Deriving the vis viva equation

(1) Energy $$E = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

(2) $$\frac{1}{2}mv_a^2 - \frac{1}{2}mv_p^2 = \frac{GMm}{r_a} - \frac{GMm}{r_p}$$

(3) Angular Momentum $$v_p = \frac{r_a}{r_p}v_a$$

Divide (2) by *m* and substitute (3) in for v_p

$$\frac{1}{2}\left(1 - \frac{r_a^2}{r_p^2}\right)v_a^2 = \frac{GM}{r_a} - \frac{GM}{r_p} = \frac{1}{2}\left(\frac{r_p^2 - r_a^2}{r_p^2}\right)v_a^2$$
$$\frac{1}{2}v_a^2 = GM\left(\frac{r_p - r_a}{r_a r_p}\right)\frac{r_p^2}{r_p^2 - r_a^2}$$

Cancelling a factor of r_p and a factor of $(r_p - r_a)$
$$\frac{1}{2}v_a^2 = GM\frac{r_p}{r_a(r_p + r_a)}$$

But the geometry of an ellipse is such that the length of the major axis ($2a$) is the sum of $r_p + r_a$
$$r_p = 2a - r_a$$
$$\frac{1}{2}v_a^2 = GM\frac{2a - r_a}{r_a 2a} = GM\left(\frac{1}{r_a} - \frac{1}{2a}\right)$$

(1) above can be written again with this velocity v_a
$$E = m\left[GM\left(\frac{1}{r_a} - \frac{1}{2a}\right)\right] - \frac{GMm}{r_a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Removing the $1/r_a$ terms (that now cancel each other) and rearranging (multiplying by 2 and dividing out *m*) one gets the final result:
$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$