Earth 460 Classnotes (Set 7)

III.) Wave Equation and Basic Propagation Theory

A.) Vibrating String (Simple One-Dimensional Wave Equation)

1.) When a string is deflected from its equilibrium position, tension in the string acts to restore the string back to its original shape.

2.) Using a version of Newton's second law of motion (i.e., $\mathbf{F} = m\mathbf{a}$), we can relate the tension in a string element to its motion.

3.) The result of this derivation is the one-dimensional wave equation:

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

where $\psi(x,t)$ is the string displacement and v is the wave velocity.

Note: the wave velocity v is dependent on the physical properties of the string; $v = \tau/\mu$ where τ is the tension and μ is the string mass per unit length (the one-dimensional equivalent of density).

4.) The general solution of the wave equation (called the d'Alembert solution) is $\psi(x,t) = \psi_1(x - vt) + \psi_2(x + vt)$.

a.) $\psi_1(x - vt)$ is the disturbance moving in the $+x$ direction.

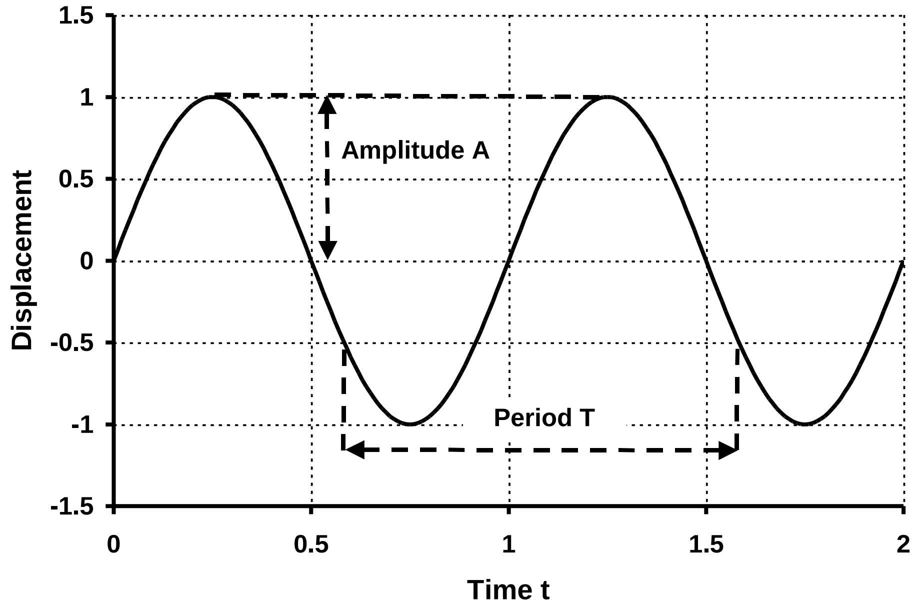
b.) $\psi_2(x + vt)$ is the disturbance moving in the $-x$ direction.

c.) ψ_1 and ψ_2 can have arbitrary functional forms

5.) Of particular interest is the harmonic solution given by $\psi(x,t) = A\cos\left[(2\pi/\lambda)(x - vt)\right]$ (a wave form with a single frequency component).

a.) A is the wave amplitude.

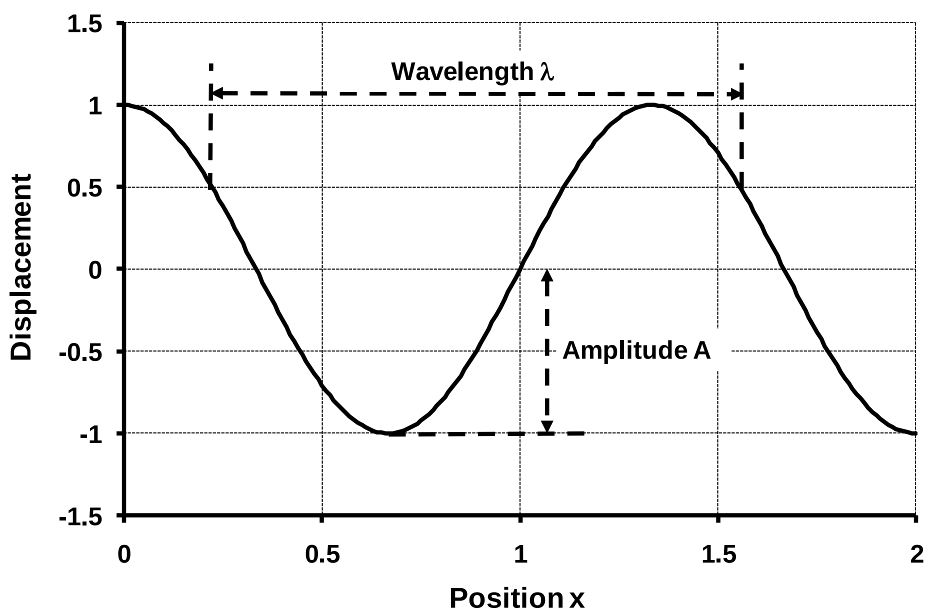
b.) Observing the temporal variations of the wave at a fixed spatial location.



1.) The harmonic waveform is repeated with period T

2.) The frequency $f = 1/T$ is the number of successive waveforms that pass the observation point per unit time.

c.) Looking at the spatial variations of the wave at a fixed time.



1.) The wavelength λ is the distance between successive repetition of the waveform.

2.) The wave number k is the number of waveforms per unit length.

d.) Conversion between cyclic and angular quantities

1.) Angular frequency: $\omega = 2\pi/T = 2\pi f$

2.) Angular wave number $\kappa = 2\pi/\lambda = 2\pi k$

e.) The argument of the cosine waveform (i.e., $(2\pi/\lambda)(x - vt) = (\kappa x - \omega t)$) is called the phase. As the wave propagates, phase can be used as follows:

1.) Determine the change in phase over time at a fixed observation point x.

