# Solution to Practice 1h

**B1(a)** Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \right\}$. Then $\mathcal{A}$ is a basis for $\mathbb{R}^3$ if and only if $\mathcal{A}$ is a linearly independent spanning set for $\mathbb{R}^3$. To see if $\mathcal{A}$ is a spanning set for $\mathbb{R}^3$, let $\vec{x} \in \mathbb{R}^3$, and we will look for a solution to the equation

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 + t_3 \\ -t_1 + t_2 + 2t_3 \\ 3t_1 - t_2 - 4t_3 \end{bmatrix}$$

This is equivalent to looking for solutions to the system

$$\begin{array}{rrrl} t_1 & +2t_2 & +t_3 & = x_1 \\ -t_1 & +t_2 & +2t_3 & = x_2 \\ 3t_1 & -t_2 & -4t_3 & = x_3 \end{array}$$

To solve this system, we need to row reduce the augmented matrix for the system:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & x_1 \\ -1 & 1 & 2 & x_2 \\ 3 & -1 & -4 & x_3 \end{array}\right] \begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & x_1 \\ 0 & 3 & 3 & x_1 + x_2 \\ 0 & -4 & -7 & -3x_1 + x_3 \end{array}\right] \begin{array}{l} \\ (1/3)R_2 \\ \\ \end{array}$$
$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & (1/3)x_1 + (1/3)x_2 \\ 0 & -4 & -7 & -3x_1 + x_3 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 4R_2 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & x_1 \\ 0 & 1 & 1 & (1/3)x_1 + (1/3)x_2 \\ 0 & 0 & -3 & (-5/3)x_1 + (4/3)x_2 + x_3 \end{array}\right]$$

This last matrix is in row echelon form, and we see that there are no bad rows, so there is a solution. This means that $\vec{x}$ is in Span $\mathcal{A}$. And since this is true for any $\vec{x} \in \mathbb{R}^3$, we see that $\mathcal{A}$ is a spanning set for $\mathbb{R}^3$.

Now, to see that $\mathcal{A}$ is linearly independent, we need to look for solutions to the equation

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

This is simply a special case of the equation we looked at above, with $x_1$, $x_2$, and $x_3$ all set equal to zero. And so, proceeding as above, we would end up looking at the following row echelon form matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

From this matrix, we see that the solution set does not have any parameters, and thus there is only one solution (namely $t_1 = t_2 = t_3 = 0$), which means that $\mathcal{A}$ is linearly independent.

And since $\mathcal{A}$ is a linearly independent spanning set for $\mathbb{R}^3$, $\mathcal{A}$ IS a basis for $\mathbb{R}^3$.

Alternate Solution: Back in Section 2.3, Theorem 2.3.5 stated that a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{v}$ is $n$. Using this theorem, we would still need to row reduce the matrix $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 3 & -1 & -4 \end{bmatrix}$ to $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ (using the same steps as above), and then simply noting that the rank is 3 would prove that $\mathcal{A}$ is a basis for $\mathbb{R}^3$.

**B1(b)** Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$. Then $\mathcal{B}$ is not a basis for $\mathbb{R}^3$, because $\mathcal{B}$ is not linearly independent. To see this, let's look at the solutions to the equation

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 + 5t_3 \\ 3t_1 + t_3 \\ t_1 + t_2 + 3t_4 \end{bmatrix}$$

This is equivalent to looking for solutions to the following homogeneous system:

$$\begin{array}{rrrrl} t_1 & -t_2 & +5t_3 & & = 0 \\ 3t_1 & & +t_3 & & = 0 \\ t_1 & +t_2 & & +3t_4 & = 0 \end{array}$$

To solve this system, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 5 & 0 \\ 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & -1 & 5 & 0 \\ 0 & 3 & -14 & 0 \\ 0 & 2 & -5 & 3 \end{bmatrix} \begin{array}{l} \\ 2R_2 \\ 3R_3 \end{array}$$
$$\sim \begin{bmatrix} 1 & -1 & 5 & 0 \\ 0 & 6 & -28 & 0 \\ 0 & 6 & -15 & 9 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & -1 & 5 & 0 \\ 0 & 6 & -28 & 0 \\ 0 & 0 & 13 & 9 \end{bmatrix}$$

This last matrix is in row echelon form, and so we see that the rank of the coefficient matrix is 3. As there are 4 variables, this means that our system has infinitely many solutions. As such, $\mathcal{B}$ is linearly dependent.

