Lecture 1n

Linear Mappings

(pages 227-228)

Let's return now to a concept we developed in our study of $\mathbb{R}^n$–linear mappings. Because, as with bases, linear mappings will be of much greater use in the general world of vector spaces than they were in simply the $\mathbb{R}^n$ spaces. We will, of course, start with a definition.

<u>Definition</u>: If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over $\mathbb{R}$, a function $L : \mathbb{V} \to \mathbb{W}$ is a **linear mapping** if it satisfies the linearity properies
(L1) $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y})$
(L2) $L(t\mathbf{x}) = tL(\mathbf{x})$
for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $t \in \mathbb{R}$. If $\mathbb{W} = \mathbb{V}$, then L may be called a **linear operator**.

Note that these two properties can be combined into one statement: $L(t\mathbf{x}+\mathbf{y}) = tL(\mathbf{x}) + L(\mathbf{y})$.

Example: The mapping $L : M(2,2) \to P_2$ defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$ is a linear mapping, because

$$\begin{aligned}
L(t\mathbf{x}+\mathbf{y}) &= L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\
&= L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) \\
&= (ta_1+a_2+tb_1+b_2) + (ta_1+a_2+tc_1+c_2)x + (ta_1+a_2+td_1+d_2)x^2 \\
&= (ta_1+tb_1) + (ta_1+tc_1)x + (ta_1+td_1)x^2 + (a_2+b_2) + (a_2+c_2)x + (a_2+d_2)x^2 \\
&= t((a_1+b_1) + (a_1+c_1)x + (a_1+d_1)x^2) + ((a_2+b_2) + (a_2+c_2)x + (a_2+d_2)x^2) \\
&= tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\
&= tL(\mathbf{x}) + L(\mathbf{y})
\end{aligned}$$

Example: The mapping $M : P_3 \to P_3$ defined by $M(a+bx+cx^2+dx^3) = a^2 + abx + acx^2 + adx^3$ is not linear. Consider, for example, that

$$M(1+2x+x^2+2x^3) = 1+2x+x^2+2x^3 \quad \text{and} \quad M(4-x+2x^2-x^3) = 16-4x+8x^2-4x^3$$

and so we have that

$$\begin{aligned}
M(1+2x+x^2+2x^3) + M(4-x+2x^2-x^3) &= (1+2x+x^2+2x^3) + (16-4x+8x^2-4x^3) \\
&= 17-2x+19x^2-2x^3
\end{aligned}$$

but that

$$\begin{aligned} M((1+2x+x^2+2x^3)+(4-x+2x^2-x^3)) &= M(5+x+3x^2+x^3) \\ &= 25+5x+15x^2+5x^3 \end{aligned}$$

and so we have that $M(1+2x+x^2+2x^3)+M(4-x+2x^2-x^3) \neq M((1+2x+x^2+2x^3)+(4-x+2x^2-x^3))$. And this means that M is not closed under addition, so it is not a linear mapping.

It happens that M is also not closed under scalar multiplication. Consider, for example, that $M(1+2x+x^2+2x^3) = 1+2x+x^2+2x^3$, so $5M(1+2x+x^2+2x^3) = 5(1+2x+x^2+2x^3) = 5+10x+5x^2+10x^3$, but $M(5(1+2x+x^2+2x^3)) = M(5+10x+5x^2+10x^3) = 25+50x+25x^2+50x^3$, so $5M(1+2x+x^2+2x^3) \neq M(5(1+2x+x^2+2x^3))$.

Students often wonder how to tell at a glance if something is linear or not. In general, if you only end up doing linear combinations with your entries (add, multiply by a constant), then it is probably linear. But if the definition involves multiplication, roots or exponents, then the mapping is probably not linear. But this only provides you with a guess–you still need to prove it!