Earth 460 Classnotes (Set 1)

I.) Time Series Analysis – Continuous Signals

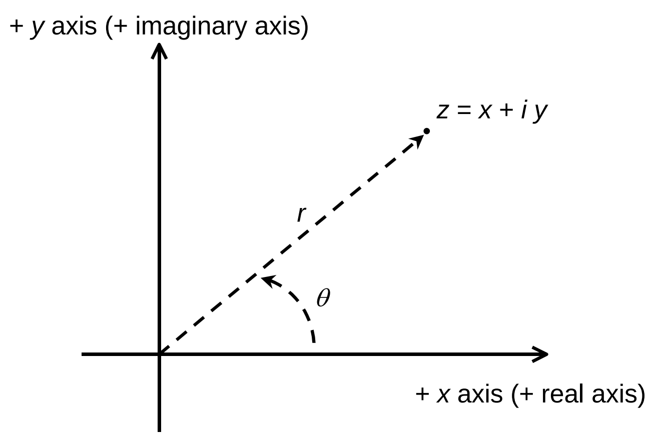
A.) Complex Numbers (A Brief Review)

1.) $z = x + i\,y$ is a complex number where x and y are real numbers and $i = \sqrt{-1}$.

a.) $x = \mathrm{Re}(z)$ is the real part of z.

b.) $y = \mathrm{Im}(z)$ is the imaginary part of z.

2.) Geometrical (polar) representation of complex numbers



z - plane (complex plane)

a.) $r = \sqrt{x^2 + y^2}$ is the absolute value or modulus of z (i.e., $r = \mathrm{mod}(z) = |z|$).

b.) $\theta = \tan^{-1}(y/x)$ is the argument or phase modulus of z (i.e., $\theta = \arg(z)$). θ is positive in the counterclockwise direction with respect to the $+x$ (positive real) axis.

c.) From the geometry, the real and imaginary parts of z can be expressed as $x = r\cos\theta$ and $y = r\sin\theta$. Hence, $z = r\,(\cos\theta + i\sin\theta)$.

3.) Complex conjugate of z

a.) $\bar{z} = x - i\,y$ is the complex conjugate of $z = x + i\,y$.

b.) $z \cdot \bar{z} = (x + i\,y)(x - i\,y) = x^2 + y^2 = r^2$ (a real-valued quantity) c.) Rationalizing the complex fraction (u/v) where $u = a + i\,b$ and $v = c + i\,d$:

$$\frac{u}{v} = \frac{a+ib}{c+id} = \frac{u\bar{v}}{v\bar{v}} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

This is the standard procedure for complex fractions.

Example: $\dfrac{2+i3}{1-i} = \dfrac{(2+i3)(1+i)}{(1-i)(1+i)} = \dfrac{(2\cdot 1+3\cdot(-1))+i(3\cdot 1+2\cdot(-1))}{1^2+(-1)^2} = \dfrac{-1+i5}{2}$

4.) Complex exponential function

a.) $e^{i\theta} = \cos\theta + i\sin\theta$ where θ is a real number.

i.) $\left|e^{i\theta}\right| = 1$

ii.) $e^{in\pi} = (-1)^n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.

iii.) $e^{i\pi/2} = i$

iv.) $z = r\,e^{i\theta}$

v.) Combined with $e^{-i\theta} = \cos\theta - i\sin\theta$, it can be shown that $\cos\theta = \left(e^{i\theta} + e^{-i\theta}\right)/2$ and $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$.

b.) For e^{z} where z is a complex number:

$$e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$$

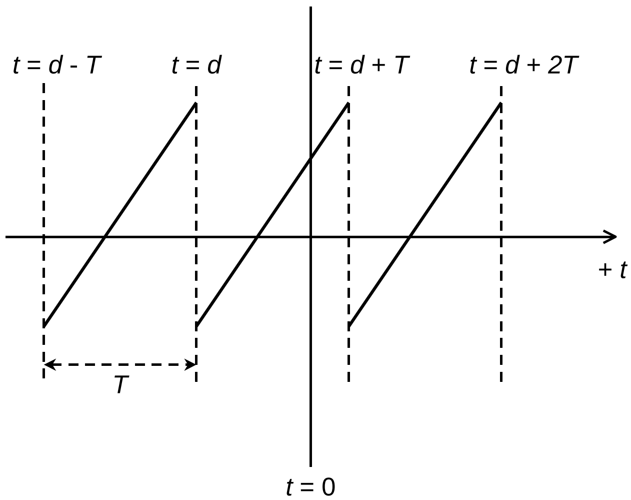
Example: $e^{\pi(1+i)} = e^{\pi}e^{i\pi} = e^{\pi}(\cos\pi + i\sin\pi) = -e^{\pi}$

