Solution to Practice 2i

B4(a) First, we verify that the system is inconsistent by row reducing its augmented matrix:

$$\left[\begin{array}{rr|r} 1 & -1 & 4 \\ 3 & 2 & 5 \\ 1 & -6 & 10 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & 4 \\ 0 & 5 & -7 \\ 0 & -5 & 6 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & 4 \\ 0 & 5 & -7 \\ 0 & 0 & -1 \end{array}\right]$$

The last row verifies that there is no solution to our system. So, we now begin to calculate $\vec{x} = (A^T A)^{-1} A^T \vec{b}$. The first step is to calculate $A^T A$:

$$A^T A = \left[\begin{array}{rrr} 1 & 3 & 1 \\ -1 & 2 & -6 \end{array}\right] \left[\begin{array}{rr} 1 & -1 \\ 3 & 2 \\ 1 & -6 \end{array}\right] = \left[\begin{array}{rr} 11 & -1 \\ -1 & 41 \end{array}\right].$$

Next, we use the matrix inverse algorithm to find $(A^T A)^{-1}$:

$$\left[\begin{array}{rr|rr} 11 & -1 & 1 & 0 \\ -1 & 41 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rr|rr} 1 & -41 & 0 & -1 \\ 11 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rr|rr} 1 & -41 & 0 & -1 \\ 0 & 450 & 1 & 11 \end{array}\right]$$
$$\sim \left[\begin{array}{rr|rr} 1 & -41 & 0 & -1 \\ 0 & 1 & 1/450 & 11/450 \end{array}\right] \sim\sim \left[\begin{array}{rr|rr} 1 & 0 & 41/450 & 1/450 \\ 0 & 1 & 1/450 & 11/450 \end{array}\right].$$

So $(A^T A)^{-1} = \frac{1}{450}\left[\begin{array}{rr} 41 & 1 \\ 1 & 11 \end{array}\right]$. Next we calculate

$$(A^T A)^{-1} A^T = \frac{1}{450}\left[\begin{array}{rr} 41 & 1 \\ 1 & 11 \end{array}\right] \left[\begin{array}{rrr} 1 & 3 & 1 \\ -1 & 2 & -6 \end{array}\right] = \frac{1}{450}\left[\begin{array}{rrr} 40 & 125 & 35 \\ -10 & 25 & -65 \end{array}\right] = \frac{1}{90}\left[\begin{array}{rrr} 8 & 25 & 7 \\ -2 & 5 & -13 \end{array}\right].$$

And finally we have that $\vec{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{90}\left[\begin{array}{rrr} 8 & 25 & 7 \\ -2 & 5 & -13 \end{array}\right] \left[\begin{array}{r} 4 \\ 5 \\ 10 \end{array}\right] = \frac{1}{90}\left[\begin{array}{r} 227 \\ -113 \end{array}\right].$

So $\vec{x} = \left[\begin{array}{r} 227/90 \\ -113/90 \end{array}\right]$ is the vector that minimizes $||\vec{b} - A\vec{x}||$.

B4(b) First, we verify that the system is inconsistent by row reducing its augmented matrix:

$$\left[\begin{array}{rr|r} 1 & 1 & 7 \\ 1 & -1 & 4 \\ 1 & 3 & 14 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 1 & 7 \\ 0 & -2 & -3 \\ 0 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 1 & 7 \\ 0 & -2 & -3 \\ 0 & 0 & 4 \end{array}\right]$$

The last row verifies that there is no solution to our system. So, we now begin to calculate $\vec{x} = (A^T A)^{-1} A^T \vec{b}$. The first step is to calculate $A^T A$:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}.$$

Next, we use the matrix inverse algorithm to find $(A^TA)^{-1}$:

$$\left[\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 3 & 11 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 0 & 8 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1/3 & 0 \\ 0 & 1 & -1/8 & 1/8 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 11/24 & -1/8 \\ 0 & 1 & -1/8 & 1/8 \end{array}\right]$$

So $(A^TA)^{-1} = \begin{bmatrix} 11/24 & -1/8 \\ -1/8 & 1/8 \end{bmatrix} \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}$. Next we calculate

$$(A^TA)^{-1}A^T = \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 8 & 14 & 2 \\ 0 & -6 & 6 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 4 & 7 & 1 \\ 0 & -3 & 3 \end{bmatrix}.$$

And finally we have that $\vec{x} = (A^TA)^{-1}A^T\vec{b} = \frac{1}{12} \begin{bmatrix} 4 & 7 & 1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 14 \end{bmatrix} =$
$\frac{1}{12} \begin{bmatrix} 70 \\ 30 \end{bmatrix} = \begin{bmatrix} 35/6 \\ 15/6 \end{bmatrix}$.

So $\vec{x} = \begin{bmatrix} 35/6 \\ 15/6 \end{bmatrix}$ is the vector that minimizes $||\vec{b} - A\vec{x}||$.