Lecture 1i

Obtaining a Basis from an Arbitrary Finite Spanning Set

(pages 209-211)

Now that we know how to see if a set is a basis, we are left with the problem of trying to find a set that we think might be a basis. In general there are two ways to go about this problem. Either we start with a linearly independent set and add more independent vectors until we finally span the vector space, or we start with a spanning set and remove vectors until we end up with a linearly independent set. We will focus our attention first on this latter method, since we frequently define a vector space as the span of a set of vectors. Which means that the very definition of our vector space provides us with a spanning set. And if that set is already linearly independent, then we already have a basis.

But if not, then one of the vectors in our spanning set can be written as a linear combination of the other members. To see this, suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a set of vectors that is linearly dependent. Then there is a non-trivial solution to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

And this means that there is a solution with at least one $t_i \neq 0$. Then $\mathbf{v}_i$ can be written as a linear combination of the other vectors as follows:

$t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k = -t_i\mathbf{v}_i \Rightarrow$
$(-t_1/t_i)\mathbf{v}_1 + \cdots + (-t_{i-1}/t_i)\mathbf{v}_{i-1} + (-t_{i+1}/t_i)\mathbf{v}_{i+1} + \cdots + (-t_k/t_i)\mathbf{v}_k = \mathbf{v}_i$

Definition: When one element of a set can be written as a linear combination of the other members of the set, we say that this element is a **dependent member** of the set.

Okay, so now we see that if our spanning set is linearly dependent, then it has a dependent member. To get our basis, we will remove the dependent members from our spanning set. But we need to verify that we will still have a spanning set for our vector space afterwards. And that's what the following theorem will show. (Note: This result does appear in the text, but not as a theorem, which is why I have not numbered the theorem here.)

Theorem 4.3.a: If $\mathcal{T} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a spanning set for a non-trivial vector space $\mathbb{V}$, and if there is some $\mathbf{v}_i \in \mathcal{T}$ that can be written as a linear combination of the other $\mathbf{v}_j$; that is, if there are scalars $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_k$ such that

$$\mathbf{v}_i = t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k$$

then $\mathcal{T} \setminus \{\mathbf{v}_i\}$ is also a spanning set for $\mathbb{V}$.

Proof of Theorem 4.3.a: Suppose that $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a spanning set for a non-trivial vector space $\mathbb{V}$, and suppose there are scalars $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k$ such that $\mathbf{v}_i = t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k$. Let $\mathbf{x} \in \mathbb{V}$. Then, since $\mathcal{T}$ is a spanning set for $\mathbb{V}$, we know there are scalars $a_1, \ldots, a_k$ such that

$$\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$$

Plugging in our known value for $\mathbf{v}_i$, we see that

$$\begin{aligned}
\mathbf{x} &= a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} \\
&\quad + a_i(t_1\mathbf{v}_1 + \cdots + t_{i-1}\mathbf{v}_{i-1} + t_{i+1}\mathbf{v}_{i+1} + \cdots + t_k\mathbf{v}_k) \\
&\quad + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_k\mathbf{v}_k \\
&= (a_1 + a_it_1)\mathbf{v}_1 + \cdots + (a_{i-1} + a_it_{i-1})\mathbf{v}_{i-1} + (a_{i+1} + a_it_{i+1})\mathbf{v}_{i+1} + \cdots + (a_k + a_it_k)\mathbf{v}_k
\end{aligned}$$

And so we have written $\mathbf{x}$ as a linear combination of the elements of $\mathcal{T} \setminus \{\mathbf{v}_i\}$. And since we can do this for all $\mathbf{x} \in \mathbb{V}$, we see that $\mathcal{T} \setminus \{\mathbf{v}_i\}$ is a spanning set for $\mathbb{V}$.

So how do we use this to find a basis? Well, we'll start with our spanning set $\mathcal{T}$, and if it is linearly independent, then we already have a basis, so we are done. If not, we find a dependent member $\mathbf{v}_i$ of $\mathcal{T}$, and then we look to see if $\mathcal{T} \setminus \{\mathbf{v}_i\}$ is linearly independent. If it is, then by Theorem 4.3.a we know that it is also a spanning set, so it is the basis we are looking for. If it is linearly dependent, then we will again find a dependent member $\mathbf{v}_j$, and then we will look to see if $\mathcal{T} \setminus \{\mathbf{v}_i, \mathbf{v}_j\}$ is linearly independent. If it is, then by a second use of Theorem 4.3.a we know that it is still a spanning set, and so it is the basis we are looking for. If it isn't, then we continue the process until we eventually end up with a linearly independent set. $\mathcal{T}$ only started with k elements, so at the extreme we would eventually end up with a single element set, which is of course linearly independent, and as such we know that this process will definitely produce a linearly independent set eventually. Let's see it in action:

Example: If $\mathcal{T} = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} \right\}$, determine a subset of $\mathcal{T}$ that is a basis for Span $\mathcal{T}$.

