Solution to Practice 1j

B9(a) Let $\mathcal{A} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ Then given any $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathbb{S}$, we have that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

And since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{S}$, we see that Span $\mathcal{A} = \mathbb{S}$, so $\mathcal{A}$ is a spanning set for $\mathbb{S}$. Next, we want to show that $\mathcal{A}$ is linearly independent. To that end, we want to look for solutions to the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This is equivalent to the system

$$\begin{aligned} t_1 \qquad &= 0 \\ t_2 &= 0 \end{aligned}$$

(Note that I left out the equations $0 = 0 + 0$, as they do not contain any information.) This clearly shows that $t_1 = t_2 = 0$ is the only solution to the equation, and thus that our set $\mathcal{A}$ is linearly independent. Since $\mathcal{A}$ is a linearly independent spanning set for $\mathbb{S}$, $\mathcal{A}$ is a basis for $\mathbb{S}$. And since $\mathcal{A}$ has two elements, we see that the dimension of $\mathbb{S}$ is 2.

B9(b) To find a guess for the basis, let's look at a typical polynomial $p(x) = p_0 + p_1x + p_2x^2 \in P_2$. Then $(1 + x^2)p(x) = (p_0 + p_1x + p_2x^2) + (p_0x^2 + p_1x^3 + p_2x^4) = p_0 + p_1x + (p_0 + p_2)x^2 + p_1x^3 + p_2x^4$. By setting each of p_0, p_1, and p_2 equal to 1 (with the other two equal to zero), we can obtain a guess at a basis for $\mathbb{S}$: $\mathcal{B} = \{1 + x^2, x + x^3, x^2 + x^4\}$.

To prove that $\mathcal{B}$ is a basis for $\mathbb{S}$, let's first show that $\mathcal{B}$ is a spanning set for $\mathbb{S}$. Since we can rewrite $\mathcal{B}$ as $\{(1 + x^2)1, (1 + x^2)x, (1 + x^2)x^2\}$, we see that the elements of $\mathcal{B}$ are in $\mathbb{S}$, so we know that the span of $\mathcal{B}$ is contained in $\mathbb{S}$. So all that is left is to show that every element of $\mathbb{S}$ is in Span $\mathcal{B}$. To that end, let $a(x) \in \mathbb{S}$. Then there is some $b_0 + b_1x + b_2x^2 \in P_2$ such that $a(x) = (1 + x^2)(b_0 + b_1x + b_2x^2)$. This means that $a(x) = b_0(1 + x^2) + b_1x(1 + x^2) + b_2x^2(1 + x^2) = b_0(1 + x^2) + b_1(x + x^3) + b_2(x^2 + x^4)$. As such, there are scalars $b_0, b_1, b_2 \in \mathbb{R}$ such that $a(x) = b_0(1 + x^2) + b_1(x + x^3) + b_2(x^2 + x^4)$, so $a(x) \in$ Span $\mathcal{B}$. And this completes the proof that $\mathcal{B}$ is a spanning set for $\mathbb{S}$.

Now we need to show that $\mathcal{B}$ is linearly independent. To do this, we need to look for solutions to the equation

$$t_1(1+x^2)+t_2(x+x^3)+t_3(x^2+x^4)=0+0x+0x^2+0x^3+0x^4$$

Performing the calculation on the left, we see that we are looking for $t_1, t_2, t_3 \in \mathbb{R}$ such that

$$t_1+t_2x+(t_1+t_3)x^2+t_2x^3+t_3x^4=0+0x+0x^2+0x^3+0x^4$$

Setting the coefficients equal to each other, we see that we are looking for solutions to the following homogeneous system:

$$\begin{array}{cccc} t_1 & & & =0 \\ & t_2 & & =0 \\ t_1 & & +t_3 & =0 \\ & t_2 & & =0 \\ & & t_3 & =0 \end{array}$$

The first equation tells us that $t_1 = 0$, the second and fourth equations tell us that $t_2 = 0$ and the fifth equation tells us that $t_3 = 0$, so we quickly see that the only solution to our equation is $t_1 = t_2 = t_3 = 0$. This means that $\mathcal{B}$ is linearly independent.

And since $\mathcal{B}$ is a linearly independent spanning set for $\mathbb{S}$, we have that $\mathcal{B}$ is a basis for $\mathbb{S}$. And since $\mathcal{B}$ has three elements, we know that the dimension of $\mathbb{S}$ is 3.

B9(c) First, we note that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = x_1+0-x_3$, so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$ if and only if $x_1 - x_3 = 0$, which happens if and only if $x_1 = x_3$. So, this means that $\mathbb{S} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$.

Now, let $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. We first note that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are in $\mathbb{S}$, so Span $\mathcal{C}$ is contained in $\mathbb{S}$. Thus, to see that $\mathcal{C}$ is a spanning set for $\mathbb{S}$, we need to show that every element of $\mathbb{S}$ is in Span $\mathcal{C}$. To that end, let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_1 \end{bmatrix} \in \mathbb{S}$. Then there are scalars $y_1, y_2 \in \mathbb{R}$ such that $y_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{y}$. This means that $\vec{y} \in$ Span $\mathcal{C}$, and thus $\mathcal{C}$ is a spanning set for $\mathbb{S}$.

