

Earth 460 Classnotes (Set 1)

I.) Time Series Analysis – Continuous Signals

A.) Complex Numbers (A Brief Review)

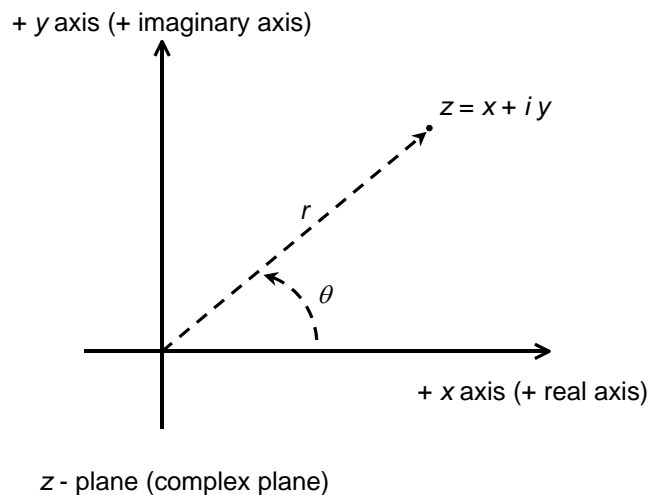
1.) $z = x + iy$ is a complex number where x and y are real numbers and

$$i = \sqrt{-1}.$$

a.) $x = \text{Re}(z)$ is the real part of z .

b.) $y = \text{Im}(z)$ is the imaginary part of z .

2.) Geometrical (polar) representation of complex numbers



a.) $r = \sqrt{x^2 + y^2}$ is the absolute value or modulus of z (i.e.,
 $r = \text{mod}(z) = |z|$).

b.) $\theta = \tan^{-1}(y/x)$ is the argument or phase modulus of z (i.e., $\theta = \arg(z)$). θ is positive in the counterclockwise direction with respect to the $+x$ (positive real) axis.

c.) From the geometry, the real and imaginary parts of z can be expressed as $x = r \cos \theta$ and $y = r \sin \theta$. Hence, $z = r (\cos \theta + i \sin \theta)$.

3.) Complex conjugate of z

a.) $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$.

b.) $z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = r^2$ (a real-valued quantity)

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c.) Rationalizing the complex fraction (u/v) where $u = a + ib$ and

$$v = c + id:$$

$$\frac{u}{v} = \frac{a + ib}{c + id} = \frac{u\bar{v}}{v\bar{v}} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

This is the standard procedure for complex fractions.

$$\text{Example: } \frac{2 + i3}{1 - i} = \frac{(2 + i3)(1 + i)}{(1 - i)(1 + i)} = \frac{(2 \cdot 1 + 3 \cdot (-1)) + i(3 \cdot 1 + 2 \cdot (-1))}{1^2 + (-1)^2} = \frac{-1 + i5}{2}$$

4.) Complex exponential function

a.) $e^{i\theta} = \cos\theta + i\sin\theta$ where θ is a real number.

i.) $|e^{i\theta}| = 1$

ii.) $e^{in\pi} = (-1)^n$ for $n = 0, \pm 1, \pm 2, \pm 3, \dots$.

iii.) $e^{i\pi/2} = i$

iv.) $z = r e^{i\theta}$

v.) Combined with $e^{-i\theta} = \cos\theta - i\sin\theta$, it can be shown that

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2 \text{ and } \sin\theta = (e^{i\theta} - e^{-i\theta})/2i.$$

b.) For e^z where z is a complex number:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

