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## Directed Polymers in Random Environment

### Chapter 4

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## 1 Semimartingale Approach and Localization Transition

In this section, we will study semimartingales to deduce a phase transition of the localization of the polymer.

### 1.1 Semimartingale Decomposition

If  $(a_n)_{n \geq 0}$  is a sequence (random or not), we write  $\Delta a_n := a_n - a_{n-1}$  for  $n \geq 1$ . We will use the following fact: **Doob's Decomposition:** Any  $(\mathcal{F}_n)$  process  $X = (X_n)_{n \geq 0} \subset L^1(\mathbb{P})$  can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), n \geq 1$$

where  $M(X)$  is an  $(\mathcal{F}_n)$  martingale and  $A(X)$  is predictable, i.e.,  $A_n(X)$  is  $\mathcal{F}_{n-1}$  measurable, and  $A_0(X) = 0$ . To obtain such processes, we compute their increments

$$\Delta A_n = \mathbb{P}(\Delta X_n | \mathcal{F}_{n-1}), \quad \Delta M_n = \Delta X_n - \mathbb{P}(\Delta X_n | \mathcal{F}_{n-1}).$$

Then  $A_n = \sum_{t=1}^n \Delta A_t$ ,  $M_n = X_0 + \sum_{t=1}^n \Delta M_t$ . The processes  $M_n(X)$ ,  $A_n(X)$  are called the martingale part and the compensator of  $X$  respectively.

**Remark 1.1.** If  $N$  is a square integrable martingale, then  $N^2 \subset L^1(\mathbb{P})$  and the compensator  $A(N^2)$ , denoted by  $\langle N_n \rangle$  is given by

$$\Delta \langle N \rangle_n = \mathbb{P}[N_n^2 - N_{n-1}^2 | \mathcal{F}_{n-1}] = \mathbb{P}[(N_n - N_{n-1})^2 | \mathcal{F}_{n-1}] = \mathbb{P}[(\Delta N_n)^2 | \mathcal{F}_{n-1}] \quad (1.1.1)$$

We are interested in the process  $X_n = -\log(W_n)$ . Because  $W_n$  is a martingale, then  $X_n$  is a submartingale. In particular  $\mathbb{P}(\Delta X_n | \mathcal{F}_{n-1}) \geq 0$ . Therefore,  $A_n$  is an increasing process. To obtain  $M_n, A_n$ , we introduce

$$U_n := P_n^{\beta, \omega}[e^{\beta \omega(n, S_n) - \lambda(\beta)}] - 1$$

Recall the relation

$$Z_{n+m} = Z_n \times P_n^{\beta, \omega}(Z_m \circ \theta_{n, S_n}) \quad (1.1.2)$$

Using this, we have  $W_n/W_{n-1} = 1 + U_n$ . Then we write  $W_n = \prod_{j=1}^n (1 + U_j) \Rightarrow X_n = -\sum_{t=1}^n \log(1 + U_t)$ . Finally, we decompose each term  $-\log(1 + U_t) = \Delta M_t + \Delta A_t$ , with  $\Delta A_t = -\mathbb{P}(\log(1 + U_t) | \mathcal{F}_{t-1})$ ,  $\Delta M_t = -\log(1 + U_t) + \mathbb{P}(\log(1 + U_t) | \mathcal{F}_{t-1})$ . As stated in [1], a key role in the asymptotics of the model is in the random variable defined on  $(\Omega, \mathcal{F}, \mathbb{P})$

$$I_n := \sum_{x \in \mathbb{Z}^d} P_{n-1}^{\beta, \omega}(S_n = x)^2 \quad (1.1.3)$$

