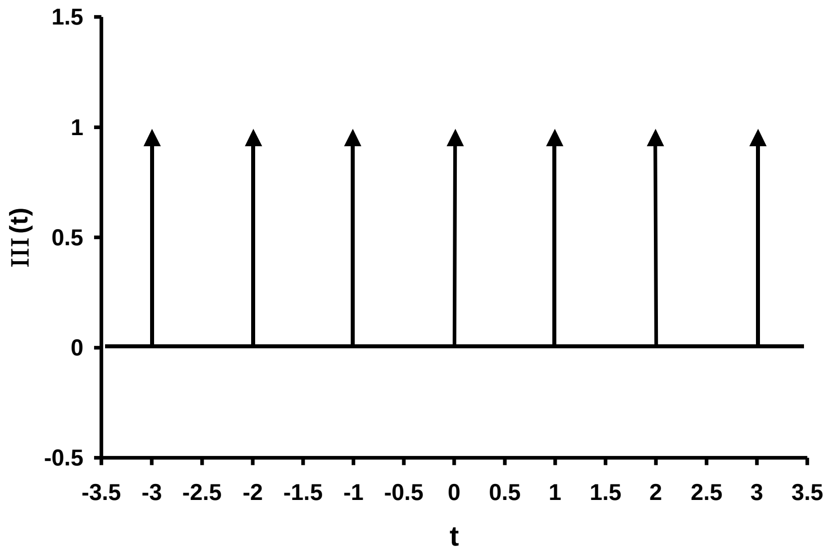
Earth 460 Classnotes (Set 5)

II.) Time Series Analysis – Discrete Signals

A.) Sampling

1.) Sampling or Replicating Function (Dirac Delta Comb) $\mathrm{III}(t)$

a.) The sampling function is an infinite series of Dirac delta functions uniformly spaced at a unit interval: $\mathrm{III}(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$



b.) Properties of $\mathrm{III}(t)$

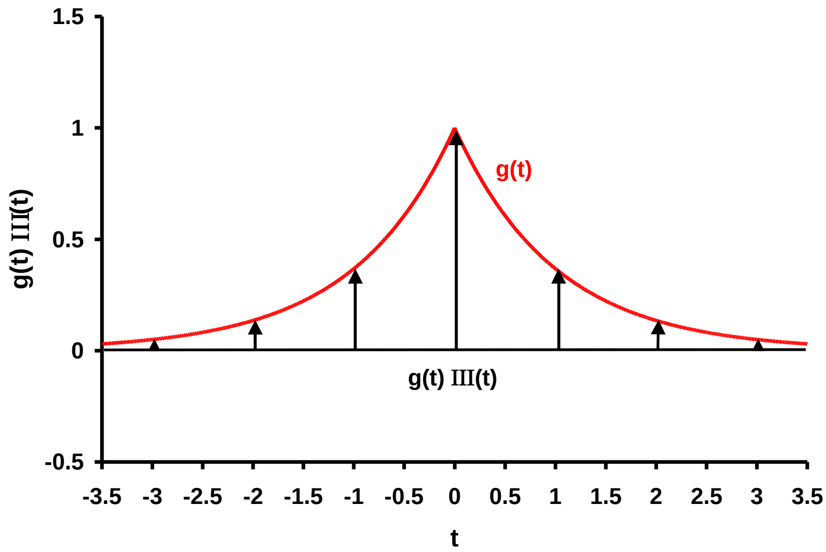
1.) $\mathrm{III}(at) = \dfrac{1}{|a|} \sum_{n=-\infty}^{+\infty} \delta(t-n/a)$

2.) $\mathrm{III}(t+n) = \mathrm{III}(t)$ where n is an integer (i.e., $\mathrm{III}(t)$ is periodic with unit period)

c.) The periodic sampling property follows from a generalization of the delta function: $g(t)\,\mathrm{III}(t) = \sum_{n=-\infty}^{+\infty} g(n)\,\delta(t-n)$.

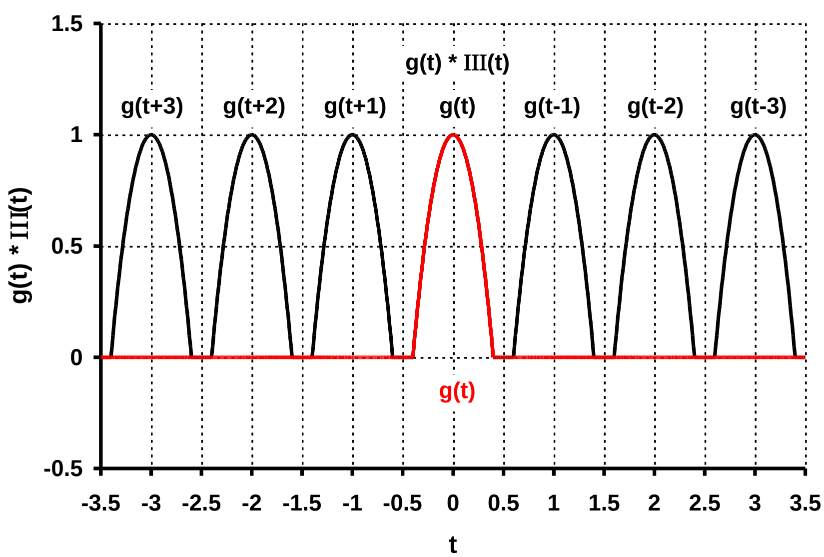
1.) $g(t)$ is sampled at a unit interval.

2.) Information about $g(t)$ in the interval between integers is removed, but $g(t)$ values at integer values are retained.



d.) Periodic replication property results from the convolution of $g(t)$ with $\mathrm{III}(t)$: $g(t) * \mathrm{III}(t) = \sum_{n=-\infty}^{+\infty} g(t) * \delta(t-n) = \sum_{n=-\infty}^{+\infty} g(t-n)$.