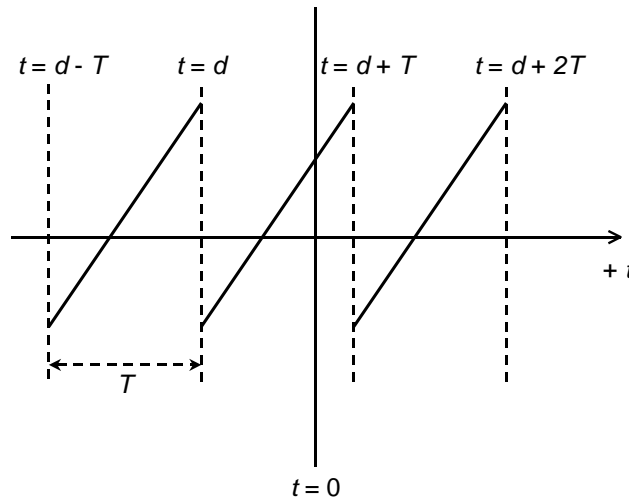
$$\text{Example: } e^{\pi(1+i)} = e^\pi e^{i\pi} = e^\pi (\cos \pi + i \sin \pi) = -e^\pi$$

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B.) Spectral Analysis - Fourier Series

1.) Definition

- a.) A Fourier series can be obtained for any periodic function $g(t)$ that satisfies given conditions.
- i.) $g(t)$ is a periodic functions with period T if $g(t) = g(t + nT)$ where $n = 0, \pm 1, \pm 2, \pm 3, \dots$.



- ii.) $g(t)$ has a finite number of extrema points and discontinuities over a single period.
- iii.) $\int_{-T/2}^{T/2} |g(t)| dt < \infty$.
- b.) The Fourier series of $g(t)$ is a summation of harmonic function given by

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right) \right]$$

- i.) $a_0/2$ is the DC bias term (corresponding to frequency $f = 0$).

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ii.) $\left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right]$ is the harmonic component

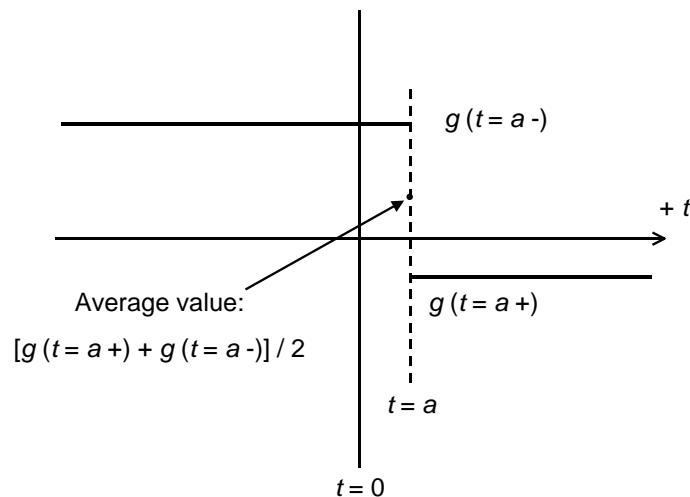
corresponding to the frequency $f_n = n/T$ ($n = 1, 2, 3, \dots$). The magnitude of this term is a measure of its relative information content about $g(t)$.

c.) Using a mathematical property called orthogonality, the coefficients of the Fourier series can be shown to be the following:

$$a_n = \frac{2}{T} \int_d^{d+T} g(t) \cos\left(\frac{2\pi nt}{T}\right) dt, \text{ with } a_0 = \frac{2}{T} \int_d^{d+T} g(t) dt$$