We can see this random variable as the probability that two independent random walks intersect. More precisely, on the product space  $\Omega_{traj}^2$  we consider the probability measure  $(P_n^{\beta, \omega})^{\otimes 2}$  and we will consider in this space two independent random walks  $(S, \tilde{S})$ , with  $S, \tilde{S}$  have the same law  $P_n^{\beta, \omega}$ . The path  $S, \tilde{S}$  are called replica. So, we can write  $I_n$  as

$$I_n = (P_n^{\beta, \omega})^{\otimes 2}(S_n = \tilde{S}_n) \quad (1.1.4)$$

Hence,  $\sum_{k=1}^n I_k$  is the expected amount of overlap up to time  $n$  of two independent polymers in the same (fixed) environment. The following theorem relates this expression with  $W_n$ :

**Theorem 1.2.** *Let  $\beta \neq 0$ . Then*

$$A := \{W_\infty = 0\} = B := \left\{ \sum_{n=1}^{\infty} I_n = \infty \right\} \quad \mathbb{P} \text{ a.s.} \quad (1.1.5)$$

Moreover, if  $\mathbb{P}(W_\infty = 0) = 1$ , there exist  $c_1, c_2 \in (0, \infty)$  depending on  $\beta, \mathbb{P}$  such that  $\mathbb{P} \text{ a.s.}$

$$c_1 \sum_{k=1}^n I_k \leq -\log(W_n) \leq c_2 \sum_{k=1}^n I_k \quad (1.1.6)$$

and also

$$\lim_{n \rightarrow \infty} \frac{-\log(W_n)}{A_n} = 1 \quad \mathbb{P} \text{ a.s.} \quad (1.1.7)$$

*Proof.* In order to prove (1.1.5),(1.1.6), it's enough to show the following:

$$\{W_\infty = 0\} \subset \left\{ \sum_{n=1}^{\infty} I_n = \infty \right\} \quad \mathbb{P} \text{ a.s} \quad (1.1.8)$$

and that there exists  $c_1, c_2 \in (0, \infty)$  such that

$$\left\{ \sum_{n=1}^{\infty} I_n = \infty \right\} \subset \{ (1.1.6) \text{ holds} \} \quad \mathbb{P} \text{ a.s} \quad (1.1.9)$$

To convince of that, note that using (1.1.8) we only need to show that  $\mathbb{P}(B \setminus A) = 0$ . If  $\mathbb{P}(A) = 1$  this is clear, and if  $\mathbb{P}(A) = 0$ , then  $W_\infty > 0$   $\mathbb{P}$  a.s, and in that case,  $\mathbb{P}(B) = 0$ , otherwise,  $\mathbb{P}(\{(1.1.6) \text{ holds}\}) > 0$ , but this cannot happen if  $-\log(W_\infty) < \infty$  because  $\sum_{n \geq 1} I_n < \infty$ . Recall that  $\Delta M_n = -\log(1 + U_n) + \mathbb{P}(\log(1 + U_n) | \mathcal{F}_{n-1})$ . Then

$$\mathbb{P}[(\Delta M_n) | \mathcal{F}_{n-1}] = \mathbb{P}[\log^2(1 + U_n) | \mathcal{F}_{n-1}] - \mathbb{P}[\log(1 + U_n) | \mathcal{F}_{n-1}]^2 \leq \mathbb{P}[\log^2(1 + U_n) | \mathcal{F}_{n-1}]$$

Using (1.1.1), we deduce that

$$\langle M \rangle_n \leq \mathbb{P}[\log^2(1 + U_n) | \mathcal{F}_{n-1}] \quad (1.1.10)$$

Now we claim that there exists a constant  $c \in (0, \infty)$  such that

$$\frac{1}{c} I_n \leq \Delta A_n \leq c I_n, \quad \Delta \langle M \rangle_n \leq c I_n \quad (1.1.11)$$

The proof of the claim will be postponed.

