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Directed Polymers in Random Environment

Chapter 2

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Abstract

These notes are based on the book "Directed Polymers in Random Environment"[1],and are intended for self-study and understand better these topics.

1 Thermodynamics and phase transition

In this section, we prove some properties related to the transition function and the polymer measure.After that, we define the Free Energy,an important concept that will be used will be used throughout these notes.

Definition 1.1. *We define* $\lambda(\beta) := \log \mathbb{P}(\exp[\beta\omega(n,x)]) < \infty$ *for all* β *by the assumptions in the notations section.*

Remark 1.2. *We will only consider* $\beta \geq 0$.

Definition 1.3. *Given* $i \geq 1, x \in \mathbb{Z}^d$,*the shift operator* $\theta_{i,x} : \Omega \to \Omega$ *is defined by*

$$(\theta_{i,x}\omega)(t,y) := \omega(i+t, x+y)$$

If $n, m \geq 1, x \in \mathbb{Z}^d$,*then*
$Z_m \circ \theta_{n,x}(\omega) = Z_m(\theta_{n,x}\omega, \beta) = P_x(exp[\sum_{t=1}^m \beta\omega(t+n, S_t)])$ *is the transition function of a polymer of size* m *that starts at the position* x *at time* n.

Note that

$$P_x(exp[\sum_{t=1}^{m} \beta\omega(t+n, S_t)]) = P(exp[\sum_{t=1}^{m} \beta\omega(t+n, S_{t+n})]|S_n = x) = P(exp[\sum_{t=n+1}^{m+n+1} \beta\omega(t, S_t)]|S_n = x)$$

$$= P(exp[\beta(H_{n+m}(S) - H_n(S))]|S_n = x)$$

So,

$$Z_m \circ \theta_{n,x}(\omega) = P(\exp[\beta(H_{n+m}(S) - H_n(S))]|\mathcal{F}_n) \tag{1.0.1}$$

in the event $\{S_n = x\}$.Thus, we can write

$$\begin{aligned} Z_{n+m} &= P(exp[\beta H_{n+m}(S)]) = P(e^{\beta[H_{n+m}(S)-H_n(S)]} \cdot e^{\beta H_n(S)}) = P(e^{\beta H_n(S)} P(e^{\beta[H_{n+m}(S)-H_n(S)]}|\mathcal{F}_n)) \\ &= P(e^{\beta H_n(S)} Z_m \circ \theta_{n,S_n}) \end{aligned}$$

Now we write

$$\begin{aligned} Z_n \times P_n^{\beta,\omega}(Z_n \circ \theta_{n,S_n}) &= Z_n \times \sum_x P_n^{\beta,\omega}(S_n = x) Z_m \circ \theta_{n,x} = Z_n \sum_x \frac{e^{\beta H_n(x)} P(S_n = x) Z_m \circ \theta_{n,x}}{Z_n} \\ &= \sum_x e^{\beta H_n(x)} Z_m \circ \theta_{n,x} P(S_n = x) = P(e^{\beta H_n(S)} Z_m \circ \theta_{n,S_n}) \end{aligned}$$

We have obtained the important relation

$$Z_{n+m} = Z_n \times P_n^{\beta,\omega}(Z_m \circ \theta_{n,S_n}) \tag{1.0.2}$$

Proposition 1.4. *The measure $P_n^{\beta\omega}$ is a Markov Chain, with transition probabilities*

$$P_n^{\beta,\omega}(S_{i+1} = y|S_i = x) = \frac{1}{Z_{n-i} \circ \theta_{i,x}} e^{\beta\omega(i+1,y)} Z_{n-i-1} \circ \theta_{i+1,y} P(S_1 = y|S_0 = x) \tag{1.0.3}$$

if $0 \le i < n$, *and if* $i \ge n$,

$$P_n^{\beta,\omega}(S_{i+1} = y|S_i = x) = P(S_1 = y|S_0 = x)$$

Proof. We use the telescopic property. For the path $x = (x_0 = 0, x_1, \cdots x_n)$ we have

$$\begin{aligned} \prod_{i=1}^{n-1} \frac{e^{\beta\omega(i+1,x+1)} Z_{n-i-1} \circ \theta_{i+1,x+1} P(S_1 = x_{i+1}|S_0 = x_i)}{Z_{n-i} \circ \theta_{i,x}} &= \frac{e^{\sum_{i=1}^n \beta\omega(i,x_i)}}{Z_n} \prod_{i=0}^{n-1} P(S_1 = x_{i+1}|S_0 = x_i) \\ &= \frac{e^{\beta H_n(x)}}{Z_n} P(S_1 = x_1, S_2 = x_2, \cdots, S_n = x_n) \\ &= P_n^{\beta,\omega}(S_1 = x_1, \cdots S_n = x_n) \end{aligned} \tag{1.0.4}$$

