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NEAR OPPOSITES: KANT AND MILL

1. Reorientation

WE pick up our story in the eighteenth century, with Immanuel Kant. There was, of course, considerable philosophical activity in Antiquity after Aristotle and through the Middle Ages, but not much of it directly focused on mathematics.[1]

The seventeenth century saw major revolutions in science and mathematics, through people like René Descartes, Isaac Newton, and Gottfried Wilhelm Leibniz. Kant was in a position to take the philosophical measure of the new scientific developments. The demands of the emerging physics led to the development of new branches of mathematics and to new conceptions of the traditional branches. The major innovations included new methods of analysis linking geometry with algebra and arithmetic (Pierre Fermat and Descartes), and the development of the calculus (Newton and Leibniz) for the study of gravitation and motion. The latter required notions of continuity, derivative, and limit, none of which smoothly fitted into previous mathematical paradigms. (See

[1] It is not uncommon for sequences in the history of philosophy to jump from Aristotle to the so-called 'modern period', with Bacon or Hobbes, or even Descartes. Courses in the history of mathematics often have a similar gap, perhaps lightly filled in. The erroneous implication is that very little of substance occurred during those two millennia. In this book the justifications for the gap are limitations on space and my competence, and the fact that we are exploring direct precursors to contemporary positions in the philosophy of mathematics.

Mancosu 1996 for a lucid treatment of mathematics and its philosophy during the seventeenth century.)

At the time there were two major schools of philosophy. On the European continent, *rationalists* like Descartes, Baruch Spinoza, and Leibniz were Plato's natural heirs. They emphasized the role of reason, as opposed to sensory experience, in obtaining knowledge. Extreme versions of the view have it that *all* knowledge is, or ideally ought to be, based on reason. The rationalist model for knowledge-gathering is mathematics—mathematical demonstration in particular. For example, Spinoza's *Ethics* has the same format as Euclid's *Elements*, containing 'propositions' and 'demonstrations'. Much of Descartes's philosophical work is an attempt to give science the same degree of certainty as mathematics. Science is supposed to be founded on philosophical first-principles. Descartes attempted a mathematical-style derivation of the laws of motion.

Empiricism, the main opposition to rationalism, is an attempt to base knowledge, or the materials from which knowledge is based, on experience from the five senses. During the period in question the major writers were John Locke, George Berkeley, David Hume, and Thomas Reid, all of whom lived in the British Isles. A common empiricist theme is that *anything* we know about the world must ultimately come from neutral and dispassionate observation. The only access to the universe is through our eyes, ears, and so on. Empiricists sometimes present an image of the mind as a blank tablet on which information is imprinted, via the senses. We are passive observers sifting through the incoming data, trying to make sense of the world around us.

There is no substantial, detailed philosophical account of mathematics during this period. The rationalists, of course, admired mathematics, and Descartes and Leibniz were themselves major mathematicians. Empiricists tended to downplay the importance of mathematics, perhaps because it does not easily fit their mould of knowledge-gathering. Berkeley launched a sustained attack on the supposed rigour of the infinitesimal calculus (see Jesseph 1993). However, given the role of mathematics in the sciences, empiricists had to provide some account of it.

Scattered philosophical remarks about mathematics reveal a surprising amount of agreement between the two major schools. Both rationalists and empiricists took mathematics to be about physical magnitudes, or extended objects. The objects are encountered empirically. The two schools differed over the mind's access to the *ideas* of extended objects and over the status of the reasoning about those ideas. Descartes, for example, held that we have clear and distinct perception of 'pure extension' that underlies physical objects, and he held that we can reason directly about this pure extension. This view attests to the rationalist conviction that the human intellect is a powerful tool for reasoning—mathematically—to substantial a priori conclusions about the physical world.

Empiricists took mathematical ideas to be derived from experience, perhaps following Aristotle. Our idea of the number six, for example, comes from our experience with groups of six objects. The idea of 'triangle' comes from looking at triangular-shaped objects. For the empiricist there is no substantial 'pure extension' underlying perceived objects. There are only the perceived objects. What you see is what you get.

Despite these and other differences, a typical empiricist might agree with a typical rationalist that, once the relevant ideas are obtained, the pursuit of mathematical knowledge is independent of any further experience. The mathematician contemplates how the various mathematical ideas relate to each other. For example, in his *Treatise on Human Nature*, Hume referred to the truths of arithmetic and algebra as 'relations of ideas' and distinguished these from 'matters of fact and existence', which we learn empirically. Geometry is an empirical science, presumably concerned with generalizations from experience. A decade later, in his popular *An Enquiry Concerning Human Understanding*, Hume claimed that arithmetic, algebra, and geometry alike all concern (mere) relations among ideas, and so are not empirical. The common ground between the schools is that, in at least some sense, mathematical truths are a priori, or independent of experience. The main dispute is over the extent to which sensory experience is needed to obtain or grasp the relevant ideas and to study them.

Mathematical truth at least appears to have a certain necessity attached to it. How could $5 + 7$ not be 12? How could the prime factorization theorem be false? Rationalism provides a smooth account of this, along roughly Platonic lines. There is no contingency in the mentally grasped mathematical ideas, like pure extension, that underlie physical objects. We may, of course, err in our grasp of mathematical ideas or in attempting a demonstration, but, if properly carried out, the methodology of mathematics delivers only necessary truths. Of course this perspective is not available to the empiricists, and they do not have such a straightforward explanation of the seeming necessity of mathematics. Some of them might hold that basic mathematical propositions are true by definition, a conclusion that a rationalist would find disappointing since it leaves mathematics without substance. Hume notes that we cannot imagine or conceive of the negations of typical mathematical theorems, but this seems to be a weak hold on the necessity of mathematics. Is it only a contingent psychological limitation that prevents us from conceiving things in any other way?

The use of the new mathematics in science brought new force to the problems of the applicability of mathematics to the physical world. Here empiricism did better. According to that school, mathematical ideas are read off from properties of observed objects, and mathematicians study the relations among these ideas. That is, empiricists held that the mathematician indirectly studies certain physical relations between observed physical objects. This explanation is not available to a rationalist. Her problem is to show how the innately grasped, eternal mathematical entities relate to the objects we perceive in the world around us and study in science. Our empiricist thus follows Aristotle, with a straightforward account of the *match* between observed physical objects and their mathematical counterparts, while our rationalist follows Plato, with a straightforward account of the *mismatch* between the objects of the senses and their mathematical counterparts, like perfect circles and triangles, and perhaps large numbers.

2. Kant

The clash between rationalism and empiricism provides a central motivation for Kant's attempt at a synthesis that captures the most plausible features of each. The result was a heroic attempt to explain or accommodate the necessity of mathematics and the a priori nature of mathematical truth, while explaining or accommodating the place of mathematics in the empirical sciences and, in particular, the applicability of mathematics to the observed physical world. Kant's problem was to show how mathematics is knowable a priori and yet is applicable universally—to all experience—with incorrigible certainty. His views on mathematics are not a separable component of his overall philosophy. On the contrary, references to mathematics occur throughout his philosophical writing. Thus, an important key to understanding Kant is to understand his views on mathematics.

The reader should note that, even if the following sketch does suggest some themes of Kant's complex and subtle philosophy of mathematics, it barely scratches the surface. Moreover, there is much disagreement among scholars (see the items mentioned at the end of this chapter, for a start). The tentative interpretations suggested below are based on some of their work, and I have tried either to take note of the major disagreements or to steer clear of them. However, it is inevitable that parts of any interpretation will be at odds with some of the prominent scholarship.

