Mixed FEM for Brinkman Flow in Free Fluid over Porous Medium

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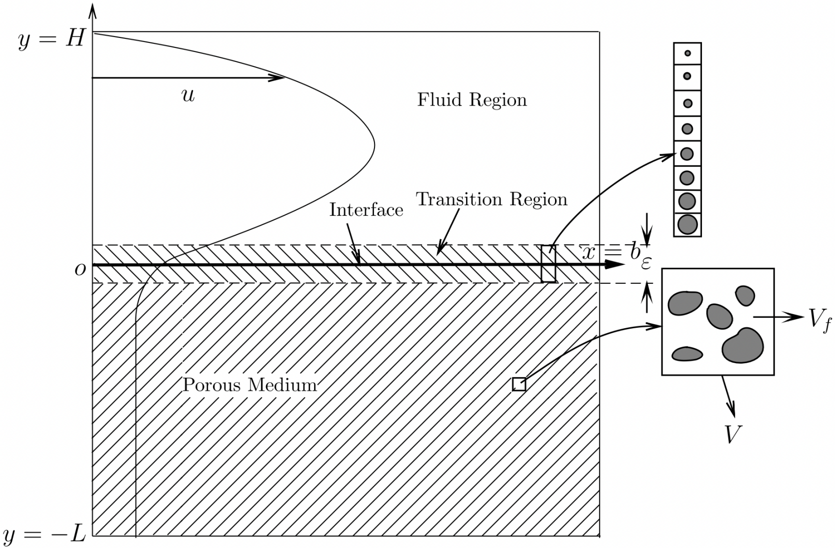
Brinkman Equation

We consider the one-domain approach based on the macroscopic Brinkman equation for a viscous flow in porous medium:

$$\nabla p - \nabla \cdot (\mu_{\mathrm{eff}} \nabla \boldsymbol{u}) + \mu \boldsymbol{K}^{-1} \boldsymbol{u} = \boldsymbol{f} \tag{1}$$

where $\boldsymbol{u}$ and p stand for the velocity and pressure, $\boldsymbol{f}$ is related to body forces, and $\boldsymbol{K}$ is the permeability tensor of Darcy's law.
We study the behavior of the solution of the (1) when the thickness ϵ of the transition region goes to zero.

Free fluid flow over porous medium



One-Domain Approach

Based on the Brinkman equation (1) in the porous medium, we start with description of the model by one-domain approach.

$$\nabla p - \nabla \cdot \left(\mu \phi_1 \left(\frac{y}{\epsilon} \right) \nabla \boldsymbol{u} \right) + \mu \boldsymbol{K}^{-1} \phi_2 \left(\frac{y}{\epsilon} \right) \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega \qquad (2)$$

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \\ &\boldsymbol{u} = 0 \quad \text{on } \ y = H \\ &\nabla \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \ y = -L, \end{aligned}$$

where the small parameter ϵ is denoted for ε / H. The variations of permeability and effective viscosity in the transition region can be obtained by choosing ϕ_1 and ϕ_2 such that

$$\phi_1 \to \begin{cases} \frac{\mu_{\text{eff}}}{\mu}, & y \to -\infty, \\ 1, & y \to +\infty, \end{cases} \qquad \phi_2 \to \begin{cases} 1, & y \to -\infty, \\ 0, & y \to +\infty. \end{cases}$$

So, we could choose ϕ_1 and ϕ_2 as

$$\phi_1\left(\frac{y}{\epsilon}\right) = \frac{1}{2}\left(1 - \frac{\mu_{\text{eff}}}{\mu}\right)\tanh\left(\frac{\theta y}{\epsilon}\right) + \frac{1}{2}\left(1 + \frac{\mu_{\text{eff}}}{\mu}\right)$$
$$\phi_2\left(\frac{y}{\epsilon}\right) = -\frac{1}{2}\tanh\left(\frac{\theta y}{\epsilon}\right) + \frac{1}{2}$$

This allows the model (2) to approach rapidly to the Stokes-Brinkman model away from the thin transition region.

• In the free fluid region Ω_f, $\mu\phi_1$ tends to μ and ϕ_2 tends to zero rapidly for any $x \in \Omega_f$ as $\varepsilon \to 0$, hence the Darcy term in (2) is zero and (2) reduces to the Stokes equation.

• In the porous medium region Ω_p, it is easy to see that (2) reduces to the Brinkman equation (1).

