Lecture 1s

Isomorphisms of Vector Spaces

(pages 246-249)

Definition: L is said to be **one-to-one** if $L(\mathbf{u}_1) = L(\mathbf{u}_2)$ implies $\mathbf{u}_1 = \mathbf{u}_2$.

Example: The mapping $L : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $L(a, b, c, d) = (a, d)$ is not one-to-one. One counterexample is that $L(1, 2, 1, 2) = (1, 2)$ and $L(1, -1, -2, 2) = (1, 2)$, so $L(1, 2, 1, 2) = L(1, -1, -2, 2)$ even though $(1, 2, 1, 2) \neq (1, -1, -2, 2)$.

The mapping $L : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $L(a, b) = (a, b, a, b)$ is one-to-one. To prove this, we assume that $L(a_1, b_1) = L(a_2, b_2)$. That means $(a_1, b_1, a_1, b_1) = (a_2, b_2, a_2, b_2)$. From this we have $a_1 = a_2$ and $b_1 = b_2$, which means $(a_1, b_1) = (a_2, b_2)$. And so we have shown that $L(a_1, b_1) = L(a_2, b_2)$ implies $(a_1, b_1) = (a_2, b_2)$.

The mapping $L : \mathbb{R}^3 \to P_2$ defined by $L(a, b, c) = a + (a + b)x + (a - c)x^2$ is one-to-one. To prove this, we assume that $L(a_1, b_1, c_1) = L(a_2, b_2, c_2)$. That means $a_1 + (a_1 + b_1)x + (a_1 - c_1)x^2 = a_2 + (a_2 + b_2)x + (a_2 - c_2)x^2$. This means that $a_1 = a_2$, $a_1 + b_1 = a_2 + b_2$, and $a_1 - c_1 = a_2 - c_2$. Plugging the fact that $a_1 = a_2$ into the other two equations gives us $a_1 + b_1 = a_1 + b_2 \Rightarrow b_1 = b_2$ and $a_1 - c_1 = a_1 - c_2 \Rightarrow c_1 = c_2$. And since $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$, we have $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, and thus L is one-to-one.

Lemma 4.7.1: L is one-to-one if and only if $\text{Null}(L) = \{\mathbf{0}\}$.

Proof of Lemma 4.7.1: We will need to prove both directions of this if and only if statement:

$\Rightarrow$ (If L is one-to-one, then $\text{Null}(L) = \{\mathbf{0}\}$): Suppose that L is one-to-one, and let $\mathbf{x} \in \text{Null}(L)$. Then $L(\mathbf{x}) = \mathbf{0}$. But we also know that $L(\mathbf{0}) = \mathbf{0}$. Since L is one-to-one and $L(\mathbf{x}) = L(\mathbf{0})$, we have $\mathbf{x} = \mathbf{0}$. As such, we see that $\mathbf{0}$ is the only element of $\text{Null}(L)$.

$\Leftarrow$ (If $\text{Null}(L) = \{\mathbf{0}\}$, then L is one-to-one): Suppose that $\text{Null}(L) = \{\mathbf{0}\}$, and further suppose that $L(\mathbf{u}_1) = L(\mathbf{u}_2)$. Then $L(\mathbf{u}_1 - \mathbf{u}_2) = L(\mathbf{u}_1) - L(\mathbf{u}_2) = \mathbf{0}$. This means that $\mathbf{u}_1 - \mathbf{u}_2$ is in the nullspace of L. But $\mathbf{0}$ is the only element of the nullspace, so we have $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}$. By the uniqueness of the additive inverse, this means that $\mathbf{u}_1 = \mathbf{u}_2$, and this shows that L is one-to-one.

Definition: $L : \mathbb{U} \to \mathbb{V}$ is said to be **onto** if for every $\mathbf{v} \in \mathbb{V}$ there exists some $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. That is, $\text{Range}(L) = \mathbb{V}$.

