Solution to Practice 1d

B2(a) We need to see if there are coefficients t_1, t_2 and t_3 such that $t_1(1+x) + t_2(x+x^2) + t_3(1-x^3) = 1$. The left side becomes:

$$\begin{array}{cccccccc} & t_1 & + & t_1x & & & & \\ & & + & t_2x & + & t_2x^2 & & \\ + & t_3 & & & & & - & t_3x^3 \\ \hline & (t_1+t_3) & + & (t_1+t_2)x & + & t_2x^2 & - & t_3x^3 \end{array}$$

Setting the coefficients equal to each other, we see that we need to look for solutions to the system:

$$\begin{array}{rcccccl} 1: & t_1 & & & + & t_3 & = & 1 \\ x: & t_1 & + & t_2 & & & = & 0 \\ x^2: & & & t_2 & & & = & 0 \\ x^3: & & & & - & t_3 & = & 0 \end{array}$$

The last two equations tell us that $t_2 = 0$ and $t_3 = 0$. Plugging this in, the first two equations become:

$t_1 + 0 = 1 \Rightarrow t_1 = 1$ and $t_1 + 0 = 0 \Rightarrow t_1 = 0$.

Since we cannot have both $t_1 = 1$ and $t_1 = 0$, we know that there are no solutions to our system, and thus that $p(x) = 1$ is NOT in Span $\mathcal{B}$.

B2(b) We need to see if there are coefficients t_1, t_2 and t_3 such that $t_1(1+x) + t_2(x+x^2) + t_3(1-x^3) = 5x + 2x^2 + 3x^3$. In part (a), we found that the left side becomes $(t_1+t_3) + (t_1+t_2)x + t_2x^2 - t_3x^3$. Setting the coefficients equal to each other, we see that we need to look for solutions to the system:

$$\begin{array}{rcccccl} 1: & t_1 & & & + & t_3 & = & 0 \\ x: & t_1 & + & t_2 & & & = & 5 \\ x^2: & & & t_2 & & & = & 2 \\ x^3: & & & & - & t_3 & = & 3 \end{array}$$

The last two equations tell us that $t_2 = 2$ and $t_3 = -3$. Plugging these values into the first two equations, we get:

$t_1 - 3 = 0 \Rightarrow t_1 = 3$ and $t_1 + 2 = 5 \Rightarrow t_1 = 3$. Thus, we know that $t_1 = 3, t_2 = 2, t_3 = -3$ is a solution to our system, and thus that $p(x) = 5x + 2x^2 + 3x^3$ IS in Span $\mathcal{B}$.

B2(c) We need to see if there are coefficients t_1, t_2 and t_3 such that $t_1(1+x) + t_2(x+x^2) + t_3(1-x^3) = 3 + x^2 - 4x^3$. In part (a), we found that the left side becomes $(t_1+t_3) + (t_1+t_2)x + t_2x^2 - t_3x^3$. Setting the coefficients equal to each other, we see that we need to look for solutions to the system:

$$\begin{array}{rccccccc}
1: & t_1 & & & + & t_3 & = & 3 \\
x: & t_1 & + & t_2 & & & = & 0 \\
x^2: & & & t_2 & & & = & 1 \\
x^3: & & & & - & t_3 & = & -4
\end{array}$$

The last two equations tell us that $t_2 = 1$ and $t_3 = 4$. Plugging these values into the first two equations, we get:

$t_1 + 4 = 3 \Rightarrow t_1 = -1$ and $t_1 + 1 = 0 \Rightarrow t_1 = -1$. Thus, we know that $t_1 = -1, t_2 = 1, t_3 = 4$ is a solution to our system, and thus that $q(x) = 3 + x^2 - 4x^3$ IS in Span $\mathcal{B}$.

