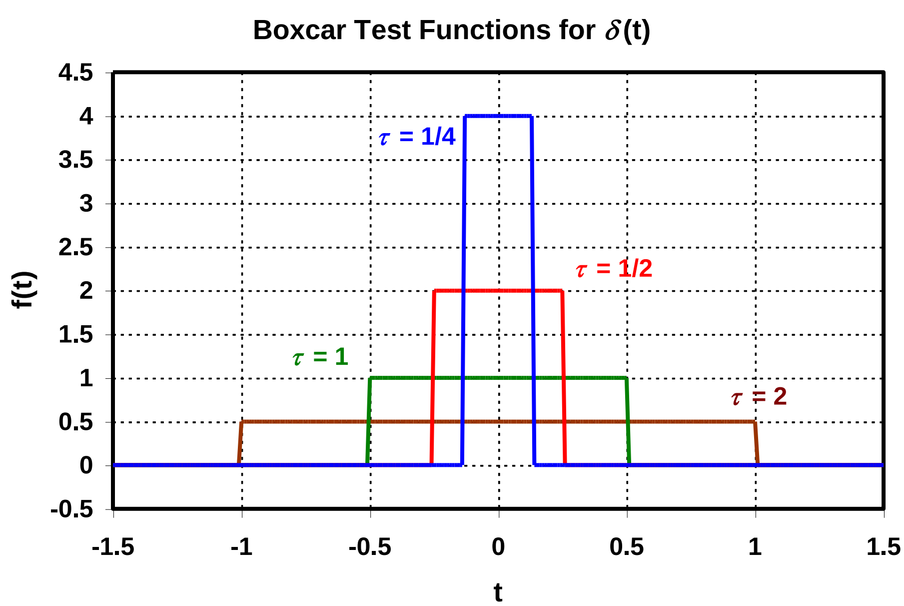
3.) Properties of $\delta(t)$ from its definition

a.) $\delta(t)=0$ for all $t\neq 0$

b.) $\int_{-\infty}^{+\infty}\delta(t)\,dt=1$

4.) Approximation of $\delta(t)$ with well behaved test functions

a.) Consider the sequence of unit area boxcar functions $\tau^{-1}\prod(t/\tau)$ as $\tau\to 0^{+}$. (Note: $\tau^{-1}\prod(t/\tau)=\begin{cases}\tau^{-1}, & |t|\le\tau/2\\ 0, & |t|>\tau/2\end{cases}$)



b.) Using these test functions, we can show that

$$\int_{-\infty}^{t}\delta(s)\,ds=\begin{cases}0, & t<0\\ 1/2, & t=0\\ 1, & t>0\end{cases}$$

5.) Sifting property of $\delta(t)$

Consider $\int_{-\infty}^{+\infty} g(t)\delta(t)dt$ in terms of the boxcar test functions:

$$\int_{-\infty}^{+\infty} g(t)\tau^{-1}\Pi(t/\tau)dt = \tau^{-1}\int_{-\tau/2}^{+\tau/2} g(t)dt = \tau^{-1}\left[\int_{-\infty}^{+\tau/2} g(t)dt - \int_{-\infty}^{-\tau/2} g(t)dt\right]$$

$= \tau^{-1}\left[h(+\tau/2) - h(-\tau/2)\right]$ where $h(\tau) = \int_{-\infty}^{\tau} g(t)dt \rightarrow \frac{d}{dt}h(\tau) = g(\tau)$.

As $\tau \to 0^+$, the limiting value of this expression is

$$\lim_{\tau\to 0^+}\left[\frac{h(+\tau/2) - h(-\tau/2)}{\tau}\right] = \frac{d}{d\tau}h(\tau)\Big|_{\tau=0} = g(0)$$

This process establishes the sifting property of $\delta(t)$ as follows:

$$\int_{-\infty}^{+\infty} g(t)\delta(t)dt = g(0)$$

a.) $\int_{-\infty}^{+\infty} g(t)\delta(t-a)dt = g(a)$

b.) $\int_{-\infty}^{+\infty} g(t-a)\delta(t)dt = g(-a)$

Examples:

