Solution to Practice 2e

B2(a): We are looking for the set of vectors that satisfy $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0$, which is the same as $3x_1 + 2x_2 + x_3 = 0$. Replacing the variable x_1 with the parameter s and the variable x_2 with the parameter t, we see that the general solution to this equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ -3s - 2t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis for $\mathbb{S}^{\perp}$.

B2(b): We are looking for the set of vectors that satisfy $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = 0$ and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. This is the same as looking for solutions to the homogeneous system

$$\begin{array}{rrrl} x_1 & +2x_2 & -4x_3 & = 0 \\ -x_1 & +2x_2 & +2x_3 & = 0 \end{array}$$

To solve this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & -4 \\ -1 & 2 & 2 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 4 & -2 \end{bmatrix} \begin{matrix} \\ (1/4)R_2 \end{matrix}$$
$$\sim \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1/2 \end{bmatrix}$$

So our system is equivalent to the system

$$\begin{array}{rrrl} x_1 & & -3x_3 & = 0 \\ & x_2 & -(1/2)x_3 & = 0 \end{array}$$

Replacing the variable x_3 with the parameter t, we see that the general solution to this equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3t \\ (1/2)t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1/2 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 3 \\ 1/2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{S}^{\perp}$.

B2(c): We are looking for the set of vectors that satisfy $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$, and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. This is the same as looking for solutions to the homogeneous system

$$\begin{array}{rrrl} x_1 & +2x_2 & -1x_3 & = 0 \\ 2x_1 & +x_2 & +x_3 & = 0 \\ -x_1 & +x_2 & +x_3 & = 0 \end{array}$$

To solve this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & 3 & 0 \end{bmatrix} (-1/3)R_2$$
$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{array}{l} \\ \\ (1/3)R_3 \end{array}$$
$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 + R_3 \\ R_2 + R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the only solution to our system is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, which means that $\mathbb{S}^{\perp} = \{\vec{0}\}$, and thus the empty set is the basis for $\mathbb{S}^{\perp}$.

B2(d): We are looking for the set of vectors that satisfy $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = 0$, and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}$. This is the same as looking for solutions to the homogeneous system

$$\begin{array}{rrrrl} 2x_1 & +x_2 & -x_3 & & = 0 \\ x_1 & +2x_2 & +x_3 & +x_4 & = 0 \\ & -x_2 & +3x_3 & +x_4 & = 0 \end{array}$$

To solve this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & -1 & 3 & 1 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix} R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & -1 & 3 & 1 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & -3 & -3 & -2 \end{bmatrix} -R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & -3 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -12 & -5 \end{bmatrix} \begin{matrix} \\ \\ (-1/12)R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 5/12 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 + 3R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 0 & 7/12 \\ 0 & 1 & 0 & 3/12 \\ 0 & 0 & 1 & 5/12 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1/12 \\ 0 & 1 & 0 & 3/12 \\ 0 & 0 & 1 & 5/12 \end{bmatrix}$$

So our system is equivalent to the system

$$\begin{array}{llll} x_1 & & & +(1/12)x_4 & = 0 \\ & x_2 & & +(3/12)x_4 & = 0 \\ & & x_3 & +(5/12)x_4 & = 0 \end{array}$$

Replacing the variable x_4 with the parameter t, we see that the general solution to this equation is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (-1/12)t \\ (-3/12)t \\ (-5/12)t \\ t \end{bmatrix} = t \begin{bmatrix} -1/12 \\ -3/12 \\ -5/12 \\ 1 \end{bmatrix} = t \begin{bmatrix} -1 \\ -3 \\ -5 \\ 12 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -1 \\ -3 \\ -5 \\ 12 \end{bmatrix} \right\}$ is a basis for $\mathbb{S}^\perp$.

D3 Suppose that $\vec{s} \in \mathbb{S}$, and $\vec{t} \in \mathbb{S}^\perp$. Then $\vec{s} \cdot \vec{t} = 0$, so $\vec{s} \in (\mathbb{S}^\perp)^\perp$. This shows that $\mathbb{S} \subset (\mathbb{S}^\perp)^\perp$. To see that $(\mathbb{S}^\perp)^\perp \subset \mathbb{S}$, let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal basis for $\mathbb{S}$ and $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{S}^\perp$, so that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Now, for any vector $\vec{x} \in (\mathbb{S}^\perp)^\perp$, there are scalars $x_1, \dots, x_n$ such that

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

Moreover, since $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an *orthonormal* basis for $\mathbb{R}^n$, we know that the coefficients x_i are $\vec{x} \cdot \vec{v}_i$. But, since $\vec{x} \in (\mathbb{S}^\perp)^\perp$, we know that $\vec{x} \cdot \vec{v} = 0$ for any $\vec{v} \in \mathbb{S}^\perp$. Thus, we specifically have $\vec{x} \cdot \vec{v}_i = 0$ for $k+1 \le i \le n$. This means that $x_i = 0$ for $k+1 \le i \le n$, so we can write

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k$$

Since $\vec{x}$ is a linear combination of the vectors in our basis $\{\vec{v}_1, \dots, \vec{v}_k\}$ for $\mathbb{S}$, we see that $\vec{x} \in \mathbb{S}$, as desired.