Lecture 1f

Subspaces

(pages 201-203)

It is rare to show that something is a vector space using the defining properties. Instead, most things we want to study actually turn out to be a subspace of something we already know to be a vector space.

Definition: Suppose that $\mathbb{V}$ is a vector space, and that $\mathbb{U}$ is a subset of $\mathbb{V}$. If $\mathbb{U}$ is a vector space, using the same definition of addition and scalar multiplication as $\mathbb{V}$, then $\mathbb{U}$ is called a **subspace** of $\mathbb{V}$.

Example: Is P_2 a subspace of P_3? Yes! Since every polynomial of degree up to 2 is also a polynomial of degree up to 3, P_2 is a subset of P_3. And we already know that P_2 is a vector space, so it is a subspace of P_3.

However, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since the elements of $\mathbb{R}^2$ have exactly two entries, while the elements of $\mathbb{R}^3$ have exactly three entries. That is to say, $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$. Similarly, $M(2,2)$ is not a subspace of $M(2,3)$, because $M(2,2)$ is not a subset of $M(2,3)$.

Why are subspaces important? Well, it turns out a subspace inherits most of the vector space axioms from its parent vector space. For example, if $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ in $\mathbb{V}$, then this will continue to be true in $\mathbb{U}$. That's because this property is actually a property of the definition of addition, not of $\mathbb{V}$ or $\mathbb{U}$. And since our definition of subspace requires that $\mathbb{U}$ use the same definition of addition, we see that addition in $\mathbb{U}$ must also be commutative. Similarly, we immediately get $\mathbb{U}$ satisfies any properties that only use the definition of addition and scalar multiplication. These properties are:

V2: addition is associative
V5: addition is commutative
V7: scalar multiplication is associative
V8: scalar addition is distributive
V9: scalar multiplication is distributive
V10: scalar multiplicative identity

In fact, given any subset (but not necessarily a vector space) $\mathbb{W}$ of a vector space $\mathbb{V}$, we know that properties V2, V5, V7, V8, V9, and V10 will hold in $\mathbb{W}$. So, if we want to prove that $\mathbb{W}$ is itself a vector space, we only need to look at properties V1, V3, V4, and V6. Now properties V1 and V6 were trivial when showing that $\mathbb{R}^n$, $M(m,n)$, and P_n were vector spaces, but this property becomes much more important when we are looking at subspaces. Consider the following:

Example: Let $\mathbb{V} = M(2,2)$, and let $\mathbb{W}$ be the subset of $M(2,2)$ consisting of matrices with at most one non-zero entry. So, for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are all elements of $\mathbb{W}$. But let's look at the sum $A + B$. We know that $A + B \in \mathbb{V}$, and even that $A + B = B + A$, since A and B are elements of $\mathbb{V}$. But it turns out that $A + B \notin \mathbb{W}$, since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$$

and thus, $A + B$ has more than one non-zero entry.

And so, when we are considering properties V1 and V6, we are not so much concerned with the existence of $\mathbf{x} + \mathbf{y}$ and $s\mathbf{x}$, as we are that these elements are contained in our smaller set. Similarly, our concern with V3 and V4 is not that $\mathbf{0}$ and $-\mathbf{x}$ exist and satisfy their respective properties, but rather that they are actually elements of $\mathbb{W}$. But, it turns out that this key fact follows from property V6 (closure under scalar multiplication). How's that? Well, let $\mathbf{x}$ be any element of $\mathbb{W}$, and suppose that we have already shown that $\mathbb{W}$ is closed under scalar multiplication. Then we know that $s\mathbf{x} \in \mathbb{W}$ for any $s \in \mathbb{W}$. But this means that $s\mathbf{x} \in \mathbb{W}$ for $s = 0$ and $s = -1$. And by Theorem 4.2.1, we know that $0\mathbf{x} = \mathbf{0}$ and $(-1)\mathbf{x} = -\mathbf{x}$. (Again, we get this by using the fact that $\mathbb{V}$ is already known to be a vector space.) So we see that $\mathbf{0} \in \mathbb{W}$ and $-\mathbf{x} \in \mathbb{W}$, as desired.

There is one fine detail I skipped over, though. And that is the fact that I assumed that $\mathbf{x}$ is an element of $\mathbb{W}$. What's wrong with that, you wonder? Well, it only works if $\mathbb{W}$ actually contains elements! That is, we cannot have $\mathbb{W} = \emptyset$. Recall that the empty set can never be a vector space, since any vector space must contain at least a zero vector.

I'd like to summarize this discussion with a theorem, but the theorem I would like to state is in fact taken to be the definition of a subspace in the text, so instead I'll summarize these results with the following alternate definition of a subspace.

<u>Definition</u>: Suppose that $\mathbb{V}$ is a vector space. Then $\mathbb{U}$ is a subspace of $\mathbb{V}$ if is satisfies the following three properties: S0: $\mathbb{U}$ is a non-empty subset of $\mathbb{V}$
S1: $\mathbf{x} + \mathbf{y} \in \mathbb{U}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{U}$ ($\mathbb{U}$ is closed under addition)
S2: $t\mathbf{x} \in \mathbb{U}$ for all $\mathbf{x} \in \mathbb{U}$ and $t \in \mathbb{R}$ ($\mathbb{U}$ is closed under scalar multiplication)

Example: Show that $\mathbb{U} = \{a + bx + cx^2 \in P_2 \mid a = b = c\}$ is a subspace of P_2.

