1. Before being de-platformed, the number of tweets that former President Trump sent on a random day might be modeled by $X \sim Poisson(\lambda)$. You've heard his average tweet rate was 8 tweets/day, but you believe it might be lower. You plan to collect a sample of size 1 and reject H_0 if $X_1 \leq 3$. Find the Type I Error rate, and the Type II Error rate if $\lambda = 6.4$. On problems 1 and 2b of this assignment, you must write hypotheses and include a beautifully-labeled diagram in your answer with two pmfs/pdfs and a rejection fence. Include the R code and output for your diagram.

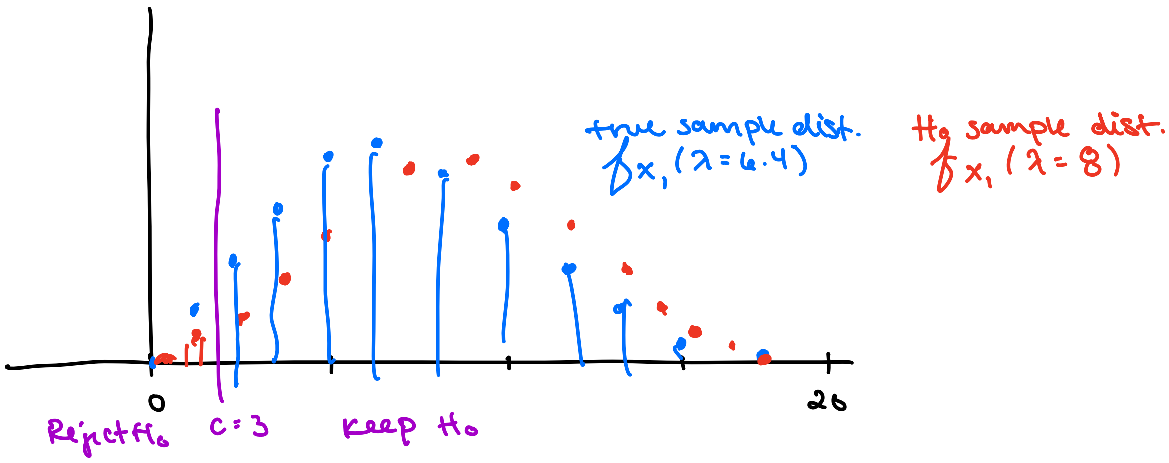
$x \sim Poisson(\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$ is # of tweets per day.

Let λ be the average tweet rate per day.

$H_0: \lambda = 8 \qquad H_1: \lambda < 8$

Decision Rule: If $X_1 \leq 3$ reject H_0 for sample size $n=1$.

Estimator: $\hat{\lambda} = X_1 \qquad$ Sample Distribution: $\hat{\lambda} \sim Poisson(\lambda)$



$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ true}) = P(X_1 \leq 3 \mid \lambda = 8)$

$$\left[\hat{\lambda} = X_1 \Longrightarrow \hat{\lambda} \sim \frac{8^x}{x!} e^{-8}\right]$$

$$= \sum_{x=0}^{3} \frac{8^x}{x!} e^{-8} = 8e^{-8}$$

$= $ >ppois(3, 8, lower = T)

$\boxed{.042 = \alpha}$

$\beta = P(\text{type 2 error}) = P(\text{keep } H_0 \mid H_1 \text{ true}) = P(X_1 > 3, \lambda = 6.4)$

$= \sum_{x=4}^{\infty} \frac{6.4^x}{x!} e^{-6.4} \qquad = 1 -$ <ppois(3, 6.4, T)

$\boxed{.881 = \beta}$

2. Suppose $X \sim Exp(\lambda)$ where X is modeled by $f(x;\lambda) = \lambda e^{-\lambda x}$, where $x, \lambda > 0$. You draw a sample of size n and plan to use the statistic $X_{\min}$ to decide between two hypotheses.

a. Show that $X_{\min}$ also has an exponential distribution and determine what parameter it is based on (instead of λ).

