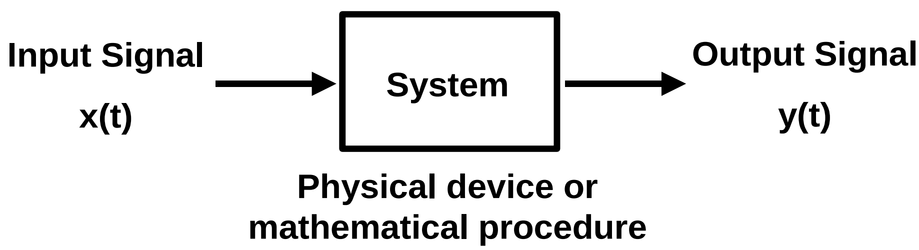
Earth 460 Classnotes (Set 3)

I.) Time Series Analysis – Continuous Signals

D.) Linear Systems, Convolution and Filters

1.) A system/operator/filter defines the relationship between an input signal $x(t)$ and output signal $y(t)$. We will denote this process by $x(t) \Rightarrow y(t)$.



2.) Linear Systems and Convolution

a.) Consider the two input-output signal pairs for a system: $x_1(t) \Rightarrow y_1(t)$ & $x_2(t) \Rightarrow y_2(t)$

b.) For a linear system, $ax_1(t) + bx_2(t) \Rightarrow ay_1(t) + by_2(t)$ where a and b are constants.

c.) This type of response is called the superposition principle

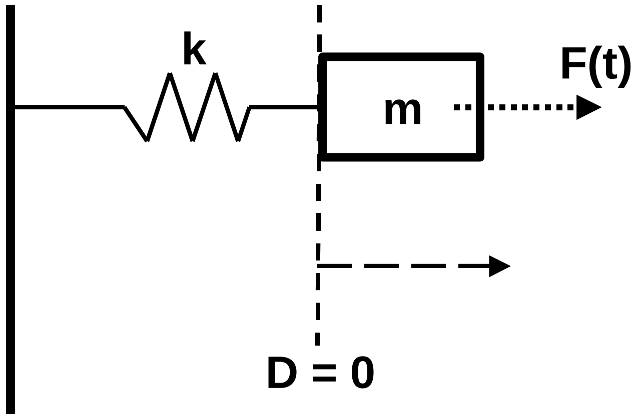
d.) If the input $x(t) = \delta(t)$, the resulting output $y(t) = I(t)$ is called the impulse response of the system.

e.) Example of a linear system: A mass-spring system

Input: $F(t)$ - applied external force

Output: $D(t)$ - displacement of the mass from its rest position

System is characterized by the mass m and spring constant k



1.) Relationship between the input and output is

$$D(t)=\int_{-\infty}^{t} m\omega_0 \sin\left[\omega_0(t-s)\right]F(s)ds \text{ where } \omega_0=\sqrt{k/m}$$

2.) If $F(t)=\delta(t)$, then $D(t)=I(t)=m\omega_0\sin(\omega_0 t)H(t)$ - the impulse response of the spring-mass system.

3.) For this system, $D(t)=\int_{-\infty}^{+\infty} I(t-s)F(s)ds$. This form is called a Faltung or convolution integral.

f.) In general, the relationship between the input and output signal of any linear system is defined ay a convolution integral:

$$y(t)=\int_{-\infty}^{+\infty} I(t-s)x(s)ds = I(t)*x(t)$$

g.) This type of system has the following characteristics:

1.) The system is time-invariant; the response of the system does not vary with time. Hence, $x(t+T)\Rightarrow y(t+T)$ for all values of time shift T and input signals $x(t)$.

2.) Information in a given frequency component does not interact or combine with information in other frequency components:

$$x(t)=A\cos(2\pi f_0 t)\Rightarrow y(t)=B\cos(2\pi f_0 t+\phi)$$

h.) Convolution "smears" or "spreads out" the input signal (i.e., the time duration of the output signal ≥ the time duration of the input signal).

i.) The mechanics of convolution are commonly described in terms of "flip" (i.e., $I(s) \to I(-s)$) and "shift" (i.e., $I(-s) \to I(t-s)$) operations.

j.) Properties of the convolution operation/integral

1.) Commutative: $x(t) * y(t) = y(t) * x(t)$ (i.e., can change the order inside the integral - $\int_{-\infty}^{+\infty} x(t-s)\, y(s)\, ds = \int_{-\infty}^{+\infty} y(t-s)\, x(s)\, ds$).

2.) Associative: $x(t) * [y(t) * z(t)] = [x(t) * y(t)] * z(t)$

3.) Distributive over addition:

$$x(t) * [y(t) + z(t)] = [x(t) * y(t)] + [x(t) * z(t)]$$

4.) Conservation of area: $\int_{-\infty}^{+\infty} [x(t) * y(t)]\, dt = \left[\int_{-\infty}^{+\infty} x(t)\, dt \right] \left[\int_{-\infty}^{+\infty} y(t)\, dt \right]$

Example:

$$[a\Pi(at)] * [e^{-bt} H(t)] = \int_{-\infty}^{+\infty} [a\Pi(as)] [e^{-b(t-s)} H(t-s)]\, ds$$

where $a\Pi(as) = \begin{cases} a, & -1/2a \le s \le +1/2a \\ 0, & \text{otherwise} \end{cases}$

and $e^{-b(t-s)} H(t-s) = \begin{cases} e^{-b(t-s)}, & (t-s) \ge 0 \to s \le t \\ 0, & (t-s) < 0 \to s > t \end{cases}$

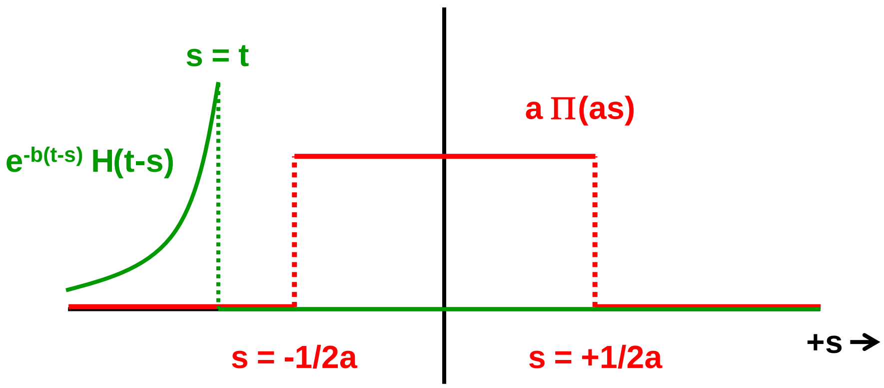
Determine the nature of the integrand by plotting the component signals and considering the possibility of different cases.

