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Directed Polymers in Random Environment

Chapter 3

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Abstract

These notes are based on the book "Directed Polymers in Random Environment"[2],and are intended for self-study and understand better these topics.

1 Martingale Approach and the L^2 region.

In this section we will study the partition function Z_n.Recall that $\lambda(\beta) = \log \mathbb{P}(e^{\beta\omega})$,so $\mathbb{P}(e^{\beta\omega}) = e^{\lambda(\beta)}$.Therefore,
$\mathbb{P}(Z_n) = \mathbb{P}(P[e^{\beta H_n}]) = P(\mathbb{P}(exp\{\sum_{i=1}^n \omega(i,S_i)\})) = P(e^{n\lambda(\beta)}) = e^{n\lambda(\beta)}$,where we used the fact that the environment is i.i.d under $\mathbb{P}$. Because this calculation, we will consider the normalized partition function

$$W_n := \frac{Z_n}{e^{n\lambda(\beta)}}, n \geq 1 \tag{1.0.1}$$

Now,fix a path x,and consider $\overline{\xi_n} = \overline{\xi_n}(x) := e^{\beta H_n(x) - n\lambda(\beta)}$.Then, for that path, $H_n(x)$ is a random walk.In particular, ξ_n is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ given by $\mathcal{F}_n := \sigma(\omega(j,x) : 1 \leq j \leq n, x \in \mathbb{Z}^d)$.Then, if $x_1, \cdots x_k$ are paths and $a_1, \cdots a_k$ are scalars, $\sum_{i=1}^k a_k \overline{\xi_n}(x_i)$ is also a martingale. Because $P(\overline{\xi_n}) = \sum_{x\in\mathbb{Z}^d} P(x)\overline{\xi_n}(x)$ then $W_n = P(\overline{\xi_n})$ also is a positive martingale.

Recall now the martingale convergence theorem,that basically says that if a martingale has L^1 norm uniformly bounded,then the limit exists $\mathbb{P}$ almost surely.In our case, the limit $\lim_{n\to\infty} W_n$ will be called W_∞. A natural question is ask if $W_\infty = 0$ or not. In fact,note first that by

$$Z_{n+m} = Z_n \times P_n^{\beta,\omega}(Z_m \circ \theta_{n,S_n}) \tag{1.0.2}$$

we can write

$$W_{n+m} = P(\overline{\xi_n} \times W_m \circ \theta_{n,S_n})$$

Taking $m \to \infty$ we deduce

$$\begin{aligned} W_\infty &= \lim_{m\to\infty} W_m = P(\overline{\xi_n} \times \lim_{m\to\infty} W_m \circ \theta_{n,S_n}) = P(\overline{\xi_n} \times W_\infty \circ \theta_{n,S_n}) \\ &= \sum_x P(S_n = x)\overline{\xi_n}(x) \times W_\infty \circ \theta_{n,x} = \sum_x P(S_n = x) e^{\beta H_n(x) - n\lambda(\beta)} \times W_\infty \circ \theta_{n,x} \\ &= Z_n e^{-n\lambda(\beta)} \sum_x \frac{1}{Z_n}(S_n = x) e^{\beta H_n(x)} \times W_\infty \circ \theta_{n,x} = W_n \times \sum_x P_n^{\beta,\omega}(S_n = x) \times W_\infty \circ \theta_{n,x} \end{aligned}$$

Because $W_n > 0$ for all n,we have

$$\{W_\infty = 0\} = \bigcap_{x\in\mathbb{Z}^d : P(S_n = x)} \{W_\infty \circ \theta_{n,x} = 0\} \in \mathcal{F}_n$$

Thus,the event $\{W_\infty = 0\}$ belongs to the tail σ algebra,and we can use the Kolmogorov 0-1 law to deduce the following theorem:

Theorem 1.1. *The limit*

$$W_\infty = \lim_{n\to\infty} W_n$$

exists $\mathbb{P}-$ *a.s.Moreover,the limit is strictly positive,or is zero* $\mathbb{P}$ *almost surely.*

It's natural introduce some terminology that distinguish this dichotomy:

Definition 1.2. *We say that the polymer is in the weak disorder when* $W_\infty > 0\ -a.s$ *and in the strong disorder if* $W_\infty = 0\ -a.s$

This definition is an analogous of the high/low temperature defined in the last section. In fact, there will be a similar statement related to a critical value.

