Lecture 1c

Addition and Scalar Multiplication of Polynomials

(pages 193-194)

Long before you were introduced to $\mathbb{R}^n$ and matrices, you were introduced to polynomials. And somewhere along the way you were taught addition and scalar multiplication of polynomials, although perhaps not using those terms.

<u>Definition</u>: If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $q(x) = b_0 + b_1 x + \cdots + b_n x^n$ are both polynomials of degree less than or equal to n, and if t is a scalar (that is, $t \in \mathbb{R}$), then there are polynomials $(p+q)(x)$ and $(tp)(x)$ of degree less than or equal to n defined as follows:

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$(tp)(x) = (ta_0) + (ta_1)x + \cdots + (ta_n)x^n$$

Examples: $3(2+5x-4x^2) = ((3)(2)) + ((3)(5))x + ((3)(-4))x^2 = 6 + 15x - 12x^2$

When adding polynomials, I find it best to line up the corresponding entries in columns:

$$(1+2x+6x^2+5x^3) + (3-x-9x^2+3x^3) = \begin{array}{rrrr} 1 & +2x & +6x^2 & +5x^3 \\ 3 & -x & -9x^2 & +3x^3 \\ \hline 4 & +x & -3x^2 & +8x^3 \end{array}$$

This is particularly useful when some of the coefficients are zero:

$$(3-x^2+8x^4)+(2+8x-4x^2+7x^3-5x^5) = \begin{array}{rrrrrr} 3 & & -x^2 & & +8x^4 & \\ 2 & +8x & -4x^2 & +7x^3 & & -5x^5 \\ \hline 5 & +8x & -5x^2 & +7x^3 & +8x^4 & -5x^5 \end{array}$$

When scalar multiplication and addition are combined, I distribute the scalar first, and then line it up in columns to add:

$$6(1-3x-5x^2)-2(9-x^2) = (6-18x-30x^2)+(-18+2x^2) = \begin{array}{rrr} 6 & -18x & -30x^2 \\ -18 & & +2x^2 \\ \hline -12 & -18x & -28x^2 \end{array}$$

So why are we talking about polynomials? Because addition and scalar multiplication of polynomials satisfy the same set of useful properties that we got for $\mathbb{R}^n$ and matrices!

Theorem 4.1.1: Let $p(x)$, $q(x)$, and $r(x)$ be polynomials of degree at most n and let $s, t \in \mathbb{R}$. Then

(1) $p(x) + q(x)$ is a polynomial of degree at most n (closed under addition)
(2) $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ (addition is associative)
(3) The polynomial $0 = 0 + 0x + \cdots + 0x^n$ (called the **zero polynomial**), satisfies $p(x) + 0 = p(x) = 0 + p(x)$ for any polynomial $p(x)$ (additive identity)
(4) For each polynomial $p(x)$, there exists an additive inverse, denoted $(-p)(x)$, with the property that $p(x) + (-p)(x) = 0$. In particular, $(-p)(x) = -1p(x)$ (additive inverses)
(5) $p(x) + q(x) = q(x) + p(x)$ (addition is commutative)
(6) $sp(x)$ is a polynomial of degree at most n (closed under scalar multiplication)
(7) $s(tp(x)) = (st)p(x)$ (scalar multiplication is associative)
(8) $(s + t)p(x) = sp(x) + tp(x)$ (scalar addition is distributive)
(9) $s(p(x) + q(x)) = sp(x) + sq(x)$ (scalar multiplication is distributive)
(10) $1p(x) = p(x)$ (scalar multiplicative identity)

The proof of this theorem follows easily from the definitions of addition and scalar multiplication. We will not concern ourselves with it just now, but we will take another look at this theorem soon.