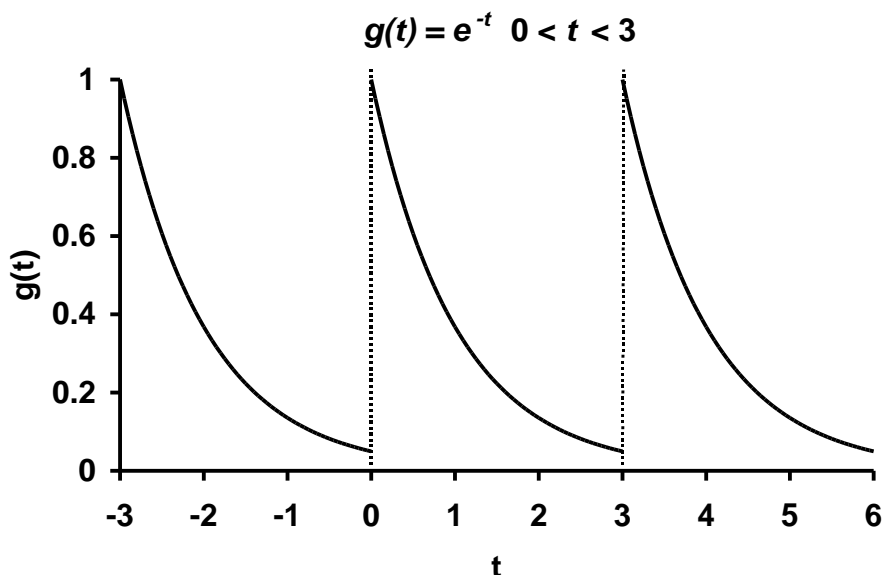
$$b_n = \frac{2}{T} \int_d^{d+T} g(t) \sin\left(\frac{2\pi nt}{T}\right) dt$$

d.) The Fourier series expansion converges to $g(t)$ where $g(t)$ is continuous and to the average of the right and left hand limits at the discontinuities.



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Example: $g(t) = e^{-t}$, $0 < t < 3$



$$a_0 = \frac{2}{T} \int_d^{d+T} g(t) dt = \frac{2}{3} \int_0^3 e^{-t} dt \quad (\text{using } \int e^{at} dt = e^{at}/a \text{ with } a = -1)$$

$$= \frac{2e^{-t}}{-3} \Big|_0^3 = 2(1 - e^{-3})/3$$

$$a_n = \frac{2}{T} \int_d^{d+T} g(t) \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{2}{3} \int_0^3 e^{-t} \cos\left(\frac{2\pi nt}{3}\right) dt$$

$$(\text{using } \int e^{at} \cos bt \, dt = \frac{e^{at}(a \cos bt + b \sin bt)}{a^2 + b^2} \text{ with } a = -1 \text{ and } b = 2\pi n/3)$$

$$= \frac{2}{3} \frac{e^{-t} [-\cos(2\pi nt/3) + (2\pi n/3) \sin(2\pi nt/3)]}{(-1)^2 + (2\pi n/3)^2} \Big|_0^3$$

(sine term is zero and reverse limits)

$$= \frac{2}{3} \frac{9}{9 + 4\pi^2 n^2} [1 - e^{-3} \cos(2\pi n)] \quad (\text{using } \cos(2\pi n) = 1)$$

$$= \frac{6(1 - e^{-3})}{9 + 4\pi^2 n^2}$$

$$b_n = \frac{2}{T} \int_d^{d+T} g(t) \sin\left(\frac{2\pi nt}{T}\right) dt = \frac{2}{3} \int_0^3 e^{-t} \sin\left(\frac{2\pi nt}{3}\right) dt$$

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$$\left(\text{using } \int e^{at} \sin bt \, dt = \frac{e^{at}(a \sin bt - b \cos bt)}{a^2 + b^2} \text{ with } a = -1 \text{ and } b = 2\pi n/3\right)$$

$$= \frac{2}{3} \frac{e^{-t} \left[-\sin(2\pi nt/3) - (2\pi n/3) \cos(2\pi nt/3) \right]}{(-1)^2 + (2\pi n/3)^2} \Bigg|_0^3$$

(sine term is zero and reverse limits)

$$= \frac{2}{3} \frac{9}{9 + 4\pi^2 n^2} (2\pi n/3) [1 - e^{-3} \cos(2\pi n)] \quad (\text{using } \cos(2\pi n) = 1)$$