2.) Observe the spatial movement with time t of the waveform point specified by $(\kappa x - \omega t) = \text{constant}$.

B.) Elastic Wave Equation (Seismic waves)

1.) We will consider seismic wave propagation in a homogeneous, isotropic, perfectly elastic medium.

2.) The elastic wave equation is obtained by analyzing the behavior of a volume element in the medium. Using the elastic constitutive relationships and a version of Newton's second law of motion, we get the following expressions for the particle displacement $\mathbf{u}(\mathbf{x},t)$

$$\rho \frac{\partial^2 u_i(\mathbf{x},t)}{\partial t^2} = (k + \mu/3)\frac{\partial \theta(\mathbf{x},t)}{\partial x_i} + \mu \nabla^2 u_i(\mathbf{x},t) \qquad i = 1, 2, 3$$

$$\text{where } \theta(\mathbf{x},t) = \frac{\partial u_1(\mathbf{x},t)}{\partial x_1} + \frac{\partial u_2(\mathbf{x},t)}{\partial x_2} + \frac{\partial u_3(\mathbf{x},t)}{\partial x_3},$$

$$\nabla^2 u_i(\mathbf{x},t) = \frac{\partial^2 u_i(\mathbf{x},t)}{\partial x_1^2} + \frac{\partial^2 u_i(\mathbf{x},t)}{\partial x_2^2} + \frac{\partial^2 u_i(\mathbf{x},t)}{\partial x_3^2},$$

$u_i(\mathbf{x},t)$ = component of particle displacement in x_i direction,

k and μ are bulk and shear moduli, respectively, and ρ is density.

3.) This results can be separated into two distinct body wave types

a.) Compressional or P-wave

1.) This wave type involves the propagation of a cubical dilatation (i.e., volumetric change) – pressure event.

2.) P-wave equation:

$$\frac{\partial^2\theta(\mathbf{x},t)}{\partial t^2} = \alpha^2\ \nabla^2\theta(\mathbf{x},t)$$

where $\alpha = \sqrt{(k+4\mu/3)/\rho}$ is the P-wave velocity.

3.) Particle displacement is parallel to the direction of wave propagation.

b.) Shear or S-wave

1.) This wave type is a pure rotational disturbance involving pure shear stress and strain; there is no volumetric change.

2.) S-wave equation:

$$\frac{\partial^2\left(\bar{\nabla}\times\mathbf{u}\right)}{\partial t^2} = \beta^2\ \nabla^2\left(\bar{\nabla}\times\mathbf{u}\right)$$

where $\beta = \sqrt{\mu/\rho}$ is the S-wave velocity.

3.) Particle displacement is perpendicular to the direction of wave propagation.

C.) Electromagnetic (EM) Wave Equation (Radio waves / GPR)

1.) We will consider radio wave propagation in a homogeneous, isotropic, perfectly insulating (i.e., non-conducting) medium.

2.) Using Maxwell's equations and electrical constitutive relationships, we obtain the following equation for the coupled electrical and magnetic fields ($\mathbf{E}$ and $\mathbf{B}$, respectively).

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = c^2\ \ \nabla^2\mathbf{E} \text{ and } \frac{\partial^2\mathbf{B}}{\partial t^2} = c^2\ \ \nabla^2\mathbf{B}$$

where $c = 1/\sqrt{\mu\varepsilon}$ is the EM wave velocity,

ε is the dielectric permittivity and μ is the magnetic permeability.

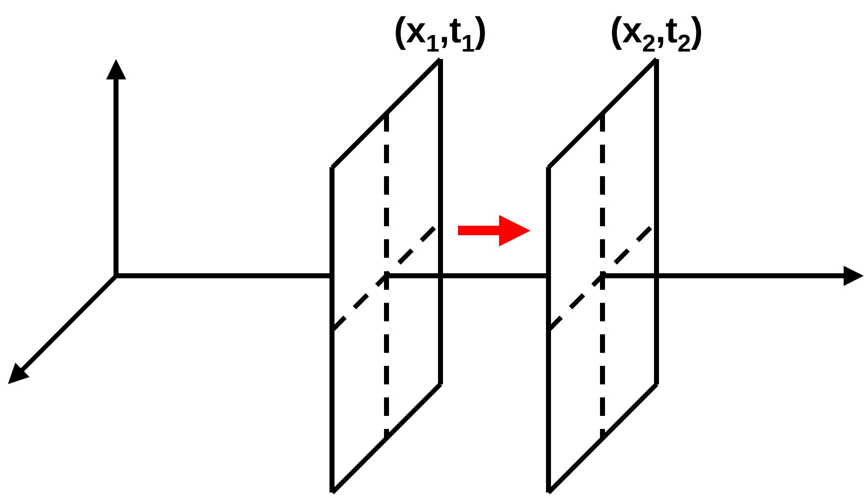
D.) Plane Wave Solution

1.) We will consider a wave propagating in the $+x$ direction. The disturbance ψ is a function of x and t ; ψ is independent of y and z.

2.) The three-dimensional wave equation reduces to the one-dimensional wave equation $\frac{\partial^2\psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2\psi(x,t)}{\partial x^2}$ with the general solution $\psi(x,t) = f(x - vt)$.

3.) The quantity $x - vt$ is the phase for this wave. The surface comprising the points satisfying the relationship $x - vt = \text{constant}$ is a called a wavefront.