The most intriguing and problematic feature of Kant's philosophy of mathematics is his thesis that the truths of geometry, arithmetic, and algebra are 'synthetic a priori', founded on 'intuition'. The key notions are thus a priori knowledge, the analytic–synthetic distinction, and the faculty of intuition.

For Kant, a universal proposition (in the form 'All *S* are *P*') is *analytic* if the predicate concept (*P*) is contained in the subject concept (*S*); otherwise the proposition is *synthetic*. For example, 'all bachelors are unmarried' is analytic if the concept of being unmarried is contained in the concept of bachelor. 'All men are mortal' is analytic if the concept of mortality is contained in the concept of man. Since being male is (presumably) not part of the concept of being President, 'all Presidents are male' is synthetic.

As we know now, not every proposition has a subject–predicate form, and so by contemporary lights Kant's definition of *analyticity* is unnatural and stifling. He does recognize other forms of judgement, suggesting that the application of the analytic–synthetic distinction to negative judgements is straightforward (*Critique of Pure Reason*, A6/B11), but he does not say very much else. What of hypothetical propositions like 'if it is raining now, then either it is raining or it is snowing'? This is not the place to suggest improvements or extensions of Kant's distinction, but we do need to examine its basis.

The metaphysical status of Kantian analytic truths turns on the nature of *concepts*. We need not delve further into this, other than to note that Kant's thesis presupposes that concepts have parts (at least metaphorically), since otherwise we cannot speak of one concept 'containing' another. The relevant issues here are epistemic. Kant believed that the parts of concepts are grasped through a mental process of conceptual analysis. For example, when presented with a proposition in the form 'All *S* is *P*' we analyse the subject concept *S* to see if the predicate *P* is among the parts. We come to know that 'all bachelors are unmarried' by analysing 'bachelor' and learning that it contains 'unmarried'. In short, whatever concepts are, Kant held that anyone who grasps one is in a position to perform the analysis and determine its components. Conceptual analysis uncovers what is already implicit in concepts: 'Analytic judgements could . . . be called *elucidatory*. For they do not through the predicate add anything to the concept of the subject; rather they only dissect the concept, breaking it up into its component concepts which had already been thought in it' (*Critique of Pure Reason*, B11). Thus, conceptual analysis does not yield new knowledge about the world. In a sense, it tells us nothing, or nothing new.

It is straightforward that analytic truths are knowable a priori. Let *A* be an analytic truth. Anyone who has grasped the concepts expressed in *A* is in a position to determine their parts and thus the truth of *A*. No particular experience of the world is necessary, beyond what is needed to grasp the requisite concepts.

Kant noted that a few mathematical propositions are analytic. Consider, for example, 'all triangles have three angles' or perhaps 'all triangles have three sides'[2] or 'all triangles are self-identical'. However, Kant held that almost all mathematical propositions are synthetic. Conceptual analysis alone does not determine that $7 + 5 = 12$, or that between any two points a straight line can be drawn, or that a straight line is the shortest distance between two points. Inspection of the concepts corresponding to '7', '5', '12', addition, identity, point, and line will not reveal the truth of these propositions.

To see why Kant thought that conceptual analysis is not sufficient to establish many mathematical propositions, we attend

[2] The concept expressed by the English word 'triangle' contains the concept of being 'three-angled'. Does it also contain the concept of 'three-sided'? The German word for 'triangle' is '*Dreieck*', or 'three-cornered'. Presumably, that concept includes 'three-angled', but, again, does it include 'three-sided'?

to Kant's epistemology. He held that synthetic propositions are knowable only via 'intuition', and so we must turn to that notion.

Kantian intuition has two features, although scholars disagree on the relative importance of each. First, intuitions are *singular*, in the sense that they are modes of representing individual objects. Indeed, intuition is essential for knowledge of individual objects. By contrast, conceptual analysis is not singular and only produces general truths. We know from conceptual analysis that all bachelors are unmarried, but we do not thereby learn that there are any bachelors, nor do we get acquainted with any. In discussing the ontological argument for the existence of God, Kant argued that we cannot learn about the existence of anything by conceptual analysis alone (*Critique of Pure Reason*, B622–3). To adapt this thesis to mathematics, suppose that someone wants to show that there is a prime number greater than 100. In typical mathematical fashion, she assumes that every natural number over 100 is composite and derives a contradiction. So *perhaps* she has established an analytic truth that it is not the case that all numbers over 100 are composite. But we only get the existence of a prime if we know that *there are* natural numbers greater than 100. As far as conceptual analysis goes, it seems that we still have the option to reject the existential assumption.[3] Similarly, we only know that a diagonal of a square is incommensurable with its side if we know that there are squares and that squares have diagonals. Conceptual analysis does not establish this. According to Kant, we need intuition to represent *numbers* (or numbered groups of objects) and geometric figures, and to learn things about them. *A fortiori*, conceptual analysis cannot deliver the (potential) infinity of number and of space (see Friedman 1985).

So one reason for taking mathematics to be synthetic is that it

[3] It is hard to be definite about this example since, as far as I know, Kant does not speak of demonstration in arithmetic. He does allow that some laws in arithmetic are analytic, but perhaps we do need intuition to determine that not every prime number greater than 100 is composite. The point here is that he would surely hold that we need intuition to establish the *existence* claim. In contemporary logical systems, 'it is not that case that all x are P' entails that 'there is an x such that not-P' In symbols, $\neg\forall xPx$ entails $\exists x\neg Px$. To engage in a barbarous anachronism, if the foregoing interpretation of Kant is correct, he would regard this inference as involving intuition. That is, the inference in question might lead from an analytic truth to a synthetic one. See Posy 1984 for an insightful account of the proper logic to attribute to Kant.

deals with individual objects like numbered groups of things, geometric figures, and even space itself—which Kant took to be singular and apprehended by intuition. However, his views are deeper than this.

In a famous, or infamous, passage, Kant argued that sums are synthetic:

> It is true that one might at first think that the proposition 7 + 5 = 12 is a merely analytic one that follows, by the principle of contradiction, from the concept of a sum of seven and five. Yet if we look more closely, we find that the concept of the sum of 7 and 5 contains nothing more than the union of the two numbers into one; but in [thinking] that union we are not thinking in any way at all what that single number is that unites the two. In thinking merely that union of seven and five, I have by no means already thought the concept of twelve; and no matter how long I dissect my concept of such a possible sum, still I shall never find in it that twelve. We must go beyond these concepts and avail ourselves of the intuition corresponding to one of the two: e.g., our five fingers or . . . five dots. In this way we must gradually add, to the concept of seven, the units of the five given in intuition . . . In this way I see the number 12 arise. That 5 *were to be added* to 7, this I had indeed already thought in the concept of a sum = 7 + 5, but not that this sum is equal to the number 12. Arithmetic propositions are therefore always synthetic. We become aware of this all the more distinctly if we take larger numbers. For then it is very evident that . . . we can never find the . . . sum by merely dissecting our concepts, i.e., without availing ourselves of intuition. (*Critique of Pure Reason*, B15–16)

Recall that for Kant conceptual analysis does not yield new knowledge. Rather, it just reveals what is implicit in the concepts. Here Kant asserts that addition does yield new knowledge, and so is synthetic.

Kant held that, even though most mathematical propositions are synthetic, they are knowable a priori—independent of sensory experience. How can this be? Whether the motivation comes from mathematics or not, much of Kant's general philosophy is devoted to showing how synthetic a priori propositions are possible. How can there be a priori truths that are not grounded in conceptual analysis?