Alternate Solution: Back in Section 2.3, Theorem 7 told us that all bases of a vector space needed to have the same number of elements. Since the standard basis for $\mathbb{R}^3$ contains 3 elements, we know that all bases for $\mathbb{R}^3$ must contain 3 elements. Since $\mathcal{B}$ does not contain 3 elements, it cannot be a basis for $\mathbb{R}^3$.

**B1(c)** Let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} \right\}$. Then $\mathcal{A}$ is a basis for $\mathbb{R}^3$ if and only if $\mathcal{A}$ is a linearly independent spanning set for $\mathbb{R}^3$. To see if $\mathcal{A}$ is a spanning set for $\mathbb{R}^3$, let $\vec{x} \in \mathbb{R}^3$, and we will look for a solution to the equation

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 + t_3 \\ 2t_1 - t_2 + t_3 \\ t_1 + 7t_3 \end{bmatrix}$$

his is equivalent to looking for solutions to the system

$$\begin{array}{rrrl} t_1 & -t_2 & +t_3 & = x_1 \\ 2t_1 & -t_2 & +t_3 & = x_2 \\ t_1 & & +7t_3 & = x_3 \end{array}$$

To solve this system, we need to row reduce the augmented matrix for the system:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & x_1 \\ 2 & -1 & 1 & x_2 \\ 1 & 0 & 7 & x_3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & x_1 \\ 0 & 1 & -1 & -2x_1 + x_2 \\ 0 & 1 & 6 & -x_1 + x_3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$
$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & x_1 \\ 0 & 1 & -1 & -2x_1 + x_2 \\ 0 & 0 & 7 & x_1 - x_2 + x_3 \end{array}\right]$$

This last matrix is in row echelon form, and we see that there are no bad rows, so there is a solution. This means that $\vec{x}$ is in Span $\mathcal{C}$. And since this is true for any $\vec{x} \in \mathbb{R}^3$, we see that $\mathcal{C}$ is a spanning set for $\mathbb{R}^3$.

Now, to see that $\mathcal{C}$ is linearly independent, we need to look for solutions to the equation

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$$

This is simply a special case of the equation we looked at above, with $x_1$, $x_2$, and $x_3$ all set equal to zero. And so, proceeding as above, we would end up looking at the following row echelon form matrix:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

From this matrix, we see that the solution set does not have any parameters, and thus there is only one solution (namely $t_1 = t_2 = t_3 = 0$), which means that $\mathcal{C}$ is linearly independent.

And since $\mathcal{C}$ is a linearly independent spanning set for $\mathbb{R}^3$, $\mathcal{C}$ IS a basis for $\mathbb{R}^3$.

<u>Alternate Solution</u>: Back in Section 2.3, Theorem 2.3.5 stated that a set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{v}$ is $n$. Using this theorem, we would still need to row reduce the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 7 \end{bmatrix}$ to $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix}$ (using the same steps as above), and then simply noting that the rank is 3 would prove that $\mathcal{C}$ is a basis for $\mathbb{R}^3$.

**B1(d)** Let $\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then $\mathcal{D}$ is not a basis for $\mathbb{R}^3$, because $\mathcal{D}$ is not a spanning set for $\mathbb{R}^3$. To see this, we try to find a solution to the equation

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_1 \end{bmatrix}$$

Equating the first components we get that $t_1 = 1$, but equating the third components gives us $t_1 = 3$. Since we cannot have both $t_1 = 1$ and $t_1 = 3$, we see that there are no solutions. As such, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not in the span of $\mathcal{D}$, and thus $\mathcal{D}$ is not a spanning set for $\mathbb{R}^3$.

<u>Alternate Solution</u>: Back in Section 2.3, Theorem 7 told us that all bases of a vector space needed to have the same number of elements. Since the standard basis for $\mathbb{R}^3$ contains 3 elements, we know that all bases for $\mathbb{R}^3$ must contain 3 elements. Since $\mathcal{D}$ does not contain 3 elements, it cannot be a basis for $\mathbb{R}^3$.