As the product is arbitrary,we conclude □

Remark 1.5. *Using that* $P_{n-i}^{\beta,\theta_{s,x}\omega}(S_1 = y-x) = P_{n-i}^{\beta,\theta_{s,x}\omega}(S_1 = y-x|S_0 = 0)$ *with the equation* (1.0.3) *to obtain the identify*

$$P_n^{\beta,\omega}(S_{i+1} = y|S_i = x) = P_{n-i}^{\beta,\theta_{i,x}\omega}(S_1 = y-x) \tag{1.0.5}$$

Remark 1.6. *The chain is not time-homogeneous. Also, it depends of the time* n*.Thus, we will consider the complete chain* $(P_n^{\beta,\omega})_{n\geq 1}$*. We also have the identities* $(0 \leq m, n)$

$$P_{m+n}^{\beta,\omega}(S_{[1,n]} = \cdot|S_n = y) = P_n^{\beta,\omega}(S_{[1,n]} = \cdot|S_n = y)$$
$$P_{m+n}^{\beta,\omega}(S_{[n,n+m]} = y + \cdot|S_n = y) = P_n^{\beta,\theta_{n,y}\omega}(S_{[0,m]} = \cdot)$$

1.1 Free Energy

We introduce the concept of free energy.

Definition 1.7. *The free energy is denoted by*

$$p_n = p_n(\omega, \beta) := \frac{1}{n}\log(Z_n(\omega, \beta)) \tag{1.1.1}$$

We want to study the limit $\lim_{n\to\infty} p_n$. In particular, its existence, and if it depends of ω. The following theorem answer this question.

Theorem 1.8. $\lim_{n\to\infty} p_n = p(\beta) = \underset{n}{sup}\frac{1}{n}\mathbb{P}(\log(Z_n(\omega, \beta)))$ $\mathbb{P}$ *a.s and in* L^p*, for all* $p \in [1, \infty)$

Proof. First we state a known lemma:

Lemma 1.9. *Suppose that* $(u_n)_{n\geq 1}$ *is a super-additive sequence,that is,* $u_{n+m} \geq u_n + u_m$ *for* $n, m \geq 1$*.Then*

$$\lim_{n\to\infty}\frac{u_n}{n} = \underset{m\geq 1}{sup}\frac{u_m}{m} \in \mathbb{R}\cup\{+\infty\}$$

Step 1: We prove that $\lim_{n\to\infty} \mathbb{P}(p_n) = \underset{n\in\mathbb{N}}{sup}\mathbb{P}(p_n) \in \mathbb{R}$.

Recall the equation (1.0.2). By the Jensen inequality,

$\log(Z_{n+m}) \geq \log(Z_n) + \sum_x P_n^{\beta,\omega}(S_n = x)\log(Z_n(\theta_{n,x}\omega))$.Taking expectations, we deduce $\mathbb{P}(\log(Z_{n+m})) \geq \mathbb{P}(\log(Z_n)) + \sum_x \mathbb{P}([P_n^{\beta,\omega}(S_n = x)] \times \log(Z_m \circ \theta_{n,x}))$.

Note that $Z_m \circ \theta_{n,x}$ has environments $\omega(i+n, S_i + x)$, while $P_n^{\beta,\omega}(S_n = x)$ has environments $\omega(i, S_i)$,so the product inside the expectations on the right side of the last equation are independent. So we have

$\mathbb{P}(\log(Z_{n+m})) \geq \mathbb{P}(\log(Z_n)) + \mathbb{P}(\log(Z_m \circ \theta_{n,x})) \sum_x \mathbb{P}([P_n^{\beta,\omega}(S_n = x)])$.