A replication of $g(t)$ is generated at unit intervals along the t axis (i.e., a periodic function).



e.) Fourier transform of $\mathrm{III}(t)$

Using a sequence of test functions, it can be shown that $\mathrm{III}(t) \Leftrightarrow \mathrm{III}(f)$.

2.) The Effects of Sampling a Continuous Signal

a.) $g(t)$ is sampled at a uniform interval Δt. The discrete time signal $d(t)$ is given by $d(t) = \sum_{n=-\infty}^{+\infty} g(n\Delta t)\,\delta(t - n\Delta t)$. Using

$\mathrm{III}(t/\Delta t) = \Delta t \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta t)$, then $d(t) = (\Delta t)^{-1} g(t)\,\mathrm{III}(t/\Delta t)$.

b.) The Fourier transform of the uniformly sampled time signal

1.) Using the scaling theorem: $\mathrm{III}(t/\Delta t) \Leftrightarrow \Delta t\ \mathrm{III}(\Delta t f)$

2.) With $g(t) \Leftrightarrow G(f)$ and the convolution relationship, we have

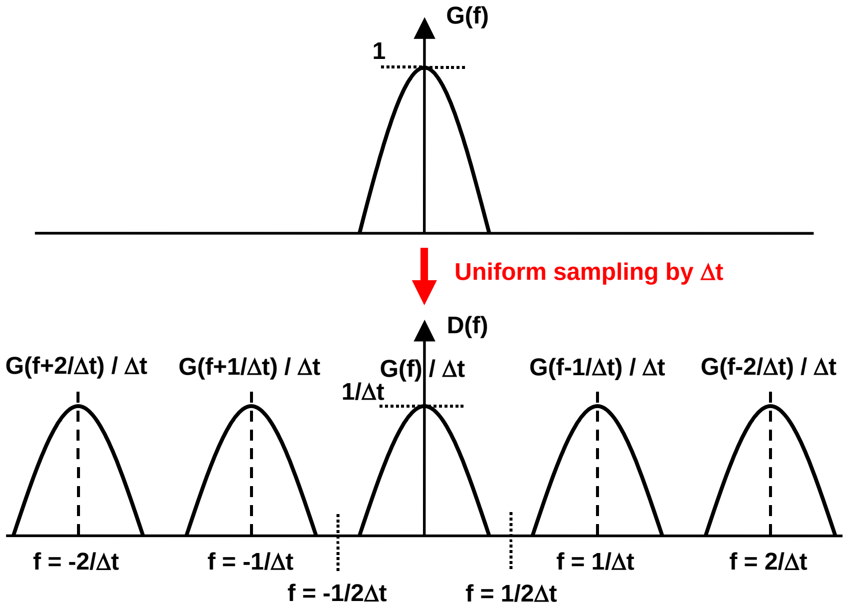
$$d(t) = (\Delta t)^{-1} g(t)\,\mathrm{III}(t/\Delta t) \Leftrightarrow D(f) = G(f) * \mathrm{III}(\Delta t f)$$

3.) Using $\mathrm{III}(\Delta t f) = (\Delta t)^{-1} \sum_{n=-\infty}^{+\infty} \delta(f - n/\Delta t)$, then

$$D(f) = (\Delta t)^{-1} G(f) * \left[\sum_{n=-\infty}^{+\infty} \delta(f - n/\Delta t)\right] = (\Delta t)^{-1} \sum_{n=-\infty}^{+\infty} G(f) * \delta(f - n/\Delta t)$$

$$= (\Delta t)^{-1} \sum_{n=-\infty}^{+\infty} G(f - n/\Delta t)$$

4.) Sampling a time signal at a uniform interval Δt causes the original Fourier transform of the continuous signal to replicate itself in the frequency domain at an interval of $1/\Delta t$ (i.e., a periodic function with a fundamental period $1/\Delta t$) and be scaled by a factor of $1/\Delta t$



(Note: If $|G(f)| > 0$ for $|f| > 1/2\Delta t$, then overlapping of replications occurs.)

3.) The Effects of Changing the Sampling Interval Δt

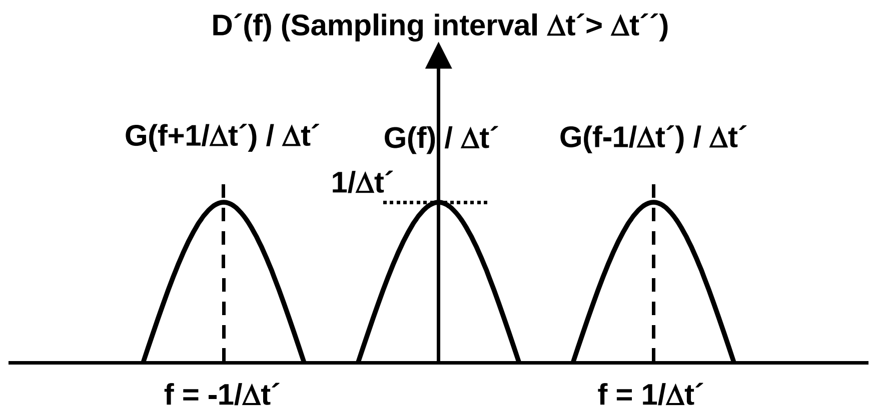
a.) As Δt is decreased, $g(t)$ is sampled on a finer interval. Hence, more information about $g(t)$ is retained.