A second feature of Kantian intuition is that it yields *immediate* knowledge. As indicated by the passage about 7 + 5, for humans at least, intuition is tied to sense perception. A typical intuition would be the perception that underlies the judgement that my right hand contains five fingers.

Of course, this sort of intuition is empirical and the knowledge it produces is contingent. We do not learn mathematics that way. Kant held that there is a form of intuition that yields a priori knowledge of necessary truths. This 'pure' intuition delivers the *forms of possible empirical intuitions*. That is, pure intuition is an awareness of the spatio-temporal form of ordinary sense perception. The idea is that pure intuition reveals the presuppositions of unproblematic, empirical knowledge of spatio-temporal objects. For example, Euclidean geometry concerns the ways humans necessarily perceive space and spatial objects. We apprehend objects in three dimensions, enclose regions with straight lines, and so on. Arithmetic concerns the ways humans have to perceive objects in space and time, locating and distinguishing objects and counting them. Arithmetic and geometry thus describe the framework of perception. As Jaakko Hintikka (1967: §18) put it, for Kant the 'existence of the individuals with which mathematical reasoning is concerned is due to the process by means of which we come to know the existence of individuals in general'. Kant held that this process is sense perception. So 'the structure of mathematical reasoning is due to the structure of our apparatus of perception'.

Recall that, for Descartes, 'pure extension' is perceived directly *in* physical objects (at least metaphorically). In contrast, Kant took pure intuition to concern the forms of possible human *perception*. These forms are not in the physical objects themselves, but, in a sense, they are supplied by the human mind. We structure our perceptions in a certain way.

Here is a passage from the *Critique of Pure Reason* that highlights the nature of geometric a priori intuitions and the necessity of mathematics. Apparently, Kant takes philosophy to be the activity of conceptual analysis, and he makes a contrast with mathematics:

> Mathematics provides the most splendid example of a pure reason successfully expanding itself on its own, without the aid of experience ... *Philosophical* cognition is *rational cognition from concepts*. *Mathematical* cognition is rational cognition from the *construction* of concepts. But *to construct* a concept means to exhibit a priori the intuition corresponding to it. Hence construction of a concept requires a *non-empirical* intuition.

> Consequently, this intuition, as intuition, is an *individual* object; but as the construction of a concept (a universal presentation), it must nonetheless express . . . its universal validity for all possible intuitions falling under the same concept. Thus, I construct a triangle by exhibiting the object corresponding to this concept either through imagination alone in pure intuition or . . . also on paper, and hence also in empirical intuition. But in both cases I do exhibit the object completely a priori, without having taken the model for it from any experience. The individual figure drawn there is empirical, and yet serves to express the concept without impairing the concept's universality. For in dealing with this empirical intuition one takes account only of the action of constructing the concept—to which many determinations are . . . inconsequential: e.g., the magnitude of the sides and of the angles—and one thus abstracts from all these differences that do not change the concept of triangle . . . [P]hilosophical cognition contemplates the particular only in the universal. Mathematical cognition, on the other hand, contemplates the universal in the . . . individual; yet it does so nevertheless a priori and by means of reason. (*Critique of Pure Reason*, B741–2)

We thus encounter a recurring theme in the history of the philosophy of mathematics, abstraction (see ch. 3, §4).

One might think of Kant's pure intuition and the process of abstraction as exhibiting typical or paradigmatic instances of given concepts. Beginning with the *concept* of triangle, for example, intuition supplies us (a priori) with a typical triangle. Similarly, starting with the concept of number, intuition produces a typical number. After this, the mathematician works with the intuited instances. However, as indicated toward the end of the passage, this is probably not what Kant had in mind. There may be a typical point or line, but there simply is no typical or paradigmatic triangle. Any given triangle, either imagined or on paper, must either be acute, right, or obtuse, and either scalene, isosceles, or equilateral, and so any given triangle cannot represent all triangles. Moreover, as Gottlob Frege (1884: §13) later pointed out, this crude abstraction does not have a prayer of application to arithmetic. Each natural number has properties unique to it and it alone, and so no natural number can represent all natural numbers.

Kant's remark that 'in dealing with [an] empirical intuition one takes account only of the action of constructing the concept' indicates a connection to a common technique in deductive reasoning.

Suppose that a geometer is engaged in a geometric demonstration about isosceles triangles. She draws one such triangle and reasons with it. In the subsequent text our geometer invokes only properties of all isosceles triangles, and does not use any other features of the drawn triangle, such as the exact size of the angles or whether the base is shorter or longer than the other sides. If successful, the conclusions hold of all isosceles triangles. This technique is common in mathematics. A number theorist might begin 'let n be a prime number' and proceed to reason with the 'example' n, using only properties that hold of all prime numbers. If she shows that n has a property P, she concludes that all prime numbers have property P, perhaps reminding the reader that n is 'arbitrary'.

This practice corresponds to a rule of inference in contemporary logical systems, sometimes called 'generalization' or 'universal introduction'. In systems of natural deduction, the rule is that from a formula in the form $\Phi(c)$ (i.e. a predicate Φ holds of an individual c) one can infer $\forall x\Phi(x)$ (i.e. Φ holds of everything), provided that the constant c does not occur in the formula $\forall x\Phi(x)$ or in any premiss that $\Phi(c)$ rests on. The restrictions on the use of the rule guarantee that the singular term c is indeed arbitrary. It could be any number. However, this rule of inference was outside the scope of logic as Kant knew it. Kant notoriously claimed that logic had no need to go much beyond the Aristotelian syllogisms. In interpreting Kant, Hintikka (1967) takes 'inferences' like the generalization rule to be the essential component of mathematical intuition. That is, any demonstration that makes essential use of this rule has a synthetic conclusion—even if its premisses are analytic. In contemporary frameworks, the rule of generalization invokes a singular term, the 'arbitrary' constant introduced into the text. After a fashion, this fits the feature that Kantian intuition deals with individual objects. According to this interpretation, if Kant had learned some contemporary logic, he would either retract his main thesis that mathematics is synthetic, or, more likely, he would claim that by the light of (our) logic, a valid inference can have analytic premisses and a synthetic conclusion, just because one of our rules of inference invokes a singular term (see also note 3 above).

Of course, Kant tied intuition to sense perception or, in the case of pure intuition, to the forms of sense perception, and the rule of generalization has nothing specific to do with either of these. The rule is completely general. Hintikka downplays Kant's theses that intuitions are immediate and that they are tied to perception or its form. He criticizes Kant for having too narrow a view of the extent of 'intuition'. Most commentators do not follow Hintikka here, and try to delimit a more direct role for immediacy and the forms of perception in Kant's philosophy of mathematics (see, e.g. Parsons 1969, and the Postscript in the reprint, Parsons 1983: Essay 5). Most scholars have Kant holding that the *axioms* of geometry are synthetic, and so the status of the logic is irrelevant.

