Lecture 1o

Range and Nullspace

(pages 228-230)

There are two subspaces that we affiliate with linear mappings: the range and the nullspace. We will define them first, and then prove that they are subspaces.

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{R}$. The **range** of a linear mapping $L : \mathbb{V} \to \mathbb{W}$ is defined to be the set

$$\text{Range}(L) = \{L(\mathbf{x}) \in \mathbb{W} \mid \mathbf{x} \in \mathbb{V}\}$$

Example: Let $L : \mathbb{R}^2 \to M(2,2)$ be the linear mapping defined by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Then we see that $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the range of L, because $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. But $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in the range of L. To see this, we note that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is in the range of L only if there is a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ whose entries a and b satisfy $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Setting the entries equal to each other, we see that this is equivalent to the system

$$a = 1 \quad b = 2 \quad b = 3 \quad a = 4$$

But since we cannot simultaneously have $a = 1$ and $a = 4$ (or $b = 2$ and $b = 3$), we see that our system is inconsistent. And since there are no such a and b, we have that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in the range of L.

Example: Determine whether or not $\mathbf{y} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$ is in the range of $L : P_3 \to \mathbb{R}^2$, where L is defined by $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + 2b + c + 2d \\ 3a + 4b - c - 2d \end{bmatrix}$. If it is, find a vector $\mathbf{x}$ such that $L(\mathbf{x}) = \mathbf{y}$.

First, we look to see if $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ is in the range of L. That is, we want to know if there is a polynomial $\mathbf{x} = a + bx + cx^2 + dx^3$ whose coefficients a, b, c and d satisfy $\begin{bmatrix} a + 2b + c + 2d \\ 3a + 4b - c - 2d \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$. Setting the components equal to each other, we see that this is equivalent to the system

$$\begin{array}{rrrrl} a & +2b & +c & +2d & = 5 \\ 3a & +4b & -c & -2d & = -7 \end{array}$$

We solve this system by row reducing its augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 2 & 1 & 2 & 5 \\ 3 & 4 & -1 & -2 & -7 \end{array}\right] \; R_2 - 3R_1 \quad \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 2 & 5 \\ 0 & -2 & -4 & -8 & -22 \end{array}\right]$$

Our matrix is now in row echelon form, and since there are no bad rows, we know that our system is consistent. This means that there is a solution, and thus that $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ IS in the range of L. Now we need to find $\mathbf{x}$ such that $L(\mathbf{x}) = \mathbf{y}$, which simply means finding a solution to our system. So, we will continue our row reduction until we reach reduced row echelon form.

$$\left[\begin{array}{rrrr|r} 1 & 2 & 1 & 2 & 5 \\ 0 & -2 & -4 & -8 & -22 \end{array}\right] \; (-1/2)R_2 \quad \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 2 & 5 \\ 0 & 1 & 2 & 4 & 11 \end{array}\right] \; R_1 - 2R_2$$
$$\sim \left[\begin{array}{rrrr|r} 1 & 0 & -3 & -6 & -17 \\ 0 & 1 & 2 & 4 & 11 \end{array}\right]$$

So we have that our system is equivalent to the system

$$\begin{array}{rrrrl} a & & -3c & -6d & = -17 \\ & b & +2c & +4d & = 11 \end{array}$$

Replacing the variable c with the parameter s and the variable d with the parameter t, we see that the general solution to our system is

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -17 + 3s + 6t \\ 11 - 2s - 4t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -17 \\ 11 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

This provides us a list of all possible a, b, c, and d. By picking specific values for s and t, we can get a vector $\mathbf{x}$ such that $L(\mathbf{x}) = \mathbf{y}$. And the easiest possible values for s and t are $s = 0$ and $t = 0$, which gives us $a = -17$, $b = 11$, $c = 0$ and $d = 0$. And so we have that $L(-17 + 11x) = \begin{bmatrix} 5 \\ -7 \end{bmatrix}$.

Definition: The **nullspace** of L is the set of all vectors in $\mathbb{V}$ whose image under L is the zero vector $\mathbf{0}_{\mathbb{W}}$. We write

$$\mathrm{Null}(L) = \{\mathbf{x} \in \mathbb{V} \mid L(\mathbf{x}) = \mathbf{0}_{\mathbb{W}}\}$$

Example: Let $L : M(2,3) \to P_1$ be the linear mapping defined by $L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = (a+b+c)+(d+e+f)x$. Then $\begin{bmatrix} 1 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$ is in the nullspace of L, since $L\left(\begin{bmatrix} 1 & 1 & -1 \\ -3 & 0 & 3 \end{bmatrix}\right) = (1+1-2)+(-3+0+3)x = 0+0x$. But $\begin{bmatrix} 1 & 2 & 1 \\ -3 & 5 & 3 \end{bmatrix}$ is not in the nullspace of L, since $L\left(\begin{bmatrix} 1 & 2 & 1 \\ -3 & 5 & 3 \end{bmatrix}\right) = (1+2+1)+(-3+5+3)x = 4+5x \neq 0+0x$.

You'll note that it is much easier to check if a vector is in the nullspace than to check if it is in the range!

Recall that one of the reasons we undertook a study of general vector spaces (instead of focusing on each one individually) is that it gives us the opportunity to prove statements that are true for all vector spaces. Or, as below, a statement that is true for all linear mappings.

<u>Theorem 4.5.1</u>: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Then
(1) $L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$
(2) Null(L) is a subspace of $\mathbb{V}$
(3) Range(L) is a subspace of $\mathbb{W}$.

<u>Proof of Theorem 4.5.1</u>: (1) Let $\mathbf{x}$ be any element of $\mathbb{V}$. Then we have

$$\begin{aligned} L(\mathbf{0}_{\mathbb{V}}) &= L(0\mathbf{x}) \\ &= 0L(\mathbf{x}) \\ &= \mathbf{0}_{\mathbb{W}} \end{aligned}$$

(2) I will leave the proof of part 2 as a practice problem.

(3) To see that Range(L) is a subspace of $\mathbb{W}$, we first want to note that Range(L) is explicitly defined as a subset of $\mathbb{W}$. Moreover, Range(L) is non-empty, since part (1) tells us that $L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$, and thus we have that $\mathbf{0}_{\mathbb{W}} \in$ Range(L). Next, we check to see if Range(L) is closed under addition. So, let $\mathbf{w}_1$ and $\mathbf{w}_2$ be elements of Range(L). This means that there are elements $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathbb{V}$ such that $L(\mathbf{v}_1) = \mathbf{w}_1$ and $L(\mathbf{v}_2) = \mathbf{w}_2$. But note that $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$. This means that $\mathbf{v}_1 + \mathbf{v}_2$ is an element of $\mathbb{V}$ such that $L(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$, and thus $\mathbf{w}_1 + \mathbf{w}_2 \in$ Range(L). Now that we have shown that Range(L) is closed under addition, let's show that it is closed under scalar multiplication. To that end, let $\mathbf{w} \in$ Range(L) and $s \in \mathbb{R}$. Then there is $\mathbf{v} \in \mathbb{V}$ such that $L(\mathbf{v}) = \mathbf{w}$. But this means that $L(s\mathbf{v}) = sL(\mathbf{v}) = s\mathbf{w}$. So, we have found $s\mathbf{v} \in \mathbb{V}$ such that $L(s\mathbf{v}) = s\mathbf{w}$, so $s\mathbf{w} \in$ Range(L).