Example: The mapping $L : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $L(a, b, c, d) = (a, d)$ is onto. To prove this, let $(s_1, s_2) \in \mathbb{R}^2$. Then $(s_1, 0, 0, s_2) \in \mathbb{R}^4$ is such that $L(s_1, 0, 0, s_2) =$

(s_1, s_2). (Note: I could have also used $(s_1, 1, 1, s_2)$ or (s_1, s_1, s_2, s_2), or countless other vectors from $\mathbb{R}^4$. We need to be able to find *at least* one vector $\mathbf{u}$ such that Ł$(\mathbf{u}) = \mathbf{v}$, not *exactly* one.)

The mapping $L : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $L(a, b) = (a, b, a, b)$ is not onto. One counterexample is that there is no $(a, b) \in \mathbb{R}^2$ such that $L(a, b) = (1, 2, 3, 4)$, since we would need to have $a = 1$ and $a = 3$, which is not possible.

The mapping $L : \mathbb{R}^3 \to P_2$ defined by $L(a, b, c) = a + (a + b)x + (a - c)x^2$ is onto. To prove this, let $s_0 + s_1 x + s_2 x^2 \in P_2$. To show that L is onto, we need to find $(a, b, c) \in \mathbb{R}^3$ such that $L(a, b, c) = s_0 + s_1 x + s_2 x^2$. That is, we need to find $a, b, c \in \mathbb{R}$ such that $a + (a + b)x + (a - c)x^2 = s_0 + s_1 x + s_2 x^2$. Setting the coefficients equal to each other, we see that we need $a = s_0$, $a + b = s_1$, and $a - c = s_2$. Plugging $a = s_0$ into the other two equations, we get that $s_0 + b = s_1 \Rightarrow b = s_1 - s_0$, and $s_0 - c = s_2 \Rightarrow c = s_0 - s_2$. And so we have found $(s_0, s_1 - s_0, s_0 - s_2) \in \mathbb{R}^3$ such that $L(s_0, s_1 - s_0, s_0 - s_2) = s_0 + (s_0 + (s_1 - s_0))x + (s_0 - (s_0 - s_2))x^2 = s_0 + s_1 x + s_2 x^2$.

Theorem 4.7.2: The linear mapping $L : \mathbb{U} \to \mathbb{V}$ has an inverse mapping $L^{-1} : \mathbb{V} \to \mathbb{U}$ if and only if L is one-to-one and onto.

Proof of Theorem 4.7.2: First, let's suppose that L has an inverse mapping. To see that L is one-to-one, we assume that $L(\mathbf{u}_1) = L(\mathbf{u}_2)$. Then $\mathbf{u}_1 = L^{-1}(L(\mathbf{u}_1)) = L^{-1}(L(\mathbf{u}_2)) = \mathbf{u}_2$, and so we see that L is one-to-one. To see that L is onto, let $\mathbf{v} \in \mathbb{V}$. Then $L^{-1}(\mathbf{v}) \in \mathbb{W}$ is such that $L(L^{-1}(\mathbf{v})) = \mathbf{v}$, and this means that L is onto.

Now, let's suppose that L is one-to-one and onto, and let's try to define a mapping $M : \mathbb{V} \to \mathbb{U}$ that will turn out to be L^{-1}. There are two things we need to do to define any mapping from $\mathbb{V}$ to $\mathbb{U}$. The first is to make sure we define M for every element of $\mathbb{V}$. So, let $\mathbf{v} \in \mathbb{V}$. Then, since L is onto, we know that there is some $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. So if we say $M(\mathbf{v}) = \mathbf{u}$ such that $L(\mathbf{u}) = \mathbf{v}$, then we are defining M for every element of $\mathbb{V}$. But the other thing we have to do when defining a mapping is to make sure that we don't send our vector $\mathbf{v}$ to two different vectors in $\mathbb{U}$. However, because L is one-to-one, we know that if $L(\mathbf{u}_1) = \mathbf{v}$ and $L(\mathbf{u}_2) = \mathbf{v}$, then $L(\mathbf{u}_1) = L(\mathbf{u}_2)$, so $\mathbf{u}_1 = \mathbf{u}_2$. And so, we can define $M(\mathbf{v})$ to be the unique vector $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. And this mapping M satisfies $M(L(\mathbf{u}) = M(\mathbf{v}) = \mathbf{u}$ and $L(M(\mathbf{v})) = L(\mathbf{u}) = \mathbf{v}$, which means that $M = L^{-1}$.

