Solution to Practice 2g

B3(a) We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1}\vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1}\vec{w}_2$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \tfrac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \tfrac{1+0-1}{1^2+0^2+1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \tfrac{0}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\end{aligned}$$

So $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, and $\mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2}\vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2}\vec{w}_3$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \tfrac{2+0+1}{1^2+0^2+1^2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{2-2-1}{1^2+(-1)^2+(-1)^2} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \tfrac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \tfrac{1}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 5/6 \\ 5/3 \\ -5/6 \end{bmatrix}
\end{aligned}$$

And so we have that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5/6 \\ 5/3 \\ -5/6 \end{bmatrix} \right\}$.

B3(b) We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1}\vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1}\vec{w}_2$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \tfrac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \tfrac{0+1+2}{0^2+1^2+1^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \tfrac{3}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}
\end{aligned}$$

So $\vec{v}_2 = \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$, but to simplify our calculations we can multiply $\vec{v}_2$ by 2, and we get $\mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2}\vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2}\vec{w}_3$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} - \tfrac{0+0-2}{0^2+1^2+1^2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{6+0-2}{2^2+(-1)^2+1^2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{4}{6} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 5/3 \\ 5/3 \\ -5/3 \end{bmatrix} \\
&= \tfrac{5}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
\end{aligned}$$

And so we have that $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.

B3(c) We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1} \vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1} \vec{w}_2$. Doing the calculations, we get

$$
\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1+0-1+1}{1^2+0^2+(-1)^2+1^2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2/3 \\ 1 \\ 4/3 \\ 2/3 \end{bmatrix}
\end{aligned}
$$

So $\vec{v}_2 = \begin{bmatrix} 2/3 \\ 1 \\ 4/3 \\ 2/3 \end{bmatrix}$, but to simplify our calculations we can multiply $\vec{v}_2$ by 3, and we get $\mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2} \vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2} \vec{w}_3$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{2+0-1+1}{1^2+0^2+(-1)^2+1^2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \tfrac{4+0+4+2}{2^2+3^2+4^2+2^2} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{2}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \tfrac{10}{31} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 24/33 \\ -30/33 \\ 15/33 \\ -9/33 \end{bmatrix} \\
&= \tfrac{3}{33} \begin{bmatrix} 8 \\ -10 \\ 5 \\ -3 \end{bmatrix}
\end{aligned}$$

And so we have that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ -10 \\ 5 \\ -3 \end{bmatrix} \right\}$.

B3(d) We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1} \vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1} \vec{w}_2$. Doing the calculations, we get

$$
\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \tfrac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \tfrac{1+0+0+2}{1^2+0^2+0^2+1^2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \tfrac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 1/2 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

So we get $\mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2} \vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2} \vec{w}_3$. Doing the calculations, we get

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{0+0+0+1}{1^2+0^2+0^2+1^2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0-2+0+1}{(-1)^2+2^2+0^2+1^2} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4/6 \\ -4/6 \\ 1 \\ 4/6 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -2 \\ -2 \\ 3 \\ 2 \end{bmatrix}$$

And so we have that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 3 \\ 2 \end{bmatrix} \right\}$.

B4(a) First we will find an orthogonal basis, and then we will turn that into our orthonormal basis. To that end, we already found an orthogonal basis for this subspace in B3(a), so we know note that

$||\vec{v}_1|| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$||\vec{v}_2|| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

$||\vec{v}_3|| = \sqrt{(5/6)^2 + (5/3)^2 + (-5/6)^2} = \sqrt{(25 + 100 + 25)/36} = 5\sqrt{6}/6$

And this means that $\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \frac{6}{5\sqrt{6}} \begin{bmatrix} 5/6 \\ 5/3 \\ -5/6 \end{bmatrix} \right\}$

$= \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ is an orthonormal basis for our subspace.

B4(b) First we will find an orthogonal basis, and then we will turn that into our orthonormal basis. We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1} \vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1} \vec{w}_2$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} - \tfrac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} - \tfrac{2+2-9}{1^2+1^2+3^2} \left[\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right] \\
&= \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \tfrac{5}{11} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 27/11 \\ 27/11 \\ -18/11 \end{bmatrix} \\
&= \tfrac{9}{11} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}
\end{aligned}$$

So $\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$, and $\mathbb{S}_2 = \text{Span}\left\{ \left[\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right], \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2} \vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2} \vec{w}_3$. Doing the calculations, we get

$$
\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} - \tfrac{3+3-12}{1^2+1^2+3^2} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{9+9+8}{3^2+3^2+(-2)^2} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} + \tfrac{6}{11} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{26}{22} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

