Lecture 3f

Polar Form

(pages 399-402)

In the previous lecture, we saw that we can visualize a complex number as a point in the complex plane. This turns out to be remarkably useful, but we need to think about things a bit differently. Instead of thinking of the point (x, y) in terms of its x_1 and x_2 components, we instead want to describe a point in terms of (a) its distance from the origin, and (b) how much it is rotated away from the positive real axis.

Definition: Given a complex number $z = x + yi$, we define the **modulus** of z (denoted $|z|$) to be the real number

$$|z| = r = \sqrt{x^2 + y^2}$$

Definition: If $|z| \neq 0$, let θ be the angle measure counterclockwise from the positive x-axis such that $x = r\cos\theta$ and $y = r\sin\theta$. The angle θ is called an **argument** of z.

Note that we say "an" argument of z, since θ is only unique up to a multiple of 2π. As such, every complex number has an infinite number of arguments. We actually *will* make use of this fact!

Definition: If $r = |z|$ and if θ is an argument for z, then the **polar form** for z is $r(\cos\theta + i\sin\theta)$.

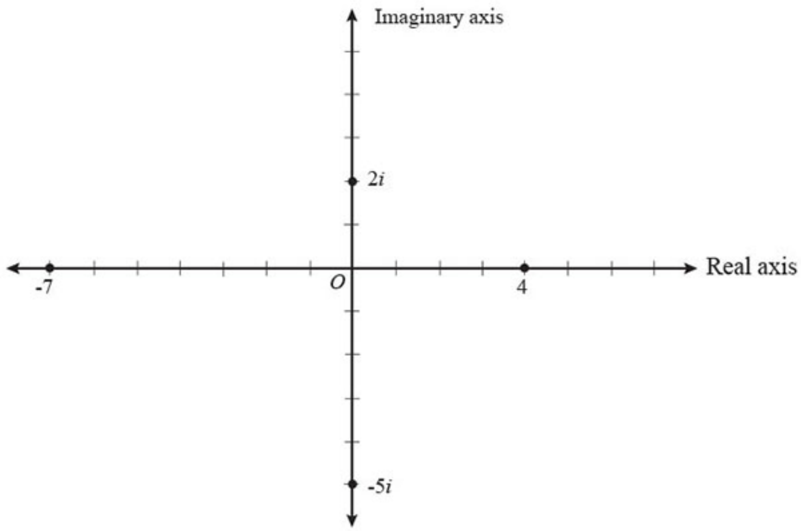
Example: To determine a polar form for $1+i$, we first compute $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Next, we need to find θ such that $1 = \sqrt{2}\cos\theta$ and $1 = \sqrt{2}\sin\theta$. That is, we want $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$. Setting $\theta = \pi/4$ satisfies these equations. So a polar form for $1 + i$ is $\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$.

The steps to determine a polar form for $1-i$ are quite similar. First we compute $r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Next, we need to find θ such that $1 = \sqrt{2}\cos\theta$ and $-1 = \sqrt{2}\sin\theta$. That is, we want $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. Since our cosine value is positive while our sine value is negative, we know we are looking for a θ in the fourth quadrant. So we can set $\theta = 7\pi/4$ to satisfy our equations. This gives the polar form $\sqrt{2}(\cos(7\pi/4) + i\sin(7\pi/4))$ for $1 - i$. But we might just have easily have chosen the θ value of $-\pi/4$ though, giving us the polar form $\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$ for $1 - i$.

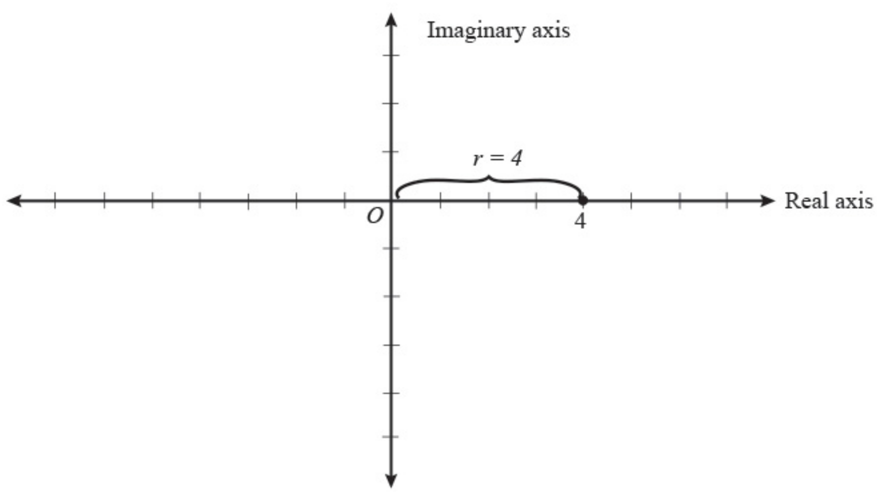
In general, you will do well to visualize the point $x + iy$ when you are trying to determine an argument for it, instead of relying purely on calculations. This is particularly true when looking for a polar form for a purely real or purely imaginary number.