Two-Domain Approach

For the standard two-domain approach, the Darcy-Stokes equations with appropriate interface conditions are written as:

$$\text{Darcy model:} \begin{cases} \mu K^{-1}\bar{\boldsymbol{u}}_{\boldsymbol{D}} + \nabla p_D = \boldsymbol{f} & \text{in } \Omega_p \\ \nabla \cdot \bar{\boldsymbol{u}}_{\boldsymbol{D}} = 0 & \text{in } \Omega_p \\ \bar{\boldsymbol{u}}_{\boldsymbol{D}} \cdot \boldsymbol{n}_{\boldsymbol{D}} = \boldsymbol{g}_{\boldsymbol{p}} & \text{on } \Gamma_D^d \end{cases} \tag{3}$$

$$\text{Stokes model:} \begin{cases} \nabla p_S - \mu \Delta \overline{\boldsymbol{u}}_S = \boldsymbol{f} & \text{in } \Omega_f, \\ \nabla \cdot \overline{\boldsymbol{u}}_S = 0 & \text{in } \Omega_f, \\ (\mu \nabla \overline{\boldsymbol{u}}_S - p\boldsymbol{I}) \cdot \boldsymbol{n}_S = \boldsymbol{g}_n & \text{on } \Gamma_S^n, \\ \overline{\boldsymbol{u}}_S = \boldsymbol{g}_d & \text{on } \Gamma_S^d \end{cases} \tag{4}$$

$$\text{Interface conditions:} \begin{cases} (\overline{\boldsymbol{u}}_S - \overline{\boldsymbol{u}}_D) \cdot \boldsymbol{n}_S = 0 \\ p_S - p_D = \mu \boldsymbol{n}_S \cdot \nabla \overline{\boldsymbol{u}}_S \cdot \boldsymbol{n}_S \\ -\boldsymbol{n}_S \cdot \nabla \overline{\boldsymbol{u}}_S \cdot \tau = \frac{\alpha}{\sqrt{K}} \overline{\boldsymbol{u}}_S \cdot \tau \end{cases} \tag{5}$$

If we consider a simplified one-dimensional case, we assume that $\boldsymbol{u} = (u(y), 0), \boldsymbol{f} = (f(y), 0)$. Thus, for the one-domain approach, the problem (2) reduces to:

$$\frac{\partial p}{\partial x} - \frac{d}{dy}\left(\mu\phi_1\left(\frac{y}{\epsilon}\right)\frac{du}{dy}\right) + \frac{\mu}{K}\phi_2\left(\frac{y}{\epsilon}\right)u = f(y), \quad y \in (-L, H)$$

$$u(H) = 0, \quad \frac{du}{dy}(-L) = 0$$

Let u_S and u_B be the solutions for $y > 0$ and $y < 0$. We consider the asymptotic expansions of u_S in the free fluid region and u_B in porous medium respectively:

$$\begin{aligned} u_S &= u_S^0 + \epsilon u_S^1 + \epsilon^2 u_S^2 + \cdots \\ u_B &= u_B^0 + \epsilon u_B^1 + \epsilon^2 u_B^2 + \cdots \end{aligned}$$

For two-domain approach, we assume that $\overline{\boldsymbol{u}}_S = (\bar{u}_S(y), 0)$ in Stokes domain and $\overline{\boldsymbol{u}}_D = (\bar{u}_D(y), 0)$ in Darcy domain. Then we suppose that the corresponding solutions $\bar{u}_S$ and $\bar{u}_D$ satisfy:

$$\frac{\partial p}{\partial x} - \mu \frac{d^2 \bar{u}_S}{dy^2} = f(y), \quad 0 < y < H$$

$$\frac{\partial p}{\partial x} + \frac{\mu}{K} \bar{u}_D = f(y), \quad -L < y < 0$$

$$\frac{d\bar{u}_S}{dy}\left(0^+\right) = \frac{\sqrt{\frac{\mu_{\text{eff}}}{\mu}}}{\sqrt{K}} \bar{u}_S\left(0^+\right)$$

Theorem 1

Let $I_f = (0, H)$, and $I_p = (-L, 0)$, then we have

$$\left\| u_S^0 - \bar{u}_S \right\|_{L^2(I_f)} \leqslant O(K)$$

$$\left\| u_B^0 - \bar{u}_D \right\|_{L^2(I_p)} \leqslant 0(K) + \left(O\left(K^{\frac{3}{4}}\right) + O\left(K^{\frac{5}{4}}\right) \right) \left(1 - e^{-2L\sqrt{\frac{\mu}{K\mu_{eff}}}} \right)^{\frac{1}{2}}$$

