7. Portfolio Theory and Diversification

Work in Progress

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1 Introduction

The economic growth of a country depends heavily on investment that firms make in a variety of projects. However, investment requires capital at a cost that is not higher than the benefits that firms can derive from their investment. Firms may abandon profitable investment opportunities if investors do not provide capital or if the cost of capital is too high. Within this context, a key objective of the financial theory is to examine how the cost of capital is determined through an interaction between firms and investors.

The financial theory provides a variety of models in which investors' demand for risky assets plays a key role in determining the prices of securities (e.g. stocks and bonds) issued by firms. Most of these models use probability to represent uncertainty associated with pay-offs for different assets. Perhaps the most well-known framework is the **mean-variance analysis** pioneered by the Nobel Laureate Harry Markowitz. The mean-variance analysis posits that investors select risky assets on the basis of means and variances of returns. This framework was then extended by others to form the **Capital Asset Pricing Model**. This session builds upon our earlier discussion on probability theory to introduce the mean-variance analysis. We will also study simple portfolio optimisation problems and its implementation in `R`.

2 Intended Learning Outcomes

By the end of this session, students should be able to

1. understand the benefits of diversification,
2. comments on the importance of correlation between asset returns for diversification,
3. generate a simple stochastic process, and

4 use `R` to compute the minimum variance portfolio and the tangency portfolio.

3 Rational Choice Model

As most of the finance literature builds upon the rational choice model under uncertainty, we provide a very short description of this model. In finance (as in many other subjects), rationality means making reasonably consistent choices. Within this framework, *"[r]easonableness is a property of patterns of choices, not of individual choices. There is nothing irrational in wanting to go home but there is something amiss in wanting both to go home and not to go home"* (Allingham 2002, p. 3).

3.1 Rational behaviour

How do people make decisions under uncertainty? This is a hard question and it is safe to say that even after decades of research, we still do not have the correct answer to this question. Readers interested in formation of preferences and choices may consult Dolan and Sharot (2011). In the mainstream finance, an investor is modelled as a rational agent with well-defined preferences that can be represented by a utility function $u(x)$, for all levels of wealth x. Investors' taste for risk is captured in the shape of their utility curve. For example, for a risk averse investor, $u'(x) > 0$ and $u''(x) < 0$ (Danthine and Donaldson 2014, Chapter 4). Roughly speaking, it means that investors prefer more wealth to less, but their marginal utility of wealth is diminishing.

The future wealth of economic agents depends on assets that they include in their portfolios. These assets then can be viewed as vehicles (or time machines) that transform current wealth into future wealth. The future wealth, however, is not certain. To model uncertainty, assets' returns are considered as random variables. Economic agents are then assumed to *maximise their expected utility* given their utility function and the probability distribution of returns for different assets. The diminishing utility, $u''(x) < 0$, can be used to derive agents *risk aversion*.

If R represents a random vector of returns and x_0 is the amount to be invested, then an investor's choice of assets is based on the maximisation of expected utility (Ruppert and Matteson 2015, chapter 16):

$$E[u(x)] = E[u(x_0(1 + R))] \tag{1}$$

The role played by the probability theory in the above framework is obvious: the maximisation of expected utility depends on the probability distribution of returns. We now provide a brief introduction to the mean-variance portfolio analysis in which risky assets are chosen on the basis of means and variances. For detail, see Francis and Kim (2013) or Danthine and Donaldson (2014).

4 Returns as Random Variables

Financial theory considers returns as random variables. Various probability models are used to approximate the evolution of returns over time. Let us consider two simple models.

4.1 Normally Distributed Single-Period Log Return

A simple, and not very accurate, model posits that single-period log returns $r_t = log(1 + R_t)$ are independent and are normally distributed with mean μ and variance σ^2. That is, for each time period t, $r_t \sim N(\mu, \sigma^2)$. For example, suppose we have two portfolios, Portfolio Y and Portfolio X. We assume that the log return on Portfolio Y is normally distributed with $\mu_Y = 5$ and $\sigma_Y = 5$, while the log return on Portfolio X is normally distributed with $\mu_X = 5$ and $\sigma_X = 2$.

Figure 1 shows the probability distributions of Portfolio Y and Portfolio X (both centred at 5). However, notice that the dispersion of returns around the mean for Portfolio Y is greater than that of Portfolio X. This larger dispersion is captured by $\sigma_Y = 5$ compared to $\sigma_X = 2$. Within the mean-variance framework, a rational risk-averse economic agent will choose Portfolio Y over Portfolio X because that latter offers same reward (measured by expected returns) for lower risk (measured by standard deviations). Note the use of `dnorm()` to plot the normal curves.

```
df <- tibble(x = seq(-15,30, length.out = 500)) %>%
  mutate(y1 = dnorm(x, mean = 5,sd = 5),
         y2 = dnorm(x, mean = 5,sd = 2))




ggplot(df) +
  geom_path(aes(x,y1, color = "Portfolio Y")) +
```

```r
  geom_path(aes(x, y2, color = "Portfolio X")) +
  labs(x = "Log Return Y", y = "f(y)") +
  theme(axis.title=element_text(size=8),
        legend.title=element_blank())
```

