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Stochastic Calculus Notes

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References

1 Stochastic Integration of Predictable Processes

This is the core chapter of these notes,were we define the integral $\int_0^t X(s)dY(s)$,where Y is a cadlag semimartingale, and X is a locally bounded predictable process.As previous steps, we construct also the integral with respect to L^2-martingales and local L^2-martingales. As an observation, we won't need the right continuity of the filtration $\{\mathcal{F}_t\}$ until we define the integral with respect to a semimartingale. In that case, it's only needed for guaranteeing that the semimartingale has a decomposition whose local martingale part is a local L^2- martingale.

1.1 Square-integrable martingale integrator

Throughout this section, we consider a fixed probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$.M is a square-integrable cadlag martingale relative to the filtration $\{\mathcal{F}_t\}$.As usual, we assume that the probability space and the filtration are complete.Right-continuity of the filtration $\{\mathcal{F}_t\}$ is not assumed unless specifically stated.

1.1.1 Predictable processes

Definition 1.1.

i) A predictable rectangle is a subset of $\mathbb{R}_+ \times \Omega$ *of the type* $(s,t] \times \mathcal{F}$*,where* $0 \leq s < t < \infty$ *and* $F \in \mathcal{F}_s$*,or of the type* $\{0\} \times F_0$*,where* $F_0 \in \mathcal{F}_0$*.The collection of all rectangles is denoted by* $\mathcal{R}$*.We also include* $\emptyset \in \mathcal{R}$*.*

ii) The $\sigma-$*algebra generated by* $\mathcal{R}$ *is called the predictable* $\sigma-$*algebra* $\mathcal{P}$*.Note that* $\mathcal{P} \subset \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$*.*

iii) Any $\mathcal{P}-$*measurable function from* $\mathbb{R}_+ \times \Omega$ *into* $\mathbb{R}$ *is called a predictable process.*

Remark 1.2. *In the exercises is proved that predictable processes are not only adapted to* $\{\mathcal{F}_t\}$ *but also to* $\{\mathcal{F}_{t-}\}$*.This gives the name 'predictable', since mathematically, the information immediately prior to* t *is represented by* $\mathcal{F}_{t-}$*.*

Lemma 1.3. *The following* σ*-algebras on* $\mathbb{R}_+ \times \Omega$ *are all equal to* $\mathcal{P}$*.*

a) The $\sigma-$*algebra generated by all continuous adapted processes.*

b) The $\sigma-$*algebra generated by all left-continuous adapted processes.*

c) The $\sigma-$*algebra generated by all adapted caglad processes(that is, left continuous, with right limits).*

Proof. First,we prove that left-continuous adapted processes are in $\mathcal{P}$.This implies that $\sigma-$algebra in $a), b), c)$ are contained in $\mathcal{P}$.
Let X be a left-continuous,adapted process.Let

$$X_n(t,\omega) := X_0(\omega)\mathbb{1}_{\{0\}}(0) + \sum_{i=1}^{\infty} X_{i2^{-n}}(\omega)\mathbb{1}_{(i2^{-n},(i+1)2^{-n}]}(t)$$

Then for $B \in \mathcal{B}_{\mathbb{R}}$,

$$\{(t,\omega) : X_n(t,\omega) \in B\} = \{0\} \times \{\omega : X_0(\omega) \in B\} \cup \bigcup_{i=0}^{\infty} \big\{(i2^{-n}, (i+1)2^{-n}] \times \{\omega : X_{i2^{-n}} \in B\}\big\} \in \mathcal{P}$$

because it's a countable union of events in $\mathcal{R}$.Thus X_n is $\mathcal{P}-$measurable. As $X_n(t,\omega) \to X(t,\omega)$ for each (t,ω) as $n \to \infty$, we deduce that $X \in \mathcal{P}$.
Now we prove the converse contention.First observe that the indicator of a rectangle is itself an adapted cadlag process, and by definition this subclass generated $\mathcal{P}$.Thus the $\sigma-$algebra in $c)$ contains $\mathcal{P}$.By the same reasoning, the $\sigma-$algebra in $b)$ contains $\mathcal{P}$.It only remains to

show that the $\sigma-$algebra in $a)$ contains $\mathcal{P}$.Is enough to check that the indicators of rectangles are pointwise limits of continuous adapted processes.
If $X = \mathbb{1}_{\{0\}\times F_0}$ for $F_0 \in \mathcal{F}_0$,let

$$g_n(t) := \begin{cases} 1 - nt, & 0 \leq t < 1/n \\ 0, t \geq 1/n \end{cases}$$

and then define $X_n(t,\omega) := \mathbb{1}_{F_0} \times g_n(t)$.As $g_n(t)$ is continuous , then also is $X_n(t)$.We can write

$$X_n(t) = \begin{cases} g_n(t)\mathbb{1}_{F_0}, & 0 \leq t < 1/n \\ 0, & t \geq 1/n \end{cases}$$

So X_n is adapted, because $F_0 \in \mathcal{F}_t$ for each t.Since $X_n(t,\omega) \to X(t,\omega)$ for each (t,ω), we conclude that $\{0\} \times F_0$ belongs to the $\sigma-$algebra in $a)$.
If $X = \mathbb{1}_{(u,v]\times F}, F \in \mathcal{F}_u$.Let

$$h_n(t) := \begin{cases} n(t-u), & u \leq t < u + 1/n \\ 1 & u + 1/n \leq t < v \\ 1 - n(t-v), & v \leq t \leq v + 1/n \\ 0, & t < u \ or \ t > v + 1/n \end{cases}$$

Consider n large enough so that $1/n < v-u$ and define $X_n(t,\omega) := \mathbb{1}_F(\omega)h_n(t)$.The function $h_n(t)$ is zero before u, grows linearly up to 1 in $u+1/n$, stays at 1 between $u+1/n$ and v, and then decreases linearly up to $v+1/n$.This function converges almost surely to $\mathbb{1}_{(u,v]}$,so $X_n \to X$ almost surely. By this same definition, $X_n(t) = h_n(t)\mathbb{1}_F$ with $h_n(t) \neq 0$ when $t \geq u$.As $F \in \mathcal{F}_u$, then $X_n(t)$ is $\mathcal{F}_t$-measurable,so is also holds for X_t. □

Definition 1.4. *Given a square-integrable cadlag martingale M, we define its Doléans measure μ_M on the predictable $\sigma-$algebra $\mathcal{P}$ by*

$$\mu_M(A) := E\int_{[0,\infty)} \mathbb{1}_A(t,\omega)d[M]_t(\omega), \quad A \in \mathcal{P}. \tag{1.1}$$

We integrate first the function $t \to \mathbb{1}_A(t,\omega)$ respect to the Lebesgue-Stieltjes measure $\Lambda_{[M](\omega)}$ of the nondecreasing right-continuous function $t \to [M]_t(\omega)$.The resulting integral is measurable function of ω, which is then averaged over the probability space $(\Omega, \mathcal{F}, P)$.Recall that the convention for the measure $\Lambda_{[M](\omega)}\{0\}$ is

$$\Lambda_{[M](\omega)}\{0\} = [M]_0(\omega) - [M]_{0-}(\omega) = 0 - 0 = 0$$

Consequently integrals over $(0, \infty)$ and $[0, \infty)$ coincide in (1.1).
Also, note that

$$\mu_M([0,T] \times \Omega) = E([M]_T) = E(M_T^2 - M_0^2) < \infty \tag{1.2}$$

For all $T < \infty$.So the measure μ_M is $\sigma-$finite.

Example 1.5. *(Brownian motion).If $M = B$,standard Brownian motion, recall that $[B]_t = t$.Then*

$$\mu_B(A) = E \int_{[0,\infty)} \mathbb{1}_A(t, \omega) dt = m \otimes P(A)$$

where m is Lebesgue measure on $\mathbb{R}_+$.

Example 1.6. *(Compensated Poisson process). Let N be a homogeneous rate α Poisson process on $\mathbb{R}_+$ with respect to the filtration $\{\mathcal{F}_t\}$.Let $M_t = N_t - \alpha t$.We claim that the Doléans measure of μ_M is $\alpha m \otimes P$, where m is Lebesgue measure on $\mathbb{R}_+$.Recall that $[M] = N$.For a predictable rectangle $A = (s, t] \times F, F \in \mathcal{F}_s$,*

$$\begin{aligned} \mu_M(A) &= E \int_{[0,\infty)} \mathbb{1}_A(u, \omega) d[M]_u(\omega) = E \int_{[0,\infty)} \mathbb{1}_F(\omega) \mathbb{1}_{(s,t]}(u) dN_u(\omega) \\ &= E[\mathbb{1}_F(N_t - N_s)] \overset{indep.incr}{=} E[\mathbb{1}_F] E[N_t - N_s] = P(F)\alpha(t-s) = \alpha m \otimes P(A) \end{aligned}$$

Both measures assign zero value to the set $\{0\} \times \mathcal{F}_0$.So both measures agree on $\mathcal{R}$,so they must agree also in $\mathcal{P}$.

Definition 1.7. *For predictable processes X, we define the L^2-norm over the set $[0,T] \times \Omega$ under the measure μ_M by*

$$||X||_{\mu_M,T} := \left(\int_{[0,T]\times\Omega} |X|^2 d\mu_M \right)^{1/2} = \left(E \int_{[0,T]} |X(t,\omega)|^2 d[M]_t(\omega) \right)^{1/2} \tag{1.3}$$

Let $\mathcal{L}_2 = \mathcal{L}_2(M, \mathcal{P})$ denote the collection of all predictable processes X such that $||X||_{\mu_M,T} < \infty$ for every $T < \infty$.We define a metric on $\mathcal{L}_2$ defined by $d_{\mathcal{L}_2}(X,Y) := ||X - Y||_{\mathcal{L}_2}$,where

$$||X||_{\mathcal{L}_2} := \sum_{k=1}^{\infty} 2^{-k} (1 \wedge ||X||_{\mu_M,k}) \tag{1.4}$$

We use the same conventions as in the definition of the space $\mathcal{L}_2(B)$ from the last chapter.In particular, to satisfy the condition $d_{\mathcal{L}_2}(X,Y) = 0 \leftrightarrow X = Y$,we consider two processes X and Y in $\mathcal{L}_2$ as equal if

$$\mu_M\{(t,\omega) : X(t,\omega) \neq Y(t,\omega)\} = 0 \tag{1.5}$$

We say that the processes X and Y are μ_M-equivalent if (1.5) holds.

Example 1.8. *Suppose X is a predictable and bounded on bounded time intervals, in other words there exist constants $C_T < \infty$ such that , for almost every ω and $R < \infty, |X_t(\omega)| \leq C_T$ for $0 \leq t \leq T$.Then $X \in \mathcal{L}_2(M, \mathcal{P})$, because*

$$E\int_{[0,T]} X(s)^2 d[M]_s \leq C_T^2 E([M]_T) = C_T^2 E(M_T^2 - M_0^2) < \infty.$$

1.1.2 Construction of the stochastic integral.

In this section we construct the process $(X \cdot M)_t = \int_{(0,t]} X dM$ for integrands $X \in \mathcal{L}_2$.As in the Itô integral we define an explicit formula for a class of simple processes, and then use an approximation argument.

Recall the definition of simple predictable processes

$$X(t,\omega) = \xi_0(\omega)(t) + \sum_{i=1}^{n-1} \xi_i(\omega) \mathbb{1}_{(t_i,t_{i+1}]}(t) \tag{1.6}$$

We set $t_1 = t_0 = 0$,so the formula for X covers the interval $[0, t_n]$ without leaving a gap in the origin.Using Lemma 1.3 and the left continuity of X as in (1.6),we get that this process is predictable.

Definition 1.9. *For a simple predictable process of the type (1.6), the stochastic integral is the process $X \cdot M = I(X)$ defined by*

$$(X \cdot M)_t(\omega) = \sum_{i=1}^{n-1} \xi_i(\omega)(M_{t_{i+1}\wedge t}(\omega) - M_{t_i \wedge t}(\omega) \tag{1.7}$$

Of course, we need to check that the integral does not depend on the representation of the process, and that the integral is linear.The proof is the same as lemma 1.6 in the last chapter.Now we state the analogous of lemma 1.7 in the last chapter.

Lemma 1.10. *Let $X \in \mathcal{S}_2$.Then $X \cdot M$ is a square-integrable cadlag martingale.If M is continuous, then so $X \cdot M$.These isometries hold: for all $t > 0$,*

$$E[(X \cdot M)_t^2] = \int_{[0,t]\times\Omega} X^2 d\mu_M \tag{1.8}$$

and

$$||X \cdot M||_{\mathcal{M}_2} = ||X||_{\mathcal{L}_2} \tag{1.9}$$

Proof. We only prove the things that change with respect to lemma 1.6 from last chapter.The cadlag property for each fixed ω follows from the definition (1.7),as in the continuity of $X \cdot M$ if M is continuous.The martingale property is the same as in the analogous lemma for Brownian motion integral.

We prove (1.8).We square

$$(X \cdot M)_t^2 = \sum_{i=1}^{n-1} \xi_i^2 (M_{t\wedge t_{i+1}} - M_{t\wedge t_i})^2 + 2\sum_{i<j} \xi_i \xi_j (M_{t\wedge t_{i+1}} - M_{t\wedge t_i})(M_{t\wedge t_{ij+1}} - M_{t\wedge t_j})$$

When we take expectations, the second sum vanish when we take conditional expectation respect to $\mathcal{F}_{t_j}$.We compute the expectation of the first sum. We use the fact that $M^2 - [M]$ is a martingale,and that $E[(M_{t\wedge t_{i+1}} - M_{t\wedge t_i})^2 | \mathcal{F}_{t_i}] = E[M^2_{t\wedge t_{i+1}} - M^2_{t\wedge t_i} | \mathcal{F}_{t_i}]$

$$\begin{aligned}
\sum_{i=1}^{n-1} E[\xi_i^2 (M_{t\wedge t_{i+1}} - M_{t\wedge t_i})^2] &= \sum_{i=1}^{n-1} E[\xi_i^2 E[(M_{t\wedge t_{i+1}} - M_{t\wedge t_i})^2 | \mathcal{F}_{t_i}]] = \sum_{i=1}^{n-1} E[\xi_i^2 E[M^2_{t\wedge t_{i+1}} - M^2_{t\wedge t_i} | \mathcal{F}_{t_i}]] \\
&= \sum_{i=1}^{n-1} E[\xi_i^2 E[[M]_{t\wedge t_{i+1}} - [M]_{t\wedge t_i} | \mathcal{F}_{t_i}]] = \sum_{i=1}^{n-1} E[\xi_i^2 [M]_{t\wedge t_{i+1}} - [M]_{t\wedge t_i}] \\
&= \sum_{i=1}^{n-1} E[\xi_i^2 \int_{[0,t]} \mathbb{1}_{(t_i,t_{i+1}]}(s) d[M]_s \\
&= E\left[\int_{[0,t]} \left(\xi_0^2 \mathbb{1}_{\{0\}}(s) + \sum_{i=1}^{n-1} \xi_i^2 \mathbb{1}_{(t_i,t_{i+1}]}(s)\right) d[M]_s\right] \\
&= E\left[\int_{[0,t]} \left(\xi_0 \mathbb{1}_{\{0\}}(s) + \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{(t_i,t_{i+1}]}(s)\right)^2 d[M]_s\right] \\
&= \int_{[0,t]\times\Omega} X^2 d\mu_M
\end{aligned}$$

This equality for each $t > 0$ also proves (1.9). $\square$

We come to the approximation step.

Lemma 1.11. *For any $X \in \mathcal{L}_2$,there exists a sequence $X_n \in \mathcal{S}_2$ such that $||X_n - X||_{\mathcal{L}_2} \to 0$.*

Proof. Let $\tilde{\mathcal{L}}_2$ denote the class of $X \in \mathcal{L}_2$ for which this approximation is possible.Clearly, $\mathcal{S}_2 \subset \tilde{\mathcal{L}}_2$.In particular, all the processes of the form

$$X(t, \omega) = \sum_{i=0}^{n} c_i \mathbb{1}_{R_i}(t, \omega) \tag{1.10}$$

where $\{c_i\}$ are constants, and $\{R_i\}$ are time-bounded predictable rectangles, are in $\mathcal{S}_2$.

Step 1. Let $G \in \mathcal{P}$ arbitrary and $c \in \mathbb{R} \setminus \{0\}$, and set $X = c\mathbb{1}_G$.We claim that $X \in \tilde{\mathcal{L}}_2$.Note that $c\mathbb{1}_G \in \mathcal{L}_2$ because

$$||c\mathbb{1}_G||_{\mu_M, T} = |c| \cdot \mu(G \cap ([0,T] \times \Omega))^{1/2} < \infty$$

due to (1.2).

