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Stochastic Calculus Notes

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1 Martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t\}$.We assume that $\{\mathcal{F}_t\}$ is complete but not right-continuous,unless so specified.

Definition 1.1. *A real-valued stochastic process $M = \{M_t : t \in \mathbb{R}_+\}$ is a submartingale adapted to $\{\mathcal{F}_t\}$ if each M_t is integrable and*

$$E[M_t|\mathcal{F}_s] \geq M_s \text{ for all } t < s$$

M is a supermartingale if -M is a submartingale.Clearly, M is a martingale if it is both a submartingale and a supermartingale.Also, M is square integrable if $E[M_t^2] < \infty$ for all $t \in \mathbb{R}_+$.

Proposition 1.2.

i) If M is a martingale and ϕ is a convex function such that $\phi(M_t)$ is integrable for all t, then $\phi(M_t)$ is a submartingale.

ii) If M is a submartingale and ϕ a nondecreasing convex function such that $\phi(M_t)$ is integrable for all t, then $\phi(M_t)$ is a submartingale.

Proof.

i) By Jensen's inequality, we have

$$E[\phi(M_t)|\mathcal{F}_s] \geq \phi(E[M_t]|\mathcal{F}_s) = \phi(M_s)$$

ii) By Jensen's inequality, we have

$$E[\phi(M_t)|\mathcal{F}_s] \geq \phi(E[M_t]|\mathcal{F}_s) \geq \phi(M_s)$$

Where in the last inequality we used that ϕ is nondecreasing.

□

We will need the notion of uniform integrability.

Definition 1.3. *Let $\{X_\alpha : \alpha \in A\}$ a collection of random variables in some probability space $(\Omega, \mathcal{F}, P)$. They are uniformly integrable if*

$$\lim_{M\to\infty} \underset{\alpha\in A}{sup} E[|X_\alpha| \cdot 1\{|X|_\alpha \geq M\}] = 0$$

This condition is equivalent to request the following two conditions:

i) $\sup_\alpha E[|X_\alpha|] < \infty$

ii) Given $\epsilon > 0$,there exists some $\delta > 0$ such that for every $B \in \mathcal{F}$ with $P(B) < \delta$,

$$\sup_\alpha \int_B |X_\alpha| \leq \epsilon$$

We will use the following two lemmas.

Lemma 1.4. *Let X be an integrable random variable on $(\omega, \mathcal{F}, P)$. Then the collection of random variables*

$$\{E[X|\mathcal{A}] : \mathcal{A} \textit{ is a sub} - \sigma \textit{ algebra of} \mathcal{F}\}$$

is uniformly integrable

Lemma 1.5. *Suppose $X_n \to X$ in L^1 on a probability space $(\omega, \mathcal{F}, P)$. Then there exits a subsequence $\{n_j\}$ such that $E[X_{n_j}|\mathcal{A}] \to E[X|\mathcal{A}]$ almost surely.*

Proof. Note that $|E[X_n|\mathcal{A}] - E[X|\mathcal{A}]| \leq E[|X_n - X||\mathcal{A}]$.Also,

$$E[E[|X_n - X|]|\mathcal{A}] = E[|X_n - X|] \to 0$$

So $E[X_n|\mathcal{A}] \to E[X|\mathcal{A}]$ in L^1,and the claim follows. □

Proposition 1.6. *Suppose M is a right-continuous submartingale with respect to the filtration $\{\mathcal{F}_t\}$. Then M is a submartingale also with respect to $\{\mathcal{F}_{t+}\}$*

Proof. Let $s < t$ and n such that $n > (t-s)^{-1}$.$M_t \vee c$ is a submartingale, so

$$E[M_t \vee c|\mathcal{F}_{s+n^{-1}}] \geq M_{s+n^{-1}} \vee c$$

Applying $E[|\mathcal{F}_{s+}]$ and using that $\mathcal{F}_{s+} \subset \mathcal{F}_{s+n^{-1}}$,

$$E[M_t \vee c|\mathcal{F}_{s+}] \geq E[M_{s+n^{-1}} \vee c|\mathcal{F}_{s+}] \tag{1.1}$$

Using the bounds

$$c \leq M_{s+n^{-1}} \vee c \leq E[M_t \vee c|\mathcal{F}_{s+n^{-1}}]$$

and Lemma 1.4 we deduce that $\{M_{s+n^{-1}} \vee c\}$ is uniformly integrable,and in particular they are uniformly bounded in L^1.The right-continuity give us for fixed c, $M_{s+n^{-1}} \vee c \to M_s \vee c$ almost surely, so we have convergence in L^1 too.Using Lemma 1.5 we can obtain a subsequence $\{n_j\}$ such that

$$E[M_{n+n_j^{-1}} \vee c|\mathcal{F}_{s+}] \to E[M_s \vee c|\mathcal{F}_{s+}]$$

This together with (1.1) implies

$$M_s \leq M_s \vee c = E[M_s \vee c\mathcal{F}_{s+}] \leq E[M_t \vee c|\mathcal{F}_s]$$

So $M_s \vee c \leq E[M_t \vee c|\mathcal{F}_{s+}]$.Now,take $c \to -\infty$ and by dominated convergence finally obtain that

$$M_s \leq E[M_t|\mathcal{F}_{s+}]$$

□

There is sort of converse of this result,found in [2].

Proposition 1.7. *Suppose the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, that is , $(\Omega, \mathcal{F}, P)$ is complete, $\mathcal{F}_0$ contains all null events, and $\mathcal{F}_{t+} = \mathcal{F}_t$.Let M be a submartingale such that $t \to EM_t$ is right-continuous. Then there exists a cadlag modification of M that is an $\{\mathcal{F}_t\}$-submartingale.*

1.1 Optional stopping

We want to extend the submartingale property from deterministic times to stopping times.

Lemma 1.8. *Let M be a submartingale. Let σ, τ be two stopping times whose values lie in an ordered countable set $\{s_1 < s_2 < \cdots\} \cup \{\infty\} \subset [0, \infty]$ where $s_j \nearrow \infty$.Then for any $T < \infty$,*

$$E[M_{\tau\wedge T}|\mathcal{F}_\sigma] \geq M_{\sigma\wedge\tau\wedge T} \tag{1.2}$$

Proof. Fix n so that $s_n \leq T < s_{n+1}$.First observe that $M_{\tau\wedge T}$ is integrable

$$|M_{\tau\wedge T}| = \sum_{i=1}^{n} 1\{\tau = s_i\}|M_{s_i}| + 1\{\tau > s_n\}|M_T| \leq \sum_{i=1}^{n} |M_{s_i}| + |M_T|$$

a finite sum of integrable random variables.

The second property is verify that $M_{\sigma\wedge\tau\wedge T}$ is $\mathcal{F}_\sigma-$measurable. To do this, it's enough to prove for $B \in \mathcal{B}(\mathbb{R}^d)$ that $\{M_{\sigma\wedge\tau\wedge T} \in B\} \cap \{\sigma \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

Let s_j the highest value such that $s_j \leq t$.If there is not such s_j,the $t < s_1$,so $\{\sigma \leq t\} \subset \{\sigma < s_1\} = \emptyset \in \mathcal{F}_t$,because $\sigma \geq s_1$.Otherwise,

$$\{M_{\sigma\wedge\tau\wedge T}\} \cap \{\sigma \leq t\} = \cup_{i=1}^{j}\{\sigma \wedge \tau = s_i\} \cap \{M_{s_i\wedge T} \in B\} \cap \{\sigma \leq t\} \in \mathcal{F}_t$$

because $s_i \leq t$ and $\sigma \cap \tau$ is a stopping time.

