Lecture 3p

Eigenvectors

(pages 417-418)

Our next goal in this course is to use complex numbers to help us diagonalize REAL matrices. That is to say, we want to look at diagonalizing matrices with real entries, but consider them as complex matrices, so that we can find all the roots to their characteristic polynomial. So, we should first look at how to diagonalize any complex matrix. And in order to do that, we need to look at eigenvalues and eigenvectors over $\mathbb{C}$.

Definition: Let $L : \mathbb{C}^n \to \mathbb{C}^n$ be a linear mapping. If for some $\lambda \in \mathbb{C}$ there exists a non-zero vector $\vec{z} \in \mathbb{C}^n$ such that $L(\vec{z}) = \lambda\vec{z}$, then λ is an **eigenvalue** of L and $\vec{z}$ is called an **eigenvector** of L that corresponds to λ. Similarly, a complex number λ is an eigenvalue of an $n \times n$ matrix A with complex entries with corresponding eigenvector $\vec{z} \in \mathbb{C}^n$, $\vec{z} \neq \vec{0}$, if $A\vec{z} = \lambda\vec{z}$.

As you can see, the definition remains unchanged from the one we used in $\mathbb{R}$, except that our vectors, matrices, and scalars are now in $\mathbb{C}$ instead of $\mathbb{R}$. Moreover, this is yet another place where the theory we developed in $\mathbb{R}$ translates directly to our situation in $\mathbb{C}$. And so we use the exact same techniques to find eigenvalues, eigenvectors, and to diagonalize matrices that we did in $\mathbb{R}$. Of course, simply doing these calculations with the complex numbers can make things different. Let's look at an example.

Example: Let $A = \begin{bmatrix} 2 & 2-i \\ -1+i & -1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

As before, the columns of P will be made from eigenvectors of A, and so we need to find these eigenvectors. But to find the eigenvectors, we first need to find the eigenvalues. These will be the solutions to the characteristic polynomial det $(A - \lambda I)$.

$$\begin{aligned} \det\,(A - \lambda I) &= \det \begin{bmatrix} 2-\lambda & 2-i \\ -1+i & -1-\lambda \end{bmatrix} \\ &= (2-\lambda)(-1-\lambda) - (-1+i)(2-i) \\ &= -2 - 2\lambda + \lambda + \lambda^2 - (-2 + i + 2i - i^2) \\ &= \lambda^2 - \lambda + (-1 - 3i) \end{aligned}$$

We can use the quadratic formula to find the roots of this polynomial:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(-1-3i)}}{2(1)} = \frac{1 \pm \sqrt{5 + 12i}}{2}$$

But what is $\sqrt{5+12i}$? We need to use de Moivre's Formula to find this. Note that the use of "$\pm$" is still valid here, since de Moivre's Formula tells us that the two square roots of any complex number z will be $2\pi/2 = \pi$ apart from each other, which is the same as multiplying by -1. So, we only need to use de Moivre's Formula to find one root of $5+12i$. But we still need to put it into polar form first. To that end, we see that $r = \sqrt{5^2+12^2} = \sqrt{169} = 13$. And this means that we need θ such that $\cos\theta = 5/13$ and $\sin\theta = 12/13$. Since our cosine and sine values are both positive, θ is in the first quadrant, so we can set θ equal to either $\cos^{-1}(5/13)$ or $\sin^{-1}(12/13)$. And we then have:

$$(5+12i)^{1/2} = (13)^{1/2}(\cos((\cos^{-1}(5/13))/2) + i\sin((\cos^{-1}(5/13)/2) = 3+2i$$

And so we have that

$$\lambda = \frac{1 \pm (3+2i)}{2} = 2+i \text{ or } -1-i$$

Now that we have the eigenvalues for A, we need to find the eigenspaces for these eigenvalues.

For $\lambda = 2+i$, we need to find the nullspace of $\begin{bmatrix} 2-(2+i) & 2-i \\ -1+i & -1-(2+i) \end{bmatrix} =$ $\begin{bmatrix} -i & 2-i \\ -1+i & -3-i \end{bmatrix}$, which we do by row reducing our matrix.

$$\begin{bmatrix} -i & 2-i \\ -1+i & -3-i \end{bmatrix} \begin{matrix} -iR_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & 1+2i \\ -1+i & -3-i \end{bmatrix} \begin{matrix} \\ R_2+(1-i)R_1 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 1+2i \\ 0 & 0 \end{bmatrix}$$

And so we see that our eigenspace consists of all solutions to to the equation $z_1 + (1+2i)z_2 = 0$, which we can write as Span$\left\{\begin{bmatrix} -1-2i \\ 1 \end{bmatrix}\right\}$.

For $\lambda = -1-i$, we need to find the nullspace of $\begin{bmatrix} 2-(-1-i) & 2-i \\ -1+i & -1-(-1-i) \end{bmatrix} =$ $\begin{bmatrix} 3+i & 2-i \\ -1+i & i \end{bmatrix}$, which again means row reducing our matrix.

$$\begin{bmatrix} 3+i & 2-i \\ -1+i & i \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} -1+i & i \\ 3+i & 2-i \end{bmatrix} \begin{matrix} (1/2)(-1-i)R_1 \\ \end{matrix}$$
$$\sim \begin{bmatrix} 1 & \frac{1}{2}-\frac{i}{2} \\ 3+i & 2-i \end{bmatrix} \begin{matrix} \\ R_2+(-3-i)R_1 \end{matrix} \sim \begin{bmatrix} 1 & \frac{1}{2}-\frac{i}{2} \\ 0 & 0 \end{bmatrix}$$

And so we see that our eigenspace consists of all solutions to to the equation $z_1 + ((1/2)-(1/2)i)z_2 = 0$, which we can write as Span$\left\{\begin{bmatrix} -1+i \\ 2 \end{bmatrix}\right\}$.

Now that we know the eigenvalues and their eigenspaces, we know that the matrix P whose columns are the basis vectors for our eigenspace is an invertible matrix such that $P^{-1}AP = D$, where D is a diagonal matrix whose diagonal entries are the eigenvalues corresponding to the eigenvectors used in P. In this particular example, this means that $P = \begin{bmatrix} -1-2i & -1+i \\ 1 & 2 \end{bmatrix}$ and
$D = \begin{bmatrix} 2+i & 0 \\ 0 & -1-i \end{bmatrix}$.

And that was just a 2×2 matrix! Since diagonalizing general complex matrices is not our goal, we will satisfy ourselves with this example and move on to looking at matrices with real entries.