Lecture 3a

The Arithmetic of Complex Numbers

(pages 395-396)

The real numbers, being the numbers that we find in the real world, do a great many things for us. But there are some things they can't do. The complex numbers were developed to be a number system in which polynomials can be completely factored. That is, if a polynomial is of degree n, then it will have n roots (counting multiplicity) in the complex numbers. In the real numbers it might not have had any! In the end, it wasn't so complicated to create the complex numbers, as it turns out that we only needed to add the solution to one polynomial to get all of them: a solution to the polynomial $x^2 = -1$.

Definition: A **complex number** is a number of the form $z = x + yi$, where $x, y \in \mathbb{R}$ and i is an element such that $i^2 = -1$. The set of all complex numbers is denoted by $\mathbb{C}$.

One of the first things we want to remember about the complex numbers is that they contain the real numbers. That is to say, any real number x can be thought of as a complex number, simply with the y in $+yi$ set equal to zero. As we extend definitions from the real numbers to the complex numbers, we will make note of the fact that in the case when we end up dealing with real numbers, the definitions are the same.

Definition: If $z = x + yi$, we say that the **real part** of z is x, and we write $\mathrm{Re}(z) = x$, and we say that the **imaginary part** of z is y, and we write $\mathrm{Im}(z) = y$. If $\mathrm{Im}(z) = 0$, then z is a real number, and we sometimes say that z is **purely real**. If $\mathrm{Re}(z) = 0$, the we say that z is **purely imaginary**.

The part of a complex number attached to i is called "imaginary" since it doesn't exist in the real world. But do not discount these numbers are mathematical nonsense, since they do lead to many useful results in our real world. It would take several courses to explore the full power of complex numbers, so our goal for this course will simply be to familiarize you with them, and make some natural extensions of our study of real vector spaces to the new complex numbers. Of course, the first thing we need to do is to define our basic arithmetic steps.

Definition: **Addition** of complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is defined by

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

As promised, we see that if $y_1 = 0$ and $y_2 = 0$, we have that $(x_1 + 0i) + (x_2 + 0i) = (x_1 + x_2) + 0i$, so addition on real numbers is unchanged. We also see that addition is basically done componentwise, as if $\mathbb{C}$ is simply a two dimensional vector space. We will explore that possibility later. For now, we also need to define multiplication, and here we see more parallels with P_1 than with $\mathbb{R}^2$.

Definition: **Multiplication** of complex numbers $z_1 = x_1+y_1i$ and $z_2 = x_2+y_2i$ is defined by

$$\begin{aligned} z_1z_2 &= (x_1 + y_1i)(x_2 + y_2i) \\ &= x_1x_2 + x_1y_2i + x_2y_1i + y_1y_2i^2 \\ &= x_1x_2 + (x_1y_2 + x_2y_1)i + (y_1y_2)(-1) \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i \end{aligned}$$

Again, in the case when z_1 and z_2 are actually real numbers, we have $y_1 = y_2 = 0$, and we get that $(x_1 + 0i)(x_2 + 0i) = (x_1x_2 - 0) + (0 + 0)i = x_1x_2 + 0i$, so multiplication of real numbers is still the same. Also worth noting is the ease with which we can multiply a complex number $z_2 = x_2+y_2i$ by a real number x_1, since our formula gives us $x_1(x_2+y_2i) = (x_1x_2-0)+(x_1y_2+0)i = x_1x_2+x_1y_2i$. So multiplying by x_1 is the same as distributing by x_1.

Examples: Using our definitions, we can compute the following:

(a) $(4 + 5i) + (3 - 2i) = (4 + 3) + (5 - 2)i = 7 + 3i$

(b) $(-3 + 2i) + 4(2 - i) = (-3 + 2i) + (8 - 4i) = (-3 + 8) + (2 - 4)i = 5 - 2i$

(c) $(5+2i)(-3-6i) = -15-30i-6i-12i^2 = (-15+12)+(-30-6)i = -3-36i$

(d) $(1 + i)(2 + 5i) + (2 + i)(7 - 2i) = (2 + 5i + 2i + 5i^2) + (14 - 4i + 7i - 2i^2) = ((2 - 5) + (5 + 2)i) + ((14 + 2) + (-4 + 7)i) = (-3 + 7i) + (16 + 3i) = (-3 + 16) + (7 + 3)i = 13 + 10i$

Note that, as usual, we will write $(x_1 + y_1i) - (x_2 + y_2i)$ to mean $(x_1 + y_1i) + (-1)(x_2 + y_2i)$, which is the same as $(x_1 + y_1i) + (-x_2 - y_2i)$.