Remark 1.3. *We can prove that* $W_\infty > 0 \Rightarrow p(\beta) = \lambda(\beta)$*. Indeed, we have a.s*

$$p(\beta) = \lim_{n\to\infty} \frac{1}{n} \log(Z_n(\beta,\omega)) = \lambda(\beta) + \lim_{n\to\infty} \frac{1}{n} \log(W_n) = \lambda(\beta)$$

because $\log(W_n) \to \log(W_\infty) < \infty$

Now we state the analogous of Theorem 1.19,chapter 2:

Proposition 1.4. *There exists $\overline{\beta}_c = \overline{\beta}_c(\mathbb{P}, d) \in [0, \infty]$ such that*

$$\begin{cases} W_\infty > 0 & \beta \in (0, \overline{\beta}_c) \\ W_\infty = 0 & \beta > \overline{\beta}_c \end{cases}$$

Proof. Let $\delta \in (0,1)$ arbitrary.The idea is prove the result for $(W_n^\delta)_n$ for every δ.Note that since $\mathbb{P}(W_n) = 1$,then $(W_n^\delta)_n$ is uniformly integrable (e.g. Hölder). We have then the a.s convergence of $W_n^\delta \to W_\infty^\delta$ and $\mathbb{P}(W_n^\delta) \to \mathbb{P}(W_\infty^\delta)$, which is 0 in the strong disorder case,and strictly positive in the weak one. Is enough to prove that the map $\beta \to \mathbb{P}(W_n^\delta)$ in non-increasing on $\mathbb{R}_+$.This implies that the map $\beta \to \mathbb{P}(W_\infty^\delta)$ also is non-increasing on $\mathbb{R}_+$.To finish,define $\overline{\beta}_c := \inf\{\beta \geq 0 : \mathbb{P}(W_n^\delta) = 0\} \in [0, \infty]$.We compute

$$\begin{aligned} \frac{d}{d\beta}\mathbb{P}(W_n^\delta) &= \mathbb{P}\left(\frac{d}{d\beta}W_n^\delta\right) = \delta\mathbb{P}\left(W_n^{\delta-1}\frac{d}{d\beta}Z_n e^{-n\lambda}\right) = \delta\mathbb{P}\left(W_n^{\delta-1}\left[e^{-n\lambda}\frac{d}{d\beta}Z_n - n\lambda' e^{-n\lambda}Z_n\right]\right) \\ &= \delta\mathbb{P}\left(W_n^{\delta-1}\left[e^{-n\lambda}P(H_n\overline{\xi_n}) - n\lambda' P(\overline{\xi}_n)\right]\right) = \delta\mathbb{P}(W_n^{\delta-1}P[\{H_n - n\lambda'\}\overline{\xi}_n]) \\ &= \delta P(\mathbb{P}[\overline{\xi}_n W_n^{\delta-1}\{H_n - n\lambda'\}]) \end{aligned} \tag{1.0.3}$$

Now, we apply the FGK-Harris inequality as follows: we consider the measure $\overline{\xi}_n d\mathbb{P}$ for fixed path x, the decreasing function (on ω) $W_n^{\delta-1}$ and the non-decreasing one $H_n - n\lambda'$. Then we obtain

$$\frac{d}{d\beta}\mathbb{P}(W_n^\delta) = \delta P(\mathbb{P}[\overline{\xi}_n W_n^{\delta-1}\{H_n - n\lambda'\}]) \leq \delta P(\mathbb{P}[\overline{\xi}_n W_n^{\delta-1}]\mathbb{P}[\overline{\xi}_n\{H_n - n\lambda'\}]) = 0$$

The last equality is by

$$\mathbb{P}[\omega e^{\beta\omega}] = \lambda'(\beta)e^{\lambda(\beta)} \tag{1.0.4}$$

□

Because the last theorem and Theorem 1.19,chapter 2,it's expected that $\beta_c = \overline{\beta}_c$,but at the moment this question remains open. However, if $d \in \{1, 2\}$,the equality is satisfied.Note that always $\overline{\beta}_c \leq \beta_c$.In effect,if $\beta = \beta_c + \epsilon$ with $\epsilon > 0$,then $\beta \geq \overline{\beta}_c$,otherwise $W_\infty(\beta) > 0 \Rightarrow p(\beta) = \lambda(\beta)$ (by Remark 1.3) $\Rightarrow \beta \leq \beta_c$,a contradiction.

Question 1. $\beta_c = \overline{\beta}_c$?

Question 2. *What happens at $\overline{\beta}_c$?*

It's expected that $W_\infty(\overline{\beta}_c) = 0$

1.1 The second moment method and the L^2 region

We recall some knows facts about the return probability of the simple random walk,denoted by π_d.More precisely,

$$\pi_d := P(S_n = 0 \text{ for some } n \geq 1) \tag{1.1.1}$$

Then $\pi_d = 1$ if $d \in \{1,2\}$ and < 1 for $d \geq 3$.In fact, $\pi_d < 1$ if $d \geq 3$.For more information,see [5] Now we define the important L2 condition:

Definition 1.5 (L2 condition)**.** *Given $d \geq 3$,we say that β satisfies the L2 condition if*

$$\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta) < \log(1/\pi_d) \tag{1.1.2}$$

