# Solution to Practice 1n

**B1(a)** $L(t(a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)) = L((ta_1+a_2)+(tb_1+b_2)x+(tc_1+c_2)x^2) = \begin{bmatrix} ta_1+a_2 \\ tb_1+b_2 \\ tc_1+c_2 \end{bmatrix} = t\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = tL(a_1+b_1x+c_1x^2) + L(a_2+b_2x+c_2x^2)$

**B1(b)** $L(t(a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)) = L((ta_1+a_2)+(tb_1+b_2)x+(tc_1+c_2)x^2) = (tb_1+b_2)+2(tc_1+c_2) = tb_1+2tc_1+b_2+2c_2 = t(b_1+2c_1)+(b_2+2c_2) = tL(a_1+b_1x+c_1x^2)+L(a_2b_2x+c_2x^2)$

**B1(c)** $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = 0 = t(0)+0 = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

**B1(d)** $L\left(t\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & 0 \\ 0 & tb_1+b_2 \end{bmatrix}\right) = (ta_1+a_2)+(ta_1+a_2+tb_1+b_2)x+(tb_1+b_2)x^2 = (ta_1+(ta_1+tb_1)x+tb_1x^2)+(a_2+(a_2+b_2)x+b_2x^2) = t(a_1+(a_1+b_1)x+b_1x^2)+(a_2+(a_2+b_2)x+b_2x^2) = tL\left(\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}\right)$

**B1(e)** $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = \begin{bmatrix} (ta_1+a_2)-(tb_1+b_2) & (tb_1+b_2)-(tc_1+c_2) \\ (tc_1+c_2)-(td_1+d_2) & (td_1+d_2)-(ta_1+a_2) \end{bmatrix} = t\begin{bmatrix} a_1-b_1 & b_1-c_1 \\ c_1-d_1 & d_1-a_1 \end{bmatrix} + \begin{bmatrix} a_2-b_2 & b_2-c_2 \\ c_2-d_2 & d_2-a_2 \end{bmatrix} = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

**B1(f)** $L\left(t\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ta_1+a_2 & tb_1+b_2 \\ tc_1+c_2 & td_1+d_2 \end{bmatrix}\right) = (ta_1+a_2+tb_1+b_2+tc_1+c_2+td_1+d_2)x^2 = ta_1x^2+a_2x^2+tb_1x^2+b_2x^2+tc_1x^2+c_2x^2+td_1x^2+d_2x^2 = t(a_1+b_1+c_1+d_1)x^2+(a_2+b_2+c_2+d_2)x^2 = tL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

**B1(g)** Using properties of the transpose (Theorem 3.1.2), we have that $T(tA+B) = (tA+B)^T = (tA)^T + B^T = tA^T + B^T = tT(A) + T(B)$.

**B2(a)** $L$ is not linear. Consider, for example, that $L(1) = L(1+0x+0x^2+0x^3) = \sqrt{1^2+0^2+0^2+0^2} = 1$ and $L(x) = L(0+x+0x^2+0x^3) = \sqrt{0^2+1^2+0^2+0^2} = 1$, so $L(1)+L(x) = 2$, but $L(1+x) = L(1+x+0x^2+0x^3) = \sqrt{1^2+1^2+0^2+0^2} = \sqrt{2}$, so $L(1)+L(x) \neq L(1+x)$.

**B2(b)** $M$ is not linear. Consider, for example, that $L(1+2x+3x^2) = 2^2 - 4(1)(3) = -8$ and $L(2+3x+4x^2) = 3^2 - 4(2)(4) = -23$, so $L(1+2x+3x^2)+$

$L(2+3x+4x^2) = -8-23 = -31$. But $L(1+2x+3x^2+2+3x+4x^2) = L(3+5x+7x^2) = 5^2-4(3)(7) = -59$. So $L(1+2x+3x^2)+L(2+3x+4x^2) \neq L(1+2x+3x^2+2+3x+4x^2)$.

**B2(c)** $N$ is linear. $N\left(t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = N\left(\begin{bmatrix} tx_1+y_1 \\ tx_2+y_2 \\ tx_3+y_3 \end{bmatrix}\right)$ $= \begin{bmatrix} tx_1+y_1-tx_3-y_3 & 0 \\ tx_2+y_2 & tx_2-y_2 \end{bmatrix} = \begin{bmatrix} tx_1-tx_3 & 0 \\ tx_2 & tx_2 \end{bmatrix} + \begin{bmatrix} y_1-y_3 & 0 \\ y_2 & y_2 \end{bmatrix} =$ $t\begin{bmatrix} x_1-x_3 & 0 \\ x_2 & x_2 \end{bmatrix} + \begin{bmatrix} y_1-y_3 & 0 \\ y_2 & y_2 \end{bmatrix} = tN\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + N\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$

**B2(d)** $L$ is linear. $L(tA+B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}(tA+B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}tA + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}B =$ $t\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}A + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}B = tL(A)+L(B)$

**B2(e)** $T$ is not linear. Consider, for example, that $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1+2x+6x^2$ and $T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = 2+6x+24x^2$, so $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) =$ $(1+2x+6x^2)+(2+6x+24x^2) = 3+8x+30x^2$. But $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) =$ $T\left(\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}\right) = 3+15x+105x^2$. And so we see that $T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) +$ $T\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right)$

**D3(a)** Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Suppose that $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is linearly independent. Consider the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}_{\mathcal{V}}$$

Since $L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$, we have that $L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = \mathbf{0}_{\mathcal{W}}$. But, using the linearity properties of $L$, this means that

$$t_1L(\mathbf{v}_1) + \cdots + t_kL(\mathbf{v}_k) = \mathbf{0}_{\mathcal{W}}$$

Since the set $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is linearly independent, the only solution to this equation is $t_1 = \cdots = t_k = 0$. This means we have shown that the only solution to the equation $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}_{\mathcal{V}}$ is $t_1 = \cdots = t_k = 0$. Which means that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

**D3(b)** There are many examples of this. One easy thing to do is to map one of your vectors to the zero vector, since any set containing the zero vector is linearly dependent. For example, let $\mathbb{V} = \mathbb{W} = \mathbb{R}^3$, and let $L : \mathbb{V} \to \mathbb{W}$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$. Then $L$ is linear, since $L\left(t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} tx_1 + y_1 \\ tx_2 + y_2 \\ tx_3 + y_3 \end{bmatrix}\right) = \begin{bmatrix} tx_1 + y_1 \\ tx_2 + y_2 \\ 0 \end{bmatrix} = t\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix} = tL\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)$. We note that the set $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is linearly independent, but the set $\left\{L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ is linearly dependent.