$X \sim Exp(\lambda) \qquad f(x;\lambda) = \lambda e^{-\lambda x}$

$$\hat{\lambda} = \min x_i \implies f_{\hat{\lambda}}(x) = \frac{n!}{(n-1)!}\left[F_x(x)\right]^{0} f_x(x)\left[1 - F_x(x)\right]^{n-1} \quad (i=1)$$

$$= n f_x(x)\left(1 - F_x(x)\right)^{n-1}$$

$$\left[F_x(x) = \int_0^x \lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_0^x = -e^{-\lambda x} - 1 = \underline{1 - e^{-\lambda x}}\right]$$

$$f_{\hat{\lambda}}(x) = n\lambda e^{-\lambda x}\left(1 - (1 - e^{-\lambda x})\right)^{n-1}$$

$$= n\lambda e^{-\lambda x}\left(e^{-\lambda x}\right)^{n-1}$$

$$= n\lambda\left(e^{-\lambda x}\right)^{n}$$

$\implies f_{X_{\min}} = n\lambda e^{-n\lambda x}$ is exponential w/ parameter $\underline{\underline{n\lambda}}$

b. You plan to test H_0: $\lambda = 2$ vs. H_1: $\lambda > 2$ via the rule: If $X_{\min} < c$, reject H_0; else keep H_0. What must c be if you want your Type II error rate to be 0.08 when the true value of λ is 6? Assume that $n = 5$.

true

WANT $\beta = P(\text{Keep } H_0 \mid H_1 \text{ true}) = .08$

true sample dist: $f_{X_{\min}}(x; n\lambda = n6)$

H_0 dist: $f_{X_{\min}}(x; n\lambda = n2)$

$$.08 = \int_0^c 5 \cdot 6 e^{-5 \cdot 6 \cdot x} dx = \int_0^c 30 e^{-30x} dx$$

$$= -e^{-30x} \Big|_0^c = -e^{-30c} + 1 \Longrightarrow 1 - e^{-30c}$$

Solving for c... $1 - e^{-30c} = .08 \Longrightarrow e^{-30c} = .92$

$$\ln(.92) = -30c \Longrightarrow \frac{-\ln(.92)}{30} = c$$

3. Let $Y \sim \chi_n^2$.

a. Show that $E[Y^k] = \dfrac{2^k \Gamma\left(\frac{n}{2} + k\right)}{\Gamma\left(\frac{n}{2}\right)}$ if k is a real number with $k > -\dfrac{n}{2}$.

We know: $E[\chi_n^2] = n = E[Y]$

$$E[Y^k] = \int y^k f_{\chi_n^2}(u)$$

$$= \int u^k \cdot \frac{u^{n/2-1} \cdot e^{-u/2}}{2^{n/2} \cdot \Gamma\left(\frac{n}{2}\right)}$$

$$= \int \frac{u^{n/2+k-1} e^{-u/2}}{2^{n/2} \Gamma\left(\frac{n}{2}\right)}$$

Gamma

$$\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

let $\lambda = \frac{1}{2}$, $r = \frac{n}{2} + k$

$$\Longrightarrow \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}+k}}{\Gamma\left(\frac{n}{2}+k\right)} x^{\frac{n}{2}+k-1} e^{-x/2}$$

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int \frac{x^{n/2+k-1} e^{-x/2}}{2^{n/2+k} \Gamma\left(\frac{n}{2}+k\right)} = 1$$

$$\frac{2^k}{2^k}\int \frac{x^{n/2+k-1}e^{-x/2}}{2^{n/2+k}\Gamma\left(\frac{n}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\int \frac{x^{n/2+k-1}e^{-x/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} = \frac{2^k\Gamma\left(\frac{n}{2}+k\right)}{\Gamma\left(\frac{n}{2}\right)} \checkmark$$

c. Using part a, prove that $E(T_n) = 0$ and $Var(T_n) = \frac{n}{n-2}$ (when $n > 2$).
(Hint: Think of T_n as a product of two RVs.)

$$T_n = \frac{Z}{\sqrt{U/n}}, \quad U = \chi_n^2, \quad Z = N(0,1)$$

$$\Longrightarrow T_n = Z \cdot (U/n)^{-1/2}$$

$$* \; E[T_n] = \underbrace{E[Z]\cdot E[(U/n)^{-1/2}]}_{\text{independent}} = 0 \text{ because } E[Z] = 0 \checkmark$$

$$* \; Var(T_n) = Var(Z\cdot(U/n)^{-1/2}) = E[Z^2\cdot\underbrace{(U/n)^{-1}}_{\frac{n}{U}}] - E[Z\cdot(U/n)^{-1/2}]^2$$

$$= \cancel{E[Z^2]}E\left[\frac{n}{U}\right] + 0 \text{ (from above)}$$

$$Var(Z) = E[Z^2] - \underbrace{E[Z]^2}_{0} = 1$$

$$\Longrightarrow E[Z^2] = 1$$

$$\text{Now } Var(T_n) = E\left[\frac{n}{U}\right] = nE[(\chi_n^2)^{-1}]$$

$$f_{\chi_n^2} = f(u) = \frac{u^{n/2-1}e^{-u/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}$$

$$= n\int_0^\infty u^{-1}\,\frac{u^{n/2-1}\,e^{-u/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$$

$$= n\int \frac{u^{n/2-2}\,e^{-u/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)}$$