Given $\epsilon > 0$, fix n large enough so that $2^{-n} < \epsilon/2$.Let $G_n = G \cap ([0,n] \times \Omega)$.Consider the restricted $\sigma-$algebra

$$\mathcal{P}_n = \{A \subset \mathcal{P} : A \subset [0,n] \times \Omega\} = \{B \cap ([0,n] \times \Omega) : B \in \mathcal{P}\}$$

$\mathcal{P}_n$ is generated by the collection $\mathcal{R}_n$ of predictable rectangles that lie in $[0,n] \times \Omega$.$\mathcal{R}_n$ is a semialgebra in the space $[0,n] \times \Omega$.The algebra $\mathcal{A}_n$ generated by $\mathcal{R}_n$ is the collection of all finite disjoint unions of members of $\mathcal{R}_n$.Restricted to $[0,n] \times \Omega, \mu_M$ is a finite measure.By a general fact of measure theory, there exists $R \in \mathcal{A}_n$ such that $\mu_M(G_n \triangle R) < |c|^{-2}\epsilon^2/4$.We can write $R = R_1 \cup \cdots R_p$ as a finite disjoint union of time-bounded predictable rectangles.Let $Z = c\mathbb{1}_R$.By the disjointness,

$$Z = c\mathbb{1}_R = \sum_{i=1}^{p} c\mathbb{1}_{R_i}$$

so Z is of type (1.10), and a member of $\mathcal{S}_2$.We estimate the $\mathcal{L}_2$ distance between Z and X.

$$||Z - X||_{\mathcal{L}_2} \leq \sum_{k=1}^{n} 2^{-k} ||c\mathbb{1}_R - c\mathbb{1}_G||_{\mu_M, k} + 2^{-n} \leq \sum_{k=1}^{n} 2^{-k} |c| \left(\int_{[0,k] \times \Omega} |\mathbb{1}_R - \mathbb{1}_G|^2 d\mu_M \right)^{1/2} + \epsilon/2$$

$$\leq |c| \left(\int_{[0,n] \times \Omega} |\mathbb{1}_R - \mathbb{1}_{G_n}|^2 d\mu_M \right)^{1/2} + \epsilon/2 = |c| \mu_M(G_n \triangle R)^{1/2} + \epsilon/2 < \epsilon$$

where we used that $|\mathbb{1}_R - \mathbb{1}_{G_n}| = \mathbb{1}_{R\triangle G_n}$.We proved that for each $\epsilon > 0$ there exists some $Z \in \mathcal{S}_2$ such that $||Z - X||_{\mathcal{L}_2} < \epsilon$.Consequently, $c\mathbb{1}_G \in \tilde{\mathcal{L}}_2$.

Observe that $\tilde{\mathcal{L}}_2$ is closed under addition,that is, if $X, Y \in \tilde{\mathcal{L}}_2$, and $X_n, Y_n \in \mathcal{S}_2$ such that $||X_n - X||_{\mathcal{L}_2} \to 0, ||Y_n - Y||_{\mathcal{L}_2} \to 0$, the by triangle inequality ,

$$||X_n + Y_n - (X + Y)||_{\mathcal{L}_2} \leq ||X_n - X||_{\mathcal{L}_2} + ||Y_n - Y||_{\mathcal{L}_2} \to 0$$

Thus $X + Y \in \tilde{\mathcal{L}}_2$.Using this and the fact that $c\mathbb{1}_G \in \tilde{\mathcal{L}}_2$,we conclude that simple functions of the type

$$X = \sum_{i=1}^{n} c_i \mathbb{1}_{G_i}, \quad c_i \in \mathbb{R}, G_i \in \mathcal{P} \tag{1.11}$$

lie in $\tilde{\mathcal{L}}_2$.

Step 2. Let X be an arbitrary process in $\mathcal{L}_2$.Given $\epsilon > 0$, pick n so that $2^{-n} < \epsilon/2$.Take simple functions X_m of the type (1.11) such that $|X - X_m| \leq |X|$ and $X_m(t, \omega) \to X(t, \omega)$ for all (t, ω).Since $X \in L^2([0, n] \times \Omega, \mathcal{P}_n, \mu_M)$,by dominated convergence for $1 \leq k \leq n$

$$\limsup_{m\to\infty} ||X - X_m||_{\mu_M,k} \leq \lim_{m\to\infty} \left(\int_{[0,n]\times\Omega} |X - X_m|^2 d\mu_M \right)^{1/2} = 0$$

Consequently,

$$\limsup_{m\to\infty} ||X - X_m||_{\mathcal{L}_2} \leq \sum_{k=1}^{n} 2^{-k} \limsup_{m\to\infty} ||X - X_m||_{\mu_M,k} + \epsilon/2 = \epsilon/2.$$

Fix m large enough such that $||X - X_m||_{\mathcal{L}_2} < \epsilon/2$.Using Step 1, there exists some $Z \in \mathcal{S}_2$ such that $||X_m - Z|| \leq \epsilon/2$.Using triangle inequality, we get $||X - Z||_{\mathcal{L}_2} < \epsilon$.This proves that an arbitrary process $X \in \mathcal{L}_2$ can be approximated by simple predictable processes in the $\mathcal{L}_2-$distance.

□

Now we define the stochastic integral.

Definition 1.12. *Let M be a square-integrable cadlag martingale on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$.For any predictable process $X \in \mathcal{L}_2(M, P)$,the stochastic integral $I(X) = X \cdot M$ is the square-integrable cadlag martingale that satisfies*

$$\lim_{n\to\infty} ||X \cdot M - X_n \cdot M||_{\mathcal{M}_2} = 0$$

for every sequence $X_n \in \mathcal{S}_2$ such that $||X_n - X||_{\mathcal{L}_2} \to 0$.The process $I(X)$ is unique up to indistinguishability.If M is continuous, then so is $X \cdot M$.

Justification of the definition.

*Existence:*Let $X \in \mathcal{L}_2$.We know that there exists a sequence $X_n \in \mathcal{S}_2$ such that $||X - X_n||_{\mathcal{L}_2} \to 0$.In particular, X_n is a Cauchy sequence in $\mathcal{L}_2$.By the isometry, then $X_n \cdot M$ is a Cauchy sequence in the space $\mathcal{M}_2$,whose limit is also in this space by the completeness.(the limit is a continuous martingale if M also is one).The limit process $Y = \lim_{n\to\infty} X_n \cdot M$ we call $X \cdot M = I(X)$.

*Uniqueness:*This is a classic argument using triangle inequality and the isometry (see lemma 1.7 from last chapter). □

Remark 1.13. *Recall that $||X_n - X||_{\mathcal{L}_2} \to 0$ is equivalent to*

$$\int_{[0,t]\times\Omega} |X_n - X|^2 d\mu_M \to 0 \quad \forall\ T < \infty$$

Convergence in $\mathcal{M}_2$ is equivalent to $L^2(P)$ convergence at each fixed time t: for martingales $N^{(j)}, N \in \mathcal{M}_2$,

$$||N^{(j)} - N||_{\mathcal{M}_2} \to 0$$

if and only if

$$E[(N_t^j - N_t)^2] \to 0 \quad \forall\ t \geq 0.$$

This two properties allows us to extend the isometry to the general integral.

Proposition 1.14. *Let $M \in \mathcal{M}_2$ and $X \in \mathcal{L}_2(M, \mathcal{P})$.Then we have the isometries*

$$E[(X \cdot M)_t^2] = \int_{[0,t]\times\Omega} X^2 d\mu_M \quad \forall\ t \geq 0, \tag{1.12}$$

and

$$||X \cdot M||_{\mathcal{M}_2} = ||X||_{\mathcal{L}_2} \tag{1.13}$$

In particular, if $X, Y \in \mathcal{L}_2(M, \mathcal{P})$ are μ_M-equivalent in the sense (1.5),then $X \cdot M$ an $Y \cdot M$ are indistinguishable.

1.1.3 Properties of the stochastic integral.

We prove some basic properties of the stochastic integral defined in Definition 1.12.The properties that take the form of an equality between two stochastic integrals are interpreted in the sense that the two processes are indistinguishable.Since the stochastic integral are cadlag processes, it's enough to check almost surely equality for all fixed times .Recall that the limit $X_n \cdot M \to X \cdot M$ satisfies the properties in lemma 1.49 from chapter 2.

Remark 1.15. *(Some notation)*

To avoid possible confusions in the future, here are some conventions about notation.

i) *A product of functions of t, ω and (t, ω) is regarded in the usual sense,that is, if X is a process, Z is a random variable and f is a function on $\mathbb{R}_+$, then fZX is the process whose value at (t, ω) is $f(t)Z(\omega)X(t, \omega)$.Also, we do not distinguish in notation between the function $t \to f(t)$ on $\mathbb{R}_+$ and the function $(t, \omega) \to f(t)$ on $\mathbb{R}_+ \times \Omega$.*

ii) *When $X_n \in \mathcal{S}_2$, approximate $X \in \mathcal{L}_2$, we write X_n in the form*

$$X_n(t, \omega) = \xi_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{n} \xi_i(\omega)\mathbb{1}_{(t_i, t_{i+1}]}(t) \tag{1.14}$$

iii) *We introduce the familiar integral notation through the definition*

$$\int_{(s,t]} XdM := (X \cdot M)_t - (X \cdot M)_s \quad \textit{for } 0 \leq s \leq t. \tag{1.15}$$

We write the integral in any of the following forms:

$$\int_{(s,t]} XdM = \int_{(s,t]} X_u dM_u = \int_{(s,t]} X_u(\omega) dM_u(\omega)$$

When the martingale is continuous, we can also write

$$\int_s^t XdM$$

because including or excluding the endpoints of the interval make no difference(see the exercises).

iv) *Since $(X \cdot M)_0 = 0$ for any stochastic integral,*

$$\int_{(0,t]} Xdm = (X \cdot M)_t$$

We will consider the interval $(0,t]$ instead of $[0,t]$ because the integral does not take in consideration any jump of the martingale at the origin.

v) An integral of the type

$$\int_{(u,v]} G(s,\omega)d[M]_s(\omega)$$

is interpreted as a path-by-path Lebesgue-Stieltjes integral.

Proposition 1.16.

a) Linearity:

$$(\alpha X + \beta Y) \cdot M = \alpha(X \cdot M) + \beta(Y \cdot M)$$

b) For any $0 \leq u \leq v$,

$$\int_{(0,t]} \mathbb{1}_{[0,v]} X dM = \int_{(0,v\wedge t]} X dM \tag{1.16}$$

and

$$\int_{(0,t]} \mathbb{1}_{(u,v]} X dM = (X \cdot M)_{t\wedge v} - (X \cdot M)_{t\wedge u} = \int_{(u\wedge t, v\wedge t]} X dM \tag{1.17}$$

The inclusion or exclusion of the origin in the interval $[0,v]$ is immaterial because a process og the type $\mathbb{1}_{\{0\}}(t)X(t,\omega)$ for $X \in \mathcal{L}_2(M,\mathcal{P})$is μ_M-equivalent to the identically zero process, and hence has zero stochastic integral.

c) For $s < t$, we have a conditional form of the isometry:

$$E[((X \cdot M)_t - (X \cdot M)_s^2 | \mathcal{F}_s] = E\left[\int_{(s,t]} X_u^2 d[M]_u | \mathcal{F}_s\right] \tag{1.18}$$

in particular, $(X \cdot M)_t - \int_{(0,t]} X_u^2 d[M]_u$ is a martingale.

Proof.

a) Simply take limits in the linearity for $X \in \mathcal{S}_2$.

b) If $X_n \in \mathcal{S}_2$ approximate X,then

$$\mathbb{1}_{[0,v]}(t)X_n(t) = \xi_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{(t_i\wedge v, t_{i+1}\wedge v]}(t)$$

are simple predictable processes that approximate $\mathbb{1}_{[0,v]}X$.

$$((\mathbb{1}_{[0,v]}X_n)\cdot M)_t = \sum_{i=1}^{k-1} \xi_i (M_{t_{i+1}\wedge t\wedge v} - M_{t_i\wedge t\wedge v})$$

Letting $n \to \infty$ along a suitable subsequence gives in the limit the almost surely equality

$$((\mathbb{1}_{[0,v]}X)\cdot M)_t = (X\cdot M)_{t\wedge v}$$

this proves (1.16).The second equality follows from $\mathbb{1}_{(u,v]}X = \mathbb{1}_{[0,v]}X - \mathbb{1}_{[0,u]}X$ and the linearity of the integral.

c) First we check the equality for a simple process X_n. Let $s < t$.If $s \geq t_k$, then $t_i \wedge s = t \wedge t_i = t_i$, so both sides in (1.18) are zero. Otherwise, fix an index $1 \leq m \leq k-1$ such that $t_m \leq s < t_{m+1}$.Then

$$\begin{aligned}
(X_n\cdot M)_t - (X_n\cdot M)_s &= \sum_{i=1}^{k-1} \xi_i (M_{t_{i+1}\wedge t} - M_{t_i\wedge t}) - \sum_{i=1}^{k-1} \xi_i (M_{t_{i+1}\wedge s} - M_{t_i\wedge s}) \\
&= \xi_i (M_{t_{i+1}\wedge t} - M_s) + \sum_{i=m+1}^{k-1} \xi_i (M_{t_{i+1}\wedge t} - M_{t_i\wedge t}) \\
&= \sum_{i=m}^{k-1} \xi_i (M_{u_{i+1}\wedge t} - M_{u_i\wedge t})
\end{aligned}$$

where we defined $u_m = s, u_i = t_i$ for $i > m$.After squaring,

$$\begin{aligned}
((X_n\cdot M)_t - (X\cdot M)_s)^2 &= \sum_{i=m}^{k-1} \xi_i^2 (M_{u_{i+1}\wedge t} - M_{u_i\wedge t})^2 \\
&+ 2 \sum_{m\leq i<j<k} \xi_i \xi_j (M_{u_{i+1}\wedge t} - M_{u_i\wedge t})(M_{u_{j+1}\wedge t} - M_{u_j\wedge t})
\end{aligned}$$

We claim that the conditional expectation respect to $\mathcal{F}_s$ in the second sum vanishes. Note that since $i < j, u_{i+1} \leq u_j$,and both ξ_i, ξ_j are $\mathcal{F}_{u_j}-$measurable.

$$\begin{aligned}
&E[\xi_i\xi_j(M_{u_{i+1}\wedge t} - M_{u_i\wedge t})(M_{u_{j+1}\wedge t} - M_{u_j\wedge t})|\mathcal{F}_s] = \\
&= E[\xi_i\xi_j(M_{u_{i+1}\wedge t} - M_{u_i\wedge t}) \underbrace{E[(M_{u_{j+1}\wedge t} - M_{u_j\wedge t})|\mathcal{F}_{u_j}]}_{=0}] = 0
\end{aligned}$$

by the martingale property.Now we compute the conditional expectation of the square.Recall that $M^2-[M]$ is a martingale and $E[(M_{t\wedge u_{i+1}}-M_{t\wedge u_i})^2|\mathcal{F}_{u_i}]=E[M^2_{t\wedge u_{i+1}}-M^2_{t\wedge u_i}|\mathcal{F}_{u_i}]$

$$
\begin{aligned}
\sum_{i=m}^{k-1}E[\xi_i^2(M_{u_{i+1}\wedge t}-M_{u_{t_i}\wedge t})^2|\mathcal{F}_s] &= \sum_{i=m}^{k-1}E[\xi_i^2E[M^2_{t\wedge u_{i+1}}-M^2_{t\wedge u_i}|\mathcal{F}_{u_i}]|\mathcal{F}_s] \\
&= \sum_{i=m}^{k-1}E[\xi_i^2E[[M]_{t\wedge u_{i+1}}-[M]_{t\wedge u_i}|\mathcal{F}_{u_i}]|\mathcal{F}_s] \\
&= E[\xi_i^2([M]_{t\wedge u_{i+1}}-[M]_{t\wedge u_i}|\mathcal{F}_{u_i})|\mathcal{F}_s] \\
&\overset{*}{=} \sum_{i=m}^{k-1}E\left[\xi_i^2\int_{(s,t]}\mathbb{1}_{(u_i,u_{i+1}]}(u)d[M]_u|\mathcal{F}_s\right] \\
&= E\left[\int_{(s,t]}\left(\xi_0\mathbb{1}_{\{0\}}+\sum_{i=1}^{k-1}\xi^2\mathbb{1}_{(t_i,t_{i+1}]}(u)\right)d[M]_u|\mathcal{F}_s\right] \\
&= E\left[\int_{(s,t]}\left(\xi_0\mathbb{1}_{\{0\}}+\sum_{i=1}^{k-1}\xi\mathbb{1}_{(t_i,t_{i+1}]}(u)\right)^2d[M]_u|\mathcal{F}_s\right] \\
&= E\left[\int_{(s,t]}X_n^2(u,\omega)^2d[M]_u(\omega)|\mathcal{F}_s\right]
\end{aligned}
$$

We replaced the $u_i's$ inside the $d[M]_u$ integral with $t_i's$ because for $u\in(s,t]$, $\mathbb{1}_{(u_i,u_{i+1}]}(u)=\mathbb{1}_{(t_i,t_{i+1}]}(u)$(if $i>m$ this is clear, and the case $i=m$,we use that $s\le t_m$).In $*$ we used that intervals $(u_i,u_{i+1}]\subset(s,t]$.

Next take $X\in\mathcal{L}_2$,and pick $X_n\in\mathcal{S}_2, X_n\to X$ in $\mathcal{L}_2-$norm.Take $A\in\mathcal{F}_s$ arbitrary. We proved in the previous calculation that

$$E[((X_n\cdot M)_t-(X_n\cdot M)_s)^2\mathbb{1}_A]=E\left[\mathbb{1}_A\int_{(s,t]}X_n^2(u)d[M]_u\right]$$

We rewrite the last equation as

$$E[(X_n\cdot M)_t^2\mathbb{1}_A]-E[(X_n\cdot M)_s^2\mathbb{1}_A]=\int_{(s,t]\times A}X_n^2d\mu_M$$

Now we use that $(X_n\cdot M)_t\to(X\cdot M)_t$ in $L^2(P)$ and $X_n\to X$ in $L^2((0,t]\times\Omega,\mu_M)$ .This gives us the same equation as above with X_n replaced by X,and we are done.