Case 1: $t < -1/2a$

In this case, there is no overlapping non-zero components →

$\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right] = 0$ for all s →

$$\int_{-\infty}^{+\infty}\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right]ds = \int_{-\infty}^{+\infty} 0\,ds = 0.$$

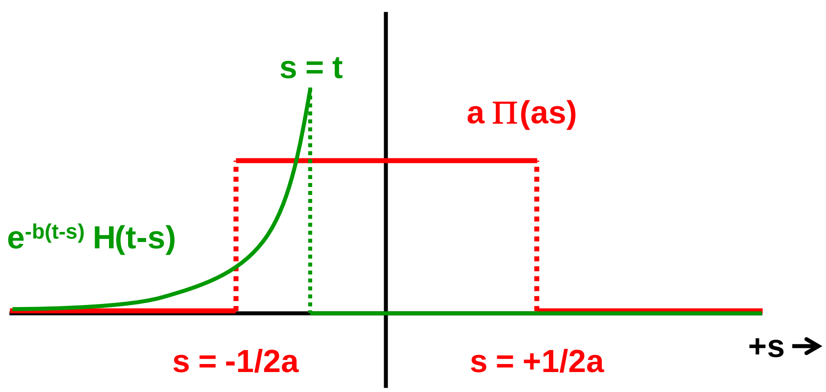


Case 2: $-1/2a \le t \le +1/2a$

$$\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right] = \begin{cases} ae^{-b(t-s)}, & -1/2a \le s \le t \\ 0, & \text{otherwise} \end{cases} \rightarrow$$

$$\int_{-\infty}^{+\infty}\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right]ds = \int_{-1/2a}^{t} ae^{-b(t-s)}ds = ae^{-bt}\int_{-1/2a}^{t} e^{bs}ds$$

$$= ae^{-bt}\left.\frac{e^{bs}}{b}\right|_{-1/2a}^{t} = \frac{ae^{-bt}}{b}\left(e^{bt} - e^{-b/2a}\right) = \frac{a}{b}\left(1 - e^{-b/2a}e^{-bt}\right)$$

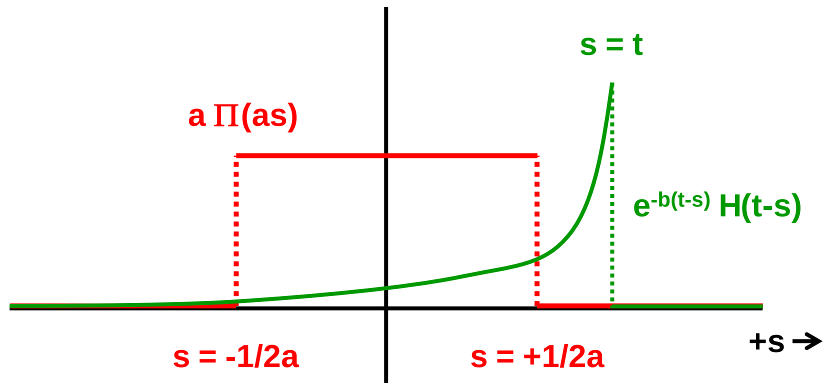


Case 3: $+1/2a < t$

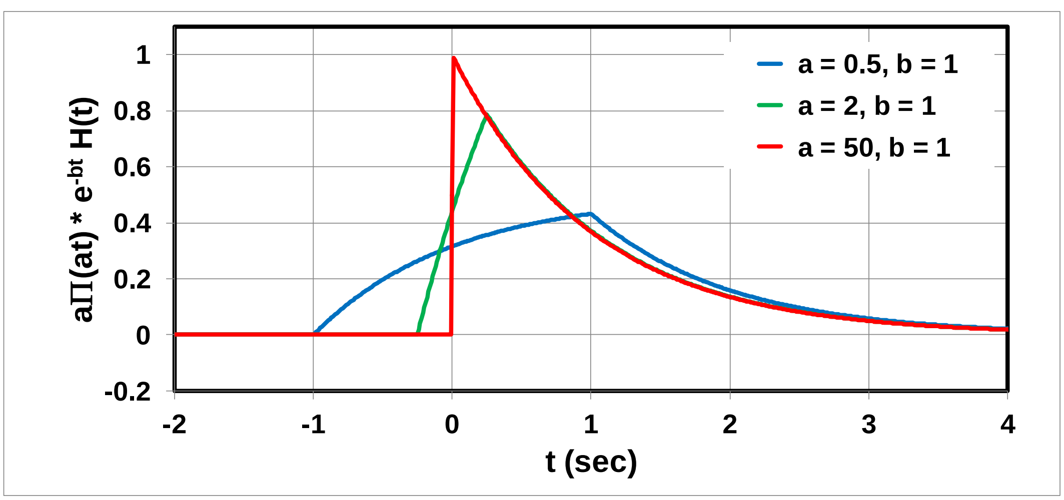
$$\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right] = \begin{cases} ae^{-b(t-s)}, & -1/2a \le s \le +1/2a \\ 0, & \text{otherwise} \end{cases} \rightarrow$$

$$\int_{-\infty}^{+\infty}\left[a\Pi(as)\right]\left[e^{-b(t-s)}H(t-s)\right]ds = \int_{-1/2a}^{+1/2a} ae^{-b(t-s)}ds = ae^{-bt}\int_{-1/2a}^{+1/2a} e^{bs}ds$$

$$= ae^{-bt}\left.\frac{e^{bs}}{b}\right|_{-1/2a}^{+1/2a} = \frac{ae^{-bt}}{b}\left(e^{+b/2a} - e^{-b/2a}\right) = \frac{a}{b}\left(e^{+b/2a} - e^{-b/2a}\right)e^{-bt}$$



$$\therefore\quad \left[a\Pi(at)\right]*\left[e^{-bt}H(t)\right] = \begin{cases} 0, & t < -1/2a \\ \frac{a}{b}\left(1 - e^{-b/2a}e^{-bt}\right), & -1/2a \le t \le +1/2a \\ \frac{a}{b}\left(e^{+b/2a} - e^{-b/2a}\right)e^{-bt}, & +1/2a < t \end{cases}$$



Earth 460 Classnotes (Set 3)

k.) Convolution and delta functions

1.) $\delta(t)*g(t)=\int_{-\infty}^{+\infty} g(t-s)\delta(s)ds = g(t)$

2.) $\delta(t-a)*g(t)=\int_{-\infty}^{+\infty} g(t-s)\delta(s-a)ds = g(t-a)$

l.) Fourier transform of the convolution integral

1.) Let $x(t)\Leftrightarrow X(f)$ and $y(t)\Leftrightarrow Y(f)$, then

a.) $x(t)*y(t)\Leftrightarrow X(f)Y(f)$

b.) $x(t)y(t)\Leftrightarrow X(f)*Y(f)$

2.) For a linear system with input signal $x(t)\Leftrightarrow X(f)$ and output signal $y(t)\Leftrightarrow Y(f)$, then $y(t)=I(t)*x(t)\Leftrightarrow Y(f)=T(f)\,X(f)$ where $T(f)$ is the transfer function for the system. Note that $I(t)\Leftrightarrow T(f)$.

m.) Amplitude and phase spectra relationships for linear systems

1.) Start with the following Fourier transform pairs:

Input signal: $x(t)\Leftrightarrow X(f)=A_X(f)\,e^{i\theta_X(f)}$

Linear system: $I(t)\Leftrightarrow T(f)=A_T(f)\,e^{i\theta_T(f)}$

Output signal: $y(t)\Leftrightarrow Y(f)=A_Y(f)\,e^{i\theta_Y(f)}$

where $A(f)$ and $\theta(f)$ are the respective amplitude and phase spectra.