Note that $\lambda_2(0) = 0$ and $\lambda_2'(\beta) = 2(\lambda'(2\beta) - \lambda'(\beta)) > 0$,since λ' is increasing.Therefore, $\lambda_2(\beta)$ is increasing in $\mathbb{R}_+$, and if $d \geq 3$, the L2 condition es equivalent to $\beta < \beta_{L^2}$,with

$$\beta_{L^2} := \inf\{\beta \geq 0 : \lambda_2 \leq \log(1/\pi_d)\} \tag{1.1.3}$$

Theorem 1.6. *Suppose that $d \geq 3$ and the L2 condition is satisfied.Then,$W_\infty > 0$ a.s. In particular, $p = \lambda$ if $\beta \leq \beta_{L^2}$*

Proof. Let's consider the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$ and the probability measure $P^{\otimes 2}(dx, d\tilde{x})$.This is the distribution of two independent walks $S, \tilde{S}$.Note that

$$W_n^2 = P\frac{Z_n^2}{e^{2n\lambda(\beta)}} = P(e^{\beta H_n(S)})P(e^{\beta H_n(\tilde{S})})e^{-2n\lambda(\beta)} = P^{\otimes 2}(e^{\beta[H_n(S)-H_n(\tilde{S})]-2n\lambda(\beta)})$$

Now we compute $\mathbb{P}(W_n^2)$:

$$\begin{aligned}\mathbb{P}(W_n^2) \overset{Fubini}{=} P^{\otimes 2}\mathbb{P}\left[\prod_{t=1}^{n} e^{\beta[\omega(t,S_t)+\omega(t,\tilde{S}_t)]-2\lambda(\beta)}\right] &= P^{\otimes 2}\left[\prod_{t=1}^{n}\mathbb{P}\left(e^{\beta[\omega(t,S_t)+\omega(t,\tilde{S}_t)]-2\lambda(\beta)}(1_{S_t=\tilde{S}_t} + 1_{S_t\neq\tilde{S}_t})\right)\right] \\ &= P^{\otimes 2}\left[\prod_{t=1}^{n} e^{\lambda(2\beta)]-2\lambda(\beta)}(1_{S_t=\tilde{S}_t}) + 1_{S_t\neq\tilde{S}_t}\right] \\ &= P^{\otimes 2}\left[\prod_{t=1}^{n} e^{\lambda_2(\beta)1_{S_t=\tilde{S}_t}}\right] = P^{\otimes 2}\left[e^{\lambda_2(\beta)N_n}\right]\end{aligned}$$

where $N_n = N_n(S, \tilde{S}) := \sum_{t=1}^{n} 1_{S_t=\tilde{S}_t}$ is the number of intersections of $S, \tilde{S}$ up to time n.Taking $n \to \infty$, we have $N_n \to \infty$,and by the Monotone Convergence Theorem,

$$\mathbb{P}(W_n^2) \to P^{\otimes 2}[e^{\lambda_2(\beta)N_\infty}]$$

Also note that if $X_n = S_n - \tilde{S}_n$,then N_∞ is the number of visits to zero of the symmetric random walk X_n (this walk is not a nearest-neighbor one).Then N_∞ is geometrically distributed with success probability π_d. Then,we have

$$P^{\otimes 2}[e^{\lambda_2(\beta)N_\infty}] = \sum_{k=0}^{\infty}(1-\pi_d)\pi_d^k e^{k\lambda_2} = \begin{cases} \frac{1-\pi_d}{1-\pi_d e^{\lambda_s}} & \pi_d e^{\lambda_2} < 1 \\ +\infty & \pi_d e^{\lambda_2} \geq 1 \end{cases}$$

Therefore, $\sup_n \mathbb{P}(W_n^2) < \infty \Leftrightarrow \lambda_2 + \log(\pi_d) < 0 \Leftrightarrow \lambda_2 < \log(1/\pi_d)$.In that case, the martingale W_n is bounded in L^2.This implies not only the convergence to W_∞,but also convergence in L^1.Thus, $\mathbb{P}(W_\infty) = 1$ and necessarily W_∞ must be strictly positive.