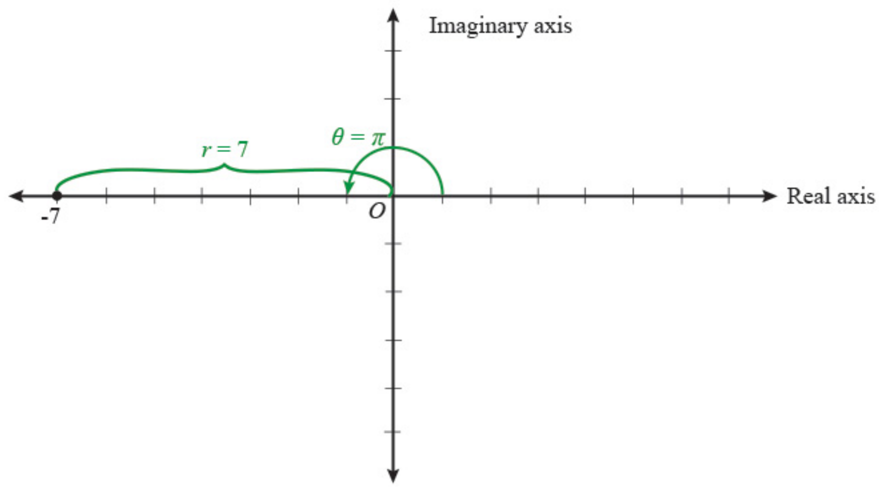
Example: Find a polar form for 4, -7, $2i$, and $-5i$.

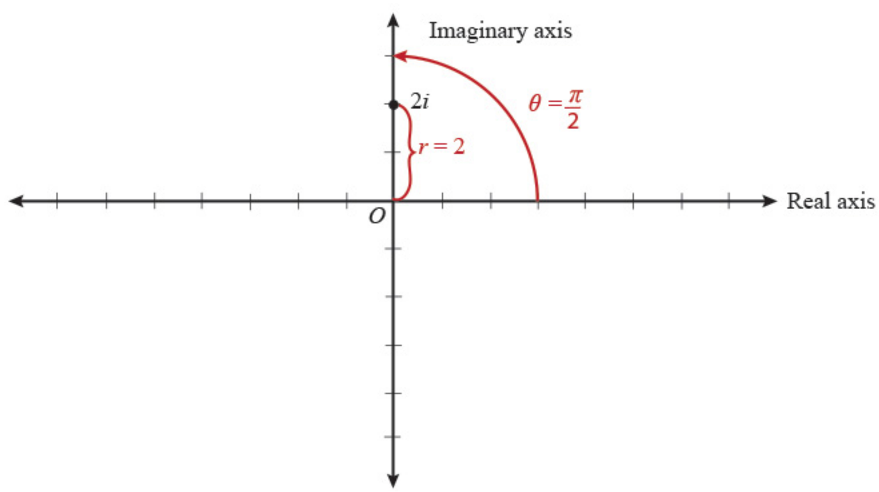
First, let's graph these points.

