Xiamen University

Finite Elements Method Course

Brinkman Equation

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2021-06-23

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Chapter 1

Introduction

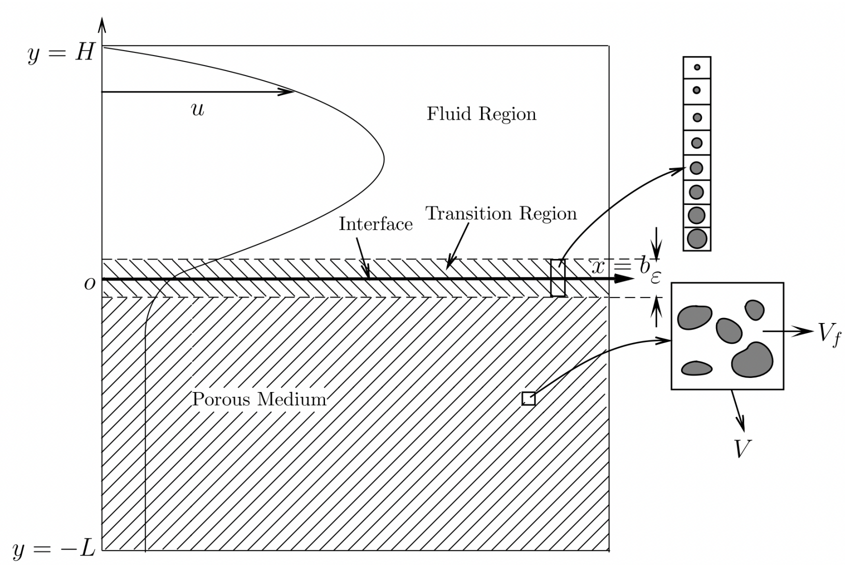
Transport phenomena involving free fluid over porous medium have been encountered in a wide range of industrial applications. The main modeling problem lies in coupling conservation equations in both free fluid region and porous medium, which leads to the modeling of fluid-porous interface condition. Generally, there are two different formulations to deal with the problem: the so-called one-domain and two-domain approaches.

For two domains, in the free fluid region, the flow could be modeled by the Stokes equation. In the porous medium, the flow could be represented by Brinkman equation.

For the two-domain approach, the difficulty is to determine an effective interface condition that connects the two different models in each domain. One of the interface condition was proposed by Beavers and Joseph

$$\left.\frac{\partial u_S}{\partial y}\right|_{\Gamma^+} = \frac{\alpha}{\sqrt{K_p}}\left(u_S|_{\Gamma^+} - u_D|_{\Gamma^-}\right)$$

In the one-domain approach, a single set of transport equation is used for the entire domain consisting the porous medium and free fluid regions and the interface which is a continuous transition zone of thickness ε, across which the physical variables undergo strong but continuous variations. The interface can also be viewed as an ideal representation of a region with continuous spatial variation of the macroscopic properties, such as permeability, porosity.



In this paper, we mainly consider the one-domain approach based on the macroscopic Brinkman equation for a viscous flow in porous medium:

$$\nabla p - \nabla \cdot (\mu_{\text{eff}} \nabla \boldsymbol{u}) + \mu \boldsymbol{K}^{-1} \boldsymbol{u} = \boldsymbol{f} \tag{1.1}$$

where $\boldsymbol{u}$ and p stand for the velocity and pressure, $\boldsymbol{f}$ is related to body forces, $\boldsymbol{K}$ is the permeability tensor of Darcy's law. From the viewpoint of the volume averaging method, the effective viscosity is $\mu_e ff = \mu/\phi_p$, where ϕ_p is the porosity of porous medium. Let $\phi_p = V_f/V$, where V is a representative elementary volume in porous medium and V_f stands for the volume of fluid phase contained in V.