But $Z_m \circ \theta_{n,x}$ has the same law as Z_m.Therefore the last equation is equivalent to

$$\mathbb{P}(\log(Z_{n+m})) \geq \mathbb{P}(\log(Z_n)) + \mathbb{P}(\log(Z_m) \sum_x \mathbb{P}([P_n^{\beta,\omega}(S_n = x)])$$
$$= \mathbb{P}(\log(Z_n)) + \mathbb{P}(\log(Z_m)$$

Thus, $\log(Z_n)$ is super-additive, and by the lemma,

$\lim_{n\to\infty} \frac{1}{n}\mathbb{P}(\log(Z_n)) = \underset{n\in\mathbb{N}}{sup}\mathbb{P}(\log(Z_n))$.

Now we prove that the limit is finite. Recall that $\lambda(\beta) = \log(\mathbb{P}(e^{\beta\omega}))$.For a path S,

$$\mathbb{P}(e^{\beta H_n(S)}) = \mathbb{P}(exp(\beta \sum_{i=1}^{n} \omega(i, S_i)) \overset{independence}{=} \mathbb{P}(e^{\beta\omega})^n = e^{n\lambda(\beta)}$$

Using Fubini,

$$\mathbb{P}(Z_n) = P(\mathbb{P}(e^{\beta H_n(S)})) = \mathbb{P}(e^{n\lambda(\beta)}) = e^{n\lambda(\beta)}$$

Finally,using the Jensen inequality,

$$\mathbb{P}(p_n(\omega, \beta)) = \frac{1}{n}\mathbb{P}(\log(Z_n)) \leq \frac{1}{n}\log(\mathbb{P}(Z_n)) = \lambda(\beta)$$

We conclude that $p(\beta) \leq \lambda(\beta)$.This is called the *annealed bound*

Step 2: We use and concentration inequality:

Theorem 1.10 (Concentration inequality of the free energy)**.** *[3]*
Assuming $\mathbb{P}(exp(\beta\omega(n, x))) < \infty\, \forall\beta$

$$\mathbb{P}(|p_n - \mathbb{P}(p_n)| \geq r) \leq \begin{cases} 2exp(-nC^2 r) & r \in [0, 1] \\ 2exp(-nCr) & r \geq 1 \end{cases}$$

for some constant $C > 0$

Using this result and Borel Cantelli, we conclude that $\limsup_{n\to\infty} |p_n - \mathbb{P}(p_n)| \leq \epsilon$, $\mathbb{P}$ $a.s.$ But $\epsilon > 0$ is arbitrary, so $limsup_{n\to\infty}|p_n - \mathbb{P}(p_n)| = 0$.But by Step 1, we knew that

$\mathbb{P}(p_n) \to p(\beta)$.Thus $p_n \to p(\beta)$, $\mathbb{P}$ $a.s.$ Now remains to prove the convergence in L^p.To do this,we recall that $E(Z) = \int_0^\infty P(Z > r)dr$ if $Z \geq 0$. We have

$$\begin{aligned}\mathbb{P}(|p_n - \mathbb{P}(p_n)|^p) &= \int_0^\infty \mathbb{P}(|p_n - \mathbb{P}(p_n)| > r^{1/p})dr \overset{Theorem\ 1.10}{\leq} 2\int_0^\infty exp[-nC(r^{1/p} \wedge r^{2/p})]dr \\ &\leq 2\int_0^\infty exp(-nCr^{2/p})dr + 2\int_1^\infty exp(-nCr^{1/p}) = C'n^{-p/2} + 2\int_1^\infty exp(-nCr^{1/p}) \\ &\approx C'n^{-p/2} + \frac{e^{-n}}{n}\end{aligned}$$

Here, $C' = \int_0^\infty exp(-cv^{2/p})dv < \infty$ after the change of variables $r = n^{-p/2}v$.Taking $n \to \infty$ we finish.