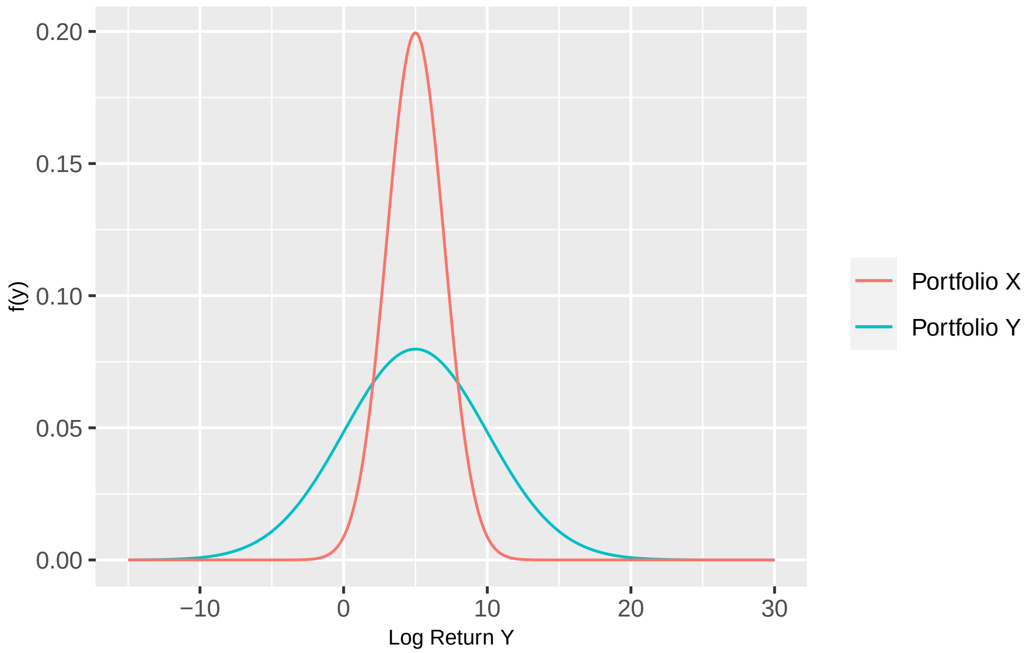


Figure 1: Normal Distribution and Log Return

```r
rm(df)
```

4.2 Returns for FTSE 100 and Normal Distribution

Is normal distribution a good model for log returns? Simple answer is no. Figure 2 shows the empirical density plot for returns for FTSE 100. We also show a normal distribution with mean and standard deviations equal to those of the empirical distribution. Although the return distribution looks symmetrical, it is not normal (see Panel 1). However, when we plot a normal curve with mean equal to the median of returns, and standard deviation equal to the median absolute deviation (Panel 2), then the normal does approximate the empirical density of returns slightly better (Ruppert and Matteson 2015, p. 51).

```r
# Open Index data

indexRt <- read_csv(file = "indices.csv")
```

```
## Parsed with column specification:
## cols(
##   date = col_date(format = ""),
##   GSPC = col_double(),
##   FTSE = col_double(),
##   HSI = col_double(),
##   N100 = col_double()
## )
```

```
# Compute Returns and select FTSE 1nd EuroNext

indexRt <- indexRt %>%
  mutate(FTSE = log(FTSE) - log(lag(FTSE))) %>%
  select(date, FTSE) %>%
  drop_na()

c(mean(indexRt$FTSE),sd(indexRt$FTSE))
```

```
## [1] -2.260632e-05  1.175137e-02
```

```
fig1 <- ggplot(indexRt) +
  geom_density(aes(FTSE), color = "steelblue") +
  stat_function(fun = dnorm,
              args = list(mean = mean(indexRt$FTSE),
                          sd = sd(indexRt$FTSE)),
              color = "red") +
  labs(title = "Panel 1")



fig2 <- ggplot(indexRt) +
  geom_density(aes(FTSE), color = "steelblue") +
  stat_function(fun = dnorm,
              args = list(mean = median(indexRt$FTSE),
                          sd = mad(indexRt$FTSE, center = median(indexRt$FTSE))),
```

```
                 color = "red") +
  labs(title = "Panel 2")



grid.arrange(fig1, fig2)
```

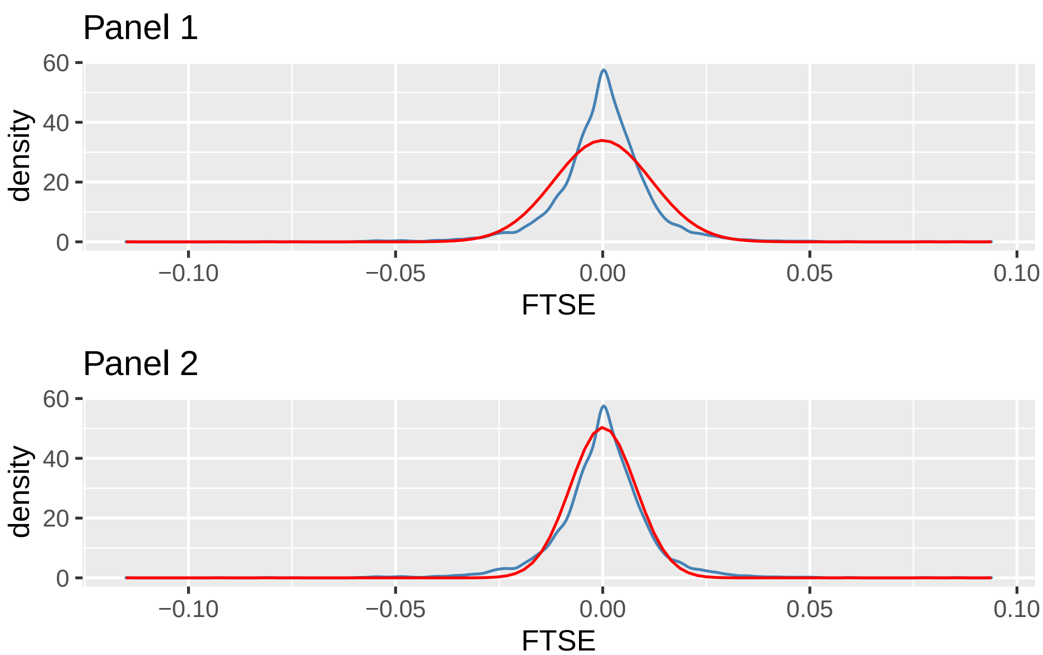


Figure 2: FTSE 100 Returns and Normal Distribution

```
rm(indexRt)
```

4.3 Stochastic Process

Investors usually hold assets for several periods and, consequently, they are interested in how returns will evolve over time. In financial theory, the random evolution of returns is modelled using various **stochastic processes**. A good source for learning stochastic processes using `R` is Dobrow (2016).

A stochastic process X_t is a sequence of random variables indexed by time. The stochastic process begins with a value X_0 and then at each time t moves up or down depending on a random draw from a distribution. For example, we may consider tossing a coin and recording the outcome for a given period of time (say every minute). In this example, the time is countable, and so the process is a discrete time stochastic process. Alternatively, we may study the evolution of temperature or stock prices at every possible instant. This is an example of a continuous time stochastic process.