2.) Since $y(t)=I(t)*x(t)\Leftrightarrow Y(f)=T(f)\,X(f)$, then

$$A_Y(f)\,e^{i\theta_Y(f)}=A_X(f)\,e^{i\theta_X(f)}\cdot A_T(f)\,e^{i\theta_T(f)}=\left[A_X(f)\cdot A_T(f)\right]e^{i\left[\theta_X(f)+\theta_T(f)\right]}$$

3.) Amplitude and phase spectra relationships

a.) $A_Y(f)=A_X(f)\cdot A_T(f)$

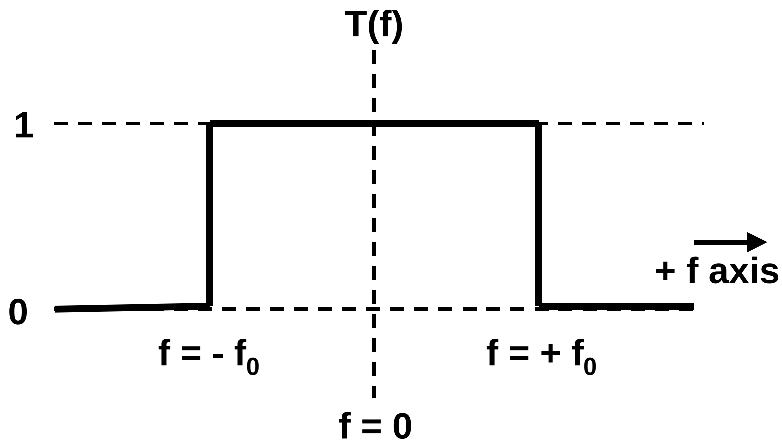
b.) $\theta_Y(f)=\theta_X(f)+\theta_T(f)$

3.) Simple Filters

a.) Ideal frequency filter that pass specified frequency components without phase distortions

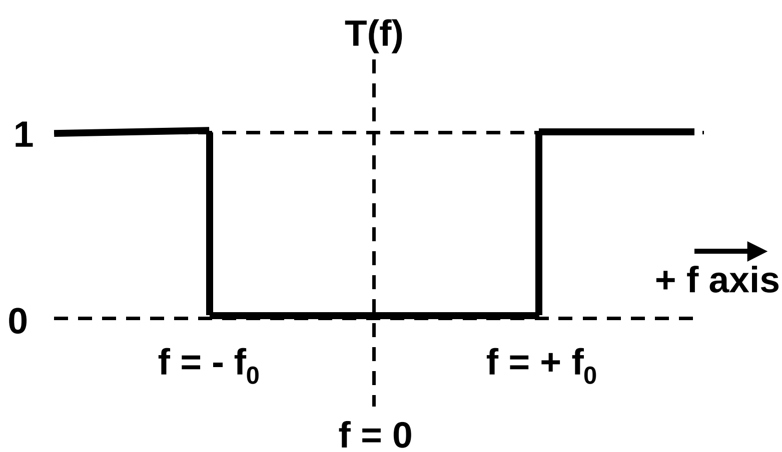
1.) Low-pass (or high-cut) filter: $T_{Low}(f) = \begin{Bmatrix} 1, & |f| \le f_0 \\ 0, & |f| > f_0 \end{Bmatrix} = \Pi(f/2f_0)$

Impulse response: $I_{Low}(t) = 2f_0 \,\mathrm{sinc}(2f_0 t)$



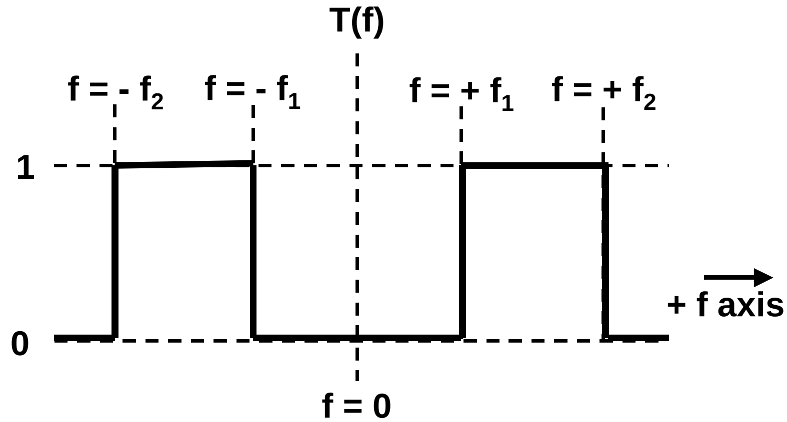
2.) High-pass (or low-cut) filter: $T_{High}(f) = \begin{Bmatrix} 1, & |f| \ge f_0 \\ 0, & |f| < f_0 \end{Bmatrix} = 1 - \Pi(f/2f_0)$

Impulse response: $I_{High}(t) = \delta(t) - 2f_0 \,\mathrm{sinc}(2f_0 t)$



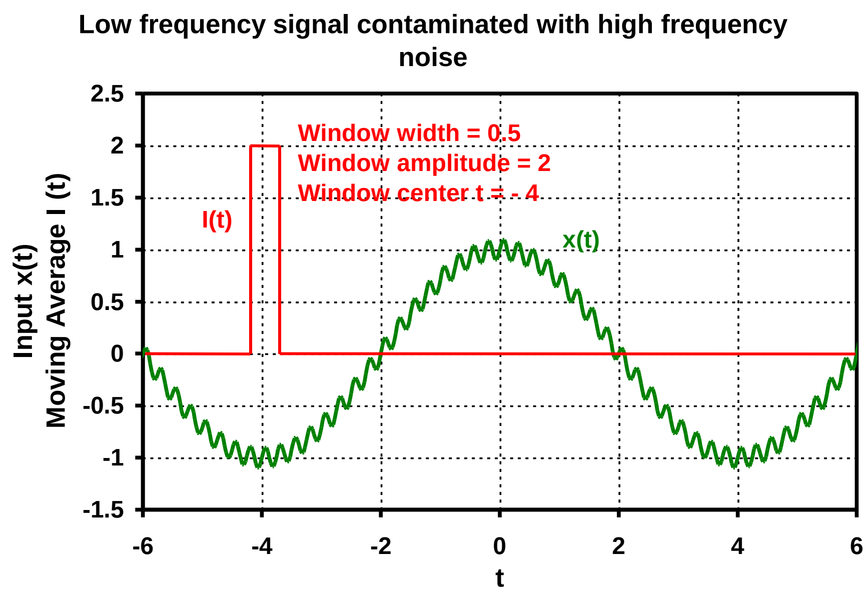
3.) Band-pass filter: $T_{Band}(f) = \left\{ \begin{matrix} 1, & f_2 \geq |f| \geq f_1 \\ 0, & |f| > f_2 \text{ or } |f| < f_1 \end{matrix} \right\} = \prod(f/2f_2) - \prod(f/2f_1)$

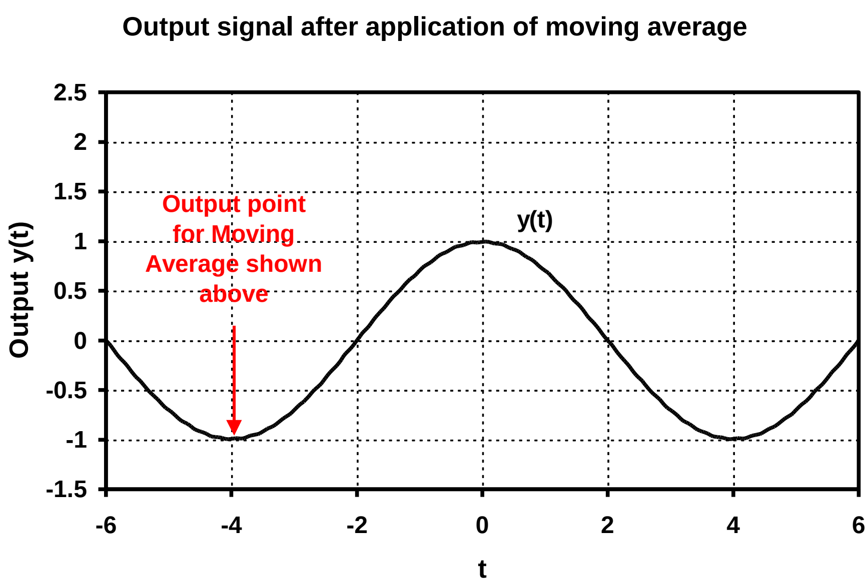
Impulse response: $I_{Band}(t) = 2f_2 \operatorname{sinc}(2f_2 t) - 2f_1 \operatorname{sinc}(2f_1 t)$



b.) Moving average filter

1.) A moving average filter gives mean value of an input signal within a specified time window; this average value   The output signal is generated by translating the window along the input signal.