□

Given stopping times σ,τ we can define various stochastic intervals.For example these sets are in $\mathcal{P}$:

$$[0\tau]=\{(t,\omega)\in\mathbb{R}_+\times\Omega:0\le t\le\tau(\omega)\},$$

and

$$(\sigma,\tau] = \{(t,\omega) \in \mathbb{R}_+ \times \Omega : \sigma(\omega) < t \leq \tau(\Omega)\}.$$

The path $t \to \mathbb{1}_{[0,t]}(t,\omega) = \mathbb{1}_{[0,\tau(\omega)]}(t)$ is adapted and left-continuous with right-limits(because it's the indicator function).Hence by Lemma 1.3 $\mathbb{1}_{[0,\tau]}$ is $\mathcal{P}-$measurable.The same goes for $\mathbb{1}_{(\sigma,\tau]}$.If X is a predictable process, then so is the product $\mathbb{1}_{[0,\tau]}X$.

Recall the stopped process M^τ defined by $M^\tau_t := M_{\tau\wedge t}$.If $M \in \mathcal{M}_2$, then also $M^\tau \in \mathcal{M}_2$ because we have

$$E[M^2_{t\wedge\tau}] \leq 2E[M^2_t] + E[M^2_0]$$

Lemma 1.17. *Let $M \in \mathcal{M}_2$ and τ a stopping time. Then for any $\mathcal{P}-$measurable nonnegative function Y,*

$$\int_{\mathbb{R}_+\times\Omega} Y d\mu_{M^\tau} = \int_{\mathbb{R}_+\times\Omega} \mathbb{1}_{[0,\tau]} Y d\mu_M \tag{1.19}$$

Proof. Consider first a nondecreasing cadlag function G on $[0,\infty)$.For $u > 0$, define the stopped function G^u by $G^u(t) := G(u \wedge t)$.We claim that the Lebesgue Stieltjes measures satisfy

$$\int_{(0,\infty)} h d\Lambda_{G^u} = \int_{(0,\infty)} \mathbb{1}_{(0,u]} h \Lambda_G$$

for every nonnegative Borel function h.To check this, consider first an interval $(s,t]$,

$$\Lambda_{G^u}(s,t] = G^u(t) - G^u(s) = G(u\wedge t) - G(u \wedge s) = \Lambda_G((s,t] \cap (0,u]).$$

This implies that the measures Λ_{G^u} and $\Lambda_G(\cdot \cap (0,u])$ coincide on all Borel sets of $(0,\infty)$.We extend this equality to $[0,\infty)$ if we set $G(0-) = G(0)$ so that the measure of $\{0\}$ is zero under both measures.

Now fix ω and apply the preceding.We have $[M]^\tau = [M^\tau]$,and so

$$\int_{[0,\infty)} Y(s,\omega) d[M^\tau]_s(\omega) = \int_{[0,\infty)} Y(s,\omega) d[M]^\tau_s(\omega) = \int_{[0,\infty)} \mathbb{1}_{[0,\tau(\omega)](s) Y(s,\omega)} d[M]_s(\omega)$$

where we applied the equality above with $h = Y, [M] = G$.Taking expectations we conclude. ∎

Remark 1.18. *The last lemma implies that the measure μ_{M^τ} is absolutely continuous with respect to μ_M, and furthermore that $\mathcal{L}_2(M,\mathcal{P}) \subset \mathcal{L}_2(M^\tau,\mathcal{P})$.*

Proposition 1.19. *Let $M \in \mathcal{M}_2, X \in \mathcal{L}_2$, and let τ be a stopping time.*

a) *Let Z be a bounded $\mathcal{F}_\tau$-measurable random variable. Then $Z\mathbb{1}_{(\tau,\infty)}X$ and $\mathbb{1}_{(\tau,\infty)}X$ are both members of $\mathcal{L}_2(M,\mathcal{P})$, and*

$$\int_{(0,t]} Z\mathbb{1}_{(\tau,\infty)}XdM = Z\int_{(0,t]} \mathbb{1}_{(\tau,\infty)}XdM \tag{1.20}$$

b) *The integral behaves as follows under stopping:*

$$((\mathbb{1}_{[0,\tau]}X)\cdot M)_t = (X\cdot M)_{\tau\wedge t} = (X\cdot M^\tau)_t \tag{1.21}$$

c) *Let also $N \in \mathcal{M}_2$ and $Y \in \mathcal{L}_2(N,\mathcal{P})$. Suppose that there is a stopping time σ such that $X_t(\omega) = Y_t(\omega)$ and $M_t(\omega) = N_t(\omega)$ for $0 \le t \le \sigma(\omega)$. Then $(X\cdot M)_{\sigma\wedge t} = (Y\cdot N)_{t\wedge\sigma}$ for all $t \ge 0$.*

Remark 1.20. *Equation (1.21) implies that τ can appear in any subset of the three locations. For example,*

$$(X\cdot M)_{\tau\wedge t} = (X\cdot M)_{\tau\wedge\tau\wedge t} = (X\cdot M^\tau)_{\tau\wedge t} = (X\cdot M^\tau)_{\tau\wedge\tau\wedge t} = ((\mathbb{1}_{[0,\tau]}X)\cdot M^\tau)_{\tau\wedge t} \tag{1.22}$$

Proof. a) $Z\mathbb{1}_{(\tau,\infty)}$ is $\mathcal{P}-$measurable because it is a caglad adapted process (the process is equal to Z if $t > \tau$,otherwise it vanishes.If $t > \tau$, then $\mathcal{F}_\tau \subset \mathcal{F}_t$ so Z is $\mathcal{F}_t$-measurable).Multiplying $X \in \mathcal{L}_2(M,\mathcal{P})$ by something bounded and $\mathcal{P}-$measurable creates a process in $\mathcal{L}_2(M,\mathcal{P})$.

Assume first $\tau = u$ is a deterministic time.Let X_n an approximate sequence of X.Then

$$\mathbb{1}_{(u,\infty)}X_n = \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{(u\vee t_i, u\vee t_{i+1}]}$$

approximates $\mathbb{1}_{(u,\infty)}X$ in $\mathcal{L}_2$, and

$$Z\mathbb{1}_{(u,\infty)}X_n = \sum_{i=1}^{k-1} Z\xi_i \mathbb{1}_{(u\vee t_i, u\vee t_{i+1}]}$$

are elements of $\mathcal{S}_2$ that approximate $Z\mathbb{1}_{(u,\infty)}X$ in $\mathcal{L}_2$.Their integrals are

$$((Z\mathbb{1}_{(u,\infty)}X_n)\cdot M)_t = \sum_{i=1}^{k-1} Z\xi_i(M_{(u\vee t_{i+1})\wedge t} - M_{(u\vee t_i)\wedge t}) = Z((\mathbb{1}_{(u,\infty)}X_n)\cdot M)_t$$

Letting $n \to \infty$ along a suitable subsequence gives almost sure convergence of both sides of this equality to the corresponding terms in (1.20) at time t, in the case $\tau = u$. Now we consider τ a general stopping time.Define τ^m by

$$\tau^m := \begin{cases} i2^{-m}; & if\ (i-1)2^{-m} \leq \tau < i2^{-m} \text{ for some } 1 \leq i \leq 2^m m \\ \infty, & if\ \tau \geq m \end{cases}$$

We have $\tau^m \searrow \tau$ pointwise as $m \nearrow \infty$, and $\mathbb{1}_{(\tau^m,\infty)} \nearrow \mathbb{1}_{(\tau,\infty)}$.Both

$$\mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} \text{and} \mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} Z$$

are $\mathcal{F}_{i2^{-m}}-$measurable for each i(The former by the definition of a stopping time and the latter by Exercise 9.The use the first part to each such random variable with $u = i2^{-m}$.

$$\begin{aligned}
(Z\mathbb{1}_{(\tau^m,\infty)}X) \cdot M &= \left(\sum_{i=1}^{2^m m} \mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} Z \mathbb{1}_{(i2^{-m},\infty)} X \right) \cdot M \\
&= \sum_{i=1}^{2^m m} (\mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} Z \mathbb{1}_{(i2^{-m},\infty)} X) \cdot M \\
&= \sum_{i=1}^{2^m m} \mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} Z (\mathbb{1}_{(i2^{-m},\infty)} X) \cdot M \\
&= Z \sum_{i=1}^{2^m m} \mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} (\mathbb{1}_{(i2^{-m},\infty)} X) \cdot M \\
&= Z \sum_{i=1}^{2^m m} (\mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} \mathbb{1}_{(i2^{-m},\infty)} X) \cdot M \\
&= Z \left(\sum_{i=1}^{2^m m} \mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} \mathbb{1}_{(i2^{-m},\infty)} X \right) \cdot M \\
&= Z((\mathbb{1}_{(\tau^m,\infty)}) \cdot M)
\end{aligned}$$

In the second and sixth equality we used the linearity of the integral. In the third equality we used the property with $\mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}} Z$,and in the fifth equality we used the property with $\mathbb{1}_{\{(i-1)2^{-m} \leq \tau < i2^{-m}\}}$.Taking $m \to \infty$,we conclude part a).

b) We prove the first equality in (1.21).Let $\tau_n = 2^{-n}(\lfloor 2^n \tau \rfloor + 1)$ the usual discrete approximation that converges down to τ as $n \to \infty$.Let $l(n) = \lfloor 2^n t \rfloor + 1$.Since $\tau \geq k2^{-n}$

iff $\tau_n \geq (k+1)2^{-n}$,we use the telescopic property in the first equality below(because $\tau \geq k2^{-n}$ implies $\tau \geq j2^{-n}$ for all $j \leq k$).

$$\begin{aligned}
(X \cdot M)_{\tau_n \wedge t} &= \sum_{k=0}^{l(n)} \mathbb{1}_{\tau \geq k2^{-n}}((X \cdot M)_{(k+1)2^{-n} \wedge t} - (X \cdot M)_{k2^{-n} \wedge t}) \\
&\overset{(1.17)}{=} \sum_{k=0}^{l(n)} \mathbb{1}_{\tau \geq k2^{-n}} \int_{(0,t]} \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]} X dM \\
&\overset{(1.20)}{=} \sum_{k=0}^{l(n)} \int_{(0,t]} \mathbb{1}_{\tau \geq k2^{-n}} \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]} X dM \\
&= \int_{(0,t]} \left(\mathbb{1}_{\{0\}} X + \sum_{k=0}^{l(n)} \mathbb{1}_{\tau \geq k2^{-n}} \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]} X \right) dM \\
&= \int_{(0,t]} \mathbb{1}_{[0,\tau_n]} X dM
\end{aligned}$$

in the last equality we used that for $s \in [0,t]$,

$$\mathbb{1}_{[0,\tau_n]}(s,\omega) = \mathbb{1}_{\{0\}}(s) + \sum_{k=0}^{l(n)} \mathbb{1}_{\tau \geq k2^{-n}}(\omega) \mathbb{1}_{(k2^{-n},(k+1)2^{-n}]}(s)$$

To see this,note that $s \in [0,t]$ iff $s = 0$ or $s \in (k2^{-n}, (k+1)2^{-n}]$ form some $k \in 0, 1, \cdots l(n)$.Also by definition, $\tau_n = 2^{-n}(j+1)$ for some $j \in \mathbb{N}$,thus $s \leq \tau_n$ iff $\tau_n \geq (k+1)2^{-n}$ iff $\tau \geq k2^{-n}$.We have obtained

$$(X \cdot M)_{\tau_n \wedge t} = \int_{(0,t]} \mathbb{1}_{[0,\tau_n]} X dM$$

Taking $n \to \infty$,by right continuity, $(X \cdot M)_{\tau_m \wedge t} \to (X \cdot M)_{\tau \wedge t}$.We show now that the right hand side converges to $((\mathbb{1}_{0,\tau} X) \cdot M)_t$.It's suffices to show (by the isometry (1.12)) that

$$\lim_{n \to \infty} \int_{[0,t] \times \Omega} |\mathbb{1}_{[0,\tau_n]} X - \mathbb{1}_{[0,\tau]} X|^2 d\mu_M = 0$$

We use dominated convergence: first notice that the integrand vanishes when $n \to \infty$, because

$$\mathbb{1}_{[0,\tau_n]} - \mathbb{1}_{[0,\tau]} = \begin{cases} 0, & \tau(\omega) = \infty \\ \mathbb{1}_{\{\tau(\omega) < t \leq \tau_n(\omega)\}}, & \tau(\omega) < \infty \end{cases}$$

and $\tau_n \searrow \tau$.The integrand is bounded by X^2 and $X \in \mathcal{L}_2$,so $\int_{[0,t]\times\Omega} |X|^2 d\mu_M < \infty$ This gives us the first equality in (1.21).

Now we prove the second equality.Let $X_n \in \mathcal{S}_2$ be an approximating sequence to $X \in \mathcal{L}_2(M, \mathcal{P})$.By Remark 1.18, we know that $X \in \mathcal{L}_2(M^\tau, \mathcal{P})$, and the process X_n also approximates X under this metric. We get

$$(X_n \cdot M^\tau)_t = \sum_i \xi_i (M^\tau_{t_{i+1}\wedge t} - M^\tau_{t_i \wedge t}) = \sum_i \xi_i (M_{t_{i+1}\wedge t \wedge \tau} - M_{t_i \wedge t \wedge \tau}) = (X_n \cdot M)_{t\wedge\tau}$$

By definition, $(X_n \cdot M^\tau)_t$ converges to $(X \cdot M^\tau)_t$ in L^2 as $n \to \infty$.To prove that $(X_n \cdot M)_{\tau\wedge t} \to (X \cdot M)_{\tau\wedge t}$,define $Y_n(t) := (X_n \cdot M)_t - (X \cdot M)_t$.This in an L^2- martingale, with $Y_n(0) = 0$.We apply the inequality

$$E|M_\rho| \le 2E[M_T^+] - E[M_0]$$

to the submartingale $Y_n^2(t)$ to get

$$E[Y_n^2(t\wedge\tau)] \le 2E[Y_n^2(t)] = 2E\left[((X_n \cdot M)_t - (X \cdot M)_t)^2\right]$$

and this expectation vanishes when $n \to \infty$ by the definition of $X \cdot M$.Consequently,

$$(X_n \cdot M)_{t\wedge t} \to (X \cdot M)_{\tau\wedge t} \; in \; L^2$$

This completes the second equality in (1.21), and hence part b).

c) Since $\mathbb{1}_{[0,\sigma]}X = \mathbb{1}_{[0,\sigma]}Y$ and $M^\sigma = N^\sigma$,

$$(X \cdot M)_{t\wedge\sigma} = ((\mathbb{1}_{[0,\sigma]}X) \cdot M^\sigma)_t = ((\mathbb{1}_{[0,\sigma]}Y) \cdot N^\sigma)_t = (Y \cdot N)_{t\wedge\sigma}$$

□

We compute some integrals using the last proposition.

Example 1.21. *i) Let $\sigma \le \tau$ be two stopping times, and ξ a bounded $\mathcal{F}_\sigma-$measurable random variable. Define $X = \xi \mathbb{1}_{(\sigma,\tau]}$, or more explicitly,*

$$X_t(\omega) = \xi(\omega)\mathbb{1}_{(\sigma(\omega),\tau(\omega]}(t).$$

This is an adapted caglad process.Hence X is predictable.Let M be an L^2-martingale.Pick a constant $C \geq |\xi(\omega)|$.Then for any $T < \infty$,

$$\int_{[0,T]\times\Omega} X^2 d\mu_M = E(\xi^2([M]_{\tau\wedge T} - [M]_{\sigma\wedge T})) \leq C^2 E[[M]_{\tau\wedge T}]$$
$$= C^2[M^2_{\tau\wedge T}] \leq C^2 E[M^2_T] < \infty$$

where we used that M^2 is submartingale.Thus $X \in \mathcal{L}_2(M,\mathcal{P})$.By (1.20) and (1.21),

$$\begin{aligned} X \cdot M &= (\xi \mathbb{1}_{(\sigma,\infty)}\mathbb{1}_{[0,\tau]}) \cdot M = \xi((\mathbb{1}_{(\sigma,\infty)}\mathbb{1}_{[0,\tau]}) \cdot M) = \xi((\mathbb{1}_{[0,\tau]} - \mathbb{1}_{[0,\sigma]}) \cdot M) \\ &= \xi((\mathbf{1} \cdot M)^\tau - (\mathbf{1} \cdot M)^\sigma) = \xi(M^\tau - M^\sigma) \end{aligned}$$

Above, $\mathbf{1}$ is the function identically one.

ii) Now consider a sequence of stopping times $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots, \sigma_i \nearrow \infty$,and random variables $\{\eta_i : i \geq 1\}$ such that ξ_i is $\mathcal{F}_{\sigma_i}-$measurable.Suppose that $C = \sup_{i,\omega} |\xi_i(\omega)| < \infty$.Let

$$X(t) := \sum_{i=1}^{\infty} \eta_i \mathbb{1}_{(\sigma_i,\sigma_{i+1}]}(t)$$

As before,X is a bounded caglad process, thus $X \in \mathcal{L}_2(M,\mathcal{P})$ for any L^2- martingale M.Let

$$X_n(t) := \sum_{i=1}^{n} \eta_i \mathbb{1}_{(\sigma_i,\sigma_{i+1}]}(t)$$

By the example above and the additivity of the integral, we know that

$$X_n \cdot M = \sum_{i=1}^{n} \xi_i (M^{\sigma_{i+1}} - M^{\sigma_i})$$

Clearly $X_n \to X$ pointwise, and since $|X_n - X| \leq 2C$,

$$\int_{[0,T]\times\Omega} |X_n - X|^2 d\mu_M \to 0$$

for any $T < \infty$ by dominated convergence.Consequently $X_n \to X$ in $\mathcal{L}_2(M,\mathcal{P})$,and by the isometry, $X_n \cdot M \to X \cdot M$ in $\mathcal{M}_2$.We deduce that

$$X \cdot M = \sum_{i=1}^{\infty} \eta_i (M^{\sigma_{i+1}} - M^{\sigma_i})$$

The last property we consider is the linearity of the integral of a single process with respect to two different martingales.

