Solution to Practice 2k

B1(a) Let $p(x) = 1 - 3x^2$ and $q(x) = 1 + x + 2x^2$. Then we have

$$\begin{array}{lll} p(0) = 1 - 3(0) = 1 & p(1) = 1 - 3(1) = -2 & p(2) = 1 - 3(4) = -11 \\ q(0) = 1 + 0 + 2(0) = 1 & q(1) = 1 + 1 + 2(1) = 4 & q(2) = 1 + 2 + 2(4) = 11 \end{array}$$

So $\langle p, q\rangle = (1)(1) + (-2)(4) + (-11)(11) = -127$

B1(b) Let $p(x) = 3 - x$ and $q(x) = -2 - x - x^2$. Then we have

$$\begin{array}{lll} p(0) = 3 - 0 = 3 & p(1) = 3 - 1 = 2 & p(2) = 3 - 2 = 1 \\ q(0) = -2 - 0 - 0 = -2 & q(1) = -2 - 1 - 1 = -4 & q(2) = -2 - 2 - 4 = -8 \end{array}$$

So $\langle p, q\rangle = (3)(-2) + (2)(-4) + (1)(-8) = -22$

B1(c) Let $p(x) = 1 - 5x + 2x^2$. Then we have

$$p(0) = 1 - 5(0) + 2(0) = 1 \quad p(1) = 1 - 5(1) + 2(1) = -2 \quad p(2) = 1 - 10 + 2(4) = -1$$

So $||p||^2 = \langle p, p\rangle = (1)(1)+(-2)(-2)+(-1)(-1) = 6$, and thus $||1-5x+2x^2|| = \sqrt{6}$.

B1(d) Let $p(x) = 73x + 73x^2$. Then we have

$$p(0) = 73(0) + 73(0) = 0 \quad p(1) = 73(1) + 73(1) = 146 \quad p(2) = 73(2) + 73(4) = 438$$

So $||p||^2 = \langle p, p\rangle = (0)(0) + (146)(146) + (438)(438) = 213160 = 146/sqrt10$

A3(a)(i): Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$, so that $\mathbb{S} = \text{Span}\{A_1, A_2, A_3\}$. Then we set $B_1 = A_1$, and

$$\begin{aligned}
B_2 &= A_2 - \tfrac{\langle A_2, B_1\rangle}{\langle B_1, B_1\rangle} B_1 \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \tfrac{1+0+0+1}{1^2+0^2+(-1)^2+1^2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \tfrac{2}{3} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 & 1 \\ 2/3 & 1/3 \end{bmatrix}
\end{aligned}$$

Because of the bilinearilty of all inner products, we can multiply our resulting "vector" by a scalar without affecting orthogonality. So, I choose to pull out a factor of 1/3 and set $B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$. Then we have

$$\begin{aligned}
B_3 &= A_3 - \tfrac{\langle A_3, B_1\rangle}{\langle B_1, B_1\rangle} B_1 - \tfrac{\langle A_3, B_2\rangle}{\langle B_2, B_2\rangle} B_2 \\
&= \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - \tfrac{2+0-1-1}{1^2+0^2+(-1)^2+1^2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \tfrac{2+0+2-1}{1^2+3^2+2^2+1^2} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - 0 \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \tfrac{1}{5} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \\
&= \tfrac{3}{5} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}
\end{aligned}$$

And so we can set $B_3 = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$.

By the Gram-Schmidt Procedure, we know that $\{B_1, B_2, B_3\}$ is an orthogonal basis for Span$\{A_1, A_2, A_3\}$. But the problem asks us to find an orthonormal basis. As such, we note that

$$\begin{aligned}
||B_1|| &= \sqrt{\langle B_1, B_1\rangle} = \sqrt{1^2 + 0^2 + (-1)^2 + 1^2} = \sqrt{3} \\
||B_2|| &= \sqrt{\langle B_2, B_2\rangle} = \sqrt{1^2 + 3^2 + 2^2 + 1^2} = \sqrt{15} \\
||B_3|| &= \sqrt{\langle B_3, B_3\rangle} = \sqrt{3^2 + (-1)^2 + 1^2 + (-2)^2} = \sqrt{15}
\end{aligned}$$

So we have that $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{1}{\sqrt{15}} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \frac{1}{\sqrt{15}} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{S}$.