Now we use the fact that a square integrable martingale converges a.s on the event  $\{\langle M \rangle_\infty < \infty\}$  and (1.1.11) to conclude (1.1.8)  $\mathbb{P}$  a.s :

$$\begin{aligned} \left\{ \sum_{n \geq 1} I_n < \infty \right\} &\subset \{A_\infty < \infty, \langle M \rangle_\infty < \infty\} \subset \{A_\infty < \infty, \lim_{n \rightarrow \infty} M_n \text{ exists and is finite}\} \\ &\subset \left\{ \lim_{n \rightarrow \infty} \log W_n \text{ exists and is finite} \right\} = \{W_\infty > 0\} \end{aligned}$$

Now we prove (1.1.9). By (1.1.11), it's enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{n \rightarrow \infty} -\frac{\log W_n}{A_n} = 1 \right\}, \quad \mathbb{P} \text{ a.s} \quad (1.1.12)$$

To show this, let's suppose that  $A_\infty = \infty$ . If  $\langle M \rangle_\infty < \infty$ , then  $\lim_{n \rightarrow \infty} M_n$  exists and is finite a.s. Therefore, (1.1.12) holds. Now, if  $\langle M \rangle_\infty = \infty$ , using the Law of Large Numbers for

martingales, we deduce that  $M_n/\langle M \rangle_n \rightarrow 0$  a.s on the event  $\{\langle M \rangle_\infty = \infty\}$ . In this case, we see that

$$-\frac{\log W_n}{A_n} = \frac{M_n + A_n}{A_n} = 1 + \frac{M_n}{\langle M \rangle_n} \frac{\langle M \rangle_n}{A_n} \rightarrow 1 \quad \mathbb{P} \text{ a.s}$$

by (1.1.11), because  $\langle M \rangle_n \leq c^2 A_n$ .  $\square$

To prove the claim (1.1.11), we need the following lemma:

**Lemma 1.3.** *Let  $e_i, 1 \leq i \leq m$  be positive, non constant i.i.d random variables on a probability space  $(H, \mathcal{G}, \mathbb{P})$  such that*

$$\mathbb{P}(e_1) = 1, \quad \mathbb{P}(e_1^3 + \log^2 e_1) < \infty$$

*For  $\{\alpha_i\}_{1 \leq i \leq m} \subset [0, \infty)$  such that  $\sum_{1 \leq i \leq m} \alpha_i = 1$ , define a centered random variable  $U > -1$  by  $U = \sum_{1 \leq i \leq m} \alpha_i e_i - 1$ . Then there exists a constant  $c \in (0, \infty)$ , independent of  $m$  and  $\{\alpha_i\}_{1 \leq i \leq m}$  such that*

$$\frac{1}{c} \sum_{1 \leq i \leq m} \alpha_i^2 \leq \mathbb{P} \left[ \frac{U^2}{U+2} \right] \quad (1.1.13)$$

$$\frac{1}{c} \sum_{1 \leq i \leq m} \alpha_i^2 \leq -\mathbb{P}[\log(1+U)] \leq c \sum_{1 \leq i \leq m} \alpha_i^2 \quad (1.1.14)$$

$$\mathbb{P}[\log^2(1+U)] \leq c \sum_{1 \leq i \leq m} \alpha_i^2 \quad (1.1.15)$$

*Proof.* We will use constants  $c_1, c_2, \dots$  for constants independent of  $\{\alpha_i\}_{1 \leq i \leq m}$ . We have

$$\mathbb{P}[U^2] = c_1 \sum_{1 \leq i \leq m} \alpha_i^2, \quad \mathbb{P}[U^3] \leq c_2 \sum_{1 \leq i \leq m} \alpha_i^2$$

Now we deduce the first inequality as follows:

$$\begin{aligned} c_1 \sum_{1 \leq i \leq m} \alpha_i^2 &= \mathbb{P} \left[ \frac{U}{\sqrt{2+U}} \sqrt{2+U} \cdot U \right] \leq \mathbb{P} \left[ \frac{U^2}{2+U} \right]^{1/2} \mathbb{P}[2U^2 + U^3]^{1/2} \\ &\leq c_3 \mathbb{P} \left[ \frac{U^2}{2+U} \right]^{1/2} \left( \sum_{1 \leq i \leq m} \alpha_i^2 \right)^{1/2} \end{aligned}$$

