Solution to Practice 3v

A3(a) We see that $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \cdot \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix} = -i + 0 + i = 0$, so $\vec{u}$ and $\vec{v}$ are orthogonal.

A3(b) Let $\mathbb{S} = \text{Span}\{\vec{u}, \vec{v}\}$. Our result from part (a) says that $\mathcal{B} = \{\vec{u}, \vec{v}\}$ is an orthogonal basis for $\mathbb{S}$. Then we have

$$\text{proj}_{\mathbb{S}}\vec{w} = \frac{\langle \vec{w}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u} + \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}$$

Let's compute all the inner products:

$\langle \vec{w}, \vec{u} \rangle = \vec{w} \cdot \overline{\vec{u}} = \begin{bmatrix} 1+i \\ 2+i \\ 3+i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix} = 1 + i - 3i - i^2 = 2 - 2i$

$\langle \vec{u}, \vec{u} \rangle = 1^2 + 0^2 + 1^2 = 2$

$\langle \vec{w}, \vec{v} \rangle = \vec{w} \cdot \overline{\vec{v}} = \begin{bmatrix} 1+i \\ 2+i \\ 3+i \end{bmatrix} \cdot \begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix} = -i - i^2 + 2 + i + 3 + i = 6 + i$

$\langle \vec{v}, \vec{v} \rangle = 1^2 + 1^2 + 1^2 = 3$

So we see that $\text{proj}_{\mathbb{S}}\vec{w} = \frac{2-2i}{2}\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + \frac{6+i}{3}\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - i + 2i + (1/3)i^2 \\ 0 + 2 + (1/3)i \\ i - i^2 + 2 + (1/3)i \end{bmatrix} =$
$\begin{bmatrix} (2/3) + i \\ 2 + (1/3)i \\ 3 + (4/3)i \end{bmatrix}$

B3(a) We see that $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} 1+i \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ -2i \\ 0 \end{bmatrix} = 1 + 2i + i^2 - 2i + 0 = 0$, so $\vec{u}$ and $\vec{v}$ are orthogonal.

B3(b) Let $\mathbb{S} = \text{Span}\{\vec{u}, \vec{v}\}$. Our result from part (a) says that $\mathcal{B} = \{\vec{u}, \vec{v}\}$ is an orthogonal basis for $\mathbb{S}$. Then we have

$$\text{proj}_{\mathbb{S}}\vec{w} = \frac{\langle \vec{w}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u} + \frac{\langle \vec{w}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}$$

Let's compute all the inner products:

$$\langle \vec{w}, \vec{u} \rangle = \vec{w} \cdot \overline{\vec{u}} = \begin{bmatrix} 2+i \\ 2-i \\ 1-2i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1 \\ 2 \end{bmatrix} = 2-2i+i-i^2+2-i+2-4i = 7-6i$$

$\langle \vec{u}, \vec{u} \rangle = 1^2+1^2+1^2+2^2 = 7$

$$\langle \vec{w}, \vec{v} \rangle = \vec{w} \cdot \overline{\vec{v}} = \begin{bmatrix} 2+i \\ 2-i \\ 1-2i \end{bmatrix} \cdot \begin{bmatrix} 1+i \\ -2i \\ 0 \end{bmatrix} = 2+2i+i+i^2-4i+2i^2+0 = -1-i$$

$\langle \vec{v}, \vec{v} \rangle = 1^2+(-1)^2+2^2 = 6$

So we see that

$$\begin{aligned}
\text{proj}_{\mathbb{S}}\vec{w} &= \tfrac{7-6i}{7} \begin{bmatrix} 1+i \\ 1 \\ 2 \end{bmatrix} + \tfrac{-1-i}{6} \begin{bmatrix} 1-i \\ 2i \\ 0 \end{bmatrix} \\
&= \tfrac{1}{7} \begin{bmatrix} 7+7i-6i-6i^2 \\ 7-6i \\ 14-12i \end{bmatrix} + \tfrac{1}{6} \begin{bmatrix} -1+i-i+i^2 \\ -2i-2i^2 \\ 0 \end{bmatrix} \\
&= \tfrac{1}{7} \begin{bmatrix} 13+i \\ 7-6i \\ 14-12i \end{bmatrix} + \tfrac{1}{6} \begin{bmatrix} -2 \\ 2-2i \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (32/21)+(1/7)i \\ (4/3)-(25/21)i \\ 2-(12/7)i \end{bmatrix}
\end{aligned}$$

B4(a) First, we will use the Gram-Schmidt Procedure to find an *orthogonal* basis.