1. $g(t) = 3\cos(2\pi t)$, then $\int_{-\infty}^{+\infty} g(t)\delta(t)dt = \int_{-\infty}^{+\infty} 3\cos(2\pi t)\delta(t)dt = 3\cos(0) = 3$

2. $g(t) = e^{-|t|}$, then $\int_{-\infty}^{+\infty} g(t)\delta(t+3)dt = \int_{-\infty}^{+\infty} e^{-|t|}\delta(t+3)dt = e^{-|-3|} = e^{-3}$ (i.e., $a = -3$)

6.) Scaling property of $\delta(t)$

$$\delta(at) = |a|^{-1}\delta(t)$$

Example: $\delta(-t) = |-1|^{-1}\delta(t) = \delta(t) \rightarrow \delta(t)$ is an even function

7.) $g(t)\delta(t)=g(0)\delta(t)$

$g(t)\delta(t-a)=g(a)\delta(t-a)$

8.) The Fourier transform of $\delta(t)$ and inverse Fourier transform of $\delta(f)$

a.) Using the boxcar test function:

$$\mathbf{F}\left[\tau^{-1}\Pi(t/\tau)\right]=\int_{-\infty}^{+\infty}\tau^{-1}\Pi(t/\tau)e^{-2\pi ift}dt=\mathrm{sinc}(\tau f)$$

b.) As $\tau\to 0^+$, $\mathrm{sinc}(\tau f)\to\mathrm{sinc}\,0=1$ for all values of f.

c.) Therefore, in the limit we have the Fourier transform pair $\delta(t)\Leftrightarrow 1$.

d.) This result is consistent with the sifting property of $\delta(t)$:

$$\int_{-\infty}^{+\infty}\delta(t)e^{-2\pi ift}dt=e^{-2\pi if0}=1$$

e.) Conversely, we can show that $1\Leftrightarrow\delta(f)$

f.) Amplitude and phase spectra:

For $\delta(t)$, $A(f)=1$ & $\theta(f)=0$ (i.e., contains all frequency components)

For 1, $A(f)=\delta(f)$ & $\theta(f)=0$ (i.e., contains only dc frequency component)

b.) Cosine and sine functions ($\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$)

1.) Using the complex-valued exponential,

$$\cos(2\pi f_0 t)=\frac{1}{2}\left(e^{2\pi f_0 it}+e^{-2\pi f_0 it}\right)\text{ and }\sin(2\pi f_0 t)=\frac{1}{2i}\left(e^{2\pi f_0 it}-e^{-2\pi f_0 it}\right)$$

2.) Starting with $1\Leftrightarrow\delta(f)$ and using the shifting property (i.e., $g(t)e^{+2\pi ait}\Leftrightarrow G(f-a)$) with $a=f_0$, then $e^{+2\pi f_0 it}\Leftrightarrow\delta(f-f_0)$

3.) Similarly, we can get $e^{-2\pi f_0 it}\Leftrightarrow\delta(f+f_0)$

4.) Combining these results, we get the following Fourier transform pairs:

$$\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$$\sin(2\pi f_0 t) \Leftrightarrow \frac{1}{2i}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

