Lecture 3q

Diagonalizing a Real Matrix over $\mathbb{C}$

(pages 418-419)

Let's look at what happens when we diagonalize a matrix with real entries over $\mathbb{C}$.

Example: Let's diagonalize the matrix $A = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}$ over $\mathbb{C}$. First, we need to find the eigenvalues, by finding the roots of the characteristic polynomial $\det(A - \lambda I)$.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 4 \\ -2 & -1-\lambda \end{bmatrix} \\ &= (3-\lambda)(-1-\lambda) + 8 \\ &= -3 - 3\lambda + \lambda + \lambda^2 + 8 \\ &= \lambda^2 - 2\lambda + 5 \end{aligned}$$

We can use the quadratic formula to find the roots of this equation:

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

So, the eigenvalues for A are $\lambda = 1 + 2i$ and $\lambda = 1 - 2i$. Now let's find the eigenspaces for these eigenvalues.

For $\lambda = 1+2i$, the eigenspace is the nullspace of $A - \lambda I = \begin{bmatrix} 3-(1+2i) & 4 \\ -2 & -1-(1+2i) \end{bmatrix} = \begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix}$. To find a basis for the nullspace, we need to row reduce the matrix.

$$\begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix} \; R_1 \updownarrow R_2 \sim \begin{bmatrix} -2 & -2-2i \\ 2-2i & 4 \end{bmatrix} \; (-1/2)R_1$$
$$\sim \begin{bmatrix} 1 & 1+i \\ 2-2i & 4 \end{bmatrix} \; R_2 + (-2+2i)R_1 \sim \begin{bmatrix} 1 & 1+i \\ 0 & 0 \end{bmatrix}$$

So, the eigenvectors for $\lambda = 1 + 2i$ satisfy the equation $z_1 + (1+i)z_2 = 0$, or $z_1 = -(1+i)z_2$. This means that the eigenspace is Span $\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 1-2i$, the eigenspace is the nullspace of $A - \lambda I = \begin{bmatrix} 3-(1-2i) & 4 \\ -2 & -1-(1-2i) \end{bmatrix} =$

$\begin{bmatrix} 2+2i & 4 \\ -2 & -2+2i \end{bmatrix}$. To find a basis for the nullspace, we need to row reduce the matrix.

$$\begin{bmatrix} 2+2i & 4 \\ -2 & -2+2i \end{bmatrix} \ R_1 \updownarrow R_2 \ \sim \begin{bmatrix} -2 & -2+2i \\ 2+2i & 4 \end{bmatrix} \ (-1/2)R_1$$

$$\sim \begin{bmatrix} 1 & 1-i \\ 2+2i & 4 \end{bmatrix} \ R_2 + (-2-2i)R_1 \ \sim \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

So, the eigenvectors for $\lambda = 1-2i$ satisfy the equation $z_1 + (1-i)z_2 = 0$, or $z_1 = -(1-i)z_2$. This means that the eigenspace is Span$\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}$.

So we have that $\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1+2i$, and $\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1-2i$, and this means that the matrix $P = \begin{bmatrix} -1-i & -1+i \\ 1 & 1 \end{bmatrix}$ is such that $P^{-1}AP = D = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}$.

To a certain extent, this situation is unsatisfying. While we couldn't diagonalize this matrix over $\mathbb{R}$, diagonalizing to a matrix with complex entries isn't really what we want either. But there are some useful things we can notice from this example. The first is that the eigenvalues are complex conjugates of each other. This is not surprising, since the eigenvalues of a matrix with real entries are the solutions to a polynomial with real coefficients, and Theorem 9.1.2 already pointed out that complex solutions to polynomials over $\mathbb{R}$ will come in conjugate pairs. But there is something new to realize at this point, and that is that the eigenvectors for $\lambda = 1-2i$ are vector conjugates of the eigenvectors for $\lambda = 1+2i$. So, not only do the eigenvalues come in conjugate pairs, but the eigenvectors come in conjugate pairs as well. We'll prove this for all matrices with real entries right after we introduce the following definition.

Definition: Let A be an $m \times n$ matrix. We define the **complex conjugate of** A, $\overline{A}$, by

$$\left(\overline{A}\right)_{ij} = \overline{(A)_{ij}}$$

Theorem 9.4.1 Suppose that A is an $n \times n$ matrix with real entries and that $\lambda = a + bi$, $b \neq 0$, is an eigenvalue of A with corresponding eigenvector $\vec{z}$. Then $\overline{\lambda}$ is also an eigenvalue, with corresponding eigenvector $\overline{\vec{z}}$.

Proof of Theorem 9.4.1: Let A be an $n \times n$ matrix with real entries, and let λ be an eigenvalue of A with corresponding eigenvector $\vec{z}$. Then $A\vec{z} = \lambda\vec{z}$, and taking the complex conjugate of both sides gives us:

$$\overline{A\vec{z}} = \overline{\lambda\vec{z}}$$

First, let's look at $\vec{w} = \overline{A\vec{z}}$. Then $\vec{w} \in \mathbb{C}^n$, and $w_k = \overline{\left(\sum_{j=1}^{n} a_{jk} z_k\right)}$. Since one of the properties of the complex conjugate is that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, and another is that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, we see that $w_k = \sum_{j=1}^{n} \overline{a_{jk}}\ \overline{z_k}$. But this means that $\vec{w} = \overline{A}\ \overline{\vec{z}}$. That is to say, we have shown that $\overline{A\vec{z}} = \overline{A}\ \overline{\vec{z}}$.