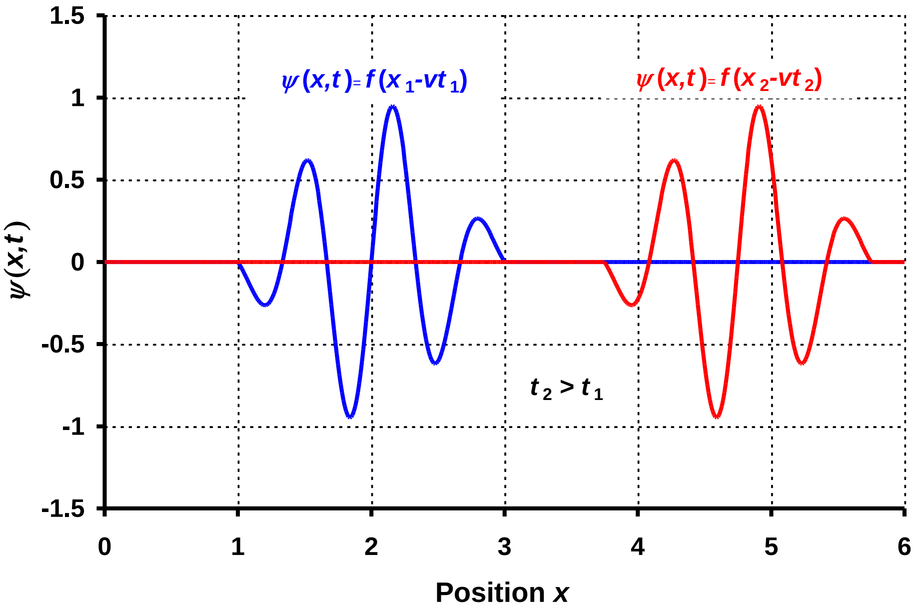
4.) For this case, $x - vt = \text{constant}$ defines a planar wavefront perpendicular to the x axis. As t increases, this plane travels in the $+x$ direction.



$$x_1 - v\,t_1 = x_2 - v\,t_2 = \text{constant}$$

$$t_2 > t_1 \rightarrow x_2 > x_1$$

5.) The shape of the disturbance does not change as it propagates, just translates along the $+x$ direction.



6.) The amplitude of the wave does not vary as it propagates.

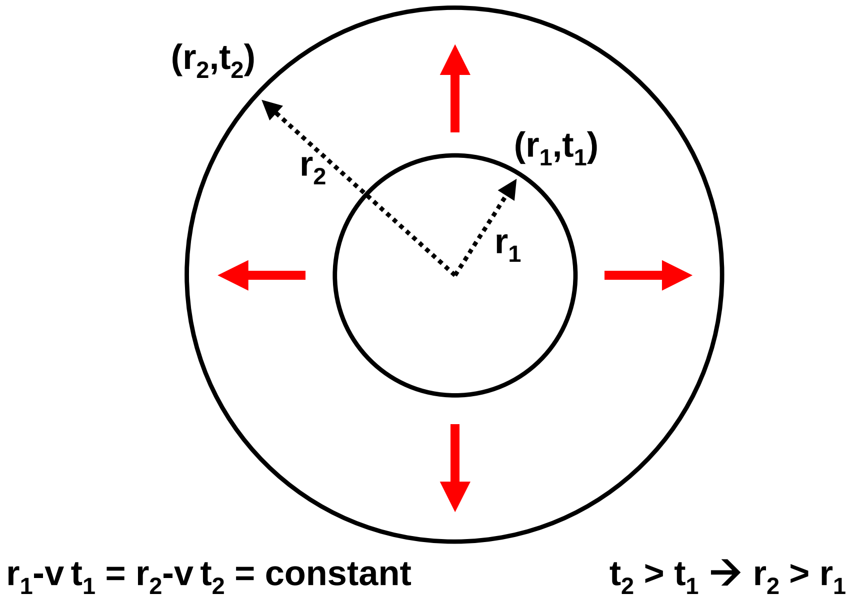
E.) Spherical Wave Solution

1.) Let us consider a wave propagating uniformly outwards from a point source.

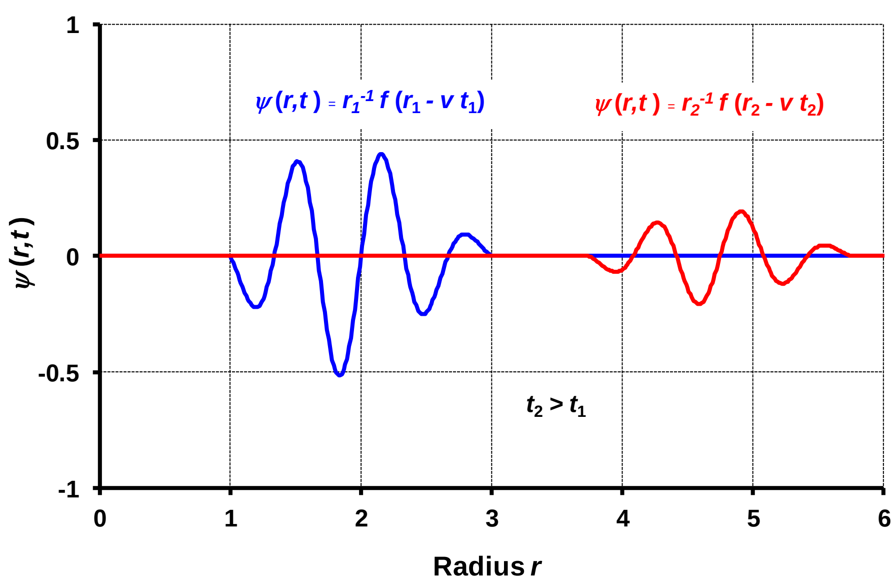
2.) The wave equation for this event can be rewritten in spherical coordinates as $\frac{\partial^2 \psi(r,t)}{\partial t^2} = \frac{v^2}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi(r,t)}{\partial r} \right)$

3.) The solution for this outward propagating wave is $\psi(r,t) = \frac{1}{r} f(r - vt)$ where $(r - vt)$ is the phase for this wave.

4.) The wave front $(r - vt) = \text{constant}$ is a spherical surface with radius r about the point source location. As t increases, the spherical surface expands outwards.



