Lecture 3b

The Complex Conjugate

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Everything we study comes with its own special operations. With matrices, we have the determinant. With polynomials, we get the idea of factoring polynomials, and of evaluating polynomials. The special operation for complex numbers is called conjugation.

Definition: The **complex conjugate** of the complex number $z = x + yi$ is $z = x - yi$, and is denoted $\overline{z}$.

Example: $\overline{1+2i} = 1 - 2i$ $\qquad$ $\overline{3-4i} = 3 + 4i$ $\qquad$ $\overline{5} = 5$

Theorem 9.1.1 (Properties of the Complex Conjugate) For complex numbers $z_1 = x_1 + y_1 i$ and z_2 we have

(1) $\overline{\overline{z_1}} = z_1$
(2) z_1 is purely real if and only if $\overline{z_1} = z_1$
(3) z_1 is purely imaginary if and only if $\overline{z_1} = -z_1$
(4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
(5) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
(6) $\overline{z_1^n} = \overline{z_1}^n$
(7) $z_1 + \overline{z_1} = 2\text{Re}(z_1) = 2x_1$
(8) $z_1 - \overline{z_1} = i2\text{Im}(z_1) = i2y_1$
(9) $z_1\overline{z_1} = x_1^2 + y_1^2$

The proofs of these properties are easy consequences of the definition of conjugation. Which makes them great practice! I'll prove properties (1), (2) and (4) now.

Proof of Theorem 9.1.1 (1): Let $z_1 = x_1 + y_1 i$ be a complex number. Then $\overline{z_1} = \overline{x_1 + y_1 i} = x_1 - y_1 i$. And this means that $\overline{\overline{z_1}} = \overline{x_1 - y_1 i} = x_1 + y_1 i = z_1$, as desired.

Proof of Theorem 9.1.1 (2): Suppose $z_1 = x_1 + y_1 i$ is a complex number that is purely real. Then $y_1 = 0$ and $z_1 = x_1$. Moreover, we see that $\overline{z_1} = x_1 - y_1 i = x_1 - 0i = x_1 = z_1$. So if z_1 is purely real, then $\overline{z_1} = z_1$.

Now suppose that z_1 is a complex number such that $\overline{z_1} = z_1$. Then $x_1 + y_1 i = x_1 - y_1 i$, so $(x_1 + y_1 i) - (x_1 - y_1 i) = 0$. This means that $2y_1 i = 0$, which can only happen if $y_1 = 0$. So we see that if $\overline{z_1} = z_1$, then $y_1 = 0$, which means that z_1 is purely real.

Proof of Theorem 9.1.1 (4): Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be complex numbers. Then we see that

$$
\begin{aligned}
\overline{z_1 + z_2} &= \overline{(x_1 + y_1 i) + (x_2 + y_2 i)} \\
&= \overline{(x_1 + x_2) + (y_1 + y_2)i} \\
&= (x_1 + x_2) - (y_1 + y_2)i \\
&= (x_1 + x_2) + (-y_1 - y_2)i \\
&= (x_1 - y_1 i) + (x_2 - y_2 i) \\
&= \overline{x_1 + y_1 i} + \overline{x_2 + y_2 i} \\
&= \overline{z_1} + \overline{z_2}
\end{aligned}
$$

We have already noticed that complex conjugation does not affect real numbers. This will become of great importance during our study of the complex numbers, as it turns out that when we generalize many of the things we did in the real numbers, we will need to introduce complex conjugation to the mix in order to have our formulas turn out correctly. I like to think that secretly we have been conjugating these numbers all the time, and just didn't realize it because it doesn't make a difference in the real numbers. But it will make a difference now, so be on the lookout for conjugation throughout the rest of the course.