4.3.1 Normal Random Walk

A simple model for log returns is the **normal random walk**. In this model, log return at each time t is assumed to be independent of returns at other time periods and is assume to be normally distributed with mean μ and standard deviation σ. In this model, starting at time $t = 0$, the t period log return can be written as (Ruppert and Matteson 2015, p. 8):

$$r_t = r_0 + r_1 + r_2 + \cdots + r_t, \qquad t \geq 1 \tag{2}$$

It can be shown that in the above model, the standard deviation of r_t is $\sqrt{t}\sigma$. So, the volatility measured by standard deviation increases with time (but not in a linear fashion). We can simulate the above stochastic process in `R` as follows. Suppose the process begins with $r_0 = 0$. Then, in each period, a random term drawn from $N(\mu = 0, \sigma = 1)$ is added to the process. Let us simulate 10 possible evolutions of random returns over 100 time periods.

We begin by creating a dataframe that has the `time` variable of length 100.

```
df2 <- tibble(time = seq(1,100, length.out = 100))
```

We now create 20 evolutions of the return process using the `replicate()` function. Each process adds a random draw to the process for each of the 100 periods.

```
# bind_rows() from dplyr is used to combine rows of different dataframes


set.seed(1234)
random_walks <- bind_rows(
  replicate(20,
            df2 %>% mutate(y = cumsum(rnorm(100, 0,1)),
                           sd1p = sqrt(time),
                           sd1n = -sqrt(time)),
            simplify=FALSE), .id="SampNum")
```

The 20 evolutions of the normal random walk are shown in Figure 3.

```r
ggplot(random_walks) +
  geom_line(aes(x = time,y= y, group = SampNum,
            color = SampNum), size = 0.3) +
  theme(legend.position="none") +
  geom_line(aes(x = time, y = sd1p), color = "red", size = 1) +
  geom_line(aes(x = time, y = sd1n) , color = "red", size = 1)
```

Figure 3: A Normal Random Walk

```r
rm(df2,random_walks)
```

4.4 Exercise

Use the above code to show 20 evolutions of a random walk in which $r_t \sim N(0, 4)$. Compare this stochastic process to the one visualised above.

```
# Your code here
```

5 Portfolio Selection Problem With Two Risky Assets

In this section, we assume that rational *mean-variance optimising* investors have access to two risky assets. We further assume that means, variances and correlations of assets' joint distribution are available to all investors. Our discussion below, including the notations, is based on Francis and Kim (2013). The key idea is that mean-variance optimising agents choose a portfolio that offers best reward (expected return) for a given level of risk (standard deviation).

5.1 Expected Return and Variance of a Portfolio

Suppose there are two risky assets: stocks issued by Dorian PLC (Asset 1) and stocks issued by Ionion plc (Asset 2). The returns of the two assets have means and variances as shown in Table 1.

Table 1: Expected Returns and Standard Deviations of Returns

	Dorian plc	**Ionian plc**
Expected Return	$\mu_1 = 2\%$	$\mu_2 = 3\%$
Variance	$\sigma_{11} = 16$	$\sigma_{22} = 25$

Figure **??**.

Let w_1 denote the fraction of wealth invested in Asset 1 and $w_2 = (1 - w_1)$ denote the fraction invested in Asset 2. Given the expected return for two assets and portfolio weights, the expected return for the portfolio is

$$\mathbb{E}(R_p) == \mu_p = w_1\mu_1 + (1 - w_1)\mu_2 \tag{3}$$

So, the expected return for the portfolio is simply the weighted average of expected returns of assets in the portfolio. The expected return of an asset (individual or portfolio) represents the *reward* that investors expect from the asset. High expected return means high reward.

In the mean-variance analysis, investors do not choose on the basis of reward only; they also consider the *risk* associated with assets' returns. Risk in the mean-variance analysis is represented by the standard deviation of asset returns. Recall that standard deviation is the square root of variance. High variance means high risk. Given the expected returns and variances, the mean-variance analysis seeks to evaluate assets that investors should include in their portfolios.

Let σ_{11} and σ_{22} denote the variances of returns for Asset 1 and Asset 2, respectively, and σ_{12} denotes the covariance of returns. The correlation between the two returns is

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}} \tag{4}$$

The variance of a portfolio, σ_p^2 of two assets with w_1 invested in Asset 1 and $w_2 = (1 - w_1)$ invested in Asset 2 is shown in Equation :

$$\sigma_p^2 = w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12} = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \sigma_{ij} \tag{5}$$

The standard deviation of the portfolio is simply the square root of the above variance. That is,

$$\sigma_p = \sqrt{\sigma_p^2} \tag{6}$$

5.2 Correlation and Diversification

In this section, we show how the covariance term in Equation determines the **risk-return** trade-off. We plot a set of portfolios available to investors for two different correlations between assets' returns. Specifically, we consider the two cases $\rho_{12} = 0.25$ and $\rho_{12} = -0.25$

```
# Save expected returns and standard deviations


R1 <- 2 # Expected return for Asset 1
R2 <- 3 # Expected return for Asset 2


S1 <- 16 # Variance return for Asset 1
S2 <- 25 # Variance return for Asset 2
```

We now store two possible correlations and then create a dataframe with portfolio weights, expected returns and variances.

```
# Two possible correlations


Corr1 <- 0.25

Corr2 <- -0.25


cov1 <- Corr1*sqrt(S1)*sqrt(S2)

cov2 <- Corr2*sqrt(S1)*sqrt(S2)


# To keep things simple, w is in [0,1]


two_assets <- tibble(w = seq(0,1, length.out = 500)) %>%

  mutate(Rp1 = R1*w + R2*(1-w),

         Sp1 = sqrt(w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*cov1),

         Rp2 = R1*w + R2*(1-w),

         Sp2 = sqrt(w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*cov2))
```

We now use the dataframe `two_assets` to plot the standard deviations and expected returns for portfolios with different weights of Asset 1 and Asset 2. Plots for two different correlations is in Figure 4. Let us call these curves the *opportunity set* or the *feasible set*.

```
fig_25 <- ggplot(two_assets) +

  geom_point(aes(Sp1, Rp1), color = "steelblue") +

  geom_point(aes(Sp2, Rp2), color = "purple") +

  labs(x = "Portfolio Standard Deviation",

       y = "Portfolio Expected Return") +

  geom_point(aes(x = 3.572588, y = 2.737475), size = 3, color = "red") +

  geom_point(aes(x = 4.078664, y = 2.737475), size = 3, color = "red") +

  annotate("text", x = 3.5, y = 2.8, label = "X")+

  annotate("text", x = 4.2, y = 2.7, label = "Y")


fig_25
```