**B1(e)** Let $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \right\}$. Then $\mathcal{E}$ is not a basis for $\mathbb{R}^3$ because it is not linearly independent. (It also is not a spanning set, but showing that it is linearly dependent will suffice.) To see that $\mathcal{E}$ is linearly dependent, note that

$$(1) \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As such, $t_1 = t_2 = t_3 = 0$ is not the only solution to the equation

$$t_1 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**B2** Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & -6 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & -6 \\ 4 & 6 \end{bmatrix} \right\}$. To see that $\mathcal{B}$ is a basis for $M(2,2)$, let us first show that $\mathcal{B}$ is a spanning set for $M(2,2)$. To that end, we let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any element of $M(2,2)$, and we look to see if there is a solution to the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = t_1 \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} + t_3 \begin{bmatrix} 3 & -6 \\ 4 & 5 \end{bmatrix} + t_4 \begin{bmatrix} 3 & -6 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 + 3t_3 + 3t_4 & -2t_1 - 3 \\ t_1 + 3t_2 + 4t_3 + 4t_4 & t_1 + 4 \end{bmatrix}$$

This is equivalent to looking for solutions to the following system:

$$\begin{array}{rrrrl} t_1 & +2t_2 & +3t_3 & +3t_4 & = a \\ -2t_1 & -3t_2 & -6t_3 & -6t_4 & = b \\ t_1 & +3t_2 & +4t_3 & +4t_4 & = c \\ t_1 & +4t_2 & +5t_3 & +6t_4 & = 0 \end{array}$$

To determine if there are solutions to this system, we need to row reduce its augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 2 & 3 & 3 & a \\ -2 & -3 & -6 & -6 & b \\ 1 & 3 & 4 & 4 & c \\ 1 & 4 & 5 & 6 & d \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 3 & a \\ 0 & 1 & 0 & 0 & 2a+b \\ 0 & 1 & 1 & 1 & -a+c \\ 0 & 2 & 2 & 3 & -a+d \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 3 & a \\ 0 & 1 & 0 & 0 & 2a+b \\ 0 & 0 & 1 & 1 & -3a-b+c \\ 0 & 0 & 2 & 3 & -5a-2b+d \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - 2R_3 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 2 & 3 & 3 & a \\ 0 & 1 & 0 & 0 & 2a+b \\ 0 & 0 & 1 & 1 & -3a-b+c \\ 0 & 0 & 0 & 1 & a-2c+d \end{array}\right]$$

This last matrix is in row echelon form, and since the row echelon form can not have any bad rows, we see that there is always a solution to the system. And this means that $\mathcal{B}$ is a spanning set for $M(2,2)$.

Now we need to see that $\mathcal{B}$ is linearly independent. To do this, we need to look for solutions to the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} + t_3 \begin{bmatrix} 3 & -6 \\ 4 & 5 \end{bmatrix} + t_4 \begin{bmatrix} 3 & -6 \\ 4 & 6 \end{bmatrix}$$

This is the same equation we were looking at above, with zero substituted in for each of $a$, $b$, $c$, and $d$. As such, to find the solutions to this equation, we look at the row echelon matrix we got above:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

From this matrix, we see that the solution set does not have any parameters, and thus there is only one solution (namely $t_1 = t_2 = t_3 = t_4 = 0$), which means that $\mathcal{B}$ is linearly independent.

**B3(a)** Let $\mathcal{A} = \{1 + x + x^2, 1 - x^2\}$. Then $\mathcal{A}$ is not a basis for $P_2$, because it is not a spanning set for $P_2$. To see this, we will show that the polynomial $p(x) = 1$ is not in the span of $\mathcal{A}$. To that end, we will try to find scalars $t_1$ and $t_2$ such that

$$1 = t_1(1 + x + x^2) + t_2(1 - x^2) = (t_1 + t_2) + t_1 x + (t_1 - t_2)x^2$$

Setting the coefficients equal to each other, we see that this is equivalent to looking for solutions to the following system:

$$\begin{array}{lll} t_1 & +t_2 & = 1 \\ t_1 & & = 0 \\ t_1 & -t_2 & = 0 \end{array}$$

From the second equation we see that $t_1 = 0$. Plugging this into the third equation, we get $0 - t_2 = 0$, so $t_2 = 0$. Plugging these values into the first equation gives us $0 + 0 = 1$, which is not true. As such, there is not a solution to our equation, which means that $p(x) = 1$ is not in the span of $\mathcal{A}$.