Gamma

$$\frac{\lambda^r}{\Gamma(r)}\,x^{r-1}e^{-\lambda x}$$

let $\lambda = \frac{1}{2}$, $r = \frac{n}{2}-1$

$$\Longrightarrow \int \frac{x^{\frac{n}{2}-2}\,e^{-x/2}}{2^{n/2-1}\,\Gamma\left(\frac{n}{2}-1\right)} = 1$$

$$\Longrightarrow \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{2}{2}\int \frac{x^{\frac{n}{2}-2}\,e^{-x/2}}{2^{n/2-1}\,\Gamma\left(\frac{n}{2}-1\right)} = 1$$

$$\Longrightarrow \int \frac{x^{\frac{n}{2}-2}\,e^{-x/2}}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} = \underline{\underline{\frac{\Gamma\left(\frac{n}{2}-1\right)}{2\,\Gamma\left(\frac{n}{2}\right)}}}$$

So $\mathrm{Var}(T_n) = n\left(\frac{\Gamma\left(\frac{n}{2}-1\right)}{2\,\Gamma\left(\frac{n}{2}\right)}\right)$

$\Gamma(x+1) = x\,\Gamma(x)$

let $x = u-1 \longrightarrow$

$\Gamma((u-1)+1) = (u-1)\,\Gamma(u-1)$

$\longrightarrow \Gamma(u) = (u-1)\,\Gamma(u-1) \longrightarrow \dfrac{\Gamma(u)}{\Gamma(u-1)} = (u-1)$

$$\Longrightarrow \frac{n\,\Gamma\left(\frac{n}{2}-1\right)}{2\,\Gamma\left(\frac{n}{2}\right)} = \frac{1}{\left(\frac{n}{2}-1\right)}\cdot\frac{n}{2} = \frac{n}{2\left(\frac{n}{2}-1\right)} = \frac{n}{n-2}\ \checkmark$$

b. Using part a, prove that $E(Y) = n$ and $Var(Y) = 2n$, as claimed in class.

We know: $$E[Y^k] = \frac{2^k \Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2})}$$

If $k=1$ ... $$E[Y^1] = E[X_n^2] = \frac{2\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})}$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$= \frac{2(\frac{n}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2})} = n \checkmark$$

$$Var(Y) = E[Y^2] - \overbrace{E[Y]^2}^{=n^2}$$

if $k=2 \longrightarrow$ $$E[(X_n^2)^2] = \frac{2^2\Gamma(\frac{n}{2}+2)}{\Gamma(\frac{n}{2})}$$

$$= \frac{4\Gamma(\frac{n}{2}+1+1)}{\Gamma(\frac{n}{2})}$$

$$= 4\left(\frac{n}{2}+1\right)\boxed{\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})}} \rightarrow = \frac{n}{2}$$

$$= 4\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)$$

So $$Var(Y) = (2n+4)\left(\frac{n}{2}\right) - n^2$$

$$= n^2 + 2n - n^2$$

$$= 2n \checkmark$$

4. Prove these two useful facts about the F distribution:

a. $E(F_{m,n}) = \dfrac{n}{n-2}$ (when $n > 2$)

$F_{m,n} = \dfrac{V/m}{U/n}$ where $U = \chi_n^2$, $V = \chi_m^2$ $\qquad E[\chi_n^2] = n$

$E[F_{m,n}] = E\left[\dfrac{V/m}{U/n}\right] = E\left[\dfrac{\chi_m^2/m}{\chi_n^2/n}\right] = E\left[\dfrac{\chi_m^2}{m} \cdot \dfrac{n}{\chi_n^2}\right]$

$= \dfrac{n}{m} E\left[\chi_m^2 \cdot (\chi_n^2)^{-1}\right] \overset{\text{independent}}{=} \dfrac{n}{m} \underbrace{E[\chi_m^2]}_{=m} E[(\chi_n^2)^{-1}]$