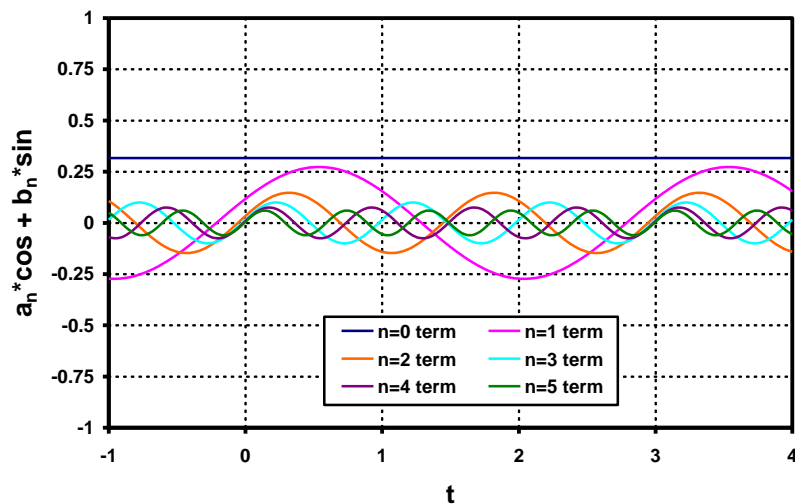
$$= \frac{4\pi n(1 - e^{-3})}{9 + 4\pi^2 n^2}$$

$$\therefore g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right]$$

$$= \frac{(1 - e^{-3})}{3} + \sum_{n=1}^{\infty} \left[\frac{6(1 - e^{-3})}{9 + 4\pi^2 n^2} \cos\left(\frac{2\pi nt}{3}\right) + \frac{4\pi n(1 - e^{-3})}{9 + 4\pi^2 n^2} \sin\left(\frac{2\pi nt}{3}\right) \right]$$

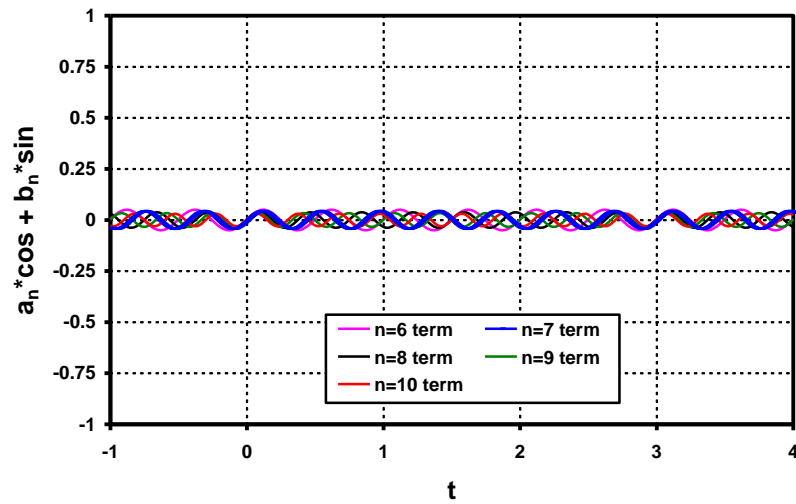
Plots of the harmonic components:

Fourier Series Terms



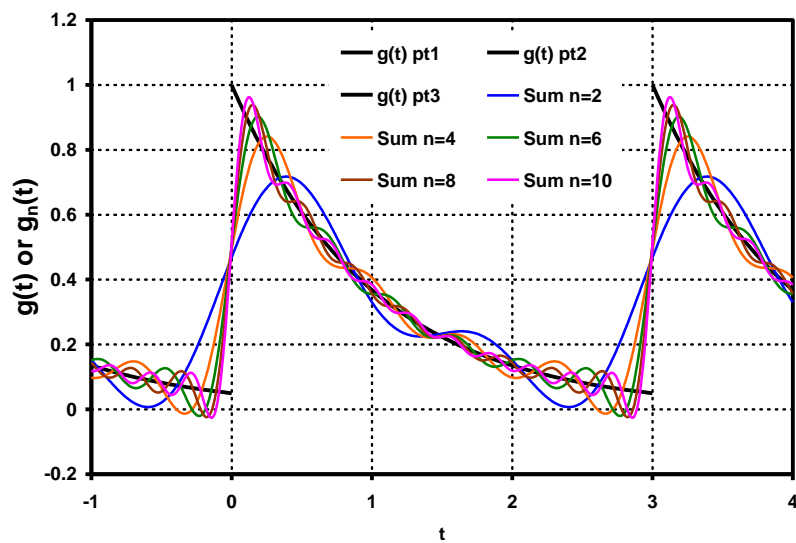
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Fourier Series Terms