Definition: If $\mathbb{U}$ and $\mathbb{V}$ are vector spaces over $\mathbb{R}$, and if $L : \mathbb{U} \to \mathbb{V}$ is a linear, one-to-one, and onto mapping, then L is called an **isomorphism** (or a vector space isomorphism), and $\mathbb{U}$ and $\mathbb{V}$ are said to be **isomorphic**.

Note: The reason that we include the alternate name "vector space isomorphism" is that there are lots of different definitions for an isomorphism in the world of mathematics. You may have encountered a definition that only requires a function that is one-to-one and onto (but not linear). The word "isomorphism" comes from the Greek words meaning "same form", so the definition of an isomorphism depends on what form you want to have be the same. In linear algebra, linear combinations are an important part of the form of a vector space, so we add the requirement that our function preserve linear combinations. You'll find that different areas of study have a different concept of form. You get used to it...

Example: We've seen that the linear mapping $L : \mathbb{R}^3 \to P_2$ defined by $L(a,b,c) = a + (a+b)x + (a-c)x^2$ is both one-to-one and onto, so L is an isomorphism, and $\mathbb{R}^3$ and P_2 are isomorphic.

<u>Theorem 4.7.3</u>: Suppose that $\mathbb{U}$ and $\mathbb{V}$ are finite-dimensional vector spaces over $\mathbb{R}$. Then $\mathbb{U}$ and $\mathbb{V}$ are isomorphic if and only if they are of the same dimension.

<u>Proof of Theorem 4.7.3</u>: We will need to prove both directions of this if and only if statement:

$\Rightarrow$ (If $\mathbb{U}$ and $\mathbb{V}$ are isomorphic, then they have the same dimension): Suppose that $\mathbb{U}$ and $\mathbb{V}$ are isomorphic, let L be an isomorphism from $\mathbb{U}$ to $\mathbb{V}$, and let $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $\mathbb{U}$. Then I claim that $\mathcal{C} = \{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is a basis for $\mathbb{V}$. First, lets show that $\mathcal{C}$ is linearly independent. To that end, let $t_1, \dots, t_n \in \mathbb{R}$ be such that

$$t_1 L(\mathbf{u}_1) + \cdots + t_n L(\mathbf{u}_n) = \mathbf{0}_{\mathbb{V}}$$

then

$$L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n) = \mathbf{0}_{\mathbb{V}}$$

and so we see that $t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n \in \text{Null}(L)$. But, since L is an isomorphism, L is one-to-one, so we know that $\text{Null}(L) = \{\mathbf{0}_{\mathbb{U}}\}$. This means that

$$t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n = \mathbf{0}_{\mathbb{U}}$$

Since the vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ are a basis for $\mathbb{U}$, they are linearly independent, so we know that $t_1 = \cdots = t_n = 0$. As such, we've shown that the vectors $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ are also linearly independent. Now we need to show that they are a spanning set for $\mathbb{V}$. That is, given any $\mathbf{v} \in \mathbb{V}$, we need to find $s_1, \dots, s_n \in \mathbb{R}$ such that

$$s_1 L(\mathbf{u}_1) + \cdots + s_n L(\mathbf{u}_n) = \mathbf{v}$$

Since L is an isomorphism, we know that L is onto, so there is a $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. And since $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a basis for $\mathbb{U}$, we know that there are $r_1, \dots, r_n \in \mathbb{R}$ such that

$$r_1\mathbf{u}_1 + \cdots + r_n\mathbf{u}_n = \mathbf{u}$$

Then

$$\begin{aligned} \mathbf{v} &= L(\mathbf{u}) \\ &= L(r_1\mathbf{u}_1 + \cdots + r_n\mathbf{u}_n) \\ &= r_1L(\mathbf{u}_1) + \cdots + r_nL(\mathbf{u}_n) \end{aligned}$$

and so we see that $\mathbf{v}$ is in the span of $\mathcal{C} = \{L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\}$. And since we have shown that $\mathcal{C}$ is a linearly independent spanning set for $\mathbb{V}$, we know that it is a basis for $\mathbb{V}$. And this means that the dimension of $\mathbb{V}$ is n, which is the same as the dimension of $\mathbb{U}$.