B.) Spectral Analysis - Fourier Series

1.) Definition

a.) A Fourier series can be obtained for any periodic function $g(t)$ that satisfies given conditions.

i.) $g(t)$ is a periodic functions with period T if $g(t) = g(t + nT)$ where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$.



ii.) $g(t)$ has a finite number of extrema points and discontinuities over a single period.

iii.) $\int_{-T/2}^{T/2} |g(t)| dt < \infty$.

b.) The Fourier series of $g(t)$ is a summation of harmonic function given by

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right]$$

i.) $a_0/2$ is the DC bias term (corresponding to frequency $f = 0$).

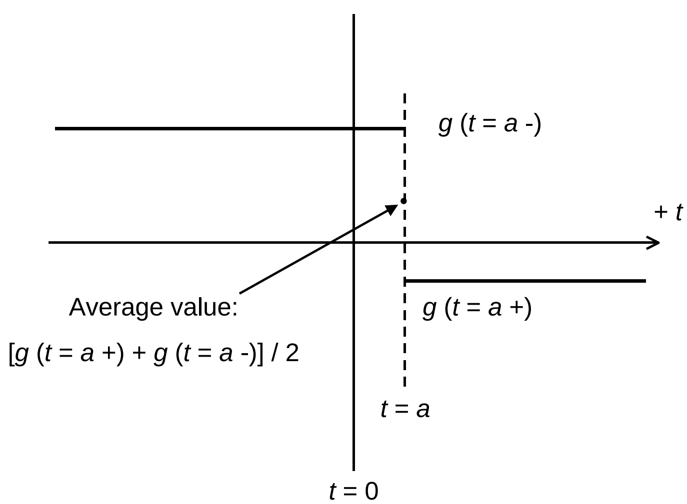
ii.) $\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$ is the harmonic component corresponding to the frequency $f_n = n/T \ (n = 1, 2, 3, \cdots)$. The magnitude of this term is a measure of its relative information content about $g(t)$.

c.) Using a mathematical property called orthogonality, the coefficients of the Fourier series can be shown to be the following:

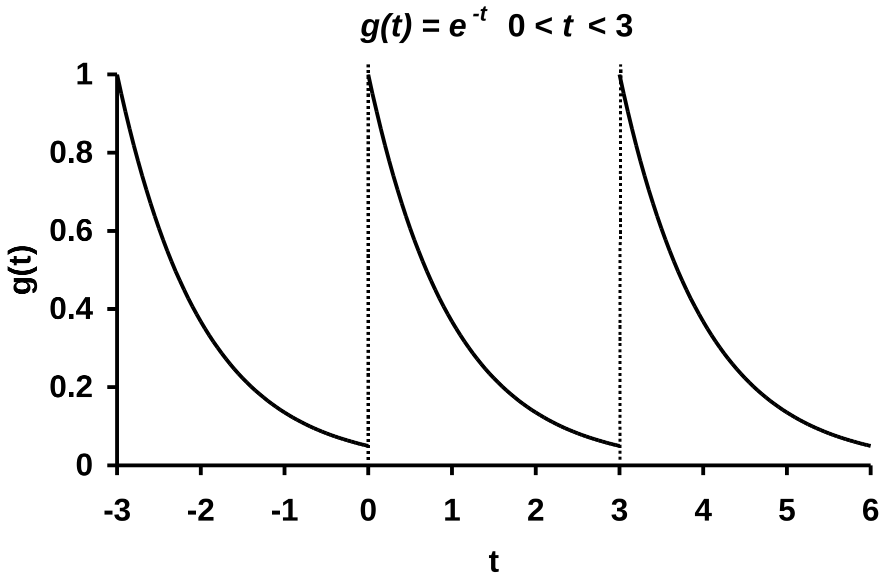
$$a_n = \frac{2}{T}\int_d^{d+T} g(t)\cos\left(\frac{2\pi n t}{T}\right)dt \text{ , with } a_0 = \frac{2}{T}\int_d^{d+T} g(t)dt$$

$$b_n = \frac{2}{T}\int_d^{d+T} g(t)\sin\left(\frac{2\pi n t}{T}\right)dt$$

d.) The Fourier series expansion converges to $g(t)$ where $g(t)$ is continuous and to the average of the right and left hand limits at the discontinuities.