Because both $E[M_{\tau\wedge T}|\mathcal{F}_\sigma]$ and $M_{\sigma\wedge\tau\wedge T}$ are $\mathcal{F}_\sigma$-measurable, (1.2) follows from

$$E\{1_A E[M_{\tau\wedge T}]|\mathcal{F}_\sigma\} \geq E\{1_A M_{\sigma\wedge\tau\wedge T}\}$$

for $A \in \mathcal{F}_\sigma$.By the definition of conditional expectation, it's reduced to show that

$$E[1_A M_{\tau\wedge T}] \geq E[1_A M_{\sigma\wedge\tau\wedge T}]$$

We split $1_A = 1_{\{A\cap\sigma\leq T\}} + 1_{\{A\cap\sigma>T\}}$. If $\sigma > T, \sigma \wedge \tau \wedge T = \tau \wedge T$, so

$$E[1_{\{A\cap\sigma>T\}} M_{\sigma\wedge\tau\wedge T}] = E[1_{\{A\cap\sigma>T\}} M_{\tau\wedge T}]$$

For the case $\sigma \leq T$, we split into sub-cases.We want to prove

$$\begin{aligned} E[1_{\{A\cap\{\sigma=s_i\}\}} M_{\tau\wedge T}] &\geq E[1_{\{A\cap\{\sigma=s_i\}\}} M_{\sigma\wedge\tau\wedge T}] \\ &= E[1_{\{A\cap\{\sigma=s_i\}\}} M_{s_i\wedge\tau\wedge T}] \quad \text{for } 1 \leq i \leq n \end{aligned}$$

As $A \cap \{\sigma = s_i\} \in \mathcal{F}_{s_i}$, it's enough to check the following:

$$E[M_{\tau\wedge T}|\mathcal{F}_{s_i}] \geq M_{s_i\wedge\tau\wedge T} \quad \text{for } 1 \leq i \leq n \tag{1.3}$$

We prove (1.3) by reverse induction on i.We first consider the case $i = n$.First, we consider an auxiliary inequality.Recall that $M_{\sigma\wedge\tau\wedge T}$ is $\mathcal{F}_\sigma-$measurable, so in particular if $\sigma \equiv s_j$,then $M_{s_j\wedge\tau\wedge T}$ is $\mathcal{F}_{s_j}-$measurable. For any j,

$$\begin{aligned}
E[M_{s_{j+1}\wedge\tau\wedge T}|\mathcal{F}_{s_j}] &= E[M_{s_{j+1}\wedge\tau\wedge T}1\{\tau > s_j\} + M_{s_j\wedge\tau\wedge T}1\{\tau \leq s_j\}|\mathcal{F}_{s_j}] \\
&= E[M_{s_{j+1}\wedge T}|\mathcal{F}_{s_j}] \cdot 1\{\tau > s_j\} + M_{s_j\wedge\tau\wedge T}1\{\tau \leq s_j\} \\
&\geq M_{s_j\wedge T} \cdot 1\{\tau > s_j\} + M_{s_j\wedge\tau\wedge T}1\{\tau \leq s_j\} \\
&= M_{s_j\wedge\tau\wedge T}
\end{aligned}$$

Recall that $s_n \leq T < s_{n+1}$,so if we apply this inequality to $j = n$,we conclude that $E[M_{\tau\wedge T}|\mathcal{F}_{s_n}] \geq M_{s_n\wedge\tau\wedge T}$,and this the the case $i = n$ from (1.3).Now assuming that (1.3) holds for i, we apply the auxiliary inequality again,

$$\begin{aligned}
E[M_{\tau\wedge T}|\mathcal{F}_{s_{i-1}}] =& E[E[M_{\tau\wedge T}|\mathcal{F}_{s_i}]|\mathcal{F}_{s_{i-1}}] \geq E[M_{s_i\wedge\tau\wedge T}|\mathcal{F}_{s_{i-1}}] \\
&\geq M_{s_{i-1}\wedge\tau\wedge T}
\end{aligned}$$

And this is (1.3) for $i - 1$.We repeat this until $i = 1$,and we are done. □

To extend this result to general stopping times, we need a preliminary lemma,and some regularity conditions.

Lemma 1.9. *Let M be a submartingale with right-continuous paths and $T < \infty$.Then for any stopping time ρ that satisfies $P(\tau \leq T) = 1$,*

$$E|M_\rho| \leq 2E[M_T^+] - E[M_0]$$

Proof. Approximate ρ by ρ_n given by $\rho_n = T$ if $\rho = T$ and $\rho_n = 2^{-n}T([2^n\rho/T] + 1)$ if $\rho < T$.Then ρ_n is a stopping time and $\rho_n \searrow \rho$ as $n \to \infty$.Apply (1.2) to $\tau = \rho_n, \sigma = 0$ and taking expectations we deduce that

$$E[M_{\rho_n}] \geq E[M_0]$$

Now apply (1.2) to the submartingale $M_t^+ = M_t \vee 0$, with $\tau = T$ and $\sigma = \rho_n$ to ge

$$E[M_T^+] \geq E[M_{\rho_n}^+]$$

Using both equations,

$$E[M_{\rho_n}^-] = E[M_{\rho_n}^+] - E[\rho_n] \leq E[M_T^+] - E[M_0]$$

Thus,

$$E[|M_{\rho_n}|] = E[M_{\rho_n}^+] + E[M_{\rho_n}^-] \leq 2E[M_T^+] - E[M_0]$$

Finally,take $n \to \infty$,and use Fatou's lemma to conclude

$$E|M_\rho| \leq \liminf_{n\to\infty} E|M_{\rho_n}| \leq 2E[M_T^+] - E[M_0]$$

$\square$

Remark 1.10. *The last result says that if τ is a stopping time and $T \in \mathbb{R}_+$,the stopped process $M_{\tau\wedge T}$ is integrable.*

Now we extend from discrete to general stopping times.

Theorem 1.11. *Let M be a submartingale with right-continuous paths, and let σ, τ be two stopping times.Then for $T < \infty$,*

$$E[M_{\tau\wedge T}|\mathcal{F}_\sigma] \geq M_{\tau\wedge\sigma\wedge T} \tag{1.4}$$

Proof. Recall that the random variables $M_{\tau\wedge T}, M_{\sigma\wedge\tau\wedge T}$ are integrable,so they conditional expectation are well defined.

Approximate the stopping times defining $\sigma_n = 2^{-n}T([2^n\sigma/T]+1)$, $\tau_n = 2^{-n}T([2^n\tau/T]+1)$.Here, $\sigma_n = \infty$ if $\sigma = \infty$,and similarly with τ_n.Fix $c \in \mathbb{R}$.The function $x \to x \vee c$ is convex and nondecreasing, hence $M_t \vee c$ is also a submartingale.Applying Lemma 1.8 to this submartingale and stopping times σ_n, τ_n give

$$E[M_{\tau_n\wedge T} \vee c|\mathcal{F}_{\sigma_n}] \geq M_{\sigma_n\wedge\tau_n\wedge T} \vee c$$

Since $\sigma \leq \sigma_n, \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma_n}$,and if we apply conditional expectation both sides,

$$E[M_{\tau_n\wedge T} \vee c|\mathcal{F}_\sigma] \geq E[M_{\sigma_n\wedge\tau_n\wedge T} \vee c|\mathcal{F}_\sigma] \tag{1.5}$$

Now we want to take $n \to \infty$ in (1.5) to obtain the conclusion to the process $M_t \vee c$,and then take $c \to -\infty$ to conclude.First note that $\tau_n \wedge T \to \tau \wedge T, \sigma_n \wedge T \to \sigma \wedge T$.By the right continuity of M,

$$M_{\tau_n \wedge T} \to M_{\tau \wedge T} \text{ and } M_{\sigma_n \wedge \tau_n \wedge T} \to M_{\sigma \wedge \tau \wedge T}$$

Next,apply Lemma 1.8 to obtain

$$c \leq M_{\tau_n \wedge T} \vee c \leq E[M_T \vee c | \mathcal{F}_{\tau_n}]$$

(taking $\sigma = \tau_n, \tau = T$), and

$$c \leq M_{\sigma_n \wedge \tau_n \wedge T} \vee c \leq E[M_T \vee c | \mathcal{F}_{\tau_n \wedge \sigma_n}]$$

(taking $\sigma = \tau_n \wedge \sigma_n, \tau = T$).Recalling Lemma 1.4, the sequences $\{M_{\tau_n \wedge T} \vee c : n \in \mathbb{N}\}, \{M_{\sigma_n \wedge \tau_n \wedge T} \vee c : n \in \mathbb{N}\}$ are uniformly integrable, and the almost surely convergence of these sequence assures the L^1 convergence.Therefore, by Lemma 1.5,there exists a subsequence $\{n_j\}$ along which the conditional expectations converge almost surely

$$E[M_{\tau_{n_j} \wedge T} \vee c | \mathcal{F}_\sigma] \to E[M_{\tau \wedge T} \vee c | \mathcal{F}_\sigma]$$

and

$$E[M_{\sigma_{n_j} \wedge \tau_{n_j} \wedge T} \vee c | \mathcal{F}_\sigma] \to E[M_{\sigma \wedge \wedge T} \vee c | \mathcal{F}_\sigma]$$