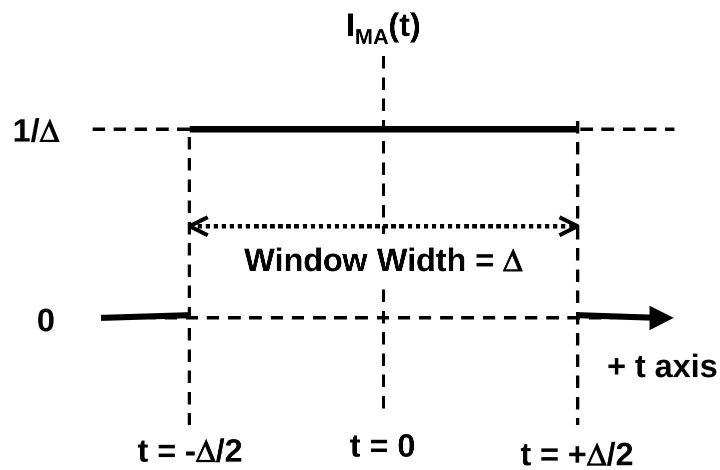


2.) For continuous signals, the mathematical relationship between the input and output signals is $y(t) = \frac{1}{\Delta} \int_{t-\Delta/2}^{t+\Delta/2} x(s)ds$ where Δ is the window width.

3.) This expression can be written in the form of a convolution integral using an appropriate boxcar function as the impulse response.

$$y(t) = \int_{-\infty}^{+\infty} \left[\Delta^{-1} \prod\left(\frac{t-s}{\Delta}\right)\right] x(s)ds = \left[\Delta^{-1} \prod(t/\Delta)\right] * x(t) \rightarrow$$

$$I_{MA}(t) = \Delta^{-1} \prod(t/\Delta)$$

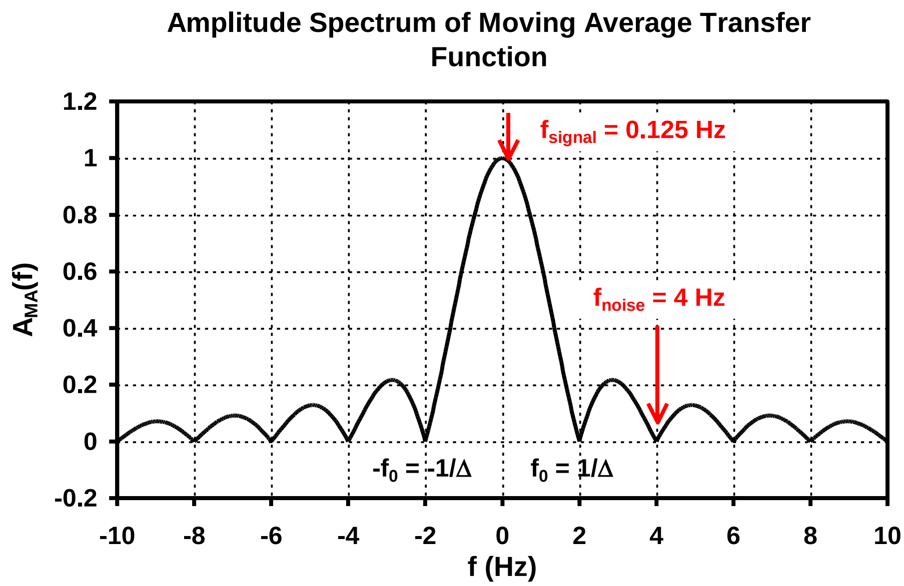


4.) From the Fourier transform, we get the transfer function for the moving average: $I_{MA}(t) = \Delta^{-1} \prod(t/\Delta) \Leftrightarrow T_{MA}(f) = \mathrm{sinc}(\Delta f)$ and

$$A_{MA}(f) = \left|\mathrm{sinc}(\Delta f)\right|$$

a.) The moving average is essentially a low-pass filter, attenuating frequency components above $f_0 = 1/\Delta$

b.) It is also a zero phase filter (i.e., $\theta_{MA}(f) = 0$); hence, $\theta_Y(f) = \theta_X(f)$



4.) Filter tapering

a.) Sharp edges in the transfer function, such as with the boxcar functions used for the simple filters above, can produce ringing (i.e., significant sidelobes) in the impulse response.

b.) This ringing means a greater degree of smearing occurs due to the convolution process.

c.) To minimize this effect, various types of tapering (e.g., linear, sine/cosine) are used in filter design.

Example: Low-pass filter with a linear taper

The untapered low-pass filter is defined by the transform pair
$2f_0 \operatorname{sinc}(2f_0 t) \Leftrightarrow \prod(f/2f_0)$.

A corresponding low-pass filter with a linear taper can be constructed by differencing two triangle functions in the frequency domain.

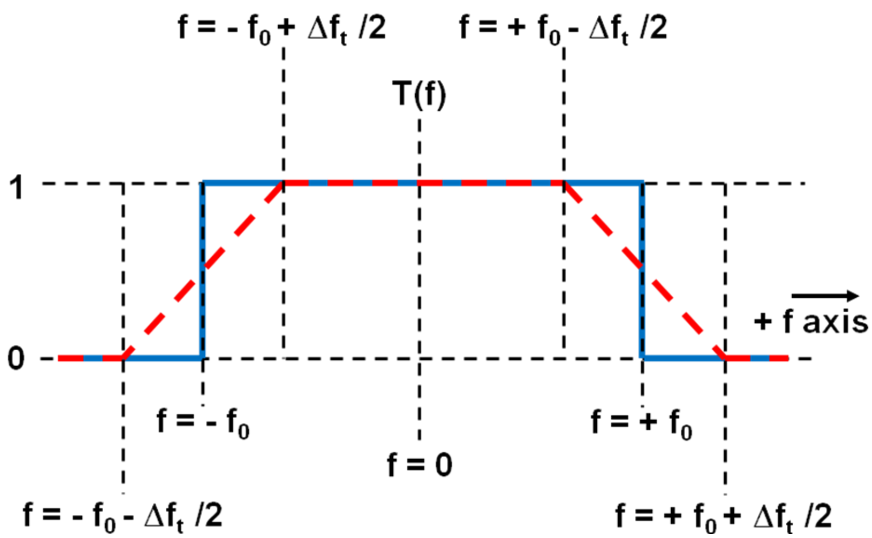
The triangle function is defined by $\Lambda(f) = \begin{cases} 1-|f|, \ |f| \le 1 = \begin{cases} 1-f, \ 0 \le f \le 1 \\ 1+f, \ -1 \le f < 0 \end{cases} \\ 0, \ |f| > 1 \end{cases}$

and forms the transform pair $\text{sinc}^2(t) \Leftrightarrow \Lambda(f)$.