B2(d) We need to see if there are coefficients t_1, t_2 and t_3 such that $t_1(1+x) + t_2(x+x^2) + t_3(1-x^3) = 1+x^3$. In part (a), we found that the left side becomes $(t_1+t_3) + (t_1+t_2)x + t_2x^2 - t_3x^3$. Setting the coefficients equal to each other, we see that we need to look for solutions to the system:

$$\begin{array}{rccccccc}
1: & t_1 & & & + & t_3 & = & 1 \\
x: & t_1 & + & t_2 & & & = & 0 \\
x^2: & & & t_2 & & & = & 0 \\
x^3: & & & & - & t_3 & = & 1
\end{array}$$

The last two equations tell us that $t_2 = 0$ and $t_3 = -1$. Plugging these values into the first two equations, we get:

$t_1 - 1 = 1 \Rightarrow t_1 = 2$ and $t_1 + 0 = 0 \Rightarrow t_1 = 0$.

Since we cannot have both $t_1 = 2$ and $t_1 = 0$, we know that there are no solutions to our system, and thus that $q(x) = 1 + x^3$ is NOT in Span $\mathcal{B}$.

B3(a) We need to see if there are any non-trivial solutions to the equation:

$t_1x^2 + t_2x^3 + t_4(x^2 + x^3 + x^4) = 0$. Doing the calculation on the left, we get:

$$\begin{array}{ccccccc}
t_1x^2 & & & & & & \\
 & & t_2x^3 & & & & \\
t_3x^2 & + & t_3x^3 & + & t_3x^4 & & \\
\hline
(t_1+t_3)x^2 & + & (t_2+t_3)x^3 & + & t_3x^4 & = & 0
\end{array}$$

Setting the coefficients equal to 0, we see that this is equivalent to looking for non-trivial solutions to the homogeneous system

$$\begin{array}{cccccc}
t_1 & & + & t_3 & = & 0 \\
 & t_2 & + & t_3 & = & 0 \\
 & & & t_3 & = & 0
\end{array}$$

The last equation tells us that $t_3 = 0$. Plugging this fact into the first two equations, we get that $t_1 = 0$ and $t_2 = 0$. Since $t_1, t_2, t_3 = 0$ is the only solution to our system, there are no non-trivial solutions, and thus the set IS linearly independent.

B3(b) We need to see if there are any non-trivial solutions to the equation:

$t_1(1+(1/2)x^2)+t_2(1-(1/2)x^2)+t_3(x+(1/6)x^3)+t_4(x-(1/6)x^3)=0$. Doing the calculation on the left, we get:

$$\begin{array}{ccccccccc} t_1 & & & + & (1/2)t_1x^2 & & & & \\ t_2 & & & - & (1/2)t_2x^2 & & & & \\ & & t_3x & & & + & (1/6)t_3x^3 & & \\ & & t_4x & & & - & (1/6)t_4x^3 & & \\ \hline (t_1+t_2) & + & (t_3+t_4)x & + & (1/2)(t_1-t_2)x^2 & + & (1/6)(t_3-t_4)x^3 & = & 0 \end{array}$$

Setting the coefficients equal to 0, we see that this is equivalent to looking for non-trivial solutions to the homogeneous system

$$\begin{array}{ccccccccc} t_1 & + & t_2 & & & & & = & 0 \\ & & & & t_3 & + & t_4 & = & 0 \\ (1/2)t_1 & - & (1/2)t_2 & & & & & = & 0 \\ & & & & (1/6)t_3 & - & (1/6)t_4 & = & 0 \end{array}$$

To solve this system, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/6 & -1/6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}.$$

From the row echelon form, we see that the system has a unique solution. As such, there are no non-trivial solutions, and thus the set IS linearly independent.

B3(c) We need to see if there are any non-trivial solutions to the equation:

$t_1(1+x+x^3)+t_2(x+x^3+x^5)+t_3(1-x^5)=0$.

