To Do

Read Sections 7.1 – 7.3.

Assignment 5 is due Monday December 5.

See detailed information regarding Tutorial Test 3 (Wednesday November 30) posted on Learn.

Last Class

(1) Comparison of Two Means, Unequal Variances

(2) Analysis of a Paired Experiment

(3) Pairing and Experimental Design

Today's Class

(1) Multinomial Likelihood Function

(2) Likelihood Ratio Goodness of Fit Test

(3) Pearson Goodness of Fit Test

Smarties Experiment

Is the distribution of colours equal in boxes of Smarties?

Data

Colour	Observed Frequency	Expected Frequency
Red	238	?
Green	210	?
Yellow	216	?
Blue	157	?
Purple	235	?
Brown	184	?
Orange	242	?
Pink	219	?
Total	1692	1692

Observed and Expected Frequencies

Colour	Observed Frequency	Expected Frequency
Red	238	1692/8 = 211.5
Green	210	211.5
Yellow	216	211.5
Blue	157	211.5
Purple	235	211.5
Brown	184	211.5
Orange	242	211.5
Pink	219	211.5
Total	1692	1692

What would you conclude based on comparing the observed and expected frequencies?

Questions of Interest

How do we test the hypothesis that the distribution of different colours in boxes of Smarties is equal?

We need a model!

Joint Distribution

Let Y_j = number of Smarties of colour j,

j=1,2,…,8

and

θ_j = P(randomly selected Smartie is of colour j),

j=1,2,…,8.

What is the joint distribution of $Y_1, Y_2, \ldots, Y_8$?

Joint Distribution

The joint distribution of $Y_1, Y_2, \ldots, Y_8$ is

$$P\left(Y_1 = y_1, Y_2 = y_2, \ldots, Y_8 = y_8\right)$$

$$= \frac{n!}{y_1! y_2! \cdots y_8!} \theta_1^{y_1} \theta_2^{y_2} \cdots \theta_8^{y_8}$$

$$0 < \theta_j < 1, \ \sum_{j=1}^{8} \theta_j = 1; \ y_j = 0, 1, \ldots \text{ and } \sum_{j=1}^{8} y_j = n$$

Multinomial Likelihood Function

The likelihood function based on the data $y_1, y_2, \ldots, y_8$ is

$$
\begin{aligned}
&L(\theta_1, \theta_2, \ldots, \theta_8) \\
&= P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_8 = y_8; \theta_1, \theta_2, \ldots, \theta_8) \\
&= \frac{n!}{y_1!\, y_2! \cdots y_8!} \theta_1^{y_1} \theta_2^{y_2} \cdots \theta_8^{y_8}
\end{aligned}
$$

$$0 < \theta_j < 1, \ \sum_{j=1}^{8} \theta_j = 1$$

Multinomial Likelihood Function

More simply (ignoring constants)

$$L(\theta_1, \theta_2, \ldots, \theta_8) = \theta_1^{y_1} \theta_2^{y_2} \cdots \theta_8^{y_8}$$

$$0 < \theta_j < 1, \ \sum_{j=1}^{8} \theta_j = 1$$

The hypothesis of interest is

$$H_0 : \theta = \left(\frac{1}{8}, \frac{1}{8}, \ldots, \frac{1}{8} \right)$$

Maximum Likelihood Estimates

For Multinomial data $y_1, y_2, \ldots, y_k$ the likelihood function is

$$L(\theta_1, \theta_2, \ldots, \theta_k) = \theta_1^{y_1} \theta_2^{y_2} \cdots \theta_k^{y_k} \quad 0 < \theta_j < 1, \ \sum_{j=1}^{k} \theta_j = 1$$

The maximum likelihood estimate of θ_j is

$$\hat{\theta}_j = \frac{y_j}{n}, \quad j = 1, 2, \ldots, k$$

and the maximum likelihood estimator of θ_j is

$$\tilde{\theta}_j = \frac{Y_j}{n}, \quad j = 1, 2, \ldots, k$$

Likelihood Ratio Test Statistic

For testing $$H_0 : \theta = \theta_0 = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k} \right)$$

we use the likelihood ratio test statistic

$$\Lambda(\theta_0) = -2 \log \left[\frac{L(\theta_0)}{L(\tilde{\theta})} \right]$$

where $$\tilde{\theta} = \left(\frac{Y_1}{n}, \frac{Y_2}{n}, \cdots, \frac{Y_k}{n} \right)$$

and $$\theta_0 = \left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k} \right)$$

Likelihood Ratio Test Statistic

$$\frac{L(\theta_0)}{L(\tilde{\theta})} = \left[\prod_{j=1}^{k}\left(\frac{1}{k}\right)^{Y_j}\right] \div \left[\prod_{j=1}^{k}\left(\frac{Y_j}{n}\right)^{Y_j}\right]$$

$$= \prod_{j=1}^{k}\left(\frac{n/k}{Y_j}\right)^{Y_j} = \prod_{j=1}^{k}\left(\frac{E_j}{Y_j}\right)^{Y_j}$$

$$\text{where } E_j = n\left(\frac{1}{k}\right) = \frac{n}{k}$$

Note: Note that Y_j is the observed number and E_j is the expected number of observations in category j if H_0 is true.