Plots of the partial summations:

Fourier Series Summations



(Note the formation of the Gibbs phenomenon at the discontinuities)

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2.) Exponential Form of the Fourier Series

a.) Using $\cos(2\pi nt/T) = (e^{2\pi int/T} + e^{-2\pi int/T})/2$ and

$\sin(2\pi nt/T) = (e^{2\pi int/T} - e^{-2\pi int/T})/2i$, the Fourier Series defined above can be rewritten as

$$\begin{aligned} g(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2} \right) e^{(2\pi int/T)} + \left(\frac{a_n + ib_n}{2} \right) e^{(-2\pi int/T)} \right] \\ &= \sum_{n=-\infty}^{\infty} c_n e^{(2\pi int/T)} \end{aligned}$$

where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$ for $n \geq +1$ and $c_n = \frac{a_n + ib_n}{2}$ for $n \leq -1$.

This is the exponential form of the Fourier series.

b.) It can be shown that the coefficients c_n can be defined as

$$c_n = \frac{1}{T} \int_d^{d+T} g(t) e^{-2\pi int/T} dt$$

c.) For a real-valued time signal $g(t)$, c_0 is real-valued and c_n ($n = \pm 1, \pm 2, \pm 3, \dots$) are complex valued.

1.) $\text{Re}(c_n)$ is an even sequence: $\text{Re}(c_n) = \text{Re}(c_{-n})$

2.) $\text{Im}(c_n)$ is an odd sequence: $-\text{Im}(c_n) = \text{Im}(c_{-n})$

Example: $g(t) = e^{-t}$, $0 < t < 3$

$$c_n = \frac{1}{T} \int_d^{d+T} g(t) e^{-2\pi int/T} dt = \frac{1}{3} \int_0^3 e^{-t} e^{-2\pi in t/3} dt = \frac{1}{3} \int_0^3 e^{-[1+(2\pi in/3)]t} dt$$

Using $\int e^{at} dt = e^{at}/a$ with $a = -[1+(2\pi in/3)]$,

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$$c_n = \frac{1}{3} \frac{e^{-[1+(2\pi in/3)]t}}{[1+(2\pi in/3)]} \Big|_0^3 = \frac{1}{3} \frac{1 - e^{-(3+2\pi in)}}{[1+(2\pi in/3)]} = \frac{1}{3} \frac{1 - e^{-3} e^{-2\pi in}}{[1+(2\pi in/3)]}$$

Using $e^{-2\pi in} = 1$,

$$c_n = \frac{1}{3} \frac{1 - e^{-3}}{[1+(2\pi in/3)]} = \frac{(3-2\pi in)(1 - e^{-3})}{9 + 4\pi^2 n^2}$$

$$\therefore g(t) = \sum_{n=-\infty}^{\infty} c_n e^{(2\pi in t/T)} = \sum_{n=-\infty}^{\infty} \frac{(3-2\pi in)(1 - e^{-3})}{9 + 4\pi^2 n^2} e^{(2\pi in t/3)}$$

3.) Amplitude and Phase Spectra

a.) The coefficients c_n represent the frequency spectrum of the periodic time signal $g(t)$.