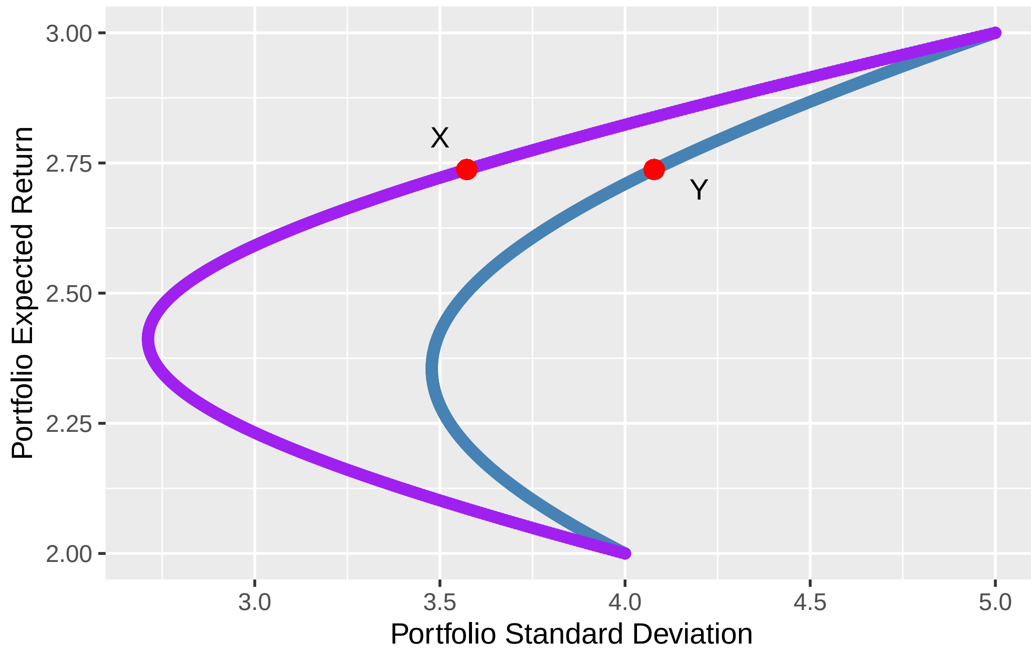


Figure 4: Risk-Return Trade-off and Correlation

```
rm(list = ls())
```

Figure 4 shows the impact of correlation on risk-return trade-offs facing investors. The purple curve shows portfolios with the correlation -0.25, while the blue curve shows the portfolios with correlation 0.25. Portfolios in the purple curve (e.g. portfolio Y) earn the same expected return but with lower standard deviation than portfolios in the blue curve (e.g. portfolio X).

5.3 Exercise

Use the above code and show plots of risk-return for two assets assuming the the correlations are -1 and 1. You will see that when the correlation is -1 we can construct a portfolio with zero risk (measured by standard deviation). Also, when the correlation is 1, there is not benefit of diversification.

```
# Your code.
```

```
rm(list = ls())
```

5.4 Plots of Multiple Frontiers using `purrr`

This is an optional section to show you that we can use `R` efficiently to produce portfolio frontiers for multiple correlations with only a few lines of code. We use the `purrr` package, also included in the `tidyverse`. You can skip this for the moment and come back to it at a later stage.

5.4.1 Writing Your Own Functions

Let us start by creating a few simple functions that will help us understand the structure of a function.

```
## Our first function - mutiply a number by 2
myFunc1 <- function(x){
  x*2
}

myFunc1(2)
```

```
## [1] 4
```

```
#myFunc1("s")
```

```
## Our second function - mutiply a number by 2 after checking

myFunc2 <- function(x){
  if (is.character(x)){
    print("Cannot multiply character with a number")
  } else {
    x*2
  }
}

myFunc2(2)
```

```
## [1] 4
```

```r
#myFunc2("s")
```

We now remove our two functions from the `R` environment.

```r
rm(list = ls())
```

5.4.2 A Function to Generate Portfolios

We now write a function to construct the opportunity set for given values of expected return and variances. The argument of the function is the correlation between returns. This will be provided when we call the function.

```r
# Write a function
opportunity.set <- function(x){
# Specify expected returns and variances
R1 <- 2 # Expected return for Asset 1
R2 <- 3 # Expected return for Asset 2

S1 <- 16 # Variance return for Asset 1
S2 <- 25 # Variance return for Asset 2
# Dataframe containing Return and Risk
two_assets <- tibble(w = seq(0,1, length.out = 500)) %>%
  mutate(Rp = R1*w + R2*(1-w),
         Sp = sqrt(w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*x*sqrt(S1)*sqrt(S2)))
}
```

Let us now call the function. This should create a dataframe. You can now see the power of functions. We can create dataframes with any correlation between two assets' returns.

```r
df <- opportunity.set(1)

rm(df)
```

Let us now create a dataframe that contains returns for several correlations. We will use `map_df()` function from the `purrr` package. We will not spend a lot of time studying `purrr` at this stage.

```r
# Create a list of correlations

correlations <- list(-1, -0.75, -0.5,-0.25,0, 0.25,0.5, 0.75, 1)

# Apply your function to the list of correlations

portfolios <- map_df(correlations, opportunity.set)
```

A little bit more work is needed. We know that our function creates 500 observations for each correlations. So, we can create a variable that will indicate the correlation that a particular risk-return row belongs to.

```r
portfolios <- portfolios %>%
  mutate(Correlation = as.factor(rep(c(-1, -0.75, -0.5,-0.25,0, 0.25,0.5, 0.75, 1),
                                     each = 500)))
```

Figure 5 shows the portfolio frontiers for 9 different correlations. It is clear from this figure that as correlation between assets goes down, risk that investors have to bear for a given level of expected return also goes down. This highlights that **diversification** reduces risk but only if assets included in your portfolio are not perfectly positively correlated.

```r
ggplot(portfolios, aes(Sp, Rp, color = Correlation)) +
  geom_point(alpha = 0.60, size = 0.8) +
  labs(x = "Portfolio Standard Deviation",
       y = "Portfolio Expected Return")
```

```r
rm(portfolios, correlations, opportunity.set)
```