Lemma 1.22. *For a predictable process Y,*

$$\left(\int_{[0,T]\times\Omega} |Y|^2 d\mu_{\alpha M+\beta N}\right)^{1/2} \leq |\alpha| \left(\int_{[0,T]\times\Omega} |Y|^2 d\mu_M\right)^{1/2} + |\beta| \left(\int_{[0,T]\times\Omega} |Y|^2 d\mu_N\right)^{1/2}$$

Proof. Note that

$$[\alpha M+\beta N] = \alpha^2[M] + 2\alpha\beta[M,N] + \beta^2[N]$$

Now we apply the Kunita-Watanabe inequality ,

$$\begin{aligned}\int_{[0,T]} |Y_s|^2 d[\alpha M+\beta N]_s &= \alpha^2 \int_{[0,T]} |Y_2|^2 d[M]_s + 2\alpha\beta \int_{[0,T]} |Y_s|^2 d[M,N]_s + \beta^2 \int_{[0,T]} |Y_s|^2 d[N]_2 \\ &\leq \alpha^2 \int_{[0,T]} |Y_2|^2 d[M]_s + 2|\alpha||\beta| \left(\int_{[0,T]} |Y_2|^2 d[M]_s\right)^{1/2} \left(\int_{[0,T]} |Y_2|^2 d[N]_s\right)^{1/2} \\ &+ \beta^2 \int_{[0,T]} |Y_s|^2 d[N]_2\end{aligned}$$

This holds for fixed ω.Now take expectations and apply Cauchy-Schwarz in the middle term to conclude. $\square$

Proposition 1.23. *Let $M,N \in \mathcal{M}_2, \alpha, \beta \in \mathbb{R}$, and $X \in \mathcal{L}_2(M,\mathcal{P}) \cap \mathcal{L}_2(N,\mathcal{P})$.Then $X \in \mathcal{L}_2(\alpha M+\beta N,\mathcal{P})$, and*

$$X\cdot(\alpha M+\beta N) = \alpha(X\cdot M) + \beta(X\cdot N) \tag{1.23}$$

Proof. The last lemma shows that $X \in \mathcal{L}_2(\alpha M+\beta N,\mathcal{P})$.Replace the measure μ_M in the proof of Lemma 1.11 with the measure $\tilde{\mu} = \mu_M + \mu_N$.This gives a sequence of simple predictable processes X_n such that

$$\int_{[0,T]\times\Omega} |X - X_n|^2 d(\mu_M + \mu_N) \to 0$$

for each $T < \infty$.This combined with the previous lemma says that $X_n \to X$ simultaneously in $\mathcal{L}_2(M,\mathcal{P}), \mathcal{L}_2(N,\mathcal{P})$ and $\mathcal{L}_2(\alpha M+\beta N,\mathcal{P})$.(1.23) holds for simple predictable processes, and the general conclusion holds taking limits. $\square$

1.2 Local square-integrable martingale integrator

We want to extend the stochastic integral when M is a local L^2- martingale.

Definition 1.24. *Given a local square-martingale M,let $\mathcal{L}(M,\mathbb{P})$ denote the class of predictable processes X which have the following property: there exists a sequence of stopping times $0 \le \tau_1 \le \tau_2 \le \cdots \le \tau_k \le \cdots$ such that*

i) $P(\tau_k \nearrow \infty) = 1$

ii) M^{τ_k} *is a square-integrable martingale for each k, and*

iii) the process $\mathbb{1}_{[0,\tau_k]} X \in \mathcal{L}_2(M^{\tau_k}, \mathcal{P})$ *for each k.*

The sequence $\{\tau_k\}$ is called a localizing sequence for the pair (X, M).

Note that by definition, for each $k, Y^k := (\mathbb{1}_{[0,\tau_k]} X) \cdot M^{\tau_k}$ is an element in $\mathcal{M}_2$.

Lemma 1.25. *Let $M \in \mathcal{M}_{2,loc}$ and let X be a predictable process. Suppose σ, τ are two stopping times such that M^σ, M^τ are cadlag L^2-martingales, $\mathbb{1}_{[0,\sigma]} X \in \mathcal{L}_2(M^\sigma, \mathcal{P})$ and $\mathbb{1}_{[0,\tau]} X \in \mathcal{L}_2(M^\tau, \mathcal{P})$.Let*

$$Z_t := \int_{(0,t]} \mathbb{1}_{[0,\sigma]} X dM^\sigma \text{ and } W_t := \int_{(0,t]} \mathbb{1}_{[0,\tau]} X dM^\tau$$

denote the stochastic integrals, which are cadlag L^2-martingales. Then

$$Z_{t\wedge\sigma\wedge\tau} = W_{t\wedge\sigma\wedge\tau}$$

in the sense that both processes are indistinguishable.

Proof. We will use (1.21),(1.22), $(M^\sigma)^\tau = (M^\tau)^\sigma = M^{\tau\wedge\sigma}$ and $\mathbb{1}_{[0,\sigma]} X, \mathbb{1}_{[0,\tau]} X$ both lie in $\mathcal{L}_2(M^{\sigma\wedge\tau}, \mathcal{P})$.

$$\begin{aligned} Z_{t\wedge\sigma\wedge\tau} = ((\mathbb{1}_{[0,\sigma]} X) \cdot M^\sigma)_{t\wedge\sigma\wedge\tau} = ((\mathbb{1}_{[0,\tau]} \mathbb{1}_{[0,\sigma]} X) \cdot (M^\sigma)^\tau)_{t\wedge\sigma\wedge\tau} = ((\mathbb{1}_{[0,\tau]} \mathbb{1}_{[0,\sigma]} X) \cdot (M^\tau)^\sigma)_{t\wedge\sigma\wedge\tau} \\ = ((\mathbb{1}_{[0,\tau]} X) \cdot (M^\tau))_{t\wedge\sigma\wedge\tau} = W_{t\wedge\sigma\wedge\tau} \end{aligned}$$

□

Let Ω_0 be the following event:

$$\Omega_0 := \{\omega : \tau_k(\omega) \nearrow \infty \text{ as } k \nearrow \infty, \text{ and for all } (k,m), Y^k_{t\wedge\tau_k\wedge\tau_m}(\omega) = Y^m_{t\wedge\tau_k\wedge\tau_m}(\omega) \ \forall\ t \in \mathbb{R}_+\} \tag{1.24}$$

Note that $P(\Omega_0) = 1$ because $P(\tau_k \nearrow \infty) = 1$, the previous lemma, and because there are countable many (k, m).This implies that on Ω_0, for t with $t \le \tau_m \wedge \tau_k$, then $Y^k_t = Y^m_t$.

Definition 1.26. *Let $M \in \mathcal{M}_{2,loc}, X \in \mathcal{L}(M,\mathcal{P})$, and let $\{\tau_k\}$ be a localizing sequence for (X,M).Define the event Ω_0 as above. The stochastic integral $X \cdot M$ is the cadlag local L^2-martingale defined as follows: on the event Ω_0 set*

$$(X \cdot M)_t(\omega) := ((\mathbb{1}_{[0,\tau_k]}X) \cdot M^{\tau_k})_t(\omega) = Y_t^k(\omega) \ \text{ for any } k \text{ such that } \tau_k(\omega) \geq t \tag{1.25}$$

Outside Ω_0,set $(X \cdot M)_t(\omega) \equiv 0$ for all t.

This definition is independent of the localizing sequence $\{\tau_k$ in the sense that using any other localizing sequence of stopping times gives a process indistinguishable from $X \cdot M$ defined above.

Justification of the definition. The process $X \cdot M$ is cadlag on any bounded interval $[0,T]$ for the following reasons. Fix $\omega \in \Omega$.If $\omega \notin \Omega_0$, then the path is constant in time. If $\omega \in \Omega_0$, pick k such that $\tau_k(\omega) > T$, and note that the path $t \to (X \cdot M)_t(\omega)$ coincides with the cadlag path $t \to Y_t^k(\omega)$ on the interval $[0,T]$.As this process is a cadlag on each interval $[0,T]$, then it is a cadlag on $\mathbb{R}_+$.

As $(X \cdot M)_t = Y_t^k$ if $t \leq \tau_k$, then $(X \cdot M)_{t\wedge\tau_k} = Y^k_{t\wedge\tau_k} = (Y_t^k)^{\tau_k}$, and this a L^2- martingale. Thus $(X \cdot M)_t$ is a local L^2- martingale with localizing sequence $\{\tau_k\}$.This shows that $X \cdot M$ is a cadlag local L^2- martingale.

To prove the independence of the localizing sequence, suppose $\{\sigma_j\}$ is another localizing sequence of stopping times for $X \cdot M$.Let

$$W_t^j = \int_{(0,t]} \mathbb{1}_{[0,\sigma_j]} X dM^{\sigma_j}$$

. As before we define an event Ω_1 with $P(\Omega_1) = 1$ and on Ω_1 we define

$$W_t(\omega) := W_t^j(\omega) \ \text{ for j such that } \sigma_j(\omega) \geq t \tag{1.26}$$

By Lemma 1.25, the processes $W^j_{t\wedge\sigma_j\wedge\tau_k}, Y^k_{t\wedge\sigma_j\wedge\tau_k}$ are indistinguishable.Let Ω_2 be the set of $\omega \in \Omega_0 \cap \Omega_1$ for which

$$W^j_{t\wedge\sigma_j\wedge\tau_k} = Y^k_{t\wedge\tau_k\wedge\sigma_j} \ \text{ for all } (j,k), t \in \mathbb{R}_+$$

$P(\Omega_2) = 1$ because its an intersection of countable many subsets of probability one.Now pick $\omega \in \Omega_2$, and pick $t \leq \tau_k \wedge \sigma_j$. Then

$$(X \cdot M)_t(\omega) = Y_t^k = Y^k_{t\wedge\sigma_j\wedge\tau_k} = W^j_{t\wedge\sigma_j\wedge\tau_k} = W_t^j = W_t(\omega)$$

This proves that the processes $(X \cdot M), W$ are indistinguishable, so the definition of $X \cdot M$ does not depend on the localizing sequence used. $\square$

Remark 1.27. *a) The value X_0 does not affect the stochastic integral because $\mu_Z(\{0\} \times \Omega) = 0$ for any L^2-martingale Z.If a predictable X is given by $\tilde{X}_t = \mathbb{1}_{(0,\infty)}(t)X_t$, then $\mu_Z(X \neq \tilde{X}) = 0$.In particular, $\{\tau_k\}$ is a localizing sequence for (X, M) iff it is a localizing sequence for $(\tilde{X}, M)$, and $X \cdot M = \tilde{X} \cdot M$ if a localizing sequence exists. Also, in part iii) of Definition 1.24 we can equivalently require that $\mathbb{1}_{(0,\tau_k]}X$ lies in $\mathcal{L}_2(M^{\tau_k}, \mathcal{P})$.*

b) If the local L^2- martingale M has continuous paths , then so do M^{τ_k}, hence $X \cdot M$ has continuous paths.

Property $iii)$ in Definition 1.24 follows from this stronger assumption

$$\text{there exist stopping times } \{\sigma_k\}\text{such that}\sigma_k \nearrow \infty \text{ almost surely and}\mathbb{1}_{(0,\sigma_k]}X \\ \text{is a bounded process for each k.} \tag{1.27}$$

If M is any local L^2- martingale with localizing sequence $\{\nu_k\}$, and assume X is a predictable process that satisfies (1.27).A bounded process is in $\mathcal{L}_2(Z, \mathcal{P})$ for any L^2-martingale Z, consequently $\mathbb{1}_{(0,\tau_k]}X \in \mathcal{L}_2(M^{\nu_k}, \mathcal{P})$(the time origin is left out because $\mathbb{1}_{[0,\tau_k]}X$ cannot be bounded unless X_0 is bounded).By the last remark, the conclusion extends to $\mathbb{1}_{[0,\tau_k]}X$.Thus the stopping times $\tau_k = \sigma_k \wedge \nu_k$ localize the pair (X, M), and the integral $X \cdot M$ is well defined.

The next proposition lists some common types of predictable processes that satisfy (1.27).In certain cases demonstrating the existence of stopping times may require a right-continuous filtration. Then one replaces $\{\mathcal{F}_t\}$ with $\{\mathcal{F}_{t+}\}$, and this can be done without losing any cadlag martingales (or local martingales,see proposition 1.6 in chapter 2).Recall also the definition

$$X_T^*(\omega) := \underset{0\leq t\leq T}{sup} |X_t(\omega)| \tag{1.28}$$

which is $\mathcal{F}_T-$ measurable for any left- or right- continuous process X, provided we make the filtration complete(see the observation after the definition of (1.28)).

Proposition 1.28. *The following cases are examples of processes with stopping times $\{\sigma_k\}$ that satisfy condition (1.27).*

i) X is predictable, and for each $T < \infty$ there exists a constant $C_T < \infty$ such that, with probability one, $X_t \leq C_T$ for all $0 < t \leq T$. Take $\sigma_k = k$.

ii) X is adapted and has almost surely continuous paths. Take

$$\sigma_k = \inf\{t \geq 0 : |X_t| \geq k\}.$$

iii) X is adapted, and there exists an adapted cadlag process Y such that $X(t) = Y(t-)$ for all $t > 0$. Take

$$\sigma_k = \inf\{t \geq 0 : |Y(t)| \geq k \text{ or } |Y(t-)| \geq k\}$$

iv) X is adapted, has almost surely left-continuous paths ,and $X_T^* < \infty$ almost surely for each $T < \infty$. Assume the underlying filtration $\{\mathcal{F}_t\}$ is right-continuous. Take

$$\sigma_k = \inf\{t \geq 0 : |X_t| \geq k\}$$

Proof. i) There is nothing to prove.

ii) By corollary 1.21 from chapter 1,σ_k is stopping time for each k.A continuous path $t \to X_t(\omega)$ is bounded on compact time intervals. Hence for almost every $\omega, \sigma_k(\omega) \nearrow \infty$.Again by continuity, $|X_s| \leq k$ for $0 < s \leq \sigma_k$.
Note that if $|X_0| > k$, then $\sigma_k = 0$, so we cannot claim $\mathbb{1}_{[0,\sigma_k]}|X_0| \leq k$.This is why boundedness cannot be required at time zero.

iii) By lemma 1.20 in chapter 1,σ_k is stopping time for each k. A cadlag path is locally bounded just like a continuous path, and so $\sigma_k \nearrow \infty$. If $\sigma_k > 0$, then $|X(t)| = |Y(t-)| < k$ for $t < \sigma_k$, and by left continuity, $|X(\sigma_k)| = |Y(\sigma_k-)| \leq k$. Note that $|Y(\sigma_k)| \leq k$ may fail.

iv) By lemma 1.18 in chapter 1, σ_k is a stopping time since we assume $\{\mathcal{F}_t\}$ is right-continuous.As in case iii), by left continuity,$|X_s| \leq k$ for $0 < s \leq \sigma_k$.Given ω such that $X_T^*(\omega) < \infty$ for all $T < \infty$, we can choose $k_T > \sup_{0\leq t\leq T} |X_t(\omega)|$ and then $\sigma_k(\omega) \geq T$

for all $k \geq k_T$.We have found for every $T > 0$ a k_T such that $\sigma_k(\omega) \geq T$ for $k \geq k_T$.Thus $\sigma_k \nearrow \infty$ almost surely.

$\square$

Remark 1.29. *Category ii) is a special case of iii), and this is a special case of iv). Notice that every cadlag X satisfies $X(t) = Y(t-)$ for the cadlag process $Y(t) := X(t+)$, but this fail to be adapted. In fact, Y is adapted if $\{\mathcal{F}_t\}$ is right-continuous.But then we find ourselves in Category iv).*

From lemma 1.26 in chapter 1 and case *ii*) above we deduce this useful corollary(a continuous local martingale is a local L^2- martingale).