Now we can take limits in (1.5) to conclude that

$$E[M_{\tau \wedge T} \vee c | \mathcal{F}_\sigma] \geq E[M_{\sigma \wedge \tau \wedge T} \vee c | \mathcal{F}_\sigma] = M_{\sigma \wedge \tau \wedge T} \vee c \geq M_{\sigma \wedge \tau \wedge T}$$

where we used that $M_{\sigma \wedge \tau \wedge T}$ is $\mathcal{F}_{\sigma \wedge \tau \wedge T}-$measurable because M is right-continuous and hence, progressively measurable (by Lemma 1.13 from first chapter), therefore is $\mathcal{F}_\sigma-$measurable. Finally,taking $c \to -\infty$ we obtain $M_{\tau \wedge T} \vee c \to M_{\tau \wedge T}$ point-wise, and for $c \leq 0, |M_{\tau \wedge T} \vee c| \leq |M_{\tau \wedge T}|$,so by dominated convergence,

$$\lim_{c \to -\infty} E[M_{\tau \wedge T} | \mathcal{F}_\sigma] = E[M_{\tau \wedge T} | \mathcal{F}_\sigma]$$

this completes the proof. $\square$

Corollary 1.12. *Suppose M is a right-continuous submartingale and τ is a stopping time. Then the stopped process $M^\tau = \{M_{\tau \wedge t} : t \in \mathbb{R}_+\}$ is a submartingale with respect to the* *filtration* $\{\mathcal{F}_t\}$*.If M is also a martingale, then* M^τ *is a martingale.Finally, if M is an* L^2-*martingale, then so is* M^τ.

Proof. Take $T = t, \sigma = s < t$ in (1.4) to obtain $E[M_{\tau\wedge t}|\mathcal{F}_s] \geq M_{\tau\wedge s}$.If M is a martingale, apply the last result to both M and -M.Finally,if M is an L^2-martingale, apply Lemma 1.9 to the submartingale M^2 and thus deduce the same result for M^τ. □

Corollary 1.13. *Suppose M is a right-continuous submartingale. Let* $\{\sigma(u) : u \geq 0\}$ *n a nondecreasing* $[0,\infty)$*-valued process such that* $\sigma(u)$ *is a bounded stopping time for each u.Then* $\{M_{\sigma(u)} : u \geq 0\}$ *is a submartingale with respect to the filtration* $\{\mathcal{F}_{\sigma(u)} : u \geq 0\}$

Proof. For $u < v$ and $T \geq \sigma(v)$,we have $\sigma(u)\wedge\sigma(v)\wedge T = \sigma(u), \sigma(v)\wedge T = \sigma(v)$, and using (1.4) give us

$$E[M_{\sigma(v)}|\mathcal{F}_{\sigma(u)}] \geq M_{\sigma(u)}.$$

As in the last corollary, if M is a martingale,apply this result to both M and -M, and if M is an L^2-martingale, utilize Lemma 1.9 in the submartingale M^2 to deduce

$$E[M^2_{\sigma(u)}] \leq 2[M_T^2] + E[M_0^2]$$

□

Remark 1.14. *The last corollary has the following implications:*

i) Using $\sigma(t) = \tau\wedge t$*, then* M^τ *is also a submartingale with respect to* $\{\mathcal{F}_{\tau\wedge t}\}$

ii) Using $\sigma(t) = \tau + t$ *for a bounded stopping time* τ*, then the process* $\tilde{M}_t := M_{\tau+t} - M_\tau$ *is an* L^2 *martingale with respect to* $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau+t}$ *if M is an* L^2*-martingale.*

1.2 Inequalities and limits

Lemma 1.15. *Let M be a submartingale,*$0 < T < \infty$*, and H a finite subset of* $[0,T]$*.Then for* $r > 0$*,*

$$P\left(\max_{t\in H} M_t \geq r\right) \leq \frac{1}{r}E[M_T^+]$$
$$P\left(\min_{t\in H} M_t \leq -r\right) \leq \frac{1}{r}E[M_T^+ - E[M_0]]$$

Proof. Let $\sigma = \min\{t \in H : M_t \geq r\}$,with the interpretation that $\sigma = \infty$ if $M_t < r$ for all $t \in H$.Now we use (1.4) with $\tau = T$,

$$E[M_T] \geq E[M_{\sigma\wedge T}] = E[M_\sigma 1\{\sigma < \infty\}] + E[M_T 1\{\sigma = \infty\}]$$

so $E[M_\sigma 1\{\sigma < \infty\}] \leq E[M_T 1\{\sigma < \infty\}]$, from which

$$\begin{aligned} rP\left(\max_{t\in H} M_t \geq r\right) &\leq rP(\sigma < \infty) \leq E(M_\sigma 1\{\sigma < \infty\}) \leq E[M_T 1\{\sigma < \infty\}] \\ &\leq E[M_T^+ 1\{\sigma < \infty\}] \leq E[M_T^+] \end{aligned}$$

And we obtain the first inequality.To probe the second one, let $\tau = \min\{t \in H : M_t \leq -r\}$ and utilize (1.4) with $\sigma = 0$ to deduce

$$E[M_0] \leq E[M_{\tau\wedge T}] = E[M_\tau 1\{\tau < \infty\}] + E[M_T 1\{\tau = \infty\}]$$

from which

$$\begin{aligned} -rP\left(\min_{t\in H} M_t \leq -r\right) &= -rP(\tau < \infty) \geq E[M_\tau 1\{\tau < \infty\}] \\ &\geq E[M_0] - E[M_T 1\{\tau = \infty\}] \geq E[M_0] - E[M_T^+] \end{aligned}$$

□

Now we generalize to uncountable suprema and infima.

Theorem 1.16. *Let M be a right-continuous submartingale and $0 < T < \infty$.Then for $r > 0$,*

$$P\left(\sup_{0\leq t\leq T} M_t \geq r\right) \leq \frac{1}{r}E[M_T^+] \tag{1.6}$$

$$P\left(\inf_{0\leq t\leq T} M_t \leq -r\right) \leq \frac{1}{r}E[M_T^+ - E[M_0]] \tag{1.7}$$

Proof. Let H be a countable dense subset of $[0, T]$ that contains $0, T$, and let $H_1 \subset H_2 \subset H_3 \subset \cdots$ be finite sets such that $H = \cup_n H_n$.We apply the last lemma to the sets $H_n's$.Let $b < r$.By right-continuity,

$$P\left(\sup_{0\leq t\leq T} M_t > b\right) = P\left(\sup_{t\in H} M_t > b\right) = \lim_{n\to\infty} P\left(\sup_{t\in H_n} M_t > b\right) \leq \frac{1}{b}E[M_T^+]$$

Take $b \nearrow r$ and conclude the first inequality. The second one is analogous. □

Definition 1.17. *Suppose that X has either left- or right-continuous paths with probability 1. In that case, define*

$$X_T^*(\omega) := \sup_{0\le t\le T} |X_t(\omega)| \tag{1.8}$$

We verify that X_T^* is $\mathcal{F}_T$ measurable for each T. Define $U := \sup_{s\in R} |X_s|$, when R is a dense countable subset of $[0, T]$ that contains the endpoints. Then U is $\mathcal{F}_T$ measurable, because it's the supremum of countable many random variables $\mathcal{F}_T$-measurable.On every left- or right-continuous path, $U = X_T^*$,so they are equal almost surely. Also, all the zero-probability events lie in $\mathcal{F}_T$ by the completeness assumption.This implies that X_T^* is also $\mathcal{F}_T-$measurable.

