Lecture 1g

Span and Linear Independence in Vector Spaces

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One of the common ways to define a subspace is to think of it as the set of all linear combinations of a set of vectors. First, let's note that this IS a subspace!

Theorem 4.2.2 If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$, and $\mathbb{S}$ is the set of all possible linear combinations of these vectors,

$$\mathbb{S} = \{t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \mid t_1, \dots, t_k \in \mathbb{R}\}$$

then $\mathbb{S}$ is a subspace of $\mathbb{V}$.

Proof of Theorem 4.2.2: Let's look at our three properties:

S0: Since $\mathbb{V}$ is closed under addition and scalar multiplication, we know that every $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$ is an element of $\mathbb{V}$, and thus $\mathbb{S}$ is a subset of $\mathbb{V}$. And $\mathbb{S}$ is not empty since, at the least, $\mathbf{v}_1 \in \mathbb{S}$.

S1: Let $\mathbf{x} = s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k$ and $\mathbf{y} = t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$ be elements of $\mathbb{S}$. Then

$$\begin{aligned} \mathbf{x}+\mathbf{y} &= (s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k) + (t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) & \\ &= s_1\mathbf{v}_1 + t_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k + t_k\mathbf{v}_k & \text{by V2 and V5} \\ &= (s_1+t_1)\mathbf{v}_1 + \cdots + (s_k+t_k)\mathbf{v}_k & \text{by V8} \end{aligned}$$

And so we see that $\mathbf{x}+\mathbf{y} \in \mathbb{S}$.

S2: Let $\mathbf{x} = s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k$ be an element of $\mathbb{S}$, and let $t \in \mathbb{R}$. Then

$$\begin{aligned} t\mathbf{x} &= t(s_1\mathbf{v}_1 + \cdots + s_k\mathbf{v}_k) & \\ &= t(s_1\mathbf{v}_1) + \cdots + t(s_k\mathbf{v}_k) & \text{by V9} \\ &= (ts_1)\mathbf{v}_1 + \cdots + (ts_k)\mathbf{v}_k & \text{by V7} \end{aligned}$$

And so we see that $t\mathbf{x} \in \mathbb{S}$.

And since properties S0, S1, and S2 hold, $\mathbb{S}$ is a subspace of $\mathbb{V}$.

Example: The set of all diagonal 2×2 matrices is a vector space, since it is the set of all possible linear combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $M(2,2)$.

As we did with $\mathbb{R}^n$ and $M(m,n)$, we will call the set of all linear combinations the span.

Definition: If $\mathbb{S}$ is the subspace of the vector space $\mathbb{V}$ consisting of all linear combinations of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{V}$, then $\mathbb{S}$ is called the subspace **spanned** by $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, and we say that the set $\mathcal{B}$ **spans** $\mathbb{S}$. The set $\mathcal{B}$ is called a **spanning set** for the subspace $\mathbb{S}$. We denote $\mathbb{S}$ by

$$\mathbb{S} = \text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span } \mathcal{B}$$

We also extend our definition of linear dependence and linear independence to general vector spaces.

Definition: If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$, then $\mathcal{B}$ is said to be linearly independent if the only solution to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

is $t_1 = \cdots = t_k = 0$; otherwise, $\mathcal{B}$ is said to be **linearly dependent**.

I won't go over examples of spanning sets and linear independence, since we've already looked at these extensively in our studies of $\mathbb{R}^n$, $M(m,n)$, and P_n. But we do have an opportunity now to prove a statement about linear independence in ALL vector spaces.

Theorem 4.2.a: Any set that contains the zero vector is linearly dependent.

Proof: Let $\mathbb{V}$ be a vector space, and let $\mathcal{A} = \{\mathbf{0}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ be a set of vectors from $\mathbb{V}$ that contains the zero vector. To see that $\mathcal{A}$ is linearly dependent, we need to find a non-trivial solution to the equation

$$t_0\mathbf{0} + t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_k\mathbf{x}_k = \mathbf{0}$$

I claim that setting $t_0 = 1$, and $t_1 = t_2 = \cdots = t_k = 0$ is such a solution. First, we note that the scalar multiplicative identity property (V10) tells us that $1\mathbf{0} = \mathbf{0}$, so setting $t_0 = 1$ means we can replace $t_0\mathbf{0}$ with $\mathbf{0}$. Next we note that, by Theorem 4.2.1, $0\mathbf{x}_i = \mathbf{0}$ for all $1 \leq i \leq k$, so setting $t_1 = t_2 = \cdots = t_k = 0$ means we can replace all the $t_i\mathbf{x}_i$ with $\mathbf{0}$. And so, our equation becomes

$$\mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$$

which is true thanks to repeated uses of the additive identity property (V3).