Corollary 1.30. *For any continuous local martingale M and continuous, adapted process X, the stochastic integral $X \cdot M$ is well defined.*

Example 1.31. *Let's us repeat Example 1.21 without boundedness assumptions. Let $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots, \sigma_i \nearrow \infty$ are stopping times, η_i is a finite $\mathbb{F}_{\sigma_i}-$measurable random variable for $i \geq 1$, and*

$$X_t := \sum_{i=1}^{\infty} \eta_i \mathbb{1}_{(\sigma_i, \sigma_{i+1}]}(t).$$

By construction X is a caglad process, and satisfies the hypotheses of case iii) of Proposition 1.28.Fix $M \in \mathcal{L}_{2,loc}$ and let $\{\rho_k\}$ be a localizing sequence for M.Define

$$\zeta_k = \begin{cases} \sigma_j, & if \ \max_{1\leq i\leq j-1} |\eta_i| \leq k < |\eta_j| \ for\ some\ j \\ \infty, & if \ |\eta| \leq k \ for\ all\ i. \end{cases}$$

Clearly $\zeta_k \nearrow \infty$.Also, ζ_k is a stopping time due to

$$\{\zeta_k \leq t\} = \bigcup_{j=1}^{\infty} \left(\left\{ \max_{1\leq i\leq j-1} |\eta_i| \leq k < |\eta_j| \right\} \cap \{\sigma_j \leq t\} \right)$$

Thus the stopping time $\tau_k = \rho_k \wedge \zeta_k$ is a stopping time that localize the pair (X, M).
Let $\eta_i^{(k)} = (\eta_i \wedge k) \vee (-k)$ the truncation. Observe that if $\sigma_i \geq \tau_k$,then $\sigma_{i+1} \geq \sigma_i \geq \tau_k$, so $(\sigma_i \wedge \tau_k, \sigma_{i+1}, \wedge \tau_k] = \emptyset$.Therefore this interval is non empty iff $\sigma_i < \tau_k \leq \zeta_k$.As $\zeta_k = \sigma_j$ for some j(or ∞),this implies $\zeta_k \geq \sigma_{i+1}$, which happens iff $\eta_l = \eta_l^{(k)}$ for $1 \leq l \leq i$.Hence

$$\mathbb{1}_{[0,\tau_k]}(t) X_t = \sum_{i=1}^{\infty} \eta_i^{(k)} \mathbb{1}_{(\sigma_i \wedge \tau_k, \sigma_{i+1}, \tau_k]}(t)$$

This process is bounded, so by Example 1.21,

$$((\mathbb{1}_{[0,\tau_k]}X) \cdot M^{\tau_k})_t = \sum_{i=1}^{\infty} \eta_i^{(k)}(M_{\sigma_{i+1}\wedge\tau_k\wedge t} - M_{\sigma_i\wedge\tau_k\wedge t})$$

Take k for which $t \leq \tau_k$. We get

$$(X \cdot M)_t = \sum_{i=1}^{\infty} \eta_i(M_{\sigma_{i+1}\wedge t} - M_{\sigma_i\wedge t})$$

We use the integral notation

$$\int_{(s,t]} XdM = (X \cdot M)_t - (X \cdot M)_s$$

Now we prove some properties of this stochastic integral.Note that expectations and conditional expectations of $(X \cdot M)_t$ do not necessarily exist any more so we cannot even contemplate their properties.

Proposition 1.32. *Let $M, N \in \mathcal{M}_{2,loc}, X \in \mathcal{L}(M, \mathcal{P})$, and let τ be a stopping time.*

a) Linearity continues to hold: if $Y \in \mathcal{L}(M, \mathcal{P})$, then

$$(\alpha X + \beta Y) \cdot M = \alpha(X \cdot M) + \beta(Y \cdot M)$$

b) Let Z be a bounded $\mathcal{F}_\tau-$measurable random variable. Then $Z\mathbb{1}_{(\tau,\infty)}X$ and $\mathbb{1}_{(\tau,\infty)}X$ are both members of $\mathcal{L}(M, \mathcal{P})$, and

$$\int_{(0,t]} Z\mathbb{1}_{(\tau,\infty)}XdM = Z\int_{(0,t]} \mathbb{1}_{(\tau,\infty)}XdM \tag{1.29}$$

Furthermore,

$$((\mathbb{1}_{[0,\tau]}X) \cdot M)_t = (X \cdot M)_{t\wedge\tau} = (X \cdot M^\tau)_t \tag{1.30}$$

c) Let $Y \in \mathcal{L}(N, \mathcal{P})$.Suppose $X_t(\omega) = Y_t(\omega)$ and $M_t(\omega) = N_t(\omega)$ for $0 \leq t \leq \tau(\omega)$. Then

$$(X \cdot M)_{\tau\wedge t} = (Y \cdot N)_{\tau\wedge t}$$

d) Suppose $X \in \mathcal{L}(M, \mathcal{P}) \cap \mathcal{L}(N, \mathcal{P})$. Then for $\alpha, \beta \in \mathbb{R}, X \in \mathcal{L}(\alpha M + \beta N, \mathcal{P})$ and

$$X \cdot (\alpha M + \beta N) = \alpha(X \cdot M) + \beta(X \cdot N).$$

Proof. The linearity is direct consequence of the definition of the stochastic integral.
Now we prove (1.29) Let $\{\sigma_k\}$ be a localizing sequence for the pair $(X \cdot M)$.Then $\{\sigma_k\}$ is also a localizing sequence for the pairs $(\mathbb{1}_{(\tau,\infty)}X, M)$ and $(Z\mathbb{1}_{(\tau,\infty)}X, M)$.Given $\omega, t,$ pick k large enough so that $t \leq \sigma_k(\omega)$.Then by the definition of the stochastic integrals for localized processes,

$$Z((\mathbb{1}_{(\tau,\infty)}X) \cdot M)_t(\omega) = Z((\mathbb{1}_{[0,\sigma_k]}\mathbb{1}_{(\tau,\infty)}X) \cdot M^{\sigma_k})_t(\omega)$$

and

$$(Z(\mathbb{1}_{(\tau,\infty)}X) \cdot M)_t(\omega) = ((\mathbb{1}_{[0,\sigma_k]}Z\mathbb{1}_{(\tau,\infty)}X) \cdot M^{\sigma_{jk}})_t(\omega)$$

and the right-hand side of both equations are equal due to (1.20) applied to the L^2-martingale M^{σ_k} and the process $\mathbb{1}_{[0,\sigma_k]}X$ instead of X.This proves (1.29).
Next we check (1.30).The sequence $\{\sigma_k\}$ also works for $(\mathbb{1}_{[0,\tau]}X, M)$.If $t \leq \sigma_k(\omega)$, then

$$((\mathbb{1}_{[0,\tau]}X) \cdot M)_t = ((\mathbb{1}_{[0,\sigma_k]}\mathbb{1}_{[0,\tau]}X) \cdot M^{\sigma_k})_t \overset{(1.21)}{=} ((\mathbb{1}_{[0,\sigma_k]}X) \cdot M^{\sigma_k})_{t\wedge\tau} = (X \cdot M)_{\tau\wedge t}$$

The first and last equalities come from the definition of the stochastic integral.This verifies the first equality in (1.30).The second equality is similar,

$$((\mathbb{1}_{[0,\tau]}X) \cdot M)_t = ((\mathbb{1}_{[0,\sigma_k]}\mathbb{1}_{[0,\tau]}X) \cdot M^{\sigma_k})_t = ((\mathbb{1}_{[0,\sigma_k]}X) \cdot (M^{\tau_k})^{\sigma_k})_t = (X \cdot M^{\tau})_t.$$

Part c) follows immediatly from the definition, noting that $X\mathbb{1}_{[0,\sigma_k]} = Y\mathbb{1}_{[0,\sigma_k]}$ and $M^{\sigma_k} = N^{\sigma_k}$ for $0 \leq t \leq \tau$,and part d) also follows from the definition. $\square$

If X is a cadlag process, define the caglad process by $X_-(0) := X(0), X_-(t) := X(t-)$ for $t > 0$.
The following proposition allow us to obtain $(X \cdot M)_t$ as limit of Riemann sums for caglad paths.

Proposition 1.33. *Let X be an adapted process and $M \in \mathcal{M}_{2,loc}$.Suppose $0 = \tau_0^n \leq \tau_1^n \leq \cdots$ are stopping times such that for each $n, \tau_i^n \to \infty$ almost surely as $i \to \infty$, and $\delta_n = \sup_i(\tau_{i+1}^n - \tau_i^n)$ tends to zero almost surely as $n \to \infty$.Define the process*

$$R_n(t) := \sum_{i=0}^{\infty} X(\tau_i^n)(M(\tau_{i+1}^n) - M(\tau_i^n)) \tag{1.31}$$

a) Assume X is left-continuous and satisfies (1.27). Then for each fixed $T < \infty$ and $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |R_n(t) - (X\cdot M)_t| > \epsilon\right) = 0$$

That is, R_n converges to $X \cdot M$ in probability, uniformly on compacts time intervals.

b) If X is a cadlag process, then R_n converges to $X_- \cdot M$ in probability, uniformly on compact time intervals.

Proof. Since $X_- = X$ for a left continuous process, we can prove parts $a), b)$ simultaneously.First assume $X_0 = 0$.By left- or right-continuity X is progressively measurable, and therefore $X(\tau_i^n)$ is $\mathcal{F}_{\tau_i^n}-$measurable on the event $\tau_i^n < \infty$.Define

$$Y_n(t) := \sum_{i=1}^{\infty} X(\tau_i^n)\mathbb{1}_{(\tau_i^n,\tau_{i+1}^n]}(t) - X_-(t)$$

By the hypotheses and Example 1.31, $Y_n \in \mathcal{L}(M,\mathcal{P})$, and its integral is

$$Y_n \cdot M = R_n - X_- \cdot M$$

Consequently, we need to show that $Y_n \cdot M \to 0$ in probability, uniformly on compacts.
Let $\{\sigma_k\}$ be a localizing sequence for (X_-, M) such that $\mathbb{1}_{(0,\sigma_k]}X_-$ is bounded.In part a), this is a hypothesis.In part b), we use that X_- is a caglad and part iii) in Proposition 1.28.As explained there $X_-(t)$ is bounded for $0 < t \le \sigma_k$(the same may not happen with X,altought is bounded for $0 < t < \sigma_k$).
As que assumed $X_-(0) = X(0) = 0$,there exist constans b_k such that $|X_-(t)| \le b_k$ for $0 \le t \le \sigma_k$.Consider the truncations $X^{(k)} = (X \wedge b_k) \vee (-b_k)$ and

$$Y_n^{(k)}(t) = \sum_{i=0}^{\infty} X^{(k)}(\tau_i^{(n)})\mathbb{1}_{(\tau_i^n,\tau_i^{n+1}]}(t) - X_-^{(k)}(t)$$

We have the equality

$$\mathbb{1}_{[0,\sigma_k]}Y_n(t) = \mathbb{1}_{[0,\sigma_k]}Y_n^{(k)}(t) \tag{1.32}$$

Because for each i,

$$\mathbb{1}_{[0,\sigma_k]}X^{(k)}(\tau_i^{(n)})\mathbb{1}_{(\tau_i^n,\tau_i^{n+1}]}(t) = \mathbb{1}_{[0,\sigma_k]}X(\tau_i^{(n)})\mathbb{1}_{(\tau_i^n,\tau_i^{n+1}]}(t)$$

and this last equality holds because both sides vanish unless $\tau_i^n < t \le \sigma_k$ and in that case, $|X(\tau_i)^n| \le b_k$ as $\tau_i^n < \sigma_k$.Thus $\{\sigma_k\}$ is a localizing sequence for (Y_n, M).On the event

$\{\sigma_k > T\}$, for $0 \leq t \leq T$, using the definition of the local stochastic integral and $)b,c)$ in Proposition 1.19,

$$(Y_n \cdot M)_t = ((\mathbb{1}_{[0,\sigma_k]} Y_n) \cdot M)_t \overset{(1.32)}{=} ((\mathbb{1}_{[0,\sigma_k]} Y_n^{(k)}) \cdot M)_t = ((Y_n^k) \cdot M^{\sigma_k})_t$$

Fix $\epsilon > 0$.We apply this inequality valid for a submartingale $\{M_t : t \in [0,T]\}$

$$P\left(\sup_{0\leq t\leq T} M_t \geq r\right) \leq \frac{1}{r} E[M_T^+] \tag{1.33}$$

to the cadlag submartingale $(Y_n^k \cdot M^{\sigma_k})^2$, isometry

$$E[(X \cdot B)_t^2] = E\int_0^t X_s^2 ds \quad \text{for all } t \geq 0 \tag{1.34}$$

to get

$$\begin{aligned}
P\left(\sup_{0\leq t\leq T} |(Y_n \cdot M)_t| \geq \epsilon\right) &\leq P(\sigma_k \leq T) + P\left(\sup_{0\leq t\leq T} |(Y_n^{(k)} \cdot M^{\sigma_k})_t| \geq \epsilon\right) \\
&= P\left(\sigma_k \leq T\right) + P\left(\sup_{0\leq t\leq T} |(Y_n^{(k)} \cdot M^{\sigma_k})_t|^2 \geq \epsilon^2\right) \\
&\leq P\left(\sigma_k \leq T\right) + \epsilon^{-2} E\left((Y_n^{(k)} \cdot M^{\sigma_k})_T^2\right) \\
&= P\left(\sigma_k \leq T\right) + \epsilon^{-2} \int_{[0,T]\times\Omega} |Y_n^{(k)}(t,\omega)|^2 \mu_{M^{\sigma_k}}(dt, d\omega)
\end{aligned}$$

Let $\epsilon_1 > 0$, and let k large enough so that $P(\sigma_k \leq T) < \epsilon_1$.As $n \to \infty, Y_n^k(t,\omega) \to 0$ for all t, if ω is such that the path $s \to X_-(s,\omega)$ is left-continuous and the assumption $\delta_n(\omega) \to 0$ holds.This excludes at most a zero probability set of $\omega's$, and so this convergence happens $\mu_{M^{\sigma_k}}-$almost everywhere. By the bound $|Y_n^{(k)}| \leq 2b_k$, by dominated convergence

$$\int_{[0,T]\times\Omega} |Y_n^{(k)}(t,\omega)|^2 \mu_{M^{\sigma_k}}(dt, d\omega) \to 0 \ as \ n \to \infty$$

Letting $n \to 0$ in the inequality above, we conclude that

$$\limsup_{n\to\infty} P\left(\sup_{0\leq t\leq T} |(Y_n \cdot M)_t| \geq \epsilon\right) \leq \epsilon_1$$

As $\epsilon_1 > 1$ is arbitrary, this limit must be equal to zero.

Now suppose that $\tilde{X}$ satisfies the hypotheses of the proposition, but $\tilde{X}$ is not identically zero.Then the convergence que have proved is valid for $X_t = \mathbb{1}_{(0,\infty)}(t)\tilde{X}_t$.As the value at $t = 0$ does not affect the stochastic integral, $X \cdot M = \tilde{X} \cdot M$.Let

$$\tilde{R}_n = \sum_{i=0}^{\infty} \tilde{X}(\tau_i^n)(M(\tau_{i+1}^n \wedge t) - M(\tau_i^n \wedge t))$$

The conclusion follows for $\tilde{X}$ if we can show that

$$\sup_{0\le t<\infty} |\tilde{R}_n(t) - R_n(t)| \to 0 \; as \; n \to \infty$$

Since $\tilde{R}_n(t) - R_n(t) = \tilde{X}(0)(M(\tau_1^n \wedge t) - M(0)$, we have the bound

$$\sup_{0\le t<\infty} |\tilde{R}_n(t) - R_n(t)| \le |\tilde{X}(0)| \sup_{0\le t\le \delta_n} |M(t) - M(0)|$$

The last quantify vanishes when $n \to \infty$ under the assumption that $\delta_n \to 0$ and the right continuity of M.This concludes the proof in the general case. □

1.3 Semimartingale integrator

Recall the semimartingale terminology. A cadlag semimartingale is a process Y that can be written as $Y_t = Y_0 + M_t + V_t$, where M is a cadlag local martingale,V is a cadlag FV process, and $M_0 = V_0 = 0$.To obtain a stochastic integral, we need that M is a local L^2-martingale. If we assume the filtration $\{\mathcal{F}_t\}$ is right-continuous and complete, then by corollary 1.28 in chapter 2, we can always select a decomposition so that M is a local $L^2 - martingale$.In this section we will work with a filtration that satisfies the usual conditions, unless one work with a semimartingale Y for which is known that M can be chosen a local L^2- martingale.Recall also that if g is a function of bounded variation on $[0, T]$, the Lebesgue-Stieltjes measure Λ_g of g exists as a signed Borel measure on $[0, T]$.
In this section the integrands will be predictable processes X that satisfy the condition

$$\begin{aligned}&\text{there exist stopping times } \{\sigma_n\} \text{ such that } \sigma_n \nearrow \infty \text{ almost surely}\\ &\text{and } \mathbb{1}_{(0,\sigma_n]}X \text{ is a bounded process for each } n.\end{aligned} \tag{1.35}$$

Observe that we ask the boundedness of $\mathbb{1}_{(0,\sigma_n]}X$ and not of $\mathbb{1}_{[0,\sigma_n]}X$, because X_0 might not be bounded.