Doing the calculation on the left, we get:

$$\begin{array}{cccc} t_1 & +t_1x & +t_1x^3 & \\ & t_2x & +t_2x^3 & +t_2x^5 \\ t_3 & & & -t_3x^5 \\ \hline (t_1+t_3) & +(t_1+t_2)x & +(t_1+t_2)x^3 & +(t_2-t_3)x^5 \end{array}$$

Setting the coefficients equal to zero, we see that this is equivalent to looking for non-trivial solutions to the homogeneous system

$$\begin{array}{rrrl} t_1 & & +t_3 & = 0 \\ t_1 & +t_2 & & = 0 \\ & t_2 & -t_3 & = 0 \end{array}$$

(Note that I only wrote the equation $t_1 + t_2 = 0$ once.)

To solve this system, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

From the row echelon form, we see that the system has a unique solution. As such, there are no non-trivial solutions, and thus the set IS linearly independent.

B3(d) We need to see if there are any non-trivial solutions to the equation:

$t_1(1 - 2x + x^4) + t_2(x - 2x^2 + x^5) + t_3(1 - 3x + x^3) = 0.$

Doing the calculation on the left, we get:

$$\begin{array}{cccccc} t_1 & -2t_1x & & & +t_1x^4 & \\ & t_2x & -2t_2x^2 & & & +t_2x^5 \\ t_3 & -3t_3x & & +t_3x^3 & & \\ \hline (t_1+t_3) & +(-2t_1+t_2-3t_3)x & -2t_2x^2 & +t_3x^3 & +t_1x^4 & +t_2x^5 \end{array}$$

Setting the coefficients equal to zero, we see that this is equivalent to looking for non-trivial solutions to the homogeneous system

$$\begin{array}{rrrl} t_1 & & +t_3 & = 0 \\ -2t_1 & +t_2 & -3t_3 & = 0 \\ & -2t_2 & & = 0 \\ & & t_3 & = 0 \\ t_1 & & & = 0 \\ & t_2 & & = 0 \end{array}$$

The third and last equation both tell us that $t_2 = 0$, while the fourth equation tells us that $t_3 = 0$ and the fifth equation tells us that $t_1 = 0$. As such, the only possible solution is the trivial solution $t_1 = t_2 = t_3 = 0$, which means that our set IS linearly independent.

B3(e) We need to see if there are any non-trivial solutions to the equation:

$t_1(1 + 2x + x^2 - x^3) + t_2(2 + 3x - x^2 + x^3 + x^4) + t_3(1 + x - 2x^2 + 2x^3 + x^4) + t_4(1 + 2x + x^2 + x^3 - 3x^4) + t_5(4 + 6x - 2x^2 + 5x^4) = 0$

Doing the calculation on the left, we get:

$$\begin{array}{rr}
t_1 & +2t_1x \\
2t_2 & +3t_2x \\
t_3 & +t_3x \\
t_4 & +2t_4x \\
4t_5 & +6t_5x \\
\hline
(t_1+2t_2+t_3+t_4+4t_5) & +(2t_1+3t_2+t_3+2t_4+6t_5)x
\end{array}$$

$$\begin{array}{rr}
+t_1x^2 & -t_1x^3 \\
-t_2x^2 & +t_2x^3 \\
-2t_3x^2 & +2t_3x^3 \\
+t_4x^2 & +t_4x^3 \\
-2t_5x^2 & \\
\hline
+(t_1-t_2-2t_3+t_4-2t_5)x^2 & +(-t_1+t_2+2t_3+t_4)x^3
\end{array}$$

$$\begin{array}{r}
+t_2x^4 \\
+t_3x^4 \\
-3t_4x^4 \\
+5t_5x^4 \\
\hline
+(t_2+t_3-3t_4+5t_5)x^4
\end{array}$$

Setting the coefficients equal to zero, we see that this is equivalent to looking for non-trivial solutions to the homogeneous system

$$\begin{array}{rrrrrl}
t_1 & +2t_2 & +t_3 & +t_4 & +4t_5 & =0 \\
2t_1 & +3t_2 & +t_3 & +2t_4 & +6t_5 & =0 \\
t_1 & -t_2 & -2t_3 & +t_4 & -2t_5 & =0 \\
-t_1 & +t_2 & +2t_3 & +t_4 & & =0 \\
 & t_2 & +t_3 & -3t_4 & +5t_5 & =0
\end{array}$$