□

Remark 1.11. *The bound $p(\beta) \leq \lambda(\beta)$ is not optimal in general. For example,if $\omega \sim N(0,1)$ we have $\lambda(\beta) = \frac{\beta^2}{2}$,and for each path $S, H_n(S) \sim N(0,n)$.Remember that if $X \sim N(0,1)$ we have the bound*

$$\frac{x}{1+x^2}g(x) \leq P(X \geq x) \leq (\frac{1}{x} \wedge \sqrt{2\pi})g(x), x \geq 0 \tag{1.1.2}$$

where $g(x) = \frac{exp(-x^2/2)}{\sqrt{2\pi}}$.Thus, for $a > 0$ we have

$$\begin{aligned}\sum_{n\geq 1}\mathbb{P}(\max_s H_n(S) > na) \leq \sum_{n\geq 1}\sum_{S:|S|=n}\mathbb{P}(H_n(S) > na) \leq \sum_{n\geq 1}(2d)^n\frac{1}{a\sqrt{n}}exp(-na^2/2) \\ = \sum_{n\geq 1}\frac{1}{a\sqrt{n}}\left(\frac{2d}{exp(a^2/2)}\right)^n < \infty\end{aligned}$$

if $\frac{2d}{exp(a^2/2)} < 1 \leftrightarrow a > \sqrt{2\log(2d)}$.Using Borel-Cantelli, we conclude that if $a > \sqrt{2\log(2d)}$, $\mathbb{P}$ $a.s$

$$\frac{Z_n}{n} \leq \frac{\beta}{n}\max_s H_n(S) \leq \beta a, \ \ n \text{ big enough}$$

So, $p(\beta) \leq \beta\sqrt{2\log(2d)} < \lambda(\beta) = \beta^2/2$ for β big enough.

With this in mind,we look to improve the annealed bound.

Proposition 1.12. *Fixed ω,we have*

1. *The function $\beta \to p_n(\omega,\beta)$ is convex, and $p_n(\omega,0) = 0$.*

2. *The function $\beta \to \frac{p_n(\omega,\beta)}{\beta}$ is increasing.*

3. The function $\beta \to \frac{p_n(\omega,\beta)+\log(2d)}{\beta}$ *is decreasing.*

Also, $p(\beta)$ *satisfies* $1-2-3$.

Proof. 1. Note first that $\beta \to p_n(\omega,\beta)$ is C^∞. We have

$$\frac{d}{d\beta}np_n = \frac{d}{d\beta}\log(\frac{1}{Z_n}\frac{d}{d\beta}P(e^{\beta H_n})Z_n) = \frac{1}{Z_n}\frac{d}{d\beta}P(e^{\beta H_n}) = \frac{1}{Z_n}P\left(H_n e^{\beta H_n}\right) = P_n^{\omega,\beta}(H_n) \tag{1.1.3}$$

To prove the last equality,we note that

$$P_n^{\omega,\beta}(H_n) = \frac{1}{Z_n}\sum_x H_n(x)e^{\beta H_n(x)}P(x) = \frac{P(H_n e^{\beta H_n})}{Z_n}$$

So,

$$\begin{aligned}\frac{d^2}{d\beta^2}np_n &= \frac{d}{d\beta}\frac{P(H_n e^{\beta H_n})}{Z_n} = \frac{Z_n\frac{d}{d\beta}P(H_n e^{\beta H_n}) - \frac{d}{d\beta}Z_n P(H_n e^{\beta H_n})}{Z_n^2}\\ &= \frac{P(H_n^2 e^{\beta H_n})}{Z_n} - \left(\frac{P(H_n e^{\beta H_n})}{Z_n}\right)^2 = Var_{P_n^{\beta,\omega}}(H_n) > 0\end{aligned} \tag{1.1.4}$$

Therefore, $p_n(\omega,\beta)$ is convex.Also,is clear that $p_n(\omega,\beta)=0$

2. Recall a known fact from convex functions. If f is a convex function, and $a \in \mathbb{R}$,then the map $x \to \frac{f(x)-f(a)}{x-a}$ is non-decreasing.In our case,taking $a=0$, we have that $\frac{p_n(\omega,\beta)}{\beta}$ is non-decreasing.