Let us consider one more passage where Kant further expounds the difference between mathematics and the conceptual analysis of 'philosophy':[4]

> Philosophy keeps to universal concepts only. Mathematics can accomplish nothing with the mere concept but hastens at once to intuition, in which it contemplates the concept *in concreto*, but yet not empirically; rather, mathematics contemplates the concept only in an intuition that it exhibits a priori—i.e., an intuition that it has constructed . . . Give to a philosopher the concept of a triangle, and let him discover in his own way what the relation of the sum of its angles to a right angle might be. He now has nothing but the concept of a figure enclosed within three straight lines and—with this figure—the concept of likewise three angles. Now, no matter how long he meditates on this concept, he will uncover nothing new. He can dissect and make distinct the concept of a straight line, or of an angle, or of the number three, but he cannot arrive at any other properties that are in no way connected with these concepts. But now let the geometer take up this question. He begins immediately by constructing a triangle. He . . . extends one side of this triangle and thus obtains two adjacent angles that together are equal to two right angles . . . He now divides the external angle by drawing a line parallel to the opposite side of the triangle; and he sees that there arises here an external adjacent angle that is equal to an internal one; etc. In this manner he arrives, by a chain of inferences but always guided by intuition, at a completely evident and at the same time universal solution of the question. (*Critique of Pure Reason*, B743–5)

Here Kant refers to the standard Euclidean proof that the sum of angles in a triangle is two right angles (180°), found in Book 1,

[4] On the concept of triangle, see note 2 above.

Proposition 32 of Euclid's *Elements*. Kant's perspective is suggestive. As noted above, conceptual analysis does not produce new knowledge but only uncovers what is implicit in concepts. It merely 'dissects' or 'makes distinct' the parts that are already there. By contrast, mathematics does produce new knowledge. Its conclusions are not implicit in the concepts. Intuition supplies us with examples of objects, or groups of objects, that exhibit the concepts in question. That is, intuition produces geometric figures or numbered collections of objects. This is only a scant beginning, however. With the examples alone the mathematician cannot get much beyond what would be available from conceptual analysis. So far, all she knows about the examples is that they have the given concepts in question, and thus any other concepts contained in them. Mathematics reveals new knowledge through an a priori mental process of *construction*. The mathematician works on and *acts on* the given examples, following rules implicit in 'pure intuition'.

Hintikka (1967: §8) points out that Kant's paradigm is Euclid's *Elements*, and it is worth a brief look at the structure of a typical Euclidean demonstration. It starts with an 'enunciation' of a general proposition, which states what is to be established. Proposition 32 of Book 1 reads (in part), 'In any triangle . . . the three interior angles . . . are equal to two right angles'. Then Euclid assumes that a particular figure, satisfying the hypothesis of the proposition, has been drawn. This is called the 'setting out' or *ecthesis*. For Kant, this setting-out involves intuition, as above. Intuition provides instances exhibiting the given concepts. (See the left part of Fig. 4.1.) The crucial third part of the demonstration is where the figure is completed by drawing certain *additional* lines, circles, points, and so on. In the example below, this would be the extension of the line segment AB to AD and the segment BE parallel to AC. (See the right

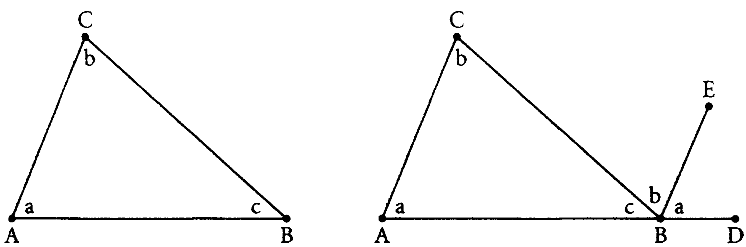

Fig. 4.1. Proof that the sum of the angles in a triangle is two right angles

half of Fig. 4.1.)[5] Perhaps these auxiliary constructions are the essence of the pure intuition involved in mathematics. The geometer (Euclid in this case) produces things that were not there before. Then Euclid proceeds with the *proof*, or *apodeixis*, which consists of a series of inferences concerning the completed figure. In the example at hand, we notice that the angle ∠CAB is equal to ∠EBD (by a previous theorem) and that ∠ACB is equal to ∠CBE. Thus, the three angles of the triangle total two right angles.

In the quoted passage, Kant said that inferences are 'always guided by intuition'. Intuition is involved in *reading* the diagrams, and thus revealing facts about the original triangle. The final, 'proof' part of the demonstration yields synthetic knowledge.[6]

Next consider what Kant says about $7 + 5 = 12$. Once again, conceptual analysis does not yield the sum, since nothing in the concepts of seven and five gives us the number twelve. To get the sum, we 'avail ourselves of the intuition corresponding to one of the two: e.g., our five fingers or . . . five dots.' This corresponds to the setting-out in a Euclidean demonstration. We need an example of a collection of five objects. This, however, is not sufficient, since we still do not have the sum. So we 'gradually add, to the concept of seven, the units of the five given in intuition'. This crucial step, where we keep 'adding' a unit, corresponds to the auxiliary construction. The mathematician thereby produces the numbers 8, 9, 10, 11, and finally she sees 'the number 12 arise'. She thus *constructs* something that is not implicit in the original concept of the sum of 7 and 5, nor in the examples supplied by intuition. Charles Parsons (1969) points out that whenever 'Kant speaks about this subject, he claims that number, and therefore arithmetic, involves *succession* in a crucial way'.

[5] Proposition 32 is: 'In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.' So the setting-out would be the triangle ABC along with the line segment BD. The auxiliary construction is the line segment BE parallel to AC.

[6] This matter is related to one of the so-called logical 'gaps' in Euclid's *Elements*. Suppose that we have a line that goes from the interior of a circle to the exterior. Euclid assumed that there is a point where the line intersects the circle. By modern lights, this does not follow from the postulates, axioms, and definitions. One must explicitly add a principle of continuity. However, if we think of Euclid's inference as 'guided by intuition' then perhaps there is no gap. From this perspective, the continuity of the circle and line is intuited, and is not logical or analytic.

Here we see how arithmetic deals with potential infinity. We intuit that we can always keep counting.

To be sure, there is an important difference between our geometric and arithmetic examples. With simple sums there is nothing that corresponds to the 'proof' stage of a Euclidean demonstration. Once the 'auxiliary constructions' are complete, we have the sum and so we are done. Kant suggested that arithmetic has no axioms (e.g. *Critique of Pure Reason*, B204–6). This might mean that he held that there are no arithmetic demonstrations.[7] Nevertheless, the similarities between arithmetic and geometry are striking. In both cases, *construction* is essential to mathematical progress.

To pursue this interpretation, or reconstruction, of Kant's account of mathematics, we would need to focus on the nature of mathematical construction. The idea is that pure intuition allows us to discover (a priori) the *possibilities* for constructive activity. The Euclidean postulates delimit possible constructions in space. For example, any line segment can be extended indefinitely, or in Euclid's words, the geometer can 'produce a finite straight line continuously in a straight line' (Postulate 2). In arithmetic, a corresponding principle is that any number can be extended to the next number. This is used in the discussion of $7 + 5 = 12$. On this interpretation, postulates tell us what the mathematician can *do*.[8] This makes mathematics primarily a mental activity, and its subject-matter is possible human mental activity (see Parsons 1984). We will encounter the idea of mathematical construction again with some versions of intuitionism—perhaps the twentieth-century philosophy of mathematics closest to Kant (see ch. 7).

We have to tighten the connection between this a priori pure intuition and ordinary sense perception, or empirical intuition. As above, pure intuition delimits the forms of perception. One interpretation is that mathematical construction reveals the *possibilities*

[7] The arithmetic theorems in Book 10 of Euclid's *Elements* are explicitly interpreted in geometrical terms. Some commentators do attribute an axiomatic foundation for arithmetic to Kant. Incidentally, I do not know what Kant would make of a *difference*, like $12 - 5 = 7$. One might think that auxiliary construction is not needed here. Once we have a grasp of the concept of twelve objects, we can 'dissect' it to determine the difference. On the other hand, perhaps the very act of *partitioning* the collection is a construction involving intuition.