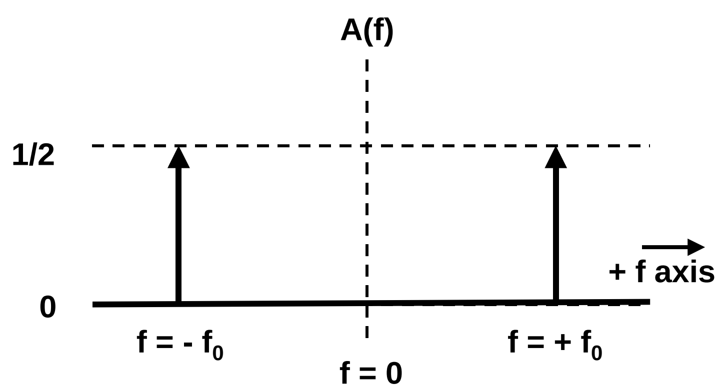
5.) Amplitude and phase spectra

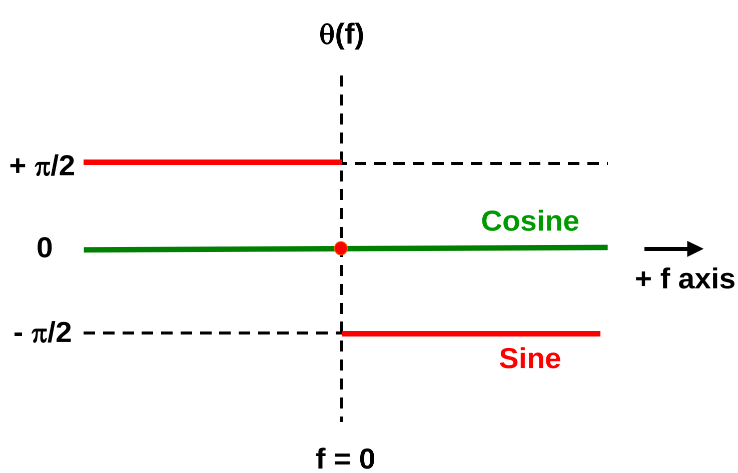
a.) For $\cos(2\pi f_0 t)$: $A(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$ & $\theta(f) = 0$

b.) For $\sin(2\pi f_0 t)$: $A(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$ &

$\theta(f) = -(\pi/2)\mathrm{sgn}(f)$

(Both contain only $\pm f_0$ components, they differ only in terms of phase)

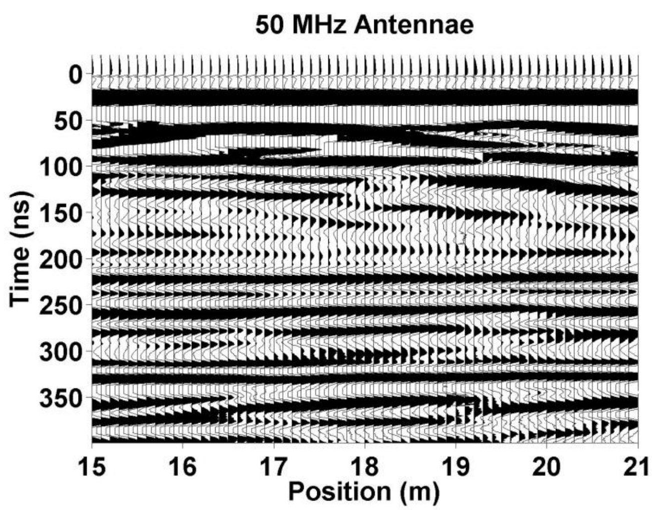


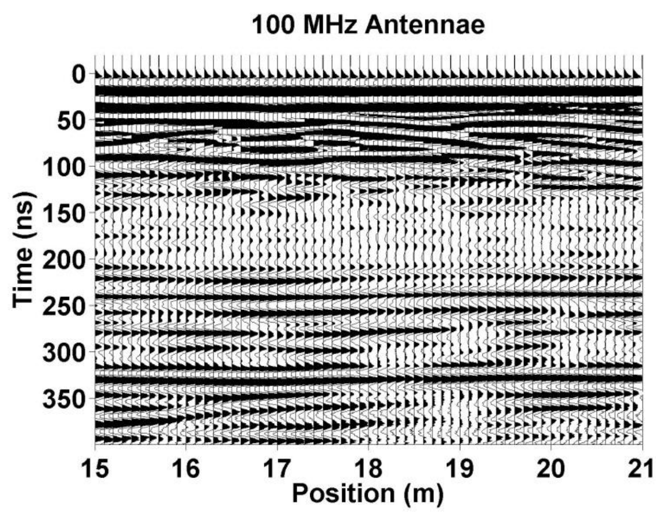


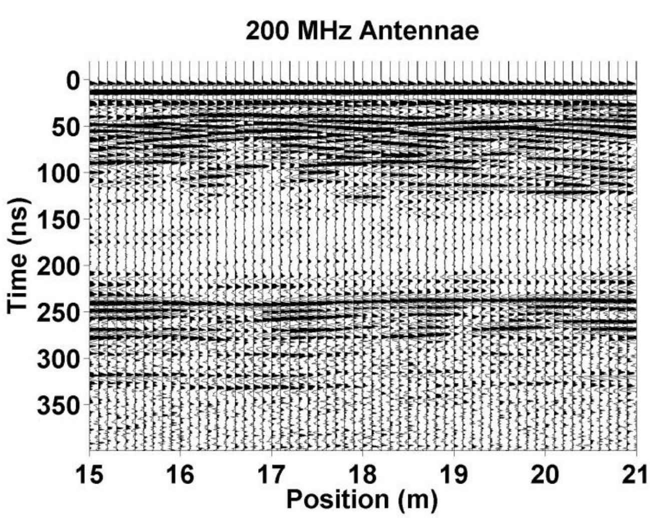
8.) Applications of Fourier Transforms to Reflection Data (Spectral Analysis)

