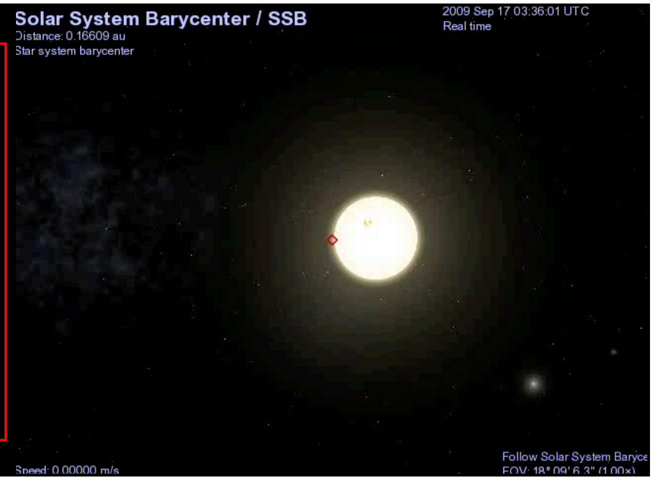
PHYS 275 Planets

Fall 2017

Lecture 3 Gravity and Orbits



Film/Lecture on Wed 20 Sept

"Wonders of Life: Home"

50 min. BBC (2013)

Tells the story of how Earth became the rich, life-supporting biosphere that it is today. Discover how rock accreted the ingredients necessary for life - oxygen and water - which over time enabled life to evolve.

Professor Fich is away on this date – please hand in assignments to box outside of room PHY211

Today's Class

Complex motions with just a bit of simple physics!

1. Background review of F=ma ½ mv² mv²/r conservation of angular momentum and circular motion with gravity
   - Centripetal acceleration
   - Frame of reference
2. Newton's Law of gravity
3. Elliptical motion

Review of Motion

Force, mass, acceleration $F = ma$ $a = \frac{F}{m}$

Constant acceleration $v = v_0 + at$ $x = x_0 + v_0 t + \frac{1}{2}at^2$

$v^2 = v_0^2 + 2a(x - x_0)$ $x - x_0 = \frac{1}{2}(v_0 + v)t$

Example: constant gravity $F = -mg$ $y = y_0 + v_0 t - \frac{1}{2}gt^2$

Sample problem: ball is thrown upwards at 15.0 m/s from a height of 2.0 meters. How long is it in the air (i.e. before hitting the ground)?

$0 = 2.0 + 15.0\,t - \frac{1}{2}\times 9.8t^2$

$t = -0.13 \text{ or } +3.2 \text{ seconds}$

Frame of Reference and Centre of Mass

Linear momentum $p = mv$ $\vec{F} = \frac{d\vec{p}}{dt}$

Centre of Mass $x_{CoM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

Motions (collisions/interactions) in the Centre of Mass frame do not affect the velocity of the CoM.
The CoM frame of reference is a useful one for motions where there are no outside influences....

Calculus and motion

Instantaneous velocity and acceleration

$$v = \frac{dx}{dt} \quad \vec{v} = \frac{d\vec{r}}{dt} \quad \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} \quad \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

Work and Kinetic Energy $K = \frac{1}{2}mv^2$ $\Delta K = K_f - K_i = W = \vec{F} \cdot \vec{d}$

$$\Delta K = W = \int \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x\, dx + \int_{y_i}^{y_f} F_y\, dy + \int_{z_i}^{z_f} F_z\, dz$$

Potential Energy

$$\Delta U = U_f - U_i = -W = -\int_{x_i}^{x_f} F(x)dx$$

For example, when gravity is constant: $U_f - U_i = -\int_{y_i}^{y_f} -mg dy$

$$U_f - U_i = mg(y_f - y_i)$$

NOTE: the potential increases as y increases. And there is no zeropoint for the potential. We could choose $U_i = U(y_i) = 0$
If we also choose $y_i = 0$ then $U = mgy$

Total Energy

If there are no external forces and the internal forces are all conservative (e.g. no friction) then there is a simple relationship between the potential and kinetic energy:

$$\Delta K = W = -\Delta U$$

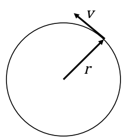
$$K_f - K_i = -(U_f - U_i)$$

$$K_f + U_f = K_i + U_i = E_{mec}$$

The total (mechanical) energy is a constant (does not change)

Uniform Circular Motion

"Uniform" means "at constant speed"