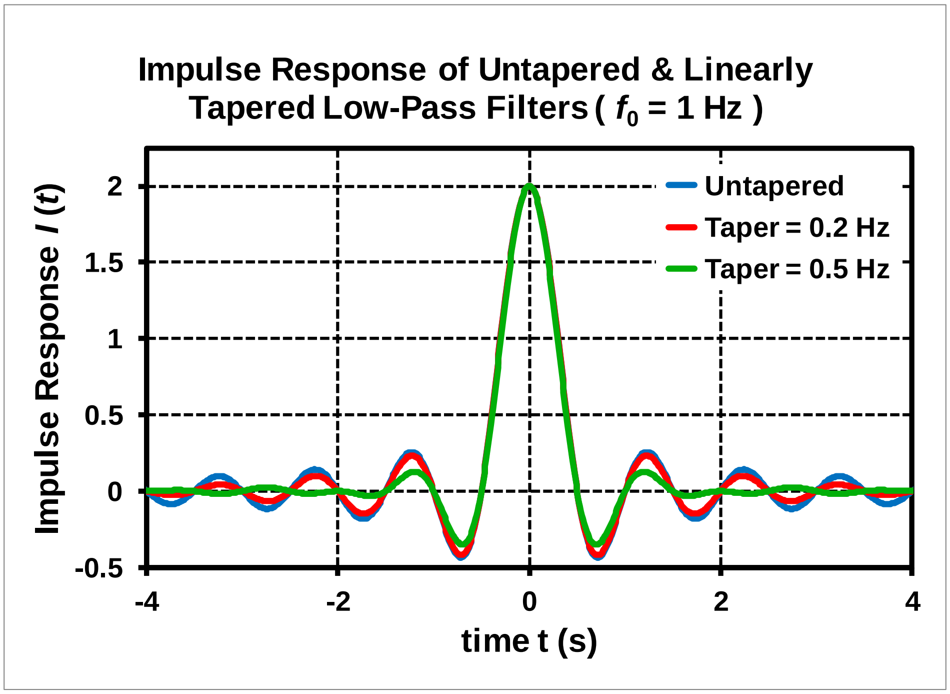
The tapered lowpass filter is described by the transform pair

$$(\Delta f_t)^{-1}\left\{\left[f_0+(\Delta f_t/2)\right]^2 \text{sinc}^2\left\{\left[f_0+(\Delta f_t/2)\right]t\right\}-\left[f_0-(\Delta f_t/2)\right]^2 \text{sinc}^2\left\{\left[f_0-(\Delta f_t/2)\right]t\right\}\right\}$$
$$\Leftrightarrow (\Delta f_t)^{-1}\left\{\left[f_0+(\Delta f_t/2)\right]\Lambda\left\{f/\left[f_0+(\Delta f_t/2)\right]\right\}-\left[f_0-(\Delta f_t/2)\right]\Lambda\left\{f/\left[f_0-(\Delta f_t/2)\right]\right\}\right\}$$

where Δf_t is the taper width.

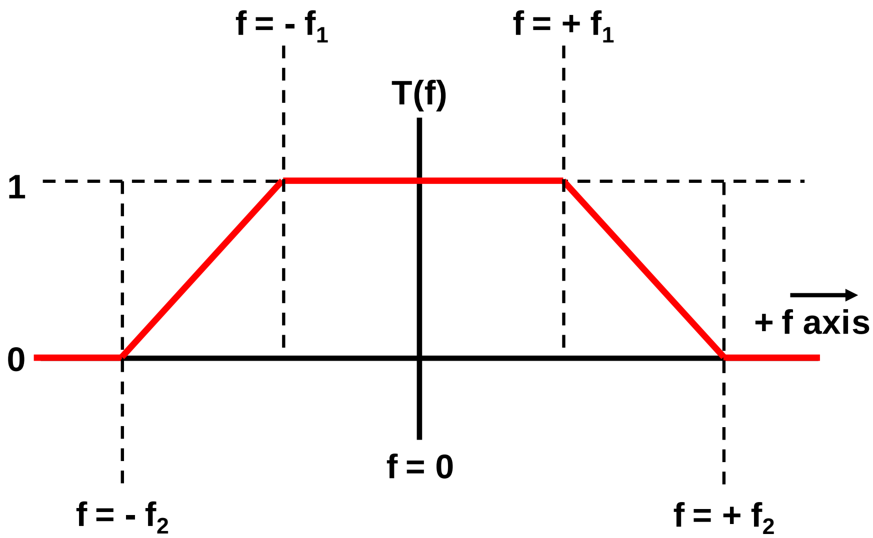


Let us compare the impulse response of the untapered low-pass filter ($f_0 = 1\,\text{Hz}$) with two linearly tapered versions ($\Delta f_t = 0.2$ Hz & $\Delta f_t = 0.5$ Hz). It can be seen that increasing the taper width Δf_t progressively attenuates the sidelobes compared to the untapered case while preserving the shape of the central lobe.



The impulse response and transfer functions for this linearly tapered lowpass filter can also be expressed in terms of cutoff and pass frequencies (i.e., $f_2 = f_0 + (\Delta f_t/2)$ and $f_1 = f_0 - (\Delta f_t/2)$, respectively):

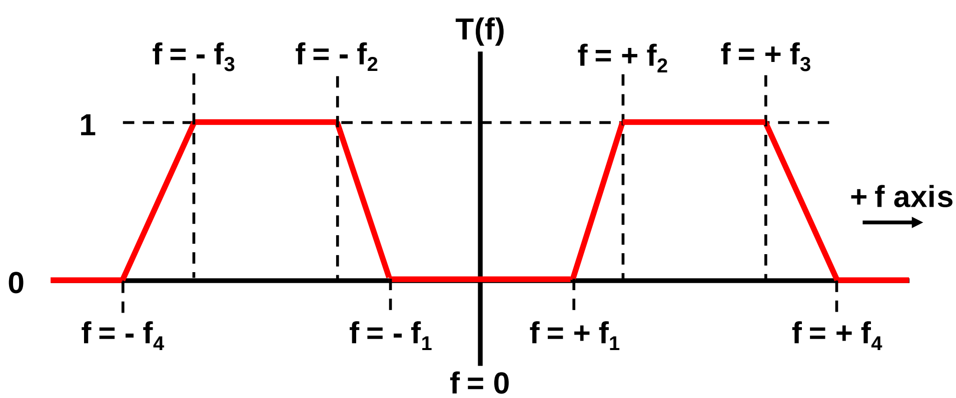
$$(f_2 - f_1)^{-1}\left[(f_2)^2 \,\mathrm{sinc}^2(f_2 t) - (f_1)^2 \,\mathrm{sinc}^2(f_1 t)\right] \Leftrightarrow (f_2 - f_1)^{-1}\left[f_2 \,\Lambda(f/f_2) - f_1 \,\Lambda(f/f_1)\right]$$



This form can be used to define the impulse response and transfer functions for a linearly tapered bandpass filter:

$$\left\{(f_4-f_3)^{-1}\left[(f_4)^2\,\mathrm{sinc}^2(f_4\,t)-(f_3)^2\,\mathrm{sinc}^2(f_3\,t)\right]\right\}-\left\{(f_2-f_1)^{-1}\left[(f_2)^2\,\mathrm{sinc}^2(f_2\,t)-(f_1)^2\,\mathrm{sinc}^2(f_1\,t)\right]\right\}$$
$$\Leftrightarrow\left\{(f_4-f_3)^{-1}\left[f_4\,\Lambda(f/f_4)-f_3\,\Lambda(f/f_3)\right]\right\}-\left\{(f_2-f_1)^{-1}\left[f_2\,\Lambda(f/f_2)-f_1\,\Lambda(f/f_1)\right]\right\}$$

where f_1 and f_2 cutoff and pass frequencies, respectively, at the low frequency limit; f_3 and f_4 pass and pass frequencies, respectively, at the high frequency limit.