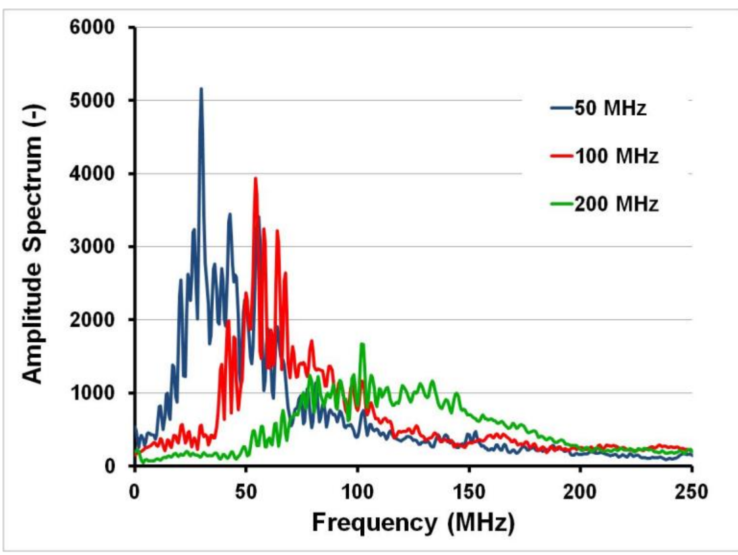
a.) Comparison of Frequency Content

Example: GPR reflection profiling data using antennae with differing design frequencies (Adair Quarry, Wiarton Ontario).