5.5 Minimum Variance Portfolio

Let us work with Equation for the variance of a portfolio. We can use this equation to determine the **minimum-variance portfolio** by first writing $w_2 = 1 - w_1$, and then differentiating with respect to w_1 and setting it equal to 0. This is the *first order condition* used in optimisation problems. We have

$$\frac{\partial \sigma_p^2}{\partial w_1} = \frac{\partial}{\partial w_1}\bigg(w_1^2\sigma_{11} + (1-w_1)^2\sigma_{22} + 2w_1(1-w_1)\sigma_{12}\bigg) = 0$$

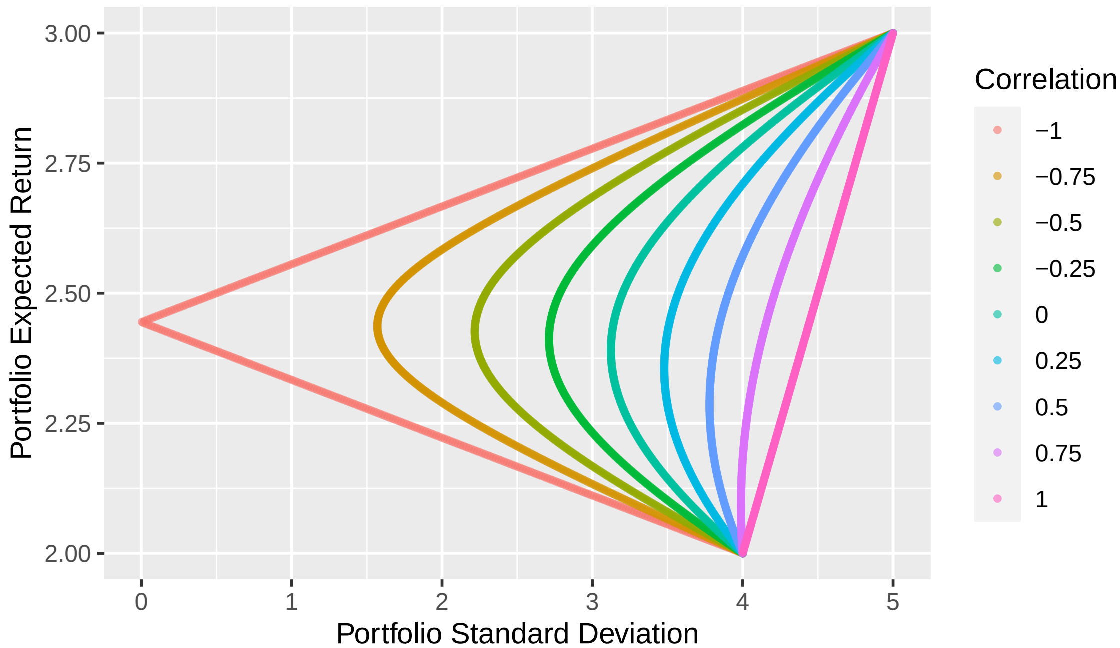


Figure 5: Portfolio Returns with Different Correlations

This obtains the following equation for the weight w_1^M of Asset 1 that leads to the portfolio with the lowest possible variance (risk) (Francis and Kim 2013, pp.118):

$$w_1^M = \frac{\sigma_{22} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}} \tag{7}$$

You may be interested in the second-order condition to check that the above is the minimum value.

The portfolio that invests w_1^M in Asset 1 and $1 - w_1^M$ in Asset 2 is called the **minimum-variance portfolio**. We will see the significance of this portfolio in a moment. Let us now assume $R_1 = 2\%$, $R_2 = 3$, $\sigma_{11} = 16$, and $\sigma_{22} = 25$, and $\sigma 12 = 5$. Note that $\sigma 12 = 5$ leads to a correlation of $\rho_{12} = 0.25$ for these two assets.

```
R1 <- 2 # Expected return for Asset 1
R2 <- 3 # Expected return for Asset 2


S1 <- 16 # Variance return for Asset 1
S2 <- 25 # Variance return for Asset 2




# Correlaton 0.25
```

```r
Corr1 <- 0.25
cov1 <- Corr1*sqrt(S1)*sqrt(S2)


w1min <- (S2 - cov1)/(S1 + S2 - 2*cov1)


two_assets <- tibble(w = seq(0,1, length.out = 500)) %>%
  mutate(Rp1 = R1*w + R2*(1-w),
         Sp1 = sqrt(w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*cov1))


# Expected return for the minimum variance portfolio
Rpmin <- R1*w1min + R2*(1-w1min)


# Standard deviation for the minimum variance portfolio
Sdmin <-  sqrt(w1min^2*S1+ (1-w1min)^2*S2 + 2*w1min*(1-w1min)*cov1)
```

Figure 6 plots the portfolio frontier and highlights the minimum-variance portfolio.

```r
ggplot(two_assets) +
  geom_point(aes(Sp1, Rp1), color = "steelblue") +
  labs(x = "Portfolio Standard Deviation",
       y = "Portfolio Expected Return") +
  geom_point(aes(x = Sdmin, y = Rpmin), size = 3, color = "red") +
  annotate("text", x = Sdmin+0.3, y = Rpmin,
           label = "Minimum-Variance Portfolio", size = 4) +
  geom_point(aes(x = 3.73468, y = 2.60), size = 3, color = "green") +
  geom_point(aes(x = 3.73468, y = 2.11), size = 3, color = "purple") +
  annotate("text", x = 3.73468, y = 2.65,
           label = "A", size = 4) +
  annotate("text", x = 3.73468, y = 2.15,
           label = "B", size = 4)
```