We consider the model (2) with Dirichlet boundary Γ_d and Neumann boundary Γ_n such that $\partial\Omega = \Gamma_d \cup \Gamma_n$ as follows:

$$\nabla p - \nabla \cdot \left(\mu\phi_1\left(\frac{y}{\epsilon}\right)\nabla \boldsymbol{u}\right) + \mu \boldsymbol{K}^{-1}\phi_2\left(\frac{y}{\epsilon}\right)\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega,$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega,$$

$$(\mu\phi_1\nabla\boldsymbol{u} - p\boldsymbol{I}) \cdot \boldsymbol{n} = \boldsymbol{g}_n \quad \text{on } \Gamma_n,$$

$$\boldsymbol{u} = \boldsymbol{g}_d \text{ on } \Gamma_d,$$

where $\boldsymbol{f} \in \left(L^2(\Omega)\right)^2, \mathrm{g}_n \in \left(L^2(\Gamma_n)\right)^2, \mathrm{g}_d \in \left(L^2(\Gamma_d)\right)^2, \boldsymbol{K} = K\boldsymbol{I}$ with a constant K, and assume that $\epsilon = \varepsilon/H = O(\sqrt{K})$.

Let

$$H_{\Gamma_d}(\mathrm{div},\Omega) = \{\boldsymbol{v} \in H(\mathrm{div},\Omega) : \boldsymbol{v} = 0 \text{ on } \Gamma_d\}$$

and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$$

The weak formulation of the above problem could be written as:

$$\begin{cases} \text{find}(\boldsymbol{u}, p) \in H(\mathrm{div},\Omega) \times L_0^2(\Omega) \\ a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) + \langle \boldsymbol{g}_n, \boldsymbol{v} \rangle_{\Gamma_n} \quad \forall \boldsymbol{v} \in H_{\Gamma_d}(\mathrm{div},\Omega) \\ b(\boldsymbol{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \end{cases}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = (\mu\phi_1 \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \left(\mu \boldsymbol{K}^{-1}\phi_2 \boldsymbol{u}, \boldsymbol{v}\right)$$

and

$$b(\boldsymbol{v}, p) = -(p, \nabla \cdot \boldsymbol{v})$$

Numerical Result 1

We consider the one-domain model (2) in the domain

$$\Omega = [0,1] \times [-1,1] \text{ with } \Gamma_n = [0,1] \times \{-1\} \text{ and } \Gamma_d = \partial\Omega \backslash \Gamma_n$$

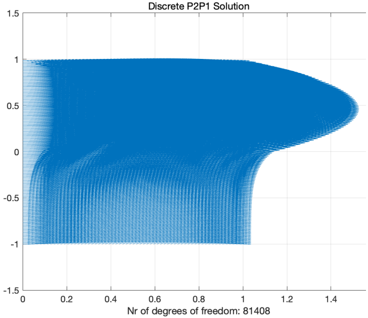
The transition region is chosen as $\epsilon = \sqrt{K} \to 0$, and the free fluid region and the porous medium region lie in $\Omega_f = [0,1] \times (0,1]$ and $\Omega_p = [0,1] \times [-1,0)$, respectively.

Let $\boldsymbol{f} = 0, \boldsymbol{g}_n = 0$ and $\boldsymbol{g}_d = \begin{pmatrix} g_d \\ 0 \end{pmatrix}$ with

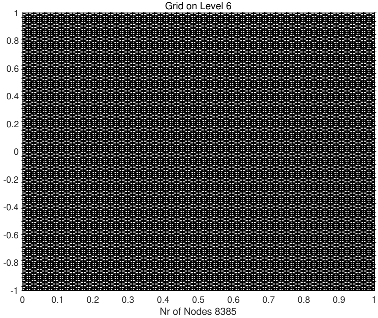
$$g_d = \begin{cases} -\frac{1}{2}\left(y^2 - 1\right) + \left(\frac{1}{2} - K\right)(y-1)/\left(\sqrt{K\phi_p} + 1\right) & \text{if } y > 0 \\ \Phi(0)e^{y\sqrt{\phi_p/K}} - \frac{K}{\mu}\tanh\left(y\sqrt{\phi_p/K}\right) & \text{if } y \leqslant 0 \end{cases}$$

In this example we use the porosity $\phi_p = 0.4$, the viscosity $\mu = 1$ and the parameter $\theta = 100$ in the functions ϕ_1 and ϕ_2.