3. Using part 1, we have

$$\frac{d}{d\beta}\frac{1}{\beta}(p_n+\log(2d)) = \frac{-1}{\beta^2}(p_n+\log(2d)) + \frac{1}{n\beta}P_n^{\beta,\omega}(H_n) = \frac{1}{n\beta^2}h(P_n^{\beta,\omega}) \tag{1.1.5}$$

Here, $h(\mu) := \sum_x \mu(x)\log(\mu(x)) \leq 0$ is the entropy of the measure μ.To prove the last equality,we compute

$$\begin{aligned}
\frac{1}{n\beta^2}h(P_n^{\beta,\omega}) &= \frac{1}{n\beta^2}\sum_x P_n^{\beta\omega}(x)\log(P_n^{\beta,\omega}(x)) \\
&= \frac{1}{n\beta^2}\sum_x \left(\frac{exp(\beta H_n(x))P(x)}{Z_n}\right)\log\left(\frac{exp(\beta H_n(x))P(x)}{Z_n}\right) \\
&= \frac{1}{n\beta^2}\sum_x \left(\frac{exp(\beta H_n(x)P(x))}{Z_n}\right)[\beta H_n(x)+\log(P(x))-\log(Z_n)] \\
&= \frac{1}{n\beta}\sum_x \frac{H_n(x)exp(\beta H_n(x))P(x)}{Z_n} - \frac{1}{\beta^2}\log(2d) - \frac{1}{\beta^2}p_n \\
&= \frac{-1}{\beta^2}(p_n+\log(2d)) + \frac{1}{n\beta}P_n^{\beta,\omega}(H_n) \qquad (1.1.6)
\end{aligned}$$

□

Proposition 1.13. *We have the bound*

$$p(\beta) \le \beta \inf_{b\in[0,\beta]} \frac{\lambda(b)-\log(2d)}{b} - \log(2d) \qquad (1.1.7)$$

Therefore, under the condition

$$(T): \quad \beta\lambda'(\beta) - \lambda(\beta) > \log(2d) \qquad (1.1.8)$$

We have $p(\beta) < \lambda(\beta)$.

More precisely, if there exists a positive root β_1 from the equation $\beta\lambda'(\beta) = \lambda(\beta)+\log(2d)$,then for all $\beta > \beta_1$ the following holds:

$$p(\beta) \le \frac{\beta}{\beta_1}(\lambda(\beta_1)+\log(2d)) - \log(2d) < \lambda(\beta) \qquad (1.1.9)$$

Proof. Let $g(\beta) := \beta\lambda'(\beta) - \lambda(\beta), f(\beta) := \frac{\lambda(\beta)+\log(2d)}{\beta}$.

The function λ is smooth,and

$$\lambda'' = \frac{\mathbb{P}(\omega^2 e^{\beta\omega}) - \mathbb{P}(\omega e^{\beta\omega})}{\mathbb{P}(e^{\beta\omega})}$$

If $d\mathbb{Q} = e^{\beta\omega}d\mathbb{P}/\mathbb{P}(e^{\beta\omega})$,then $\lambda'' = Var_{\mathbb{Q}}(\omega) > 0$.Then λ is convex.We have $g'(\beta) = \lambda'(\beta) + \beta\lambda''(\beta) - \lambda'(\beta) = \beta\lambda''(\beta)$.So, $g'(\beta)$ has the same sign as β. In particular, $g' > 0$ in $\mathbb{R}_+$,that is, g is increasing in $\mathbb{R}_+$.Let's define $\lambda^*(u) := \underset{\beta}{sup}(\beta u - \lambda(\beta))$, then $g(\beta) = \lambda^*(\lambda'(\beta))$. On the other hand,

$$f'(\beta) = \frac{1}{\beta^2}[-\lambda(\beta) - \log(2d) + \beta\lambda'(\beta)] = \frac{1}{\beta^2}[g(\beta) - \log(2d)]$$

Let's write $\mathbb{P}(p_n(\omega,\beta)+\log(2d)) = \beta\frac{1}{\beta}\mathbb{P}(p_n(\omega,\beta)+\log(2d))$.Using Proposition 1.12,this equals

$$\beta \inf_{b\in[0,\beta]} \frac{1}{b}\mathbb{P}(p_n(\omega,b)+\log(2d)) \leq \beta \inf_{b\in[0,\beta]} f(b)$$

In the last inequality,we used that $\mathbb{P}(p_n(\omega,\beta)) \leq \lambda(\beta)$.We conclude that

$$\mathbb{P}(p_n(\omega,\beta)+\log(2d)) \leq \beta \inf_{b\in[0\beta]} f(b) \quad \text{Taking } n\to\infty$$

$$p(b)+\log(2d) \leq \beta \inf_{b\in[0\beta]} \frac{1}{b}\lambda(b)+\log(2d) \tag{1.1.10}$$

This gives us the first bound.