Now we state the important Doob's inequality:

Theorem 1.18. *(Doob's inequality) Let M be a nonnegative right-continuous submartingale and* $0 < T < \infty$*.Then for* $1 < p < \infty$

$$E[\sup_{0\le t\le T} M_t^p] \le \left(\frac{p}{p-1}\right)^p E[M_T^p] \tag{1.9}$$

Proof. As M is nonnegative, $M_T^* = \sup_{0\le t\le T} M_t$

Step 1 We show that

$$P(M_T^* > r) \le \frac{1}{r} E[M_T 1\{M_T^* \ge r\}], \quad r > 0$$

Let $\tau = \inf\{t > 0 : M_t > r\}$, an $\{\mathcal{F}_{t+}\}$-stopping time by Proposition 1.6. By right-continuity, $M_\tau \ge r$ when $\tau < \infty$.Also, $M_T^* > r$ implies $\tau \le T$.This says that $M_T^* > r \to \tau < \infty \to M_\tau \ge r$ and so

$$rP(M_T^* > r) \le E[M_\tau 1\{M_T^* > r\}] \le E[M_\tau 1\{\tau \le T\}]$$

Since M is a submartingale with respect to $\{\mathcal{F}_{t+}\}$by Proposition 1.6 (and τ is an $\{\mathcal{F}_{t+}\}$-stopping time), we can apply Theorem 1.11 to obtain

$$\begin{aligned} E[M_\tau 1\{\tau \le T\}] &= E[M_{\tau\wedge T}] - E[M_T 1\{\tau > T\}] \le E[M_T] - E[M_T 1\{\tau > T\}] \\ &= E[M_T 1\{\tau \le T\}] \le E[M_T 1\{\tau \le T\}] \le E[M_T 1\{M_T^* \ge r\}] \end{aligned}$$

in the last step we used that $\tau \le T \to M_T^* \ge r$.That concludes the first step.

Step 2 Let $0 < b < \infty$. We use the identity $E[(M_T^* \wedge b)^p] = \int_0^b pr^{p-1}P[M_T^* > r]dr$ and Hölder inequality:

$$E[(M_T^* \wedge b)^p] = \int_0^b pr^{p-1}P[M_T^* > r]dr \leq \int_0^b pr^{p-2}E[M_T 1\{M_T^* \geq r\}]dr$$
$$\overset{Fubini}{=} E[M_T \int_0^{b \wedge M_T^*} pr^{p-2}dr] = \frac{p}{p-1}E[M_T(b \wedge M_T^*)^{p-1}]$$
$$\overset{Holder}{\leq} \frac{p}{p-1}E[M_T^p]^{1/p}E[(b \wedge M_T^*)^p]^{\frac{p-1}{p}}$$

Now we divide by $E[(b \wedge M_T^*)^p]^{\frac{p-1}{p}}$ (which is finite by the truncation) to obtain

$$E[(M_T^* \wedge b)^p]^{1/p} \leq \frac{p}{p-1}E([M_T^p]^{1/p}$$

Raising to the $p - th$ power both sides, and taking $b \to \infty$ to conclude the claim by monotone convergence.

$\square$

Corollary 1.19. *Let M be a nonnegative right-continuous submartingale and τ a bounded stopping time .Then for $1 < p < \infty$,*

$$E\left[\left(\underset{0 \leq t \leq \tau}{sup} M_t\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p E[M_\tau^p] \tag{1.10}$$

Proof. Consider the stopped process $M_{t\wedge\tau}$ and T with $\tau \leq T$.Then, $M_{\tau\wedge T} = M_\tau$and $\sup_{0\leq t\leq T} M_{\tau\wedge t} = \sup_{0\leq t\leq \tau} M_t$.We apply the Doob's inequality no $M_{\tau\wedge t}$. $\square$

For completeness, we state the Martingale Convergence Theorem.

Theorem 1.20. *Let M be a right-continuous submartingale such that*

$$\underset{t\in\mathbb{R}_+}{sup} E(M_t^+) < \infty$$

Then there exists a random variable M_∞ integrable and $M_t(\omega) \to M_\infty(\omega)$ as $t \to \infty$ for almost every ω

When the limit M_∞ exist, the question is if the complete process $\{M_t : t \in [0, \infty]\}$ is a martingale,that is, if $E(M_\infty|\mathcal{F}_t) = M_t$.The following theorem characterizes this condition.

Theorem 1.21. *Let $M = \{M_t : t \in \mathbb{R}_+\}$ be a right-continuous martingale. The following are equivalent:*

i) The collection $\{M_t : t \in \mathbb{R}_+\}$ is uniformly integrable.

ii) There exists an integrable random variable M_∞ such that

$$\lim_{t\to\infty} E|M_t - M_\infty| = 0$$

iii) There exists an integrable random variable M_∞ such that

$$M_t(\omega) \overset{t\to\infty}{\to} M_\infty(\omega) \quad \textit{almost surely}$$
$$E(M_\infty|\mathcal{F}_t) = M_t \quad \textit{for all } t \in \mathbb{R}_+$$

iv) There exists an integrable random variable Z such that $M_t = E(Z|\mathcal{F}_t)$ for all $t \in \mathbb{R}_+$.

We state a direct corollary from the last theorem.

Corollary 1.22.

i) For $Z \in L^1(P), E(Z|\mathcal{F}_t) \overset{t\to\infty}{\to} E(Z|\mathcal{F}_\infty)$ both almost-surely and in L^1.

ii) For $A \in \mathcal{F}_\infty, , E(1_A|\mathcal{F}_t) \overset{t\to\infty}{\to} E(1_A|\mathcal{F}_\infty)$ both almost-surely and in L^1.

Proof. Part ii) follows from part i). To prove part i),define $M_t = E(Z|\mathcal{F}_t)$.Then, part iv) of Theorem 1.21, there exist a limit M_∞ almost surely and a limit M'_∞ in L^1.But as the $M'_t s$ are uniformly integrable, then the convergence is also in L^1,so $M'_\infty = M_\infty$.By construction, M_∞ is $\mathcal{F}_\infty-$measurable.We need to conclude that $M_\infty = E(Z|\mathcal{F}_\infty)$.For $A \in \mathcal{F}_s$ we have for $t > s$ and by L^1 convergence

$$E[1_A Z] = E[1_A M_t] \to E[1_A M_\infty]$$

By a standard argument, extend $E[1_A Z] = E[1_A M_\infty]$ to all $A \in \mathcal{F}_\infty$.This concludes the proof. □

1.3 Local martingales and semimartingales

For a stopping time τ and a process $X = \{X_t : t \in \mathbb{R}_+\}$,the stopped process X^τ is defined by $X^\tau_t = X_{\tau\wedge t}$.

Definition 1.23. *Let $M = \{M_t : t \in \mathbb{R}_+\}$ be a process adapted to a filtration $\mathbb{F}_t$.M is a local martingale if there exists a sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots$ such that $P(\tau_k \nearrow \infty) = 1$ and for each k, M^{τ_k} is a martingale with respect to $\{\mathcal{F}_t\}$.M is a local square-integrable martingale (local L^2-martingale from now)if for each k,M^{τ_k} is a square-integrable martingale.In both cases we say $\{\tau_k\}$ is a localizing sequence for M.*

Remark 1.24. *If we consider a local martingale $\{M_t : t \in [0,T]\}$, then it's enough to consider a nondecreasing sequence σ_n with $\sigma_n \geq T$ for large enough n almost surely.In that case, if we define $\tau_n = \sigma_n 1\{\sigma_n < T\} + \infty 1\{\sigma_n \geq T\}$,we recover the original definition.*

Lemma 1.25. *Suppose M is a local martingale and σ is an arbitrary stopping time.Then M^σ is also a local martingale.Similarly, if M is a local L^2-local martingale, it's also M^σ.In both cases, if $\{\tau_k\}$ is a localizing sequence, for M, it's also for M^σ.*

Proof. Let $\{\tau_k\}$ be a localizing sequence for M.Then, M^{τ_k} is a martingale for each k.By Corollary 1.12, the process $M^{\tau_k}_{\sigma\wedge t} = (M^\sigma)^{\tau_k}_t$ is a martingale.Thus the stopping times τ_k also work for M^σ.

If M^{τ_k} is an L^2-martingale, then it's also $M^{\tau_k}_{\sigma\wedge t} = (M^\sigma)^{\tau_k}_t$ by Lemma 1.9 applied to the submartingale $(M^{\tau_k})^2$,

$$E[M^2_{\sigma\wedge\tau_k\wedge t}] \leq E[M^2_{\tau_k\wedge t}] + E[M^2_0]$$

□

Lemma 1.26. *Suppose M is a cadlag local martingale,and there is a constant c such that $|M_t(\omega) - M_{t-}(\omega)| \leq c$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.Then M is a local L^2-martingale.*

Proof. Let $\tau_k \nearrow \infty$ be stopping times such that M^{τ_k} is a martingale.Let

$$\rho_k := \inf\{t \geq 0 : |M_t| \text{ or } |M_{t-}| \geq k\}$$

By the cadlag assumption, each path $t \to M_t(\omega)$ is bounded in each bounded time interval, so $\rho_k \nearrow \infty$ when $k \to \infty$.Let $\sigma_k = \rho_k \wedge \tau_k$.Then $\sigma_k \nearrow \infty$ when $k \to \infty$, and M^{σ_k} is a martingale for each k.Furthermore(make a draw if unsure about this bound),

$$|M^{\sigma_k}_t| = |M_{\tau_k\wedge\rho_k\wedge t}| \leq \sup_{0\leq s<\rho_k} |M_s| + |M_{\rho_k} - M_{\rho_k-}| \leq k + c$$

So M^{σ_k} is a bounded process ,and in particular an L^2-process. □

Recall that the usual conditions on the filtration $\{\mathcal{F}_t\}$ meant that the filtration is complete and right-continuous.