Example: $g(t) = e^{-t}, \quad 0 < t < 3$



$$a_0 = \frac{2}{T}\int_d^{d+T} g(t)\,dt = \frac{2}{3}\int_0^3 e^{-t}dt \qquad \text{(using } \int e^{at}dt = e^{at}/a \text{ with } a=-1)$$

$$= \frac{2\,e^{-t}}{-3}\bigg|_0^3 = 2\left(1-e^{-3}\right)/3$$

$$a_n = \frac{2}{T}\int_d^{d+T} g(t)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{3}\int_0^3 e^{-t}\cos\left(\frac{2\pi n t}{3}\right)dt$$

(using $\int e^{at}\cos bt\,dt = \dfrac{e^{at}\left(a\cos bt + b\sin bt\right)}{a^2+b^2}$ with $a=-1$ and $b = 2\pi n/3$)

$$= \frac{2}{3}\,\frac{e^{-t}\left[-\cos(2\pi nt/3) + (2\pi n/3)\sin(2\pi nt/3)\right]}{(-1)^2 + (2\pi n/3)^2}\Bigg|_0^3$$

(sine term is zero and reverse limits)

$$= \frac{2}{3}\,\frac{9}{9+4\pi^2 n^2}\left[1-e^{-3}\cos(2\pi n)\right] \qquad \text{(using } \cos(2\pi n)=1)$$

$$= \frac{6\left(1-e^{-3}\right)}{9+4\pi^2 n^2}$$

$$b_n = \frac{2}{T}\int_d^{d+T} g(t)\sin\left(\frac{2\pi n t}{T}\right)dt = \frac{2}{3}\int_0^3 e^{-t}\sin\left(\frac{2\pi n t}{3}\right)dt$$

(using $\int e^{at}\sin bt\,dt = \dfrac{e^{at}\left(a\sin bt - b\cos bt\right)}{a^2+b^2}$ with $a=-1$ and $b = 2\pi n/3$)

$$= \frac{2}{3}\,\frac{e^{-t}\left[-\sin(2\pi nt/3)-(2\pi n/3)\cos(2\pi nt/3)\right]}{(-1)^2+(2\pi n/3)^2}\Bigg|_0^3$$

(sine term is zero and reverse limits)

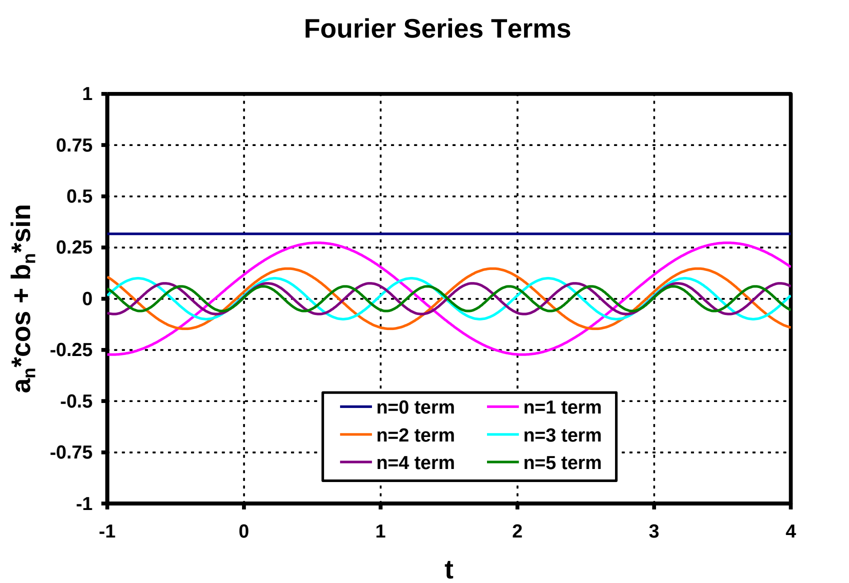
$$= \frac{2}{3}\,\frac{9}{9+4\pi^2 n^2}(2\pi n/3)\left[1-e^{-3}\cos(2\pi n)\right] \qquad \text{(using } \cos(2\pi n)=1\text{)}$$