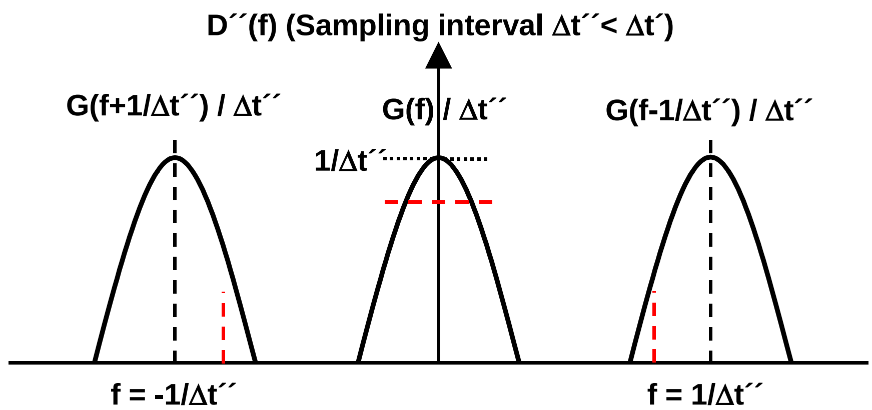
b.) Effect on the Fourier transform of the sampled signal

1.) The sampling interval is decreased from $\Delta t'$ to $\Delta t''$ (i.e., $\Delta t' > \Delta t''$).

2.) The replication interval of the Fourier transform ($1/\Delta t$) increases – the replications of $G(f)$ move farther apart.

3.) The amplitude scaling factor $1/\Delta t$ increases.





4.) The converse occurs if the sampling interval is increased.

c.) If $G(f)$ is band-limited (i.e., $G(f)=0$ for $f>f_{up}$ where f_{up} is the highest nonzero frequency component of $G(f)$), then it is possible to select a sampling interval Δt such that the replications do not overlap. This condition occurs when $f_{up}<1/2\Delta t$.

4.) Sampling Theorem

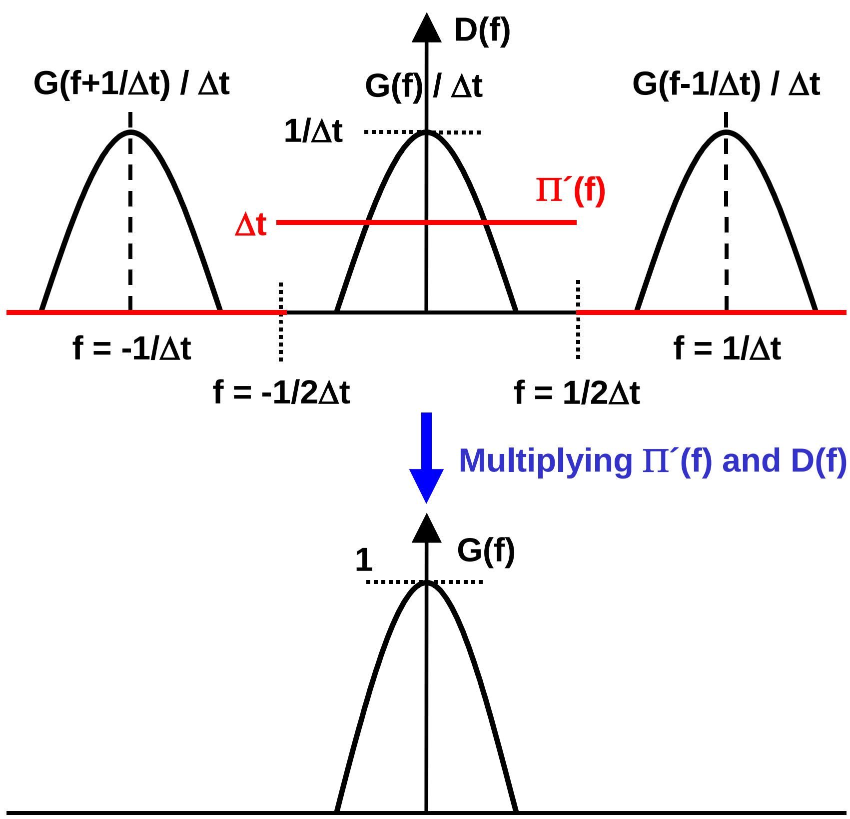
a.) By sampling $g(t)$ at a uniform interval Δt , information about the signal between the sampling points is lost. However, it is possible to completely recovery $g(t)$ after sampling under certain conditions.

b.) Consider $D(f)$, the Fourier transform of $g(t)$ after being discretely sampled at an interval Δt. Suppose that the replications of $G(f)$ do not overlap (i.e., $G(f)$ is bandlimited such that $G(f)=0$ for $f>1/2\Delta t$).

c.) Multiplying $D(f)$ by an appropriate boxcar function will reproduce $G(f)$.

That boxcar function is $\prod'(f)=\Delta t\prod(\Delta t f)=\begin{cases}\Delta t, |f|\le 1/2\Delta t\\ 0, |f|>1/2\Delta t\end{cases}$.

Hence, $\prod'(f)D(f)=\left[\Delta t\prod(\Delta t f)\right]\left[(\Delta t)^{-1}\sum_{n=-\infty}^{+\infty}G(f-n/\Delta t)\right]=G(f)$



d.) This frequency domain process is equivalent to convolving $d(t)$ with a sinc function in the time domain. Since $\text{sinc}(t/\Delta t) \Leftrightarrow \Delta t \prod(\Delta t f)$, then $g(t) = d(t) * \text{sinc}(t/\Delta t) \Leftrightarrow D(f)\Delta t \prod(\Delta t f) = G(f)$.

e.) In summary, the sampling theorem states: No information is lost by a uniform sampling Δt if the highest non-zero frequency component in the signal is less than $1/2\Delta t$.

1.) For a given sampling interval Δt, the quantity $1/2\Delta t$ is called the Nyquist frequency (i.e., $f_N = 1/2\Delta t$)

2.) This criterion is equivalent to requiring more than two samples per cycle for the highest non-zero frequency component.

3.) Since such a sampling interval gives complete information about $g(t)$, nothing further is gained by a finer sampling.