**B3(b)** Let $\mathcal{B} = \{1 + x^2, 2 - x + 2x^2, -2 + 2x\}$. Then $\mathcal{B}$ is a basis for $P_2$ if and only if $\mathcal{B}$ is a linearly independent spanning set for $P_2$. To see if $\mathcal{B}$ is a spanning set for $P_2$, let $p(x) = p_0 + p_1 x + p_2 x^2$ be any element of $P_2$, and we will look for a solution to the equation

$$\begin{aligned} p_0 + p_1 x + p_2 x^2 &= t_1(1 + x^2) + t_2(2 - x + 2x^2) + t_3(-2 + 2x) \\ &= (t_1 + 2t_2 + 2t_3) + (-t_2 + 2t_3)x + (t_1 + 2t_2)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to looking for solutions to the following system:

$$\begin{array}{rrrl} t_1 & +2t_2 & +2t_3 & = p_0 \\ & -t_2 & +2t_3 & = p_1 \\ t_1 & +2t_2 & & = p_2 \end{array}$$

To determine if this system has any solutions, we row reduce its augmented matrix:

$$\left[\begin{array}{rrr|l} 1 & 2 & -2 & p_0 \\ 0 & -1 & 2 & p_1 \\ 1 & 2 & 0 & p_2 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & -2 & p_0 \\ 0 & -1 & 2 & p_1 \\ 0 & 0 & 2 & -p_0 + p_2 \end{array}\right]$$

The matrix is now in row echelon form, and since it can not have any bad rows, we know that there is always a solution. This means that $\mathcal{B}$ is a spanning set for $P_2$.

Now we need to see that $\mathcal{B}$ is linearly independent. To do this, we need to look for solutions to the equation

$$0 + 0x + 0x^2 = t_1(1 + x^2) + t_2(2 - x + 2x^2) + t_3(-2 + 2x)$$

This is the same equation we were looking at above, with zero substituted in for each of $p_0$, $p_1$, and $p_2$. As such, to find the solutions to this equation, we look at the row echelon matrix we got above:

$$\left[\begin{array}{rrr|l} 1 & 2 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

From this matrix, we see that the solution set does not have any parameters, and thus there is only one solution (namely $t_1 = t_2 = t_3 = t_4 = 0$), which means that $\mathcal{B}$ is linearly independent.

And since $\mathcal{B}$ is a linearly independent spanning set for $P_2$, $\mathcal{B}$ IS a basis for $P_2$.

**B3(c)** Let $\mathcal{C} = \{1 + x^2, 1 + x + x^2, -x - x^2, 1 + 2x^2\}$. Then $\mathcal{C}$ is not a basis for $P_2$, because it is not linearly independent. To see this, we will look for solutions to the equation

$$\begin{aligned} 0 + 0x + 0x^2 &= t_1(1 + x^2) + t_2(1 + x + x^2) + t_3(-x - x^2) + t_4(1 + 2x^2) \\ &= (t_1 + t_2 + t_4) + (t_2 - t_3)x + (t_1 + t_2 - t_3 + 2t_4)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to looking for solutions to the following system:

$$\begin{array}{rrrrl} t_1 & +t_2 & & +t_4 & = 0 \\ & t_2 & -t_3 & & = 0 \\ t_1 & +t_2 & -t_3 & +2t_4 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Our matrix is now in row echelon form, so we see that the rank of the coefficient matrix is 3. Since this is less than the number of variables, we know that the system has an infinite number of solutions. And this means that $\mathcal{C}$ is linearly dependent.

**B3(d)** Let $\mathcal{D} = \{3 + 2 + 2x^2, 1 + x + x^2, 1 - x - x^2\}$. Then $\mathcal{D}$ is not a basis for $P_2$, because it is not linearly independent. (It also is not a spanning set, but it will suffice to show that it is not linearly independent.) To see this, we will look for solutions to the equation

$$\begin{aligned} 0 + 0x + 0x^2 &= t_1(3 + 2 + 2x^2) + t_2(1 + x + x^2) + t_3(1 - x - x^2) \\ &= (3t_1 + t_2 + t_3) + (2t_1 + t_2 - t_3)x + (2t_1 + t_2 - t_3)x^2 \end{aligned}$$

Setting the coefficients equal to each other, we see that this is equivalent to looking for solutions to the following system:

$$\begin{array}{rrrl} 3t_1 & +t_2 & +t_3 & = 0 \\ 2t_1 & +t_2 & -t_3 & = 0 \\ 2t_1 & +t_2 & -t_3 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \\ R_2 - (2/3)R_1 \\ \\ \end{array} \sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1/3 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form, and so we see that the rank of the of the coefficient matrix is 2. Since this is less than the number of variables, we know that the system has an infinite number of solutions. And this means that $\mathcal{D}$ is linearly dependent.