To prove the other two inequalities, we define a function  $\phi : (-1, \infty) \rightarrow [0, \infty)$  by  $\phi(u) = u - \log(1+u)$ , so that (recall that  $\mathbb{P}(U) = 0$ )

$$-\mathbb{P}[\log(1+U)] = \mathbb{P}[\phi(U)]$$

So,  $-\mathbb{P}[\log(1+U)] = \mathbb{P}[\phi(U)] \geq \frac{1}{4}\mathbb{P}\left[\frac{U^2}{2+U}\right] \geq \frac{1}{c}\sum_{1 \leq i \leq m} \alpha_i^2$   
because if  $u > -1$ ,  $\phi(u) \geq \frac{1}{4}\frac{u^2}{2+u}$ . This give us the first inequality in (1.1.14). To obtain the second one, note that for  $\epsilon \in (0, 1)$ ,

$$\begin{aligned}\mathbb{P}(\phi(U)) &= \mathbb{P}(\phi(U) : 1+U \geq \epsilon) + \mathbb{P}(\phi(U) : 1+U \leq \epsilon) \\ &\leq \mathbb{P}(\phi(U) : 1+U \geq \epsilon) - \mathbb{P}[\log(1+U) : 1+U \leq \epsilon]\end{aligned}$$

Note that if  $1+u \geq \epsilon$ ,  $\phi(u) \leq \frac{1}{2}u^2 \leq \frac{1}{2}(\frac{u}{\epsilon})^2$ . Then we have

$$\mathbb{P}(\phi(U) : 1+U \geq \epsilon) \leq \frac{1}{2}\epsilon^{-2}\mathbb{P}(U^2) = \frac{1}{2}e^{-2}c_1 \sum_{1 \leq i \leq m} \alpha_i^2 \quad (1.1.16)$$

Now define  $\gamma := -\mathbb{P}[\log e_1] \geq 0$  by Jensen, and take  $\epsilon > 0$  small enough such that  $\log(1/\epsilon) - \gamma \geq 1$ . Define another centered random variable by  $V := \sum_{1 \leq i \leq m} \alpha_i(\log(e_i) + \gamma)$ . Note that by Jensen inequality ( $-\log$  is convex) we have that  $\sum \alpha_i \log(e_i) \leq \log(\sum \alpha_i e_i)$ . Using that, we have

$$\begin{aligned}\{1+U \leq \epsilon\} &\subset \{V - \gamma \subset \log(1+U) \leq \log \epsilon\} \\ &\leq \{-\log(1+U) \leq -V + \gamma\} \cap \{1 \leq -V\}\end{aligned}$$

Hence, we have

$$\begin{aligned}-\mathbb{P}[\log(1+U) : 1+U \leq \epsilon] &\leq \mathbb{P}[-V + \gamma : 1 \leq -V] \leq \mathbb{P}[-V : 1 \leq -V] + \gamma\mathbb{P}[1 \leq -V] \\ &\leq (1+\gamma)\mathbb{P}[V^2] = c_4 \sum_{1 \leq i \leq m} \alpha_i^2\end{aligned}$$

where the last inequality comes from on the event  $-V \geq 1$ ,  $V^2 \geq -V \geq 1$ , and this implies that  $\mathbb{P}[-V : -V \geq 1] \leq \mathbb{P}[V^2]$ ,  $\mathbb{P}[1 \leq -V] \leq \mathbb{P}[V^2]$

This result, together with (1.1.16), deduce the second part in (1.1.14). To prove the last inequality, the argument is similar. We use the fact that  $|\log(1+u)| \leq \epsilon^{-1} \log(\epsilon^{-1})|u|$  if  $\epsilon < 1+u$ , we have that

$$\mathbb{P}[\log^2(1+U) : \epsilon \leq 1+U] \leq \epsilon^{-2} \log^2(\epsilon^{-1})\mathbb{P}[U^2]$$

On the other hand, the following holds:

$$\{1+U \leq \epsilon\} = \{V - \gamma \leq \log(1+U) \leq \log(\epsilon)\} \subset \{\log^2(1+U) \leq 2V^2 + 2\gamma^2\} \cap \{1 \leq -V\}$$

