V

Gordan's Problem

Hilbert was resolved that as a docent he would educate himself as well as his students through his choice of subjects and that he would not repeat lectures, as many docents did. At the same time, on the daily walk to the apple tree, he and Hurwitz set for themselves the goal of "a systematic exploration" of mathematics.

The first semester he prepared lectures on invariant theory, determinants and hydrodynamics, the last at the suggestion of Minkowski, who was habilitating at Bonn and showing an interest in mathematical physics. There were not many who took advantage of this earliest opportunity to hear David Hilbert. Only in the lectures on invariant theory was he able to draw the number of students required by the University for the holding of a class. "Eleven docents depending on about the same number of students," he complained disgustedly to Minkowski. In honor of his new status he had a formal picture taken. It showed a young man with glasses, a somewhat straggly moustache and already thinning hair, who looked as if he might be expected to go after what he wanted.

In Bonn, Minkowski was having his troubles. He did not find the other docents congenial, and the mathematics professor had been taken ill. "I feel his absence especially. He was the only one here to whom I could put a mathematical question, or with whom I could speak at all on a mathematical subject." Whenever he had the opportunity, he returned to Königsberg and joined Hilbert and Hurwitz on their daily walks.

During these years the friendship between Hilbert and Minkowski deepened. Minkowski was a frequent vacation guest at Rauschen. Receiving the photograph of Hilbert after one of the Rauschen visits, Minkowski wrote, "If I had not seen you in it so stately and dignified, I would otherwise have had to think of the outlandish impression which you made on

me in your Rauschen outfit and hairstyle at our brief meeting this summer." He added, musingly: "That we, although so close, could not at all open up to one another was for me more than a little surprising."

In their correspondence they continued to address each other by the formal pronoun "Sie"; but Hilbert, sending Minkowski a reprint of his first published work – the paper which Klein had presented to the Leipzig Academy the previous year – inscribed it: "To his friend and colleague in the closest sense . . . from the author."

That first year as a docent, Hilbert made none of the trips which he had so optimistically planned in order to compensate himself for the isolation of Königsberg. Later he was to recall the years in the "security" of his native city as a time of "slow ripening." The second semester he gave the lectures on determinants and hydrodynamics which he had originally hoped to give the first semester. He began to plan lectures on spherical harmonics and numerical equations. In spite of the variety of his lectures, his own published work continued to be entirely in the field of algebraic invariants; but he also interested himself in questions in other fields.

Finally, at the beginning of 1888, he felt that he was at last ready to take the trip which he had so long promised himself. He drew up an itinerary which would allow him to call on 21 prominent mathematicians, and in March he set out. In his letters to Minkowski he jokingly referred to himself as "an expert invarianttheory man." Now he went first to Erlangen, where the "king of the invariants" held his court.

Paul Gordan was an impressive personality among the mathematicians of the day. Twenty-five years older than Hilbert, he had come to science rather late. His merchant father, while recognizing the son's unusual computational ability, had refused for a long time to concede his mathematical ability. A one-sided, impulsive man, Gordan was to leave a curiously negative mark upon the history of mathematics; but he had a sharp wit, a deep capacity for friendship, and a kinship with youth. Walks were a necessity of life to him. When he walked by himself, he did long computations in his head, muttering aloud. In company he talked all the time. He liked to "turn in" frequently. Then, sitting in some cafe in front of a foaming stein of the famous Erlangen beer, surrounded by young people, a cigar always in his hand, he talked on, loudly, with violent gestures, completely oblivious of his surroundings. Almost all of the time he talked about the theory of algebraic invariants.

It had been Gordan's good fortune to enter this theory just as it moved onto a new level. The first years of development had been devoted to determining the laws which govern the structure of invariants; the next concern had been the orderly production and enumeration of the invariants, and this was Gordan's meat. Sometimes a piece of his work would contain nothing but formulas for 20 pages. "Formulas were the indispensable supports for the formation of his thoughts, his conclusions and his mode of expression," a friend later wrote of him. Gordan's strength, however, in the invention and execution of the formal algebraic processes was considerable. At the beginning of his career, he had made the first break-through in a famous invariant problem. For this he had been awarded his title as king of the invariants. The general problem, which was still unsolved and now the most famous problem in the theory, was called in his honor "Gordan's Problem." This was the problem which Hermite had discussed with Hilbert and Study in Paris.

"Gordan's Problem" was far removed from the "solving for x" with which algebra had begun so many centuries before. It was a sophisticated "pure mathematical" question posed, not by the physical world, but by mathematics itself. The internal structure of all invariant forms was by this time known. Although there would be certain ambiguities and repetitions, different invariant forms of specified order and degree could be written down and counted, at least in principle. The next question was of a quite different nature, for it concerned the totality of invariants. Was there a *basis*, a finite system of invariants in terms of which all other invariants, although infinite in number, could be expressed rationally and integrally?