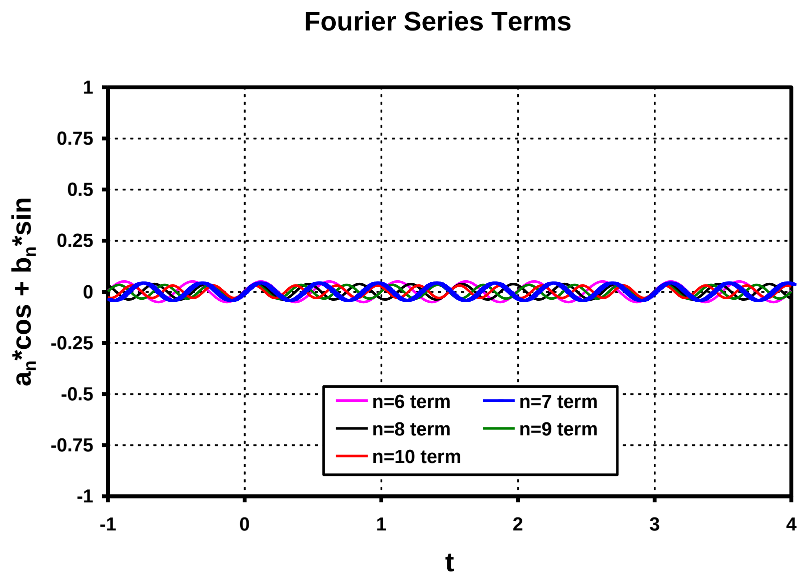
$$= \frac{4\pi n\left(1-e^{-3}\right)}{9+4\pi^2 n^2}$$

$$\therefore \quad g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi nt}{T}\right) + b_n\sin\left(\frac{2\pi nt}{T}\right)\right]$$

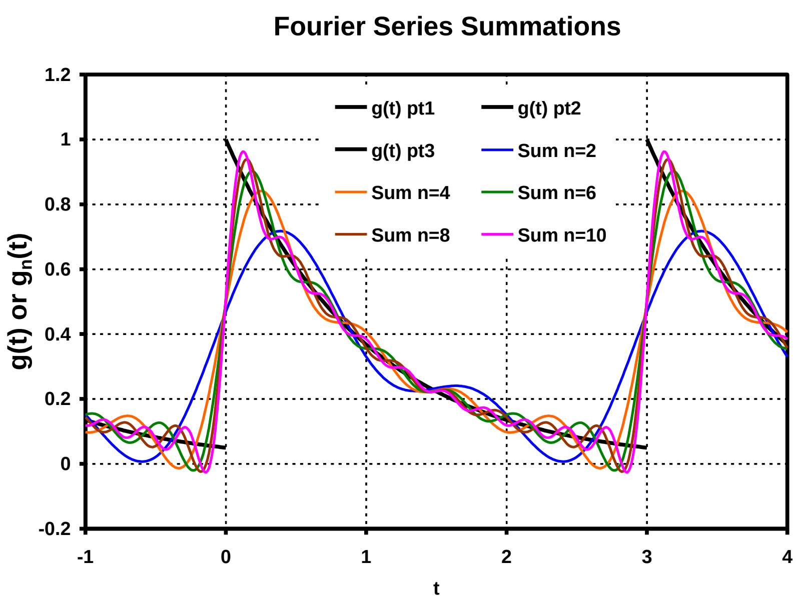
$$= \frac{\left(1-e^{-3}\right)}{3} + \sum_{n=1}^{\infty}\left[\frac{6\left(1-e^{-3}\right)}{9+4\pi^2 n^2}\cos\left(\frac{2\pi nt}{3}\right) + \frac{4\pi n\left(1-e^{-3}\right)}{9+4\pi^2 n^2}\sin\left(\frac{2\pi nt}{3}\right)\right]$$

Plots of the harmonic components:





Plots of the partial summations:



(Note the formation of the Gibbs phenomenon at the discontinuities)

2.) Exponential Form of the Fourier Series

a.) Using $\cos(2\pi nt/T) = \left(e^{2\pi int/T} + e^{-2\pi int/T}\right)/2$ and $\sin(2\pi nt/T) = \left(e^{2\pi int/T} - e^{-2\pi int/T}\right)/2i$, the Fourier Series defined above can be rewritten as

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{(2\pi int/T)} + \left(\frac{a_n + ib_n}{2}\right)e^{(-2\pi int/T)}\right]$$
$$= \sum_{n=-\infty}^{\infty} c_n\, e^{(2\pi int/T)}$$

where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$ for $n \geq +1$ and $c_n = \frac{a_n + ib_n}{2}$ for $n \leq -1$.