Chapter 2

Implementation

2.1 The one-domain approach

Based on the Brinkman equation (1.1) in the porous medium, we start with description of the model by one-domain approach. The two dimensional domain $\Omega_f = (0, b) \times (0, H)$ is occupied by free fluid, $\Omega_f = (0, b) \times (-L, 0)$ is occupied by a porous medium, and the permeable sharp interface is $\Gamma = (0, b) \times \{0\}$.

$$\nabla p - \nabla \cdot \left(\mu \phi_1 \left(\frac{y}{\epsilon} \right) \nabla \boldsymbol{u} \right) + \mu \boldsymbol{K}^{-1} \phi_2 \left(\frac{y}{\epsilon} \right) \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega \tag{2.1}$$

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \\ &\boldsymbol{u} = 0 \quad \text{on } y = H \\ &\nabla \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } y = -L \end{aligned}$$

where the small parameter ε is denoted for ε / H. The variations of permeability and effective viscosity in the transition region can be obtained by choosing ϕ_1 and ϕ_2 as follows

$$\begin{aligned} \phi_1 \left(\tfrac{y}{\epsilon} \right) &= \tfrac{1}{2} \left(1 - \tfrac{\mu_{\text{eff}}}{\mu} \right) \tanh \left(\tfrac{\theta y}{\epsilon} \right) + \tfrac{1}{2} \left(1 + \tfrac{\mu_{\text{eff}}}{\mu} \right) \\ \phi_2 \left(\tfrac{y}{\epsilon} \right) &= -\tfrac{1}{2} \tanh \left(\tfrac{\theta y}{\epsilon} \right) + \tfrac{1}{2} \end{aligned}$$

where $\theta > 0$ is an adjustable parameter. We choose ϕ_1, ϕ_2 such that

$$\phi_1 \to \begin{cases} \frac{\mu_{\text{ eff}}}{\mu}, & y \to -\infty \\ 1, & y \to +\infty, \end{cases} \qquad \phi_2 \to \begin{cases} 1, & y \to -\infty \\ 0, & y \to +\infty \end{cases}$$

This allows the model (2.1) to approach rapidly to the Stokes–Brinkman model away from the thin transition region.

2.2 Analysis of one-domain approach

We first consider a simplified one-dimensional case. In this case, we assume that the functions are x -independent except for the pressure, the normal velocity and the normal part of body force are zero, i.e., $\boldsymbol{u} = (u(y), 0), \boldsymbol{f} = (f(y), 0)$. Thus, the problem (2.1) reduces to:

$$\frac{\partial p}{\partial x} - \frac{d}{dy} \left(\mu \phi_1 \left(\frac{y}{\epsilon} \right) \frac{du}{dy} \right) + \frac{\mu}{K} \phi_2 \left(\frac{y}{\epsilon} \right) u = f(y), \quad y \in (-L, H) \tag{2.2}$$

$$u(H) = 0, \quad \frac{du}{dy}(-L) = 0 \tag{2.3}$$

where $\frac{\partial p}{\partial x}$ actually only depends on y, and K is the permeability of the porous medium in x -direction. For simplicity, K is assumed to be a constant.

Let u_S and u_B be the solutions of problem (2.2) for $y > 0$ and $y < 0$. First, we consider the outer expansions of u_S in the free fluid region and u_B in porous medium respectively:

$$\begin{aligned} u_S &= u_S^0 + \epsilon u_S^1 + \epsilon^2 u_S^2 + \cdots \\ u_B &= u_B^0 + \epsilon u_B^1 + \epsilon^2 u_B^2 + \cdots \end{aligned}$$

We show the error estimates between the leading order solution of the one-domain approach and the standard Darcy-Stokes model of two-domain approach with BJS interface condition. We also assume that $\overline{\boldsymbol{u}}_S = (\bar{u}_S(y), 0)$ in the Stokes domain and $\overline{\boldsymbol{u}}_D = (\bar{u}_D(y), 0)$ in the Darcy domain. Then the corresponding solutions $\bar{u}_S$ and $\bar{u}_D$ satisfy

$$\frac{\partial p}{\partial x} - \mu \frac{d^2 \bar{u}_S}{dy^2} = f(y), \quad 0 < y < H$$

$$\frac{\partial p}{\partial x} + \frac{\mu}{K} \bar{u}_D = f(y), \quad -L < y < 0$$

$$\frac{d\bar{u}_S}{dy}\left(0^+\right) = \frac{\sqrt{\frac{\mu_{\text{eff}}}{\mu}}}{\sqrt{K}} \bar{u}_S\left(0^+\right)$$