6.) Its amplitude is scaled by a factor r^{-1} as the wave propagates outward.



F.) Signal Amplitude and Gain Functions

1.) Geometrical Spreading / Spherical Divergence

a.) Local wave amplitude on a wave front is related to local energy density ξ.

b.) The total energy E is obtained by a surface integration of the energy density along the wave front. If the energy density is uniform over the wave front, then $E = \xi \cdot \tilde{A}$ where $\tilde{A}$ is the surface area of the wave front.

c.) For a non-lossy medium (i.e., no conversion of wave energy to heat), the total energy remains constant as the wave propagates.

d.) Consider a spherical wavefront propagating outward from a point source in a non-lossy medium. The total energy is given by $E = \xi\left(4\pi r^2\right)$.

1.) As the wave front propagates outward from radius r_1 to $r_2\left(> r_1\right)$,

$$E = \xi_1\left(4\pi r_1^2\right) = \xi_2\left(4\pi r_2^2\right) \quad \Rightarrow \quad \frac{\xi_2}{\xi_1} = \frac{r_1^2}{r_2^2}$$

. Hence, energy density decreases as r^{-2}.

2.) The wave amplitude $A \propto \sqrt{\xi}$. This relationship implies $\frac{A_2}{A_1} = \frac{r_1}{r_2}$.

Hence amplitude decreases as r^{-1}.

e.) This decrease in energy density and amplitude due to changing wave front surface area is called geometrical spreading or spherical divergence.

Note: there is no such effect for plane waves because their surface area remains constant during propagation.

2.) Attenuation / Absorption

a.) In a lossy medium, wave energy is irreversibly converted into heat energy as the wave propagates.

b.) This conversion decreases the energy density along the wave front independent of geometrical spreading effects.

c.) Absorption mechanisms change the form of the wave equation. For example, the EM wave equation in a conductive medium is

$$\nabla^2 \mathbf{E} = \varepsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} + \sigma \mu \frac{\partial \mathbf{E}}{\partial t} \text{ and } \nabla^2 \mathbf{B} = \varepsilon \mu \frac{\partial^2 \mathbf{B}}{\partial t^2} + \sigma \mu \frac{\partial \mathbf{B}}{\partial t}$$

where σ is the electrical conductivity of the medium. The $\partial/\partial t$ term is ohmic loss term due to the occurrence of conductive currents.

d.) The effects of absorption on wave amplitude are manifested as an exponential scaling term. For a plane wave,

$$A(x) = A_0 \, e^{-\alpha x}$$

where x is the distance of wave travel, A_0 is the wave amplitude at $x = 0$, $A(x)$ is the wave amplitude after propagating distance x and α is the absorption coefficient.

e.) Theoretical formulations and experimental evidence suggest that $\alpha \propto f$. This linear dependence implies that absorption increases with frequency and leads to a preferential loss of the higher frequency signal component with propagation distance.

1.) For an EM wave in a conductive medium,

$$\alpha = \omega \left\{ \left(\frac{\mu \varepsilon}{2} \right) \left[\left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2} \right)^{1/2} - 1 \right] \right\}^{1/2} \text{ where } \omega = 2\pi f .$$

2.) For a seismic wave in an anelastic medium, $\alpha = \omega/(2vQ)$ where v is the wave velocity and Q is the quality factor.

f.) The relative importance of geometrical spreading and absorption on wave propagation vary with frequency and distance. As frequency and or distance increase, the relative importance of absorption increases and becomes dominant over spreading.

3.) Other Factors that Affect Wave Amplitude

a.) Transmission losses at interfaces

b.) Scattering due to heterogeneities

c.) Interference with other waves (e.g., multiples)

Note:

Changes in signal amplitude and energy due to physical processes (such as these above) or filtering are sometimes expressed in terms decibels (dB).

A decibel is a logarithmic unit that quantifies the ratio of two values of a physical parameter.

For amplitude related values, such as voltages and currents:

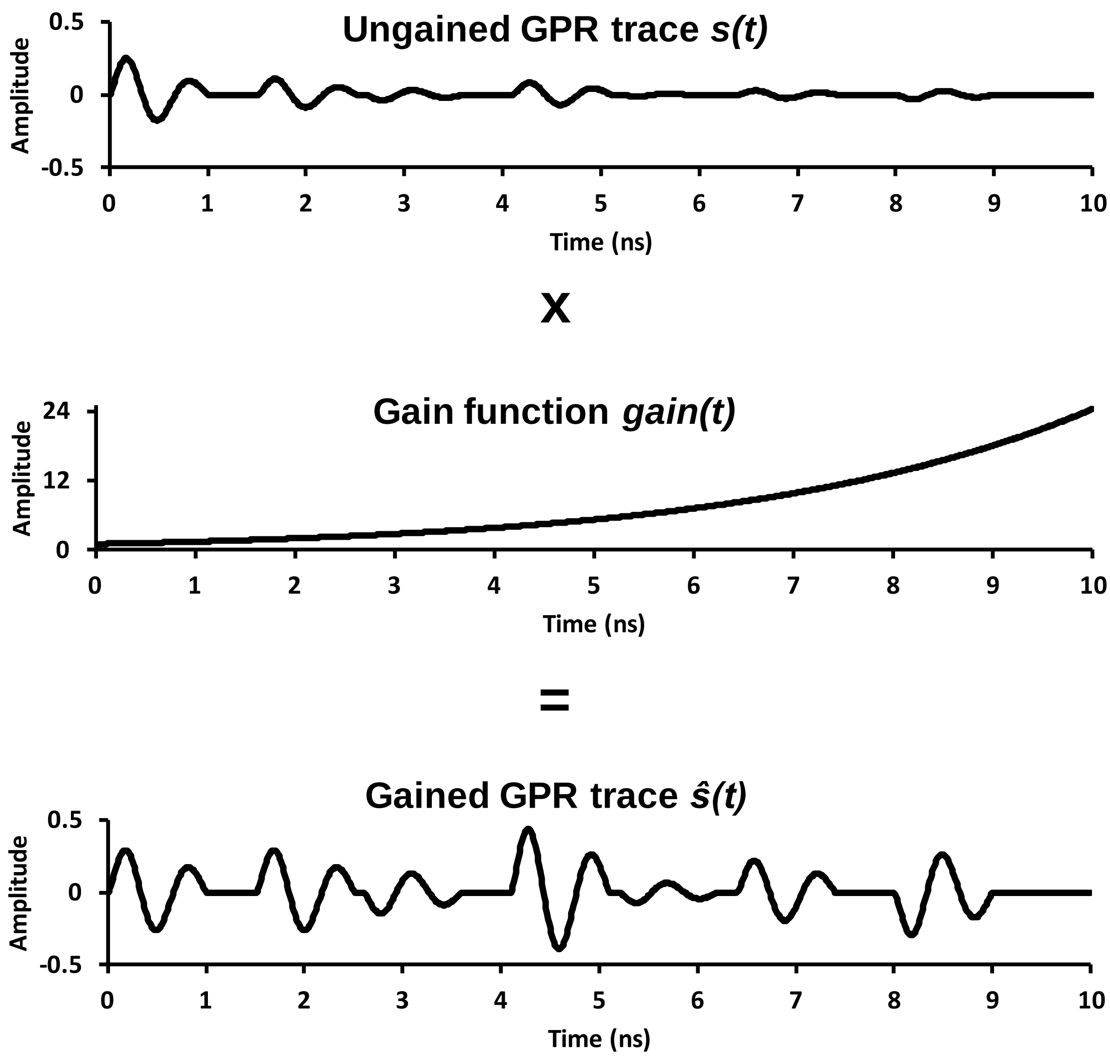
$$\left\{\frac{v_2}{v_1}\text{ in dB}\right\} = 20\,\log_{10}\left(\frac{v_2}{v_1}\right)$$