Definition 1.34. *Let Y be a cadlag semimartingale, and X be a predictable process that satisfies (1.35).Then we define the integral of X with respect to Y as the process*

$$\int_{(0,t]} X_s dY_s = \int_{(0,t]} X_s dM_s + \int_{(0,t]} X_s \Lambda_V(ds) \tag{1.36}$$

Here $Y = Y_0 + M + V$ is some decomposition of Y into a local L^2- martingale M and an FV process V,

$$\int_{(0,t]} X_s dM_s = (X \cdot M)_t$$

is the stochastic integral defined in the previous section, and

$$\int_{(0,t]} X_s \Lambda_V(ds) = \int_{(0,t]} X_s dV s_s$$

is the path-by-path Lebesgue-Stieltjes integral of X with respect to the function $s \to V_s$.The process $\int XdY$ thus defined is unique up to indistinguishability and it is a semimartingale. We write $X \cdot Y$ and $\int XdY$ interchangeably.

Justification of the definition First we check that the integral does not depend on the representation of the semimartingale Y.Suppose $Y = Y_0 + \tilde{M} + \tilde{V}$ is another decomposition of Y with martingale $\tilde{M}$ and an FV process $\tilde{V}$.We need to show that

$$\int_{(0,t]} X_s dM_s + \int_{(0,t]} X_s \Lambda_V(ds) = \int_{(0,t]} X_s dM_s + \int_{(0,t]} \tilde{X}_s \Lambda_{\tilde{V}}(ds)$$

by Proposition 1.32 d) and the additivity of the Lebesgue-Stieltjes measures, this is equivalent to

$$\int_{(0,t]} X_s d(M - \tilde{M})_s = \int_{(0,t]} X_s \Lambda_{\tilde{V}-V}(ds) \tag{1.37}$$

From $Y = Y_0 + M + V = Y_0 + \tilde{M} + \tilde{V}$, we get $M - \tilde{M} = \tilde{V} - V$, and this process is both a local L^2- martingale and an FV process.The equality of (1.37) is consequence of the next proposition.

Proposition 1.35. *Suppose Z is a cadlag local L^2- martingale and an FV process.Let X be a predictable process that satisfies (1.35).Then for almost every ω,*

$$\int_{(0,t]} X(s,\omega) dZ_s(\omega) = \int_{(0,t]} X(s,\omega) \Lambda_{Z(\omega)}(ds) \ \forall \ 0 \leq t < \infty \tag{1.38}$$

On the left is the stochastic integral, on the right the Lebesgue-Stieltjes integral evaluated separately for each fixed ω.

Proof. Both sides of (1.38) are right- continuous in t, so in order to prove the indistinguishability of the two processes, it's enough t0 check that for each t they agree with probability 1.

Step 1 Suppose first that Z is an L^2- martingale.Fix $0 < t < \infty$.Let

$$\mathcal{H} := \{X : X \text{ is a bounded predictable process and (1.38)holds for } t\}.$$

By linearity of both integrals, $\mathcal{H}$ is a linear space. First note that if $X = \mathbb{1}_{(u,v]}\mathbb{1}_F, F \in \mathcal{F}_u$, then $X \in \mathcal{H}$,because the left side of (1.38) equals $\mathbb{1}_F(Z_{v\wedge t} - Z_{u\wedge t})$ by definition of the stochastic integral,and it's the same of the Lebesgue-Stieltjes integral. Thus $X \in \mathcal{H}$.If $X = \mathbb{1}_{\{0\}}\mathbb{1}_F, F \in \mathcal{F}_0$, both sides in (1.38) vanish; the left side by definition of the stochastic integral, and the left side because the integral is over $(0, t]$, hence does not include the time origin.

Let X be a bounded predictable process, $X_n \in \mathcal{H}, X_1 \leq X_2 \leq \cdots \leq X_n \leq \cdots$ and $X_n \to X$ pointwise on $\mathbb{R}_+ \times \Omega$.We know that we can choose X_n as sums of indicators of predictable rectangles, and this elements are in $\mathcal{H}$ by the argument above.If we can choose a subsequence $\{n_j\}$ such that both sides of

$$\int_{(0,t]} X_n(s,\omega)dZ_s(\omega) = \int_{(0,t]} X_n(s,\omega)\Lambda_{Z(\omega)}(ds) \tag{1.39}$$

converge for almost every ω to the corresponding integrals with X, then $X \in \mathcal{H}$, and this would conclude this case.

To check this, first we analyze the right hand side.For a fixed ω, the BV function $s \to Z_s(\omega)$on $[0, t]$ can be expressed as the difference $Z_s(\omega) = f(s) - g(s)$ of two nondecreasing functions. Hence the signed measure $\Lambda_{Z(\omega)}$ is the difference of two finite positive measures: $\Lambda_{Z(\omega)} = \Lambda_f - \Lambda_g$. Then by dominated convergence (we have $-C \leq X_1 \leq X_n \leq X \leq C$ for some constant C)

$$\begin{aligned}
\lim_{n\to\infty} \int_{(0,t]} X_n(s,\omega)\Lambda_{Z(\omega)}(ds) &= \lim_{n\to\infty} \int_{(0,t]} X_n(s,\omega)\Lambda_f(ds) - \lim_{n\to\infty} \int_{(0,t]} X_n(s,\omega)\Lambda_g(ds) \\
&= \int_{(0,t]} X(s,\omega)\Lambda_f(ds) - \int_{(0,t]} X(s,\omega)\Lambda_g(ds) \\
&= \int_{(0,t]} X(s,\omega)\Lambda_{Z(\omega)}(ds)
\end{aligned}$$

To obtain the convergence in the left side, we apply the isometry and dominated convergence.We have

$$\lim_{n\to\infty} \int_{[0,T]\times\Omega} |X - X_n|^2 d\mu_Z = 0$$

Hence $||X_n - X||_{\mathcal{L}_2} \to 0$, and so $||X_n \cdot Z - X \cdot Z||_{\mathcal{M}_2} \to 0$.Then for some fixed t,by lemma 1.49 in chapter 2, there exists some subsequence $\{n_j\}$ such that $(X_{n_j} \cdot Z) \to (X \cdot Z)_j$ almost surely.Taking the limit along $\{n_j\}$ on both sides of (1.39) gives

$$\int_{(0,t]} X(s,\omega)dZ_s(\omega) = \int_{(0,t]} X(s,\omega)\Lambda_{Z(\omega)}(ds)$$

This proves that $X \in \mathcal{H}$ for every bounded and predictable process.

Step 2 Now consider a local L^2- martingale Z.By the assumption on X we may pick a localizing sequence $\{\tau_k\}$ such that Z^{τ_k} is an L^2- martingale and $\mathbb{1}_{(0,\tau_k]}X$ is bounded. Then by Step 1,

$$\int_{(0,t]} \mathbb{1}_{(0,\tau_k]}(s)X(s)dZ_s^{\tau_k} = \int_{(0,t]} \mathbb{1}_{(0,\tau_k]}(s)X(s)\Lambda_{Z^{\tau_k}}(ds) \tag{1.40}$$

We claim that on the event $\{\tau_k \geq t\}$ the left and right sides of (1.40) coincide almost surely with the corresponding sides of (1.38).

The left hand side of (1.40) coincides almost surely with $((\mathbb{1}_{[0,\tau_k]}X) \cdot Z^{\tau_k})$ due to the irrelevance of the time origin, and this equals to $(X \cdot Z)_{t\wedge\tau_k} = (X \cdot Z)_t$ on the event $\{\tau_k \geq t\}$.

On the right hand side of (1.40), we only need to observe that if $\tau_k \geq t$, then on the interval $(0,t]$, $\mathbb{1}_{(0,\tau_k]}(s)X(s)$ coincides with $X(s)$ and $Z_s^{\tau_k}$ coincides with Z_s.Therefore the integrals in (1.38) and (1.40) coincide.

The union over k of the events $\{\tau_k \geq t\}$ equals almost surely to the whole space Ω.Thus we have verified (1.38) almost surely, for fixed t.

□

To finish the justification in the definition of the stochastic integral, we have already proved that the integral $\int XdY$ does not depend on the decomposition.To prove that $\int XdY$ is a semimartingale,observe that $X \cdot M$ is a local martingale, and for fixed ω,

$$t \to F(t) := \int_{(0,t]} X_s(\omega)\Lambda_{V(\omega)}(ds)$$

has bounded variation on every compact interval. To verify this, pick a partition $0 = s_0 <$

$s_1 < \cdots < s_n = t$,

$$\sum_i |F(s_{i+1}) - F(s_i)| = \sum_i \left| \int_{(s_i, s_{i+1}]} X \Lambda_{V(\omega)} \right| \leq \sum_i \int_{(s_i, s_{i+1}]} |X||\Lambda_{V(\omega)}|$$
$$\leq \int_{(0,t]} |X||\Lambda_{V(\omega)}| < \infty$$

Thus the right hand side of (1.36) is the semimartingale decomposition of $\int X dY$. $\square$

Now we state the analogous result of Riemann sums approximation for semimartingales.Before we state a similar result for Lebesgue-Stieltjes integrals.

Lemma 1.36. *Let ν be a finite signed measure on $(0, T]$.Let f be a bounded Borel function on $[0, T]$ for which the left limit $f(t-)$ exists at all $0 < t \leq T$.Let $\pi^n = \{0 = s_1^n < \cdots < s_{m(n)}^n = T\}$ be partitions of $[0, T]$ for which $mesh(\pi^n) \to 0$. Then*

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{m(n)-1} f(s_i^n) \nu(s_i^n \wedge t, s_{i+1}^n \wedge t] - \int_{(0,t]} f(s-) \nu(ds) \right| = 0$$

In particular, for a right-continuous function $G \in BV[0, T]$,

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} \left| \sum_{i=0}^{m(n)-1} f(s_i^n) G(s_i^n \wedge t, s_{i+1}^n \wedge t] - \int_{(0,t]} f(s-) dG(s) \right| = 0$$

Proof. For each $0 \leq t \leq T$,

$$\left| \sum_{i=0}^{m(n)-1} f(s_i^n) \nu(s_i^n \wedge t, s_{i+1}^n \wedge t] - \int_{(0,t]} f(s-) \nu(ds) \right|$$
$$\leq \int_{(0,t]} \left| \sum_{i=0}^{m(n)-1} f(s_i^n) f(s_i^n) \mathbb{1}_{(s_i^n, s_{i+1}^n]}(s) - f(s-) \right| |\nu|(ds)$$
$$\leq \int_{(0,T]} \left| \sum_{i=0}^{m(n)-1} f(s_i^n) f(s_i^n) \mathbb{1}_{(s_i^n, s_{i+1}^n]}(s) - f(s-) \right| |\nu|(ds)$$

This last expression is uniform in $t \in [0, t]$, and vanishes when $n \to \infty$ by dominated convergence. $\square$

Proposition 1.37. *Let X be an adapted process and Y a cadlag semimartingale.Suppose $0 = \tau_0^n \leq \tau_1^n \leq \cdots$ are stopping times such that for each $n, \tau_i^n \to \infty$ almost surely as*

$i \to \infty$,and $\delta_n = \sup_{0\le i<\infty}(\tau^n_{i+1} - \tau^n_i)$ tends to zero almost surely as $n \to \infty$.Define

$$S_n(t) := \sum_{i=0}^{\infty} X(\tau^n_i)(Y(\tau^n_{i+1} \wedge t) - Y(\tau^n_i \wedge t)) \tag{1.41}$$

a) Assume X is left-continuous and satisfies (1.35).Then for each fixed $T < \infty$ and $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |S_n(t) - (X \cdot Y)_t| \ge 0\right) = 0$$

b) If X is an adapted cadlag process,then

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le T} |S_n(t) - (X_- \cdot Y)_t| \ge 0\right) = 0$$

Proof. Pick a decomposition $Y = Y_0 + M + V$.We get the corresponding decomposition $S_n = U_n + R_n$ by defining

$$R_n(t) := \sum_{i=0}^{\infty} X(\tau^n_i)(M_{\tau^n_{i+1}} - M^n_{\tau_i})$$

and

$$U_n(t) := \sum_{i=0}^{\infty} X(\tau^n_i)(V_{\tau^n_{i+1}} - V^n_{\tau_i})$$

By Proposition 1.33,$R_n(t) \to X_- \cdot M$.Now apply Lemma 1.36 to the Lebesgue-Stieltjes measure of $V_t(\omega)$, to deduce for almost every ω,

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| U_n(t,\omega) - \int_{(0,t]} X(s-,\omega)\Lambda_{V(\omega)}(ds) \right| = 0$$

As almost surely convergence implies convergence in probability, we are done. □

Remark 1.38. ***Matrix-valued integrands and vector-valued integrators.*** *In order to consider equations for vector-valued processes, we need to establish some conventions regarding the integrals of matrix-valued processes with vector-valued integrators. Let a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$ be given. If $Q_{i,j}(t), 1 \le i \le d, 1 \le j \le m$ are predictable processes on this space, then we regard $Q(t) = (Q_{i,j}(t))$ as a $d \times m-$ matrix valued predictable process.If $Y_1, \cdots Y_m$ are semimartingales on this space, then $Y = (Y_1, \cdots, Y_m)^T$ is an $\mathbb{R}^m-$ valued semimartingale. The stochastic integral $Q \cdot Y = \int QdY$ is the $\mathbb{R}^d-$ valued process whose ith component is*

$$(Q \cdot Y)_t = \sum_{j=1}^{m} \int_{(0,t]} Q_{i,j}(s) dY_j(s) \tag{1.42}$$

assuming that all the components are well defined.

1.4 Further properties of stochastic integrals

We state the analog of (1.32) for the semimartingale integrator.

Proposition 1.39. *Let Y and Z be semimartingales, G and H predicable processes that satisfy* (1.35)*, and let τ be a stopping time.*

a) Let U be a bounded $\mathcal{F}_\tau$-measurable random variable. Then

$$\int_{(0,t]} U\mathbb{1}_{(\tau,\infty)} G dY = U \int_{(0,t]} \mathbb{1}_{(\tau,\infty)} G dY \tag{1.43}$$

Furthermore,

$$((\mathbb{1}_{[0,\tau]}G)\cdot Y)_t = (G\cdot Y)_{\tau\wedge t} = (G\cdot Y^\tau)_t \tag{1.44}$$

b) Suppose $G_t(\omega) = H_t(\omega)$ and $Y_t(\omega) = Z_t(\omega)$ for $0 \le t \le \tau(\omega)$. Then

$$(G\cdot Y)_{\tau\wedge t} = (H\cdot Z)_{\tau\wedge t}$$

Proof. Let $Y = Y_0 + M + V$ a semimartingale decomposition for Y.

a) We write

$$\int_{(0,t]} U\mathbb{1}_{(\tau,\infty)} G dY = \int_{(0,t]} U\mathbb{1}_{(\tau,\infty)} G dM + \int_{(0,t]} U\mathbb{1}_{(\tau,\infty)} \Lambda_{V(\omega)}(ds)$$
$$\overset{(1.29)}{=} U\int_{(0,t]} \mathbb{1}_{(\tau,\infty)} G dM + U\int_{(0,t]} \mathbb{1}_{(\tau,\infty)} G\Lambda_{V(\omega)}(ds) = U\int_{(0,t]} \mathbb{1}_{(\tau,\infty)} G dY$$

As the second integral doest not depend on ω, we can put U outside this Lebesgue-Stieltjes integral.

Again, by definition and (1.30)

$$\begin{aligned}((\mathbb{1}_{[0,\tau]}G)\cdot Y)_t &= ((\mathbb{1}_{[0,\tau]}G)\cdot M)_t + \int_{(0,t]} \mathbb{1}_{[0,\tau]} G\Lambda_{V(\omega)}(ds)\\ &= (G\cdot M)_{t\wedge\tau} + \int_{(0,t\wedge\tau]} G\Lambda_{V(\omega)}(ds)\\ &= (G\cdot Y)_{t\wedge\tau}\end{aligned}$$

This gives us the first equality in (1.43),and the second is similar.

b) Suppose $Y = Y_0 + M + V, Z = Z_0 + N + W$.In particular, $Y_0 = Z_0$ so $M - N = W - V$, and this process is both a local L^2-martingale and an FV process.By Proposition 1.35,

$$\int_{(0,t]} Gd(M-N) = \int_{(0,t]} G\Lambda_{W-V(\omega)}(ds)$$

This implies that

$$(G \cdot M)_t + \int_{(0,t]} G\Lambda_{V(\omega)}(ds) = (G \cdot N)_t + \int_{(0,t]} G\Lambda_{W(\omega)}(ds)$$

In particular,

$$\begin{aligned}
(G \cdot Y)_{\tau\wedge t} &= (G \cdot M)_{\tau\wedge t} + \int_{(0,t\wedge t]} G\Lambda_{V(\omega)}(ds) \\
&= (G \cdot N)_{\tau\wedge t} + \int_{(0,t\wedge t]} G\Lambda_{W(\omega)}(ds) \\
&= (H \cdot N)_{\tau\wedge t} + \int_{(0,t\wedge t]} H\Lambda_{W(\omega)}(ds) \\
&= (H \cdot Z)_{t\wedge t}
\end{aligned}$$

□

1.4.1 Jumps of the stochastic integral

For any cadlag process Z, $\triangle Z(t) := Z(t) - Z(t-)$ denotes the jump at t.