$\Leftarrow$ (If $\mathbb{U}$ and $\mathbb{V}$ have the same dimension, then they are isomorphic): Let $\mathbb{U}$ and $\mathbb{V}$ have the same dimension, let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{U}$, and let $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{V}$. Then I claim that the linear mapping $L : \mathbb{U} \to \mathbb{V}$ defined by

$$L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n) = t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$$

is an isomorphism. I will take for granted that it is a linear mapping, and instead focus on showing that it is one-to-one and onto. To see that L is one-to-one, suppose that

$$L(r_1\mathbf{u}_1 + \cdots + r_n\mathbf{u}_n) = L(s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n)$$

Then

$$r_1\mathbf{v}_1 + \cdots + r_n\mathbf{v}_n = s_1\mathbf{v}_1 + \cdots + s_n\mathbf{v}_n$$

Bringing everything to one side, we get that

$$(r_1 - s_1)\mathbf{v}_1 + \cdots + (r_n - s_n)\mathbf{v}_n = \mathbf{0}_{\mathbb{V}}$$

But since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{V}$, it is linearly independent, so we have that $r_i - s_i = 0$ for all i, or that $r_i = s_i$ for all i. Which means that

$$r_1\mathbf{u}_1 + \cdots + r_n\mathbf{u}_n = s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n$$

and so L is one-to-one. Now let's see that L is onto. To that end, let $\mathbf{v} \in \mathbb{V}$. Then there are $s_1, \ldots, s_n \in \mathbb{R}$ such that

$$\mathbf{v} = s_1\mathbf{v}_1 + \cdots + s_n\mathbf{v}_n$$

And we then have

$$L(s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n) = s_1\mathbf{v}_1 + \cdots + s_n\mathbf{v}_n$$

so $s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n \in \mathbb{U}$ is such that $L(s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n) = \mathbf{v}$, and we see that L is onto.

For once the book has generalized more than I would, because I find the following statement to be quite important in the study of linear algebra.

Corollary to Theorem 4.7.3: If $\mathbb{U}$ is a vector space with dimension n, then $\mathbb{U}$ is isomorphic to $\mathbb{R}^n$.

That is, every finite dimensional vector space is really just the same as $\mathbb{R}^n$. Which is fabulous, because we know a lot of things about $\mathbb{R}^n$, and $\mathbb{R}^n$ is easy to work with. I'll discuss the wonders of this new revelation in the next lecture. But for now, we finish with one last theorem.

Theorem 4.7.4: If $\mathbb{U}$ and $\mathbb{V}$ are n-dimensional vector spaces over $\mathbb{R}$, then a linear mapping $L : \mathbb{U} \to \mathbb{V}$ is one-to-one if and only if it is onto.

Proof of Theorem 4.7.4: Let $\mathbb{U}$ and $\mathbb{V}$ be n-dimensional vector spaces over $\mathbb{R}$, and let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{U}$ and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{V}$.

Now suppose that L is one-to-one. To show that L is onto, I am first going to show that the set $\mathcal{C}_1 = \{L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\}$ is a basis for $\mathbb{V}$. And to do that, I first want to show that $\mathcal{C}_1$ is linearly independent. To that end, suppose that $t_1, \ldots, t_n \in \mathbb{R}$ are such that

$$t_1L(\mathbf{u}_1) + \cdots + t_nL(\mathbf{u}_n) = \mathbf{0}$$

Using the linearity of L we get

$$L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n) = \mathbf{0}$$

and so we see that $t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n$ is in the nullspace of L. But, since L is one-to-one, Lemma 4.7.1 tells us that $\text{Null}(L) = \{\mathbf{0}\}$. So we have

$$t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n = \mathbf{0}$$

At this point we use the fact that $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis for $\mathbb{U}$, and therefore is linearly independent, to see that $t_1 = \cdots = t_n = 0$. And this means that our set $\mathcal{C}_1 = \{L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\}$ is also linearly independent. We also know that $\mathcal{C}_1$ contains n vectors. This is actually not immediately obvious, since it in general one could have that $L(\mathbf{u}_i) = L(\mathbf{u}_j)$ for some $i \neq j$, but because L is one-to-one, we know that $L(\mathbf{u}_i) \neq L(\mathbf{u}_j)$ whenever $i \neq j$, because the vectors $\mathbf{u}_i$ are all distinct. An even easier way to see this is to note that the set $\mathcal{C}_1$ is linearly independent, and a linearly independent set cannot contain duplicate vectors.