Then we have the following conclusions:

- Let $I_f = (0, H)$. There holds the following error estimate

$$\left\|u_S^0 - \bar{u}_S\right\|_{L^2(I_f)} \leqslant O(K)$$

- Let $I_p = (-L, 0)$. Then we have

$$\left\|u_B^0 - \bar{u}_D\right\|_{L^2(I_p)} \leqslant 0(K) + \left(O\left(K^{\frac{3}{4}}\right) + O\left(K^{\frac{5}{4}}\right)\right)\left(1 - e^{-2L\sqrt{\frac{\mu}{K\mu_{\text{eff}}}}}\right)^{\frac{1}{2}}$$

2.3 Numerical Scheme

In this section, we present the numerical method for the one-domain model. For the one-domain approach, the Taylor-Hood finite element space is employed for discretization which satisfies the discrete $inf - sup$ condition. We consider the model (2.1) with Dirichlet boundary Γ_d and Neumann boundary Γ_n such that $\partial\Omega = \Gamma_d \cup \Gamma_n$ as follows:

$$\nabla p - \nabla \cdot \left(\mu \phi_1\left(\frac{y}{\epsilon}\right) \nabla \boldsymbol{u}\right) + \mu \boldsymbol{K}^{-1} \phi_2\left(\frac{y}{\epsilon}\right) \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega \tag{2.4}$$

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \\ &(\mu \phi_1 \nabla \boldsymbol{u} - p\boldsymbol{I}) \cdot \boldsymbol{n} = \mathbf{g}_n \quad \text{on } \Gamma_n \\ &\boldsymbol{u} = \mathbf{g}_d \quad \text{on } \Gamma_d \end{aligned}$$

where $\boldsymbol{f} \in \left(L^2(\Omega)\right)^2, \mathbf{g}_n \in \left(L^2\left(\Gamma_n\right)\right)^2, \mathbf{g}_d \in \left(L^2\left(\Gamma_d\right)\right)^2$ and $\boldsymbol{K} = K\boldsymbol{I}$ with a constant K. We choose $\epsilon = \sqrt{K}$ in our numerical experiments. Let $H_{\Gamma_d}(\text{div}, \Omega) = \{\boldsymbol{v} \in H(\text{div}, \Omega) : \boldsymbol{v} = 0 \text{ on } \Gamma_d\}$ and $L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\}$.

The weak formulation for the above problem is to find $(\boldsymbol{u}, p) \in H(\text{div}, \Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{p}) = (\boldsymbol{f}, \boldsymbol{v}) + \langle \boldsymbol{g}_n, \boldsymbol{v}\rangle_{\Gamma_n} \quad \forall \boldsymbol{v} \in H_{\Gamma_d}(\text{div}, \Omega) \\ &b(\boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega), \end{aligned}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = (\mu \phi_1 \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \left(\mu \boldsymbol{K}^{-1} \phi_2 \boldsymbol{u}, \boldsymbol{v}\right)$ and $b(\boldsymbol{v}, p) = -(p, \nabla \cdot \boldsymbol{v})$. Then the Taylor-Hood finite element space is applied for the solution of this weak formulation.

Chapter 3

Numerical Results

3.1 Example 1

We consider the one-domain model (2.1) in the domain

$$\Omega = [0,1] \times [-1,1] \text{ with } \Gamma_n = [0,1] \times \{-1\} \text{ and } \Gamma_d = \partial\Omega \backslash \Gamma_n$$