Now, let's look at $\vec{u} = \overline{\lambda\vec{z}}$. Then $\vec{u} \in \mathbb{C}^n$, and $u_k = \overline{\lambda z_k} = \overline{\lambda}\ \overline{z_k}$. Which means that $\vec{u} = \overline{\lambda}\ \overline{\vec{z}}$. This is to say, we have shown that $\overline{\lambda\vec{z}} = \overline{\lambda}\ \overline{\vec{z}}$.

Combining our results, we see that

$$\overline{A}\ \overline{\vec{z}} = \overline{A\vec{z}} = \overline{\lambda\vec{z}} = \overline{\lambda}\ \overline{\vec{z}}$$

This means that $\overline{\lambda}$ is an eigenvalue for $\overline{A}$, with corresponding eigenvector $\overline{\vec{z}}$.

It is worth noting that so far in this proof we have not made use of the fact that A is a matrix with real entries. That means that all these results hold true for any matrix A with complex entries! But, to get the result we want, we will now use the fact that A is a matrix with real entries, to see that $\overline{A} = A$, which means that $\overline{\lambda}$ is an eigenvalue for A, with corresponding eigenvector $\overline{\vec{z}}$, as desired.

Let's see how we can make use of this fact:

Example: Let's diagonalize the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$ over $\mathbb{C}$. First, we need to find the eigenvalues, by finding the roots of the characteristic polynomial $\det\,(A - \lambda I)$.

$$\begin{aligned}
\det\,(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 0 & 1 \\ 2 & 1-\lambda & 2 \\ -1 & 0 & 3-\lambda \end{bmatrix} \\
&= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} - 0 + \begin{vmatrix} 2 & 1-\lambda \\ -1 & 0 \end{vmatrix} \\
&= (3-\lambda)(1-\lambda)(3-\lambda) + (1-\lambda) \\
&= (1-\lambda)((3-\lambda)(3-\lambda) + 1) \\
&= (1-\lambda)(9 - 3\lambda - 3\lambda + \lambda^2 + 1) \\
&= (1-\lambda)(10 - 6\lambda + \lambda^2)
\end{aligned}$$

Because $1-\lambda$ is a factor of the characteristic polynomial, we know that $\lambda = 1$ is an eigenvalue of A. To find the other eigenvalues of A, we use the quadratic formula to find the roots of $10 - 6\lambda + \lambda^2$:

$$\lambda = \frac{6 \pm \sqrt{36 - 4(1)(10)}}{2(1)} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

So, the eigenvalues of A are $\lambda = 1, 3+i, 3-i$. Now let's find their eigenspaces.

The eigenspace for $\lambda = 1$ is the nullspace of $A - \lambda I = \begin{bmatrix} 3-1 & 0 & 1 \\ 2 & 1-1 & 2 \\ -1 & 0 & 3-1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, which we find by row reducing this matrix.

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 6 \\ 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenvectors of $\lambda = 1$ are the the vectors that satisfy $z_1 = 0$ and $z_3 = 0$, which we can write as Span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

The eigenspace for $\lambda = 3+i$ is the nullspace of $A - \lambda I = \begin{bmatrix} 3-(3+i) & 0 & 1 \\ 2 & 1-(3+i) & 2 \\ -1 & 0 & 3-(3+i) \end{bmatrix} = \begin{bmatrix} -i & 0 & 1 \\ 2 & -2-i & 2 \\ -1 & 0 & -i \end{bmatrix}$, which we find by row reducing this matrix.

$$\begin{bmatrix} -i & 0 & 1 \\ 2 & -2-i & 2 \\ -1 & 0 & -i \end{bmatrix} \begin{matrix} iR_1 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & i \\ 2 & -2-i & 2 \\ -1 & 0 & -i \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & i \\ 0 & -2-i & 2-2i \\ 0 & 0 & 0 \end{bmatrix} \; (1/5)(-2+i)R_2 \; \sim \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & -\frac{2}{5} + \frac{6}{5}i \\ 0 & 0 & 0 \end{bmatrix}.$$

So the eigenvectors of $\lambda = 3+i$ are the the vectors that satisfy $z_1 + iz_3 = 0$ and $z_2 - ((2/5) - (6/5)i)z_3 = 0$. If we replace the variable z_3 with the parameter α, we see that the general solution to this system is

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -i\alpha \\ ((2/5) - (6/5)i)\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -i \\ (2/5) - (6/5)i \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} -5i \\ 2 - 6i \\ 5 \end{bmatrix}$$

which we can write as Span $\left\{\begin{bmatrix} -5i \\ 2-6i \\ 5 \end{bmatrix}\right\}$.

To find the eigenspace for $\lambda = 3-i$, we note that $3-i = \overline{3+i}$, and so by Theorem 9.4.1, the eigenspace for $\lambda = 3-i$ is $\{\overline{\vec{z}} \mid \vec{z}$ is in the eigenspace of $3+i\}$, which we can write as Span $\left\{\begin{bmatrix} \overline{-5i} \\ \overline{2-6i} \\ \overline{5} \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 5i \\ 2+6i \\ 5 \end{bmatrix}\right\}$.

So we have that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1$, $\begin{bmatrix} -5i \\ 2-6i \\ 5 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 3+i$, and $\begin{bmatrix} 5i \\ 2+6i \\ 5 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = 3-i$. And this means that the matrix $P = \begin{bmatrix} 0 & -5i & 5i \\ 1 & 2-6i & 2+6i \\ 0 & 5 & 5 \end{bmatrix}$ is such that $P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3+i & 0 \\ 0 & 0 & 3-i \end{bmatrix}$