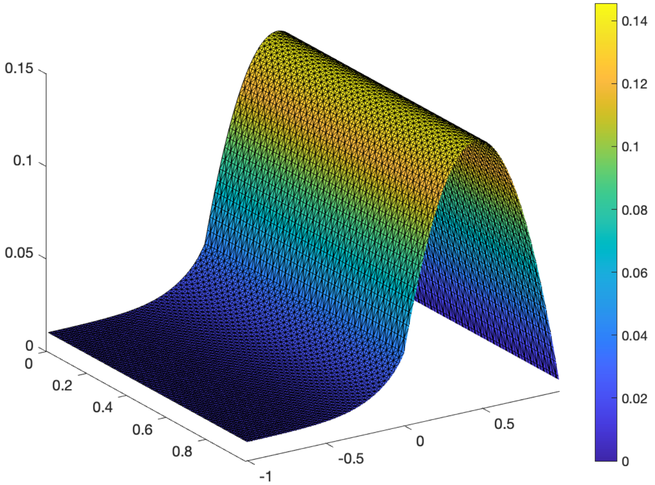
In this example, we apply mixed P2P1-FEM in uniform mesh:



Velocity Solution



Mesh



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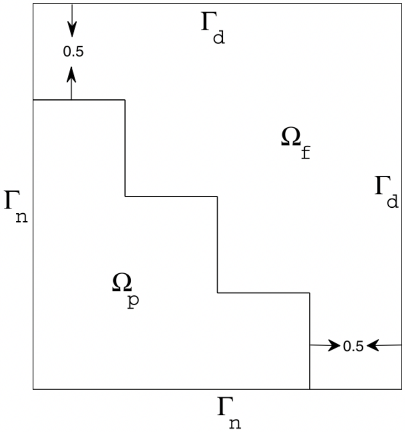
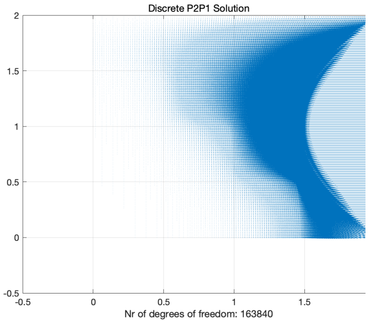
Numerical Result 2

We consider the one-domain model (2) in the domain

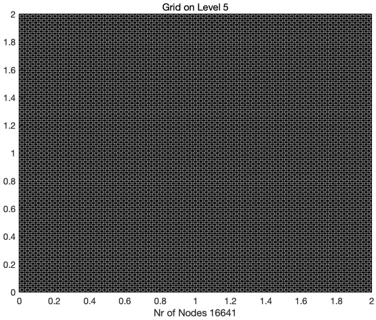
$\Omega = [0,2]\times[0,2]$ with $\Gamma_d = [0,2]\times\{2\}\cup\{2\}\times[0,2]$ and $\Gamma_n = \partial\Omega\backslash\Gamma_d$



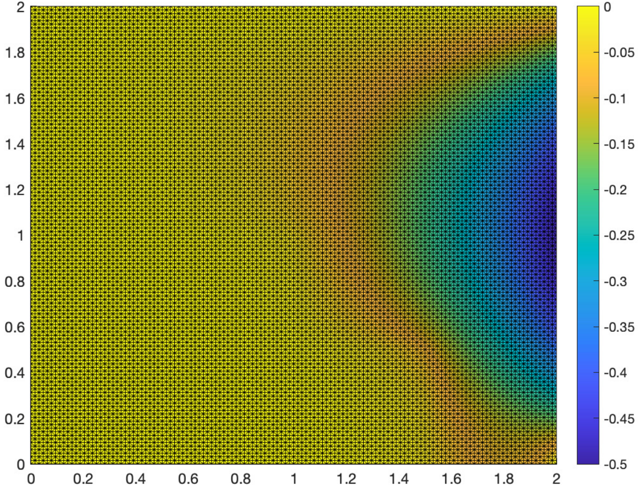
Let $\boldsymbol{g}_d = \begin{pmatrix} g_d \\ 0 \end{pmatrix}$ with $g_d = \frac{y}{2}(y-2)$ on Γ_d and $\boldsymbol{g}_n = 0$ on Γ_n.
The source term $\boldsymbol{f}$, porosity ϕ_p, viscosity μ and the parameter θ in functions ϕ_1, ϕ_2 are all chosen as the numberical example 1.



Velocity Solution



Mesh



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Thanks!

Bibliography

Huangxin Chen and Xiao-Ping Wang *A one-domain approach for modeling and simulation of free fluid over a porous medium*, Journal of Computational Physics 259 (2014) 650-671