Lecture 3m

Span, Linear Independence, and Basis

(page 413-4)

The most common way to define a subspace is as all possible linear combinations of a set of vectors. First, let us verify that this is still a subspace.

Theorem 9.3.a If $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$ over $\mathbb{C}$, and $\mathbb{S}$ is the set of all possible linear combinations of these vectors,

$$\mathbb{S} = \{\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{C}\}$$

then $\mathbb{S}$ is a subspace of $\mathbb{V}$.

Proof of Theorem 9.3.a: Let's look at our three properties:

S0: Since $\mathbb{V}$ is closed under addition and scalar multiplication, we know that every $\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ is an element of $\mathbb{V}$, and thus $\mathbb{S}$ is a subset of $\mathbb{V}$. And $\mathbb{S}$ is not empty since, at the least, $\mathbf{v}_1 \in \mathbb{S}$.

S1: Let $\mathbf{w} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ and $\mathbf{z} = \gamma_1\mathbf{v}_1 + \cdots + \gamma_k\mathbf{v}_k$ be elements of $\mathbb{S}$. Then

$$\begin{aligned}
\mathbf{w} + \mathbf{z} &= (\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k) + (\gamma_1\mathbf{v}_1 + \cdots + \gamma_k\mathbf{v}_k) & \\
&= \alpha_1\mathbf{v}_1 + \gamma_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k + \gamma_k\mathbf{v}_k & \text{by V2 and V5} \\
&= (\alpha_1 + \gamma_1)\mathbf{v}_1 + \cdots + (\alpha_k + \gamma_k)\mathbf{v}_k & \text{by V8}
\end{aligned}$$

And so we see that $\mathbf{w} + \mathbf{z} \in \mathbb{S}$.

S2: Let $\mathbf{z} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ be an element of $\mathbb{S}$, and let $\gamma \in \mathbb{C}$. Then

$$\begin{aligned}
\gamma\mathbf{z} &= \gamma(\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k) & \\
&= \gamma(\alpha_1\mathbf{v}_1) + \cdots + \gamma(\alpha_k\mathbf{v}_k) & \text{by V9} \\
&= (\gamma\alpha_1)\mathbf{v}_1 + \cdots + (\gamma\alpha_k)\mathbf{v}_k & \text{by V7}
\end{aligned}$$

And so we see that $\gamma\mathbf{z} \in \mathbb{S}$.

And since properties S0, S1, and S2 hold, $\mathbb{S}$ is a subspace of $\mathbb{V}$.

Note that this theorem is exactly the same as Theorem 4.2.2, except that our scalars are from $\mathbb{C}$, not $\mathbb{R}$. We can do this because it relies only on vector space properties. As we will find many similarities between vector spaces over $\mathbb{R}$ and vector spaces over $\mathbb{C}$, we sometime use Greek letters to indicate our scalars, to remind us that they are from $\mathbb{C}$, not $\mathbb{R}$. We have already been using the Greek letter α ("alpha") for this purpose, and in this proof I also used the Greek letter γ ("gamma"). If you are ever curious about a symbol used in the lecture, just ask!

Continuing with the theme of "vector spaces over $\mathbb{C}$ are just like vector spaces over $\mathbb{R}$", we will now define spanning sets, linear independence, and bases.

Definition: If $\mathbb{S}$ is the subspace of the vector space $\mathbb{V}$ over $\mathbb{C}$ consisting of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{V}$, then $\mathbb{S}$ is called the subspace **spanned** by $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, and we say that the set $\mathcal{B}$ **spans** $\mathbb{S}$. The set $\mathcal{B}$ is called a **spanning set** for the subspace $\mathbb{S}$. We denote $\mathbb{S}$ by

$$\mathbb{S} = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\ \mathcal{B}$$

Definition: If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space $\mathbb{V}$ over $\mathbb{C}$, then $\mathcal{B}$ is said to be **linearly independent** if the only solution to the equation

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

is $\alpha_1 = \cdots = \alpha_k = 0$; otherwise, $\mathcal{B}$ is said to be **linearly dependent**.

Definition: A set $\mathcal{B}$ of vectors in a vector space $\mathbb{V}$ over $\mathbb{C}$ is a **basis** for $\mathbb{V}$ if it is a linearly independent spanning set for $\mathbb{V}$.

And, as before, we like bases because they allow us to write every vector in $\mathbb{V}$ as a unique linear combination of the basis vectors.