Now,take $\beta_1 \in (0,\infty], \beta_1 := \inf\{\beta \geq 0 : g(\beta) \geq \log(2d)\}$.Recall that $f'(\beta) = \frac{1}{\beta^2}(g(\beta)-\log(2d))$,so $f'(\beta_1) = 0$ if β_1 is finite and f reach is minimum.Therefore,

$$\inf_{\beta'\in[0,\beta]} f(\beta') = \begin{cases} f(\beta) & \beta \leq \beta_1 \\ f(\beta_1) & \beta \geq \beta_1 \end{cases}$$

Finally,note that $\beta > \beta_1 \Leftrightarrow (T)$,and that such β the inequality is strict due to the strict convexity of λ.More precisely, f is also strictly convex and reach his unique minimum in $f(\beta_1)$ if $\beta > \beta_1$,then $p(\beta)+\log(2d) \leq \beta f(\beta_1) < \beta f(\beta) = \lambda(\beta)+\log(2d) \Rightarrow p(\beta) < \lambda(\beta)$ $\square$

Remark 1.14. *Recall that $\frac{d}{d\beta}p_n(\omega,\beta)) = \frac{1}{n}P_n^{\beta,\omega}(H_n)$.Let $\mathcal{D} = \mathcal{D}(p)$ be the set of β's such that p is differentiable in β.Because p is convex, we know that p' exists at most some countable set,and in fact, p is C^1 is $\mathcal{D}$.The next result is a standard result related to convex functions and their derivatives:*

Proposition 1.15. *For all $\beta \in \mathcal{D}$ and a.s ω,*

$$\lim_{n\to\infty} \frac{1}{n}P_n^{\beta,\omega}(H_n) = p'(\beta) = \lim_{n\to\infty} \frac{1}{n}\mathbb{P}(P_n^{\beta,\omega}(H_n)) \tag{1.1.11}$$

The next result give us information about $\lambda - p$:

Theorem 1.16. *The functions*

$$\beta \to \lambda(\beta) - \mathbb{P}(p_n(\omega,\beta))$$

$$\beta \to \lambda(\beta) - p(\beta)$$

are not decreasing in $\mathbb{R}^+$

Before giving a proof,we need the concept of positive association.

Definition 1.17. *A family* $\{X_n\}_{n=1}^k$ *of real valued random variables is positively associated if for all* $f, g : \mathbb{R}^k \to \mathbb{R}$ *bounded and increasing,* $\mathbb{E}(f(X)g(X)) \geq \mathbb{E}(f(x))\mathbb{E}(g(X))$

Theorem 1.18 (FKG-Harris inequality)**.** *[2] Any family of independent random variables are positively associated.*

Proof of Theorem 1.16: Let $q(dh) := \mathbb{P}(\omega(n,x) \in dh), \xi_n(S) := exp(\beta H_n(S))$.Recall that

$$\frac{d}{d\beta}\mathbb{P}(\log(Z_n)) = \mathbb{P}\left(\frac{1}{Z_n}P[H_n e^{\beta H_n}]\right) = \mathbb{P}\left(\frac{1}{Z_n}P[H_n\xi_n]\right) \overset{Fubini}{=} P\left[\mathbb{P}\left(\frac{H_n\xi_n}{Z_n}\right)\right]$$

Now fix a path x,and define the measure $\tilde{P}^x$ given by

$$d\tilde{P}^x = \xi_n(x)e^{-n\lambda(\beta)}d\mathbb{P} \quad \text{(observe that } \mathbb{P}[\xi_n exp\{-n\lambda(\beta)\}] = 1) \tag{1.1.12}$$