(Because  $0 \leq -\log(1+U) \leq \gamma - V \rightarrow \log^2(1+U) \leq (V - \gamma)^2 \leq 2V^2 + 2\gamma^2$ )

Therefore, we have

$$\mathbb{P}[\log^2(1+U) : 1+U \leq \epsilon] \leq 2\mathbb{P}[V^2] + 2\gamma^2\mathbb{P}[1 \leq -V] \leq c_5 \sum_{1 \leq i \leq m} \alpha_i^2$$

□

*Proof of the claim.* Recall that  $\triangle A_n = -\mathbb{P}(\log(1+U_n)|\mathcal{F}_{n-1})$  and (1.1.10). We apply the lemma with  $\{e_i\}$ ,  $\{\alpha_i\}$  and  $\mathbb{P}$  as  $\{e^{\beta\omega(n,z)-\lambda(\beta)}\}_{|z|_1 \leq n}$ ,  $\{P_{n-1}^{\beta,\omega}(S_n = z)\}_{|z|_1 \leq n}$  and  $\mathbb{P}[\cdot|\mathcal{G}_{n-1}]$ . More precisely, we use the last two equation in the lemma, and recall that in our case,  $\alpha_i^2 = I_i$  by (1.1.4) □

## 1.2 Localization versus Delocalization

The strong disorder regime present a strange behavior. On one hand, the polymer has a tendency to diffuse. On the other hand, it tends to localize in certain regions (the regions which maximizes  $H_n$ ). To study this phenomenon, we will consider the random variable  $J_n$ , which is the probability of the favorite endpoint for the polymer of size  $n$ :

$$J_n := \max_{x \in \mathbb{Z}^d} P_{n-1}^{\beta,\omega}(S_n = x) \quad (1.2.1)$$

This random variable measures how much spread is the polymer. For example, if  $\beta = 0$ , we have  $J_n = O(n^{-d/2})$ , but it should be larger if the polymer is localized in certain regions. Recalling the definition of  $I_n = \sum_{x \in \mathbb{Z}^d} P_{n-1}^{\beta,\omega}(S_n = x)^2$ , we have the following inequalities

$$J_n^2 \leq I_n \leq J_n \quad (1.2.2)$$

### Definition 1.4.

We say the polymer is **localized** if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n J_t > 0 \quad \mathbb{P} \text{ a.s} \quad (1.2.3)$$

and the polymer is **delocalized** if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n J_t = 0 \quad \mathbb{P} \text{ a.s} \quad (1.2.4)$$

Now we show that a criterion to distinguish each case, and also a phase transition:

**Theorem 1.5** (Localization Transition). *Let  $\beta \neq 0$ . Then*

- *localized if and only if  $p < \lambda$*
- *delocalized if and only if  $p = \lambda$*

*Localization occurs for all  $\beta > \beta_c$ , and delocalization for  $\beta \leq \beta_c$*

*Proof.* Note first that  $\frac{1}{n} \sum_n I_n \approx \frac{1}{n} \sum_n J_n$ , so they have the same limits.

If the polymer is localized, then  $\sum_n I_n = \infty \Rightarrow W_\infty = 0 \Rightarrow p < \lambda$ . If  $p < \lambda$ , then  $W_\infty = 0$ . Then  $\sum_n I_n/n \approx -\log(W_n)/n$ . But  $\log(W_n)/n = -\log(Z_n)/n\lambda \rightarrow \lambda - p > 0$ . Thus, the polymer is localized.

If  $p = \lambda$ , then  $\sum_n I_n < \infty$ , so the polymer is delocalized. If the polymer is delocalized, then must happen that  $p = \lambda$ , because we proved that  $p < \lambda$  implies that the polymer is localized. That concludes the proof.  $\square$

## References

- [1] Francis Comets. Directed polymers in random environments. *Lecture Notes in Mathematics*, 2175, 2017.