[8] However, Euclid's fourth postulate is 'that all right angles are equal', which does not represent a construction.

of perception in space and time. Arithmetic, for example, describes properties of perceived collections of objects. From this perspective, geometry is more problematic. On the interpretation in question, the Euclidean postulates describe possible lines that we can *see*. However, if we look down a long stretch of straight parallel lines, such as a pair of railway tracks, they appear to meet. If we rotate a circle, it appears elliptical. In short, Euclidean geometry does not always describe how space *appears*. Perception is projective, not Euclidean. Since Kant ties intuition to sense perception, and thus appearance, he must resolve this appearance–reality dichotomy. Presumably, a Kantian can somehow abstract from the various perspectives of the different observers, looking for what is common to them. A second problem is with the idealizations, a problem we have encountered before and will encounter again. One simply cannot perceive a line without breadth. With actual drawn figures (apprehended via 'empirical intuition'), two straight lines, or a line tangent to a circle, do not meet in a single point, but in a small region (determined by the thickness of the lines; see Fig. 3.2.). To resolve this problem, Kant does not have Plato's option of separating the world of geometry from the physical world we inhabit, with the latter being only a poor and imperfect exemplar of the former. That would be a lapse into rationalism, and would sever the close tie with perception.

Kant took geometry to describe *space*, and so Euclidean figures are parts of space. We cannot see a Euclidean line, since it is too thin, but it is a part of space nonetheless. Perceived objects exist in space and we only understand perception to the extent that we understand space. Geometry studies the forms of perception in the sense that it describes the infinite space that conditions perceived objects. This Euclidean space is the background for perception, and so it provides the forms of perception or, in Kantian terms, the a priori form of empirical intuition. The way we learn about space a priori is by performing constructions in pure intuition, and proving things about the results.

What is the relation between geometric figures and their drawn counterparts? No one can deny that drawn lines only approximate Euclidean lines. However, Kant refers to drawn figures, and 'empirical intuition' as part of *geometric* demonstrations, following Euclid. What, then, is the role of drawn figures in Euclidean demonstration? One account, perhaps, is that lines drawn (and grasped via empirical intuition) aid the mathematician in focusing on corresponding Euclidean lines. Constructions on the drawn figures correspond to mentally apprehended constructions in Euclidean space. Surely Kant did not think that it is necessary to actually draw a figure on paper in order to grasp a Euclidean demonstration. With some practice, one follows the text of a demonstration directly—via the mind's eye—without consulting the diagram. Similarly, Kant surely did not hold that we have to *look at* a group of five objects (such as 'our five fingers or . . . five dots') in order to calculate $7+5$. We can count mentally. In sum, drawn figures or diagrams—in empirical intuition—aid the mind in focusing on the a priori forms of perception.

It is widely agreed that Kant's philosophy of mathematics faltered on later developments in science and mathematics. The most common example cited is the rise and acceptance of non-Euclidean geometry, and its application to physics. Kant held that the parallel postulate is an a priori, necessary truth. So it could not be false, and yet, according to contemporary physics—an empirical theory—space-time is best understood as non-Euclidean. There is disagreement among scholars as to whether Kant could have allowed non-Euclidean geometry any legitimate status. Some argue that he envisioned only one kind of necessity, and thus he could have made no distinction between pure and applied geometry. If these scholars are correct, then for Kant non-Euclidean geometry is a non-starter. However, others attribute to Kant a distinction between conceptual possibility and what may be called 'intuitive' possibility. A proposition, or theory, is conceptually possible if analysis of the relevant concepts does not reveal a contradiction. Kant does allow that certain thoughts that conflict with Euclidean geometry are coherent, because the thoughts do not involve a contradiction. He mentions a plane figure enclosed by two straight lines. Since Euclidean geometry is synthetic, non-Euclidean geometry is conceptually possible.[9] Of course, non-Euclidean geometry is not *intuitively* possible, since Euclidean geometry is necessarily true.

On this interpretation of non-Euclidean geometry, a Kantian would have to allow a conceptually possible non-standard

[9] Some interpreters have Kant holding that one cannot express the *concepts* of geometry without appealing to construction in intuition—and this construction is Euclidean. So even non-Euclidean geometry presupposes the necessity of Euclidean geometry.

arithmetic—what we might call a 'non-Peano arithmetic'. For Kant, $7+5=12$ is synthetic, and so $7+5=10$ and $7+5=13$ are conceptually possible. But could we have a coherent 'pure' mathematics in which one (or even both) of these are true?

Even if the manoeuvre in question accords non-Euclidean geometry some legitimate status, perhaps as pure mathematics, it does not accommodate its use in physics. Kant wrote that (Euclidean) geometry enjoys 'objective validity only through empirical intuition, whose . . . form the pure intuition is'. Were it not for the connection to intuition, geometry would have 'no objective validity whatever, but [be] mere play . . . by the imagination or by the understanding' (*Critique of Pure Reason*, B298). Since non-Euclidean geometry presumably does sever Kant's tie with intuition, it *is* mere play. It follows from Kant's view that we know a priori that non-Euclidean geometry *cannot* be applied in physics.

A better response, perhaps, would be for a Kantian to withdraw the thesis that the parallel postulate is synthetic a priori. This special status is accorded only to those propositions that are common to Euclidean and some of the non-Euclidean geometries (i.e. all of Euclid's postulates except the fifth). Perhaps it is not a deeply entrenched part of Kant's philosophy that *Euclidean* geometry is synthetic a priori. What matters is that *geometry* is synthetic a priori, and in Kant's day the geometry was Euclidean. To avoid being embarrassed twice, our Kantian might remain on guard for future developments in physics that negate one of the other postulates or axioms. However, it is curious that a Kantian would change his views about what is knowable *a priori* in response to developments in an empirical enterprise like physics. As we saw in chapter 1, §3, a naturalist should expect to modify her philosophical views in light of developments in science and mathematics. Philosophy is a holistic enterprise. But Kant was no naturalist. He fits the mould of the school I call 'philosophy-first' in chapter 1, §2. Kant took himself to be delimiting the a priori *presuppositions* of experience, and of empirical science. The fact that physics did not conform to the strictures is deeply problematic, unless the Kantian is prepared to reject the developments in physics out of hand. Is it coherent to modify one's views about what is a priori in response to empirical science?

Other developments in mathematics also proved problematic for Kantians. For example, the important distinctions between continuity and differentiability and between uniform and pointwise continuity seem to have no basis in intuition. How do these distinctions relate to the forms of perception? Other branches of pure and applied mathematics go further in severing the tie with intuition. How can we relate complex analysis, higher-dimensional geometry, functional analysis, and set theory to the forms of perception? Many of these branches of mathematics have found application in the sciences. Indeed, many were developed in response to the needs of the sciences. Of course, Kant is not to be faulted for this since most of the developments in question came after his lifetime, but he did regard his views as providing the limits for all future science. A contemporary Kantian has a tough row to hoe.

3. Mill

Despite Kant's considerable influence, many philosophers found, and continue to find, his notion of intuition—and the concomitant thesis of synthetic a priori truth—troublesome. According to Alberto Coffa (1991), a main item on the agenda of philosophy throughout the nineteenth century was to account for the prima facie necessity and a priori nature of mathematics and logic without invoking Kantian intuition. Can we understand mathematics and logic independent of the forms of spatial and temporal intuition? From an overall empiricist perspective, there are two alternatives to the Kantian view that mathematics is synthetic a priori. One can either understand mathematics as analytic, or else understand it as empirical, and thus a posteriori. The next chapter concerns logicists, who took the former route. Some versions of formalism can also be construed as a defence of the analyticity of mathematics (see ch. 6). We now consider a radical empiricist, John Stuart Mill, who took the latter route, arguing that mathematics is empirical. He is a precursor to some influential, contemporary empiricist accounts of mathematics (see ch. 8, §2).