To solve this system, we need to row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 2 & 3 & 1 & 2 & 6 \\ 1 & -1 & -2 & 1 & -2 \\ -1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & -3 & -3 & 0 & -6 \\ 0 & 3 & 3 & 2 & 4 \\ 0 & 1 & 1 & -3 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -3 & -3 & 0 & -6 \\ 0 & 3 & 3 & 2 & 4 \\ 0 & 1 & 1 & -3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Our last matrix is in reduced row echelon form. From this, we see that the rank of the matrix is 3, which is 2 less than the number of variables, so the general solution has two parameters. As such, we know that our set is linearly dependent. To find all linear combinations of the polynomials that equal the zero polynomial, we finish finding the solution to our system. From the RREF matrix, we see that our system is equivalent to the following:

$$\begin{array}{ccccccc} t_1 & & -t_3 & & +t_5 & = 0 \\ & t_2 & +t_3 & & +2t_5 & = 0 \\ & & & t_4 & -t_5 & = 0 \end{array}$$

Replacing the variables t_3 and t_5 with the parameters s and t, we see that the general solution is:

$$t_1 = s - t \quad t_2 = -s - 2t \quad t_3 = s \quad t_4 = t \quad t_5 = t$$

As such, the list of all linear combinations of the polynomials that equal the zero polynomial is:

$$\begin{array}{l} (s-t)(1+2x+x^2-x^3) + (-s-2t)(2+3x-x^2+x^3+x^4) + s(1+x-2x^2+2x^3+x^4) \\ +t(1+2x+x^2+x^3-3x^4) + t(4+6x-2x^2+5x^4), \text{ for all } s,t \in \mathbb{R} \end{array}$$

B4 To prove that the set $\mathcal{B} = \{1, x-2, (x-2)^2, (x-2)^3\} = \{1, -2+x, 4-4x+x^2, -8+12x-6x^2+x^3\}$ is linearly independent, we need to show that the only solution to the equation

$$t_1 + t_2(-2+x) + t_3(4-4x+x^2) + t_4(-8+12x-6x^2+x^3) = 0$$

is the trivial solution. Performing the calculation on the left, we get:

$$\begin{array}{rrrr} t_1 & & & \\ -2t_2 & +t_2x & & \\ 4t_3 & -4t_3x & +t_3x^2 & \\ -8t_4 & +12t_4x & -6t_4x^2 & t_4x^3 \\ \hline (t_1-2t_2+4t_3-8t_4) & +(t_2-4t_3+12t_4)x & +(t_3-6t_4)x^2 & t_4x^3 \end{array}$$

Setting the coefficients equal to zero, we see that we are looking for solutions to the following homogeneous system of linear equations:

$$\begin{array}{ccccc}
t_1 & -2t_2 & +4t_3 & -8t_4 & =0 \\
 & t_2 & -4t_3 & +12t_4 & =0 \\
 & & t_3 & -6t_4 & =0 \\
 & & & t_4 & =0
\end{array}$$

From the last equation, we know that $t_4 = 0$. Plugging this fact into our third equation, we now see that $t_3 - 6(0) = 0$, so we have $t_3 = 0$. Plugging these two values into the second equation, we now see that $t_2 - 4(0) + 12(0) = 0$, so $t_2 = 0$. And, finally, plugging these three values into the first equation, we see that $t_1 - 2(0) + 4(0) - 8(0) = 0$, so $t_1 = 0$. As such, we have shown that the only possible solution is the trivial solution $t_1 = t_2 = t_3 = t_4 = 0$, which proves that our set is linearly independent.