□

Corollary 1.7. *Let $s = ess\sup_{\mathbb{P}} \omega(t,x)$. The function $\lambda_2(\beta)$ is increasing on $\mathbb{R}_+$, with $\lambda_2(+\infty) = -\log(\mathbb{P}(\omega(t,x) = s))$. Thus, the L2 condition holds for all $\beta \geq 0$ as soon as $\mathbb{P}(\omega(t,x) = s) > \pi_d$.*

Proof. Let q be the law of $\omega(t,x)$. By the Theorem 1.6, it's enough to prove that

$$\lim_{\beta\to\infty} \lambda_2(\beta) = \begin{cases} \infty & s = \infty \\ -\log(q(\{s\})) & s < \infty \end{cases}$$

The claim is clear for $s = \infty$,so we consider the case $s < \infty$. To prove this case,observe that when $q(\{s\}) > 0$, it' enough to prove

$$\lambda(\beta) = \beta s + \log(q(\{s\})) + \epsilon(\beta) \tag{1.1.4}$$

for some $\epsilon(\beta) \overset{\beta\to\infty}{\to} 0$. This implies that $\lambda_2(\beta) \overset{\beta\to\infty}{\to} -\log(q(\{s\}))$.To prove (1.1.4), we write for $h > 0$

$$\begin{aligned} \mathbb{P}(e^{\beta\omega} : \omega = s) := \mathbb{P}(e^{\beta\omega} 1_{\omega=s}) \leq \mathbb{P}(e^{\beta\omega}) &= \mathbb{P}(e^{\beta\omega} : \omega \in [s-h, s]) + \mathbb{P}(e^{\beta\omega} : \omega \leq s-h) \\ &= e^{\beta s}(q([s-h,s]) + e^{-\beta h} q((-\infty, s-h]) \\ &\leq e^{\beta s} q(\{s\})(q([s-h,s]) + e^{-\beta h})/q(\{s\}) \end{aligned}$$

Taking logarithms, we deduce

$$\beta s + \log(q(\{s\})) \leq \lambda(\beta) \leq \beta s + \log(q(\{s\})) + \log\left(\frac{q([s-h,s]) + e^{-\beta h}}{q(\{s\})}\right)$$

Finally observe that $\inf\left\{\log\left(\frac{q([s-h,s])+e^{-\beta h}}{q(\{s\})}\right) : h > 0\right\}$ decreases to 1 when $\beta \to +\infty$ $\square$

Example 1.8. *If $\omega \sim N(0,1)$,then $\lambda(\beta) = \beta^2/2$.Therefore, $\lambda_2(\beta) = \beta^2$.Hence, the L2 condition holds if $\beta < \sqrt{\log(1/\pi_d)}$*

Definition 1.9 (L^2 region)**.** *We call L^2 region to the set of β's such that the L2 condition (1.1.2) holds.*

1.2 Diffusive Behavior in L^2 Region

In this section we assume $d \geq 3, \beta$ in the L^2 region.The next theorem states that in this region, the environment does not change the transversal fluctuations of the polymer for large d and small β.

Theorem 1.10. *Under the assumptions of Theorem 1.6,we have*

$$\lim_{n\to\infty} P_n^{\beta,\omega}(|S_n|^2)/n = 1 \ \mathbb{P}\ a.s \tag{1.2.1}$$

and for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity

$$\lim_{n\to\infty} P_n^{\beta,\omega}[f(S_n/\sqrt{n})] = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x/\sqrt{d})exp(-|x^2|/2)dx \ \mathbb{P}\ a.s \tag{1.2.2}$$

In particular,if $Z \sim N_d(0, d^{-1}I_d)$, we have

$$P_n^{\beta,\omega}(S_n/\sqrt{n} \in \cdot) \to P(Z \in \cdot) \ \mathbb{P}\ a.s$$

Before giving a proof of this theorem, we will define a family of martingales $(M_n)_{n\geq 1}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$M_n = P(\phi(n, S_n)\bar{\xi}_n) \tag{1.2.3}$$

Where we recall that $\bar{\xi}_n = e^{\beta H_n(x) - n\lambda(\beta)}$and $\phi : \mathbb{N}\times\mathbb{Z}^d \to \mathbb{R}$ is a function for which we assume the following:

(P1) There are constants $C_i, p \in \mathbb{N}, i = 0,1,2$ such that

$$|\phi(n,x)| \leq C_0 + C_1|x|^p + C_2 n^{p/2} \text{ for all } (n,x) \in \mathbb{N}\times\mathbb{Z}^d \tag{1.2.4}$$

(P2) $\Phi_n := \phi(n, S_n)$ is a martingale on $(\Omega_{traj}, \mathcal{G}, P)$ with respect to the filtration $\mathcal{G}_n = \sigma(S_j : j \leq n)$

Note that M_n is a $(\mathcal{F}_n)$ martingale on $(\Omega, \mathcal{F}, \mathbb{P})$.In effect,if $\mathbb{P}^{\mathcal{F}_n}, P^{\mathcal{G}_n}$ denote the conditional expectations, we have

$$\begin{aligned}\mathbb{P}^{\mathcal{F}_n} M_{n+1} &= P[\phi(n+1, S_{n+1})\overline{\xi}_n \mathbb{P}^{\mathcal{F}_n} e^{\beta\omega(n+1,S_{n+1})-\lambda}] = P[\phi(n+1, S_{n+1})\overline{\xi}_n] \\ &= P[\overline{\xi}_n P^{\mathcal{G}_n}\phi(n+1, S_{n+1})] = P[\overline{\xi}_n \phi(n, S_n)] = M_n \end{aligned} \tag{1.2.5}$$

We will need the following proposition before proving Theorem 1.10.