5.) Properties of systems

a.) Stability: a system is stable if its output signal has finite energy (i.e., $\int_{-\infty}^{+\infty} \left[y(t) \right]^2 dt < +\infty$) when the input signal also has finite energy (i.e., $\int_{-\infty}^{+\infty} \left[x(t) \right]^2 dt < +\infty$). This condition means $\int_{-\infty}^{+\infty} \left[I(t) \right]^2 dt < +\infty$ and $I(t) \to 0$ as $t \to \pm\infty$.

b.) Causality: a system is causal if $I(t) = 0$ for $t < 0$. Hence, the system responds only after it is stimulated (it does not anticipate the input). For the transfer function of a causal system, $T(f) = T'(f) + iT''(f)$ $= T'(f) - i\dfrac{1}{\pi f} * T'(f) \to T''(f) = \dfrac{-1}{\pi}\int_{-\infty}^{+\infty} \dfrac{T'(s)}{f - s}\,ds$. The real and imaginary parts of $T(f)$ ($T'(f)$ and $T''(f)$, respectively) are related through the Hilbert transform.

c.) Physically-realizable: a system that is both stable and causal.

6.) Phase considerations

a.) It is sometimes necessary to preserve phase relationships between the input and output signals when applying a filter.

b.) An important phase concept in geophysics is minimum phase/delay.

c.) A minimum phase wavelet is sometimes called “front-loaded” because its energy is concentrated in the leading end of the pulse. Further, a minimum phase wavelet is causal and stable.

d.) It is commonly assumed that the wavelets for impulsive sources (e.g., explosives, weight-drops) are minimum phase. In addition, wave propagation phenomena, such as water bottom reverberation have minimum phase characteristics.

e.) Minimum phase signals have a unique relationship between their amplitude and phase spectra: $\theta(f) = \frac{-1}{\pi}\int_{-\infty}^{+\infty} \frac{\log A(s)}{f-s} ds$. $\theta(f)$ is the Hilbert transform of $\log A(f)$.

f.) The convolution of two minimum phase signals produces a minimum phase output.

g.) Minimum phase filters/linear systems are commonly used in geophysical data processing to preserve the minimum phase characteristics of the signal (e.g., seismic trace).

7.) Common applications filters in geophysical data processing

a.) Data analysis (e.g., filter panels)

b.) Noise suppression

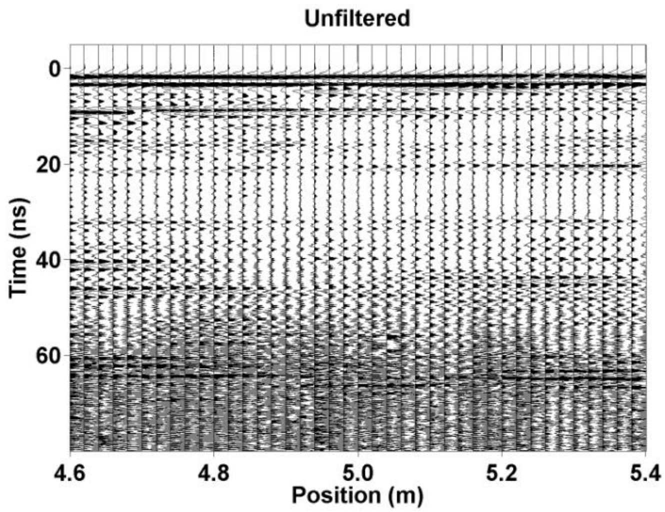
c.) Wavelet shaping filters: to obtain a wavelet of a specific form to have a uniform source wavelet covering a number of data sets acquired using various source types and differing processing. This filter is also used to match seismic profiling data with synthetic seismograms generated from well log surveys.

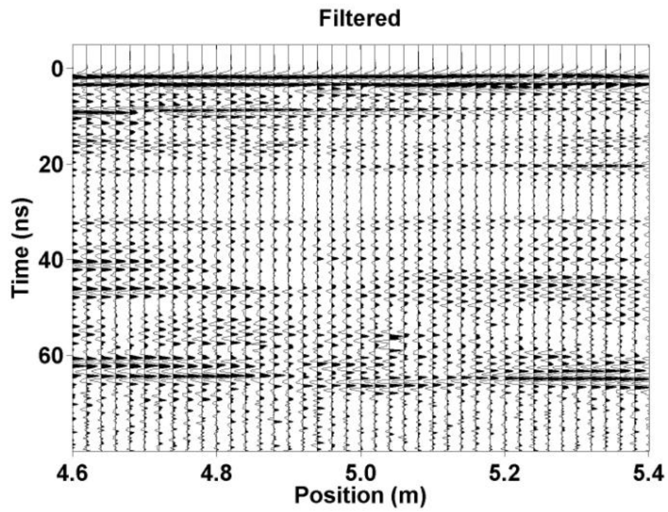
Example #1:

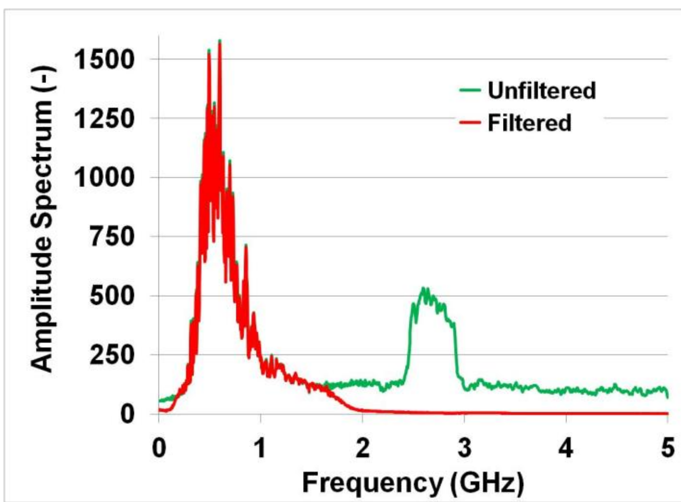
Suppression of high frequency noise in GPR reflection profiling data using a bandpass filter.

Data: GPR 900 MHz reflection profiling data from soil moisture monitoring project (Site A, Smith Farm, Waterloo Moraine, September 27, 2007)