5.) Spatial Sampling

a.) The concepts developed above for the discrete sampling of continuous time signals can be directly applied to the effect of discrete spatial sampling.

b.) For a collection of uniformly spaced seismic/GPR traces (e.g., CMP gather, reflection profile), the spatial Nyquist wave number is $k_N = 1/2\Delta x$ where Δx is the incremental offset/distance between adjacent traces.

B.) Aliasing

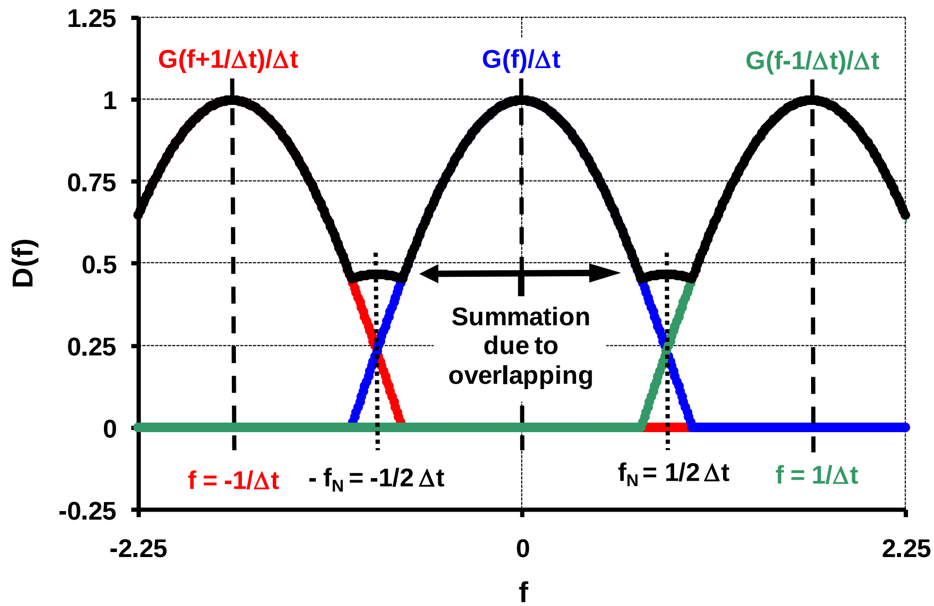
1.) Nature of Aliasing

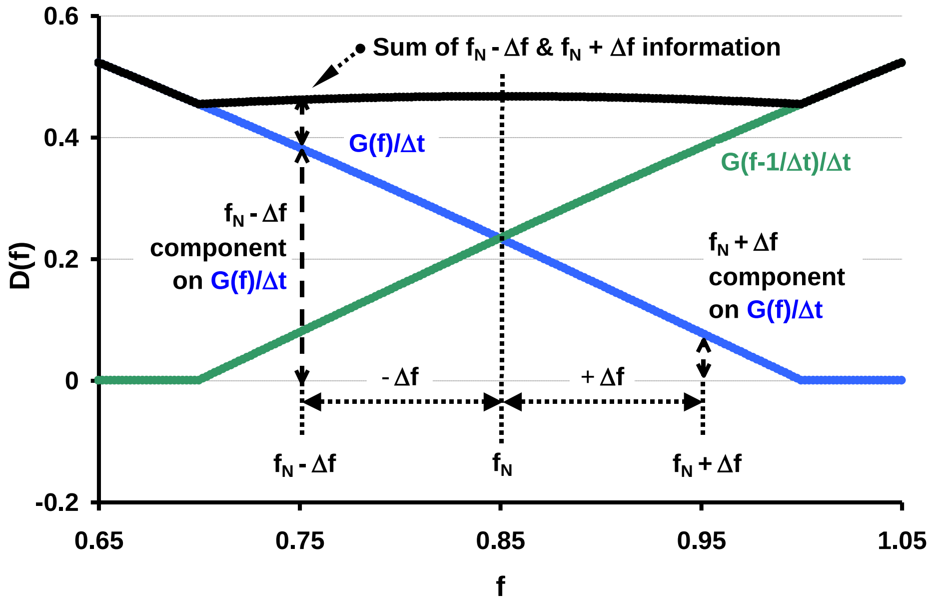
a.) Suppose a signal $g(t)$ is sampled such that is contains non-zero frequency components greater then f_N.

b.) Replications of $G(f)$ due to the sampling process overlap. This results in a summation in the overlap region that distorts the information contained in these frequency components

c.) Higher frequency from the adjacent replication contaminates lower frequency components such that the information in the $f_N + \Delta f$ component is indistinguishable from the $f_N - \Delta f$ component (the higher frequency component is "aliased"). For $D(f)$, the summation of these two components is found at $f_N - \Delta f$.

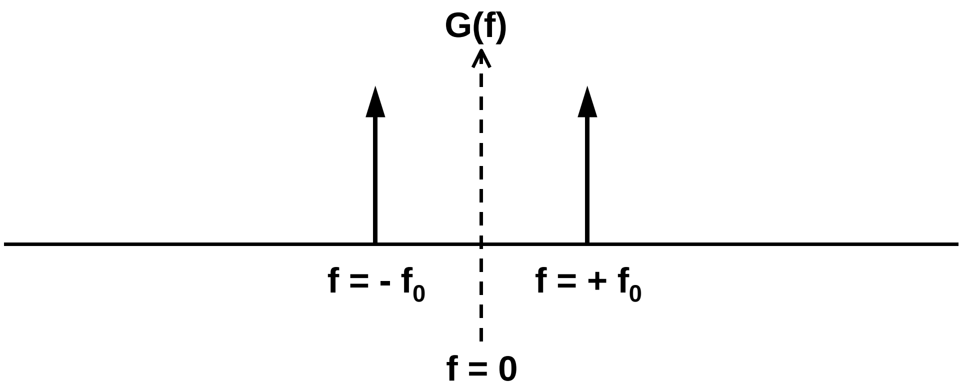
d.) This effect is equivalent to a "folding back" of the higher frequency information about f_N and combined with the lower frequencies.