Theorem 1.27. *(Fundamental Theorem of Local Martingales) Assume $\{\mathcal{F}_t\}$ is complete and right-continuous,M is a cadlag local martingale and $c > 0$.Then there exists cadlag local martingale $\tilde{M}$ and A such that the jumps of $\tilde{M}$ are bounded by c, A is a finite variation process, and $M = \tilde{M} + A$.The last lemma says that $\tilde{M}$ is an L^2-local martingale.*

Combining this theorem with the last lemma, we get the following corollary:

Corollary 1.28. *Assume $\{F_t\}$ is complete and right-continuous. Then a cadlag local martingale M can be written as a sum $M = \tilde{M} + A$ of a cadlag local L^2- martingale $\tilde{M}$ nd a local martingale A that is an FV process.*

Definition 1.29. *A cadlag process Y is a semimartingales if it can be written as*

$$Y_t = Y_0 + M_t + V_t$$

where M is a cadlag local martingale, V is a cadlag FV process, and $M_0 = V_0 = 0$.

1.4 Quadratic variation for semimartingales

We study the quadratic variation and covariation for semimartingales.Recall that for two independent Brownian motions B and Y, $[B]_t = t$ and $[B, Y] = 0$.For the Poisson process, if N is a homogeneous rate α Poisson process, and $M_t = N_t - \alpha t$,then $[M] = [N] = N$.If $\tilde{N}$ is an independent rate $\tilde{\alpha}$ Poisson process with $\tilde{M_t} = \tilde{N_t} - \tilde{\alpha}t, [M, \tilde{M}] = 0$.
The following result is an existence theorem of quadratic variation for local martingales.

Theorem 1.30. *Let M be a right-continuous local martingale with respect to the filtration $\mathcal{F}_t$.Then the quadratic variation process $[M]$ exists in the sense of*

$$\lim_{mesh(\pi)\to 0} \sum_{i=0}^{m(\pi)-1} (B_{t_{i+1}} - B_{t_i})^2 = t \quad in\ L^2(P). \tag{1.11}$$

There is a version of $[M]$ with the following properties:$[M]$ is real-valued, right-continuous, nondecreasing adapted process such that $[M]_0 = 0$.

Suppose M is an L^2- martingale.Then th convergence in (1.11) for $Y = M$ holds also in L^1,namely for any $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty} E|\sum_i (M_{t^n_{i+1}} - M^n_{t_i})^2 - [M]_t| = 0 \tag{1.12}$$

for any sequence of partitions $\pi^n = \{t^n_i\}_i$ of $[0,t]$ with mesh$(\pi^n) \overset{n\to\infty}{\to} 0$.Furthermore,

$$E([M]_t) = E(M_t^2 - M_0^2) \tag{1.13}$$

If M is continuous, the so is $[M]$.

The following result states that $[M]^\tau = [M^\tau]$ for local L^2-martingales. However,the result is valid for local martingales (a proof is of the general fact together with the proof of the theorem above is found in [1])

Lemma 1.31. *Let M be a right-continuous L^2-martingale or local L^2-martingale.Let τ be a stopping time. Then $[M]^\tau = [M^\tau]$ in the sense that these processes are indistinguishable.*

Proof. By the last theorem, both processes are right-continuous.By Lemma 1.14 from the first chapter,it's enough that for each fixed time $t, [M_t]^\tau = [M^\tau_t]$ almost surely.

Step 1 Supposes first that τ take discrete values $u_1 < u_2 < \cdots$ with $u_j \nearrow \infty$.Fix t and consider a sequence of partitions $\pi^n = \{t^n_i\}_i$ of $[0,t]$ with $mesh(\pi^n) \to 0$ as $n \to \infty$.For any u_j,

$$\sum_i (M_{u_j \wedge t^n_{i+1}} - M_{u_j \wedge t^n_i})^2 \to [M]_{u_j \wedge t} \quad \text{in probability, as } n \to \infty$$

Observe that if $u_j > t$,the random variable above is the same of all j large enough, so they are finite many of them. Therefore we can consider a sub-partition of π^n such that for every j,the convergence above is almost surely.Denote again by π^n to this sub-partition.

Fix ω at which the convergence happens. Let $u_j = \tau(\omega)$.We have

$$\begin{aligned}
[M]^\tau_t(\omega) &\overset{\textit{Theorem } 1.30}{=} \lim_{n\to\infty} \sum_i (M^\tau_{t^n_{i+1}}(\omega) - M^\tau_{t^n_i}(\omega)^2 \\
&= \lim_{n\to\infty} \sum_i (M_{\tau\wedge t^n_{i+1}}(\omega) - M_{\tau\wedge t^n_i}(\omega)^2 \\
&= \lim_{n\to\infty} \sum_i (M_{u_j\wedge t^n_{i+1}} - M_{u_j\wedge t^n_i})^2 \\
&= [M]_{u_j\wedge t}(\omega) = [M]_{\tau\wedge t}(\omega) = [M]^\tau_t(\omega)
\end{aligned}$$

Step 2 Let τ be an arbitrary stopping time, but now assume M is an L^2-martingale.Let τ_n the stopping time approximation such that $\tau_n \searrow \tau$.We apply

$$|[X]_t - [Y]_t| \leq [X-Y]_t + 2[X-Y]_t^{1/2}[Y]_t^{1/2} \quad a.s \tag{1.14}$$

to $X = M^{\tau_n}, Y = M^{\tau}$,(1.13) and Cauchy Schwartz to obtain

$$\begin{aligned} E(|[M^{\tau_n}]_t - [M^{\tau}]_t|) &\leq E([M^{\tau_n} - M^{\tau}]_t) + 2E([M^{\tau_n} - M^{\tau}]_t^{1/2}[M^{\tau}]_t^{1/2}) \\ &\leq E[(M_t^{\tau_n} - M_t^{\tau})^2] + 2E([M^{\tau_n} - M^{\tau}]_t)^{1/2}E([M^{\tau}]_t)^{1/2} \\ &= E[(M_{t\wedge\tau_n} - M_{t\wedge\tau})^2] + 2\{E(M_{\tau_n\wedge t} - M_{t\wedge\tau})^2\}^{1/2}\{E(M_{t\wedge\tau})^2\}^{1/2} \\ &\leq E[(M_{t\wedge\tau_n})^2] - E[(M_{t\wedge\tau})^2] + 2(E(M_{\tau_n\wedge t})^2 - E(M_{t\wedge\tau})^2)^{1/2}\{E(M_t)^2\}^{1/2} \end{aligned}$$

In the last step was used (1.4) two times.First we used that

$$\begin{aligned} E[(M_{t\wedge\tau_n} - M_{t\wedge\tau_n})^2] &= E(M_{t\wedge\tau_n}^2) - 2E\{E(M_{\tau_n\wedge t}|\mathcal{F}_{\tau\wedge t})M_{\tau\wedge t}\} + E(M_{\tau\wedge t}^2) \\ &= E(M_{t\wedge\tau_n}^2) - E(M_{\tau\wedge t}^2) \end{aligned}$$

In the second sum we applied (1.4) to the submartingale M^2 to get

$$E(M_{\tau\wedge t}^2)^{1/2} \leq E(M_t^2)^{1/2}$$

Using this inequality together with M is an L^2-martingale, we conclude that $[M^{\tau_n}]_t \to [M^{\tau}]_t$ in L^1 as $n \to \infty$,if we can show that

$$E(M_{\tau_n\wedge t}^2) \to E(M_{\tau\wedge t}^2)$$

But we know that $M_{\tau_n\wedge t}^2 \to M_{\tau\wedge t}^2$ almost surely by right continuity,Also by optional stopping

$$0 \leq M_{\tau_n\wedge t}^2 \leq E(M_t^2|\mathcal{F}_{\tau_n\wedge t})$$

The sequence of conditional expectations $\{E(M_t^2|\mathcal{F}_{\tau_n\wedge t})\}$ is uniformly integrable, so it is $\{M_{\tau_n\wedge t}^2\}$, and this implies the convergence in L^1.
We have shown that $[M^{\tau_n}]_t \to [M^{\tau}]_t$ in L^1 as $n \to \infty$.By Step 1, $[M^{\tau_n}]_t = [M]_t^{\tau_n} = [M]_{\tau_n\wedge t} \to [M]_{\tau\wedge t}$ by the right continuity of the process $[M]$,so $[M^{\tau}]_t = [M]_{\tau\wedge t}$ for L^2 martingales.