First we let $\vec{w}_1 = \vec{v}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{w}_1\}$. Then

$$\begin{aligned}
\vec{w}_2 &= \text{perp}_{\mathbb{S}_1}\vec{v}_2 \\
&= \vec{v}_2 - \text{proj}_{\mathbb{S}_1}\vec{v}_2 \\
&= \vec{v}_2 - \tfrac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle}\vec{w}_1
\end{aligned}$$

Let's compute the inner products we need:

$\langle \vec{v}_2, \vec{w}_1 \rangle = \vec{v}_2 \cdot \overline{\vec{w}_1} = \begin{bmatrix} 1-i \\ -2-3i \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 2+i \\ -1-i \end{bmatrix} = 1 - i - i + i^2 - 4 - 2i - 6i -$
$3i^2 + 1 + i = -9i$

$\langle \vec{w}_1, \vec{w}_1 \rangle = 1^2 + 1^2 + 2^2 + (-1)^2 + (-1)^2 + 1^2 = 9$

So we have that

$$\begin{aligned} \vec{w}_2 &= \begin{bmatrix} 1-i \\ -2-3i \\ -1 \end{bmatrix} - \left(\tfrac{-9i}{9}\right) \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix} \\ &= \begin{bmatrix} 1-i+i-1 \\ -2-3i+2i+1 \\ -1-i-1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \end{aligned}$$

So, by the Gram-Schmidt Procedure, $\mathcal{B} = \left\{ \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix}, \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \right\}$ is an orthogonal basis for $\mathcal{S}$. To find an orthonormal basis, we simply need to divide the vectors by their length. We have already computed that $||\vec{w}_1||^2 = \langle \vec{w}_1, \vec{w}_1 \rangle =$ 9, so we have that $||\vec{w}_1|| = \sqrt{9} = 3$. And we see that $||\vec{w}_2|| = \sqrt{0^2 + (-1)^2 + (-1)^2 + (-2)^2 + (-1)^2} = \sqrt{7}$. And this means that $\mathcal{C} = \left\{ \frac{1}{3} \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix}, \frac{1}{\sqrt{7}} \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \right\}$ is an orthonormal basis for $\mathcal{S}$.

B4(b) We can use the orthogonal basis $\mathcal{B}$ found in part (a) to find $\text{proj}_{\mathbb{S}}\vec{u}$.

$$\text{proj}_{\mathbb{S}}\vec{u} = \frac{\langle \vec{u}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 + \frac{\langle \vec{u}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \vec{w}_2$$

We already know that $\langle \vec{w}_1, \vec{w}_1 \rangle = 9$ and $\langle \vec{w}_2, \vec{w}_2 \rangle = ||\vec{w}_2||^2 = 7$. We compute the other two inner products as follows:

$\langle \vec{u}, \vec{w}_1 \rangle = \vec{u} \cdot \overline{\vec{w}_1} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 2+i \\ -1-i \end{bmatrix} = 1 - i - i - i^2 = 2 - 2i$

$$\langle \vec{u}, \vec{w}_2 \rangle = \vec{u} \cdot \overline{\vec{w}_2} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} = 0 + 0 - 2i + i^2 = -1 - 2i$$

So we see that

$$\begin{aligned}
\text{proj}_{\mathbb{S}} \vec{w} &= \tfrac{2-2i}{9} \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix} + \tfrac{-1-2i}{7} \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \\
&= \tfrac{1}{63} \left(7(2-2i) \begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix} + 9(-1-2i) \begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \right) \\
&= \tfrac{1}{63} \left(\begin{bmatrix} 14 + 14i - 14i - 14i^2 \\ 28 - 14i - 28i + 14i^2 \\ -14 + 14i + 14i - 14i^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 9 + 9i + 18i + 18i^2 \\ 18 + 9i + 36i + 18i^2 \end{bmatrix} \right) \\
&= \tfrac{1}{63} \begin{bmatrix} 28 + 0 \\ 14 - 42i - 9 + 27i \\ 28i + 45i \end{bmatrix} \\
&= \begin{bmatrix} 4/9 \\ (5/63) - (5/21)i \\ (73/63)i \end{bmatrix}
\end{aligned}$$