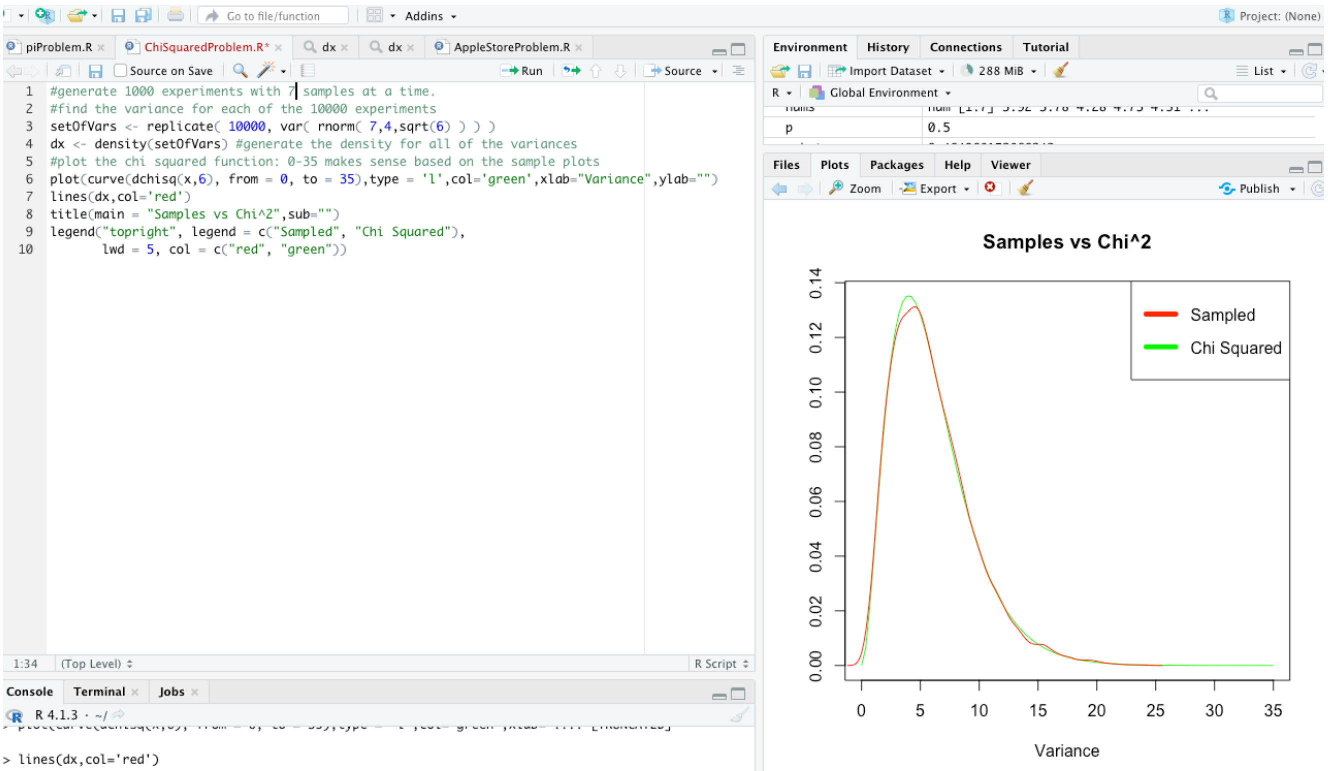
$\Longrightarrow n E[(\chi_n^2)^{-1}] = \text{Var}(T_n) = n\left(\dfrac{\Gamma\left(\frac{n}{2}-1\right)}{2\,\Gamma\left(\frac{n}{2}\right)}\right)$

From #3 part (c) $\qquad \hookrightarrow = \dfrac{n}{n-2}$ ✓

b. $F_{m,n} = \dfrac{1}{F_{n,m}}$

$F_{m,n} = \dfrac{\chi_m^2/n}{\chi_n^2/n} = \dfrac{1}{\dfrac{\chi_n^2/n}{\chi_m^2/n}} = \dfrac{\chi_n^2/n}{\chi_m^2/n} = \dfrac{1}{F_{n,m}}$

5. Students often find it hard to believe that $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. So, let's simulate this situation and see if the data agree. To do so, generate $n = 7$ numbers from $N(4, \sigma^2 = 6)$. Find the variance of these n numbers, and replicate this process a total of 10000 times. Make a density plot of the 10000 values for $\dfrac{(n-1)S^2}{\sigma^2}$. Then, in red, overlay the pdf for χ^2_{n-1}. Include your code and a sketch of your plot.



6. It is also surprising that $\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim T_{n-1}$. To empirically convince you of this, do these steps: Let $X \sim N(3, \sigma^2 = 5^2)$. Draw an iid sample of size $n = 3$. Using this sample, calculate the ratio $\dfrac{\overline{X}-\mu}{S/\sqrt{n}}$. Replicate this process 10000 times. Draw the density, and then in red, overlay the density of T_{n-1}. Include your code and a sketch of the plot.