Bandpass frequencies: 0.100 - 0.200 - 1.500 - 2.000 GHz



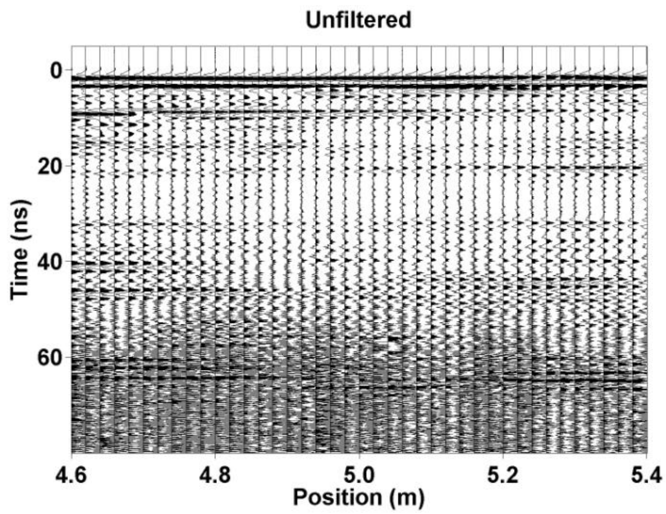


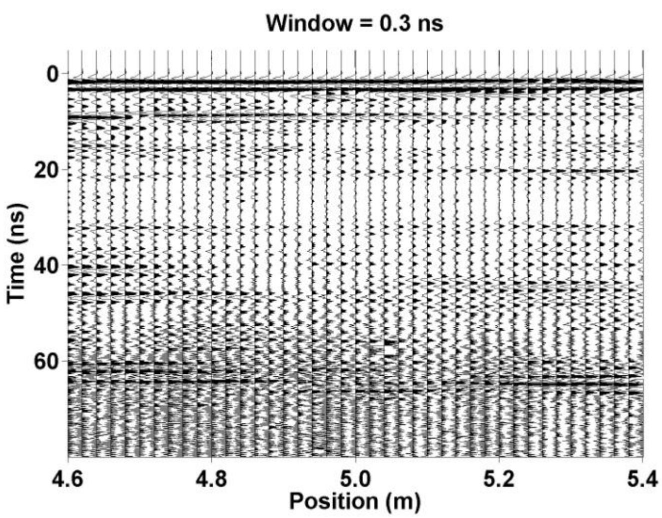


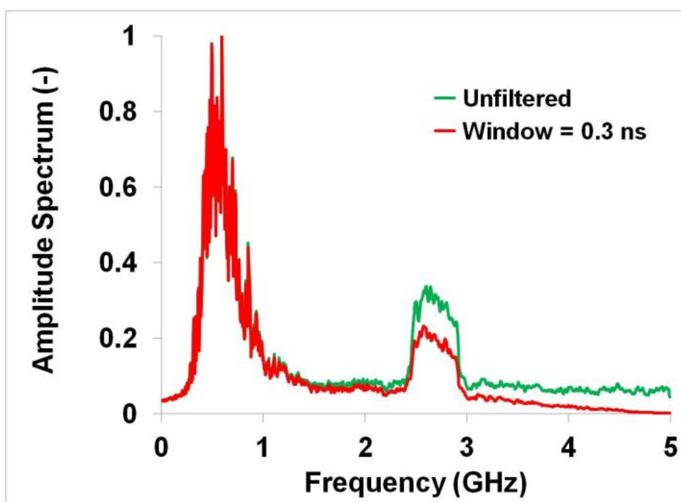
Example #2:

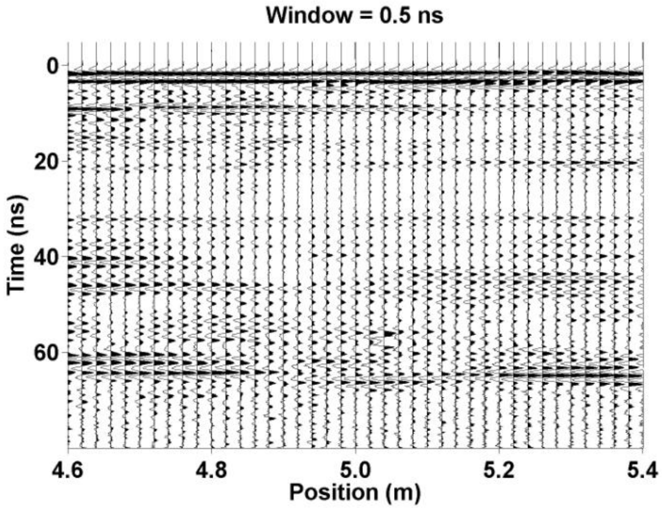
Suppression of high frequency noise in GPR reflection profiling data using a moving time average filter.

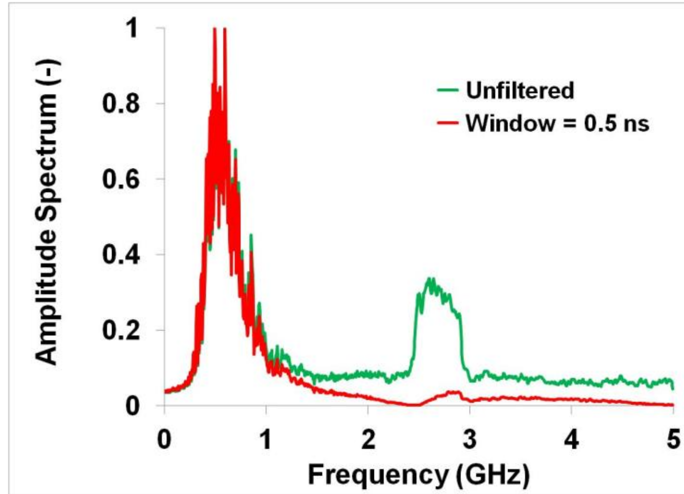
Data: GPR 900 MHz reflection profiling data from soil moisture monitoring project (Site A, Smith Farm, Waterloo Moraine, September 27, 2007)

