Solution to Practice 3m

1(a) To see if A is a spanning set for $\mathbb{C}^3$, we need to see if, for any vector $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$, there are scalars $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\alpha_1 \begin{bmatrix} 1 \\ 2+3i \\ -1-i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1-i \\ 3+2i \\ 4-i \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

To solve this vector equation, we row reduce its augmented matrix:

$$\left[\begin{array}{rr|c} 1 & 1-i & z_1 \\ 2+3i & 3+2i & z_2 \\ -1-i & 4-i & z_3 \end{array}\right] \begin{array}{l} \\ R_2 + (-2-3i)R_1 \\ R_3 + (1+i)R_1 \end{array}$$

$$\sim \left[\begin{array}{rr|r} 1 & 1-i & z_1 \\ 0 & -2+i & z_2 + (-2-3i)z_1 \\ 0 & 6-i & z_3 + (1+i)z_1 \end{array}\right] \ (1/-2+i)R_2$$

$$\sim \left[\begin{array}{rr|r} 1 & 1-i & z_1 \\ 0 & 1 & (1/-2+i)(z_2 + (-2-3i)z_1) \\ 0 & 6-i & z_3 + (1+i)z_1 \end{array}\right] \ R_3 + (-6+i)R_2$$

$$\sim \left[\begin{array}{rr|r} 1 & 1-i & z_1 \\ 0 & 1 & (1/-2+i)(z_2 + (-2-3i)z_1) \\ 0 & 0 & (-6+i/-2+i)(z_2 + (-2-3i)z_1) + z_3 + (1+i)z_1 \end{array}\right]$$

So, if there are any values of z_1, z_2, and z_3 that make $(-6+i/-2+i)(z_2 + (-2-3i)z_1) + z_3 + (1+i)z_1 \neq 0$, our last row becomes a bad row. We quickly see that $z_1 = 0$, $z_2 = 0$, $z_3 = 1$ is such a possibility, and thus $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span of A. And this means that A is not a spanning set for $\mathbb{C}^3$.

1(b) To see if A is linearly independent, we need to see if there are any non-trivial solutions to the vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2+3i \\ -1-i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1-i \\ 3+2i \\ 4-i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting $z_1 = 0$, $z_2 = 0$, and $z_3 = 0$ into our work from part (a), we see that the following row echelon form matrix is equivalent to our equation

$$\left[\begin{array}{rr|r} 1 & 1-i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

From the row echelon form matrix, we see that the rank of the coefficient matrix is 2, which is the same as the number of columns in the coefficient matrix, so our equation has a unique solution. And thus, the only solution is the trivial solution, which means that A is linearly independent.

1(c) Since A is not a spanning set for $\mathbb{C}^3$, A is not a basis for $\mathbb{C}^3$.

2(a) To see if B is a spanning set for $\mathbb{C}^3$, we need to see if, for any vector $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$, there are scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$\alpha_1 \begin{bmatrix} 1 \\ 2+i \\ 4i \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ i \\ 1+3i \end{bmatrix} + \alpha_3 \begin{bmatrix} 1-i \\ 3-i \\ 4+4i \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

To solve this vector equation, we row reduce its augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 0 & 1-i & z_1 \\ 2+i & i & 3-i & z_2 \\ 4i & 1+3i & 4+4i & z_3 \end{array}\right] \begin{array}{l} \\ R_2+(-2-i)R_1 \\ R_3+(-4i)R_1 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1-i & z_1 \\ 0 & i & 0 & z_2+(-2-i)z_1 \\ 0 & 1+3i & 0 & z_3+(-4i)z_1 \end{array}\right] \begin{array}{l} \\ (-i)R_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1-i & z_1 \\ 0 & 1 & 0 & -iz_2+(-1+2i)z_1 \\ 0 & 1+3i & 0 & z_3+(-4i)z_1 \end{array}\right] \begin{array}{l} \\ \\ R_3+(-1-3i)R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 1-i & z_1 \\ 0 & 1 & 0 & -iz_2+(-1+2i)z_1 \\ 0 & 0 & 0 & z_3+(-4i)z_1+(-1-3i)(-iz_2+(-1+2i)z_1) \end{array}\right]$$

So, if there are any values of z_1, z_2, and z_3 that make $z_3+(-4i)z_1+(-1-3i)(-iz_2+(-1+2i)z_1) \neq 0$, our last row becomes a bad row. We quickly see that $z_1 = 0$, $z_2 = 0$, $z_3 = 1$ is such a possibility, and thus $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span of B. And this means that B is not a spanning set for $\mathbb{C}^3$.

2(b) To see if B is linearly independent, we need to see if there are any non-trivial solutions to the vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ 2+i \\ 4i \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ i \\ 1+3i \end{bmatrix} + \alpha_3 \begin{bmatrix} 1-i \\ 3-i \\ 4+4i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting $z_1 = 0$, $z_2 = 0$, and $z_3 = 0$ into our work from part (a), we see that the following row echelon form matrix is equivalent to our equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the row echelon form matrix, we see that the rank of the coefficient matrix is 2, which is less than the number of columns in the coefficient matrix, so the general solution to our equation has a parameter. As such, there is more than one solution to the equation, which means that B is not linearly independent.

2(c) Since B is neither a spanning set for $\mathbb{C}^3$ nor linearly independent, B is not a basis for $\mathbb{C}^3$.