A3(a)(ii): Let $C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$. Then

$$
\begin{aligned}
\text{proj}_{\mathbb{S}} C &= \tfrac{\langle C,B_1\rangle}{\langle B_1,B_1\rangle} B_1 + \tfrac{\langle C,B_2\rangle}{\langle B_2,B_2\rangle} B_2 + \tfrac{\langle C,B_3\rangle}{\langle B_3,B_3\rangle} B_3 \\
&= \tfrac{4+0+2+1}{1^2+0^2+(-1)^2+1^2} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \tfrac{4+9-4+1}{1^2+3^2+2^2+1^2} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \\
&\quad + \tfrac{12-3-2-2}{3^2+(-1)^2+1^2+(-2)^2} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \\
&= \tfrac{7}{3} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \tfrac{2}{3} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + \tfrac{1}{3} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 5/3 \\ -2/3 & 7/3 \end{bmatrix}
\end{aligned}
$$

A3(b)(i): Let $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$, so that $\mathbb{S} = \text{Span}\{A_1, A_2, A_3\}$. Then we set $B_1 = A_1$, and

$$
\begin{aligned}
B_2 &= A_2 - \tfrac{\langle A_2,B_1\rangle}{\langle B_1,B_1\rangle} B_1 \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} - \tfrac{0-1+0+1}{1^2+1^2+0^2+1^2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} - 0 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
B_3 &= A_3 - \tfrac{\langle A_3,B_1\rangle}{\langle B_1,B_1\rangle} B_1 - \tfrac{\langle A_3,B_2\rangle}{\langle B_2,B_2\rangle} B_2 \\
&= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} - \tfrac{2+0+0+1}{1^2+1^2+0^2+1^2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \tfrac{0+0-1+1}{0^2+(-1)^2+1^2+1^2} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 0 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ -1 & -0 \end{bmatrix}
\end{aligned}
$$

By the Gram-Schmidt Procedure, we know that $\{B_1, B_2, B_3\}$ is an orthogonal basis for $\text{Span}\{A_1, A_2, A_3\}$. But the problem asks us to find an orthonormal basis. As such, we note that

$$
\begin{aligned}
||B_1|| &= \sqrt{\langle B_1, B_1\rangle} = \sqrt{1^2+1^2+0^2+1^2} = \sqrt{3}\\
||B_2|| &= \sqrt{\langle B_2, B_2\rangle} = \sqrt{0^2+(-1)^2+1^2+1^2} = \sqrt{3}\\
||B_3|| &= \sqrt{\langle B_3, B_3\rangle} = \sqrt{1^2+(-1)^2+(-1)^2+0^2} = \sqrt{3}
\end{aligned}
$$

So we have that $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 \\ -1 & -0 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{S}$.

A3(b)(ii): Let $C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$. Then

$$
\begin{aligned}
\text{proj}_{\mathbb{S}} C &= \tfrac{\langle C, B_1\rangle}{\langle B_1, B_1\rangle} B_1 + \tfrac{\langle C, B_2\rangle}{\langle B_2, B_2\rangle} B_2 + \tfrac{\langle C, B_3\rangle}{\langle B_3, B_3\rangle} B_3 \\
&= \tfrac{4+3+0+1}{1^2+1^2+0^2+1^2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \tfrac{0-3-2+1}{0^2+(-1)^2+1^2+1^2} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\
&\quad + \tfrac{4-3+2+0}{1^2+(-1)^2+(-1)^2+0^2} \begin{bmatrix} 1 & -1 \\ -1 & -0 \end{bmatrix} \\
&= \tfrac{8}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \tfrac{4}{3} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} + \tfrac{3}{3} \begin{bmatrix} 1 & -1 \\ -1 & -0 \end{bmatrix} \\
&= \begin{bmatrix} 11/3 & 3 \\ -7/3 & 4/3 \end{bmatrix}
\end{aligned}
$$

B3(a)(i): Let $p_1 = 1$ and $p_2 = x - x^2$.