Either way, we have found that $\mathcal{C}_1$ is a linearly independent set containing $\dim(\mathbb{V})$ elements of $\mathbb{V}$, so by the two-out-of-three rule (aka Theorem 4.3.4(3)), $\mathcal{C}_1$ is also a spanning set for $\mathbb{V}$, and therefore is a basis for $\mathbb{V}$.

But what I actually need is the fact that $\mathcal{C}_1$ is a spanning set for $\mathbb{V}$. Because now, given any $\mathbf{v} \in \mathbb{V}$, there are scalars $s_1, \ldots, s_n$ such that

$$s_1L(\mathbf{u}_1) + \cdots + s_nL(\mathbf{u}_n) = \mathbf{v}$$

This means that $s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n \in \mathbb{U}$ is such that

$$L(s_1\mathbf{u}_1 + \cdots + s_n\mathbf{u}_n) = s_1L(\mathbf{u}_1) + \cdots + s_nL(\mathbf{u}_n) = \mathbf{v}$$

and at long last we have shown that L is onto.

Now, we turn to the other direction of our proof. So, let us assume that L is onto, and show that L is also one-to-one. The first step in that process will be to note that, because L is onto, for every vector $\mathbf{v}_i$ in our basis $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{V}$, there is a vector $\mathbf{w}_i$ in $\mathbb{U}$ such that $L(\mathbf{w}_i) = \mathbf{v}_i$. I claim that the set $\mathcal{B}_1 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of these vectors is a basis for $\mathbb{U}$. To prove this claim, I will first show that $\mathcal{B}_1$ is linearly independent. To that end, let $t_1, \ldots, t_n \in \mathbb{R}$ be such that

$$t_1\mathbf{w}_1 + \cdots + t_n\mathbf{w}_n = \mathbf{0}$$

Then we see that

$$\begin{aligned} & L(t_1\mathbf{w}_1 + \cdots + t_n\mathbf{w}_n) = L(\mathbf{0}) = \mathbf{0}, \\ \text{so}\quad & t_1L(\mathbf{w}_1) + \cdots + t_nL(\mathbf{w}_n) = \mathbf{0}, \\ \text{so}\quad & t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n = \mathbf{0} \end{aligned}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, we have $t_1 = \cdots = t_n = 0$. And so we have shown that $\mathcal{B}_1 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is also linearly independent. And, as noted earlier, this means that $\mathcal{B}_1$ does not contain any repeated entries, which means that $\mathcal{B}_1$ has $n = \dim(\mathbb{U})$ elements. Again using the two-out-of-three rule, $\mathcal{B}_1$ is a spanning set for $\mathbb{U}$, and thus is a basis for $\mathbb{U}$. But again, it is the fact that $\mathcal{B}_1$ is a spanning set for $\mathbb{U}$ that I want to use. For now, to show that L is one-to-one, suppose that $L(\mathbf{u}) = L(\mathbf{w})$. Since $\mathcal{B}_1$ is a spanning set for $\mathbb{U}$, we know that there are scalars $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that

$$\mathbf{u} = a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n \quad \text{and} \quad \mathbf{w} = b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n$$

Then

$$L(\mathbf{u}) = L(a_1\mathbf{w}_1 + \cdots + a_n\mathbf{w}_n) = a_1L(\mathbf{w}_1) + \cdots + a_nL(\mathbf{w}_n) = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

and similarly

$$L(\mathbf{w}) = L(b_1\mathbf{w}_1 + \cdots + b_n\mathbf{w}_n) = b_1L(\mathbf{w}_1) + \cdots + b_nL(\mathbf{w}_n) = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

which means that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

and from this we get

$$(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{0}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, we have $(a_1 - b_1) = \cdots = (a_n - b_n) = 0$. And this means that $a_i = b_i$ for all $1 \leq i \leq n$. And this means that $\mathbf{u} = \mathbf{w}$, so we have shown that L is one-to-one.