Lecture 1d

Span and Linear Independence in Polynomials

(pages 194-196)

Just as we did with $\mathbb{R}^n$ and matrices, we can define spanning sets and linear independence of polynomials as well.

Definition: Let $\mathcal{B} = \{p_1(x), \ldots, p_k(x)\}$ be a set of polynomials of degree at most n. Then the **span** of $\mathcal{B}$ is defined as

$$\text{Span } \mathcal{B} = \{t_1 p_1(x) + \cdots + t_k p_k(x) \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

Example: Let $\mathcal{B} = \{1 + x + x^2, 1 + 2x + 3x^2, -5 - 5x^2\}$.

Then $-24 + 8x - 20x^2$ is in Span $\mathcal{B}$, since $4(1 + x + x^2) + 2(1 + 2x + 3x^2) + 6(-5 - 5x^2) = -24 + 8x - 20x^2$.

We also have $x^2 \in$ Span $\mathcal{B}$, since $-(1+x+x^2)+(1/2)(1+2x+3x^2)-(1/10)(-5-5x^2) = x^2$.

To see if $6 + x + 6x^2 \in$ Span $\mathcal{B}$, we need to look for a solution to the equation

$$t_1(1 + x + x^2) + t_2(1 + 2x + 3x^2) + t_3(-5 - 5x^2) = 6 + x + 6x^2$$

Performing the calculation on the left, we get

$t_1(1+x+x^2)+t_2(1+2x+3x^2)+t_3(-5-5x^2) = (t_1+t_1x+t_1x^2)+(t_2+2t_2x+3t_2x^2)+(-5t_3-5t_3x^2) \quad =$

$$\begin{array}{rrr}
t_1 & +t_1x & +t_1x^2 \\
t_2 & +2t_2x & +3t_2x^2 \\
-5t_3 & & -5t_3x^2 \\
\hline
(t_1+t_2-5t_3) & +(t_1+2t_2)x & +(t_1+3t_2-5t_3)x^2
\end{array}$$

So we are looking for $t_1, t_2, t_3, \in \mathbb{R}$ such that $(t_1 + t_2 - 5t_3) + (t_1 + 2t_2)x + (t_1 + 3t_2 - 5t_3)x^2 = 6 + x + 6x^2$. Setting the coefficients equal to each other, we find that we are looking for a solution to the following system of equations:

$$\begin{array}{rllll}
1: & t_1 & +t_2 & -5t_3 & = 6 \\
x: & t_1 & +2t_2 & & = 1 \\
x^2: & t_1 & +3t_2 & -5t_3 & = 6
\end{array}$$

To solve this system, we will row reduce the augmented matrix for the system, as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & -5 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 2 & 0 & 0 \end{array}\right] R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & -10 & 10 \end{array}\right] (-1/10)R_3 \sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + 5R_3 \\ R_2 - 5R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So we see that the solution to the system is $t_1 = 1$, $t_2 = 0$, $t_3 = -1$. This means that

$$(1 + x + x^2) + (0)(1 + 2x + 3x^2) - (-5 - 5x^2) = 6 + x + 6x^2$$

and so, yes, $6 + x + 6x^2$ IS in Span $\mathcal{B}$.

Definition: The set $\mathcal{B} = \{p_1(x), \ldots, p_k(x)\}$ is said to be **linearly independent** if the only solution to the equation

$$t_1 p_1(x) + \cdots + t_k p_k(x) = 0$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be **linearly dependent**.

Example To determine whether or not $\mathcal{B}$ is linearly independent, where

$$\mathcal{B} = \{1+2x+3x^2+4x^3, 2-x+7x^2+5x^3, -4-4x-8x^2-8x^3, -2-x+3x^2+3x^3\}$$

we need to look for non-trivial solutions to the equation

$$t_1(1+2x+3x^2+4x^3)+t_2(2-x+7x^2+5x^3)+t_3(-4-4x-8x^2-8x^3)+t_4(-2-x+3x^2+3x^3) = 0$$

Performing the calculation on the left, we get

$t_1(1+2x+3x^2+4x^3)+t_2(2-x+7x^2+5x^3)+t_3(-4-4x-8x^2-8x^3)+t_4(-2-x+3x^2+3x^3) =$

$(t_1+2t_1x+3t_1x^2+4t_1x^3)+(2t_2-t_2x+7t_2x^2+5t_2x^3)+(-4t_3-4t_3x-8t_3x^2-8t_3x^3)+(-2t_4-t_4x+3t_4x^2+3t_4x^3) \quad =$

$$\begin{array}{rrrr}
t_1 & +2t_1x & +3t_1x^2 & +4t_1x^3 \\
2t_2 & -t_2x & +7t_2x^2 & +5t_2x^3 \\
-4t_3 & -4t_3x & -8t_3x^2 & -8t_3x^3 \\
-2t_4 & -t_4x & +3t_4x^2 & +3t_4x^3 \\
\hline
(t_1+2t_2-4t_3-2t_4) & +(2t_1-t_2-4t_3-t_4)x & +(3t_1+7t_2-8t_3+3t_4)x^2 & +(4t_1+5t_2-8t_3+3t_4)x^3
\end{array}$$

So, we looking for non-trivial solutions to the equation

$$(t_1+2t_2-4t_3-2t_4)+(2t_1-t_2-4t_3-t_4)x+(3t_1+7t_2-8t_3+3t_4)x^2+(4t_1+5t_2-8t_3+3t_4)x^3 = 0+0x+0x^2+0x^3$$

Setting the coefficients equal, we see that this is equivalent to looking for solutions to the following homogeneous system of linear equations:

$$\begin{array}{rrrrl}
t_1 & +2t_2 & -4t_3 & -2t_4 & = 0 \\
2t_1 & -t_2 & -4t_3 & -t_4 & = 0 \\
3t_1 & +7t_2 & -8t_3 & +3t_4 & = 0 \\
4t_1 & +5t_2 & -8t_3 & +3t_4 & = 0
\end{array}$$

To solve this system, we row reduce the coefficient matrix as follows:

$$\left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 2 & -1 & -4 & -1 \\ 3 & 7 & -8 & 3 \\ 4 & 5 & -8 & 3 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-4R_1 \end{array} \sim \left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 0 & -5 & 4 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & -3 & 8 & 11 \end{array}\right] \begin{array}{l} \\ R_2 \updownarrow R_3 \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & -5 & 4 & 3 \\ 0 & -3 & 8 & 11 \end{array}\right] \begin{array}{l} \\ \\ R_3+5R_2 \\ R_4+3R_2 \end{array} \sim \left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 24 & 48 \\ 0 & 0 & 20 & 38 \end{array}\right] \begin{array}{l} \\ \\ (1/24)R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 20 & 38 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4-20R_3 \end{array} \sim \left[\begin{array}{rrrr} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 4. Since this is equal to the number of variables, our system has only one solution, specifically the trivial solution, which means that the set $\mathcal{B}$ is linearly independent.