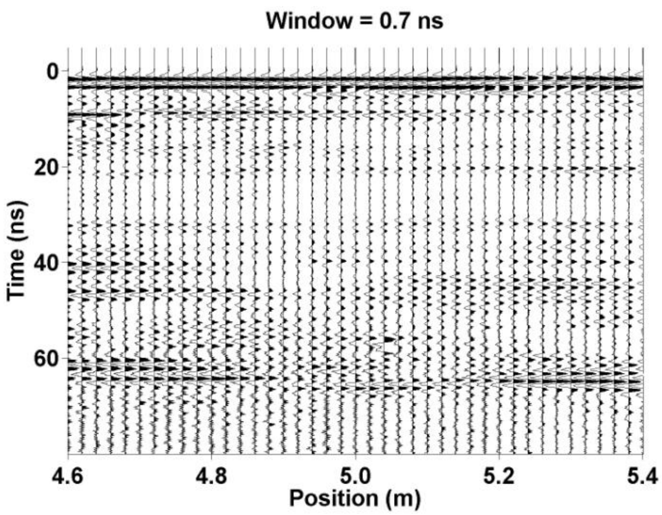


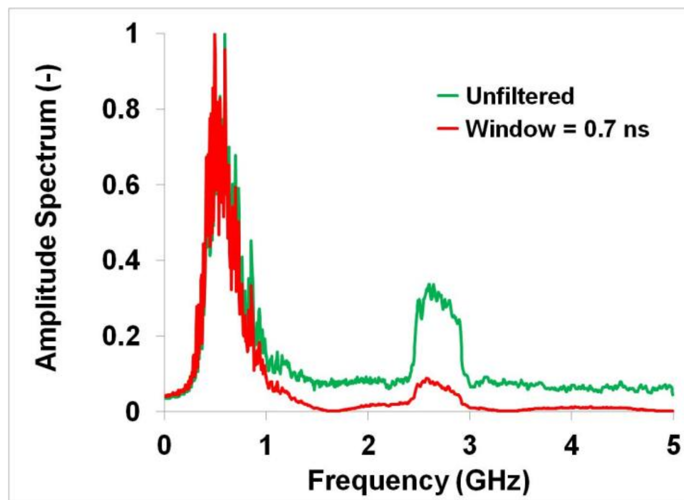


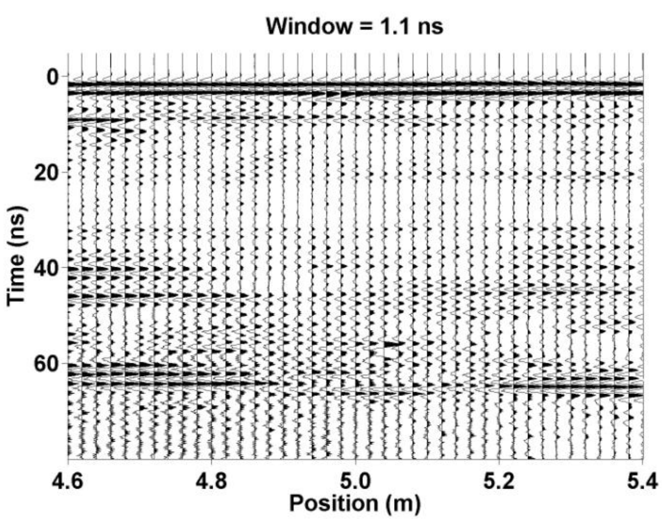


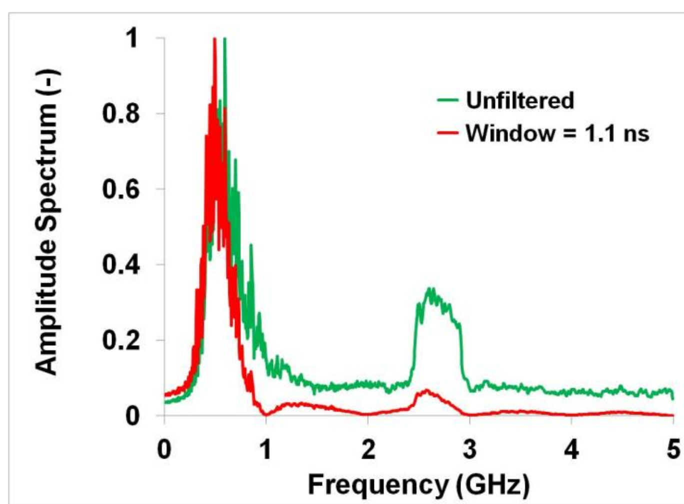

Window = 0.5 ns
Time (ns)
0
20
40
60
4.6
4.8
5.0
5.2
5.4
Position (m)

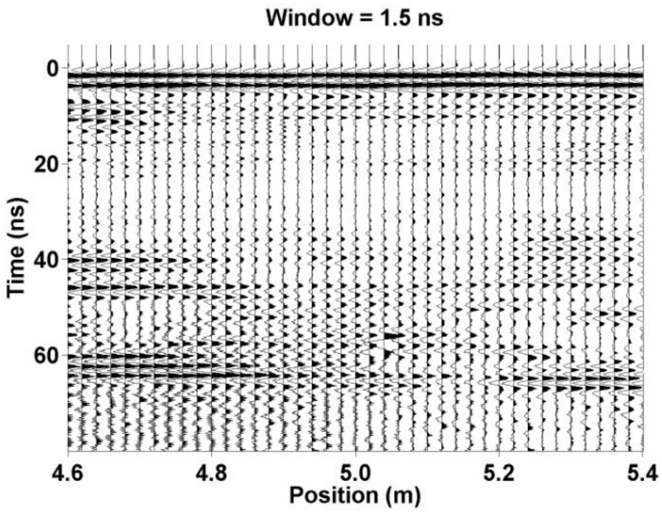

Amplitude Spectrum (-)
Frequency (GHz)
Unfiltered
Window = 0.5 ns
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5

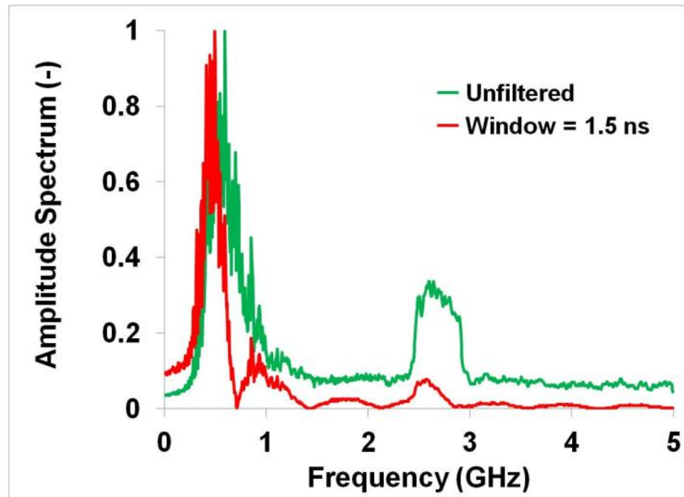

Window = 0.7 ns
Time (ns)
0
20
40
60
4.6
4.8
5.0
5.2
5.4
Position (m)

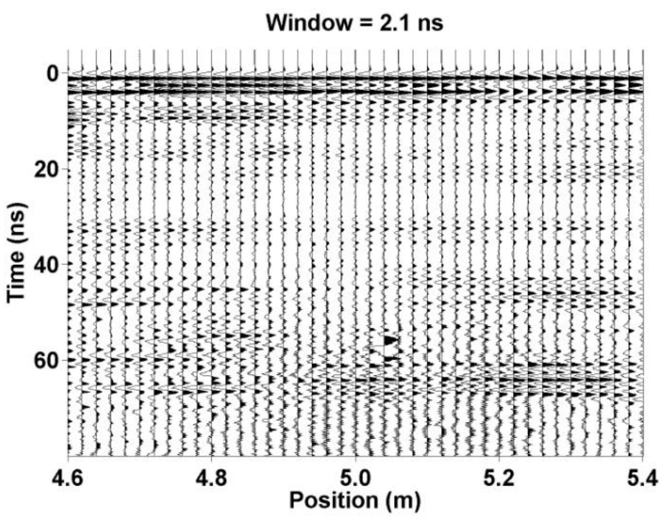

Amplitude Spectrum (-)
Frequency (GHz)
Unfiltered
Window = 0.7 ns
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5

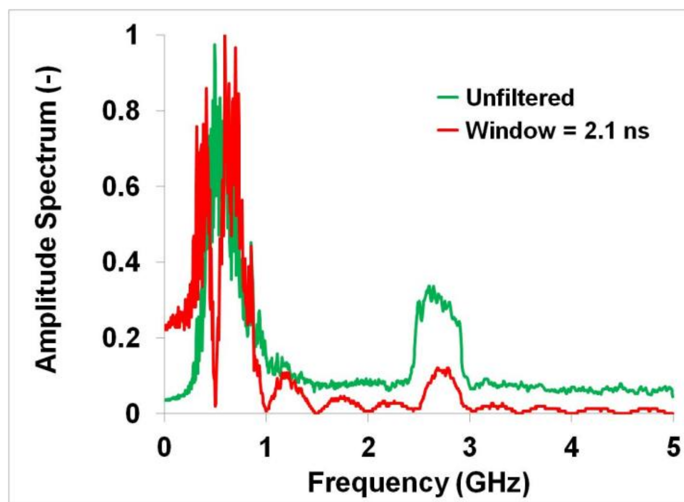

Window = 1.1 ns
Time (ns)
0
20
40
60
4.6
4.8
5.0
5.2
5.4
Position (m)


Amplitude Spectrum (-)
Frequency (GHz)
Unfiltered
Window = 1.1 ns
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5


Window = 1.5 ns
Time (ns)
0
20
40
60
4.6
4.8
5.0
5.2
5.4
Position (m)


Amplitude Spectrum (-)
1
0.8
0.6
0.4
0.2
0
Unfiltered
Window = 1.5 ns
0
1
2
3
4
5
Frequency (GHz)


Window = 2.1 ns
Time (ns)
0
20
40
60
4.6
4.8
5.0
5.2
5.4
Position (m)


Amplitude Spectrum (-)
1
0.8
0.6
0.4
0.2
0
Unfiltered
Window = 2.1 ns
0
1
2
3
4
5
Frequency (GHz)