Solution to Practice 3i

A1(a) We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & i & 1+i & 1-i \\ -2 & 1-2i & -2 & 2i \\ 2i & -2 & -2-3i & -1+3i \end{array}\right]$$

First, we need to replace R_2 with $R_2 + 2R_1$ and R_3 with $R_3 + (-2i)R_1$. These calculations are as follows:

$$\begin{array}{r|rrr|r} R_2: & -2 & 1-2i & -2 & 2i \\ 2R_1: & 2 & 2i & 2+2i & 2-2i \\ \hline R_2 + 2R_1: & 0 & 1 & 2i & 2 \end{array}$$

$$\begin{array}{r|rrr|r} R_3: & 2i & -2 & -2-3i & -1+3i \\ -2iR_1: & -2i & 2 & 2-2i & -2-2i \\ \hline R_3 + (-2i)R_1: & 0 & 0 & -5i & -3+i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{rrr|r} 1 & i & 1+i & 1-i \\ 0 & 1 & 2i & 2 \\ 0 & 0 & -5i & -3+i \end{array}\right]$$

Next, we multiply row 3 by $i/5$ and get

$$\left[\begin{array}{rrr|r} 1 & i & 1+i & 1-i \\ 0 & 1 & 2i & 2 \\ 0 & 0 & 1 & (-1/5) + (-3/5)i \end{array}\right]$$

Now we need to replace R_1 with $R_1 + (-1-i)R_3$ and $R_2 + (-2i)R_3$. These operations will leave the first two columns unchanged, and will put a zero in the third columns of rows 1 and 2. The only thing left to compute is the result in the fourth column:

$R_1 + (-1-i)R_3 : (1-i) + (-1-i)((-1/5) + (-3/5)i) = 1 - i + (1/5) + (3/5)i + (1/5)i + (3/5)i^2 = (1 + (1/5) - (3/5)) + (-1 + (3/5) + (1/5))i = (3/5) + (-1/5)i$

$R_2 + (-2i)R_3 : 2 + (-2i)((-1/5) + (-3/5)i) = 2 + (2/5)i + (6/5)i^2 = (2 - (6/5)) + (2/5)i = (4/5) + (2/5)i$

and our matrix becomes

$$\left[\begin{array}{rrr|r} 1 & i & 0 & (3/5) + (-1/5)i \\ 0 & 1 & 0 & (4/5) + (2/5)i \\ 0 & 0 & 1 & (-1/5) + (-3/5)i \end{array}\right]$$

The final row operation is to replace R_1 with $R_1 + (-i)R_2$. This operation will leave the first and third columns unchanged and will put a zero in the second column of the first row. The only thing left to compute is the result in the fourth column:

$R_1 + (-i)R_2$: $((3/5) + (-1/5)i) + (-i)((4/5) + (2/5)i) = (3/5) + (-1/5)i + (-4/5)i + (-2/5)i^2 = ((3/5) + (2/5)) + ((-1/5) + (-4/5))i = 1 - i$

and our matrix becomes

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & (4/5)+(2/5)i \\ 0 & 0 & 1 & (-1/5)+(-3/5)i \end{array}\right]$$

We now see that the solution to the system is $z_1 = 1 - i$, $z_2 = (4/5) + (2/5)i$, $z_3 = (-1/5) + (-3/5)i$.

A1(b) We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{crrr|r} 1 & 1+i & 2 & 1 & 1-i \\ 2 & 2+i & 5 & 2+i & 4-i \\ i & -1+i & 1+2i & 2i & 1 \end{array}\right]$$

First we need to replace R_2 with $R_2 + (-2)R_1$ and R_3 with $R_3 + (-i)R_1$. These calculations are as follows:

$$\begin{array}{rrrrr|r} R_2: & 2 & 2+i & 5 & 2+i & 4-i \\ -2R_1: & -2 & -2-2i & -4 & -2 & -2+2i \\ \hline R_2+(-2)R_1: & 0 & -i & 1 & i & 2+i \end{array}$$

$$\begin{array}{rrrrr|r} R_3: & i & -1+i & 1+2i & 2i & 1 \\ -iR_1: & -i & 1-i & -2i & -i & -1-i \\ \hline R_3+(-i)R_1: & 0 & 0 & 1 & i & -i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{crcr|r} 1 & 1+i & 2 & 1 & 1-i \\ 0 & -i & 1 & i & 2+i \\ 0 & 0 & 1 & i & -i \end{array}\right]$$