Proposition 1.11. *Consider the martingale $(M_n)_{n\geq 1}$ defined in 1.2.3.Suppose that $d \geq 3$ the L2 condition (1.1.2),(P1) and (P2) are satisfied.Then,there exists $\kappa \in [0, p/2)$ such that*

$$\max_{0\leq j\leq n} |M_j| = O(n^\kappa) \quad \text{as } n \to \infty \tag{1.2.6}$$

If in addition, $p < \frac{1}{2}d - 1$,then

$$\lim_{n\to\infty} M_n \text{ exists } \mathbb{P} \text{ a.s and in } L^2(\mathbb{P}) \tag{1.2.7}$$

Sketch of the proof: As is shown in [4],we have

$$\mathbb{P}(M_n^2) = O(b_n), \quad b_n = \sum_{j=1}^{n} j^{p-d/2} \tag{1.2.8}$$

Let $M_n^* := \max_{0\leq j\leq n} |M_j|$. Note that $\sqrt{b_n} = O(n^{p/2-d/4+1/2})$.If $d \geq 3$, then $\frac{1}{2} - \frac{d}{4} \leq -\frac{1}{4}$.Thus, $b_n = O(n^{p/2-1/4})$.Therefore, to prove the first part of the proposition, is enough to show that if $\delta > 0$ is small enough, then

$$M_n^* = O(n^\delta \sqrt{b_n}) \tag{1.2.9}$$

when $n \to \infty$, $\mathbb{P}$ a.s. Because the monotonicity of M_n^* and the polynomial growth of $n^\delta\sqrt{b_n}$, it's enough to prove (1.2.9) along a subsequence $\{n^k : n \in \mathbb{N}\}$ with $k \geq 2$. If $k > 1/\delta$,using the Doob's inequality for martingales ,we have

$$\mathbb{P}(M_{n^k}^* > n^{k\delta}\sqrt{b_{n^k}}) \leq \mathbb{P}(M_{n^k}^* > n\sqrt{b_{n^k}}) \leq \mathbb{P}[(M_{n^k}^*)^2]/(n^2 b_{n^k}) \leq 4\mathbb{P}[(M_{n^k})^2]/(n^2 b_{n^k}) \leq Cn^{-2}$$

Where the last inequality comes from (1.2.8).Using Borel-Cantelli, we have that

$$\mathbb{P}(M_{n^k}^* \leq n^{k\delta}\sqrt{b_{n^k}} \text{ for large enough n's}) = 1$$

That concludes the first part of the proposition.The second part comes from the Martingale Convergence Theorem.

□

Proof of Theorem 1.10. We the first part.Take $\phi(n,x) := |x|^2 - n$.In that case,$p = 2$.By Proposition 1.11, there exists $\kappa \in [0,1)$ such that

$$\begin{aligned} P_n^{\beta,\omega}[|S_n|^2] - n = \frac{1}{Z_n}\sum_x |x|^2 e^{\beta H_n(x)} P(S_n = x) - n = \frac{1}{W_n}\sum_x |x|^2 \overline{\xi}_n(x) P(S_n = x) - n \\ = P[|S_n|^2\overline{\xi}_n]/W_n - n = P[\phi(S_n,n)\overline{\xi}_n]/W_n = O(n^\kappa) \end{aligned}$$

Dividing by n and taking $n \to \infty$,we conclude the first part.

Now we sketch the second part.We will use the standard multi-index notation,that is , if $a = (a_j)_{j=1}^d$,then $|a|_1 = a_1 + \cdots a_d, x^a = x_1^{a_1} \cdots x_d^{a_d}, (\frac{\partial}{\partial x})^a = (\frac{\partial}{\partial x_1})^{a_1} \cdots (\frac{\partial}{\partial x_d})^{a_d}$ for $x \in \mathbb{R}^d$.

It's enough to prove the second part if $f(x) = x^a$. We prove this by induction in $|a|_1$.Let define

$$\begin{aligned} \phi(n,x) &:= \left(\frac{\partial}{\partial\theta}\right)^a exp(\theta\cdot x - n\rho(\theta))|_{\theta=0} \\ \psi(n,x) &:= \left(\frac{\partial}{\partial\theta}\right)^a exp(\theta\cdot x - n\frac{|\theta|^2}{2d})|_{\theta=0} \end{aligned}$$

where $\rho(\theta) := \log(\frac{1}{d}\sum_{j=1}^d cosh(\theta_j))$.