Gordan's great achievement, exactly 20 years before the meeting with Hilbert, had been to prove the existence of a finite basis for the binary forms, the simplest of all algebraic forms. Characteristically, his proof had been a computational one, based on the nature of certain elementary operations which generate invariants. Today it is dismissed as "crude computation"; but that it was, in its day, a high point in the history of invariant theory is apparent from the fact that in 20 years of effort by English, German, French and Italian mathematicians, no one had been able to extend Gordan's proof beyond binary forms, although in certain specific cases the theorem was known to be true. The title won in 1868 remained unchallenged. Just before Hilbert's arrival in Erlangen, Gordan had published the second part of his "Lectures on Invariant Theory," the plan of this work being primarily "to expound and exemplify worthily" (as a writer of the day explained) the theorem which he had proved at that time.

Hilbert had been familiar with Gordan's Problem for some time; but now, listening to Gordan himself, he seems to have experienced a phenomenon which he had not experienced before. The problem captured his imagination with a completeness that was almost supernatural.

Here was a problem which had every one of the characteristics of a great fruitful mathematical problem as he himself was later to list them:

Clear and easy to comprehend ("for what is clear and easily comprehended attracts, the complicated repels").

Difficult ("in order to entice us") *yet not completely inaccessible* ("lest it mock our efforts").

Significant ("a guidepost on the tortuous paths to hidden truths").

The problem would not let him go. He left Gordan, but Gordan's Problem accompanied him on the train up to Göttingen, where he went to visit Klein and H. A. Schwarz. Before he left Göttingen, he had produced a shorter, more simple, more direct version of Gordan's famous proof of the theorem for binary forms. It was, according to an American mathematician of the period, "an agreeable surprise to learn that the elaborate proofs of Gordan's theorem formerly current could be replaced by one occupying not more than four quarto pages."

From Göttingen, Hilbert went on to Berlin and visited Lazarus Fuchs, who was now a professor at the university there; also Helmholtz; and Weierstrass, who had recently retired. He then paid another call on Kronecker. He had a great deal of admiration for Kronecker's mathematical work, but still he found the older man's authoritarian attitude toward the nature of mathematical existence extremely distasteful. Now he discussed with Kronecker some plans for future investigations in invariant theory. Kronecker does not seem to have been much impressed. He cited a work of his own and said, Hilbert noted, "that my investigation on the subject is contained therein." They had a long talk, however, about Kronecker's ideas on what constitutes mathematical existence and his objections to Weierstrass's use of irrational numbers. "Equal is only $2 = 2$. . . . Only the discreet and singular have significance," Hilbert wrote in the little booklet in which he kept notes on the conversations with the mathematicians he visited. The importance the conversation with Kronecker had in Hilbert's mind at this time is indicated by the fact that he devoted four pages of his notebook to it — the other mathematicians visited, including Gordan, never received more than a page.

He left Kronecker, still thinking about Gordan's Problem.

Back home in Königsberg, the problem was with him in the midst of pleasure and work, even at dances, which he loved to attend. In August he went up to Rauschen, as was still his custom; and from Rauschen, on September 6, 1888, he sent a short note to the *Nachrichten* of the Göttingen Scientific Society. In this note he showed in a totally unexpected and original way how Gordan's Theorem could be established, by a uniform method, for forms in any desired number of variables.

No one was prepared for the announcement of the solution of the famous old problem, and the first reaction was almost sheer disbelief.

Since Gordan's own solution of the simplest case, the solution of the general problem had been sought in essentially the same manner, by means of the same kind of elaborate algorithmic apparatus which had been used so successfully by Gordan. With many variables and a complicated transformation group, this approach became fantastically difficult. It was not unusual for a single formula to run from page to page in the *Annalen*. "Comparable only to the formulas which describe the motion of the moon!" a later mathematician complained. In this atmosphere of absolute formalism it had occurred to Hilbert that the only way to achieve the desired proof would be to approach it from a path entirely different from the formalistic one which all investigators to date had taken and found impenetrable. He had set aside the whole elaborate apparatus and rephrased the question essentially as follows:

"If an infinite system of forms be given, containing a finite number of variables, under what conditions does a finite set of forms exist, in terms of which all the others are expressible as linear combinations with rational integral functions of the same variables for coefficients?"

The answer he came to was that such a set of forms *always* exists.