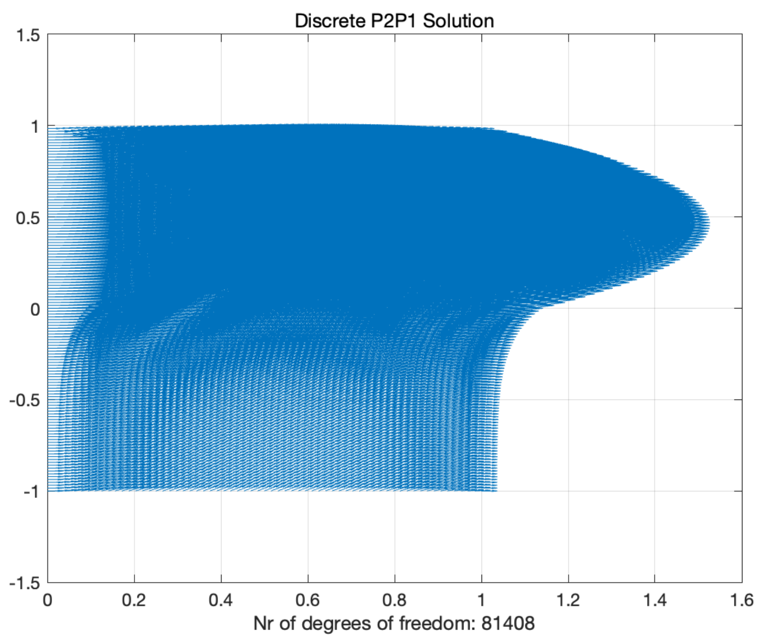
The transition region is chosen as $\epsilon = \sqrt{K} \to 0$, and the free fluid region and the porous medium region lie in $\Omega_f = [0,1] \times (0,1]$ and $\Omega_p = [0,1] \times [-1,0)$, respectively.

Let $\boldsymbol{f} = 0, \mathbf{g}_n = 0$ and $\mathbf{g}_d = \begin{pmatrix} g_d \\ 0 \end{pmatrix}$ with

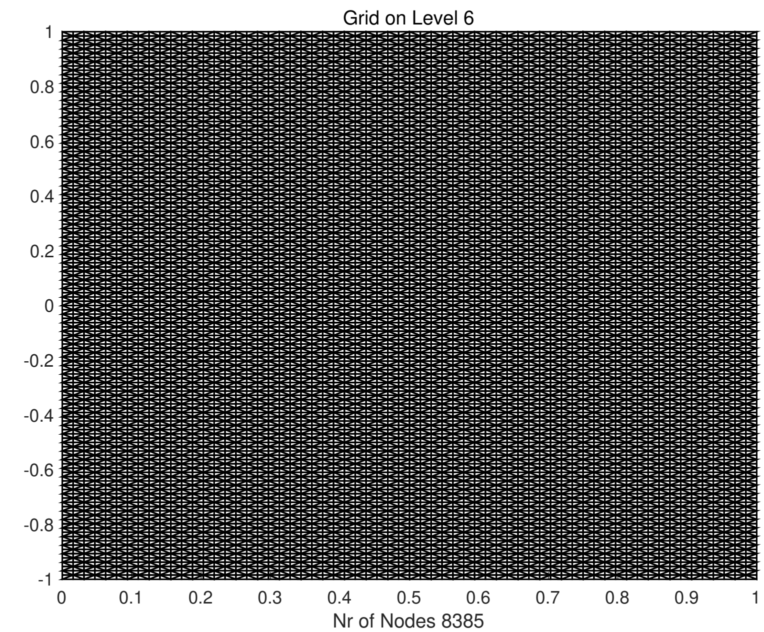
$$g_d = \begin{cases} -\frac{1}{2}\left(y^2 - 1\right) + \left(\frac{1}{2} - K\right)(y-1)/\left(\sqrt{K\phi_p} + 1\right) & \text{if } y > 0 \\ \Phi(0)e^{y\sqrt{\phi_p/K}} - \frac{K}{\mu}\tanh\left(y\sqrt{\phi_p/K}\right) & \text{if } y \leqslant 0 \end{cases}$$

In this example we use the porosity $\phi_p = 0.4$, the viscosity $\mu = 1$ and the parameter $\theta = 100$ in the functions ϕ_1 and ϕ_2. Moreover, the Tylor-Hood element is applied to this example in uniform mesh.