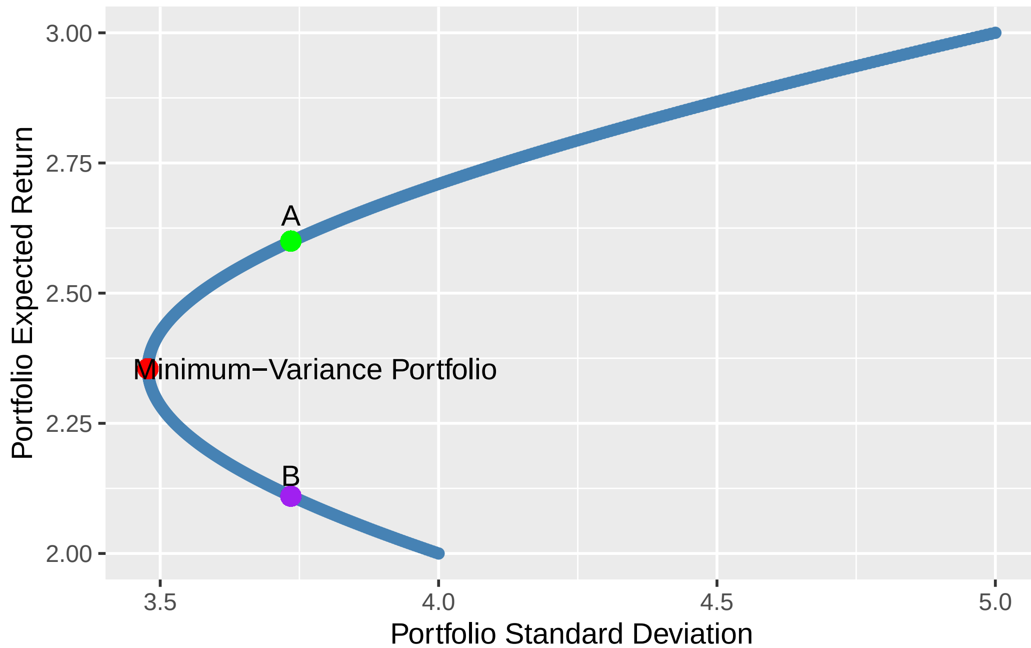


Figure 6: Portfolio Frontier and the Minimum-Variance Portfolio

5.6 Very Simple Optimisation in R

Optimisation is a big topic and has many applications in economics and finance. R has several tools for optimisation problems. Here is a very simple example in with we use the `optimize()` function to obtain the minimum-variance portfolio weight w_1.

```
port.var <- function(w) w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*cov1


optimize(port.var, c(0, 1))
```

```
## $minimum
## [1] 0.6451613
##
## $objective
## [1] 12.09677
```

Notice that the output shows the minimum value of `w` - this is the value of `w` that minimises the function specified in the `port.var` expression.

5.7 The Efficient Frontier - Without Risk-Free Asset

In Figure 6, we have a world with only two risky assets. What can we say about investors' portfolio choice assuming that they seek to maximise reward (i.e. expected return) and minimise risk (i.e. standard deviation)? It is evident that rational investors should not invest in any part of the frontier that is below the minimum-variance portfolio because for each portfolio below the minimum-variance point, there is a portfolio above this point that offer a better return for the same level of risk. For example, compare the two portfolios marked by A and B in Figure Figure 6. We call the part of the frontier above the minimum-variance portfolio the **efficient frontier** without the risk-free asset.

```
rm(list = ls())
```

5.8 Multiple Risky Assets

The analysis above contains only two risky assets. It is reasonably straightforward, but mathematically a bit more demanding, to extend the analysis to more than two assets. The key lesson that we learn from applying the above analysis to more than two assets is that diversification reduces risk when we select assets that are not perfectly correlated (lower correlation between returns leads to higher diversification benefit). We have provided `R` code using our own function and `map_df()` function from `purrr` to show portfolio frontier for three assets in Figure 7. We recommend that you come back to this after gaining a bit more experience with `R`.

```
# Write a function

set.seed(1234)

# We can extend our function and include means,
# varainces and covariances as additional arguments.

portfolio.frontier <- function(x){
  R1 <- 5
  R2 <- 6
  R3 <- 8
  S1 <- 36
```

```r
  S2 <- 64
  S3 <- 81
  C12 <- 20
  C13 <- -20
  C23 <- 0
# Dataframe containing Return and Risk
two_assets <- tibble(w1 = runif(100,min = 0,max = 0.9)) %>%
  mutate(Rp = R1*w1 + R2*w2 + R3*(1-w1-w2),
         Sp = sqrt(w1^2*S1+ w2^2*S2 +(1-w1-w2)^2*S3+
                     2*w1*w2*C12 + 2*w1*(1-w1-w2)*C13 + 2*w2*(1-w1-w2)*C23))
}

# A list of correlations
set.seed(1234)
w2 <- runif(50,min = 0,max = 0.5)

# Apply your function to the list of correlations
port_returns <- map_df(w2, portfolio.frontier)
```

```r
ggplot(port_returns, aes(Sp, Rp)) +
  geom_point(alpha = 0.15, size = 0.5, color = "steelblue") +
  labs(x = "Portfolio Standard Deviation",
       y = "Portfolio Expected Return") +
  geom_point(aes(x = 6, y = 5, color = "Asset 1"), size = 2.5)+
  geom_point(aes(x = 8, y = 6, color = "Asset 2"), size = 2.5)+
  geom_point(aes(x = 9, y = 8, color = "Asset 3"), size = 2.5) +
  theme(axis.title=element_text(size=8),
        legend.title=element_blank())
```