Usually, I would now multiply row 2 by i to get a 1 in the second column, but I noticed an excellent opportunity to establish the leading 1 in the third column, so my next row operations are to replace R_1 with $R_1 + (-2)R_3$ and R_2 with $R_2 + (-1)R_3$. These calculations are as follows:

$$\begin{array}{rrrrr|r} R_1: & 1 & 1+i & 2 & 1 & 1-i \\ -2R_3: & 0 & 0 & -2 & -2i & 2i \\ \hline R_1+(-2)R_3: & 1 & 1+i & 0 & 1-2i & 1+i \end{array}$$

$$\begin{array}{r|cccc|c} R_2: & 0 & -i & 1 & i & 2+i \\ -R_1: & 0 & 0 & -1 & -i & i \\ \hline R_2+(-1)R_3: & 0 & -i & 0 & 0 & 2+2i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 1+i & 0 & 1-2i & 1+i \\ 0 & -i & 0 & 0 & 2+2i \\ 0 & 0 & 1 & i & -i \end{array}\right]$$

Now we multiply row 2 by i, and our matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 1+i & 0 & 1-2i & 1+i \\ 0 & 1 & 0 & 0 & -2+2i \\ 0 & 0 & 1 & i & -i \end{array}\right]$$

Last, we need to replace R_1 with $R_1+(-1-i)R_2$. This calculation will leave the first, third, and fourth columns unchanged, and will make the second entry in the first row a zero. The only thing left to compute is the result in the fifth column:

$R_1+(-1-i)R_2 : (1+i)+(-1-i)(-2+2i) = 1+i+2-2i+2i-2i^2 = (1+2+2)+(1-2+2)i = 5+i$

so our matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1-2i & 5+i \\ 0 & 1 & 0 & 0 & -2+2i \\ 0 & 0 & 1 & i & -i \end{array}\right]$$

This matrix is in reduced row echelon form, so we now proceed to find the general solutions. We see that our system is equivalent to the system:

$$\begin{aligned} z_1+(1-2i)z_4 &= 5+i \\ z_2 &= -2+2i \\ z_3+iz_4 &= -i \end{aligned}$$

If we replace the variable z_4 with the parameter t, we see that the general solution is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} (5+i)-(1-2i)t \\ -2+2i \\ (-i)-(i)t \\ t \end{bmatrix} = \begin{bmatrix} 5+i \\ -2+2i \\ -i \\ 0 \end{bmatrix} + t\begin{bmatrix} -1+2i \\ 0 \\ -i \\ 1 \end{bmatrix}$$

B1(a) We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & -i & 1+3i \\ i & -1 & -1+i & 1+2i \\ 2 & 2i & 3+2i & 4 \end{array}\right]$$

First, we multiply row 2 by $-i$ to get

$$\left[\begin{array}{ccc|c} 0 & 1 & -i & 1+3i \\ 1 & i & 1+i & 2-i \\ 2 & 2i & 3+2i & 4 \end{array}\right]$$

and then we can switch row 1 and row 2 to get

$$\left[\begin{array}{ccc|c} 1 & i & 1+i & 2-i \\ 0 & 1 & -i & 1+3i \\ 2 & 2i & 3+2i & 4 \end{array}\right]$$

Now, we want to replace R_3 with $R_3 + (-2)R_1$, which we calculate as follows:

$$\begin{array}{rrrr|r} R_3: & 2 & 2i & 3+2i & 4 \\ -2R_1: & -2 & -2i & -2-2i & -4+2i \\ \hline R_3+(-2)R_1: & 0 & 0 & 1 & 2i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{ccc|c} 1 & i & 1+i & 2-i \\ 0 & 1 & -i & 1+3i \\ 0 & 0 & 1 & 2i \end{array}\right]$$

Now, we need to replace R_1 with $R_1 + (-1-i)R_3$ and R_2 with $R_2 + iR_3$. These operations will leave the first and second columns unchanged, and will change the third entries in rows 2 and 3 into zero. The only thing left to compute is the result in the fourth column:

$$R_1+(-1-i)R_3 : (2-i)+(-1-i)(2i) = 2-i-2i-2i^2 = (2+2)+(-1-2)i = 4-3i$$

$$R_2 + iR_3 : (1+3i) + (i)(2i) = 1 + 3i + 2i^2 = (1-2) + 3i = -1 + 3i.$$

and our matrix becomes

$$\left[\begin{array}{ccc|c} 1 & i & 0 & 4-3i \\ 0 & 1 & 0 & -1+3i \\ 0 & 0 & 1 & 2i \end{array}\right]$$

