Solution to Practice 2c

A1(a) The orthogonal change of coordinates matrix P is the matrix whose columns are the vectors in the orthonormal set $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \right\}$. So

$$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

A1(c) The orthogonal change of coordinates matrix P is the matrix whose columns are the vectors in the orthonormal set $\left\{ \begin{bmatrix} 1/\sqrt{11} \\ 1/\sqrt{11} \\ 3/\sqrt{11} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{110} \\ -10/\sqrt{110} \\ 3/\sqrt{110} \end{bmatrix} \right\}$.
So

$$P = \begin{bmatrix} 1/\sqrt{11} & 3/\sqrt{10} & 1/\sqrt{110} \\ 1/\sqrt{11} & 0 & -10/\sqrt{110} \\ 3/\sqrt{11} & -1/\sqrt{10} & 3/\sqrt{110} \end{bmatrix}$$

B1(a) The orthogonal change of coordinates matrix P is the matrix whose columns are the vectors in the orthonormal set $\left\{ \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix} \right\}$.
So

$$P = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}$$

B1(b) The orthogonal change of coordinates matrix P is the matrix whose columns are the vectors in the orthonormal set $\left\{ \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{30} \\ 5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix} \right\}$.
So

$$P = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 0 & 1/\sqrt{30} \\ 1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \end{bmatrix}$$

B4(a) $A^T A = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

$$= \begin{bmatrix} (4+1)/5 & (-2-2)/5 \\ (-2-2)/5 & (1+4)/5 \end{bmatrix} = \begin{bmatrix} 1 & -4/5 \\ -4/5 & 1 \end{bmatrix}$$

So A is not orthogonal. The columns of A fail to form an orthonormal set because they are not orthogonal.

B4(b) $A^T A = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ -1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

$$= \begin{bmatrix} (4+1)/5 & (-2+2)/5 \\ (-2+2)/5 & (1+4)/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So A is orthogonal.

B4(c) $A^T A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} (1+1)/4 & (1-1)/4 \\ (1-1)/4 & (1+1)/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

So A is not orthogonal. The columns of A fail to form an orthonormal set because their length does not equal 1.

B4(d) $A^T A = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} =$

$$\begin{bmatrix} (1+1+1)/3 & (1-1+0)/\sqrt{6} & (-1-1+2)/\sqrt{18} \\ (1-1+0)/\sqrt{6} & (1+1+0)/2 & (-1+1+0)/\sqrt{12} \\ (-1-1+2)/\sqrt{18} & (-1+1+0)/\sqrt{12} & (1+1+4)/6 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A is orthogonal.

B4(e) $A^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 0 \end{bmatrix} =$

$$\begin{bmatrix} (1+1+1)/3 & (1+1-1)/\sqrt{18} & (1+1+0)/\sqrt{6} \\ (1+1+-1)/\sqrt{18} & (1+1+1)/6 & (1+1+0)/\sqrt{12} \\ (1+1+0)/\sqrt{6} & (1+1+0)/\sqrt{12} & (1+1+0)/2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1/\sqrt{18} & 2/\sqrt{6} \\ 1/\sqrt{18} & 1/2 & 2/\sqrt{12} \\ 2/\sqrt{6} & 2/\sqrt{12} & 1 \end{bmatrix}$$

So A is not orthogonal. The columns of A fail to form an orthonormal set because none of the columns are orthogonal to each other, and because the second column is not a unit vector.

D1 Suppose that A and B are both orthogonal matrices. Then $(AB)^T(AB) = B^T A^T AB = B^T IB = B^T B = I$, so AB is also an orthogonal matrix.