Step 3 Now we consider L^2-local martingales. Let $\{\sigma_k\}$ be stopping times with $\sigma_k \nearrow \infty$ and M^{σ_k} is an L^2-martingale for each k.By Step 2,

$$[M^{\sigma_k \wedge \tau}]_t = [M^{\sigma_k}]_{\tau \wedge t}$$

On the event $\{\sigma_k > t\}$, throughout the time interval $[0, t], M^{\sigma_k \wedge \tau} = M^\tau, M^{\sigma_k} = M$.Hence the corresponding square sums also agree. Taking the mesh of the partition go to zero we conclude that $[M^{\sigma_k \wedge \tau}]_t = [M^\tau]_t$ and by right continuity, $[M^{\sigma_k}]_s = [M]_s$ for all $s \in [0, t]$.If we take $s = \tau \wedge t$,we get the desired equality $[M^\tau]_t = [M]_{\tau \wedge t}$

□

Theorem 1.32.

i) If M is a right-continuous L^2-martingale,then $M_t^2 - [M]_t$ is a martingale.

ii) If M is a right-continuous local L^2-martingale,then $M_t^2 - [M]_t$ is a local martingale.

Proof.

i) Let $s < t$ and $A \in \mathcal{F}_s$.Let $0 = t_0 < \cdots t_m = t$ be a partition of $[0, t]$, and assume that $s = t_l$ for some $l \in \{0, \cdots, m-1\}$.We compute

$$\begin{aligned} E[1_A(M_t^2 - M_s^2) - [M]_t - [M]_s] &= E\left[1_A\left(\sum_{i=l}^{m-1}(M_{t_{i+1}}^2 - M_{t_i}^2) - [M]_t - [M]_s\right)\right] \\ &= E\left[1_A\left(\sum_{i=l}^{m-1}(M_{t_{i+1}} - M_{t_i})^2 - [M]_t - [M]_s\right)\right] \\ &= E\left[1_A\left(\sum_{i=0}^{m-1}(M_{t_{i+1}} - M_{t_i})^2 - [M]_t\right)\right] \\ &+ E\left[1_A\left([M]_s - \sum_{i=0}^{l-1}[M]_s - (M_{t_{i+1}} - M_{t_i})^2\right)\right] \end{aligned}$$

In the second inequality we used

$$E[M_{t_{i+1}}^2 - M_{t_i}^2|\mathcal{F}_{t_i}] = E[(M_{t_{i+1}} - M_{t_i})^2|\mathcal{F}_{t_i}]$$

In the last equation, we take the mesh go to zero, and both sums vanish by the L^1-convergence of the quadratic variation for L^2-martingales (Theorem 1.30).Because $A \in \mathcal{F}_s$ is arbitrary, by the definition con conditional expectation we are done.

ii) let $X = M^2 - [M]$ for a local L^2-martingale M.Let $\{\tau_k\}$ be a localizing sequence for M.By the last part, $(M^{\tau_k})_t^2 - [M^{\tau_k}]_t$ is a martingale.Now we use that $[M^{\tau_k}]_t = [M]_{\tau_k \wedge t}$,so $M^2_{\tau_k \wedge t} - [M]_{\tau_k \wedge t} = X^{\tau_k}$ is a martingale. Therefore $\{\tau_k\}$ is a localizing sequence for X,and we are done.

$\square$

Remark 1.33. *From Theorem 1.30, the quadratic covariation $[M, N]$ of two right-continuous local martingales M and N exist.Because $[M, N]$ is the difference of two increasing processes, $[M, N]$ is a finite variation process.*

Lemma 1.34. *Let M and N be cadlag L^2- martingales or local L^2-martingales. Let τ be a stopping time. Then $[M^\tau, N] = [M^\tau, N^\tau] = [M, N]^\tau$*

Proof. $[M^\tau, N^\tau] = [M, N]^\tau$ follows from

$$[X, Y] := [\frac{1}{2}(X + Y)] - [\frac{1}{2}(X - Y)] \tag{1.15}$$

and Lemma 1.31.To prove the first equality, consider a partition of $[0, t]$.If $0 < \tau \leq t$, let l be the index such that $t_l < \tau \leq t_{l+1}$.Then

$$\sum_i (M^\tau_{t_{i+i}} - M^\tau_{t_i})(N_{t_{i+1}} - N^\tau_{t_{i+1}} + N^\tau_{t_i} - N_{t_i}) = (M_\tau - M_{t_l})(N_{t_{l+1}} - N_\tau)1\{0 < \tau \leq t\} \tag{1.16}$$

This equality is true because $\tau > t$,then $\tau \wedge t_i = t_i$ for all i,and the left sum vanishes.If $0 < \tau \leq t$,then for all indexes with $i > l$, $M^\tau_{t_{i+1}} - M^\tau_{t_i} = M_\tau - M_\tau = 0$,so the sum above vanishes for $i > l$,and if $i < l$, $(N_{t_{i+1}} - N^\tau_{t_{i+1}} + N^\tau_{t_i} - N_{t_i}) = 0$,so the only no zero term in the sum in (1.16) is when $i = l$, and is precisely the right-side of the equality (1.16).Note that if $\tau = 0$,both sides in (1.16) vanish.The equation above can be written as

$$\sum_i (M^\tau_{t_{i+i}} - M^\tau_{t_i})(N_{t_{i+1}} - N_{t_i}) = (M_\tau - M_{t_l})(N_{t_{l+1}} - N_\tau)1\{0 < \tau \leq t\} + \sum_i (M^\tau_{t_{i+i}} - M^\tau_{t_i})(N^\tau_{t_{i+1}} - N^\tau_{t_i})$$

If we let the mesh of the partition goes to zero,using the cadlag property, the equation above implies that

$$[M^\tau, N] = (M_\tau - M_{\tau_-})(N_\tau - N_\tau)1\{0 < \tau \leq t\} + [M^\tau, N^\tau] = [M^\tau, N^\tau]$$

$\square$

Theorem 1.35.

i) If M,N are right-continuous L^2-martingales,then $MN - [M,N]$ is a martingale.

ii) If M,N are right-continuous local L^2-martingales,then $MN - [M,N]$ is a local martingale.

Proof. Write $MN - [M,N] = \frac{1}{2}\{(M+N)^2 - [N+M]\} - \frac{1}{2}\{M^2 - [M]\} - \frac{1}{2}\{N^2 - [N]\}$,and apply the last lemma to the martingales/local martingales $M, N, M+N$ □

We want to extend this result to semimartingales.Before we state a lemma.

Lemma 1.36. *Let f,g be real-valued cadlag functions on $[0,T]$ and assume $f \in BV([0,T])$.Then*

$$[f,g](T) = \sum_{s\in(0,T]} (f(s) - f(s-)(g(s) - g(s-)$$

and th sum above converges absolutely

Corollary 1.37. *Let M be a cadlag local martingale,V a cadlag FV process,$M_0 = V_0 = 0$, and $Y = Y_0 + M + V$ the cadlag semimartingale.Then the cadlag quadratic variation process $[Y]$ exists and satisfies*

$$\begin{aligned}[Y]_t &= [M]_t + 2[M,V]_t + [V]_t \\ &= [M]_t + 2\sum_{s\in(0,t]} \triangle M_s \triangle V_s + \sum_{s\in(0,t]} (\triangle V_s)^2 \end{aligned} \tag{1.17}$$

Furthermore, $[Y^\tau] = [Y]^\tau$ for any stopping time τ and the covariation $[X,Y]$ exists for any pair of cadlag semimartingales.