We have the following related to these functions:

1. ϕ satisfies $(P1)-(P2)$ with $p = |a|_1$

2. $$(2\pi)^{-d/2}\int_{\mathbb{R}^d} \psi(1, x/\sqrt{d}) e^{-|x^2|/2} dx = 0 \tag{1.2.10}$$

Following [1] ,he proves that

$$\begin{aligned} \phi(n,x) &= x^a + \phi_0(n,x) \\ \psi(n,x) &= x^a + \psi_0(n,x) \end{aligned} \tag{1.2.11}$$

where

$$\begin{aligned} \phi_0(n,x) &= \sum_{\substack{|b|_1+2j\le|a|_1 \\ j\ge1}} A_a(b,j)x^b n^j \\ \psi_0(n,x) &= \sum_{\substack{|b|_1+2j=|a|_1 \\ j\ge1}} A_a(b,j)x^b n^j \end{aligned}$$

for some $A_a(b,j) \in \mathbb{R}$.In particular, ϕ_0, ψ_0 have the same coefficients for $x^b n^j$ with $|b|_1 + 2j = |a|_1$. Thus, we can write

$$(x/\sqrt{n})^a = \phi(n,x)n^{-|a_1|/2} - \psi_0(1, x/\sqrt{n}) + [\psi_0(n,x) - \psi_0(n,x)]n^{-|a_1|/2}$$

because $(x/\sqrt{n})^a = x^a n^{-|a|_1/2}$ and $\psi_0(1, x\sqrt{n}) = \psi_0(n,x)n^{-|a|_1/2}$. Therefore, we have

$$P_n^{\beta,\omega}[(S_n/\sqrt{n})^a] = \frac{1}{W_n} P[\phi(n,S_n)\bar{\xi}_n]n^{-|a_1|/2} - \frac{1}{W_n} P[\psi_0(1, S_n/\sqrt{n})\bar{\xi}_n] + \frac{1}{W_n}[\psi_0(n,x) - \psi_0(n,x)]n^{-|a_1|/2}$$

The second term converges to $(2\pi)^{-d/2}\int_{\mathbb{R}^d}(x/\sqrt{d})^a e^{-|x|^2/2}dx$ by (1.2.10),(1.2.11) and the induction hypothesis.The first term converges to zero by Theorem 1.6 and Proposition 1.11. A similar argument applies to the third term. $\square$

1.3 Local Limit Theorem in the L^2 region.

In this section we will consider the point to point partition function

$$W_n(y) := P(e^{\beta H_n(S) - n\lambda(\beta)}|S_n = y) \quad , y \in \mathbb{Z}^d \tag{1.3.1}$$

That is, we fix the last point to (n,y). To make a distinction, we usually call point to level partition function to W_n.

Let be $n \in \mathbb{N} \setminus \{0\}, x \in \mathbb{Z}^d$ such that $P(S_n = x) > 0$, so $|x|_1 \leq n$ and $|x|_1 \equiv n \pmod 2$. We write

$$x \leftrightarrow n \Leftrightarrow P(S_n = x) > 0 \tag{1.3.2}$$

In that case,

$$\mathbb{P}(W_n(x)) = P[\mathbb{P}(e^{\beta H_n(S) - n\lambda(\beta)}|S_n = y)] = P[\mathbb{P}(e^{\beta H_n(S) - n\lambda(\beta)})] = 1$$

and we have

$$W_n = \sum_y W_n(y)P(S_n = y)$$

Define the reflection operator $\theta^{\leftarrow}_{n,x}(\omega) = \omega'$, with $\omega'(u,y) := \omega(n-u, x+y)$ In this section we will write for $0 \leq k \leq n$

$$\bar{\xi}_{k,n}(x) = e^{\beta\sum_{i=k}^n \omega(i,x_i) - (n-k+1)\lambda(\beta)}$$

Note that $\overline{\xi}_{1,n} = \overline{\xi}_n$.We also write

$$\overline{\xi}^{\leftarrow}_{k,n} = e^{\beta \sum_{i=k-1}^{n-1} \omega(n-i,x_i)-(n-k+1)\lambda(\beta)}$$

Note that, because the $\omega's$ are i.i.d then $P_y(\overline{\xi}^{\leftarrow}_{1,n}) \overset{law}{=} W_n$.Also, the random variable $W_\infty \circ \theta^{\leftarrow}_{n,x}$ is well defined for all n, x, thus we have

$$P_y(\overline{\xi}^{\leftarrow}_{1,n}) - W_\infty \circ \theta^{\leftarrow}_{n,x} \to 0$$

in probability, and in L^2 in the L^2 region.