Period of motion

$$T = \frac{2\pi r}{v}$$

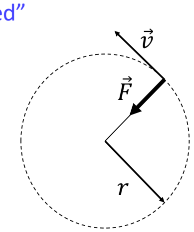
Centripetal Force

$$F_{cent} = m\frac{v^2}{r}$$

Angular momentum

$$\vec{L} = m\vec{r}\times\vec{v}$$

$$L = mrv_\theta = mr\left(r\frac{d\theta}{dt}\right) = \mathrm{m}r^2\frac{d\theta}{dt}$$



Earth's Orbit and Gravity

- The Earth takes one year to travel once around the Sun in its almost circular orbit.
- The distance from the Earth to the Sun is 1.0 Astronomical Units (A.U.)
  - An A.U. is precisely 149597870700 meters ≈ 1.50 × 10^{11} meters

How fast is the Earth moving?
How much force does this motion require?
What is the mass of the Sun?

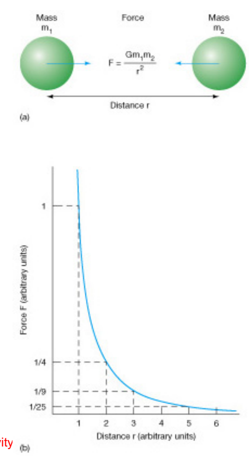
$$365.25\times24\times3600\ seconds = \frac{2\pi\times1.496\times10^{11}\ meters}{v}$$

$$v = 2.979\times10^4\ {}^{m}/_{s} = 29.79\ km/s$$

universal law of gravity: describes the forces acting between bodies due to gravity

$$\boxed{F = \frac{Gm_1m_2}{r^2}}$$

$$G = 6.67408\times10^{-11}\quad m^3kg^{-1}s^{-2}$$



Earth's Orbit and Gravity

How much force does this motion require? Mass$_{Earth}$ = 5.972 × 10^{24} kg
What is the mass of the Sun?

$$F = m\frac{v^2}{r} = 5.972\times10^{24}\frac{(2.979\times10^4)^2}{1.496\times10^{11}} = 3.543\times10^{22}N$$

$$F = 3.543\times10^{22} = \frac{GM_{Sun}M_{Earth}}{(1.496\times10^{11})^2} = \frac{6.67408\times10^{-11}\ M_{Sun}5.972\times10^{24}}{(1.496\times10^{11})^2}$$

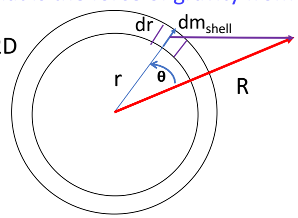
$$M_{Sun} = 1.989\times10^{30}kg$$

~333,000 M_{Earth}

Force of gravity from an extended spherically symmetric object

What is the force of gravity from a thin shell? (constant density)

in 2D



If $R \geq r$ then $F = \frac{G m_{shell} m}{R^2}$

If $R < r$ then $F = 0$

The force from the shell "parts" cancel !

This also means that an extended spherical object acts as if all of the mass is at its centre!

Need to do full 3 dimensional integral

Integral of power law (reminder)

$y = x^n \qquad \frac{dy}{dx} = n x^{n-1}$

$y = \int x^n dx \qquad y = \frac{1}{n+1} x^{n+1} + constant$

Example: $Q = \int_1^3 \frac{1}{r^3} dr \qquad Q = \left[-\frac{1}{2r^2}\right]_1^3 = -\frac{1}{18} + \frac{1}{2} = \frac{4}{9}$

Gravitational Potential Energy

$$\Delta U = U_f - U_i = -W = -\int_{x_i}^{x_f} F(x) dx$$

But force of gravity only depends on distance *r* : $\vec{F} = -\frac{G m_1 m_2}{r^2} \hat{r}$

$$\Delta U = U_f - U_i = -\int F(r) dr = -\int_{r_i}^{r_f} -\frac{G m_1 m_2}{r^2} dr = -G m_1 m_2 \left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

It is common to choose zero potential at *r*=∞ $\qquad U(r) = -\frac{G m_1 m_2}{r}$

Total (mechanical) energy with gravity varying with position – two objects (ie orbit)

$$K_f + U_f = K_i + U_i = E_{mec}$$

$$\frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2 - \frac{G m_1 m_2}{r_f} = \frac{1}{2} m_1 v_{1,i}^2 + \frac{1}{2} m_2 v_{2,i}^2 - \frac{G m_1 m_2}{r_i}$$

Complicated! There must be a way to simplify things…
Where is the orbit in this?