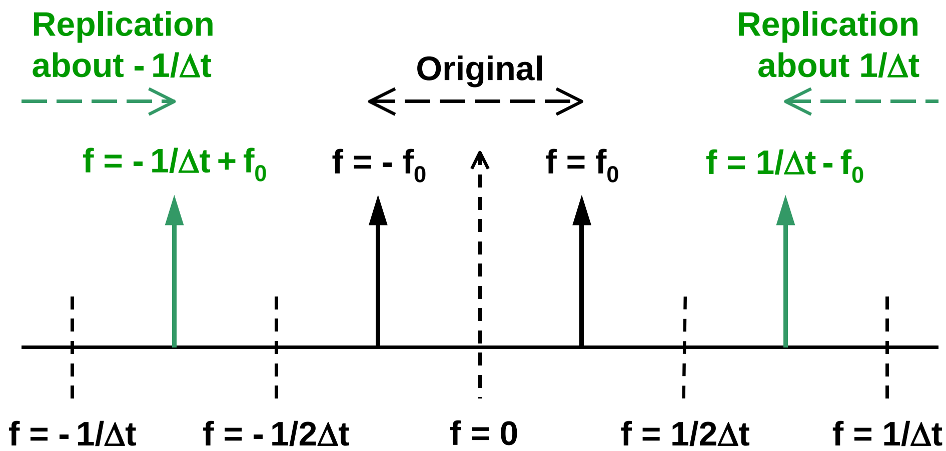
2.) Effects of Aliasing on a Specific Frequency Component

a.) Consider a simple cosine signal (i.e., a signal with a signal frequency component): $g(t) = \cos(2\pi f_0 t) \Leftrightarrow G(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$

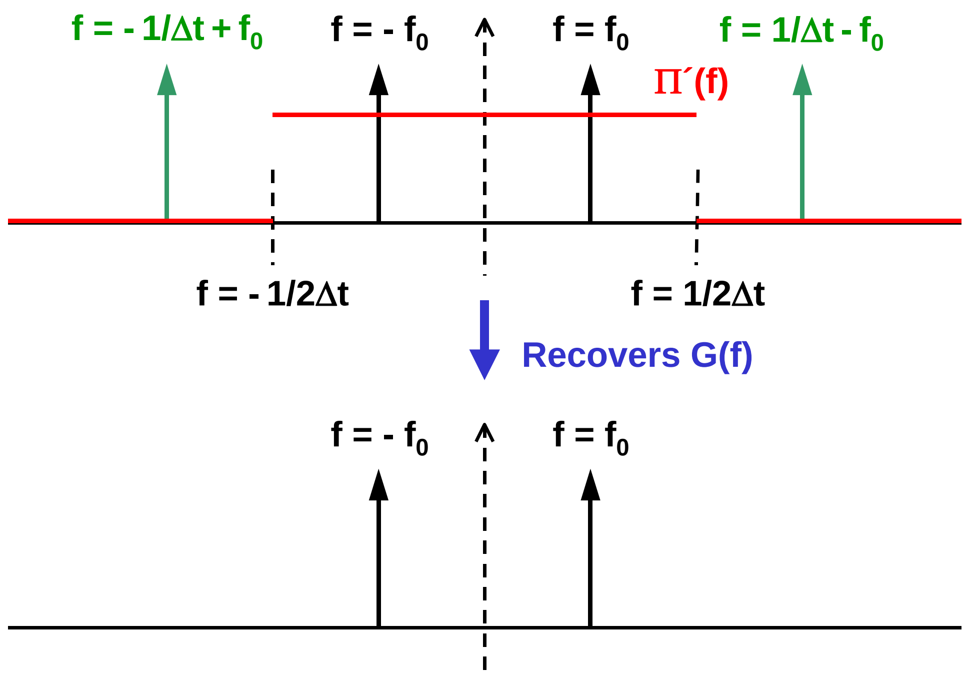


b.) Let $g(t)$ be sampled at an uniform interval Δt

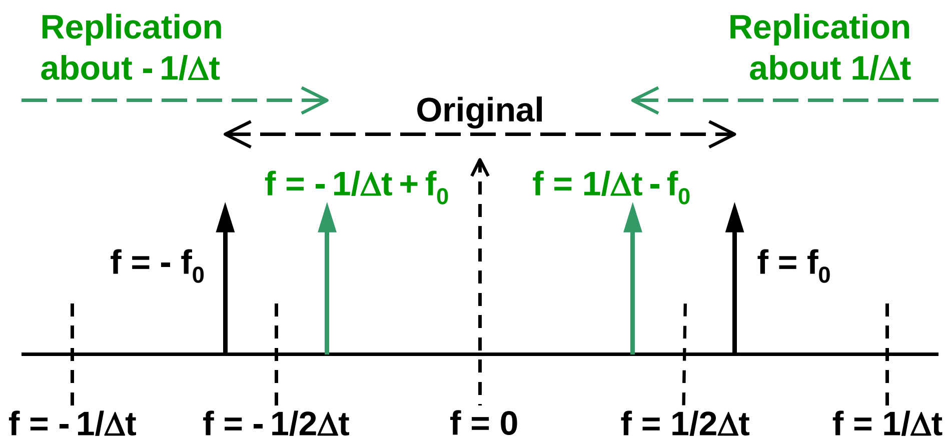
c.) Case 1: $f_N = 1/2\Delta t > f_0$ (i.e., no overlapping replications)



