Lecture 1k

Extending a Linearly Independent Subset to a Basis

(pages 213-216)

Now that we know that the vector spaces in this course have a finite number of vectors in their basis, we can proceed to extend any linearly independent subset to a basis. And the way we do so is easy–just pick a vector not already in the span, and add it.

Theorem 4.3.b: If $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, and if $\mathbf{w} \notin \text{Span}\,\mathcal{T}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is also a linearly independent set.

Proof of Theorem 4.3.b: Suppose that $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, and that $\mathbf{w} \notin \text{Span}\,\mathcal{T}$. To see that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is linearly independent, we will look for solutions to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w} = \mathbf{0}$$

Suppose, by way of contradiction, that $t_{k+1} \neq 0$. Then we can divide by $t_{k+1}$ and get

$$\frac{t_1}{t_{k+1}}\mathbf{v}_1 + \cdots + \frac{t_k}{t_{k+1}}\mathbf{v}_k + \mathbf{w} = \mathbf{0}$$

which means that

$$-\frac{t_1}{t_{k+1}}\mathbf{v}_1 - \cdots - \frac{t_k}{t_{k+1}}\mathbf{v}_k = \mathbf{w}$$

But this means we can write $\mathbf{w}$ as a linear combination of the vectors in $\mathcal{T}$, which contradicts our choice of $\mathbf{w}$ as not being in the span of $\mathcal{T}$. From this contradiction, we know that $t_{k+1} = 0$. And this turns our linear independence equation into

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + 0\mathbf{w} = \mathbf{0}$$

which is the same as

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

And since $\mathcal{T}$ is linearly independent, we know that the only solution to this equation is $t_1 = \ldots = t_k = 0$. As such, we have shown that the only solution to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w} = \mathbf{0}$$

is $t_1 = \ldots = t_k = t_{k+1} = 0$. And this means that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is linearly independent.

Now, the fact that our vector spaces have a finite basis does more than guarantee that our expansion process will come to an end. It actually tells us *when* our process will end. Because our basis will need to have exactly dim $\mathbb{V}$ elements in it. We see how to use this fact in the following example.

**Example**: (a) Produce a basis $\mathcal{B}$ for the plane $\mathcal{P}$ in $\mathbb{R}^3$ with equation $2x_1 + 4x_2 - x_3 = 0$, and (b) extend the basis $\mathcal{B}$ to a basis $\mathcal{C}$ for $\mathbb{R}^3$.

We already know that the dimension of any plane in $\mathbb{R}^3$ is 2, so to find a basis $\mathcal{B}$ for $\mathcal{P}$, we simply need to find two linearly independent vectors on our plane. But a set of two vectors is linearly independent whenever they are not a scalar multiple of each other. So we can quickly note that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ are two vectors that satisfy $2x_1 + 4x_2 - x_3 = 0$ that are not scalar multiples of each other. And thus, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathcal{P}$.

Now, we know that the dimension of $\mathbb{R}^3$ is 3, so to extend $\mathcal{B}$ to a basis $\mathcal{C}$ of $\mathbb{R}^3$, we simply need to find one vector not in the span of $\mathcal{B}$. But the span of $\mathcal{B}$ is precisely the vectors on the plane $\mathcal{P}$. So this means we are looking for any vector not on the plane. That is, we are looking for any vector that does NOT satisfy $2x_1 + 4x_2 - x_3 = 0$. The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ quickly comes to mind as such a vector, and so we have that $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

During the process of shrinking spanning sets and expanding linearly independent sets, we have discovered the following:

<u>Theorem 4.3.4</u> Let $\mathbb{V}$ be an $n$-dimensional vector space. Then

(1) A set of more than $n$ vectors in $\mathbb{V}$ must be linearly dependent.
(2) A set of fewer than $n$ vectors cannot span $\mathbb{V}$.
(3) A set with $n$ elements of $\mathbb{V}$ is a spanning set for $\mathbb{V}$ if and only if it is linearly

independent.

Course Author's note: I refer to part (3) as the "two-out-of three" rule for proving that something is a basis. By definition, a basis must be a spanning set and linearly independent. But we also know that our basis must have the correct number of elements. So, really there are three features that our basis must have: spanning, linear independence, and $n$ elements. Between the definition of a basis and part (3), we see that showing any two of these three features will guarantee that our set is a basis.