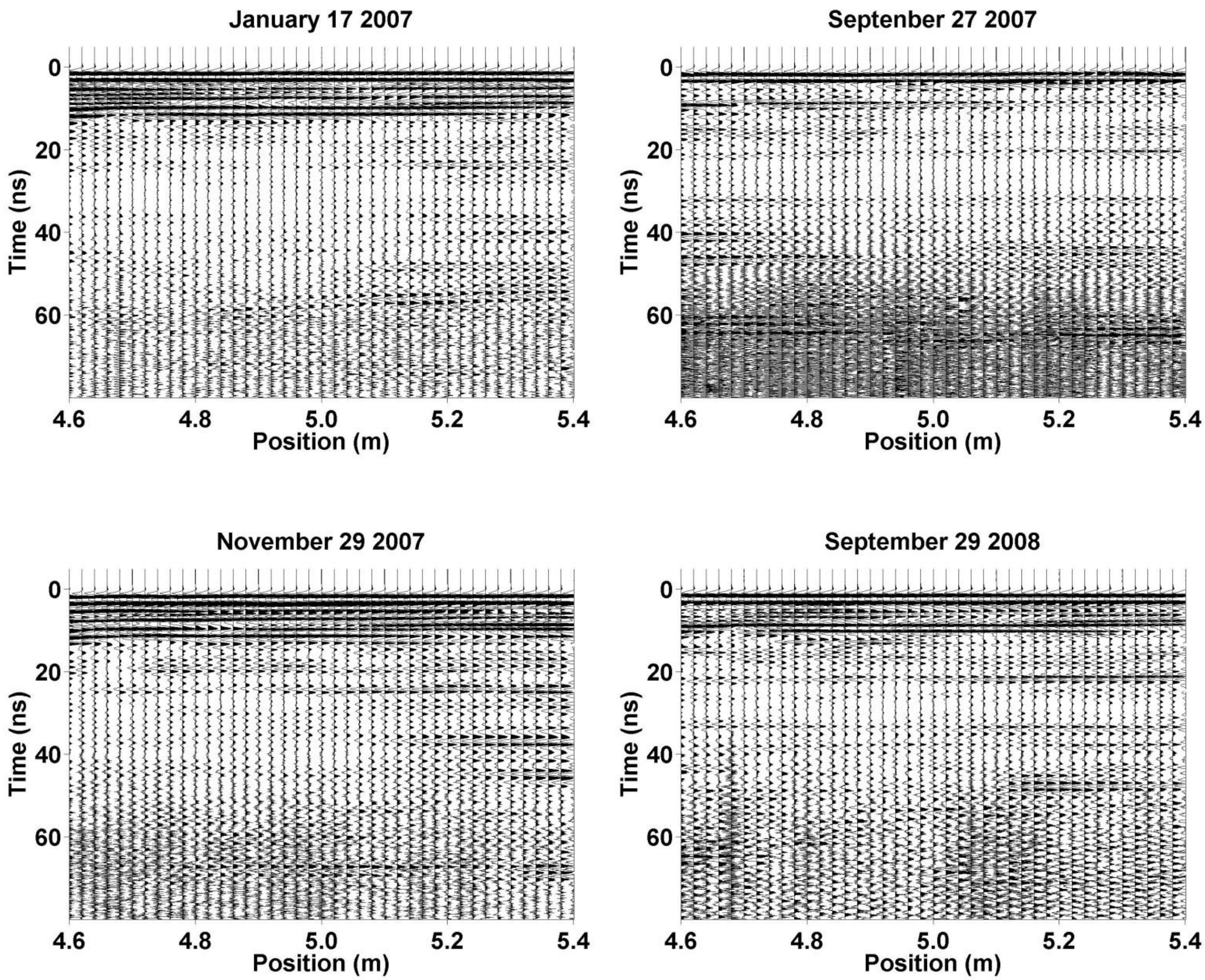


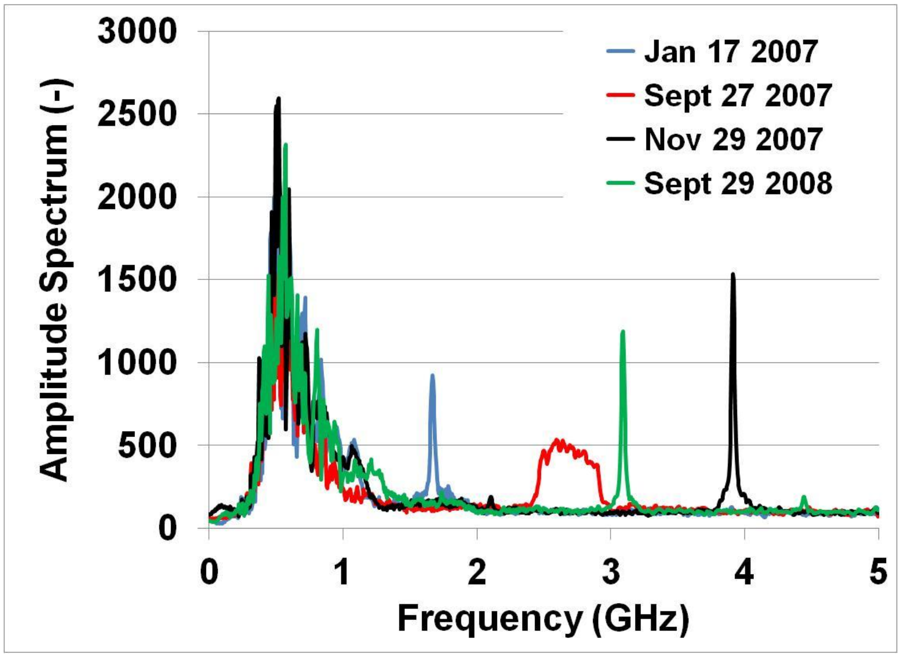




b.) Characterization of signal and noise in data

Example: GPR 900 MHz reflection profiling data from soil moisture monitoring project (Site A, Smith Farm, Waterloo Moraine)




Jan 17 2007
Sept 27 2007
Nov 29 2007
Sept 29 2008
Amplitude Spectrum (-)
Frequency (GHz)
0
500
1000
1500
2000
2500
3000
0
1
2
3
4
5