3(a) To see if C is a spanning set for $\mathbb{C}^3$, we need to see if, for any vector $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$, there are scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$\alpha_1 \begin{bmatrix} i \\ i \\ i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 2i \\ -i \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

To solve this vector equation, we row reduce its augmented matrix:

$$\left[\begin{array}{ccc|c} i & 1 & -1 & z_1 \\ i & 0 & 2i & z_2 \\ i & 3i & -1 & z_3 \end{array}\right] \begin{array}{l} -iR_1 \\ -iR_2 \\ -iR_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -i & i & -iz_1 \\ 1 & 0 & 2 & -iz_2 \\ 1 & 1i & i & -iz_3 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -i & i & -iz_1 \\ 0 & i & 2-i & -iz_2 + iz_1 \\ 0 & 1+i & 0 & -iz_3 + iz_2 \end{array}\right] \begin{array}{l} \\ -iR_2 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -i & i & -iz_1 \\ 0 & 1 & -1-2i & -z_2 + z_1 \\ 0 & 1+i & 0 & -iz_3 + iz_2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + (-1-i)R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -i & i & -iz_1 \\ 0 & 1 & -1-2i & -z_2+z_1 \\ 0 & 0 & -1+3i & -iz_3+iz_2+(-1-i)(-z_2+z_1) \end{array}\right]$$

Our matrix is now in row echelon form, and since there are no values of z_1, z_2 and z_3 which can cause there to be a bad row, we see that our system has a solution for all values of z_1, z_2 and z_3. This means that $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ is in the span of C for all $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ in $\mathbb{C}^3$, so C is a spanning set for $\mathbb{C}^3$.

3(b) To see if C is linearly independent, we need to see if there are any non-trivial solutions to the vector equation

$$\alpha_1 \begin{bmatrix} i \\ i \\ i \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 2i \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting $z_1 = 0$, $z_2 = 0$, and $z_3 = 0$ into our work from part (a), we see that the following row echelon form matrix is equivalent to our equation:

$$\left[\begin{array}{ccc|c} 1 & -i & i & 0 \\ 0 & 1 & -1-2i & 0 \\ 0 & 0 & -1+3i & 0 \end{array}\right]$$

From the row echelon form matrix, we see that the rank of the coefficient matrix is 3, which is the same as the number of columns in the coefficient matrix, we see that this equation has a unique solution. And thus, the only solution is the trivial solution, which means that C is linearly independent.

3(c) Since C is a spanning set for $\mathbb{C}^3$ and is linearly independent, C is a basis for $\mathbb{C}^3$.

4(a) To see if D is a spanning set for $\mathbb{C}^3$, we need to see if, for any vector $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$, there are scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$ such that

$$\alpha_1 \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix} + \alpha_2 \begin{bmatrix} -2i \\ 2 \\ 2-2i \end{bmatrix} + \alpha_3 \begin{bmatrix} 2i \\ 0 \\ 3i \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 2 \\ 2+i \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

To solve this vector equation, we row reduce its augmented matrix:

$$\left[\begin{array}{rrrr|l} 1 & -2i & 2i & 0 & z_1 \\ i & 2 & 0 & 2 & z_2 \\ 1+i & 2-2i & 3i & 2+i & z_3 \end{array}\right] \begin{array}{l} \\ R_2 + (-i)R_1 \\ R_3 + (-1-i)R_1 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -2i & 2i & 0 & z_1 \\ 0 & 0 & 2 & 2 & z_2 - iz_1 \\ 0 & 0 & 2+i & 2+i & z_3 + (-1-i)z_1 \end{array}\right] \begin{array}{l} \\ \\ R_3 + ((-2-i)/2)R_2 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -2i & 2i & 0 & z_1 \\ 0 & 0 & 2 & 2 & z_2 - iz_1 \\ 0 & 0 & 0 & 0 & z_3 + (-1-i)z_1 + ((-2-i)/2))(z_2 - iz_1) \end{array}\right]$$

So, if there are any values of z_1, z_2, and z_3 that make $z_3 + (-1-i)z_1 + ((-2-i)/2))(z_2 - iz_1) \neq 0$, our last row becomes a bad row. We quickly see that $z_1 = 0$, $z_2 = 0$, $z_3 = 1$ is such a possibility, and thus $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in the span of D. And this means that D is not a spanning set for $\mathbb{C}^3$.

4(b) To see if D is linearly independent, we need to see if there are any non-trivial solutions to the vector equation

$$\alpha_1 \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix} + \alpha_2 \begin{bmatrix} -2i \\ 2 \\ 2-2i \end{bmatrix} + \alpha_3 \begin{bmatrix} 2i \\ 0 \\ 3i \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 2 \\ 2+i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Substituting $z_1 = 0$, $z_2 = 0$, and $z_3 = 0$ into our work from part (a), we see that the following row echelon form matrix is equivalent to our equation

$$\left[\begin{array}{rrrr|r} 1 & -2i & 2i & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the row echelon form matrix, we see that the rank of the coefficient matrix is 2, which is less than the number of columns in the coefficient matrix, so the general solution to our equation has parameters. As such, there is more than one solution to the equation, which means that D is not linearly independent.

4(c) Since D is neither a spanning set for $\mathbb{C}^3$ nor linearly independent, D is not a basis for $\mathbb{C}^3$.