For energy/power related values, such as watts: $\left\{\frac{v_2}{v_1}\text{ in dB}\right\} = 10\,\log_{10}\left(\frac{v_2}{v_1}\right)$

4.) Gain Functions

a.) Due to the mechanisms above, wave amplitude decreases with increasing traveltime.

b.) To compensate for this decrease, a time-varying scaling called a gain function is applied to seismic and GPR data (i.e., $\hat{s}(t) = gain(t) \times s(t)$).



c.) Reasons for applying gain functions:

1.) Data display

2.) Analysis of mechanisms affecting signal amplitude d.) Types of gain functions

1.) Gain functions based on physical mechanisms

a.) Spherical divergence correction accounts for geometrical spreading effects. To compensate for spherical divergence, a linear factor is used (i.e., $gain(t) \propto t$).

b.) Also accounts for large-scale absorption effects using an exponential factor (i.e., spherical exponential correction) (i.e., $gain(t) \propto e^{\alpha t}$).

c.) A gain function incorporating both of these effects is commonly referred to as spherical exponential compensation (SEC) gain.

d.) This type of gain function preserves relative amplitude, allowing comparison of interface reflection strength (e.g., seismic bright spots associated with nature gas fields and strong water table reflections on GPR profiles).

e.) Undesired events, such as surface waves and random noise, do not undergo the spherical divergence and absorption experienced by reflected body waves. This can lead to excessive amplification of these events by spherical divergence / exponential corrections.

2.) Gain functions based on record signal strength

a.) Automatic gain control (AGC) gain functions adjust signal strength in a specified window to a preset maximum level.

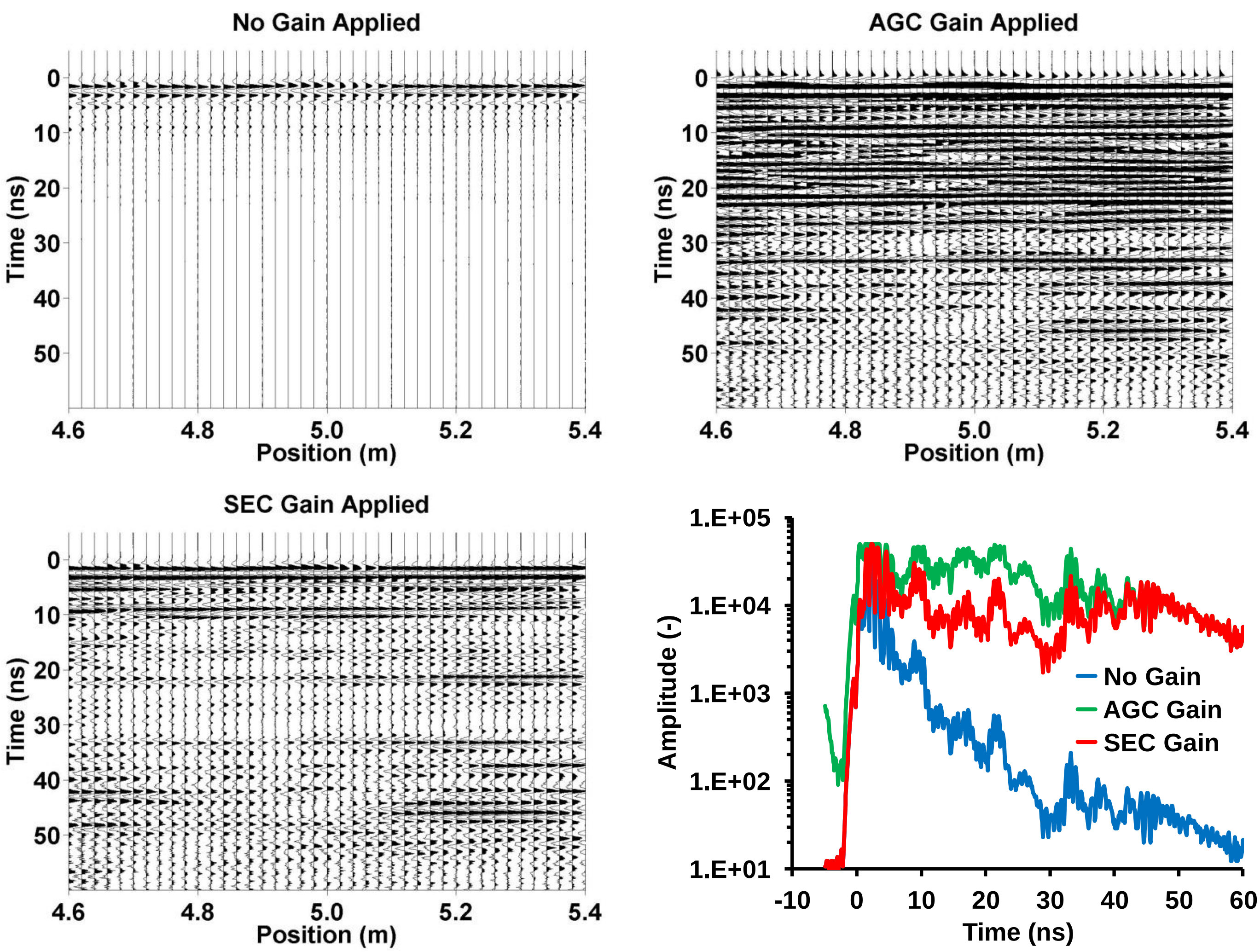
b.) Different window lengths produce different AGC gain functions.

