1. If you look at a bottle of ibuprofen, it will likely list the amount of medicine per pill (usually, 200 mg). Of course, this is only an average, and if you carefully measured the amount from pill to pill, you would get a normal distribution. The spread of this distribution is very important because giving too much or too little medicine can be dangerous. Suppose that the standard deviation in dosage is 10 mg based on current manufacturing processes. You've come up with a new way to create the pills that you believe will increase the precision of the dosage. To check this claim, you produce a bunch of pills and randomly select some to measure the dosage. You get these values: 206.5 198.9 205.2 205.8 192.0 199.5 182.5 191.9 197.6 190.7 186.8 187.3 192.0.

Avg. dosage = 200mg

$\sigma = 10$ known

$n = 13$

$\alpha = .04$

a. Conduct a hypothesis test with $\alpha = 0.04$.

σ^2 is variance of medicine/pill.

$H_0: \sigma^2 = (10)^2 \quad H_1: \sigma^2 < (10)^2 \rightsquigarrow$ reject H_0 if $P(\chi^2_{n-1} < \chi^2) < \alpha$

Assuming H_0, we know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \implies \frac{(12)S^2}{10^2} \sim \chi^2_{13-1}$

From R: $\bar{x} = 195.132$

$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 = 7.766$ – sample sd

Assuming H_0 $(\sigma^2 = 10^2)$

$\hookrightarrow \quad \chi^2_{stat}: \frac{(n-1)S^2}{\sigma^2} = \frac{(12)(7.766)^2}{10^2} \approx 7.237 \sim \chi^2$

$H_1: \sigma^2 < (10)^2 \rightarrow$ want smaller vals of χ^2

If $P(\chi^2_{n-1} < \chi^2) < \alpha \Rightarrow$ REJECT

Pval $= P(\chi^2_{12} \leq 7.237)$

>pchisq(7.237, 12)

.1584

$.1584 > .04$ we keep H_0

b. Construct a 97% two-sided CI for the standard deviation in pill dosages for the new manufacturing process.

We have $n=13$, $S_x = 14$

$1-\alpha = .97 \rightarrow \alpha = .03$

$$1-\alpha = P\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{1-(\alpha/2),\, n-1}}} \le \sigma \le \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,\, n-1}}}\right)$$

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> qchisq(1-.015, 12)
24.963
> qchisq(.015, 12)
3.910
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$$\Longrightarrow \left(\sqrt{\frac{12(7.766)^2}{24.963}}, \sqrt{\frac{12(7.766)^2}{3.910}}\right)$$

$$\Longrightarrow (5.384, 13.608)$$

Looks like the medication dosage is pretty consistent.

2. I recently attended a Padres (baseball) game that was on pace to be the shortest game in the modern era (all hope was ruined when people started scoring in the 8th inning!). I also happened to be watching TV in 2010 when the longest tennis game ever was played (11 hours, 5 minutes). All this got me thinking about the times of sporting events. For the sake of fans, commentators, and marketing departments, it is helpful to have low variability in the time it takes to complete an event, and an average game time that is long enough to entertain fans, but not so long that people get exhausted. You decide to explore the effects of various rule changes that occurred in the NHL (ice hockey!). Prior to a new rule set lauched in the early 2000s, hockey game times were known to be normally distributed with an average time of 2 hours and 36 minutes and a standard deviation of 19.2 minutes. Using a random sample of 24 games from the 2012 season (these occurred after rule changes; we'll assume they are normally distributed), you find an average time of $\bar{x} = 2.316$ hours with $s_x = 18.3$ minutes. If the goal of these changes was to decrease the average time but keep the variation the same, do you think the new rules have done it? Argue using two hypothesis tests, each with $\alpha = 0.02$. Do the variance test first, and then the mean test.

$\mu = 2\text{hr}, 36\text{min} = \underline{156\text{min}}$

$\sigma = 19.2$ min

$n = 24$

$\bar{x} = 2.316\text{hr} = 138.96\text{min}$

$S_x = 18.3\text{min}$

$\alpha = .02$

Test Variance $\qquad H_0: \sigma^2 = 19.2\text{min} \qquad H_1: \sigma^2 \neq 19.2$

Assuming H_0, $\quad \dfrac{(n-1)S^2}{\sigma^2} = \dfrac{(23)(18.3)^2}{(19.2)^2} = 20.894 \sim \chi^2_{23}$ Test Stat

To test $H_1: \sigma^2 \neq 19.2$ $\qquad$ * we reject H_0 if either $\chi^2 > \chi^2_{1-\alpha/2,\,n-1}$ or $\chi^2 < \chi^2_{\alpha/2,\,n-1}$

>qchisq(1-.01, 23)
41.638 > 20.894 →

>qchisq(.01, 23)
10.196 < 20.894 →

$\boxed{\text{We Keep } H_0\text{, the variance has remained the same.}}$

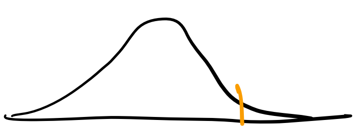
Test Mean $\qquad H_0: \mu = 156\text{min} \qquad H_1: \mu < 156\text{min} \qquad \bar{x} = 138.96\text{min}$

Assuming H_0 true: $\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim T_{n-1} = \dfrac{138.96 - 156}{19.2/\sqrt{24}} = -4.348$

σ IS KNOWN FROM ABOVE

To test H_1: * if $t < t_{\alpha,n-1}$ reject H_0

>qnorm(.02) = -2.054



-2.054

$-4.348 < -2.054$

$\boxed{\text{We reject } H_0\text{, the average game time was decreased.}}$

3. Suppose that $X \sim N(\mu, 1)$. We plan to test H_0: $\mu = 0$ vs. H_1: $\mu = 4$ using a random sample of size n. Show that the BCR will take the form $C = \{\mathbf{X} | \sum_{i=1}^{n} X_i > c\}$ where c is a constant.

