Solution to Practice 3l

(1)

S0: $\begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$ for all $z_2, z_3 \in \mathbb{C}$, so $\mathbb{U}$ is a subset of $\mathbb{C}^3$. We see that $\mathbb{U}$ is non-empty by noting that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{U}$.

S1: Let $\vec{w}, \vec{z} \in \mathbb{U}$, say $\vec{w} = \begin{bmatrix} 0 \\ w_2 \\ w_3 \end{bmatrix}$ and $\vec{z} = \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix}$. Then

$$\vec{w} + \vec{z} = \begin{bmatrix} 0 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ w_2 + z_2 \\ w_3 + z_3 \end{bmatrix}$$

where $w_2 + z_2, w_3 + z_3 \in \mathbb{C}$. So $\vec{w} + \vec{z} \in \mathbb{U}$.

S2: Let $\vec{z} \in \mathbb{U}$, say $\vec{z} = \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix}$, and $\alpha \in \mathbb{C}$ Then

$$\alpha\vec{z} = \alpha \begin{bmatrix} 0 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha z_2 \\ \alpha z_3 \end{bmatrix}$$

where $\alpha z_2, \alpha z_3 \in \mathbb{C}$. So $\vec{w} + \vec{z} \in \mathbb{U}$.

(2)

S0: $\begin{bmatrix} z_1 & iz_1 \\ 0 & z_2 \end{bmatrix} \in C(2,2)$ for all $z_1, z_2 \in \mathbb{C}$, so $\mathcal{A}$ is a subset of $C(2,2)$. To see that $\mathbb{U}$ is non-empty, we can set $z_1 = z_2 = 0$, and see that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$.

S1: Let $A, B \in \mathcal{A}$, say $A = \begin{bmatrix} a_1 & ia_1 \\ 0 & a_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & ib_1 \\ 0 & b_2 \end{bmatrix}$. Then we have

$$A+B = \begin{bmatrix} a_1 & ia_1 \\ 0 & a_2 \end{bmatrix} + \begin{bmatrix} b_1 & ib_2 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & ia_1 + ib_1 \\ 0 + 0 & a_2 + b_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & i(a_1 + b_1) \\ 0 & a_2 + b_2 \end{bmatrix}$$

and since $a_1 + b_1 \in \mathbb{C}$ and $a_2 + b_2 \in \mathbb{C}$, we see that $A + B \in \mathcal{A}$.

S2: Let $A \in \mathcal{A}$ (say $A = \begin{bmatrix} a_1 & ia_1 \\ 0 & a_2 \end{bmatrix}$) and $\alpha \in \mathbb{C}$. Then

$$\alpha A = \alpha \begin{bmatrix} a_1 & ia_1 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha(ia_1) \\ 0 & \alpha a_2 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & i(\alpha a_1) \\ 0 & \alpha a_2 \end{bmatrix}$$

and since $\alpha a_1 \in \mathbb{C}$ and $\alpha a_2 \in \mathbb{C}$, we see that $\alpha A \in \mathcal{A}$.

(3)

The easiest thing to notice is that $\vec{0} \notin \mathbb{W}$, so $\mathbb{W}$ is not a vector space, and thus is not a subspace. This uses the original definition of a subspace, although failing to have the zero vector means it can't be closed under scalar multiplication either. (e.g. $\begin{bmatrix} i \\ i \\ i \end{bmatrix} \in \mathbb{W}$, but $0 \begin{bmatrix} i \\ i \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathbb{W}$)