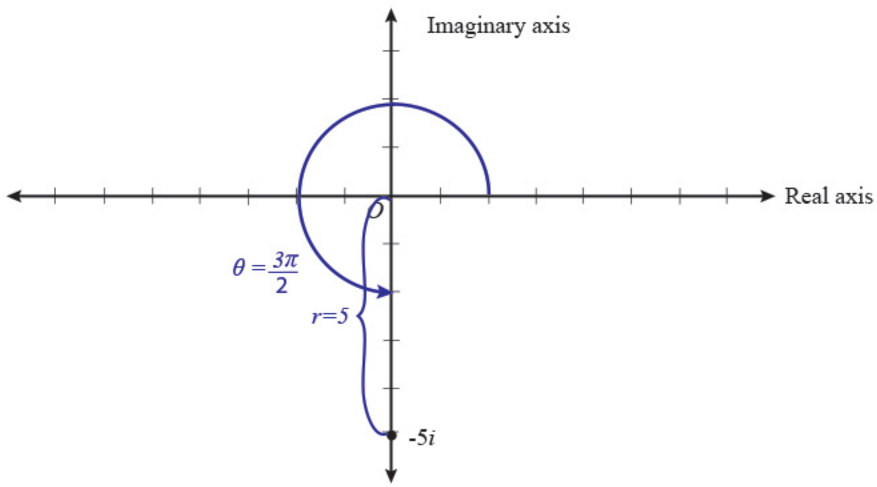


From the graph, it is obvious that a polar form for 4 is $4(\cos(0)+i\sin(0))$, a polar form for -7 is $7(\cos(\pi)+i\sin(\pi))$, a polar form for $2i$ is $2(\cos(\pi/2)+i\sin(\pi/2))$, and a polar form for $-5i$ is $5(\cos(3\pi/2)+i\sin(3\pi/2))$.




Imaginary axis
Real axis
$\theta = \pi$
$r = 7$
-7
O


Imaginary axis
Real axis
$2i$
$\theta = \frac{\pi}{2}$
$r = 2$
O



Theorem 9.1.a: If $z = r(\cos\theta + i\sin\theta)$, then $\overline{z} = r(\cos(-\theta) + i\sin(-\theta))$.

Proof of Theorem 9.1.a: Let $z = x+iy$, so $\overline{z} = x-yi$. Then $|\overline{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z| = r$. And so we see that the modulus of $\overline{z}$ is the same as the modulus for z. Since $z = r\cos\theta + ir\sin\theta$, and since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, we have

$$\begin{aligned}\overline{z} &= r\cos\theta - ir\sin\theta \\ &= r\cos(-\theta) + ir\sin(-\theta) \\ &= r(\cos(-\theta) + i\sin(-\theta))\end{aligned}$$

We can use properties of cosine and sine to help us multiply and divide complex numbers as well. Recall the following trigonometric identities:

$$\begin{aligned}\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\end{aligned}$$

We use them to get the following result: (Note that this result does appear in the text–it simply isn't listed as a theorem.)

Theorem 9.1.b For any complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, we have

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Proof of Theorem 9.1.b: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned}
z_1 z_2 &= (r_1(\cos\theta_1 + i\sin\theta_1))(r_2(\cos\theta_2 + i\sin\theta_2)) \\
&= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) \\
&= r_1r_2\cos\theta_1\cos\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2 + i^2r_1r_2\sin\theta_1\sin\theta_2 \\
&= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \\
&= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))
\end{aligned}$$

Theorem 9.1.3 For any complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, with $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Proof of Theorem 9.1.3 Remember that $z_1/z_2 = z_3$ if $z_1 = z_2z_3$, so we can see that $z_1/z_2 = (r_1/r_2)(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$ by showing that $z_1 = z_2(r_1/r_2)(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$. And we show this as follows:

$$\begin{aligned}
z_2(r_1/r_2)(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2) &= r_2(\cos(\theta_2) + i\sin(\theta_2))(r_1/r_2)(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2) \\
&= (r_2(r_1/r_2))(\cos(\theta_2+\theta_1-\theta_2) + i\sin(\theta_2+\theta_1-\theta_2)) \\
&= r_1(\cos\theta_1 + i\sin\theta_1) \\
&= z_1
\end{aligned}$$

Corollary 9.1.4 Let $z = r(\cos\theta + i\sin\theta)$ with $r \neq 0$. Then

$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$$

Example: Earlier, we found that a polar form for $1+i$ is $\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$. Now, let's find a polar form for $-\sqrt{3}+3i$. First, we find $r = \sqrt{(-\sqrt{3})^2+3^2} = \sqrt{12} = 2\sqrt{3}$. Next, we need to find θ such that $\cos\theta = -\sqrt{3}/2\sqrt{3} = -1/2$ and $\sin\theta = 3/2\sqrt{3} = \sqrt{3}/2$. Since our cosine value is negative and our sine value is positive, we know that θ is in the second quadrant, so we can use $\theta = 2\pi/3$ as an argument. This means that $2\sqrt{3}(\cos(2\pi/3) + i\sin(2\pi/3))$ is a polar form for $-\sqrt{3}+3i$.