As we saw, philosophers like Kant took themselves to be exploring the preconditions and limits of human thought and experience via methods that are independent of, and prior to, the natural sciences. They held that we need philosophy to determine the basic foundation and a priori limits of all empirical inquiry. Kant took himself to be uncovering the framework of empirical knowledge, to which our perceptions must conform. Philip Kitcher (1998) calls views like this *transcendentalism*, since they see philosophy as transcending the natural sciences. They are varieties of the view I call 'philosophy-first' in chapter 1, §2, entailing that, conceptually, philosophy comes before just about everything else—certainly before science in some foundational ordering. In Kant's view, philosophy reveals the presuppositions of empirical science.

The view now called *naturalism* opposes this foundationalism. Naturalists see human beings as entirely part of the causal order studied in science. There are no sources of philosophical knowledge that stand independent of, and prior to, the natural sciences. Willard Van Orman Quine (1981: 72) characterizes naturalism as 'the abandonment of first philosophy' and 'the recognition that it is within science itself . . . that reality is to be identified and described' (see also Quine 1969). Any epistemic faculty that the philosopher invokes must be amenable to ordinary, scientific scrutiny. Epistemology blends into cognitive psychology.[10]

Mill was one of the most consistent naturalists in the history of philosophy. Against the Kantians, he held that the human mind is thoroughly a part of nature, and thus that no significant knowledge of the world can be a priori. He developed an epistemology on that radical empiricist basis.

Mill's distinction between 'verbal' and 'real' propositions seems to be modelled after Kant's analytic–synthetic dichotomy or, better, Hume's distinction between 'relations of ideas' and 'matters of fact'. For Mill, verbal propositions are true by definition. They have no genuine content, and do not say anything about the world. Mill differs from Kant and from some other empiricists, such as Hume before him and Rudolf Carnap after, in holding that the propositions of mathematics—and most of logic—are *real* and thus synthetic and empirical. In Hume's terms, for Mill mathematics and logic concern matters of fact.

Unlike earlier and later empiricists, the fundamental epistemological inference for Mill is *enumerative induction*. We see many black crows and none of any other colour, and conclude that all crows are black and that the next crow we see will be black. All (real) knowledge of the world is *indirectly* traced back to generaliza-

[10] See chapter 1, §3 for a brief account of naturalism in the philosophy of mathematics. Maddy 1997 is a thorough treatment.

tions on observation. Mill's overall epistemology is sophisticated, and includes his famous principles of experimental enquiry in science. The epistemic connection between scientific laws and generalizations from experience is rather circuitous. However, Mill's epistemology for mathematics and logic is not as sophisticated. He held that the laws of mathematics and logic can be traced directly to enumerative induction—inferences from observation via generalizations on what is observed.

In at least one place, Mill suggests that generalizations add nothing to the force of arguments, since all important inference is from 'particulars to particulars'. Universal propositions, like 'all crows are black', are just summary records of what we have observed and what we expect to observe. For Mill, typical mathematical propositions are generalizations, and so these propositions also record and summarize experience. Mill's philosophy of mathematics is designed to show just what mathematical propositions are, in order to bring them in line with this general epistemological theme.

Let us begin with geometry. Mill rejects the existence of abstract objects, and he seeks to found geometry on observation. Thus, like Aristotle, he must account for the obvious sense in which the objects studied in geometry are not like anything we observe in the physical world. Every line we see has breadth and is not perfectly straight. Mill's writing on this is not clear, but a general outline can be made out. He held that geometric objects are approximations of actual drawn figures. Geometry concerns idealizations of possibilities of construction. The two central notions here are 'idealization' and 'possibility'. How does this unrelenting empiricist understand these concepts?

Mill takes lines without breadth and points without length to be *limit concepts*. A given line on paper may be more or less thin, depending on the quality of the ink, the sharpness of the pencil, or the resolution of the printer. We can think of the lines of geometry as the limit approached as we draw thinner and thinner, and straighter and straighter lines. Similarly, a point is the limit approached as we draw thinner and shorter line segments, and a circle is the limit approached as we draw thinner and more perfect circles.[11] Physically, of course, there are no such limits, and Mill

[11] Notice the similarity between Mill's limit concepts and the way limits, such as derivatives and integrals, are defined in contemporary analysis.

holds that geometry does not deal with existing objects. So, strictly speaking, Euclidean geometry is a work of fiction. The postulated figures are 'feigned proxies'. However, since geometric figures approximate drawn figures and natural objects, geometric propositions are true (of nature) to the extent that real figures and objects approximate the idealizations. If we measure the angles of a drawn triangle, we will find the sum to be *about* two right angles. The more straight and thin the lines of the drawn triangle are, the closer their angles become to two right angles. If we carefully draw a triangle, we will see that the three perpendicular bisectors intersect each other. If we are sloppy (but not too sloppy), we will see that the bisectors almost intersect each other. In this sense, the propositions of geometry are inductive generalizations about possible physical figures in physical space. They have been confirmed by long-standing experience.

One can question the notion of possibility that Mill invokes in his account of geometry. To focus on an example, what are we to make of the Euclidean postulate that between any two points one can draw a straight line? If this means that we can draw a breadthless line, then the postulate is not even approximately true. Indeed, we cannot even conceive of drawing a breadthless line. What instrument would we use? The talk of limits suggests that the postulate might mean that if we are given any two physical points A, B, no matter how small, and if we are given any degree of thickness d, we can draw a straight line between A and B no thicker than d. This is not much better, since we cannot regard this limit-statement as a well-established generalization from experience. How much experience do we have with really thin lines? Is the generalization even true? As far as we know, there is a lower limit to the thickness of a line we can draw and perceive. Can we draw a line thinner than the diameter of a hydrogen atom? With what material? Understood in such stark physical terms, the limit version of the Euclidean postulate is certainly false. Similarly, the theorem that every line has a perpendicular bisector is physically false, even allowing Mill's idealizations. Suppose we start with a given line segment, say two centimetres long, and bisect it. Then bisect the left half, and then bisect the left half of that, continuing as long as possible. It is simply not possible to continue this thirty times. The thirtieth line segment would have a sub-atomic length.

So in what sense is it possible to draw a line between two points or to bisect any line segment? Perhaps Mill took geometry to concern a hypothetically improved experience, in which our acuity is maximally sharp. Or perhaps a Millian could interpret the geometric axioms in terms of some distinctive *mathematical* possibility, rather than the physical possibility invoked above. The underlying thesis is that it is consistent with the mathematical laws of space, if not the physical laws of the universe, that there is no limit to the thinness of lines and no limit to the line segments that can be bisected. However, it is hard to see how Mill has the resources to make out either the hypothetically improved super-acuity or the distinctive mathematical possibilities. Remember that, for Mill, all mathematical knowledge is based on inductive generalizations from experience. So where would we learn about super-sharp acuity and mathematical possibilities?