Lemma 1.40. *Let Y be a semimartingale. Then the quadratic variation $[Y]$ exists. For almost every ω, $\triangle[Y]_t = (\triangle Y_t)^2$.*

Proof. Fix $0 < T < \infty$.By Proposition 1.37, we can pick a sequence of partitions $\pi^n = \{t_i^n\}$ of $[0,T]$ such that the process

$$S_n(t) = 2\sum_i Y_{t_i^n}(Y_{t_{i+1}^n\wedge t} - Y_{t_i^n} \wedge t)$$

To the process $S(t) = 2\int_{(0,t]} Y(s-)dY(s)$ uniformly for $t \in [0,T]$, for almost every ω.Note that

$$Y_t^2 - Y_0^2 - \sum_i (Y_{t_{i+1}^n \wedge t} - Y_{t_i}^n \wedge t)^2 = \sum_i Y^2_{t_{i+1}^n \wedge t} - \sum_i Y^2_{t_i^n \wedge t} - \sum_i (Y_{t_{i+1}^n \wedge t} - Y_{t_i}^n \wedge t)^2$$
$$= 2\sum_i Y_{t_{i+1}^n \wedge t} Y_{t_i^n \wedge t} - 2\sum_i Y^2_{t_i^n \wedge t} = 2\sum_i Y_{t_i^n \wedge t}(Y_{t_{i+1}^n \wedge t} - Y^n_{t_i \wedge t})$$
$$= 2\sum_i Y_{t_i^n}(Y_{t_{i+1}^n \wedge t} - Y^n_{t_i \wedge t}) = S_n(t)$$

Therefore the quadratic variation converges to $S(t) + Y_t^2 - Y_0^2$.

We use the following general fact:

If f, g are cadlag and $|f(x) - g(x)| \leq \epsilon$ for all x, then

$$|\triangle f(x) - \triangle g(x)| = \lim_{y \nearrow x, y < x} |f(x) - f(y) + g(x) - g(y)| \leq 2\epsilon$$

Fix ω for which the uniform convergence $S_n \to S$ holds. Then by the fact, for each $t \in (0, T]$ the jumps $\triangle S_n(t) \to \triangle S(t) = \triangle (Y^2)_t - \triangle [Y]_t$.Now we compute $\triangle S_n(t)$.If Let k the index such that $t \in (t_k^n, t_{k+1}^n]$.If $s < t$ is close enough, also $s \in (t_k^n, t_{k+1}^n]$, and then

$$\triangle S_n(t) = \lim_{s \nearrow t, s<t} (S_n(t) - S_n(s) = \lim_{s \nearrow t, s<t} \left(2\sum_i Y_{t_i^n}(Y_{t_{i+1}^n \wedge t} - Y_{t_i^n \wedge t}) - 2\sum_i Y_{t_i^n}(Y_{t_{i+1}^n \wedge s} - Y_{t_i^n} \wedge s) \right)$$
$$\lim_{s \nearrow t, s<t} (2Y_{t_k^n}(Y_t - Y_{t_k^n}) - 2Y_{t_k^n}(Y_s - Y_{t_k^n})) = 2 \lim_{s \nearrow t, s<t} Y_{t_k^n}(Y_t - Y_s) = 2Y_{t_k^n} \triangle Y_t$$

By the cadlag property, we conclude that $\triangle S_n(t) \to 2Y_{t-} \triangle Y_t = \triangle (Y^2)_t - \triangle [Y]_t$.We deduce that $\triangle [Y]_t = \triangle (Y^2)_t - 2Y_{t-} \triangle Y_t$.But

$$\triangle (Y^2)_t - 2Y_{t-} \triangle Y_t = \lim_{s \nearrow t, s<t} Y_t^2 - Y_s^2 - 2Y_s(Y_t - Y_s)$$
$$= \lim_{s \nearrow t, s<t} (Y_t - Y_s)^2 = (\triangle Y_t)^2$$

Thus $\triangle [Y]_t = (\triangle Y_t)^2$. $\square$

To prove the next theorem, we need two lemmas.

Lemma 1.41. *Let M be a cadlag local L^2- martingale and $X \in \mathcal{L}(M, \mathcal{P})$.Then for all ω in a set of probability one, $\triangle (X \cdot M)(t) = X(t) \triangle M(t)$ for all $0 < t < \infty$.*

Proof. First suppose $M \in \mathcal{M}_2, X \in \mathcal{L}_2(M, \mathcal{P})$.Pick a sequence of simple predictable processes

$$X_n(t) = \sum_{i=1}^{m(n)-1} \xi_i^n \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t)$$

such that $X_n \to X$ in $\mathcal{L}_2(M, \mathcal{P})$.By the last lemma, the processes $\triangle[M]_t$ and $(\triangle M_t)^2$ are indistinguishable. Then

$$E\left[\sum_{s\in(0,T]} |X_n(s)\triangle M_s - X(s)\triangle M_s|^2\right] = E\left[\sum_{s\in(0,T]} |X_n(s) - X(s)|^2 \triangle[M]_s\right]$$
$$\leq E\left[\int_{(0,T]} |X_n(s) - X(s)|^2 d[M]_s\right]$$

in the last inequality we used the definition of the Lebesgue-Stieltjes integral.By hypothesis this expectation vanishes when $n \to \infty$.Convergence in $L^1(P)$ implies almost surely convergence of a subsequence, so if $\{n_j\}$ is such subsequence,

$$\lim_{j\to\infty} \sum_{s\in(0,T]} |X_{n_j}(s)\triangle M_s - X(s)\triangle M_s|^2 = 0$$

In particular, on this event of full probability, for any $t \in (0, T]$

$$\lim_{j\to\infty} |X_{n_j}(t)\triangle M_t - X(t)\triangle M_t|^2 = 0 \tag{1.45}$$

On the other hand, $X_{n_j} \cdot M \to X \cdot M$ implies that along a further subsequence (which we denote also by n_j),almost surely,

$$\lim_{j\to\infty} \sup_{t\in[0,T]} |(X_{n_j} \cdot M)_t - (X \cdot M)_t| = 0 \tag{1.46}$$

Fix an ω such that both (1.44),(1.45) hold. For any $t \in (0, T]$, he uniform convergence in (1.45) implies the uniform convergence of the jumps(see the proof in last lemma) ,so for any $t \in (0, T], \triangle(X_{n_j} \cdot M)_t \to \triangle(X \cdot M)_t$.Also, since a path of $X_{n_j} \cdot M$ is a step function, $\triangle(X_{n_j} \cdot M)_t = X_{n_j(t)}\triangle M_t$(this also is proved in the last lemma).Combining these two observations with the limit in (1.44) we obtain for this fixed ω and $t \in (0, T]$

$$\triangle(X \cdot M)_t = \lim_{j\to\infty} \triangle(X_{n_j} \cdot M)_t = \lim_{j\to\infty} X_{n_j}\triangle M_t = X(t)\triangle M_t$$

Finally, consider countable many T that increase up to ∞.This proves the case $M \in \mathcal{M}_2, X \in \mathcal{L}_2(M, \mathcal{P})$.

Now in the general case, pick a sequence $\{\sigma_k\}$ that localizes (X, M), and let $X_k = \mathbb{1}_{[0,\sigma_k]}X$.Pick ω such that the definition of the integral $X \cdot M$ works and the conclusion above holds for each integral $X_k \cdot M^{\sigma_k}$.Then if $\sigma_k > t$,

$$\triangle(X \cdot M)_t = \triangle(X_k \cdot M^{\sigma_k})_t = X_k(t)\triangle M_t^{\sigma_k} = X(t)\triangle M_t$$

$\square$

Lemma 1.42. *Let f be a bounded Borel function and U a BV function on $[0,T]$.Denote the Lebesgue-Stieltjes integral by*

$$(f \cdot U)_t = \int_{(0,t]} f(s)dU(s)$$

Then $\triangle(f \cdot U)_t = f(t)\triangle U(t)$ for all $0 < t \leq T$.

Proof. By the rules concerning Lebesgue-Stieltjes integration,

$$(f \cdot U)_t - (f \cdot U)_s = \int_{(s,t]} f(r)dU(r) = \int_{(s,t)} f(r)dU(r) + f(t)\triangle U(t)$$

and

$$\left| \int_{(s,t]} f(s)dU(s) \right| \leq ||f||_\infty \Lambda_{V_U}(s,t)$$

where V_U is the total variation function of U.As $s \nearrow t, \Lambda_{V_U}(s,t) \to 0$.That proves our claim. $\square$

The next theorem follows directly from the last two lemmas.

Theorem 1.43. *Let M be a cadlag local L^2- martingale and $X \in \mathcal{L}(M, \mathcal{P})$.Then for all ω in a set of probability one, $\triangle(X \cdot M)(t) = X(t)\triangle M(t)$ for all $0 < t < \infty$.*

We introduce the following notation from the left limit of a stochastic integral:

$$\int_{(0,t)} HdY = \lim_{s \nearrow t, s<t} \int_{(s,t]} HdY \tag{1.47}$$

We have the following identity :

$$\int_{(0,t]} HdY = \int_{(0,t)} HdY + H(t)\triangle Y(t). \tag{1.48}$$

This follows from the cadlag property and the last theorem:

$$(H \cdot Y)_t - \lim_{s \nearrow t, s<t} (H \cdot Y)_s = \lim_{s \nearrow t, s<t} [(H \cdot Y)_t - (H \cdot Y)_s] = \triangle(H \cdot Y)_s = H(t)\triangle Y(t)$$

1.4.2 Convergence theorem for stochastic integrals.

We state a sort of dominated convergence for stochastic integrals.This will be used to prove the existence and uniqueness theorem for stochastic differential equations.

Theorem 1.44. *Let $\{H_n\}$ be a sequence of predictable processes and $\{G_n\}$ a sequence of nonnegative adapted cadlag processes.Assume $|H_n(t)| \leq G_n(t-)$ for all $0 < t < \infty$, and the running maximum*

$$G_n^*(T) = \sup_{0 \leq t \leq T} G_n(t)$$

converges to zero in probability , for each fixed $0 < T < \infty$.Then for any cadlag semimartingale Y,$H_n \cdot Y \to 0$ in probabibility, uniformly on compact time intervals.

Proof. Let $Y = Y_0 + M + U$ be a decomposition of Y into a local L^2- martingale, M and a FV process U. We show that both terms in $H_n \cdot Y = H_n \cdot M + H_n \cdot U$ converge to zero in probability, uniformly on compact time intervals. The FV part convergence is derived from

$$\sup_{0 \leq t \leq T} \left| \int_{(0,t)} H_n(s) dU(s) \right| \leq \sup_{0 \leq t \leq T} |H_n(s)| dV_U(s) \leq G_n^*(T) V_U(T)$$

Here V_U is the total variation measure of U.Since $V_U(T)$ is finite, the last bound converges to zero in probability.

Now we study the convergence of the stochastic integral.Pick a sequence of stopping times $\{\sigma_k\}$ that localizes M.Let

$$\rho_n := \inf\{t \geq 0 : G_n(t) \geq 1\} \wedge \inf\{t > 0 : G_n(t-) \geq 1\}.$$

By lemma 1.20 in chapter 1,these are stopping times.By left-continuity, $G_n(t-) \leq 1$ for $0 < t \leq \rho_n$.For any $T < \infty$,

$$P(\rho_n \leq T) \leq P(G_n^*(T) > 1/2) \to 0 \textit{ as } n \to \infty$$

Let $H_n^{(1)} = (H_n \wedge 1) \vee (-1)$ the truncation.If $t \leq \sigma_k \wedge \rho_n$, then $H_n \cdot M = H_n^{(1)} \cdot M^{\sigma_k}$ by part c) of Proposition 1.32.Because $H_n^{(1)}$ is a bounded process, this belongs to $\mathcal{L}_2(M^{\sigma_k}, \mathcal{P})$.Now we apply (1.33) to the submartingale $(H_n^{(1)} \cdot M^{\sigma_k})^2$ and the isometry of the stochastic integral to obtain

$$
\begin{aligned}
P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon\right) &= P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon, T\le\sigma_k\wedge\rho_n\right)+P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon,\sigma_k\wedge T\ge\rho_n\right)\\
&\le P(T\ge\sigma_k\wedge\rho_n)+P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon, T\le\sigma_k\wedge\rho_n\right)\\
&\le P(T\ge\sigma_k)+P(T\ge\rho_n)+P\left(\sup_{0\le t\le T}|H_n^{(1)}\cdot M^{\sigma_k}|_t\ge\epsilon\right)\\
&\le P(T\ge\sigma_k)+P(T\ge\rho_n)+\epsilon^{-2}E[(H_n^{(1)}\cdot M^{\sigma_k})_T^2]\\
&= P(T\ge\sigma_k)+P(T\ge\rho_n)+e^{-2}E\int_{[0,T]}|H_n^{(1)}|^2d[M^{\sigma_k}]_t\\
&\le P(T\ge\sigma_k)+P(T\ge\rho_n)+\epsilon^{-2}E\int_{[0,T]}(G_n(t-)\wedge 1)^2d[M^{\sigma_k}]_t\\
&\le P(T\ge\sigma_k)+P(T\ge\rho_n)+\epsilon^{-2}E\left([M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2\right)
\end{aligned}
$$

To prove that the expectations converges to zero when $n\to\infty$, we use the dominated convergence theorem under convergence in probabiltiy(that is, we change the almost surely convergence by probability convergence to get the same result).Observe that $|M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2|\le[M^{\sigma_k}]_T$, and this is an integrable random variable.Given $\delta>0$, pick $K>\delta$ so that

$$P([M^{\sigma_k}]_T\ge K)<\delta/2$$

Then

$$
\begin{aligned}
P\left([M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2\ge\delta\right) &= P\left([M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2\ge\delta,[M^{\sigma_k}]_T\ge K\right)\\
&\quad+P\left([M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2\ge\delta,[M^{\sigma_k}]_T\le K\right)\\
&\le P([M^{\sigma_k}]_T\ge K)+P(G_n^*(T)\ge\sqrt{\delta/K})\\
&\le \delta/2+P(G_n^*(T)\ge\sqrt{\delta/K})
\end{aligned}
$$

The last probability converges to zero when $n\to\infty$ by the convergence in probability of G_n^*.So we have the convergence in probability of $[M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2$.When we take $n\to\infty$ in

$$P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon\right)\le P(T\ge\sigma_k)+P(T\ge\rho_n)+\epsilon^{-2}E\left([M^{\sigma_k}]_T(G_n^*(T)\wedge 1)^2\right)$$

The result is

$$\limsup_{n\to\infty}P\left(\sup_{0\le t\le T}|H_n\cdot M|_t\ge\epsilon\right)\le P(\tau_k\le T)$$

and this converges to zero when $k \to \infty$.This proves that $H_n \cdot M$ converges to zero in probability, uniformly on $[0, T]$. $\square$

1.4.3 Restarting at a stopping time.

Let Y be a cadlag semimartingale and G a predictable process that satisfies the local boundedness condition (1.35).Let σ be a bounded stopping time with respect to the underlying filtration $\{\mathcal{F}_t\}$.We ask for a bounded stopping time in order to use lemma 1.9 in chapter 2.Define the new filtration $\overline{\mathcal{F}}_t := \mathcal{F}_{t+\sigma}$, and let $\overline{\mathcal{P}}$ be the predictable $\sigma-$algebra under the filtration $\{\overline{\mathcal{F}_t}\}$.In other words, $\overline{\mathcal{P}}$ is the $\sigma-$algebra generated by the sets of the type $(u, v] \times \Gamma, \Gamma \in \overline{\mathcal{F}_u}$ and $\{0\} \times \Gamma_0, \Gamma_0 \in \overline{\mathcal{F}_0}$.
Define the new processes

$$\overline{Y}(t) := Y(\sigma + t) - Y(\sigma) \;\text{ and }\; \overline{G}(t) := G(\sigma + s)$$

We subtract $Y(\sigma)$ in order to have initial value zero. It could also be defined as $\overline{Y}(t) = Y(\sigma + t)$, because only the increments matter to the integrator.

Theorem 1.45. *Let σ be a bounded stopping time with respect to $\{\mathcal{F}_t\}$.Under the filtration $\{\overline{\mathcal{F}_t}\}$,the process $\overline{G}(t)$ is predictable and $\overline{Y}$ is an adapted semimartingale. We have this equality of stochastic integrals :*

$$\begin{aligned}\int_{(0,t]} \overline{G}(s) d\overline{Y}(s) &= \int_{(\sigma,\sigma+t]} G(s) dY(s) \\ &:= \int_{(0,\sigma+t]} G(s) dY(s) - \int_{(0,\sigma]} G(s) dY(s)\end{aligned} \tag{1.49}$$

The proof comes after two lemmas.