Now we state the Local Limit Theorem:

Theorem 1.12 (Local Limit Theorem)**.** *Assume* (1.1.2)*.Then, for all $A < \infty$ and all sequence of integers $l_n \to \infty$ with $l_n = o(n^\alpha)$ for some $\alpha < 1/2$, we have*

$$P[\overline{\xi}_{1,n}|S_n = y] = P(\overline{\xi}_{1,l_n})P_y(\overline{\xi}^{\leftarrow}_{1,l_n}) + \delta^y_n \tag{1.3.3}$$

with

$$\lim_{n\to\infty} sup\{\mathbb{P}(|\delta^y_n|^2) : |y| \le An^{1/2}, n \leftrightarrow y\} = 0$$

Moreover,

$$W_n(y) = W_\infty \times (W_\infty \circ \theta^{\leftarrow}_{n,y}) + \epsilon^y_n \tag{1.3.4}$$

Where the error term $\epsilon^y_n \to 0$ uniformly on $\{y : |y| \le An^{1/2}, n \leftrightarrow y\}$

Remark 1.13.

1. *Note that* (1.3.3) *can be written as $W_n(y) = W_{l_n} \times P_y(\overline{\xi}^{\leftarrow}_{1,l_n}) + \delta^y_n$*
2. *The result says that $P^{\beta,\omega}_n(S_n = y) \simeq W_\infty \circ \theta^{\leftarrow}_{n,y} \times P(S_n = y)$.The interpretation is that the polymer measure is close to a gaussian measure, up to a factor that depend of the endpoint.*
3. *Intuition: The polymer only is affected by the environment in the endpoints, and behaves like a gaussian in the middle*

Sketch of the proof of Theorem 1.12. The details are found in [6]

Step 1: With $l \leq n/2$, we approximate $W_n(y)$ with $P[\overline{\xi}_{1,l_n}\overline{\xi}_{n-l_n,n}]$ in L^2:

$$\lim_{n\to\infty} \sup_{|y|\leq An^{1/2}} ||W_n(y) - P[\overline{\xi}_{1,l_n}\overline{\xi}_{n-l_n,n}]||_2^2 = 0$$

Step 2: Using the standard local limit theorem for random walks, we deduce that

$$\lim_{n\to\infty} ||P[\overline{\xi}_{1,l_n}\overline{\xi}_{n-l_n,n}] - P_y(\overleftarrow{\overline{\xi}}_{1,l_n})||_1 = 0 \tag{1.3.5}$$

□

1.4 Rate of Martingale Convergence

First we define two modes of convergence in distribution.Let (Y_n) be a sequence of random variables defined in a common probability space $(\Omega, \mathcal{F}, P)$,such that Y_n converges in distribution to Y.

1. The convergence is stable if for all $B \in \mathcal{F}$ with $P(B) > 0$, the conditional law of Y_n given B converges to some probability distribution depending on B.
2. The convergence is called mixing if it's stable, and the limit of conditional laws does not depend on B (and therefore, is the law of Y).

The stable convergence says that for a random variable Z defined on $(\Omega, \mathcal{F}, P)$, (Y_n, Z) converges in law to some coupling of Y, Z in an extended space. The mixing convergence says that Y_n is asymptotically independent of all event $A \in \mathcal{F}$(Recall the mixing in Ergodic Theory).more precisely, if G_n is the left hand of (1.4.2),then $P(\{G_n \leq y\} \cap A) \to P(G \leq y)P(A)$ for $A \in \mathcal{F}$

We now state the theorem,found in [3]:

Theorem 1.14. *For $d \geq 3$,there exists some $\beta_0 > 0$ such that for $|\beta| < \beta_0$:*

$$n^{\frac{d-2}{4}}(W_\infty - W_n) \Rightarrow \sigma_1 W_\infty G \text{ in distribution} \tag{1.4.1}$$

and

$$n^{\frac{d-2}{4}}\frac{(W_\infty - W_n)}{W_n} \Rightarrow \sigma_1 G \text{ in distribution} \tag{1.4.2}$$

where

$$\sigma_1^2 := \frac{d^{d/2}(1-\pi_d)}{2^{d/2}(d-2)\pi^{d/2}\pi_d} \times Var(W_\infty) \tag{1.4.3}$$

where $G \sim N(0,1)$*, which is independent of* W_∞*.Moreover, the convergence in* (1.4.1) *is stable,and the convergence in* (1.4.2) *is mixing*

The theorem is based on a Central Limit Theorem for infinite martingale arrays:

Theorem 1.15. *[3] For* $n \geq 1$*,let* $\{(S_{n,i}, \mathcal{F}_{n,i}) : i \geq 0\}$ *be a martingale defined on a probability space* $(\Omega, \mathcal{F}, P)$*, with* $S_{n,0} = 0$ *and*

$$\sup_{n,i\geq 1} P(S_{n,i}^2) < \infty$$

Let $X_{n,i} = S_{n,i} - S_{n,i-1}, i \geq 1$ *be the martingale differences, and* $S_{n,\infty} = \lim_{i\to\infty} S_{n,i}$ *be the a.s limit of* $(S_{n,i} : i \geq 0)$*.Suppose that:*