Turning now to arithmetic, Mill agrees with Plato and Aristotle that natural numbers are numbers *of* collections. He sides with Aristotle in rejecting ideal 'units' and so, for Mill, numbers are numbers of ordinary objects:

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. But though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning all things whatsoever, all objects, all existences of every kind, known to our experience. (Mill 1973: 254–5)

Thus Mill does not take a numeral to be a singular term that denotes a single object. Rather, numerals are general terms, like 'dog' or 'red'. They do not range over individual objects, but over aggregates of objects: 'Two, for instance, denotes all pairs of things, and twelve all dozens of things' (1973: 610).

What of arithmetic propositions? Mill is concerned to give an account of sums, like '5 + 2 = 7' and '165 + 432 = 597'. He says that there are only two axioms, namely, 'things which are equal to the same thing are equal to one another' and 'equals added to equals make equal sums' (1973: 610) and a definition scheme, one for each numeral which denotes the number 'formed by the addition of a unit to the number next below it'. From this, he gives a derivation of '5 + 2 = 7'. It is clear how to extend the procedure to derive any correct sum.[12]

The distinctive feature here is that, for Mill, these sums are *real*, not verbal, propositions about physical aggregates and their structural properties. Since they are real, they must ultimately be known by enumerative induction, generalization on experience. Our almost uniform experience with collecting and separating objects confirms the arithmetic sums. In an infamous passage, Mill wrote that the sum '2 + 1 = 3' involves the assumption 'that collections of objects exist, which while they impress the senses thus, °o°, may be separated into two parts, thus, o o o' (1973: 257).

Frege's *Foundations of Arithmetic* contains a sustained, bitter assault on Mill's account of arithmetic:

> What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and 2 + 1 would not be 3! What a pity that Mill did not also illustrate the physical facts underlying the numbers 0 and 1! . . . From this we can see that it is really incorrect to speak of three strokes when the clock strikes three, or to call sweet, sour, and bitter three sensations of taste . . . For none of these impresses the senses thus, °o°. (Frege 1884: §7)

Frege thus takes Mill's talk of 'arranging' in a starkly physical sense: 'Must we literally hold a rally of all the blind in Germany before we can attach any sense to the expression "the number of blind in Germany"?' (§23).

Frege's criticism is unfair. As we saw above, Mill himself mentions numbers of things which cannot be physically arranged into the vertices of a triangle. He speaks of sounds and heartbeats. So Mill must have had something more general in mind. Collecting and separating small collections of objects is *one typical instance* of the generalizations of arithmetic sums. We do mentally 'collect' and 'separate' heartbeats and clock chimes, not to mention continents and planets, even if they do not impress the senses thus °o° and cannot be physically separated thus o o o. We also collect one and even zero objects of a certain kind, when we consider how many white kings are on a chessboard or how many female US Presidents were inaugurated before 1999.

[12] As noted by Frege in another context, Mill's derivation of the sums makes essential use of the associative law.

Nevertheless, Frege is correct that the serious burden on the empiricist is to make sense of the terms 'collecting' and 'separating'. Exactly what experience is involved in the proposition that two heartbeats plus one heartbeat makes three heartbeats, or that two planets plus three planets makes five planets?

Frege also questions Mill's idea that numbers denote physical aggregates. If we think of an aggregate as a physical hunk of stuff, we will not be able to attach a number to it: 'If I place a pile of playing cards in [someone's] hands, with the words: Find the number of these, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even say of points in a game of skat. To have given him the pile in his hands is not yet to have given him completely the object he is to investigate; I must add some further word—cards, or packs, or points'. (1884: §22). In the next section, Frege wrote that a 'bundle of straw can be separated into parts by cutting all the straws in half, or by splitting it up into single straws, or by dividing it into two bundles'. He adds that 'the number word "one" . . . in the expression "one straw" signally fails to do justice to the way in which the straw is made up of cells or molecules'.

Mill (1973: 611) himself has the response to Frege here: 'When we call a collection of objects *two*, *three*, or *four*, they are not two, three, or four in the abstract; they are two, three, or four things of some particular kind; pebbles, horses, inches, pounds weight. What the name of the number connotes is, the manner in which single objects of the given kind must be put together, in order to produce that particular aggregate.' For Mill, then, an aggregate is identified with the physical hunk of stuff *together with* the units in which it is to be divided (and thus counted). The one pack of cards is not the same physical aggregate as the fifty-two individual cards, the four suits, and so on. The aggregates are located in the same place at the same time, but they are different aggregates nonetheless. Similarly, the aggregate of Frege's bundle of straw is not the same as the aggregate of the bundle of half-straws or the bundle of two half-bundles or the bundle of molecules. Although Mill rejects the existence of abstract objects, and thus holds that aggregates are material, his ontology is not as austere as one might think.

Once again, however, the burden is on the empiricist to make out this ontological category and show how it is grounded in experience. Penelope Maddy (1990: ch. 2, §2) suggests that there is a difference between seeing, say, four shoes and seeing them as two pairs. Perhaps something like this would help the Millian here (see also Burge 1977).

Frege also takes Mill to task concerning large numbers. Do we have experience of an aggregate of size 1,234,457,890, and can we distinguish it from an aggregate of size 1,234,457,891? What is the experience generalized by $1{,}234{,}457{,}890 + 6{,}792 = 1{,}234{,}464{,}682$? We can extend Frege's point, by asking how we would confirm a medium-sized sum, like $1{,}256 + 2{,}781 = 4{,}037$. Suppose we took a random sample of adults and gave each of them a pile of 1,256 marbles and a pile of 2,781 marbles and asked him to collect the two piles into one big aggregate and determine its number. Human attention being what it is, very few (if any) of our subjects would produce 4,037 as the final number. On Mill's view, do we have to regard this outcome as a disconfirmation of the sum? Suppose we used rabbits instead of marbles, and it took several months to compete the experiment? Suppose we use gallons of two liquids, where a chemical reaction or evaporation might change the volume of the aggregate? We would not get the correct results and would have to declare the sum disconfirmed. Prima facie, it seems absurd even to attempt this experiment to confirm arithmetic sums. We know what the correct sum is *before* we start the experiment. We might use the results to determine the competence of the subjects in adding and counting.

Along similar lines, Mill holds that each numeral represents collections the size of the corresponding number. This entails that there are, or could be, infinitely many objects. Do we have empirical support for this? What if we adopted a physical theory that entails that there are only finitely many physical objects. Would this disconfirm arithmetic?

The situation here is similar to the mismatch between the propositions of geometry and statements about ordinary objects. Our limited experience does not exactly match the mathematical propositions. As with geometry, a Millian might respond with talk of idealization, possibility, and approximation. The mathematical propositions—especially the definitions of the numerals—do not exactly conform to experience. They concern *possible* experience, under idealized conditions in which our attention-span is improved and any differences and interactions between the units (that might change the number over time) are ignored. Experience confirms that arithmetic propositions are approximately true of experience. However, once again, the burden is on the Millian to make out this notion of possibility.