Proof. We already know the existence properties of $[M]$.According to Lemma 1.36, the two sums in (1.17) converge absolutely. It can be proved that in that case, the process given in (1.17) is a cadlag process.Theorem 1.30 and Lemma 1.36 together imply that (1.17) is the limit in probability of sums $\sum_i (Y_{t_{i+1}} - Y_{t_i})^2$ as $mesh(\pi) \to 0$.Denote the process in (1.17) by U_t.By definition, for $s < t, U_s \leq U_t$,then this happens simultaneously for all pair of rationals $s < t$. By taking limits, and using the cadlag property, we can extend the monotonicity to all times $s < t$.Thus U is an increasing process and gives a version of $[Y]$ with nonnegative paths. This proves the existence of $[Y]$.The equality $[Y^\tau] = [Y]^\tau$ follows from applying Lemma 1.31

at each term in (1.17).The covariation $[X,Y]$ exists because the existence of $[X+Y],[X-Y]$ and (1.15). $\square$

1.5 Doob-Meyer decomposition

Throughout this section we work with a fixed probability space $(\Omega,\mathcal{F},P)$ with a filtration $\{\mathcal{F}_t\}$ with satisfies the *usual conditions.*

Definition 1.38. *The predictable σ-algebra $\mathcal{P}$ on the space $\mathbb{R}_+\times\Omega$ in the sub-$\sigma-$algebra of $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}$ generated by left-continuous adapted processes.More precisely, $\mathcal{P}$ is generated by events of the form $\{(t,\omega):X_t(\omega)\in B\}$,where X is an adapted,left-continuous process and $B\in\mathcal{B}(\mathbb{R}^d)$.Recall that such processes are progressively measurable by Lemma 1.13 from the first chapter.*

Intuitively,a predictable process permit us obtain X_t if we know X_s for $s<t$.
Any $\mathcal{P}-$measurable function $X:\mathbb{R}_+\times\Omega\to\mathbb{R}$ is called a *predictable process.*
We state a similar theorem of Theorem 1.30.

Theorem 1.39. *Assume the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions.*

i) Let M be a right-continuous square-integrable martingale.Then there is a unique predictable process $\langle M\rangle$(called the predictable quadratic covariation) such that $M^2-\langle M\rangle$ is a martingale.

ii) Let M be a right-continuous local square-integrable martingale.Then there is a unique predictable process $\langle M\rangle$ such that $M^2-\langle M\rangle$ is a local martingale.

Uniqueness above means uniqueness up to indistinguishablenessWe can define the predictable covariation by

$$\langle M,N\rangle=\frac{1}{4}\langle M+N\rangle-\frac{1}{4}\langle M-N\rangle$$

From the uniqueness from Theorem 1.39 and Theorem 1.30, if $[M]$ is predictable, then $[M]=\langle M\rangle$.

Proposition 1.40. *Assume the filtration satisfies the usual conditions.*

i) Suppose M is a continuous L^2-martingale.Then $[M]=\langle M\rangle$.

ii) Suppose M is a right-continuous L^2-martingale with stationary and independent increments: for all $s, t \geq 0, M_{s+t} - M_s$ is independent of $\mathcal{F}_s$,and has the same distribution as $M_t - M_0$.Then $\langle M \rangle_t = t \cdot E[M_1^2 - M_0^2]$.

Proof.

i) This follows because M is continuous,hence predictable, and the uniqueness part of Theorem 1.39.

ii) The deterministic,continuous function $t \to E \cdot E[M_1^2 - M_0^2]$is predictable.For any $t > 0$,and integer k,

$$E[M_{kt}^2 - M_0^2] = \sum_{i=1}^{k-1} E[M_{(j+1)t}^2 - M_{jt}^2] = \sum_{i=1}^{k-1} E[(M_{(j+1)t} - M_{jt})^2]$$
$$= kE[(M_t - M_0)^2] = kE[M_t^2 - M_0^2]$$

Using this twice, for any rational k/n,

$$E[M_{k/n}^2 - M_0^2] = kE[M_{1/n}^2 - M_0^2] = (k/n)E[M_1^2 - M_0^2]$$

Given an irrational $t > 0$,pick rationals $q_m \searrow t$.Fix $T \geq q_m$.By right-continuity of paths, $M_{q_m} \to M_t$ almost surely.Using the bound

$$0 \leq M_{q_m}^2 \leq E[M_T^2 | \mathcal{F}_{q_m}]$$

then $\{M_{q_m}^2\}_m$ is uniformly integrable.This gives the convergence $E[M_{q_m}^2] \to E[M_t^2]$,so

$$E[M_t^2 - M_0^2] = tE[M_1^2 - M_0^2].$$

The martingale property follows from

$$E[M_t^2 | \mathcal{F}_s] = M_s^2 + E[M_t^2 - M_s^2 | \mathcal{F}_s] = M_s^2 + E[(M_t - M_s)^2 | \mathcal{F}_s]$$
$$= M_s^2 + E[(M_{t-s} - M_0)^2 | \mathcal{F}_s] = M_s^2 + (t-s)[M_1^2 - M_0^2]$$

□

Example 1.41. *For a standard Brownian motion,* $\langle B \rangle_t = [B]_t = t$.
For a compensated Poisson process $M_t = N_t - \alpha t$,

$$\begin{aligned}\langle M \rangle_t &= tE[M_1^2] = tE[(N_1 - \alpha t)^2] = t(EN_1^2 - 2\alpha EN_1 + \alpha^2) \\ &= t(\alpha + \alpha^2 - 2\alpha^2 + \alpha^2) = \alpha t\end{aligned}$$

Definition 1.42. *An increasing process* A *is natural if for every bounded cadlag martingale* M

$$E\int_{(0,t]} M(s)dA(s) = E\int_{(0,t]} M(s-)dA(s) \quad for\ 0 < t < \infty. \tag{1.18}$$

Lemma 1.43. *Let* A *be an increasing process and* M *a bounded cadlag martingale.If* A *is continuous,then* (1.18) *holds.*

Proof. Recall that a cadlag path $\omega \to M(s, \omega)$ hast at most countable many discontinuities. If A is continuous, then the Lebesgue-Stieltjes measure of every singleton is zero,because $\Lambda_A(\{s\}) = A(s) - A(-s) = 0$.Therefore, the measure of countable sets is also zero.Thus

$$\int_{(0,t]} (M(s) - M(s-)dA(s) = 0$$

□

It can be proved that an increasing process is natural if and only if is predictable.

Definition 1.44. *for* $0 < u < \infty$,*let* $\mathcal{T}_u$ *be the collection of stopping times* τ *that satisfy* $\tau \leq u$.*A process is of class DL if the random variables* $\{X_\tau : \tau \in \mathcal{T}_u\}$ *are uniformly integrable for each* $0 < u < \infty$.

Lemma 1.45. *A right-continuous nonnegative submartingale is of class DL.*

Proof. Simply recall the inequality

$$0 \leq X_\tau \leq E[X_u|\mathcal{F}_\tau]$$

and the uniformly integrability of the conditional expectations. □

Now we state the main result.

Theorem 1.46. *(Doob-Meyer Decomposition) Assume the underlying filtration is complete and right-continuous.Let X be a right-continuous submartingale of class DL.Then there is an increasing natural process A,unique up to indistinguishableness, such that $X - A$ is a martingale.*

Applying this result to a right-continuous martingale M ,as M^2 is a submartingale, we can define $\langle M \rangle$ as the unique increasing, natural process such that $M_t^2 - \langle M \rangle_t$ is a martingale, given by the Doob-Meyer decomposition.

1.6 Spaces of martingales

Definition 1.47. *Given a probability space $(\omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$,let $\mathcal{M}_2$ the space of L^2-cadlag martingales on this space with respect to $\mathcal{F}_t$.The subspace of $\mathcal{M}_2$ of continuous L^2-martingales is $\mathcal{M}_2^c$.*

For $M \in \mathcal{M}_2$,we define the quantity

$$||M||_{\mathcal{M}_2} := \sum_{k=1}^{\infty} 2^{-k}(1 \wedge ||M_k||_{L^2(P)}) \tag{1.19}$$

Note that for $a, b \geq 0$, the following inequality holds :

$$1 \wedge (a, b) \leq 1 \wedge a + 1 \wedge b$$

Therefore $||M+N||_{\mathcal{M}_2} \leq ||M||_{\mathcal{M}_2} + ||N||_{\mathcal{M}_2}$ Finally,observe that as in L^p spaces, $||M-N|| = 0$ if M and N are indistinguishable. So we will consider in this case two martingales M, N as equal if they are indistinguishable,and define

$$d_{\mathcal{M}_2}(M, N) := ||M - N||_{\mathcal{M}_2} \tag{1.20}$$

this defines a metric on $\mathcal{M}_2$.