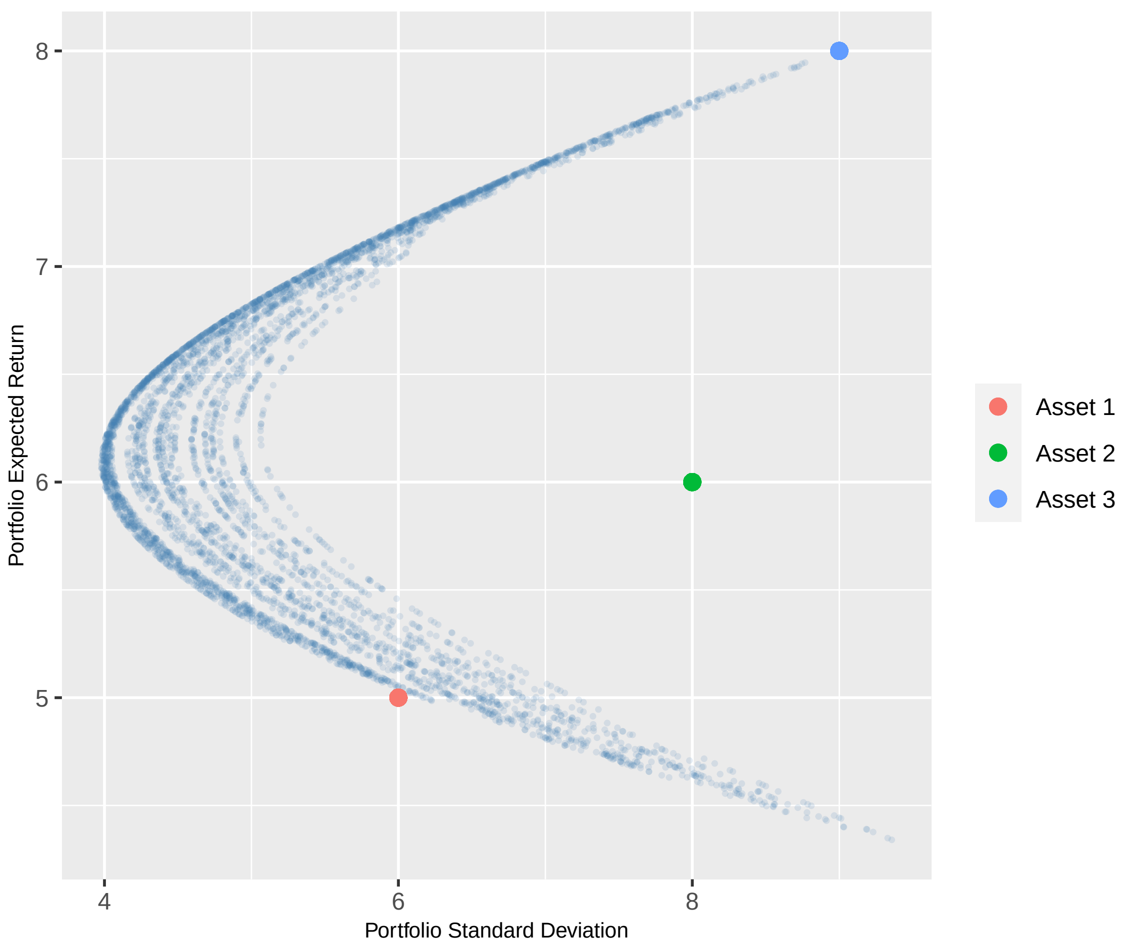


Figure 7: Portfolios for Three Assets

6 Portfolio Selection Problem - Including Risk-Free Asset

6.1 Efficient Portfolios with Risk-Free Asset

This section provides an elementary discussion on computing the optimal risky portfolio. We recommend that one you have more experience with `R`, you use optimisation tools available in `R` for the computation below. Some discussion and `R` code is available in Ruppert and Matteson (2015).

We now introduce a risk-free asset that earns R_f into the above framework with two risky assets. The analysis can be extended for combinations of a risk-free asset and the efficient frontier formed with N risky assets in a similar manner. In this section we will use matrix notations. Our aim here is to identify a portfolio R_t, called the **tangency portfolio**, which is composed of risky assets only. This tangency portfolio, when combined with the risk-free asset, offers the best possible risk-return trade-off to investors.

Let us continue to work with our 2 risky assets, namely Asset 1 and Asset 2. Define

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1 is the fraction invested in Asset 1 and the w_2 is the fraction invested in Asset 2. Let w_f be the fraction invested in the risk-free asset. We assume that the sum of weights is equal to 1. That is,

$$w_f + \mathbf{w}'\mathbf{1} = 1$$

or

$$w_f = 1 - \mathbf{w}'\mathbf{1}$$

where $\mathbf{1}$ is a column vector of 1s. We are now interested in a portfolio that contains the two risky asset with weights $\mathbf{w}$ and the risk-free asset with weight w_f such that for any given level of expected return $\mathbb{E}(R_p)$, the variance of the portfolio is minimised. Note that the risk-free asset has 0 variance and it does not co-move with the other two assets. Thus, it does not add anything to the variance of the portfolio that we are looking for.

So, with given $\mathbb{E}(R_p)$, we wish to minimise the following variance of our portfolio

$$\sigma_p^2 = w_1^2\sigma_{11}^2 + w_2^2\sigma_{22}^2 + 2w_1w_2\sigma_{12} \tag{8}$$

subject to the constraint

$$\mathbb{E}(R_p) = w_1\mu_1 + w_2\mu_2 + w_f R_f \tag{9}$$

Note that we have already assumed that weights sum to 1.

Now, weights that minimise the variance σ_p^2 are same as minimising $\frac{1}{2}\sigma_p^2$. This multiplication with $\frac{1}{2}$ will simplify our optimisation problem. With this in mind, using the above constrains, and multiplying variance with $\frac{1}{2}\sigma_p^2$, we set-up the following Lagrangian optimisation problem :

$$minimise \;\; \mathscr{L} = \frac{1}{2}\bigg(w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12}\bigg) + \lambda\bigg(\mathbb{E}(R_p) - (w_1\mu_1 + w_2\mu_2) - (1 - w_1 - w_2)R_f\bigg) \tag{10}$$

We now differentiate the $\mathscr{L}$ with respect to w_1, w_2 and λ and obtain the following first-order conditions for the minimisation problem:

$$\frac{\partial \mathscr{L}}{\partial w_1} = w_1\sigma_{11} + w_2\sigma_{12} - \lambda\bigg(\mu_1 - R_f\bigg) = 0 \tag{11}$$

$$\frac{\partial \mathscr{L}}{\partial w_2} = w_2\sigma_{22} + w_1\sigma_{12} - \lambda\bigg(\mu_2 - R_f\bigg) = 0 \tag{12}$$

$$\mathbb{E}(R_p) - (w_1\mu_1 + w_2\mu_2) - (1 - w_1 - w_2)R_f = 0 \tag{13}$$

We can write Equation and Equation in matrix form as follows:

$$\Sigma\mathbf{w} = \lambda\Theta \tag{14}$$

where

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \mu_1 - R_f \\ \mu_2 - R_f \end{bmatrix}$$

and $\mathbf{w}$ is the vector of weights for the two risky assets. Now, dividing both sides of Equation by λ and setting $\frac{\mathbf{w}}{\lambda} = \mathbf{z}$ obtains

$$\Sigma\mathbf{z} = \Theta \tag{15}$$

Finally, given that variance-covariance matrix is positive definite and has an inverse Σ^{-1}, the solution for Equation is

$$\mathbf{z} = \Sigma^{-1}\Theta \tag{16}$$

or

$$\mathbf{z} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mu_1 - R_f \\ \mu_2 - R_f \end{bmatrix} \tag{17}$$