1. *The conditional variance converges in probability: for a real random variable* $V \in [0,\infty)$*,*

$$V_{n,\infty}^2 := \sum_{i=1}^{\infty} P(X_{n,i}^2 | \mathcal{F}_{n,i-1}) \to V^2 \text{ in probability.}$$

2. *The conditional Lindeberg condition holds:*

$$\forall \epsilon > 0, \sum_{i=1}^{\infty} P(X_{n,i}^2 1_{|X_{n,i}|>\epsilon} \mathcal{F}_{n,i-1}) \to 0 \text{ in probability.}$$

3. $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$ *for all* $n, i \geq 1$*.*

Then,

$$S_{n,\infty} \to VG \text{ in distribution} \tag{1.4.4}$$

where $G \sim N(0,1)$ *and independent of* V*.If additionally* $V \neq 0$ *a.s ,then*

$$\frac{S_{n,\infty}}{V_{n,\infty}} \to G \text{ in distribution} \tag{1.4.5}$$

Moreover, the convergence in (1.4.4) *is stable, and the convergence in* (1.4.5) *is mixing.*

Sketch of the proof of Theorem 1.14. We write $n^{\frac{d-2}{4}}(W_\infty - W_n) = n^{\frac{d-2}{4}} \sum_{k=n}^{\infty} D_{k+1}$, where $D_{k+1} := W_{k+1} - W_k, k \geq n$ forms a sequence of martingale differences with respect t the sequence of filtrations $\mathcal{F}^{(n)} = (\mathcal{F}_i^{(n)})_{i\geq 0}$,where $\mathcal{F}_i^{(n)}) = \sigma(W_j : j \leq i+n)$.Using the last theorem with $X_{n,i} = X_{n+i}\mathcal{F}_{n,i} = \mathcal{F}_{n+i}$, is enough to show that

1.

$$s_n^2 := n^{\frac{d-1}{1}} \sum_{k \geq n} \mathbb{P}[D_{k+1}^2 | \mathcal{F}_k] \to \sigma_1^2 W_\infty^2 \tag{1.4.6}$$

in probability

2. The following Lindeberg condition holds:

$$\forall \epsilon > 0, n^{\frac{d-1}{2}} \sum_{k \geq n} \mathbb{P}[D_{k+1}^2 1_{n^{\frac{d-1}{4}} |D_{k+1}| > \epsilon} | \mathcal{F}_k] \to 0 \tag{1.4.7}$$

in probability

So here, $V_{n,\infty} = \sigma_1 W_n, V = \sigma_1 W_\infty \neq 0$ because we are in the L^2 region.Also, note that $W_n / W_\infty \to 1$ in probability, so we can change W_∞ by $W_{n,\infty}$ in (1.4.3).

To prove (1.4.6), it's shown in [3] that there exists $\beta_0 > 0$ such that for $|\beta| < |\beta_0|$ we have, as $n \to \infty$:

$$\mathbb{P}[(W_n - W_\infty)^4] \to 0 \tag{1.4.8}$$

$$\mathbb{P}(s_n^4) - \sigma_1^4 \mathbb{P}(W_n^4) \to 0 \tag{1.4.9}$$

$$\mathbb{P}(s_n^2 W_n^2) - \sigma_1^2 \mathbb{P}(W_n^4) \to 0 \tag{1.4.10}$$

Using the three equations we deduce that

$$\mathbb{P}[(s_n^2 - \sigma_1^2 W_n^2)^2] = \mathbb{P}(s_n^4) - 2\sigma_1^2 \mathbb{P}(s_n^2 W_n^2) + \sigma_1^4 \mathbb{P}(W_n^4) \to 0 \tag{1.4.11}$$

Thus, $s_n^2 - \sigma_1^2 W_n^2 \to 0$ in L^2.As $W_n^2 \to W_\infty^2$ in L^2, it follows that $s_n^2 \to \sigma_1^2 W_\infty^2$ in L^2.

To check (1.4.7), is verified for $q > 1$, when $|\beta| > 0$ is small enough, we have

$$\mathbb{P}(D_{k+1}^4) = O(k^{-d/q}), \quad k \geq 1$$

In that case, when β is small, $n^{d-2} \sum_{k \geq n} \mathbb{P}(D_{k+1}^4) \to 0$. This implies that $n^{d-2} \sum_{k \geq n} \mathbb{P}(D_{k+1}^4 | \mathcal{F}_k) \to 0$, and at the same, implies (1.4.7) by Cauchy Schwarz. □

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