Theorem 1.48. *Assume the underlying probability space $(\omega, \mathcal{F}, P)$ and the filtration $\{\mathcal{F}_t\}$ is complete. Let indistinguishable processes be interpreted as equal. Then $\mathcal{M}_2$ is a complete metric space under the metric $d_{\mathcal{M}_2}$.The subspace $\mathcal{M}_2^c$ is closed, and hence a complete metric space also.*

Proof. Suppose $M \in \mathcal{M}_2$ and $||M||_{\mathcal{M}_2} = 0$.Then $E[M_k^2] = 0$ for each $k \in \mathbb{N}$.Since M_t^2 is a submartingale, $E[M_t^2] \leq E[M_k^2]$ for $t \leq k$, and consequently,$E[M_t^2] = 0$ for all $t \geq 0$.In particular, for each $t, P(M_t = 0) = 1$.Now we consider $\Omega_0 = \cap_{q \in \mathbb{Q}_+}\{M_q = 0\}$, so $P(\Omega_0 = 1)$.By right-continuity, $M_t(\omega) = 0$ for all $t \geq 0$ almost surely.This proves that M is indistinguishable from 0, and therefore $||M||_{\mathcal{M}_2} = 0$ if and only if $M \equiv 0$ where $\equiv$ is the equivalence relation of indistinguishablenessThe triangle inequality was proved above, and the symmetry follows by definition. We conclude that $d_{\mathcal{M}_2}$ defines a metric in $\mathcal{M}_2$.

Now we need to prove the completeness.Let $\{M^{(n)} : n \in \mathbb{N}\}$ be a Cauchy sequence in $\mathcal{M}_2$.We need to show that exists some $M \in \mathcal{M}_2$ such that $d_{\mathcal{M}_2}(M^n, M) \to 0$.Given $t \leq k \in \mathbb{N}$, using that $(M_t^{(m)} - M_t^{(n)})^2$ is a submartingale, and the the definition (1.19) we obtain

$$\begin{aligned} 1 \wedge E[(M_t^{(m)} - M_t^{(n)})^2]^{1/2} &\leq 1 \wedge E[(M_k^{(m)} - M_k^{(n)})^2]^{1/2} \\ &\leq 2^k ||M^{(m)} - M^{(n)}||_{\mathcal{M}_2} \end{aligned}$$

This implies that $\{M_t^{(n)} : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(P)$ for each $t \geq 0$.As $L^2(P)$ is complete, there exists some $Y_t \in L^2(P)$ such that $M_t^n \to Y_t$ in $L^2(P)$.In particular, $M_t^n \to Y_t$ in $L^1(P)$, so for $s < t, A \in \mathcal{F}_s$, from the equality $E[1_A M_t^n] = E[1_A M_t^n]$ we take $n \to \infty$ to deduce that $E[1_A Y_t] = E[1_A Y_s]$, so Y_t is a martingale for each $t \geq 0$.However, we don't know if the cadlag property holds for Y_t. To handle that, we use (1.6) to obtain the inequality

$$P(\sup_{0 \leq t \leq k} |M_t^{(m)} - M_t^{(n)}| \geq \epsilon) \leq \frac{1}{\epsilon^2} E[(M_k^{(m)} - M_k^{(n)})^2] \tag{1.21}$$

We can choose a subsequence $\{n_k\}$ such that

$$P(\sup_{0 \leq t \leq k} |M_t^{(n_{k+1})} - M_t^{(n_k)}| \geq 2^{-k}) \leq 2^{-2k} \tag{1.22}$$

To verify this, take $n_0 = 1$,and if n_{k-1} has been chose, pick $n_k > n_{k-1}$ such that

$$||M^{(m)} - M^{(n)}||_{\mathcal{M}_2} \leq 2^{-3k}$$

for all $n, m \geq n_k$(such n_k exists because $M^{(n)}$ is Cauchy).Then for $m \geq n_k$,

$$1 \wedge E[(M_k^{(m)} - M_k^{(n_k)})^2]^{1/2} \leq 2^k ||M^{(m)} - M^{(n)}||_{\mathcal{M}_2} \leq 2^{-2k}$$

So $1 \wedge E[(M_k^{(m)} - M_k^{(n_k)})^2]^{1/2} = E[(M_k^{(m)} - M_k^{(n_k)})^2]^{1/2} \leq 2^{-2k}$.Now choose $\epsilon = 2^{-k}$ in (1.21) to obtain (1.22).By the Borel-Cantelli lemma, there exists Ω_1 with $P(\Omega_1) = 1$ such that for every $\omega \in \Omega_1$,

$$\sup_{0 \le t \le k} |M_t^{(n_{k+1})}(\omega) - M_t^{(n_k)}(\omega)| < 2^{-k}$$

for all but finitely many $k's$.It follows that the sequence of cadlag functions $t \to M_t^{(n_k)}$ is Cauchy under the uniform metric over any bounded time interval $[0, T]$.As cadlag functions form a complete metric space under this metric, we conclude that for each $T < \infty$ there exists a cadlag process $\{N_t^{(T)}(\omega) : 0 \le t \le T\}$ such that $M_t^{(n_k)} \to N_t^{(T)}$ uniformly on $[0, T]$,when $k \to \infty$.$N_t^{(S)}$ and $N_t^{(T)}$ mus agree if $t \in [0, S \wedge T]$, because both are the limit of the same sequence.Thus we can define one cadlag function $t \to M_t^{(\omega)}$ on $\mathbb{R}_+$,and for $\omega \in \Omega_1$, $M_t^{(n_k)}(\omega) \to M_t(\omega)$ uniformly on each bounded interval $[0, T]$.If $\omega \notin \Omega_1$,we define $M_t(\omega) = 0$. As $\mathcal{F}_t$ is complete for each $t, \Omega_1 \in \mathcal{F}_t$. In particular, $M_t(\omega)$ is $\mathcal{F}_t-$measurable. We know also that $M_t^{(n_k)} \to Y_t$ in $L^2(P)$,so $M_t = Y_t$ almost surely, so M is a martingale,and $M_t^{(n_k)} \to M_t$ in $L^2(P)$ when $k \to \infty$.To prove that $||M^{(n_k)} - M||_{\mathcal{M}_2} \to 0$,we write

$$\sum_{k=1}^{\infty} 2^{-k}(1 \wedge ||M_k^{(n_k)} - M_k||_{L^2(P)}) \le \sum_{k=1}^{l} 2^{-k}(1 \wedge ||M_k^{(n_k)} - M_k||_{L^2(P)}) + \sum_{k=l+1}^{\infty} 2^{-k}$$

For fixed l and $\epsilon > 0$,the first sum is bounded by ϵ if k is large enough, and the second sum is $\le c2^{-l-1}$.Taking $k \to \infty$, the left side sum is bounded by $c2^{-l-1}$.But $l \ge 1$ is arbitrary, so we are done.

If all $M^{(n)}$ are continuous, the uniform limit produces a continuous function M, so $\mathcal{M}_2^c$ is complete under the same metric. $\square$

By adapting the argument above from (1.21) onwards, we get this useful consequence of convergence in $\mathcal{M}_2$.

Lemma 1.49. *Suppose $||M^{(n)} - M||_{\mathcal{M}_2} \to 0$ as $n \to \infty$.Then for each $T < \infty$ and $\epsilon > 0$,*

$$\lim_{n\to\infty} P(\sup_{0 \le t \le T} |M_t^{(n)} - M_t| \ge \epsilon) = 0 \tag{1.23}$$

Furthermore ,there exists a subsequence $\{M^{(n_k)}\}$ and an event Ω_0 such that $P(\Omega_0) = 1$ and for each $\omega \in \Omega_0$ and $T < \infty$,

$$\lim_{n\to\infty} \sup_{0 \le t \le T} |M_t^{(n_k)}(\omega) - M_t(\omega)| = 0$$

When (1.23) holds for all $T < \infty$ and each $\epsilon > 0$, it is called uniform convergence in probability on compact sets.

References

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