Theorem 9.3.b **(Unique Representation Theorem)**: Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a spanning set for a vector space $\mathbb{V}$ over $\mathbb{C}$. Then every vector in $\mathbb{V}$ can be expressed in a *unique* way as a linear combination of the vectors of $\mathcal{B}$ if and only if the set $\mathcal{B}$ is linearly independent.

As with Theorem 9.3.a, the proof of this theorem can be copied straight from its counterpart in $\mathbb{R}$ (Theorem 4.3.1). I won't bother to copy it here.

Not only are the definitions for these basic linear algebra concepts the same, but the way we go about determining them is the same. Let's look at an example.

Example: Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ -2+i \end{bmatrix}, \begin{bmatrix} 1+i \\ -3-i \end{bmatrix}, \begin{bmatrix} 1+i \\ -3 \end{bmatrix}, \begin{bmatrix} 5-2i \\ -5+9i \end{bmatrix} \right\}$. Determine whether or not $\mathcal{A}$ is a spanning set for $\mathbb{C}^2$ and if it is linearly independent. Use these results to determine if $\mathcal{A}$ is a basis for $\mathbb{C}^2$.

First, let's see if $\mathcal{A}$ is a spanning set for $\mathbb{C}^2$. To do this, we will take an arbitrary element $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ of $\mathbb{C}^2$, and see if we can find scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$, such that

$$\alpha_1 \begin{bmatrix} 1 \\ -2+i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1+i \\ -3-i \end{bmatrix} + \alpha_3 \begin{bmatrix} 1+i \\ -3 \end{bmatrix} + \alpha_4 \begin{bmatrix} 5-2i \\ -5+9i \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

If we compute the linear combination on the left, we see that this is the same as checking the following vector equality:

$$\begin{bmatrix} \alpha_1 + \alpha_2(1+i) + \alpha_3(1+i) + \alpha_4(5-2i) \\ \alpha_1(-2+i) + \alpha_2(-3-i) + \alpha_3(-3) + \alpha_4(-5+9i) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

By the definition of vector equality, this equation is equivalent to the following system:

$$\begin{aligned} \alpha_1 + (1+i)\alpha_2 + (1+i)\alpha_3 + (5-2i)\alpha_4 &= z_1 \\ (-2+i)\alpha_1 + (-3-i)\alpha_2 - 3\alpha_3 + (-5+9i)\alpha_4 &= z_2 \end{aligned}$$

To solve this system, we need to row reduce its augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 1+i & 1+i & 5-2i & z_1 \\ -2+i & -3-i & -3 & -5+9i & z_2 \end{array}\right] \quad R_2 + (2-i)R_1$$
$$\sim \left[\begin{array}{cccc|c} 1 & 1+i & 1+i & 5-2i & z_1 \\ 0 & 0 & i & 3 & z_2 + (2-i)z_1 \end{array}\right]$$

This new matrix is in row echelon form, and since there are no bad rows (no matter our choice of z_1 and z_2), we see that our system does have a solution. And this means that $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is in the span of $\mathcal{A}$ for any $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{C}^2$, so $\mathcal{A}$ is a spanning set for $\mathbb{C}^2$.

Now let's determine if $\mathcal{A}$ is linearly independent. This means we need to determine if there are any non-trivial solutions to the equation

$$\alpha_1 \begin{bmatrix} 1 \\ -2+i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1+i \\ -3-i \end{bmatrix} + \alpha_3 \begin{bmatrix} 1+i \\ -3 \end{bmatrix} + \alpha_4 \begin{bmatrix} 5-2i \\ -5+9i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But this is the same equation we looked at to determine whether or not $\mathcal{A}$ is a spanning set, except that z_1 and z_2 are now set to zero. So, using the results of our previous work, we see that the solution to this equation can be found by looking at following matrix:

$$\left[\begin{array}{cccc|c} 1 & 1+i & 1+i & 5-2i & 0 \\ 0 & 0 & i & 3 & 0 \end{array}\right]$$

At this time our concern is not whether or not this equation has a solution (since we already know that the trivial solution will be a solution), but instead with how many solutions there are. We see from this row echelon form matrix that the rank of the coefficient matrix is 2, while there are 4 columns, so the general solution will contain $4 - 2 = 2$ parameters. As such, this means that the trivial solution is not the only solution, so the set $\mathcal{A}$ is not linearly independent.

And since $\mathcal{A}$ is not linearly independent, it is not a basis for $\mathbb{C}^2$.