Next, we need to show that $\mathcal{C}$ is linearly independent. To do this, we need to find the solutions to the equation

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalent to the homogeneous system

$$\begin{aligned} t_1 \quad\quad &= 0 \\ t_2 &= 0 \\ t_1 \quad\quad &= 0 \end{aligned}$$

The first and third equations tell us that $t_1 = 0$, and the second equation tells us that $t_2 = 0$, so we see that the only solution to our equation is $t_1 = t_2 = 0$. This means that $\mathcal{C}$ is linearly independent.

And since $\mathcal{C}$ is a linearly independent spanning set for $\mathbb{S}$, we have that $\mathcal{C}$ is a basis for $\mathbb{S}$. And since $\mathcal{C}$ has two elements, we know that the dimension of $\mathbb{S}$ is 2.

B9(d) Looking at the defining properties of $\mathbb{S}$, we note that $a - b = 0$ iff $a = b$, $b - c = 0$ iff $b = c$, and $c - a = 0$ iff $c = a$. Combined, we see that $a = b = c$. As such, another way of describing $\mathbb{S}$ is as $\left\{ \begin{bmatrix} a & a \\ a & d \end{bmatrix} \mid a, d \in \mathbb{R} \right\}$.

This description leads us to think that $\mathcal{D} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{S}$. We first note that $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{S}$, so Span $\mathcal{D}$ is contained in $\mathcal{S}$. So, to show that $\mathcal{D}$ is a spanning set for $\mathbb{S}$, we need to show that every element of $\mathbb{S}$ can be written as a linear combination of the elements of $\mathcal{D}$. To that end, let $A \in \mathbb{S}$. Then there are real numbers a_1 and a_2 such that $A = \begin{bmatrix} a_1 & a_1 \\ a_1 & a_2 \end{bmatrix}$. Then we can write

$$A = a_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

And thus we see that A can be written as a linear combination of the elements of $\mathcal{D}$. And this means that $\mathcal{D}$ is a spanning set for $\mathbb{S}$.

All that remains to show that $\mathcal{D}$ is a basis for $\mathbb{S}$ is to show that $\mathcal{D}$ is linearly independent. But the equation

$$t_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

clearly only has solution $t_1 = t_2 = 0$. And so we easily see that $\mathcal{D}$ is linearly independent. And since $\mathcal{D}$ is a linearly independent spanning set for $\mathbb{S}$, we have that $\mathcal{D}$ is a basis for $\mathbb{S}$. And since $\mathcal{D}$ has two elements, we know that the dimension of $\mathbb{S}$ is 2.

B9(e) Polynomials that satisfy $p(x) = p(-x)$ are known as even polynomials, in no small part because they only contain even powers of x. To see this, consider the following. If $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + p_4 x^4 + p_5 x^5$, then $p(-x) == p_0 + p_1(-x) + p_2(-x)^2 + p_3(-x)^3 + p_4(-x)^4 + p_5(-x)^5 == p_0 + p_1(-x) + p_2 x^2 + p_3(-x^3) + p_4 x^4 + p_5(-x^5) == p_0 - p_1 x + p_2 x^2 - p_3 x^3 + p_4 x^4 - p_5 x^5$. In order to have $p(x) = p(-x)$, we need to have:

$$\begin{array}{ccc} p_0 = p_0 & p_1 = -p_1 & p_2 = p_2 \\ p_3 = -p_3 & p_4 = p_4 & p_5 = -p_5 \end{array}$$

We always have $p_0 = p_0$, $p_2 = p_2$, and $p_4 = p_4$. But in order to have $p_1 = -p_1$, we need to have $p_1 = 0$, and similarly we get that $p_3 = 0$ and $p_5 = 0$.

Thus, another way to write $\mathbb{S}$ is to say $\mathbb{S} = \{p_0 + p_2 x^2 + p_4 x^4 \mid p_0, p_2, p_4 \in \mathbb{R}\}$. And this description of $\mathbb{S}$ leads us quickly to guess that $\mathcal{E} = \{1, x^2, x^4\}$ is a basis for $\mathbb{S}$.

First, we note that $1, x^2, x^4 \in \mathbb{S}$, so we know that Span $\mathcal{E}$ is contained in $\mathbb{S}$. Thus, to show that Span $\mathcal{E} = \mathbb{S}$, we need to show that every element of $\mathbb{S}$ is in Span $\mathcal{E}$. To that end, let $a(x) \in \mathbb{S}$. Then there are real numbers a_0, a_2, a_4 such that $a(x) = a_0 + a_2 x^2 + a_4 x^4$. And this shows that $a(x)$ can be written as a linear combination of the elements of $\mathcal{E}$, and thus is in Span $\mathcal{E}$. Therefore, we have that $\mathcal{E}$ is a spanning set for $\mathbb{S}$.

All that remains is to show that $\mathcal{E}$ is linearly independent. But $\mathcal{E}$ is a subset of the standard basis for P_5, and thus is a subset of a linearly independent set. This means that $\mathcal{E}$ must also be linearly independent.

And since $\mathcal{E}$ is a linearly independent spanning set for $\mathbb{S}$, we have that $\mathcal{E}$ is a basis for $\mathbb{S}$. And since $\mathcal{E}$ has three elements, we know that the dimension of $\mathbb{S}$ is 3.