Solution to Practice 1o

B3(a) For $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ to be in the range of L, there would need to be a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$\begin{bmatrix} -x_1 - 2x_2 \\ 2x_1 + x_3 \\ -2x_1 + x_2 - 2x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Setting the components equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} -x_1 & -2x_2 & & = 1 \\ 2x_1 & & +x_3 & = 1 \\ -2x_1 & +x_2 & -2x_3 & = -1 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} -1 & -2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ -2 & 1 & -2 & -1 \end{array}\right] \begin{array}{l} (-1)R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 2 & 0 & 1 & 1 \\ -2 & 1 & -2 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & -4 & 1 & 3 \\ 0 & 5 & -2 & -3 \end{array}\right] (-1/4)R_2 \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/4 & -3/4 \\ 0 & 5 & -2 & -3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 5R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/4 & -3/4 \\ 0 & 0 & -3/4 & 3/4 \end{array}\right] \begin{array}{l} \\ \\ (-4/3)R_3 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2 + (1/4)R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

And so we see that $x_1 = 1$, $x_2 = -1$, $x_3 = -1$ is the solution to our system. This means that $\mathbf{y}$ is in the range of L, and that $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ is such that $L(\mathbf{x}) = \mathbf{y}$.

B3(b) For $\mathbf{y} = 1 + x - x^2$ to be in the range of L, there would need to be a polynomial $\mathbf{x} = a + bx + cx^2$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$(-a - 2b) + (2a + c)x + (-2a + b - 2c)x^2 = 1 + x - x^2$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{cccl} -a & -2b & & = 1 \\ 2a & & +c & = 1 \\ -2a & +b & -2c & = -1 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} -1 & -2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ -2 & 1 & -2 & -1 \end{array}\right]$$

Noting that this is the same matrix we get in question B3(a), we know that its reduced row echelon form is

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

And so we see that $a = 1$, $b = -1$, $c = -1$ is the solution to our system. This means that $\mathbf{y}$ is in the range of L, and that $\mathbf{x} = 1 - x - x^2$ is such that $L(\mathbf{x}) = \mathbf{y}$.

B3(c) For $\mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ to be in the range of L, there would need to be a polynomial $\mathbf{x} = a + bx + cx^2$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$\begin{bmatrix} a \\ b \\ a+b+c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Setting the components equal to each other, we see that we need to have $a = 2$, $b = -1$, and $a + b + c = 3$. Plugging $a = 2$ and $b = -1$ into our third equation gives us $2 - 1 + c = 3$, so $c = 2$. Thus we have that $\mathbf{y}$ is in the range of L, and that $\mathbf{x} = 2 - x + 2x^2$ is such that $L(\mathbf{x}) = \mathbf{y}$.

B3(d) For $\mathbf{y} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ to be in the range of L, there would need to be a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$\begin{bmatrix} x_1 & x_2 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

Setting the entries equal to each other, we see that we need $x_1 = 1$, $x_2 = 1$, and $x_2 = 2$. Since we can't simultaneously have that $x_2 = 1$ and $x_2 = 2$, there is no solution to this equation. As such, $\mathbf{y}$ is not in the range of L.

B3(e) For $\mathbf{y} = 2 + x - x^2$ to be in the range of L, there would need to be an upper triangular matrix $\mathbf{x} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$(-a - c) + (a - 2b)x + (-2a + b + c)x^2 = 2 + x - x^2$$

Setting the coefficients equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} -a & & -c & = 2 \\ a & -2b & & = 1 \\ -2a & +b & +c & = -1 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} -1 & 0 & -1 & 2 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & 1 & -1 \end{array}\right] \begin{array}{l} (-1)R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & 1 & -1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & -2 & -1 & 3 \\ 0 & 1 & 3 & -5 \end{array}\right] R_2 \updownarrow R_3 \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & -5 \\ 0 & -2 & -1 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 2R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 5 & -7 \end{array}\right] \begin{array}{l} \\ \\ (1/5)R_3 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 1 & -7/5 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - 3R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & -7/5 \end{array}\right]$$

And so we see that $a = -3/5$, $b = -4/5$, $c = -7/5$ is the solution to our system. This means that $\mathbf{y}$ is in the range of L, and that $\mathbf{x} = \begin{bmatrix} -3/5 & -4/5 \\ 0 & -7/5 \end{bmatrix}$ is such that $L(\mathbf{x}) = \mathbf{y}$.

B3(f) For $\mathbf{y} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$ to be in the range of L, there would need to be a polynomial $\mathbf{x} = a + bx + cx^2$ such that $L(\mathbf{x}) = \mathbf{y}$. That is, we need to have

$$\begin{bmatrix} -a - 2c & 2b - c \\ -2a + 2c & -2b - c \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$$

Setting the entries equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrl} -a & & -2c & = -2 \\ & 2b & -c & = 2 \\ -2a & & +2c & = 0 \\ & -2b & -c & = -2 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} -1 & 0 & -2 & -2 \\ 0 & 2 & -1 & 2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & -1 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 + R_2 \end{array} \sim \left[\begin{array}{rrr|r} -1 & 0 & -2 & -2 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 + (1/3)R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} -1 & 0 & -2 & -2 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 4/3 \end{array}\right]$$

This last matrix is in row echelon form, and it has a bad row, so we see that our system is inconsistent. This means that $\mathbf{y}$ is not in the range of L.

Proof that Null(L) is a subspace: First we note that, by part (1) of Theorem 4.5.1, $L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$. So we have that $\mathbf{0}_{\mathbb{V}} \in \text{Null}(L)$, and thus, Null($L$) is non-empty. We also note that Null(L) is explicitly defined to be a subset of $\mathbb{V}$.

So, now let's see if Null(L) is closed under addition. Let $\mathbf{x}, \mathbf{y} \in \text{Null}(L)$. Then $L(\mathbf{x}) = \mathbf{0}_{\mathbb{W}}$ and $L(\mathbf{y}) = \mathbf{0}_{\mathbb{W}}$. From this we see that $L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}) = \mathbf{0}_{\mathbb{W}} + \mathbf{0}_{\mathbb{W}} = \mathbf{0}_{\mathbb{W}}$. So $\mathbf{x} + \mathbf{y} \in \text{Null}(L)$, as desired.

Finally, we check to see if Null(L) is closed under scalar multiplication. To that end, let $\mathbf{x} \in \text{Null}(L)$ and $s \in \mathbb{R}$. Then $L(\mathbf{x}) = \mathbf{0}_{\mathbb{W}}$. From this we see that $L(s\mathbf{x}) = sL(\mathbf{x}) = s\mathbf{0}_{\mathbb{W}} = \mathbf{0}_{\mathbb{W}}$. So $s\mathbf{x} \in \text{Null}(L)$, as desired.