Then we set $q_1 = p_1$, and we find q_2 as follows:

$$
q_2 \quad = p_2 - \tfrac{\langle p_2, q_1\rangle}{\langle q_1, q_1\rangle} q_1
$$

To computer $\langle p_2, q_1\rangle$ and $\langle q_1, q_1\rangle$, we first make the following calculations:

$$
\begin{array}{lll}
q_1(-1) = 1 & q_1(0) = 1 & q_1(1) = 1 \\
p_2(-1) = -1 - (-1)^2 = -2 & p_2(0) = 0 & p_2(1) = 1 - 1^2 = 0
\end{array}
$$

So we now compute that

$$
\begin{aligned}
q_2 \quad &= (x - x^2) - \tfrac{(-2)(1)+(0)(1)+(0)(1)}{1^2+1^2+1^2}(1) \\
&= x - x^2 + \tfrac{2}{3} \\
&= \tfrac{1}{3}(2 + 3x - 3x^2)
\end{aligned}
$$

And so we can set $q_2 = 2 + 3x - 3x^2$, and by the Gram-Schmidt Procedure we know that $\{1, 2 + 3x - 3x^2\}$ is an orthogonal basis for $\mathbb{S}$.

B3(a)(ii): Let $r = 1 + x + x^2$. Then we have

$$\text{proj}_{\mathbb{S}} r = \frac{\langle r, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 + \frac{\langle r, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$$

To compute our inner products, we first make the following calculations:

$$\begin{array}{lll} q_1(-1) = 1 & q_1(0) = 1 & q_1(1) = 1 \\ q_2(-1) = 2 - 3 - 3(-1)^2 = -4 & q_2(0) = 2 & q_2(1) = 2 + 3 - 3(1^2) = 2 \\ r(-1) = 1 - 1 + (-1)^2 = 1 & r(0) = 1 & r(1) = 1 + 1 + 1^2 = 3 \end{array}$$

So we can now find $\text{proj}_{\mathbb{S}} r$:

$$\begin{aligned} \text{proj}_{\mathbb{S}} r &= \tfrac{(1)(1)+(1)(1)+(3)(1)}{1^2+1^2+1^2}(1) + \tfrac{(1)(-4)+(1)(2)+(3)(2)}{(-4)^2+2^2+2^2}(2 + 3x - 3x^2) \\ &= \tfrac{5}{3} + \tfrac{4}{24}(2 + 3x - 3x^2) \\ &= 2 + \tfrac{1}{2}x - \tfrac{1}{2}x^2 \end{aligned}$$

B3(b)(i): Let $p_1 = 1 + x^2$ and $p_2 = x - x^2$.

Then we set $q_1 = p_1$, and we find q_2 as follows:

$$q_2 \quad = p_2 - \tfrac{\langle p_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1$$

To computer $\langle p_2, q_1 \rangle$ and $\langle q_1, q_1 \rangle$, we first make the following calculations:

$$\begin{array}{lll} q_1(-1) = 1 + (-1)^2 = 2 & q_1(0) = 1 & q_1(1) = 1 + 1^2 = 1 \\ p_2(-1) = -1 - (-1)^2 = -2 & p_2(0) = 0 & p_2(1) = 1 - 1^2 = 0 \end{array}$$

So we now compute that

$$\begin{aligned} q_2 &= (x - x^2) - \tfrac{(-2)(2)+(0)(1)+(0)(2)}{2^2+1^2+2^2}(1 + x^2) \\ &= x - x^2 + \tfrac{4}{9}(1 + x^2) \\ &= \tfrac{1}{9}(4 + 9x - 5x^2) \end{aligned}$$

And so we can set $q_2 = 4 + 9x - 5x^2$, and by the Gram-Schmidt Procedure we know that $\{1 + x^2, 4 + 9x - 5x^2\}$ is an orthogonal basis for $\mathbb{S}$.