The foundation on which this sensational proof of the existence of a finite basis of the invariant system rested was a lemma, or auxiliary theorem, about the existence of a finite basis of a module, a mathematical idea he had obtained from the study of Kronecker's work. The lemma was so simple that it seemed almost trivial. Yet the proof of Gordan's general theorem followed directly from it. The work was the first example of the characteristic quality of Hilbert's mind — what one of his pupils was to describe as "a natural naiveté of thought, not coming from authority or past experience."

When the proof of Gordan's Theorem appeared in print in December, Hilbert promptly fired off a copy to Arthur Cayley, who half a century before had laid the foundation of the theory. ("The theory of algebraic invariants," a later mathematician once wrote, "came into existence somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's jovian head. Her Athens, over which

she ruled and which she served as a tutelary and beneficent goddess, was projective geometry. From the beginning she was dedicated to the proposition that all projective coordinate systems are created equal . . . .")

"Dear Sir," Cayley replied politely from Cambridge on January 15, 1889, "I have to thank you very much for the copy of your note . . . . It [seems] to me that the idea is a most important valuable one, and that it ought to lead to a demonstration of the theorem as to invariants, but I am unable to satisfy myself as yet that you have obtained such a demonstration."

By January 30, however, having received two explanatory letters from Hilbert in the intervening time, Cayley was congratulating the young German: "My difficulty was an *a priori* one, I thought that the like process should be applicable to semi-invariants, which it seems it is not; and now I quite see. . . . I think you have found the solution of a great problem."

Hilbert had solved Gordan's Problem very much as Alexander had untied the Gordian Knot.

At Gordium [Plutarch tells us] he saw the famous chariot fastened with cords made of the rind of the cornel-tree, which whosoever should untie, the inhabitants had a tradition, that for him was reserved the empire of the world. Most authors tell the story that Alexander, finding himself unable to untie the knot, the ends of which were secretly twisted round and folded up within it, cut it asunder with his sword. But Aristobulus tells us it was easy for him to undo it, by only pulling the pin out of the pole, to which the yoke was tied, and afterwards drawing of the yoke itself from below.

To prove the finiteness of the basis of the invariant system, one did not actually have to construct it, as Gordan and all the others had been trying to do. One did not even have to show how it could be constructed. All one had to do was to prove that a finite basis, of logical necessity, *must exist*, because any other conclusion would result in a contradiction — and this was what Hilbert had done.

The reaction of some mathematicians was similar to what must have been the reaction of the Phrygians to Alexander's "untying" of the knot. They were not at all sure that he had untied it. Hilbert had not produced the basis itself, nor had he given a method of producing it. His proof of Gordan's Theorem could not be utilized to produce in actuality a finite basis of the invariant system of even a single algebraic form.

Lindemann found his young colleague's methods "unheimlich" — *uncomfortable, sinister, weird*. Only Klein seemed to recognize the power of the work — "wholly simple and, therefore, logically compelling" — and it was at this time that he decided he must get Hilbert to Göttingen at the first opportunity. Gordan himself announced in a loud voice that has echoed in mathematics long after his own mathematical work has fallen silent:

"Das ist nicht Mathematik. Das ist Theologie."

Hilbert had now publicly taken a position in the current controversy provoked by Kronecker over the nature of mathematical existence. Kronecker insisted that there could be no existence without construction. For him, as for Gordan, Hilbert's proof of the finiteness of the basis of the invariant system was simply not mathematics. Hilbert, on the other hand, throughout his life was to insist that if one can prove that the attributes assigned to a concept will never lead to a contradiction, the mathematical existence of the concept is thereby established.

In spite of the philosophical difference, Hilbert was at this time greatly under the influence of the mathematical ideas of Kronecker — in fact, the fundamental significance of his work in invariants was later to be seen as the application of arithmetical methods to algebraic problems. He sent a copy of every paper he published to Kronecker. Nevertheless, Kronecker remarked petulantly to Minkowski that he was going to stop sending papers to Hilbert if Hilbert did not send papers to him. Hilbert promptly composed a letter which managed to be formal and respectful but firm:

"I remember exactly, and my list of mailed papers also shows it clearly, that I have taken the liberty of sending you a copy of each paper without exception immediately after its publication; and you have had the kindness to send your thanks on postcards for some of the last mailings. On the other hand, most honorable professor, it has never happened that a reprint of one of your papers has arrived as a gift from you to me. When I had the honor of calling on you about a year ago, however, you mentioned that you would choose something from your papers and send it to me. Under the circumstances I believe that there must be some misunderstanding, and I write these lines to remove it as fast and as surely as possible."