This is the exponential form of the Fourier series.

b.) It can be shown that the coefficients c_n can be defined as

$$c_n = \frac{1}{T}\int_d^{d+T} g(t)e^{-2\pi int/T}dt$$

c.) For a real-valued time signal $g(t)$, c_0 is real-valued and c_n ($n = \pm 1, \pm 2, \pm 3, \cdots$) are complex valued.

1.) $\mathrm{Re}(c_n)$ is an even sequence: $\mathrm{Re}(c_n) = \mathrm{Re}(c_{-n})$

2.) $\mathrm{Im}(c_n)$ is an odd sequence: $-\mathrm{Im}(c_n) = \mathrm{Im}(c_{-n})$

Example: $g(t) = e^{-t}, \quad 0 < t < 3$

$$c_n = \frac{1}{T}\int_d^{d+T} g(t)e^{-2\pi int/T}dt = \frac{1}{3}\int_0^3 e^{-t}e^{-2\pi int/3}dt = \frac{1}{3}\int_0^3 e^{-[1+(2\pi in/3)]t}dt$$

Using $\int e^{at}dt = e^{at}/a$ with $a = -\left[1+(2\pi in/3)\right]$,

$$c_n = \frac{1}{3}\,\frac{e^{-\left[1+(2\pi i n/3)\right]t}}{-\left[1+(2\pi i n/3)\right]}\Bigg|_0^3 = \frac{1}{3}\,\frac{1-e^{-(3+2\pi i n)}}{\left[1+(2\pi i n/3)\right]} = \frac{1}{3}\,\frac{1-e^{-3}e^{-2\pi i n}}{\left[1+(2\pi i n/3)\right]}$$

Using $e^{-2\pi i n} = 1$,

$$c_n = \frac{1}{3}\,\frac{1-e^{-3}}{\left[1+(2\pi i n/3)\right]} = \frac{(3-2\pi i n)(1-e^{-3})}{9+4\pi^2 n^2}$$

$$\therefore\quad g(t) = \sum_{n=-\infty}^{\infty} c_n e^{(2\pi i n t/T)} = \sum_{n=-\infty}^{\infty} \frac{(3-2\pi i n)(1-e^{-3})}{9+4\pi^2 n^2}\, e^{(2\pi i n t/3)}$$

3.) Amplitude and Phase Spectra

a.) The coefficients c_n represent the frequency spectrum of the periodic time signal $g(t)$.

1.) The amplitude spectrum A_n of $g(t)$ is defined as

$$A_n = |c_n| = \sqrt{\left[\mathrm{Re}(c_n)\right]^2 + \left[\mathrm{Im}(c_n)\right]^2}$$

2.) The phase spectrum θ_n of $g(t)$ is defined as

$$\theta_n = \arg(c_n) = \tan^{-1}\left[\mathrm{Im}(c_n)/\mathrm{Re}(c_n)\right]$$

b.) It can be shown that for a real-valued time signal $g(t)$:

1.) The amplitude spectrum A_n is an even, non-negative function.

2.) The phase spectrum θ_n is an odd function. (Note: $\theta_0 = 0$)

c.) This is a discrete spectrum with harmonic components having frequencies $f_n = n/T$ ($n = 0, \pm 1, \pm 2, \pm 3, \cdots$).

d.) In general, $A_n \to 0$ as $n \to \pm\infty$.