Multiplying $D(f)$ by $\Pi'(f) = \Delta t \, \Pi(\Delta t f)$ returns $G(f)$



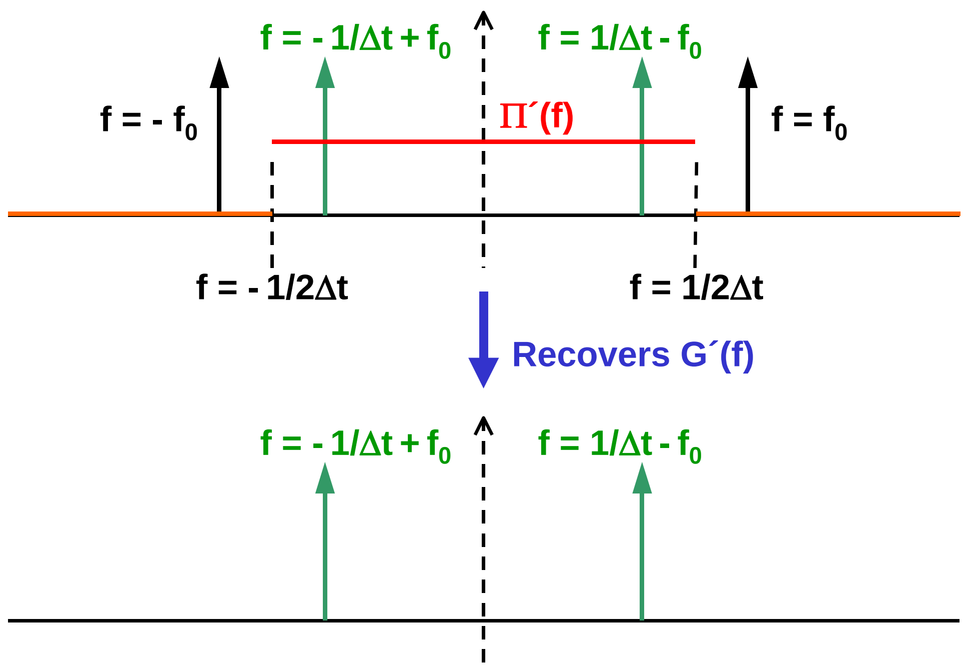
d.) Case 2: $1/\Delta t > f_0 > f_N = 1/2\Delta t$ (i.e., $G(f)$ overlapping with neighboring replication)



Multiplying $D(f)$ by $\prod'(f) = \Delta t \prod(\Delta t f)$ returns

$G'(f) = \frac{1}{2}\left[\delta\left(f - (1/\Delta t - f_0)\right) + \delta\left(f + (1/\Delta t - f_0)_0\right)\right]$ where

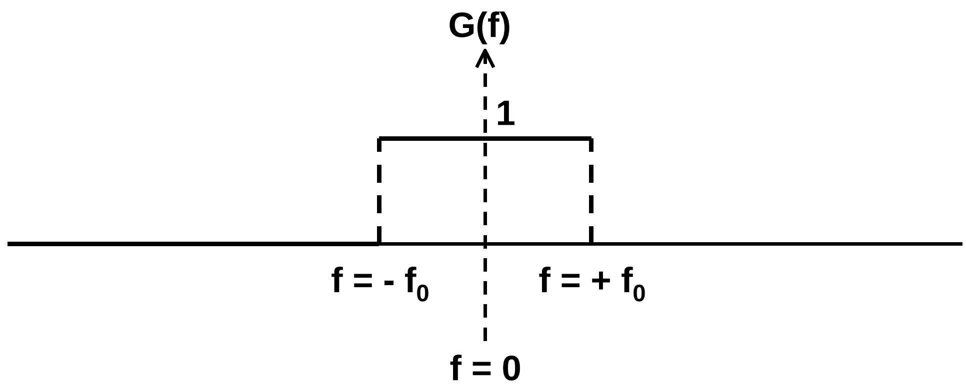
$g'(t) = \cos\left(2\pi(1/\Delta t - f_0)t\right) \Leftrightarrow G'(f)$



Note: the frequency of the aliased cosine is lower than the original signal by $2f_0 - 1/\Delta t$

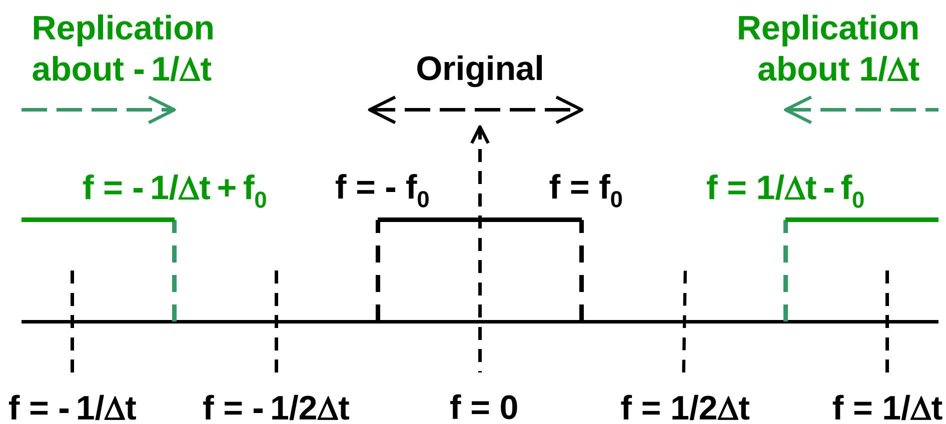
3.) Effects of Overlap due to Aliasing

a.) Consider a simple sine function $g(t) = 2f_0 \operatorname{sinc}(2f_0 t) \Leftrightarrow G(f) = \Pi(f/2f_0)$

