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Random Polymers in Random Environment

Chapter 1

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Abstract

These notes are based on the book "Directed Polymers in Random Environment"[1],and are intended for self-study and understand better these topics.

1 Introduction

In this section is introduced some notation and the basics definitions.Also, it's discussed some natural questions for the model, and finally we connect with model with a Random Walk in a Random Potential.

1.1 Notation

1. $S = ((S_n)_{n\in\mathbb{N}}, P_x)$ is a simple random walk in $\mathbb{Z}^d$ with initial point x.The increments $S_1 - S_0, S_2 - S_1, \cdots$ are independent, and $P_x(S_0 = x) = 1$.

2. In this notes, we denote by $P_x(X)$ the expectation of the random variable X respect to the measure P_x. Don't confuse with probability (but it may happen).Also,we write $P_0 = P$.

3. $\omega = \{\omega(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$ is a sequence of random variables real valued, i.i.d and non constant in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}[exp(\beta\omega(n, x)] < \infty$ for all $\beta \in \mathbb{R}$. These random variables are the *environments*.

4. We also denote by $\mathbb{P}(Y)$ the expectation of the random variable Y respect the measure $\mathbb{P}$,and $\mathbb{P}[Y;A]$ is the expectation of Y in the event A,and Y defined in $(\Omega,\mathcal{F},\mathbb{P})$.We consider here $\Omega=\mathbb{R}^{\mathbb{N}*\times\mathbb{Z}^d}$

1.2 Polymer measure

Here we introduce the main definitions relative to the polymer itself.

Definition 1.1. *Given $n>0$:*

1. *Define the polymer measure $P_n^{\beta,\omega}$ in $(\Omega,\mathcal{F})$ by*

$$P_n^{\beta,\omega}(dx):=\frac{1}{Z_n(\omega,\beta)}exp[\beta H_n(x)]P(dx),\quad ,\beta>0$$

2. *The energy of $x=(x_1,\cdots,x_n)$ in the environment ω is $H_n^{\omega}(x):=\sum_{j=1}^{n}\omega(j,x_j)$*

3. *The partition function is $Z_n(\omega,\beta):=P[exp(\beta\sum_{j=1}^{n}\omega(j,S_j))]$*

Remark 1.2.

1. *The partition function normalizes $P_n^{\beta,\omega}$ in such a way is a probability measure.*

2. *Note that we can compute explicitly the partition function.In fact,is equal to $\sum_x(2d)^{-n}exp[\beta H_n(x)]$, where x is among the $(2d)^n$ possible paths of the simple random walk up to time n.*

Interpretation: The polymer is attracted to sites where the energy is positive,and repelled to the sites where is negative.

Question of the model: Study the asymptotic of the mode, i.e, when $n\to\infty$.

1.3 Exponents and localization

Informally, we define the characteristics exponents $\chi^{||}\in[0,\frac{1}{2}],\chi^{\perp}\in[\frac{1}{2},1]$ given by

$$P_n^{\beta,\omega}(|S_n|^2)\approx n^{2\chi^{\perp}} \tag{1.3.1}$$

$$Var(log(Z_n(\omega,\beta)))\approx n^{2\chi^{||}} \tag{1.3.2}$$

It's conjectured the relation $\chi^{||}=2\chi^{\perp}-1$ (KPZ relation). There are two main possible scenarios:

1. If the dimension is big and β is small, is expected that the shape of the polymer shouldn't be affected by the impurities of the environment.Intuitively this happens because the polymer has plenty of space to move, and the impurities are insignificant. Therefore, the walk should have a diffusive behavior (i.e, $\chi^{\perp} = \frac{1}{2}$).

2. If the dimension is low of the environment is strong, the polymer has stimuli to travel and search for better places. This correspond to a localized phase $\chi^{\perp} > \frac{1}{2}$. In that case, is expected that the exponents are *universal*,that is,not depend of the details of the model,only of the dimension. In particular, in the case $d = 1$ is expected $\chi^{||} = \frac{1}{3}, \chi^{\perp} = \frac{2}{3}$

1.4 Relation between the Polymer Measure and a Random Walk in a Random Potential

In this subsection we assume $\omega(n,x) \leq 0 \, \mathbb{P} \, a.s.$ We will discuss briefly a model of random walk in random potential.