1.) The amplitude spectrum A_n of $g(t)$ is defined as

$$A_n = |c_n| = \sqrt{[\operatorname{Re}(c_n)]^2 + [\operatorname{Im}(c_n)]^2}$$

2.) The phase spectrum θ_n of $g(t)$ is defined as

$$\theta_n = \arg(c_n) = \tan^{-1}[\operatorname{Im}(c_n)/\operatorname{Re}(c_n)]$$

b.) It can be shown that for a real-valued time signal $g(t)$:

1.) The amplitude spectrum A_n is an even, non-negative function.

2.) The phase spectrum θ_n is an odd function. (Note: $\theta_0 = 0$)

c.) This is a discrete spectrum with harmonic components having frequencies $f_n = n/T$ ($n = 0, \pm 1, \pm 2, \pm 3, \dots$).

d.) In general, $A_n \rightarrow 0$ as $n \rightarrow \pm \infty$.

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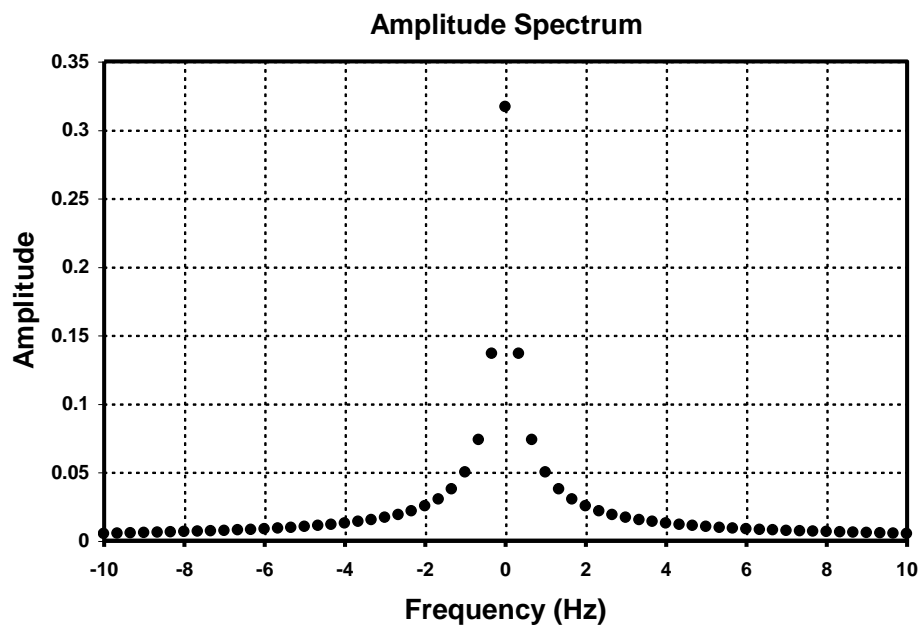
Example: $g(t) = e^{-t}$, $0 < t < 3$

$$c_n = \frac{(3 - 2\pi i n)(1 - e^{-3})}{9 + 4\pi^2 n^2} \rightarrow \text{Re}(c_n) = \frac{3(1 - e^{-3})}{9 + 4\pi^2 n^2} \text{ and } \text{Im}(c_n) = \frac{-2\pi n(1 - e^{-3})}{9 + 4\pi^2 n^2}$$

$$A_n = \sqrt{\left[\frac{3(1 - e^{-3})}{9 + 4\pi^2 n^2} \right]^2 + \left[\frac{-2\pi n(1 - e^{-3})}{9 + 4\pi^2 n^2} \right]^2} = \frac{(1 - e^{-3})}{9 + 4\pi^2 n^2} \sqrt{9 + 4\pi^2 n^2}$$

$$= \frac{(1 - e^{-3})}{\sqrt{9 + 4\pi^2 n^2}}$$

$$\theta_n = \tan^{-1} \left[\frac{-2\pi n(1 - e^{-3}) / (9 + 4\pi^2 n^2)}{3(1 - e^{-3}) / (9 + 4\pi^2 n^2)} \right] = \tan^{-1}(-2\pi n/3)$$



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