Then, with many crossings-out, he struggled to express the idea that what he had written should not be construed as expressing any other meaning than the stated one: not reproaches, but just explanations. He finally gave up, and simply signed himself, "Most respectfully, David Hilbert."

During the next two years, Hilbert, still a docent, sent two more notes to the *Nachrichten* and then in 1890 brought all his papers on algebraic forms together into a unified whole for the *Annalen*. By this time the revolutionary effect of Hilbert's work was being generally recognized and accepted. Gordan, offering another proof of one of Hilbert's theorems, was deferential to the young man — Herr Hilbert's proof was "completely correct," he wrote, and his own proof would not even have been possible "if Herr Hilbert had not utilized in invariant theory concepts which had been developed by Dedekind, Kronecker and Weber in another part of mathematics."

While Hilbert was thus involved in the purest of pure mathematics, Minkowski was moving increasingly away from it. Heinrich Hertz, two years after his discovery of the electromagnetic waves predicted by Maxwell, and still only 31 years old, had recently become professor of physics at Bonn. Minkowski, complaining of "a complete lack of half-way normal mathematicians" among his colleagues, found himself attracted more and more by Hertz and by physics. At Christmas he wrote that, contrary to his custom, he would not be spending the vacation at Königsberg:

"I do not know if I need console you though, since this time you would have found me thoroughly infected with physics. Perhaps I even would have had to pass through a 10-day quarantine period before you and Hurwitz would have admitted me again, mathematically pure and unapplied, to your joint walks."

At another time he wrote:

"The reason that I am now almost completely swimming in physical waters is because here at the moment as a pure mathematician I am the only feeling heart among wraiths. So for now," he explained, "in order to have points in common with other mortals, I have surrendered myself to magic — that is to say, physics. I have my laboratory periods at the Physics Institute; at home I study Thomson, Helmholtz and consorts. And from the end of next week on, I will even work several days a week in a blue smock in an institute for the production of physical instruments, a technician, therefore, and as practical as you can imagine!"

But the diverging of scientific interests did not affect the friendship; and, in fact, it was at this time that the two young men made the significant transition in their correspondence from the formal pronoun "Sie" to the intimate "du."

The Privatdozent years seemed to stretch out interminably. The letters were much concerned with the possibility of promotion. In 1891 Minkowski wrote that he had been told that he might be proposed for a position in Darmstadt. "But this ray of hope could easily shine so long that it shines upon mostly grey hair." That same year — apparently with special permission from the University — Hilbert was delivering his lectures on analytic functions to only one student — an American from Baltimore — a man somewhat older than the young lecturer but, in his opinion, "very sharp and extraordinarily interested." This was Fabian Franklin, an important man in invariant theory and the successor of Sylvester at Johns Hopkins.

Because there were few mathematics students at Königsberg, Hilbert attended the meetings of the natural scientists as well as those of the mathematicians. But Königsberg was surprisingly full of congenial young people. Wiechert was a docent too; and he had recently been joined by a student named Arnold Sommerfeld, with whom he was devising a harmonic analyzer. Both Wiechert and Sommerfeld were to become masters of electrodynamic theory, but when "Little Sommerfeld" heard Hilbert lecture on ideal theory, he became convinced that his interest lay entirely with the most pure and abstract mathematics. "Already," he later commented, "it was clear that a spirit of a special sort was at work."

There was lots of happy social life. Hilbert was a gay young man with a reputation as a "snappy dancer" and a "charmeur," according to a relative. He flirted, outrageously, with a great number of girls. His favorite partner for all activities, however, was Käthe Jeroseh, the daughter of a Königsberg merchant, an outspoken young lady with an independence of mind that almost matched his own.

Even after the work of 1890, Gordan's Problem still would not let Hilbert go. As a mathematician he preferred an actual construction to a proof of existence. "There is," as one mathematician has said, "an essential difference between proving the existence of an object of a certain type by constructing a tangible example of such an object, and showing that if none existed one could deduce contrary results. In the first case one has a tangible object, while in the second one has only the contradiction." He would very much have liked to produce for old Kronecker, Gordan and the rest a constructive proof of the finiteness of the basis of the invariant system. At the moment there was simply no method at hand.

In the course of the next two years, however, the nature of his work began to change. It became infused with the ideas of algebraic number fields. Again, Kronecker's ideas were important. And it was here that Hilbert found at last the powerful new tools he had been seeking. In a key work, in 1892, he took up the question of exactly what was needed to produce in actuality a full system of invariants in terms of which all the other invariants could be represented. Using as a foundation the theorem which he had earlier proved, he was able to produce what was in essence a finite means of executing the long sought construction.