c.) Since AGC gain is not based on physical mechanisms, it does not conserve relative amplitude information.

d.) AGC gain treats reflections and undesired events the same; hence, excessive amplification of undesired signals can occur.

Example: Application of AGC and SEC gain functions to GPR reflection profiling data.

Data: GPR 900 MHz reflection profiling data from soil moisture monitoring project (Site A, Smith Farm, Waterloo Moraine, June 18, 2008)



G.) Huygens' Principle

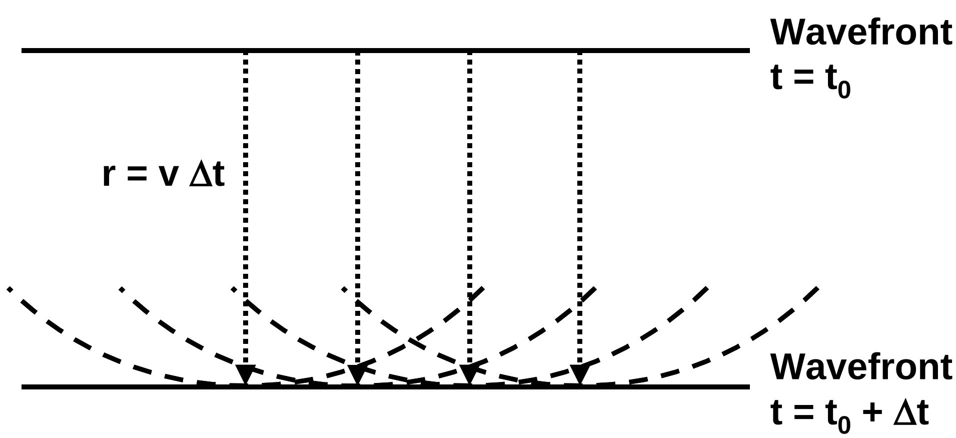
1.) Huygens' principle states that every point on a wavefront can be regarded as new sources for generating future wavefronts.

2.) Consider a wavefront at a time t_0. The wavefront at time $t_0 + \Delta t$ is constructed using Huygens' principle in the following manner.

a.) During the interval Δt, the wave energy propagates a distance $v\,\Delta t$ where v is the velocity of the medium. (Note: v may vary spatially).

b.) At all points along the original wavefront, arcs with radius $v\,\Delta t$ represent the wave propagation from these point sources.

c.) The wavefront at time $t_0 + \Delta t$ is the envelope formed by the arcs where constructive summation of the elemental wavelets occurs. Elsewhere, these elemental wavelets destructively interfere with each other.



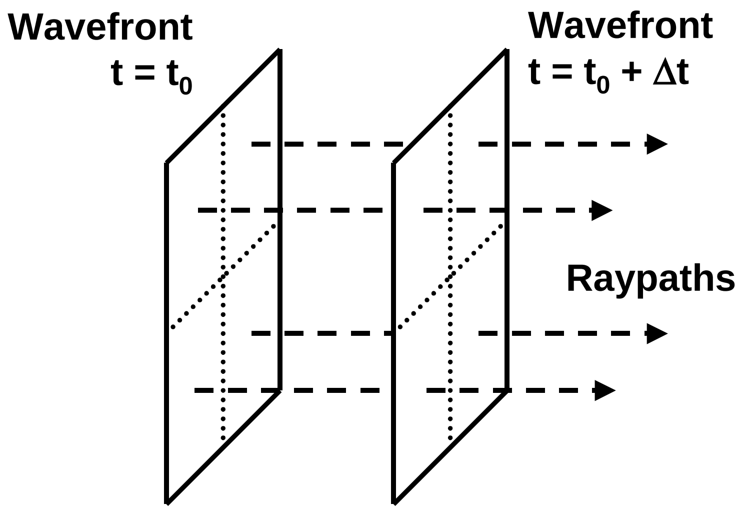
H.) Raypaths

1.) We can consider each point on a wavefront as a small packet of wave energy that travels along with the wave as it propagates through a medium.