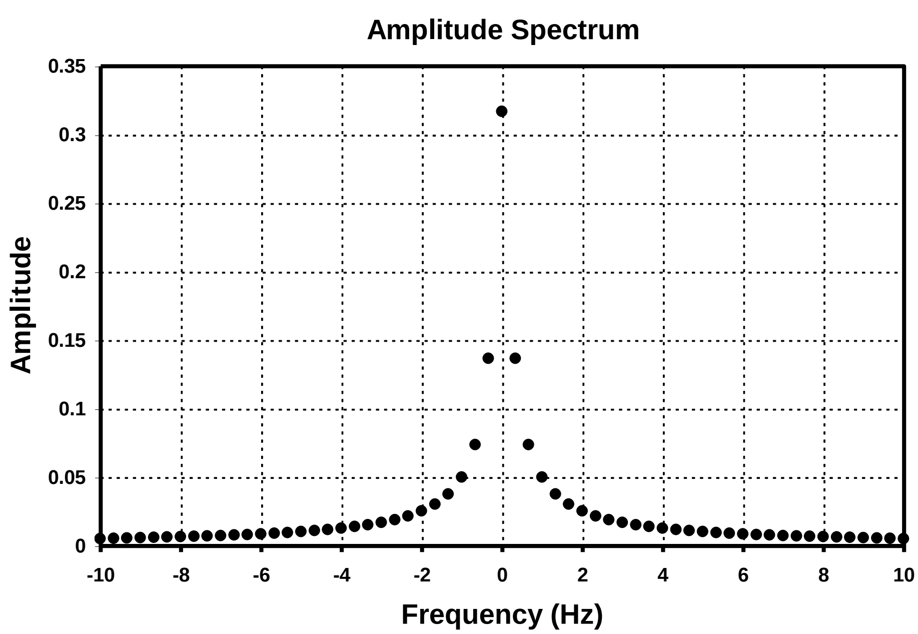
Example: $g(t) = e^{-t}, \quad 0 < t < 3$

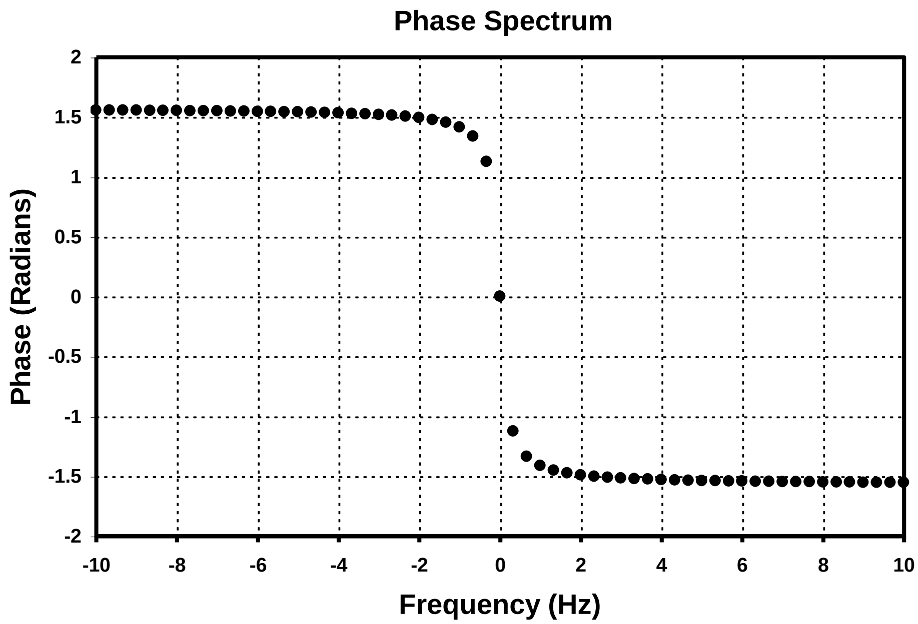
$$c_n = \frac{(3 - 2\pi i n)(1 - e^{-3})}{9 + 4\pi^2 n^2} \rightarrow \mathrm{Re}(c_n) = \frac{3(1 - e^{-3})}{9 + 4\pi^2 n^2} \text{ and } \mathrm{Im}(c_n) = \frac{-2\pi n (1 - e^{-3})}{9 + 4\pi^2 n^2}$$

$$A_n = \sqrt{\left[\frac{3(1 - e^{-3})}{9 + 4\pi^2 n^2}\right]^2 + \left[\frac{-2\pi n (1 - e^{-3})}{9 + 4\pi^2 n^2}\right]^2} = \frac{(1 - e^{-3})}{9 + 4\pi^2 n^2}\sqrt{9 + 4\pi^2 n^2}$$

$$= \frac{(1 - e^{-3})}{\sqrt{9 + 4\pi^2 n^2}}$$

$$\theta_n = \tan^{-1}\left[\frac{-2\pi n (1 - e^{-3}) / (9 + 4\pi^2 n^2)}{3(1 - e^{-3}) / (9 + 4\pi^2 n^2)}\right] = \tan^{-1}(-2\pi n/3)$$




Phase Spectrum
Phase (Radians)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-10
-8
-6
-4
-2
0
2
4
6
8
10
Frequency (Hz)