Another dimension of Mill's view, implicit in what we have seen already, is that he has moved considerably away from the received view of mathematics as highly (if not absolutely) certain and necessary. According to Mill, many mathematical propositions are not even true at all, let alone necessarily true and indubitable, and let alone a priori knowable. Mill takes seriously the problem of showing why the received view is so compelling. He asks: 'Why are mathematical certainty, and the evidence of demonstration, common phrases to express the very highest degree of assurance attainable by reason? Why are mathematics by almost all philosophers . . . considered to be independent of the evidence of experience and observation, and characterized as systems of Necessary Truth?' (Mill 1973: 224). Mill held that arithmetic *appears* to be necessary and a priori knowable because the axioms and definitions are 'known to us by early and constant experience' (1973: 256). The basic truths of arithmetic, such as the simple sums, have been confirmed from the time we began to interact with the world. This does not make them genuinely a priori. Mill agrees that simple arithmetic sums are necessary, but only in the sense that we cannot imagine things to be otherwise (the aforementioned idealizations notwithstanding). Thus, for example, we cannot imagine that a collection of objects exists, which, while they impress the senses thus, °o°, may be separated into two parts, thus, ∘∘ ∘∘, or at least not without changing the objects in some way.[13]

Mill agrees with the Kantians that the ultimate source of confidence in the axioms of arithmetic and geometry lies in the limits of what we can perceive. The axioms of mathematical theories are chosen by reflection on how we perceive the structure of the

[13] Mill's resolution of the apparent necessity and a priority of mathematics is similar to Hume's thesis about causality and 'necessary connection'. Hume suggested that our belief that one thing causes another is based on our constant experience of the two things together, to the point that when we see one of them we expect the other. See Yablo 1993 for an insightful discussion of the extent to which conceivability is a reliable guide to possibility. Some of the results of modern physics indicate that perhaps the universe operates in ways that we find inconceivable. Does this provide empirical *disconfirmation* of the reliability of spatial and temporal intuition? If in the end we cannot rely on 'early and constant experience', then what can a Millian rely on?

world. Of course, Mill agrees that these insights into perceptual intuition are reliable, in that we are not led astray by following them and assuming, for example, that the world is Euclidean and that aggregates conform to arithmetic. But he insists that the reliability of perceptual intuition concerning actual geometric and arithmetic properties of physical objects is an *empirical* matter. That is, we discover *by experience* that perceptual intuition is reliable. By self-observation we see that we cannot perceive the world in any other way and that observation continues to conform to Euclidean and arithmetic forms.

Given the paltry epistemological basis of enumerative induction, it is interesting that Mill takes his unrelenting empiricism as far as he does, presenting sophisticated philosophical accounts of Euclidean geometry and basic arithmetic. However, his philosophy of mathematics does not go very far. Mill only deals with geometry, arithmetic, and some algebra, not the branches of higher mathematics. This shortcoming is understandable in Aristotle, of course, but not so easily here, given the importance of higher mathematics in the developing sciences of Mill's day.

Even Mill's accounts of arithmetic and geometry are severely limited in their scope. His philosophy of arithmetic captures little more than simple sums and differences, what is learned in elementary school. The (perhaps ill-named) principle of *mathematical induction* is the thesis that for any property P, if P holds of 0 and if, for every natural number n, if P holds of n then P holds of $n + 1$, then P holds of all natural numbers. In symbols:

$$(P0 \ \&\ \forall x((Nx \ \&\ Px) \rightarrow Px + 1)) \rightarrow \forall x(Nx \rightarrow Px).$$

The principle of mathematical induction is a central theme of axiomatic arithmetic. It is hard to shed much light on the natural numbers without it. As far as I can tell, *enumerative* induction—generalizations from experience—provides no support for mathematical induction. What 'early and constant experience' confirms mathematical induction? Mill might respond that we cannot imagine mathematical induction to be false, and he might invoke the empirical reliability of this faculty of imagination. However, it is hard to see how mathematical induction directly bears on experience. What sorts of experience does it describe?

At this point our Millian might attempt the Euclidean manoeuvre of founding arithmetic on geometry (although this would detract from the universal applicability of arithmetic). A geometric analogue of mathematical induction is the *Archimedean* principle, that for any two line segments *a*, *b*, there is a natural number *n*, such that the *n*-fold multiplication of *a* is longer than *b*. A Millian can point out that this principle is confirmed by early and constant experience (so long as we speak of mathematical possibility and not physical possibility). A counterexample to the Archimedean principle would be a pair of line segments, one of which is infinitesimally smaller than the other. Surely, we have no direct experience with infinitesimals, even if we manage to imagine them.

Even if our Millian can relate the principle of mathematical induction to the Archimedean principle, this is not much of a straw to grasp. The completeness axiom would be a further stumbling-block. In real analysis, the principle states that every bounded set of real numbers has a least upper bound. An analogue in geometry is the Bolzano–Weierstrass property that every bounded, infinite set of points has a limit point. Since we have no experience with infinite sets of points or objects, there seems to be no basis for these principles in enumerative induction.

Let us take stock. We have noted some tough criticisms of the various notions of 'possibility' needed to sustain Mill's account of mathematics. Although these might be overcome, it seems that the burden is a difficult one. Second, and more important, Mill's decision to base all of mathematics and logic on enumerative induction is untenable. For reasons like those outlined here, contemporary empiricists do not attempt to defend Mill on these matters. Nevertheless, the main thrust of Mill's empiricism is alive today, and perhaps even well. A dedicated core of philosophers accept and defend the 'radical' aspect of Mill's empiricism, the view that logic and mathematics contain 'synthetic' or 'real' propositions and that, contra Kant, these propositions are known a posteriori, ultimately empirically.

Kitcher (1983, 1998) provides a subtle and sophisticated account of higher mathematics in a roughly Millian framework. Like Mill, Kitcher takes mathematics to relate to human abilities to construct and collect, but he is more explicit than Mill about the idealizations involved. Instead of speaking of the collecting and constructing activities of actual humans, Kitcher speaks of the activities of fictitious ideal constructors who do not share human limitations of time, space, attention-span, or even lifetime. The ideal constructors draw breadthless lines and they collect large aggregates. For example, the axioms of mathematical induction and the Bolzano–Weierstrass property represent statements of abilities allotted to the ideal constructors, corresponding to arithmetic and real analysis. These constructors deal with infinite sets of line segments and take limit points and least upper bounds of them. For Kitcher, mathematical truth—propositions about the ideal constructors—relates to truths about human abilities via more or less straightforward idealization and approximation. In the more advanced branches, such as set theory, the idealizations are very ideal indeed. For every infinite cardinal number κ, the ideal constructor can make a collection of size κ. Nevertheless, the connection with actual human construction is not forgotten.

Of course, unlike Mill, Kitcher does not rely solely on enumerative induction to ground mathematics and logic. The moves allotted to the ideal constructor are justified on the basis of the utility of the theory in the overall scientific enterprise. Kitcher is still a radical empiricist, in that the overarching goal of the entire scientific enterprise—mathematics included—is to account for experience. He joins Mill in rejecting the received view that mathematics is a priori knowable. Kitcher argues that we need experience to determine just which idealizations are useful in predicting experience and controlling the environment. Mathematics is not incorrigible, since we have to keep open the possibility of radically different idealizations, and thus radically different mathematics. In chapter 8, §2 below we consider another unrelenting empiricist, Quine, who maintains a hypothetical-deductive epistemology for all of mathematics and science, but further departs from Mill in not taking mathematics to be about real or ideal constructive activity. In the meantime, we turn to other views of mathematics, including a less radical empiricism (ch. 5, §3).

4. Further Reading

See Coffa 1991: ch. 1, and the papers in Posy 1992 for an excellent start on the wealth of scholarship on Kant's philosophy of mathematics (especially Posy's introduction). The anthology contains the above cited papers Parsons 1969, 1984, Friedman 1985, Hintikka

1967, and Posy 1984, as well as a wealth of other influential and insightful work. See also Friedman 1992. Mill's own *A System of Logic* (1973) is a readable account of his views on mathematics. The definitive secondary source is Skorupski 1989: ch. 5. See also the papers in Skorupski 1998, especially Skorupski 1998a and Kitcher 1998.