This means we set $\mathbb{S}_3 = \mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \right\}$, and we move on to the next step, where we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_3} \vec{w}_4 = \vec{w}_4 - \text{proj}_{\mathbb{S}_3} \vec{w}_4$. Doing the calculations, we get

$$
\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_4 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{\vec{w}_4 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{1+0+3}{1^2+1^2+3^2} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{3+0-2}{3^2+3^2+(-2)^2} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{4}{11} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \tfrac{1}{22} \left[\begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} \right] \\
&= \tfrac{1}{22} \begin{bmatrix} 11 \\ -11 \\ 0 \end{bmatrix} = \tfrac{11}{22} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
\end{aligned}
$$

And so we have that $\left\{ \left[\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right], \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is an orthogonal basis.

To get our orthonormal basis we first note that

$||\vec{v}_1|| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$

$||\vec{v}_2|| = \sqrt{3^2 + 3^2 + (-2)^2} = \sqrt{22}$

$||\vec{v}_3|| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$

And this means that $\mathcal{B} = \left\{ \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{22}} \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis for our subspace.

B4(c) First we will find an orthogonal basis, and then we will turn that into our orthonormal basis. We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1} \vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1} \vec{w}_2$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{0-1+0+1}{1^2+1^2+0^2+1^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

So we get $\mathbb{S}_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2} \vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2} \vec{w}_3$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{3+0+0+1}{1^2+1^2+0^2+1^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{0+0+1+1}{0^2+(-1)^2+1^2+1^2} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \tfrac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{2}{3} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 5/3 \\ -2/3 \\ 1/3 \\ -1 \end{bmatrix} \\
&= \tfrac{1}{3} \begin{bmatrix} 5 \\ -2 \\ 1 \\ -3 \end{bmatrix}
\end{aligned}$$

And so we have that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$ is an orthogonal basis. To get our orthonormal basis we first note that

$||\vec{v}_1|| = \sqrt{1^2 + 1^2 + 0^2 + 1^2} = \sqrt{3}$

$||\vec{v}_2|| = \sqrt{0^2 + (-1)^2 + 1^2 + 1^2} = \sqrt{3}$

$||\vec{v}_3|| = \sqrt{5^2 + (-2)^2 + 1^2 + (-3)^2} = \sqrt{39}$

And this means that $\mathcal{B} = \left\{ \tfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tfrac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \tfrac{1}{\sqrt{39}} \begin{bmatrix} 5 \\ -2 \\ 1 \\ -3 \end{bmatrix} \right\}$ is an orthonormal basis for our subspace.

B4(d) First we will find an orthogonal basis, and then we will turn that into our orthonormal basis. We start by setting $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_2 = \text{perp}_{\mathbb{S}_1} \vec{w}_2 = \vec{w}_2 - \text{proj}_{\mathbb{S}_1} \vec{w}_2$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_2 &= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\vec{w}_2 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-1+0+0+1+0}{1^2+1^2+0^2+1^2+0^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

So we get $\mathbb{S}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Next, we set $\vec{v}_3 = \text{perp}_{\mathbb{S}_2} \vec{w}_3 = \vec{w}_3 - \text{proj}_{\mathbb{S}_2} \vec{w}_3$. Doing the calculations, we get

$$\begin{aligned}
\vec{v}_3 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{\vec{w}_3 \cdot \vec{v}_1}{||\vec{v}_1||^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\vec{w}_3 \cdot \vec{v}_2}{||\vec{v}_2||^2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{1+1+0+2+0}{1^2+1^2+0^2+1^2+0^2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-1+0+0+2+1}{(-1)^2+0^2+1^2+1^2+1^2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/6 \\ -2/6 \\ -3/6 \\ 1/6 \\ 3/6 \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \\ 3 \end{bmatrix}
\end{aligned}$$

And so we have that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is an orthogonal basis. To get our orthonormal basis we first note that

$||\vec{v}_1|| = \sqrt{1^2 + 1^2 + 0^2 + 1^2 + 0^2} = \sqrt{3}$

$||\vec{v}_2|| = \sqrt{(-1)^2 + 0^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$

$||\vec{v}_3|| = \sqrt{1^2 + (-2)^2 + (-3)^2 + 1^2 + 3^2} = \sqrt{24}$

And this means that $\mathcal{B} = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{24}} \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \\ 3 \end{bmatrix} \right\}$ is an orthonormal basis for our subspace.