Likelihood Ratio Test Statistic

$$\Lambda(\theta_0) = -2\log\left[\frac{L(\theta_0)}{L(\tilde{\theta})}\right]$$

$$= -2\log\left[\prod_{j=1}^{k}\left(\frac{E_j}{Y_j}\right)^{Y_j}\right]$$

$$= 2\sum_{j=1}^{k} Y_j \log\left(\frac{Y_j}{E_j}\right)$$

= 2*{sum [observed * log (observed /expected)]}

Likelihood Ratio Test Statistic

$$\Lambda(\theta_0) = 2\sum_{j=1}^{k} Y_j \log\left(\frac{Y_j}{E_j}\right)$$

Why does this test statistic make sense?

When does it take on large values?

When does it take on small values?

Data

Colour	Observed Frequency y_j	Expected Frequency e_j
Red	238	1692/8 = 211.5
Green	210	211.5
Yellow	216	211.5
Blue	157	211.5
Purple	235	211.5
Brown	184	211.5
Orange	242	211.5
Pink	219	211.5
Total	1692	1692

Observed Value

The observed value of the likelihood ratio test statistic is

$$\lambda(\theta_0) = 2\sum_{j=1}^{8} y_j \log\left(\frac{y_j}{e_j}\right) = 29.9743$$

The approximate *p*-value is

p-value ≈ P(W ≥ 29.9743) ≈ 0

where W $\backsim \chi^2(7)$

Chi-squared approximation

There is very strong evidence against the hypothesis of an equal number of each colour based on these data.

Note: The Chi-squared approximation is good when n is large and the expected frequencies under H_0 are all at least five.

Pearson's Chi-squared Goodness of Fit Statistic

An alternative test statistic that was developed historically before the likelihood ratio test statistic is the "Pearson" Goodness of Fit Statistic:

$$D = \sum_{j=1}^{k} \frac{\left(Y_j - E_j\right)^2}{E_j}$$

For large *n*, *D* and Λ are asymptotically equivalent and have the same asymptotic Chi-squared distribution.

The Pearson Goodness of Fit Test is more popular than the Likelihood Ratio Test.

Previous Example: Alpha-Particle Emissions

The Poisson distribution is used to model random events in time.

In a 1910, Ernest Rutherford and Hans Geiger recorded the number of alpha-particles emitted form a polonium source during a fixed period of time (one-eighth of a minute). They made 2608 recordings.

Observed Data for 1910 Study of Alpha-Particles

Number of Alpha-Particles Detected	Frequency y_j	Expected Frequency e_j
0	57	?
1	203	?
2	383	?
3	525	?
4	532	?
5	408	?
6	273	?
7	139	?
8	45	?
9	27	?
10	10	?
11	6	?
Total	2608	2608

Alpha Particle Emissions - Test of Fit of Poisson Model

Our model is Multinomial($\theta_0, \theta_1, \ldots, \theta_{11}$) which is a function of 11 parameters.

***H*$_0$: Data fit a Poisson model or more specifically**

$$H_0 : \theta_j = \frac{\theta^j e^{-\theta}}{j!}, \ \ j = 0, 1, \ldots$$

Under *H*$_0$ there is one unknown parameter θ which must be estimated.

Therefore the degrees of freedom for the Chi-squared approximation are 11 – 1 = 10.

Observed and Expected Frequencies under assumed Poisson Model

Number of Alpha-Particles Detected	Frequency y_j	Expected Frequency e_j
0	57	54.42
1	203	210.58
2	383	407.43
3	525	525.54
4	532	508.41
5	408	393.47
6	273	253.77
7	139	140.28
8	45	67.86
9	27	29.18
10	10	11.29
11+	6	5.77
Total	2608	2607.99

Alpha Particle Emissions - Test of Fit of Poisson Model

Note that the expected frequencies are all at least five so we can use the Chi-squared approximation to obtain the *p*-value.

The observed value of the likelihood ratio statistic is

$$\lambda(\theta_0) = 2\sum_{j=0}^{11} y_j \log\left(\frac{y_j}{e_j}\right) = 14.01$$

with *p*-value ≈ P(*W* ≥ 14.01) = 0.17   where W∽χ^2((12-1) - 1 = 10) and there is no evidence against the Poisson model based on these data.

Alpha Particle Emissions - Test of Fit of Poisson Model

The observed value of the Pearson goodness of fit statistic is

$$d = \sum_{j=1}^{k} \frac{(y_j - e_j)^2}{e_j} = 12.98$$

with p-value ≈ P(*W* ≥ 12.98) = 0.22 where W∽χ^2(10).

There is no evidence against the Poisson model based on these data.

What to do if expected frequencies are not all at least 5?

Suppose we have the following table of observed frequencies and expected frequencies calculated under the null hypothesis.

Category	Observed Number	Expected Number
1	53	50
2	21	25
3	11	12.5
4	8	6.25
5	4	3.125
≥ 6	3	3.125
Total	100	100

What to do if Expected Frequencies are not all at least 5?

The last two categories have expected frequencies less than five and so it may not be appropriate to use the Chi-squared approximation.

Usually we collapse two or more adjacent categories with the smallest expected frequencies.

What to do if expected frequencies are not all at least 5?

Category	Observed Number	Expected Number
1	53	50
2	21	25
3	11	12.5
4	8	6.25
≥ 5	7	6.25
Total	100	100