Lecture 1j

Dimension

(pages 211-213)

Before we look at extending a linearly independent set into a basis, we will need a few more facts. In our technique for shrinking a spanning set to a basis, we depend on the fact that, if we get small enough, we will eventually be linearly independent. Now, we want to know that if we get big enough, we will eventually be a spanning set. And the first step towards understanding why this happens is to realize that every basis for a vector space has the same number of vectors.

Lemma 4.3.2 Suppose that $\mathbb{V}$ is a vector space and Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \mathbb{V}$. If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set in $\mathbb{V}$, then $k \leq n$. In words, this says that the number of vectors in a linearly independent set will be less than or equal to the number of vectors in a spanning set.

The proof of this Lemma is available in the text. Instead, let's focus on the main result, which follows easily from the Lemma.

Theorem 4.3.3 If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ are both bases of a vector space $\mathbb{V}$, then $k = n$.

Note that this theorem does not say the a vector space will have a unique basis, merely that the number of vectors in a basis is unique. We call this unique number of basis vectors the dimension of the vector space.

Definition: If a vector space $\mathbb{V}$ has a basis with n vectors, then we say that the **dimension** of $\mathbb{V}$ is n and write

$$\dim \mathbb{V} = n$$

The dimension of the trivial vector space $\mathbb{O} = \{\mathbf{0}\}$ is defined to be 0, consistent with our definition that the basis for $\mathbb{O}$ is the empty set. If a vector space $\mathbb{V}$ does not have a basis with finitely many elements, then $\mathbb{V}$ is called **infinite-dimensional**. We will not be studying the properties of infinite-dimensional spaces in this course.

Standard Examples:
(1) $\mathbb{R}^n$ has standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$, so dim $\mathbb{R}^n = n$.
(2) $M(m, n)$ has a standard basis consisting of the mn matrices with a 1 in one entry and a 0 in the other entries. As such, dim $M(m, n) = mn$.
(3) P_n has standard basis $\{1, x, x^2, \ldots, x^n\}$, so dim $P_n = n+1$ (count carefully!)
(4) $\mathcal{F}$, $\mathcal{F}(a, b)$, and $\mathcal{C}(a, b)$ are all infinite-dimensional spaces.

Example: In the previous lecture, we found that $\mathcal{T}_2 = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} \right\}$ is a basis for Span $\mathcal{T}$, where

$$\mathcal{T} = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} \right\}$$

Since $\mathcal{T}_2$ contains 3 matrices, this means that the dimension of Span $\mathcal{T}$ is 3.