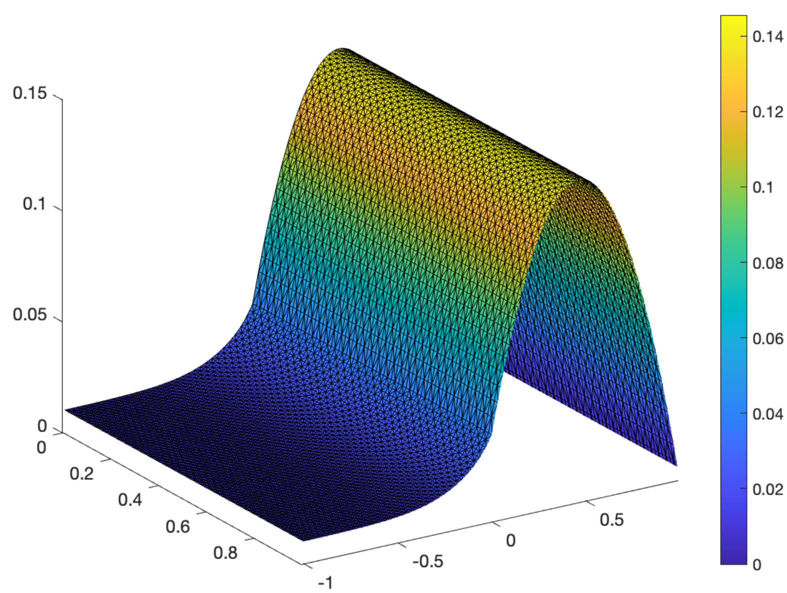
The solutions are printed as follows:



Velocity Solution



Mesh



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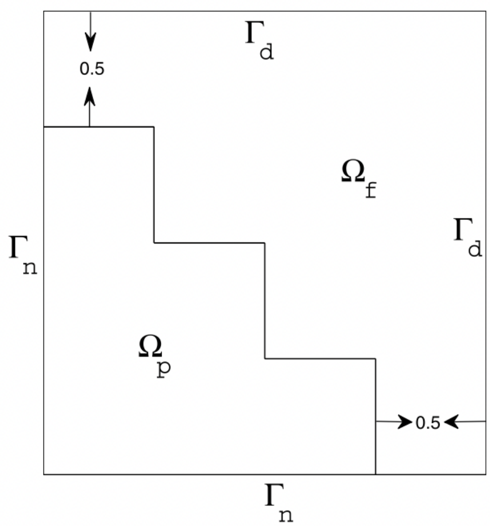
3.2 Example 2

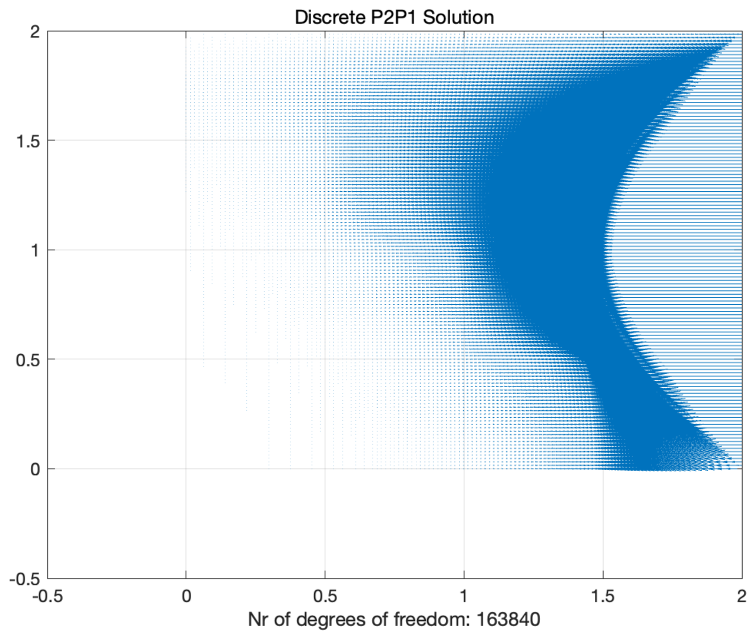
We consider the one-domain model (2.1) in the domain

$$\Omega = [0,2] \times [0,2] \text{ with } \Gamma_d = [0,2] \times \{2\} \cup \{2\} \times [0,2] \text{ and } \Gamma_n = \partial\Omega \backslash \Gamma_d$$