Finally, we need to replace R_1 with $R_1 + (-i)R_2$. This operation will leave the first and third columns unchanged, and will change the second entry in the first row to a zero. The only thing left to compute is the result in the fourth column:

$$R_1+(-i)R_2 : (4-3i)+(-i)(-1+3i) = 4-3i+i-3i^2 = (4+3)+(-3+1)i = 7-2i$$

and our matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7-2i \\ 0 & 1 & 0 & -1+3i \\ 0 & 0 & 1 & 2i \end{array}\right]$$

We now see that the solution to the system is $z_1 = 7-2i$, $z_2 = -1+3i$, $z_3 = 2i$.

B1(b) We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2+i & i & 0 & 1+i \\ i & -1+2i & 0 & 2i & -i \\ 1 & 2+i & 1+i & 2i & 2-i \end{array}\right]$$

First, we need to replace R_2 with $R_2+(-i)R_1$ and R_3 with $R_3+(-1)R_1$. These calculations are as follows:

$$\begin{array}{r|cccc|c} R_2: & i & -1+2i & 0 & 2i & -i \\ -iR_1: & -i & 1-2i & 1 & 0 & 1-i \\ \hline R_2+(-i)R_1: & 0 & 0 & 1 & 2i & 1-2i \end{array}$$

$$\begin{array}{r|cccc|c} R_3: & 1 & 2+i & 1+i & 2i & 2-i \\ -R_1: & -1 & -2-i & -i & 0 & -1-i \\ \hline R_3+(-1)R_1: & 0 & 0 & 1 & 2i & 1-2i \end{array}$$

and our matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 2+i & i & 0 & 1+i \\ 0 & 0 & 1 & 2i & 1-2i \\ 0 & 0 & 1 & 2i & 1-2i \end{array}\right]$$

We easily see that we can replace R_3 with $R_3 - R_2$ to get the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 2+i & i & 0 & 1+i \\ 0 & 0 & 1 & 2i & 1-2i \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now we simply need to replace R_1 with $R_1 + (-i)R_2$. This operation will leave the first and second columns unchanged, and will change the third entry of the first row into a zero. All that is left to compute is the result in the fourth and fifth columns:

Column 4: $0 + (-i)(2i) = -2i^2 = 2$

Column 5: $(1+i) + (-i)(1-2i) = 1 + i - i + 2i^2 = (1-2) + (1-1)i = -1$

and so our matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 2+i & 0 & 2 & -1 \\ 0 & 0 & 1 & 2i & 1-2i \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row echelon form, so we now proceed to find the general solutions. We see that our system is equivalent to the system:

$$\begin{aligned} z_1 + (2+i)z_2 + 2z_4 &= -1 \\ z_3 + 2iz_4 &= 1-2i \end{aligned}$$

If we replace the variable z_2 with the parameter s and the variable z_4 with the parameter t, we see that the general solution is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -1-(2+i)s-2t \\ s \\ (1-2i)-(2i)t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1-2i \\ 0 \end{bmatrix} + s \begin{bmatrix} -2-i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -2i \\ 1 \end{bmatrix}$$

B1(c) We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrr|r} i & 2 & -3-i & 1 \\ 1+i & 2-2i & -4 & i \\ i & 2 & -3-3i & 1+2i \end{array}\right]$$

First we multiply row 1 by $-i$ and row 3 by i, getting the matrix

$$\left[\begin{array}{rrr|r} 1 & -2i & -1+3i & -i \\ 1+i & 2-2i & -4 & i \\ -1 & 2i & 3-3i & -2+i \end{array}\right]$$

Now we need to replace R_2 with $R_2 + (-1-i)R_1$, and R_3 with $R_3 + R_1$. We can quickly compute $R_3 + R_1$ as follows:

$$\begin{array}{rrrr|r} R_3: & -1 & 2i & 3-3i & -2+i \\ R_1: & 1 & -2i & -1+3i & -i \\ \hline R_3+R_1: & 0 & 0 & 2 & -2 \end{array}$$

To compute $R_2 + (-1-i)R_1$, we will first want to compute $(-1-i)R_1$:

$$(-1-i)R_1: \quad 1(-1-i) \quad (-2i)(-1-i) \quad (-1+3i)(-1-i) \mid (-i)(-1-i)$$