Under $\tilde{P}^x$,the random variables $\omega(t,x), t \geq 0, x \in \mathbb{Z}^d$ are independent (but not i.i.d).Then we can apply the FKG-Harris inequality with functions $H_n, -(Z_n)^{-1}$.We obtain

$$\begin{aligned}\tilde{P}^x[(Z_n)^{-1}H_n] \leq \tilde{P}^x[Z_n^{-1}]\tilde{P}^x[H_n] \Leftrightarrow \mathbb{P}[(Z_n)^{-1}H_n\xi_n exp\{-n\lambda(\beta)\}] &\leq e^{-2n\lambda(\beta)}\mathbb{P}[(Z_n)^{-1}\xi_n]\mathbb{P}[H_n\xi_n] \\ \Leftrightarrow \mathbb{P}[(Z_n)^{-1}H_n\xi_n] &\leq e^{-n\lambda(\beta)}\mathbb{P}[(Z_n)^{-1}\xi_n]\mathbb{P}[H_n\xi_n] \\ &= \mathbb{P}[(Z_n)^{-1}\xi_n] \times n\lambda'(\beta)\end{aligned}$$

The last equality is result of

$$\mathbb{P}[\omega e^{\beta\omega}] = \lambda'(\beta)e^{\lambda(\beta)} \tag{1.1.13}$$

and $\mathbb{P}[H_n\xi_n] \overset{Independence}{=} n\mathbb{P}[\omega e^{\beta\omega}]$

Then we have

$$\begin{aligned}&\mathbb{P}[(Z_n)^{-1}H_n\xi_n] \leq n\lambda'(\beta)\mathbb{P}[(Z_n)^{-1}\xi_n] \\ &\quad\Rightarrow P\left[\mathbb{P}\left(\frac{H_n\xi_n}{Z_n}\right)\right] = \frac{d}{d\beta}\mathbb{P}(\log(Z_n)) \leq n\lambda'(\beta)P[\mathbb{P}(Z_n^{-1}\xi_n)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad = n\lambda'(\beta)\mathbb{P}P[Z_n^{-1}\xi_n] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad = n\lambda'(\beta)\mathbb{P}\left(\frac{P(\xi_n)}{Z_n}\right) = n\lambda'(\beta)\end{aligned}$$

$$\begin{aligned}&\therefore \frac{d}{d\beta}\frac{1}{n}\mathbb{P}[\log(Z_n)] =\leq \lambda'(\beta) \\ &\Leftrightarrow \frac{d}{d\beta}[\lambda(\beta) - \frac{1}{n}\mathbb{P}[\log(Z_n)]] \geq 0 \quad \text{if } \beta \geq 0\end{aligned}$$

That concludes our first claim.Taking limit when $n \to \infty$,we deduce the second claim. $\square$

1.2 Phase Transition

The last theorem permit us classify β in two categories.

Theorem 1.19 (Critical temperature)**.** *There exists $\beta_c = \beta_c(\mathbb{P}, d) \in [0, \infty]$ such that*

$$\begin{cases} p(\beta) = \lambda(\beta) & 0 \leq \beta \leq \beta_c \\ p(\beta) < \lambda(\beta) & \beta > \beta_c \end{cases}$$

Proof. Let $\beta_c := \inf\{\beta \geq 0 : p(\beta) < \lambda(\beta)\}$.By Theorem 1.16,the function $f(\beta) := \lambda(\beta) - p(\beta)$ satisfies $f(0) = 0$, and is non decreasing in $\beta \in [0, \infty)$.Thus, we will have $p(\beta) = \lambda(\beta)$ if $0 \leq \beta \leq \beta_c$ and $\lambda(\beta) > p(\beta)$ is $\beta > \beta_c$ $\square$

Definition 1.20. *We call high temperature region to the β's such that $p(\beta) = \lambda(\beta)$, and low temperature region when $p(\beta) < \lambda(\beta)$.*

Remark 1.21.
1. *$p(0) = 0 = \lambda(0) \Rightarrow 0$ is in the high temperature region.*
2. *The condition (T) from 1.1.8 implies $\beta_c < \infty$.*
3. *It's expected different behaviors of $P_n^{\beta,\omega}$ for both regimes.*

   Before finishing the chapter, we state some questions of the model:

   Question 1. *What phenomenon is responsible of phase transition?*

   Question 2. *Which different characteristics present the polymer measure in both regime?*

   Question 3. *If we observe a path under the polymer measure, can we decide if β is in the low temperature or high temperature region?*

   Question 4. *What happens in β_c?*

References

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