Let $\boldsymbol{g}_d = \begin{pmatrix} g_d \\ 0 \end{pmatrix}$ with $g_d = \frac{y}{2}(y-2)$ on Γ_d and $\boldsymbol{g}_n = 0$ on Γ_n. The source term $\boldsymbol{f}$, porosity ϕ_p, viscosity μ and the parameter θ in functions ϕ_1, ϕ_2 are all chosen as the numberical example 1.

The solutions are printed as follows:



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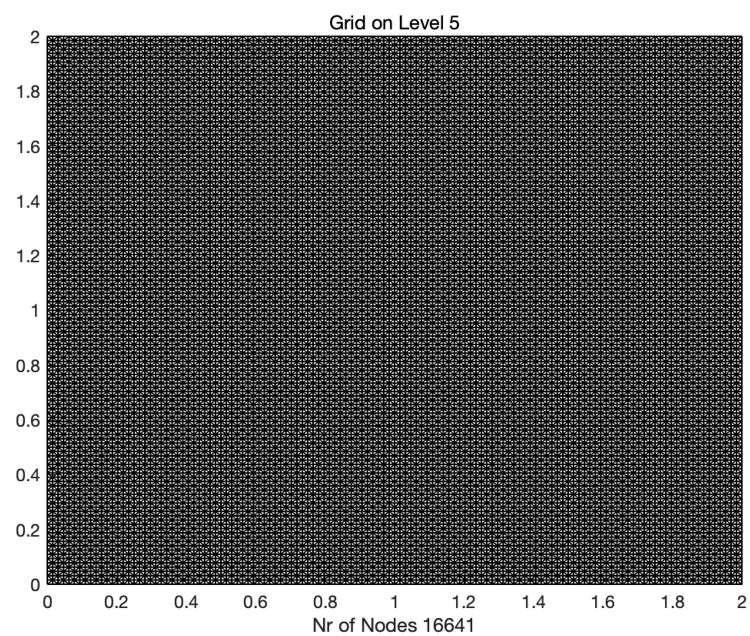

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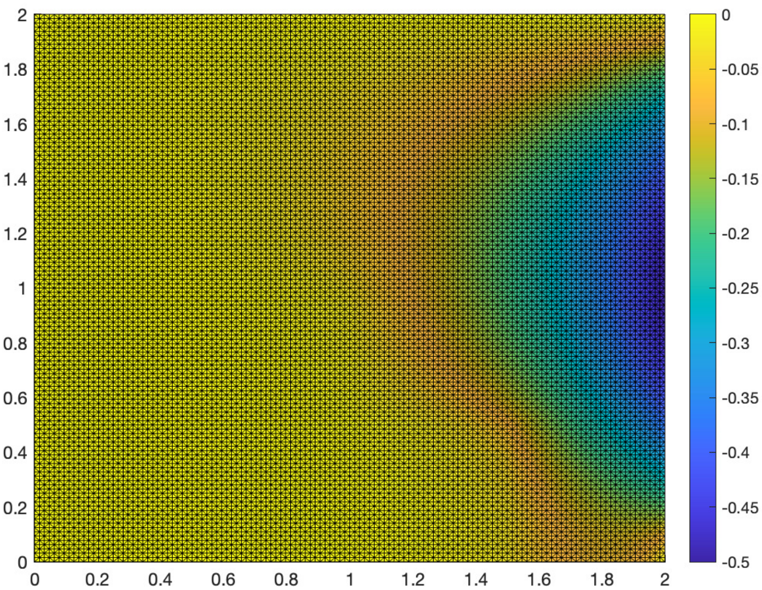

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Bibliography

[1] Huangxin Chen and Xiao-Ping Wang *A one-domain approach for modeling and simulation of free fluid over a porous medium*, Journal of Computational Physics 259 (2014) 650-671