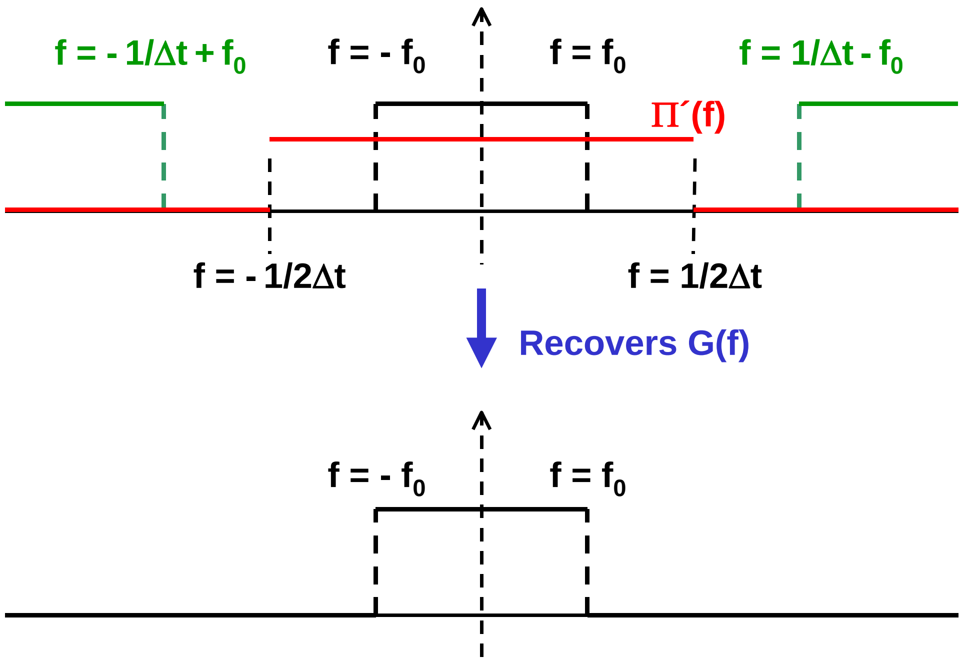


b.) Let $g(t)$ be sampled at an uniform interval Δt

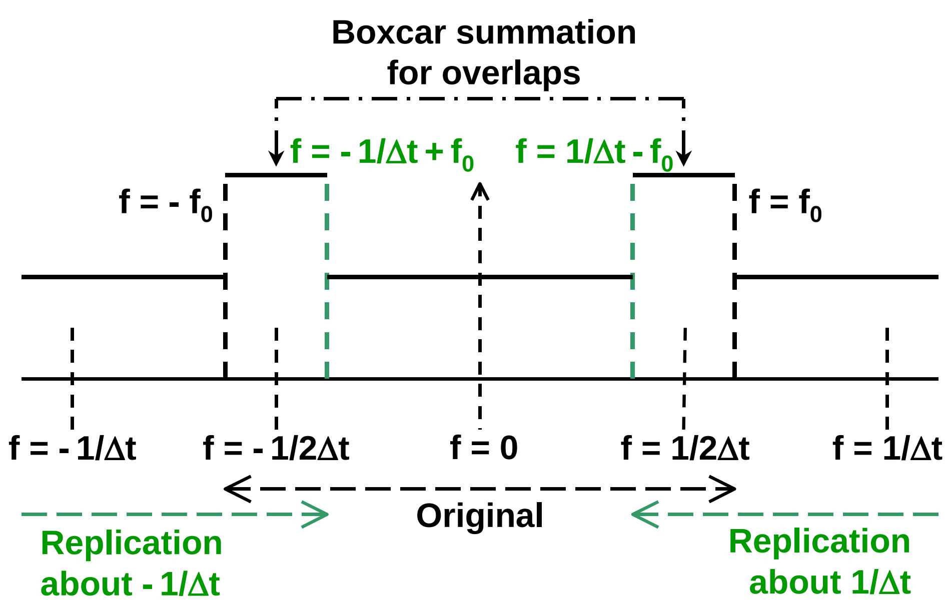
c.) Case 1: $f_N = 1/2\Delta t > f_0$ (i.e., no overlapping replications)



Multiplying $D(f)$ by $\prod'(f) = \Delta t \prod(\Delta t f)$ returns $G(f)$



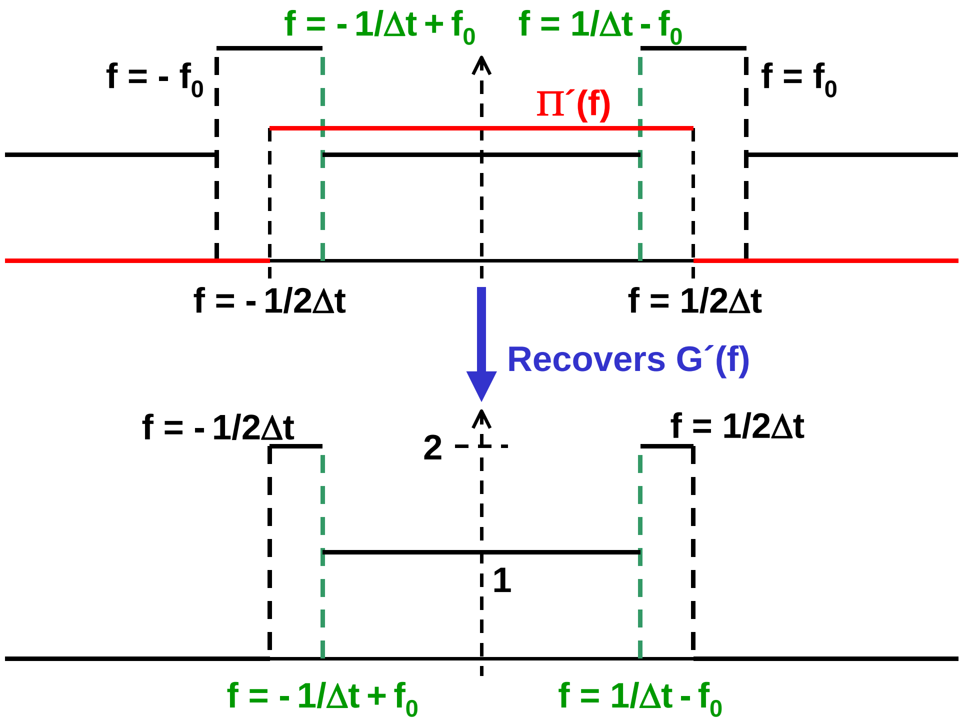
d.) Case 2: $1/\Delta t > f_0 > f_N = 1/2\Delta t$ (i.e., $G(f)$ overlapping with neighboring replication)