To show that Span $\mathcal{B}$ is the set of all polynomials of degree less than or equal to three, we need to show that any polynomial $p(x) = p_0 + p_1x + p_2x^2 + p_+3x^3$ of degree less than or equal to three can be written as a linear combination of the polynomials in $\mathcal{B}$. That is, we need to look for scalars t_1, t_2, t_3, t_4 such that

$$t_1+t_2(-2+x)+t_3(4-4x+x^2)+t_4(-8+12x-6x^2+x^3) = p_0+p_1x+p_2x^2+p_+3x^3$$

We already performed the calculation on the left, so we know that this is the same as looking for t_1, t_2, t_3, t_4 such that

$$(t_1-2t_2+4t_3-8t_4)+(t_2-4t_3+12t_4)x+(t_3-6t_4)x^2+t_4x^3 = p_0+p_1x+p_2x^2+p_+3x^3$$

Setting the coefficients equal to each other, this is equivalent to looking for solutions to the following system of equations:

$$\begin{array}{rccccc}
1: & t_1 & -2t_2 & +4t_3 & -8t_4 & =p_0 \\
x: & & t_2 & -4t_3 & +12t_4 & =p_1 \\
x^2: & & & t_3 & -6t_4 & =p_2 \\
x^3: & & & & t_4 & =p_3
\end{array}$$

From our fourth equation, we see that $t_4 = p_3$. Plugging this fact into our third equation, we now see that $t_3 - 6(p_3) = p_2$, so we have $t_3 = p_2 + 6p_3$. Plugging these two values into the second equation, we now see that $t_2 - 4(p_2 + 6p_3) + 12(p_3) = p_1$, which means $t_2 - 4p_2 - 24p_3 + 12p_3 = p_1$, so $t_2 = p_1 + 4p_2 + 12p_3$. And, finally, plugging these three values into the first equation, we see that $t_1 - 2(p_1 + 4p_2 + 12p_3) + 4(p_2 + 6p_3) - 8(p_3) = p_0$, which means $t_1 - 2p_1 - 8p_2 - 24p_3 + 4p_2 + 24p_3 - 8p_3 = p_0$, so $t_1 = p_2 + 2p_1 + 4p_2 + 8p_3$. As such, we see that there is a solution for any possible choice of p_0, p_1, p_2, p_3. This means that every function $p_0 + p_1x + p_2x^2 + p_+3x^3$ is in the span of $\mathcal{B}$, as desired.

Additional notes on the solution to B4: It can be confusing to understand the proof that Span $\mathcal{B}$ is the set of all polynomials of degree less than or equal to 3 if you have never seen such a proof before. The first thing to keep clear in your mind is this: which variables are "variable" (i.e. which variables are we solving for), and which variables are "constants". So it is important to remember in this proof that we are trying to solve for the t_i. As such, we take the p_i as constants. If you think about it, it makes sense that our values for t_i are dependent on the p_i values, since we would need to actually know the polynomial to be able to determine which scalars to use in our linear combination. As such, what I have given is essentially a proof by algorithm. The idea is that, if you give me a polynomial $p_0 + p_1x + p_2x^2 + p_+3x^3$, then I can calculate that $t_1 = p_2 + 2p_1 + 4p_2 + 8p_3$, $t_2 = p_1 + 4p_2 + 12p_3$, $t_3 = p_2 + 6p_3$, and $t_4 = p_3$ are the coefficients of the linear combination of the polynomials in $\mathcal{B}$ that will yield $p_0 + p_1x + p_2x^2 + p_+3x^3$. For example, if I wanted to write $1 + x + x^2 + x^3$ as a linear combination of $\{1, -2+x, 4-4x+x^2, -8+12x-6x^2+x^3\}$, then I use my algorithm (with $p_0 = p_1 = p_2 = p_3 = 1$) to get that $t_1 = 1+2(1)+4(1)+8(1) = 15$, $t_2 = 1 + 4(1) + 12(1) = 17$, $t_3 = 1 + 6(1) = 7$ and $t_4 = 1$. And sure enough, we see that

$$15(1)+17(-2+x)+7(4-4x+x^2)+(-8+12x-6x^2+x^3) = \begin{array}{rrrr} 15 & & & \\ -34 & +17x & & \\ 28 & -28x & +7x^2 & \\ -8 & +12x & -6x^2 & x^3 \\ \hline 1 & +x & +x^2 & +x^3 \end{array}$$