B3(b)(ii): Let $r = 1 + x + x^2$. Then we have

$$\text{proj}_{\mathbb{S}} r = \frac{\langle r, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 + \frac{\langle r, q_2 \rangle}{\langle q_2, q_2 \rangle} q_2$$

To compute our inner products, we first make the following calculations:

$$\begin{array}{lll} q_1(-1) = 1 + (-1)^2 = 2 & q_1(0) = 1 & q_1(1) = 1 + 1^2 = 1 \\ q_2(-1) = 4 - 9 - 5(-1)^2 = -10 & q_2(0) = 4 & q_2(1) = 4 + 9 - 5(1^2) = 8 \\ r(-1) = 1 - 1 + (-1)^2 = 1 & r(0) = 1 & r(1) = 1 + 1 + 1^2 = 3 \end{array}$$

So we can now find $\text{proj}_{\mathbb{S}} r$:

$$\begin{aligned} \text{proj}_{\mathbb{S}} r &= \tfrac{(1)(2)+(1)(1)+(3)(2)}{2^2+1^2+2^2}(1 + x^2) + \tfrac{(1)(-10)+(1)(4)+(3)(8)}{(-10)^2+4^2+8^2}(4 + 9x - 5x^2) \\ &= \tfrac{8}{9}(1 + x^2) + \tfrac{18}{180}(4 + 9x - 5x^2) \\ &= \tfrac{1}{90}(116 + 81x + 35x^2) \end{aligned}$$

B4(a) Let $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{w}_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$.

Then we set $\vec{v}_1 = \vec{w}_1$, and we find $\vec{v}_2$ as follows:

$$\begin{aligned} \vec{v}_2 = \vec{w}_2 - \tfrac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 & \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{(-1)(1)+3(1)(1)+2(0)(0)}{1^2+3(1^2)+2(0^2)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \tfrac{1}{2} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

So we set $\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, and then we compute $\vec{v}_3$ as follows:

$$
\begin{aligned}
\vec{v}_3 &= \vec{w}_3 - \tfrac{\langle \vec{w}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \tfrac{\langle \vec{w}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 \\
&= \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \tfrac{(-1)(1)+3(1)(1)+2(-2)(0)}{1^2+3(1^2)+2(0^2)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{(-1)(-3)+3(1)(1)+2(-2)(0)}{(-3)^2+3(1^2)+2(0^2)} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} - \tfrac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \tfrac{6}{12} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}
\end{aligned}
$$

By the Gram-Schmidt Procedure, we know that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{S}$.

B4(b) Let $\vec{b}$ be the coordinates of $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to the basis found in part (a). Then we have

$$
\begin{aligned}
b_1 &= \tfrac{\langle \vec{x}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} = \tfrac{(1)(1)+3(0)(1)+2(0)(0)}{1^2+3(1^2)+2(0^2)} = \tfrac{1}{4} \\
b_2 &= \tfrac{\langle \vec{x}, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} = \tfrac{(1)(-3)+3(0)(1)+2(0)(0)}{(-3)^2+3(1^2)+2(0^2)} = \tfrac{-3}{12} = \tfrac{-1}{4} \\
b_3 &= \tfrac{\langle \vec{x}, \vec{v}_3 \rangle}{\langle \vec{v}_3, \vec{v}_3 \rangle} = \tfrac{(1)(0)+3(0)(0)+2(0)(-2)}{0^2+3(0^2)+2((-2)^2)} = 0
\end{aligned}
$$

So the coordinates of $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to the basis found in part (a) are $\begin{bmatrix} 1/4 \\ -1/4 \\ 0 \end{bmatrix}$.