Let's consider a particle in the origin at time $t = 0$, that moves on integer times like a simple random walk, but immediately after jump, it survives or dies. If the particle is alive at time $t-1$,his survival rate at time t is $exp[\beta\omega(t,x)] \in (0,1]$.

More precisely, let $P, \mathbb{P}$ as before, and Unif the uniform law in $[0,1]$. Let $\omega = \{\omega(t,x) : (t,x) \in \mathbb{N} \times \mathbb{Z}^d\}$ fixed. In the space $\Omega_{traj} \times [0,1]$ we define the product measure $Pr := P \times Unif$ and $U : (x,u) \to u$ the projection in the second coordinate.

Let $\tau := \inf\{n \geq 1 : U > exp[\beta H_n^{\omega}(S)]\}$ the lifespan of S. We add to $\mathbb{Z}^d$ a cemetery state $\dagger$. The process is defined by

$$X_n = \begin{cases} S_n, & n < \tau \\ \dagger, & n \geq \tau \end{cases}$$

Proposition 1.3. *The process $X = (X_n)_n$ is a Markov chain (non time homogeneous) in $\mathbb{Z}^d \cup \dagger$. $\dagger$ is an absorbent state, and his transition probabilities are*

$$Pr(X_{n+1} = y | X_n = x) = \frac{1}{2d} exp[\beta\omega(n+1,y)], y \in \mathbb{Z}^d, |y-x|_1 = 1 \quad (1.4.1)$$

$$Pr(X_{n+1} = \dagger | X_n = x) = \frac{1}{2d} \sum_{z \in \mathbb{Z}^d, |z-x|_1 = 1} (1 - exp[\beta\omega(n+1),z]) \quad (1.4.2)$$

Finally, we have

$$Pr(\tau > n)) = Z_n(\omega, \beta) \tag{1.4.3}$$

$$P_n^{\beta,\omega}(x) = Pr(X_{[0,n]} = x|\tau > n) \tag{1.4.4}$$

Proof. Given $n \in \mathbb{N}$, if x is a path of size n,

$$\begin{aligned} Pr(\tau > n|S_{[0,n]} = x) &= Unif(U \leq exp[\beta H_n^\omega(x)]) \\ &= exp[\beta H_n^\omega(x)] \end{aligned} \tag{1.4.5}$$

In the other hand,we know that

$$Pr(\tau > n, S_{[0,n]} = x) = Pr(\tau > n|S_{[0,n]} = x)Pr(S_{[0,n]} = x) = (2d)^{-n}exp[\beta H_n^\omega(x)]$$

(using 1.4.5).Using the last equation,we obtain

$$Pr(\tau > n) = \sum_x Pr(\tau > n, S_{[0,n]} = x) = \sum_x (2d)^{-n}exp[\beta H_n^\omega(x)] = Z_n(\omega, \beta)$$

Therefore,

$$\begin{aligned} Pr(X_{[0,n]} = x|\tau > n) = Pr(S_{[0,n]} = x|\tau > n) &= \frac{Pr(\tau > n|S_{[0,n]} = x)Pr(S_{[0,n]} = x)}{Pr(\tau > n)} \\ &= \frac{(2d)^{-n}exp[\beta H_n^\omega(x)]}{Z_n(\omega, \beta)} = P_n^{\beta,\omega}(x) \end{aligned} \tag{1.4.6}$$

Finally,we verify the Markov property.We note that if $x_n = x, x_{n+1} = y$,

$$\begin{aligned} Pr(\tau > n+1, S_{[0,n+1]} = x_{[0,n+1]}|\tau > n, S_{[0,n]} = x_{[0,n]}) &= \frac{Pr(\tau > n+1, S_{[0,n+1]} = x_{[0,n+1]})}{Pr(\tau > n+1, S_{[0,n]} = x_{[0,n]})} \\ &= \frac{1}{2d}exp[\beta\omega(n+1, y)](\text{using } 1.4.5) \end{aligned}$$

□

References

[1] Francis Comets. Directed polymers in random environments. *Lecture Notes in Mathematics*, 2175, 2017.