Since we already know that $\mathcal{T}$ is a spanning set for Span $\mathcal{T}$, we want to check if $\mathcal{T}$ is linearly independent. To that end, we will look for solutions to the equation

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + t_2 \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} + t_3 \begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix} + t_4 \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} +$ $t_5 \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$ Performing the calculation on the right, this becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t_1 - 3t_2 - t_3 + 2t_4 - 4t_5 & -2t_1 + 7t_2 + 4t_3 - 8t_4 - t_5 \\ -3t_1 + 11t_2 + 7t_3 - 12t_4 & t_1 - 5t_2 - 5t_3 + 7t_4 + 5t_5 \end{bmatrix}$$

Setting the corresponding entries equal to each other, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrrrl} t_1 & -3t_2 & -t_3 & +2t_4 & -4t_5 & = 0 \\ -2t_1 & +7t_2 & +4t_3 & -8t_4 & -t_5 & = 0 \\ -3t_1 & +11t_2 & +7t_3 & -12t_4 & & = 0 \\ t_1 & -5t_2 & -5t_3 & +7t_4 & +5t_5 & = 0 \end{array}$$

To find the solutions to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -3 & -1 & 2 & -4 \\ -2 & 7 & 4 & -8 & -1 \\ -3 & 11 & 7 & -12 & 0 \\ 1 & -5 & -5 & 7 & 5 \end{bmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + 3R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & -3 & -1 & 2 & -4 \\ 0 & 1 & 2 & -4 & -9 \\ 0 & 2 & 4 & -6 & -12 \\ 0 & -2 & -4 & 5 & 9 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & 2 & -4 \\ 0 & 1 & 2 & -4 & -9 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & -3 & -9 \end{bmatrix} \begin{array}{l} \\ \\ (1/2)R_3 \\ \end{array} \sim \begin{bmatrix} 1 & -3 & -1 & 2 & -4 \\ 0 & 1 & 2 & -4 & -9 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 & -9 \end{bmatrix} \begin{array}{l} R_1 - 2R_3 \\ R_2 + 4R_3 \\ \\ R_4 + 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & -1 & 0 & -10 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + 3R_2 \\ \\ \\ \\ \end{array} \sim \begin{bmatrix} 1 & 0 & 5 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row echelon form, and from it we see that the rank of the coefficient matrix is 3. Since this is less than the number of variables, we know that our equation has an infinite number of solutions, which means that $\mathcal{T}$ is linearly dependent. So $\mathcal{T}$ is not a basis for Span $\mathcal{T}$. This means we need to remove at least one dependent member of $\mathcal{T}$. To find such a dependent member, we will find the general solution to our equation. From our RREF matrix, we see that our system is equivalent to the system

$$\begin{aligned} t_1 + 5t_3 - t_5 &= 0 \\ t_2 + 2t_3 + 3t_5 &= 0 \\ t_4 + 3t_5 &= 0 \end{aligned}$$

Replacing the variable t_3 with the parameter s and the variable t_5 with the parameter t, we see that the general solution to our equation is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} -5s + t \\ -2s - 3t \\ s \\ -3t \\ t \end{bmatrix}$$

Setting $s = -1$ and $t = 0$, we see that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 5\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 2\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix} + 0\begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} + 0\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$$

This means that $\begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix} = 5\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 2\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}$. And this means that $\begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix}$ is a dependent member of $\mathcal{T}$.

But notice also that setting $t = -1$ and $s = 0$ gives us

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 3\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} + 0\begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix} + 3\begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} - \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$$

This means that $\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} = -\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 3\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} + 3\begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}$.
Which means that $\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$ is also a dependent member.

Which one should we remove? Well, both actually. The easiest way to see this is to realize that we can first remove either one. Let's go ahead and remove $\begin{bmatrix} -1 & 4 \\ 7 & -5 \end{bmatrix}$, leaving us with $\mathcal{T}_1 = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} \right\}$.
By Theorem 4.3.a we know that $\mathcal{T}_1$ is a spanning set for Span $\mathcal{T}$. But since we already know that $\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} = -\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 3\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} + 3\begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}$, we know that $\mathcal{T}_1$ is not linearly independent. So let's now remove the dependent member $\begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}$ of $\mathcal{T}_1$, and consider the set $\mathcal{T}_2 = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} \right\}$.
By Theorem 4.3.a, we know that $\mathcal{T}_2$ is still a spanning set for Span $\mathcal{T}$. Now let's check to see if $\mathcal{T}_2$ is linearly independent. To that end, we will look for solutions to the equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + t_2\begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix} + t_3\begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix}$$

Performing the calculation on the right, this becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t_1 - 3t_2 - 2t_3 & -2t_1 + 7t_2 - 8t_3 \\ -3t_1 + 11t_2 - 12t_3 & t_1 - 5t_2 + 7t_3 \end{bmatrix}$$

Setting the corresponding entries equal to each other, we see that this is equivalent to looking for solutions to the system:

$$\begin{array}{rrrl} t_1 & -3t_2 & +2t_3 & = 0 \\ -2t_1 & +7t_2 & -8t_3 & = 0 \\ -3t_1 & +11t_2 & -12t_3 & = 0 \\ t_1 & -5t_2 & +7t_3 & = 0 \end{array}$$

To find the solution to this system, we row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & -8 \\ -3 & 11 & -12 \\ 1 & -5 & 7 \end{bmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + 3R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 2 & -6 \\ 0 & -2 & 5 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 + 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ \\ (1/2)R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 + 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in row echelon form, and so we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables, we know that our system has a unique solution. And so we have shown that $\mathcal{T}_2$ is linearly independent. And as we have that $\mathcal{T}_2$ is a spanning set for Span $\mathcal{T}$, we have shown that $\mathcal{T}_2 = \left\{ \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 7 \\ 11 & -5 \end{bmatrix}, \begin{bmatrix} 2 & -8 \\ -12 & 7 \end{bmatrix} \right\}$ is a basis for Span $\mathcal{T}$.