Although Hilbert was not the first to make use of indirect, non-constructive proofs, he was the first to recognize their deep significance and value

and to utilize them in dramatic and extremely beautiful ways. Kronecker had recently died; but to those who like Kronecker still declared that existence statements are meaningless unless they actually specify the object asserted to exist, Hilbert was always to reply:

"The value of pure existence proofs consists precisely in that the individual construction is eliminated by them, and that many different constructions are subsumed under one fundamental idea so that only what is essential to the proof stands out clearly; brevity and economy of thought are the *raison d'être* of existence proofs .... To prohibit existence statements . . . is tantamount to relinquishing the science of mathematics altogether."

Now, through a proof of existence, Hilbert had been able to obtain a construction. The impetus which his achievement gave to the use of existential methods can hardly be overestimated.

Minkowski was utterly delighted:

"For a long while it has been clear to me that it could be only a question of time until the old invariant question was settled by you — only the dot was lacking on the 'i'; but that it all turned out to be so surprisingly simple has made me very happy, and I congratulate you."

He was inspired to literary flight and an assortment of metaphors. The first existence proof might have got smoke in Gordan's eyes, but now Hilbert had found a smokeless gunpowder. The castle of the robber barons — Gordan and the rest — had been razed to the ground with the danger that it might never rise again. Hilbert would be doing a service to his fellow mathematicians if he would bring together the materials in this area on which one could rebuild. But he probably would not want to spend his time doing that. There were still too many other things that he was capable of doing!

Gordan himself conceded gracefully.

"I have convinced myself that theology also has its merits."

When Klein went to Chicago for what was billed as an "International Congress of Mathematicians" to celebrate the founding of the University of Chicago, he took with him a paper by Hilbert in which that young man matter-of-factly summarized the history of invariant theory and his own part in it:

"In the history of a mathematical theory the developmental stages are easily distinguished: the naive, the formal, and the critical. As for the theory of algebraic invariants, the first founders of it, Cayley and Sylvester, are together to be regarded as the representatives of the naive period: in the drawing up of the simplest invariant concepts and in the elegant applications to the solution of equations of the first degrees, they experienced the immediate joy of first discovery. The inventors and perfecters of the symbolic calculation, Clebsch and Gordan, are the champions of the second period. The critical period finds its expressions in the theorems I have listed above. . . ."

The theorems he referred to were his own.

It was a rather brash statement for a young mathematician who was still not even an Extraordinarius, but it had considerable truth in it. Cayley and Sylvester were both alive, one at Cambridge and the other at Oxford. Clebsch was dead, but Gordan was one of the most prominent mathematicians of the day. Now suddenly, in 1892, as a result of Hilbert's work, invariant theory, as it had been treated since the time of Cayley, was finished. "From the whole theory," a later mathematician wrote, "the breath went out."

With the solution of Gordan's Problem, Hilbert had found himself and his method — an attack on a great individual problem, the solution of which would turn out to extend in significance far beyond the problem itself. Now something totally unexpected occurred. The problem which had originally aroused his interest had been solved. The solution released him.

At the conclusion of his latest paper on invariants he had written: "Thus I believe the most important goals of the theory of function fields generated by invariants have been obtained." In a letter to Minkowski, he announced with even more finality: "I shall definitely quit the field of invariants."

VI

Changes

During the next three years Hilbert rose in the academic ranks, did all the things that most young men do at this time of their lives, married, fathered a child, received an important assignment, and made a decision which changed the course of his life.

This sudden series of events was set into motion by the death of Kronecker and the game of "mathematical chairs" which ensued in the German universities. Suddenly it seemed that the meager docent years might be coming to an end. Minkowski calling in Berlin on Friedrich Althoff, who was in charge of all matters pertaining to the universities, heralded the news:

"A. says . . . the following are supposed to receive paid Extraordinariats: you, I, Eberhard, and Study. I have not neglected to represent you to A. as the coming man in mathematics. . . . As to Study, in conscience I could only praise his good intentions and his diligence. A. is very devoted to you and Eberhard."

At almost the same time Hurwitz, who had been an associate professor (Extraordinarius) at Königsberg for eight years, received an offer of a full professorship from the Swiss Federal Institute of Technology in Zürich. This meant an end to the daily mathematical walks, but opened up the prospect of Hilbert's being appointed to Hurwitz's place.

"Through this circumstance," Minkowski wrote affectionately, "your frightful pessimism will have been allayed so that one dares again to venture a friendly word to you. In some weeks now, hopefully, the Privatdozent-sickness will be definitely over. You see — at last comes a spring and a summer."

In June, Hurwitz married Ida Samuels, the daughter of the professor of medicine. Hilbert had recently become engaged to Käthe Jerosch, and