Approximation: one very big and one very small object in CoM frame

From conservation of momentum we know that in any interaction between two objects the velocity of the big (mass) object is small.

$m_1 v_1 = -m_2 v_2$ If $m_1 \gg m_2$ then magnitude $v_2 \gg v_1$

If the mass difference is large enough then we can ignore the small velocity and the energy equations becomes:

$$\left[\frac{1}{2}m_2 v_{2,f}^2 - \frac{Gm_1 m_2}{r_f}\right] = \left[\frac{1}{2}m_2 v_{2,i}^2 - \frac{Gm_1 m_2}{r_i}\right] = \text{E}$$

Final energy = initial energy

Example of energy from gravity

An object starts at rest very far from the Sun and falls in towards the Sun. How fast is it moving when it is reaches a distance of 1.0 A.U. from the Sun?

SOLUTION: "very far from the Sun" as in... effectively at $r=\infty$ where the potential is zero.

$$\frac{1}{2}m_2 v_{2,f}^2 - \frac{Gm_1 m_2}{r_f} = \cancel{\frac{1}{2}m_2 v_{2,i}^2} - \cancel{\frac{Gm_1 m_2}{r_i}}$$

$$\frac{1}{2}m_2 v_{2,f}^2 - \frac{6.67\times10^{-11} 1.99\times10^{30} m_2}{1.496\times10^{11}} = 0 \qquad v_{2,f} = 42.1\ km/s$$

$$(= \sqrt{2}\times 29.79\ km/s)$$

Properties of an ellipse

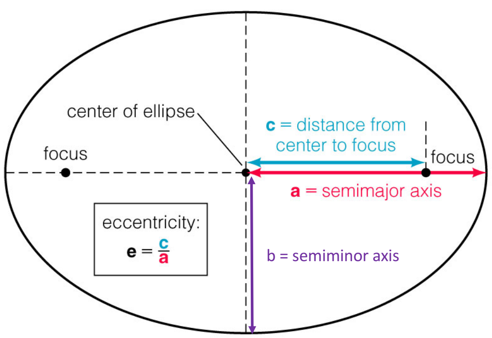
Semimajor axis is the average distance of a body from the orbit's focus

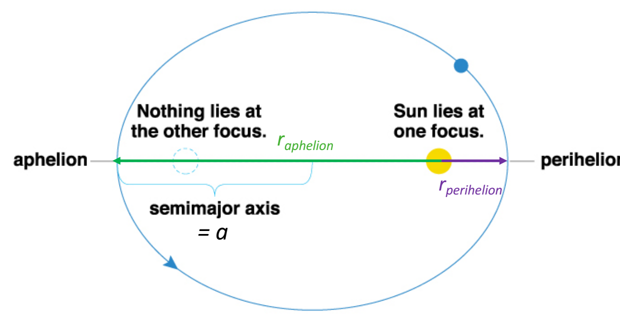
An orbit's focus is at the location of the centre of mass of the system

The distance from a focus to the ellipse is

a – c = a – ae = a(1 – e)

OR a + c = a + ae = a(1 + e)

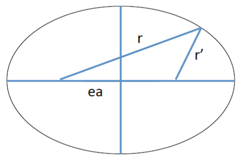




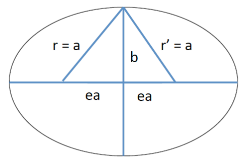
$$r_{perihelion} + r_{aphelion} = 2a$$

More properties of ellipses

r + r′ = 2a



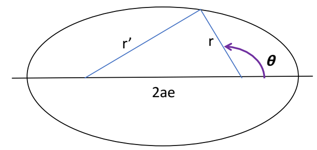
(ea)^2 + b^2 = a^2, or b^2 = (1-e^2)a^2

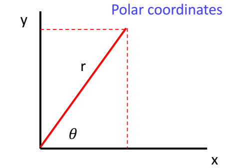


Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A polar coordinate version of the equation of an ellipse





Cosine Law

$$r'^2 = r^2 + 4a^2e^2 + 4rae\cos\theta$$

Previous slide: $r' = 2a - r$

Combine to get $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$

Elliptical Orbit ???

We are missing any consideration of angular momentum – it is conserved (always).