Let's calculate all of these entries individually. Obviously, $1(-1-i) = -1-i$. Next we find that $(-2i)(-1-i) = 2i + 2i^2 = -2+2i$. In the third column, we get $(-1+3i)(-1-i) = 1+i-3i-3i^2 = (1+3)+(1-3)i = 4-2i$. And lastly, we find that $(-i)(-1-i) = i + i^2 = -1+i$. So we now have $(-1-i)R_1$:

$$(-1-i)R_1: \quad -1-i \; -2+2i \quad 4-2i \quad -1+i \mid$$

and this lets us compute $R_2 + (-1-i)R_1$:

$$\begin{array}{r|r|r} & \begin{array}{rrr} & & \end{array} & \end{array}$$

$$\begin{array}{rrrr|r} R_2: & 1+i & 2-2i & -4 & i \\ (-1-i)R_1: & -1-i-2+2i & 4-2i & -1+i & \\ \hline R_2+(-1-i)R_1: & 0 & 0 & -2i & -1+2i \end{array}$$

So, if we replace R_2 with $R_2 + (-1-i)R_1$ and R_3 with $R_3 + R_1$ our matrix becomes:

$$\left[\begin{array}{rrr|r} 1 & -2i & -1+3i & -i \\ 0 & 0 & -2i & -1+2i \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Next we can replace R_2 with $R_2 + iR_3$. We can quickly compute this:

$$\begin{array}{rrrr|r} R_2: & 0 & 0 & -2i & -1+2i \\ iR_3: & 0 & 0 & 2i & -2i \\ \hline R_2+iR_3: & 0 & 0 & 0 & -1 \end{array}$$

and our matrix becomes

$$\left[\begin{array}{rrr|r} 1 & -2i & -1+3i & -i \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Since the second row is a bad row, we see that our system is inconsistent, and has no solutions.

B1(d) We solve this homogeneous system by row reducing its coefficient matrix:

$$\left[\begin{array}{rrrr} 1 & 1 & i & 1+i \\ i & i & -1-i & -2+i \\ 2 & 2 & 2+2i & 2 \end{array}\right]$$

First, we need to replace R_2 with $R_2 + (-i)R_1$ and R_3 with $R_3 + (-2)R_1$. These calculations are as follows:

$$\begin{array}{rrrr|r} R_2: & i & i & -1-i & -2+i \\ -iR_1: & -i & -i & 1 & 1-i \\ \hline R_2+(-i)R_1: & 0 & 0 & -i & -1 \end{array}$$

$$\begin{array}{rrrr|r} R_3: & 2 & 2 & 2+2i & 2 \\ -2R_1: & -2 & -2 & -2i & -2-2i \\ \hline R_3+(-2)R_1: & 0 & 0 & 2 & -2i \end{array}$$

and our matrix becomes

$$\begin{bmatrix} 1 & 1 & i & 1+i \\ 0 & 0 & -i & -1 \\ 0 & 0 & 2 & -2i \end{bmatrix}$$

Next, we multiply row 2 by i, to get the matrix

$$\begin{bmatrix} 1 & 1 & i & 1+i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 2 & -2i \end{bmatrix}$$

Lastly, we will replace R_1 with $R_1 + (-i)R_2$ and R_3 with $R_3 - 2R_2$, which we calculate as follows:

$$\begin{array}{r|r} R_1: \quad 1 \quad 1 \quad i & 1+i \\ -iR_2: \quad 0 \quad 0 \quad -i & -1 \\ \hline R_1+(-i)R_2: \quad 1 \quad 1 \quad 0 & i \end{array}$$

$$\begin{array}{r|r} R_3: \quad 0 \quad 0 \quad 2 & -2i \\ -2R_2: \quad 0 \quad 0 \quad -2 & 2i \\ \hline R_3+(-2)R_2: \quad 0 \quad 0 \quad 0 & 0 \end{array}$$

and our matrix becomes

$$\begin{bmatrix} 1 & 1 & 0 & i \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in reduced row echelon form, so we now proceed to find the general solution. Our work shows that our system is equivalent to the following system:

$$\begin{aligned} z_1 + z_2 + iz_4 &= 0 \\ z_3 - iz_4 &= 0 \end{aligned}$$

If we replace the variable z_2 with the parameter s and the variable z_4 with the parameter t, then we see that the general solution to our system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -s - it \\ s \\ it \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -i \\ 0 \\ i \\ 1 \end{bmatrix}$$