The weights $\mathbf{w}$ for the tangency portfolio R_t can be obtained by

$$w_i = \frac{z_i}{\Sigma_{j=1}^{n} z_j} \tag{18}$$

We revisit our earlier example of two risky assets but now with a risk-free asset $R_f = 1$. We will now calculate the weights of the two assets in the tangency portfolio, and the return and standard deviation for this portfolio.

```
R1 <- 2 # Expected return for Asset 1
R2 <- 3 # Expected return for Asset 2
```

```r
S1 <- 16 # Variance return for Asset 1
S2 <- 25 # Variance return for Asset 2

Corr1 <- 0.25

cov1 <- Corr1*sqrt(S1)*sqrt(S2)


Rf <- 1

vcoc1 <- matrix(data = c(S1,cov1,cov1,S2), nrow = 2, byrow = TRUE)

Evec <- matrix(data = c(R1, R2), nrow = 2)

vcoc1
```

```
##      [,1] [,2]
## [1,]   16    5
## [2,]    5   25
```

```r
Evec-Rf
```

```
##      [,1]
## [1,]    1
## [2,]    2
```

```r
z <- solve(vcoc1) %*% (Evec-Rf)
z
```

```
##       [,1]
## [1,] 0.040
## [2,] 0.072
```

```r
w1t <- z[1]/sum(z)
w2t <- z[2]/sum(z)
```

```r
# Expected return for the tangency portfolio
Rpt <- R1*w1t + R2*(1-w1t)
Rpt
```

```
## [1] 2.642857
```

```r
# Standard deviation for the tangency portfolio
Sdt <-  sqrt(w1t^2*S1+ (1-w1t)^2*S2 + 2*w1t*(1-w1t)*cov1)
Sdt
```

```
## [1] 3.82993
```

```r
slope <- (Rpt-Rf)/Sdt
slope
```

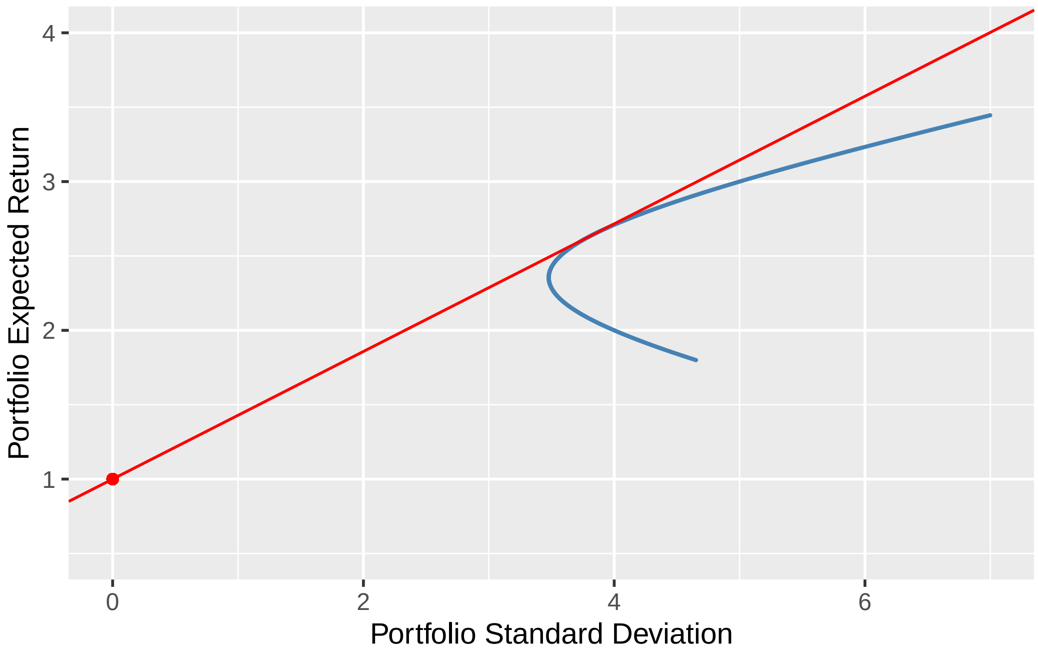
```
## [1] 0.4289522
```

```r
#


two_assets <- tibble(w = seq(-0.5,1.2, length.out = 500)) %>%
  mutate(Rp = R1*w + R2*(1-w),
         Sp = sqrt(w^2*S1+ (1-w)^2*S2 + 2*w*(1-w)*cov1))


ggplot(two_assets) +
  geom_point(aes(Sp, Rp), size = 0.09, color = "steelblue") +
  labs(x = "Portfolio Standard Deviation",
       y = "Portfolio Expected Return") +
  geom_point(aes(x=0, y = 1), color = "red") +
  geom_abline(intercept = 1, slope = slope, color = "red") +
  xlim(0, 7) +
  ylim(0.5,4)
```

```
## Warning: Removed 16 rows containing missing values (geom_point).
```



6.2 The Tangency Portfolio and the Capital Market Line

The *Capital Market Line* (CML) is the ray passing through the risk-free return R_f (on the vertical axis) and tangent to the old efficient frontier at *the tangency portfolio* of risky assets R_t. The CML is the new linear efficient frontier where efficient portfolios between R_f and R_t involves lending at the risk-free rate and efficient portfolios above R_t involves borrowing at the risk-free rate. The CML can be written as follow:

$$\mathbb{E}(R_p) = R_f + \frac{\mu_t - R_f}{\sigma_t}\sigma_p \qquad (19)$$

In our example the CML is: $\mathbb{E}(R_p) = 1 + 0.4289\sigma_p$

7 Next Step

This section provides an introduction to portfolio selection using `R` and the mean-variance analysis. We assume that key features of the distribution of returns (e.g. mean, variance and covariances) are available. In the next session, we will look at the use of the simple market model to estimate expected returns within the context of event studies.

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