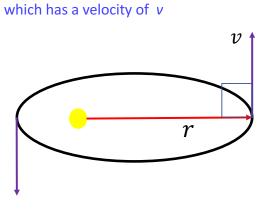
The total energy is $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$

Slight change of variables names… *M* for big mass, *m* for small mass which has a velocity of *v*

What is the angular momentum?

"Trick" – at perihelion and aphelion all of the velocity is directed perpendicular to the radius vector

$$L = mrv$$



Write energy and angular momentum at perihelion and aphelion

$$\mathrm{E} = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = \frac{1}{2}mv_p^2 - \frac{GMm}{r_p}$$

Angular momentum $L = mr_av_a = mr_pv_p \Rightarrow v_p = \frac{r_a}{r_p}v_a$

Substituting v_p into energy equation and using major axis length $2a=r_a+r_p$

Clearing up (lots of algebra) one gets: $E = -\frac{GMm}{2a}$

$$-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r} \Rightarrow \boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$ Vis viva equation

Vis viva examples

$$\boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

What is the speed of a circular orbit around the Sun at a distance of 1.0 A.U.?

$$v^2 = 6.67\times10^{-11}\times1.99\times10^{30}\left(\frac{2}{1.496\times10^{11}} - \frac{1}{1.496\times10^{11}}\right) \qquad v = 29.79\ km/s$$

What is the speed of an object in orbit around the Sun, at 1.0 A.U., that will escape from the Sun?

To escape the orbit must be infinite in size. e.g. $a = \infty$

$$v^2 = 6.67\times10^{-11}\times1.99\times10^{30}\left(\frac{2}{1.496\times10^{11}} - 0\right) \qquad v_{escape} = \sqrt{2}v_{circular}$$

$42.1\ km/s$

Another vis viva example

An object is in orbit around the Sun with its orbit perihelion at 0.40 A.U. and aphelion at 3.2 A.U. How fast is it moving at perihelion?

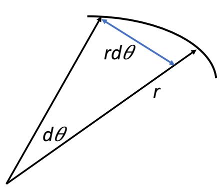
Semi-major axis given by: $r_a + r_p = 2a = 3.6\ A.U. \Rightarrow a = 1.8\ A.U.$

$$v^2 = 6.67\times10^{-11}\times1.99\times10^{30}\left(\frac{2}{0.4\times1.496\times10^{11}} - \frac{1}{1.8\times1.496\times10^{11}}\right)$$

$$v^2 = 3.94\times10^{9} \Rightarrow v = 62.8\ km/s$$

Where do Kepler's Second and Third Law come from?

Area in an ellipse = πab Angular momentum $L = mr^2\ d\theta/dt$



The area swept out by the radius vector is

$$dA = {}^{1}\!/_{2}\, r\times rd\theta = {}^{1}\!/_{2}\, r^2 d\theta$$

And the rate that the area is being swept out is

$$\frac{dA}{dt} = {}^{1}\!/_{2}\, r^2\frac{d\theta}{dt} = \frac{L}{2m}$$

A constant!!! Kepler's 2nd Law

If we integrate for the complete area/time to travel full ellipse: (where P is the Period) $\pi ab = \frac{L}{2m}P$

Kepler's 3rd Law (for m<<M)

At perihelion $v^2 = GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)$ $\qquad L = ma(1-e)v$

$$\underline{\pi ab = \pi a\times a(1-e^2)^{1/2}} = L\frac{P}{2m} = \underline{ma(1-e)\left[GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)\right]^{1/2}\frac{P}{2m}}$$

Cancel out m, square, bring together a $\qquad (1-e)^2 GM\left(\frac{2}{(1-e)} - 1\right)P^2 = 4\pi^2 a^3(1-e^2)$

Clean up the e terms (and they all cancel!!!) $\qquad P^2 = \frac{4\pi^2}{GM}a^3$

Kepler 3rd Law Orbits around the Sun

a (A.U.)	P (years)
0.7	0.59
1.0	1.0
3.0	5.2
30	165

Kepler's Law from Newton's Physics

Can find Kepler's Laws (2nd and 3rd) from Newton's Equations.

But we did have to assume an elliptical orbit…
And low mass planets…

What's the real story???

Also, we did not learn where a planet is at a given time
i.e. What is $\overrightarrow{r(t)}$? Or what is $\theta(t)$?

Next Lecture!