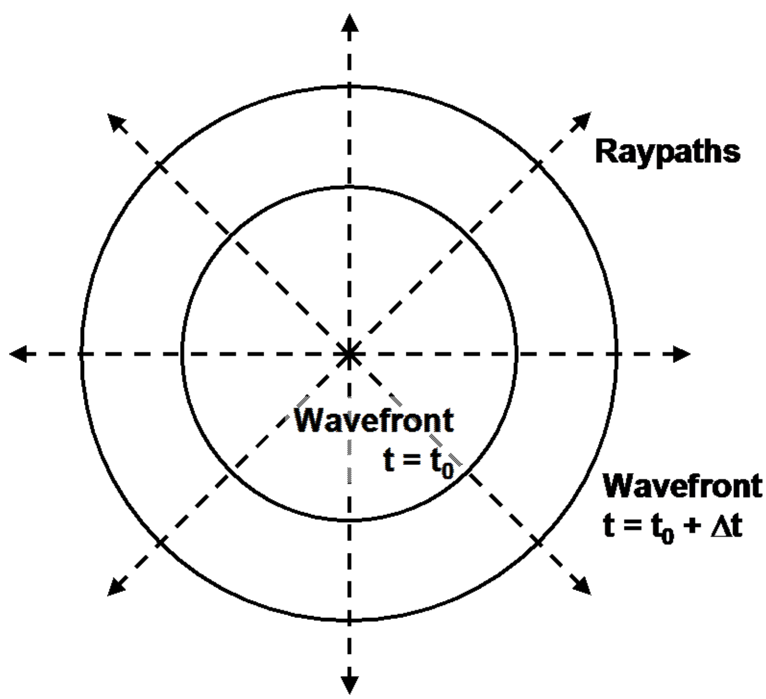
2.) The line traced by packet as it moves over time is a raypath.

3.) For an isotropic medium, the raypath is perpendicular to a wavefront at their point of intersection.

a.) For a plane wave, raypaths are parallel and oriented in the direction of wave propagation.



b.) Raypaths for a spherical wave extend out radially from the point source.



3.) A mathematical basis for the concept of raypaths (i.e., geometrical ray theory) is a version of the wave equation called the eikonal equation.

Earth 460 Classnotes (Set 7)

I.) Fermat's Principle

1.) Fermat's principle governs the geometry of raypaths.

2.) Fermat's principle can be derived from the eikonal equation.

3.) Let us consider the family of potential raypaths between two fixed points.

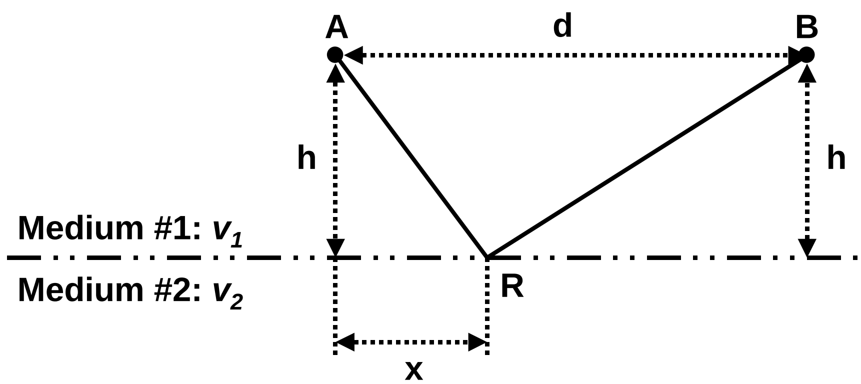
a.) The actual raypath between these two points is the path of stationary time.

b.) This conditions means that the actual raypath corresponds to either a minimum or maximum traveltime path for the potential raypath family.

c.) Because the actual raypath frequently corresponds to a minimum traveltime path, Fermat's principle is commonly referred to as the *Principle of Least Time*.

Example:

Consider the reflection of a raypath, assuming no mode conversion.



Traveltime for any reflected raypath between the fixed points A and B:

$$t_{AB} = \frac{1}{v_1}\left(l_{AR} + l_{RB}\right) = \frac{1}{v_1}\left\{\left[x^2 + h^2\right]^{1/2} + \left[\left(d-x\right)^2 + h^2\right]^{1/2}\right\}.$$

The stationary path solves the equation

$$\frac{d}{dx}t_{AB} = \frac{1}{v_1}\left\{x\left[x^2 + h^2\right]^{-1/2} - \left(d-x\right)\left[\left(d-x\right)^2 + h^2\right]^{-1/2}\right\} = 0.$$

Hence, $x\left[x^2+h^2\right]^{-1/2}=(d-x)\left[(d-x)^2+h^2\right]^{-1/2}$ →

$x^2\left[(d-x)^2+h^2\right]=(d-x)^2\left[x^2+h^2\right]$ → $x=d/2$.

The nature of this stationary path is determined by evaluating the second derivative: $\frac{d^2}{dx^2}t_{AB}=\frac{h^2}{v_1}\left\{\left[x^2+h^2\right]^{-3/2}+\left[(d-x)^2+h^2\right]^{-3/2}\right\}$. For $x=d/2$,

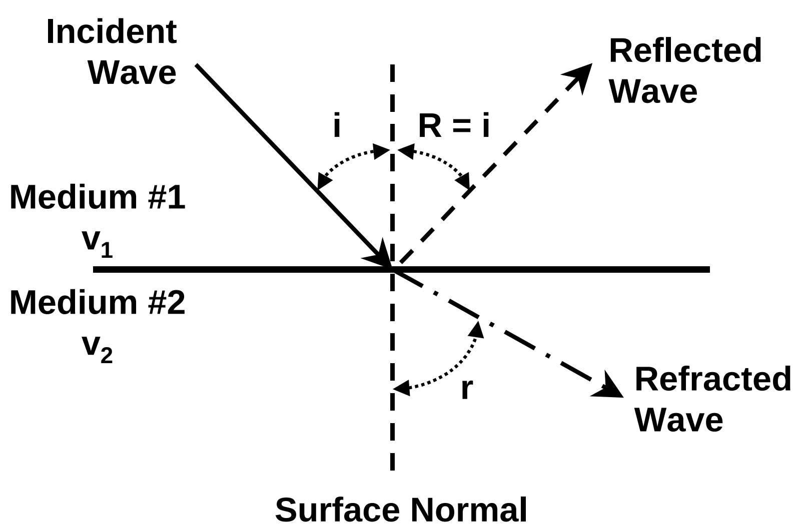
$\frac{d^2}{dx^2}t_{AB}>0$ → this solution is a minimum time travelpath.