Multiplying $D(f)$ by $\prod'(f)=\Delta t\prod(\Delta t f)$ returns

$G'(f)=2\prod(f/2f_N)-\prod(f/2(1/\Delta t-f_0))$ where

$g'(t)=4f_N\,\mathrm{sinc}(2f_N t)-2(1/\Delta t-f_0)\,\mathrm{sinc}(2(1/\Delta t-f_0)t)\Leftrightarrow G'(f)$



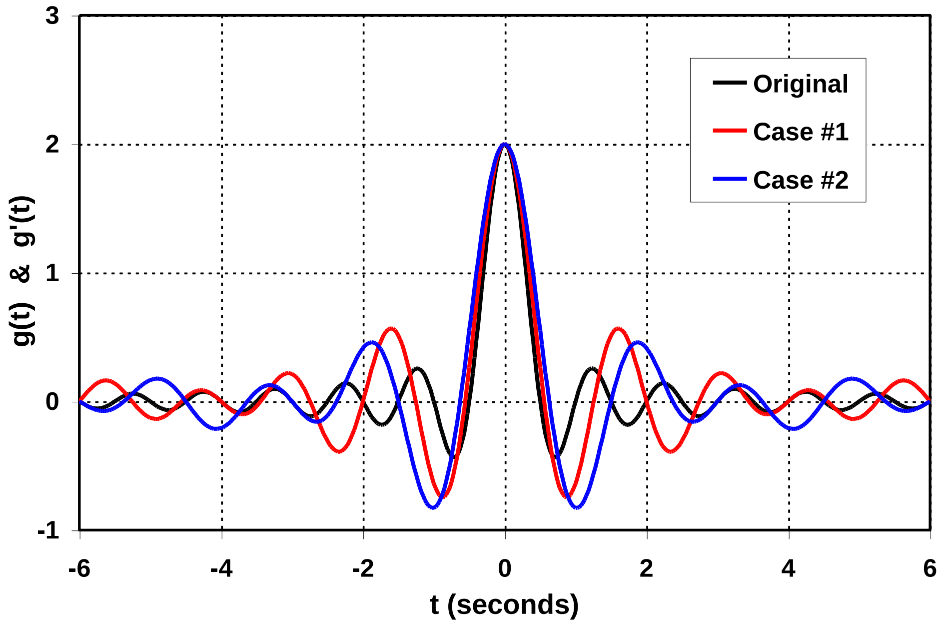
Examples:

Original signal $g(t)=2f_0\,\mathrm{sinc}(2f_0 t)$ with $f_0=1$ Hz

Aliased signal $g'(t)=4f_N\,\mathrm{sinc}(2f_N t)-2(1/\Delta t-f_0)\,\mathrm{sinc}(2(1/\Delta t-f_0)t)\Leftrightarrow G'(f)$

Case #1: $\Delta t=\dfrac{2}{3}$ s $\rightarrow$ $f_N=\dfrac{3}{4}$ Hz

Case #2: $\Delta t=\dfrac{3}{4}$ s $\rightarrow$ $f_N=\dfrac{2}{3}$ Hz



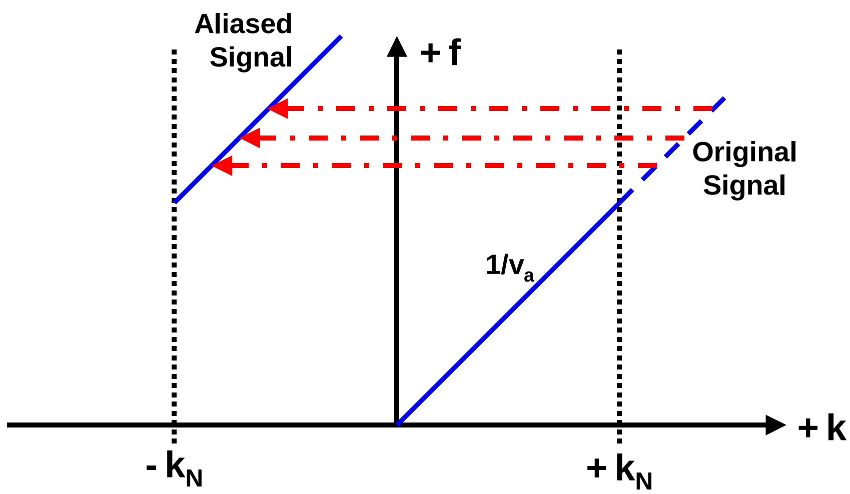
4.) Spatial Aliasing

a.) For a collection of uniformly spaced seismic/GPR traces (e.g., CMP gather, reflection profile), information having a wave number k greater than Nyquist wave number k_N will undergo spatial aliasing.

b.) If an event has an apparent horizontal velocity v_a, then its temporal frequency component f has a wave number $k = f/v_a$. If $k \geq k_N$, then this frequency component is aliased.

c.) In *f-k* space, the effects of spatial aliasing of an event causes *wrap around* of its frequency component having $k \geq k_N$.

d.) Spatial aliasing is avoided by having sufficiently small spatial sampling Δx or using a low-pass filter to remove temporal frequency components that undergo aliasing.


Aliased Signal
+ f
Original Signal
1/v_a
+ k
- k_N
+ k_N