$$L(\mu; \vec{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-(\Sigma x_i - \mu)^2/2}$$

BCR: want $\dfrac{L(\mu=0)}{L(\mu=4)} < K$ for some $K > 0$

$$= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-(\Sigma x_i)^2/2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-(\Sigma x_i - 4)^2/2}} = e^{-\frac{(\Sigma x_i)^2}{2} + \frac{(\Sigma x_i - 4)^2}{2}} = e^{\left[(\Sigma x_i - 4)^2 - (\Sigma x_i)^2\right]/2} < K$$

$$\Rightarrow \left[\frac{(\Sigma x_i - 4)^2 - (\Sigma x_i)^2}{2}\right] < \ln(K)$$

$$\Rightarrow \cancel{(\Sigma x_i)^2} - 8\Sigma x_i + 16 - \cancel{\Sigma x_i^2} < 2\ln(K)$$

$$\Rightarrow -8\Sigma x_i < 2\ln(K) + 16$$

$$\Rightarrow \Sigma x_i > 16\ln(K) + 128 = c$$

$$\rightsquigarrow \text{BCR} = C = \{\vec{x} \mid \sum_{i=1}^{n} x_i > c\} \text{ as desired.}$$

4. Suppose that $X \sim N(\overset{\mu}{\theta_1}, \overset{\sigma^2}{\theta_2})$, both unknown. We plan to test H_0: $\theta_1 = 0$, $\theta_2 = 1$ vs. H_1: $\theta_1 = 1$, $\theta_2 = 3$. Find the simplest expression you can for a BCR in this setting. Assume a random sample of size n.

$$L(\theta_1, \theta_2; \vec{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_i - \theta_1)^2 / 2\theta_2}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^n e^{-(\sum x_i - \theta_1)^2 / 2\theta_2}$$

$$\frac{L(\theta_1 = 0, \theta_2 = 1)}{L(\theta_1 = 1, \theta_2 = 3)} < K \quad \text{for } K > 0$$

$$= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-(\sum x_i)^2/2}}{\left(\frac{1}{\sqrt{2\pi 3}}\right)^n e^{-(\sum x_i - 1)^2/6}} = \left(\frac{1}{\sqrt{3}}\right)^n e^{-\frac{(\sum x_i)^2}{2} + \frac{(\sum x_i - 1)^2}{6}}$$

$$= \left(\frac{1}{\sqrt{3}}\right)^n e^{\frac{(\sum x_i - 1)^2 - 3(\sum x_i)^2}{6}} = \left(\frac{1}{\sqrt{3}}\right)^n e^{-\frac{2(\sum x_i)^2 - 2\sum x_i + 1}{6}} < K$$

$$\Rightarrow e^{\frac{-2(\sum x_i)^2 - 2\sum x_i + 1}{6}} < (\sqrt{3})^n K = K_1$$

$$\Rightarrow \frac{-2(\sum x_i)^2 - 2\sum x_i + 1)}{6} < \ln(K_1) = K_2$$

$$\Rightarrow -2(\sum x_i)^2 - 2\sum x_i < 6K_2 - 1 = K_3 \Rightarrow \sum x_i^2 + \sum x_i > K_4$$

$$\Rightarrow \sum x_i^2 + x_i > c \quad \boxed{\text{BCR is } C = \{\vec{x} \mid \sum_{i=1}^{n} x_i^2 + x_i > c}$$

5. Suppose you take a random sample of size 12 from $X \sim N(0, \sigma^2)$, whose variance is unknown. You plan to test H_0: $\sigma^2 = 1$ against H_1: $\sigma^2 = 3$. Find the BCR of size $\alpha = 0.08$.

$$P_0(\vec{X} \in C) = \alpha = .08$$

$$L(\sigma^2, \vec{x}) = \prod_{i=1}^{12} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x_i)^2/2\sigma^2} = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{12} e^{-(\sum x_i)^2/2\sigma^2}$$

$$\frac{L(\sigma^2=1)}{L(\sigma^2=3)} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^{12} e^{-(\sum x_i)^2/2}}{\left(\frac{1}{\sqrt{6\pi}}\right)^{12} e^{-(\sum x_i)^2/6}} = \frac{\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{6}}\right)^{12}} e^{-\frac{3(\sum x_i)^2}{6} + \frac{(\sum x_i)^2}{6}}$$

$$= 729\, e^{-(\sum x_i)^2/3} < K \quad \text{for} \quad K > 0$$

$$\Rightarrow -(\sum x_i)^2/3 < \ln\left(\frac{K}{729}\right)$$

$$\Rightarrow (\sum x_i)^2 > -3\ln\left(\frac{K}{729}\right) \Rightarrow \sum_{i=1}^{12} x_i^2 > c$$

w/ $c = \left(-3\ln\left(\frac{K}{729}\right)\right)^2$

We need $.08 = P_0\left(\sum_{i=1}^{12} x_i^2 > c\right)$ w/ $x_i \sim N(0, \sigma^2)$

$H_0: \sigma^2 = 1$

$\hookrightarrow x_i \sim N(0,1)$

$$\chi_n^2 = \sum_{i=1}^{n} Z_i^2$$

$$x_1 + \dots + x_{12} \sim N(0,1)$$

So $\sum_{i=1}^{12} x_i \sim \chi_{12}^2$

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> qchisq(0.08, 12, lower=F)
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$= c$

$$\boxed{c = 19.369}$$