First, we check S0: Well, $\mathbb{U}$ is specifically defined as a subset of P_2, and we see that $0 + 0x + 0x^2 \in \mathbb{U}$, so $\mathbb{U}$ is not the empty set.

Next, we check S1: Suppose $p(x), q(x) \in \mathbb{U}$. Then there are $p, q \in \mathbb{R}$ such that $p(x) = p + px + px^2$ and $q(x) = q + qx + qx^2$. And we have that

$$p(x) + q(x) = (p + px + px^2) + (q + qx + qx^2) = (p + q) + (p + q)x + (p + q)x^2$$

And so we see that $p(x) + q(x) \in \mathbb{U}$.

Finally, we check S2: Suppose $p(x) \in \mathbb{U}$ and $s \in \mathbb{R}$. Let $p \in \mathbb{R}$ be such that $p(x) = p + px + px^2$. Then we have that

$$sp(x) = s(p + px + px^2) = (sp) + (sp)x + (sp)x^2$$

And so we see that $sp(x) \in \mathbb{U}$.

And since $\mathbb{U}$ satisfies properties S0, S1, and S2, we have that $\mathbb{U}$ is a subspace of P_2.

Example: Show that $\mathcal{A} = \{a + bx \in P_1 \mid b = a^2\}$ is not a subspace of P_1.

To show that something is not a subspace, we need to show that any one of the three properties does not hold. Of course, figuring out which property fails can be tricky. S0 is so easy to check that we usually start there. In this case, $\mathcal{A}$ is obviously a subset of P_1, and we quickly see that the zero polynomial in an element of $\mathcal{A}$, so it is non-empty.

So, S0 is not the property that fails to hold. But what about S1? Well, if we had two functions in $\mathcal{A}$, say $a + a^2x$ and $b + b^2x$, then when we add them we get $(a + b) + (a^2 + b^2)x$. For this function to be in $\mathcal{A}$, we need to have that $a^2 + b^2 = (a + b)^2$. This is, of course, not true in general. It is true sometimes, however, so instead of using a blanket statement of $a^2 + b^2 \neq (a + b)^2$ to show that S1 fails, the correct course of action is to find specific values of a and b such that $a^2 + b^2 \neq (a + b)^2$, and use them as our counterexample. One possible choice is $a = 1$ and $b = 2$, and using these values I get the following proof that $\mathcal{A}$ is not a subspace of P_1:

$\mathcal{A}$ is not a subspace of P_1, since there are elements $1 + x, 2 + 4x \in \mathcal{A}$ such that $(1 + x) + (2 + 4x) = 3 + 5x \notin \mathcal{A}$, so $\mathcal{A}$ is not closed under addition.

Short and simple!

Before we move on, I want to make a couple of comments about this example. First, I want to remind you that anytime you are trying to prove that something is not a subspace because properties S1 or S2 fail, you should use specific counterexamples. Many students want to stop with the fact that $a^2 + b^2 \neq (a + b)^2$, but as I pointed out, this statement is *sometimes* true, so how do we know that it isn't true for the specific elements of $\mathcal{A}$. This particular statement would be true whenever a or b equals zero, so at a minimum we already need to verify that there are two non-zero elements in $\mathcal{A}$. Which, of course, is exactly what I did in my counterexample. Basically, continuing on a "theoretical" path is likely to be more work than just using your information to generate the needed counterexample. So, while noticing that the statement $a^2 + b^2 = (a + b)^2$ is not true was a key step in finding my counterexample, this statement does not ever need to actually appear in my proof. Use it and lose it! And yes, this means that you do not need to show any work for how you find your counterexample.

The last thing I want to point out is that S2 also fails to hold in this case. One counterexample could be that $2+4x \in \mathcal{A}$, but $5(2+4x) = 10+20x \notin \mathcal{A}$. Again, we only need to show that one of the three properties fails, but since this is an example, I thought I would show this alternate possibility.

Now, remember that the point of subspaces is that they are a quick way to show that something is a vector space!

Example: The set $\mathcal{D}$ of differentiable functions over $\mathbb{R}$ is a vector space. We see this by considering it as a subspace of $\mathcal{F}$. Then we only need to check the three subspace properties:

S0: Differentiable functions are, of course, functions, so $\mathcal{D}$ is a subset of $\mathcal{F}$. Moveover, $\mathcal{D}$ is not empty, since, for example, the zero function is differentiable.

S1: Suppose f and g are differentiable functions. Then $f+g$ is also differentiable (in fact, $(f + g)' = f' + g'$), so $f + g \in \mathcal{D}$.

S2: Suppose f is a differentiable function, and $s \in \mathbb{R}$. Then sf is also differentiable (in fact, $(sf)' = sf'$), so $sf \in \mathcal{D}$.

This same technique tells us that the set $\mathcal{C}$ of continuous functions, is a vector space, as is the set $\mathcal{C}(a, b)$ of continuous functions on the interval (a, b). Even our polynomial spaces P_n can be thought of as subspaces of $\mathcal{F}$.