J.) Snell's Law

1.) When a wave encounters an interface between two medium with contrasting physical properties (e.g., boundary between two lithologic units), some of the wave energy is reflected back into the incident medium; the remained is transmitted / refracted into the second medium.

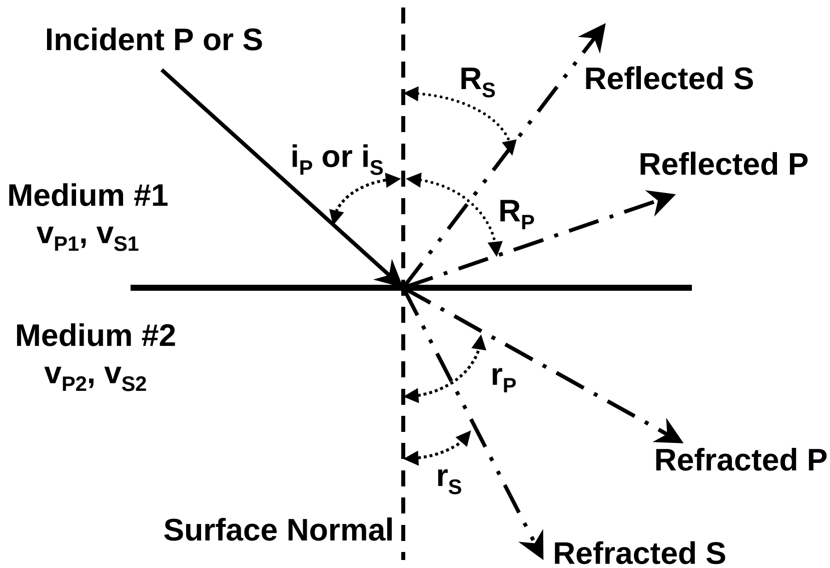
2.) The orientation of the raypaths for the incident, reflected and refracted waves are related by Snell's law.

3.) For the case where there is no mode conversion (i.e., acoustic waves, SH waves, EM waves):



$$\frac{\sin i}{v_1}=\frac{\sin R}{v_1}=\frac{\sin r}{v_2}$$

4.) For the case where there is mode conversion (i.e., elastic waves):



$$\left.\begin{array}{ll}\text{Incident P-wave:} & \dfrac{\sin i_P}{v_{P1}} \\ \text{Incident S-wave:} & \dfrac{\sin i_S}{v_{S1}}\end{array}\right\} = \frac{\sin R_P}{v_{P1}} = \frac{\sin R_S}{v_{S1}} = \frac{\sin r_P}{v_{P2}} = \frac{\sin r_S}{v_{S2}}$$

5.) The quantity $\dfrac{\sin\theta}{v}$ is identical for incident, reflected and refracted rays.

a.) This quantity $1/v$ is the *slowness*, the time it takes a wave to travel a unit distance.

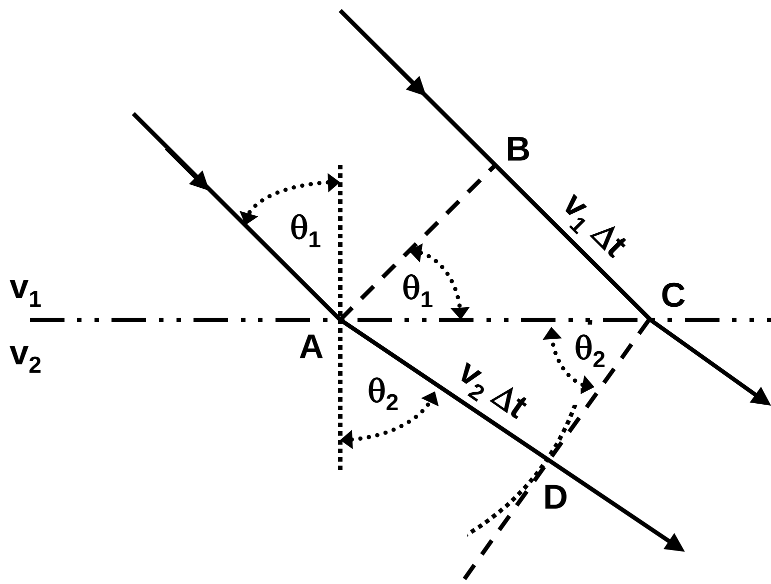
b.) $\dfrac{\sin\theta}{v}$ is the component of slowness parallel to the interface. Snell's Law implies that this component is conserved during reflection and refraction.

c.) The quantity $p = \dfrac{\sin\theta}{v}$ is called the ray parameter.

6.) Snell's Law can be derived using either Huygens' or Fermat's principle.

Example:

Using Huygens' Principle to show Snell's Law for refracted rays.



Consider two rays associated with an incident plane wave.

After the first ray reaches the interface at point A, the second ray needs to travel an additional time Δt and corresponding distance $BC = v_1 \Delta t$.

According to Huygens' Principle, the intersection point A is a point source for wave propagation into the refracting medium. This energy travels a distance $v_2 \Delta t$ while the second ray travels to the interface. The tangent line to this arc CD defines the wavefront for the refracted plane wave.

Using the right triangles ABC and ACD, we can obtain the two relationships $v_1 \Delta t = AC \sin\theta_1$ and $v_2 \Delta t = AC \sin\theta_2$

Solving for the horizontal component of the slowness gives $\dfrac{\Delta t}{AC} = \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.