We can use these polar forms to calculate the following:

$$\begin{aligned}
(1+i)(-\sqrt{3}+3i) &= (\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)))(2\sqrt{3}(\cos(2\pi/3) + i\sin(2\pi/3))) \\
&= ((\sqrt{2})(2\sqrt{3}))(\cos((\pi/4)+(2\pi/3)) + i\sin((\pi/4)+(2\pi/3)) \\
&= 2\sqrt{6}(\cos(7\pi/12) + i\sin(7\pi/12))
\end{aligned}$$

$$\begin{aligned}
(1+i)/(-\sqrt{3}+3i) &= (\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)))/(2\sqrt{3}(\cos(2\pi/3) + i\sin(2\pi/3))) \\
&= ((\sqrt{2})/(2\sqrt{3}))(\cos((\pi/4)-(2\pi/3)) + i\sin((\pi/4)-(2\pi/3)) \\
&= 1/\sqrt{6}(\cos(-\pi/12) + i\sin(-\pi/12))
\end{aligned}$$

$$
\begin{aligned}
(-\sqrt{3}+3i)/(1+i) &= (2\sqrt{3}(\cos(2\pi/3)+i\sin(2\pi/3)))/(\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4))) \\
&= ((2\sqrt{3})/(\sqrt{2}))(\cos((2\pi/3)-(\pi/4))+i\sin((2\pi/3)-(\pi/4)) \\
&= \sqrt{6}(\cos(\pi/12)+i\sin(\pi/12))
\end{aligned}
$$

Notice that this result agrees with the result we would get if we considered $(-\sqrt{3}+3i)/(1+i)$ to be $((1+i)/(-\sqrt{3}+3i))^{-1}$.

Example: Use polar form to calculate $(1-2i)(-3-4i)$.

First we will need to find the polar form of $1-2i$ and $-3-4i$. This is not as easy as in the previous example, as our arguments will not be one of our standard angles. Instead, we will use a calculator and decimal approximations for these calculations.

To find a polar form of $1-2i$, we first note that $r = \sqrt{1^2+(-2)^2} = \sqrt{5}$. And now we need to find θ such that $\cos\theta = 1/\sqrt{5}$ and $\sin\theta = -2/\sqrt{5}$. Since the cosine is positive and the sine is negative, we know that θ is in the fourth quadrant. Plugging $\theta = \sin^{-1}(-2/\sqrt{5})$ into a calculator will give us a value in the fourth quadrant, and so we see that $\theta \approx -1.11$. And so we have found that a polar form of $1-2i$ is $\sqrt{5}(\cos(-1.11)+i\sin(-1.11))$.

To find a polar form of $-3-4i$, we first note that $r = \sqrt{(-3)^2+(-4)^2} = \sqrt{9+16} = 5$. And now we need to find θ such that $\cos\theta = -3/5$ and $\sin\theta = -4/5$. Since the cosine is negative and the sine is negative, we know that θ is in the third quadrant. Unfortunately, a calculator never outputs a value in the third quadrant. So, we can have $\theta = -\cos^{-1}(-3/5)$, or $\theta = \cos^{-1}(3/5)+\pi$. I'll choose to do the first, which gives me $\theta \approx -2.21$. And so we have found that a polar form of $-3-4i$ is $5(\cos(-2.21)+i\sin(-2.21))$.

Now that we have polar forms for $1-2i$ and $-3-4i$, we can calculate that

$$
\begin{aligned}
(1-2i)(-3-4i) &= (\sqrt{5}(\cos(-1.11)+i\sin(-1.11)))(5(\cos(-2.21)+i\sin(-2.21))) \\
&= 5\sqrt{5}(\cos(-1.11-2.21)+i\sin(-1.11-2.21)) \\
&= 5\sqrt{5}(\cos(-3.32)+i\sin(-3.32))
\end{aligned}
$$

We can double check our calculation by putting things back into standard form. With a healthy amount of round-off error, we do in fact see that $5\sqrt{5}(\cos(-3.32)+i\sin(-3.32)) = -11+2i$.

Given how much effort it is to find polar form, at this point it probably does not seem worth while to put complex numbers into polar form simply to multiply and divide them. But in a couple of lectures, we will use polar form to find the n-th roots of complex numbers, and this really is the best way to solve this type of question. So, for now, consider this lecture a gentle introduction to polar form and calculations in polar form. Because polar form is not going away!