Lemma 1.46. *For any $\mathcal{P}$-measurable function G, $\overline{G}(t, \omega) = G(\sigma(\omega)+t, \omega)$ is $\overline{\mathcal{P}}-$ measurable.*

Proof. Let $\mathcal{U}$ be the space of $\mathcal{P}$-measurable functions G such that $\overline{G}$ is $\mathcal{P}$-measurable. $\mathcal{U}$ is a linear space and closed under pointwise limits since these operations preserve measurability.Since any $\mathcal{P}$-measurable function is a pointwise limit of bounded $\mathcal{P}$- measurable functions, it suffices to show that $\mathcal{U}$ contains all bounded $\mathcal{P}$-measurable functions.In fact, it's only necessary to check that $\mathcal{U}$ contains the indicators of predictable rectangles.Suppose

$\Gamma \in \mathcal{F}_u, G = \mathbb{1}_{(u,v]} \times \Gamma$.Then

$$\overline{G}(t) = \mathbb{1}_{(u,v]}(\sigma + t)\mathbb{1}_{\Gamma}(\omega) = \begin{cases} \mathbb{1}_{\Gamma}(\omega), & u < \sigma + t \leq v \\ 0, & \text{otherwise.} \end{cases}$$

For a fixed $\omega, \overline{G}(t)$ is a caglad process.By Lemma 1.3, $\mathcal{P}$-measurability of $\overline{G}$ follows if it is adapted to $\{\overline{\mathcal{F}}_t\}$.Since $\{\overline{G}(t) = 1\} = \Gamma \cap \{u < \sigma + t \leq v\} \in \overline{\mathcal{F}_t}$ by part i) in lemma 1.11 from chapter 1,because $\Gamma \cap \{u < \sigma + t\} \in \mathcal{F}_{\sigma+t}$ and $\Gamma \cap \{\sigma + t \leq v\} = \Gamma \cap \{v < \sigma + t\}^c \in \mathcal{F}_{\sigma+t}$. Now if $G = \mathbb{1}_{\{0\}} \times \Gamma, \Gamma \in \mathcal{F}_0$, then

$$\overline{G}(t) = \mathbb{1}_{\{0\}}(\sigma + t)\mathbb{1}_{\Gamma}(\omega) = \begin{cases} \mathbb{1}_{\Gamma}(\omega), & \sigma + t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

As $\sigma + t = 0$ iff $\sigma = t = 0$, we obtain $\overline{G}(t) = \mathbb{1}_{\{0\}}\mathbb{1}_{\{\sigma=0\}}\mathbb{1}_{\Gamma}(\omega)$.As again, for fixed $\omega, \overline{G}(t)$ is a caglad process.Also $\{\overline{G}(t) = 1\} = \{t = 0\} \cap \{\sigma = 0\} \cap \Gamma \in \mathcal{F}_0 \subset \mathcal{F}_\sigma$, and this gives the adaptedness. □

Lemma 1.47. *Let M be a local L^2-martingale with respect to $\{\mathcal{F}_t\}$.Then $\overline{M}_t = M_{\sigma+t} - M_\sigma$is a local L^2- martingale with respect to $\{\overline{\mathcal{F}_t}\}$.If $G \in \mathcal{L}(M, \mathcal{P})$, then $\overline{G} \in \mathcal{L}(\overline{M}, \mathcal{P})$, and*

$$(\overline{G} \cdot \overline{M})_t = (G \cdot M)_{\sigma+t} - (G \cdot M)_\sigma$$

Proof. Let $\{\tau_k\}$ localize (G, M).Let $\nu_k = (\tau_k - \sigma)^+$.For any $0 \leq t < \infty$,

$$\{\nu_k \leq t\} = \{\tau_k \leq \sigma + t\} \in \mathcal{F}_{\sigma+t} = \overline{\mathcal{F}_t} \text{ by ii) in lemma 1.11 in chapter 1}$$

This says that ν_k is a stopping time for the filtration $\{\overline{\mathcal{F}_t}\}$.If $\sigma \leq \tau_k$, then $\nu_k = \tau_k - \sigma$.Note that $(\nu_k \wedge t) = t$ iff $\nu_k \geq t$ iff $\tau_k \geq \sigma + t$ iff $\tau_k \wedge (\sigma + t) = \sigma + t$.Therefore $\sigma + (t \wedge \nu_k) = (\sigma + t) \wedge \tau_k$. Using this,

$$\overline{M}_t^{\nu_k} = \overline{M}_{t \wedge \nu_k} = M_{\sigma + (t \wedge \nu_k)} - M_\sigma = M_{(\sigma+t)\wedge\tau_k} - M_{\sigma\wedge\tau_k} = M_{\sigma+t}^{\tau_k} - M_\sigma^{\tau_k}$$

If $\sigma > \tau_k$, then $\nu_k = 0$ and $\overline{M}_t^{\nu_k} = \overline{M}_0 = M_\sigma - M_\sigma = 0$.In the case $\sigma > \tau_k, M_{\sigma+t}^{\tau_k} - M_\sigma^{\tau_k} = 0$, so in both cases we have

$$\overline{M}_t^{\nu_k} = M_{\sigma+t}^{\tau_k} - M_\sigma^{\tau_k} \tag{1.50}$$

The right-hand side in the last equation is an L^2- martingale with respect to $\overline{\mathcal{F}_t}$ by $ii)$ in remark 1.14 from chapter 2.Consequently, $\overline{M}^{\nu_k}$ is an L^2- martingale with respect to $\{\overline{\mathcal{F}}_t\}$, and hence $\overline{M}$ is a local L^2- martingale.

Next we show that $\{\nu_k\}$ localizes $(\overline{G}, \overline{M})$.We fix k temporarily and write $Z = M^{\tau_k}$.Then (1.50) shows that

$$\overline{Z}_t = Z_{\sigma+t} - Z_\sigma$$

Let $\Gamma \in \mathcal{F}_u$, then $\mathbb{1}_{(u,v]\times\Gamma}$ is $\mathcal{P}$-measurable, while $\mathbb{1}_{(u,v]\times\Gamma}(\sigma+t,\omega)$ is $\overline{\mathcal{P}}$-measurable.In the following calculation, we will use the following facts: $(u-\sigma)^+$ is a stopping time for $\{\overline{\mathcal{F}_t}\}$, and $\Gamma \in \overline{\mathcal{F}}_{(u-\sigma)^+}$.As $\overline{Z}^2 - [\overline{Z}]$ is a martingale, then $E[\mathbb{1}_\Gamma \overline{Z}^2_{(u-\sigma)^+}] = E[\mathbb{1}_\Gamma[\overline{Z}]_{(u-\sigma)^+}]$ by optional stopping.Also note that $\sigma + (u-\sigma)^+ = \sigma \wedge u$ and the martingale property is also justified under $\sigma + (u-\sigma)^+$ because $\Gamma \in \mathcal{F}_{\sigma\wedge u}$.Finally, we use that $(\sigma\wedge u, \sigma\wedge v] = (\sigma,\infty)\cap(u,v]$.Now we make the calculation

$$\begin{aligned}
\int \mathbb{1}_\Gamma(\omega)\mathbb{1}_{(u,v]}(\sigma+t)\mu_{\overline{Z}}(dt,d\omega) &= E\left[\mathbb{1}_\Gamma(\omega)\left([\overline{Z}]_{(v-\sigma)^+} - [\overline{Z}]_{(u-\sigma)^+}\right)\right] \\
&= E\left[\mathbb{1}_\Gamma(\omega)\left(\overline{Z}_{(v-\sigma)^+} - \overline{Z}_{(u-\sigma)^+}\right)^2\right] \\
&= E\left[\mathbb{1}_\Gamma(\omega)\left(Z_{\sigma+(v-\sigma)^+} - Z_{\sigma+(u-\sigma)^+}\right)^2\right] \\
&= E\left[\mathbb{1}_\Gamma(\omega)\left([Z]_{\sigma\wedge v} - [Z]_{\sigma\wedge u}\right)\right] \\
&= E\left[\mathbb{1}_\Gamma(\omega)\int \mathbb{1}_{(\sigma(\omega),\infty)}(t)\mathbb{1}_{(u,v]}(t)d[Z]_t\right] \\
&= \int \mathbb{1}_\Gamma(\omega)\mathbb{1}_{(\sigma(\omega),\infty)}(t)\mathbb{1}_{(u,v]}(t)d\mu_Z
\end{aligned}$$

We extend this identity to nonnegative predictable processes X to give, with $\overline{X}_t = X(\sigma+t)$,

$$\int \overline{X}d\mu_Z = \int \mathbb{1}_{(\sigma,\infty)}Xd\mu_Z \tag{1.51}$$

Let $T<\infty$ and apply this to $X = \mathbb{1}_{(\sigma,\sigma+T]}(t)\mathbb{1}_{(0,\tau_k]}(t)|G(t)|^2$. Then

$$\overline{X}(t) = \mathbb{1}_{[0,T]}(t)\mathbb{1}_{(0,\nu_k]}(t)|\overline{G}(t)|^2$$

and

$$\int_{[0,T]\times\Omega}\mathbb{1}_{(0,\nu_k]}(t)|\overline{G}(t)|^2d\mu_{\overline{Z}} = \int \mathbb{1}_{(\sigma,\sigma+T]}(t)\mathbb{1}_{(0,\tau_k]}(t)|G(t)|^2d\mu_Z < \infty$$

in the last inequality we used that $\{\tau_k\}$ localizes (G,M) and σ is bounded.This proves that $\{\tau_k\}$ localizes $(\overline{G},\overline{M})$.This checks $\overline{G}\in\mathcal{L}(\overline{M},\overline{\mathcal{P}})$.

To prove the last claim, fix again k, and as before, $Z = M^{\tau_k}, \overline{Z} = \overline{M}^{\nu_k}$.First consider the case where H_n is a simple $\mathcal{P}$-predictable process

$$H_n(t) = \sum_{i=0}^{m-1} \xi_i \mathbb{1}_{(u_i, u_{i+1}]}(t).$$

Let k denote the index that satisfies $u_{k+1} > \sigma \geq u_k$(if there is no such k ,then $\overline{H}_t = 0$).In that case

$$\overline{H}_n(t) = \sum_{i \geq k} \xi_i \mathbb{1}_{(u_i - \sigma, u_{i+1} - \sigma]}(t)$$

In the next computation we will use

$$\begin{cases} \sigma + (u_i - \sigma) \wedge t = (\sigma + t) \wedge u_i & for\ i > k \\ (\sigma + t) \wedge u_i = \sigma \wedge u_i = u_i & for\ i \leq k \end{cases} \tag{1.52}$$

The stochastic integral is

$$\begin{aligned}
(\overline{H}_n \cdot \overline{Z})_t &= \sum_{i \geq k} \xi_i (\overline{Z}_{t \wedge (u_{i+1} - \sigma)} - \overline{Z}_{(u_i - \sigma)^+ \wedge t}) \\
&= \sum_{i > k} \xi_i (Z_{\sigma + t \wedge (u_{i+1} - \sigma)} - Z_{\sigma + (u_i - \sigma) \wedge t}) + \xi_k (Z_{\sigma + (u_{k+1} - \sigma) \wedge t} - Z_\sigma) \\
&= \sum_{i > k} \xi_i (Z_{u_{i+1} \wedge (\sigma + t)} - Z_{u_i \wedge (\sigma + t)}) + \xi_k (Z_{u_{k+1} \wedge (\sigma + t)} - Z_\sigma) \\
&= \sum_i \xi_i (Z_{u_{i+1} \wedge (\sigma + t)} - Z_{u_i \wedge (\sigma + t)}) + \sum_i \xi_i (Z_{u_{i+1} \wedge \sigma} - Z_{\sigma \wedge u_i}) \\
&= (H_n \cdot Z)_{\sigma + t} - (H_n \cdot Z)_\sigma
\end{aligned}$$

To check the second to last equality, separate each sum in the cases $i > k, i = k, i < k$. The sums $i < k$ cancel by (1.52), the case $i > k$ in the second sum is zero, and the terms $i = k$ give $\xi_k (Z_{u_{k+1} \wedge (\sigma + t)} - Z_\sigma)$.
Now we take an arbitrary process $H \in \mathcal{L}_2(Z, \mathcal{P})$, and we want to show that

$$(\overline{H} \cdot \overline{Z})_t = (H \cdot Z)_{\sigma + t} - (H \cdot Z)_\sigma \tag{1.53}$$

Take a sequence $\{H_n\}$ of simple predictable processes such that $H_n \to H$ in $\mathcal{L}_2(Z, \mathcal{P})$.By (1.51), applied to each H_n implies that $\overline{H}_n \to \overline{H}$ in $\mathcal{L}_2(\overline{Z}, \overline{\mathcal{P}})$.By isometry, we get the convergence $\overline{H}_n \cdot \overline{Z} \to \overline{H} \cdot \overline{Z}$ and $H_n \cdot Z \to H \cdot Z$ in $\mathcal{M}_2$. By lemma 1.49 in chapter 2, we obtain in both cases convergence in probability, uniformly on compact time intervals.As (1.53) was proved for simple predictable processes, then this identify passes to the limit. Boundedness of σ is used to check that the time arguments $\sigma, \sigma+t$ in the right side of (1.53) remain bounded.This proves the L^2-martingale case.

To prove the local L^2-martingale case, we will use the identity for L^2- martingales

$$(\overline{H} \cdot \overline{Z})_t = ((\mathbb{1}_{(\sigma,\infty)} H) \cdot Z)_{\sigma+t} \tag{1.54}$$

Now given $t > 0$, recall that $\{\nu_k\}, \{\tau_k\}$ are localizing sequences for the integrals $\overline{G} \cdot \overline{M}, G \cdot M$ respectively. Take $\nu_k > t$.Then $\tau_k > \sigma + t$.We get

$$\begin{aligned}
(\overline{G} \cdot \overline{M})_t &= ((\mathbb{1}_{(0,\nu_k]} \overline{G}) \cdot \overline{M}^{\nu_k})_t \\
&= ((\mathbb{1}_{(\sigma,\infty)} \mathbb{1}_{[0,\tau_k]} G) \cdot M^{\nu_k})_{\sigma+t} \\
&= ((\mathbb{1}_{(\sigma,\infty)} G) \cdot M)_{\sigma+t} = (G \cdot M)_{\sigma+t} - (G \cdot M)_\sigma
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 1.45. The fact that $\overline{Y}$ is a semimartingale follows from Lemma 1.47 for the martingale part, and is direct for the FV part. Similarly,(1.49) follows from Lemma 1.47 for the stochastic integral, and by definition of Lebesgue-Stieltjes integral for the FV part. □

1.4.4 Stopping just before a stopping time.

Let τ be a stopping time and Y a cadlag process. The process $Y^{\tau-}$ is defined by

$$Y^{\tau-}(t) := \begin{cases} Y(0), & t = 0 \ or \ \tau = 0 \\ Y(t), & 0 < t < \tau \\ Y(\tau-), & 0 < \tau \le t \end{cases} \tag{1.55}$$

In other words, the process Y has been stopped just prior to the stopping time. This type of stopping time is useful for processes with jumps. For example, if

$$\tau = \inf\{t \ge 0 : |Y(t)| \ge r \ or \ |Y(t-)| \ge r\}$$

Then $|Y^\tau| \le r$ may fail if Y has a jump exactly at time τ, but $|Y^{\tau-}| \le r$ is true.

For continuous processes Y^τ and $Y^{\tau-}$ coincide.The general relation is

$$Y^\tau(t) = Y^{\tau-}(t) + \triangle Y(\tau) \mathbb{1}_{\{t \ge \tau\}}$$

Example 1.48. *This example shows that stopping just before a stopping time may not preserve the martingale property.Let N be a rate α Poisson process and $M_t = N_t - \alpha t$.Let τ be the time of the first jump of N.Then $N^{\tau-} = 0$, from which $M_t^{\tau-} = -\alpha t \mathbb{1}_{t<\tau} - \alpha\tau\mathbb{1}_{t\geq\tau}$.Observe that $M_0^{\tau-} = 0$, while $M_t^{\tau-} < 0$ for $t > 0$.Thus $M_t^{\tau-}$ cannot be a martingale.*

Lemma 1.49. *Let Y be a semimartingale and τ a stopping time. Then $Y^{\tau-}$ is a semimartingale.*

Proof. Let $Y = Y_0 + M + U$ be a decomposition of Y into a local L^2- martingale M and an FV bounded process U.We write

$$\begin{aligned} Y^{\tau-}(t) &= Y^{\tau}(t) - \triangle Y(\tau)\mathbb{1}_{\{t\geq\tau\}} \\ &= Y_0 + M^{\tau} + U^{\tau} - \triangle M(\tau)\mathbb{1}_{\{t\geq\tau\}} - \triangle U(\tau)\mathbb{1}_{\{t\geq\tau\}} \\ &= Y_0 + M^{\tau} + U^{\tau-} - \triangle M(\tau)\mathbb{1}_{\{t\geq\tau\}} \end{aligned}$$

Clearly, M^{τ} is a local L^2-martingale, and $U^{\tau-} - M(\tau)\mathbb{1}_{\{t\geq\tau\}}$ is an FV process. □

Now we state some properties of integrals stopped just before τ.

Proposition 1.50. *Let Y and Z be semimartingales, G and J predictable processes locally bounded in the sense of (1.35), and τ a stopping time.*

a) $(G \cdot Y)^{\tau-} = (\mathbb{1}_{[0,\tau]}) \cdot (Y^{\tau-}) = G \cdot (Y^{\tau-})$

b) If $G = J$ on $[0,\tau]$ and $Y = Z$ on $[0,\tau)$, then $(G \cdot Y)^{\tau-} = (J \cdot Z)^{\tau-}$

Proof. Part *b*) follows immediately from the first equality in *a*).So we only prove this part.We write the semimartingale decomposition of $Y^{\tau-} = Y_0 + M^{\tau} + U^{\tau-} - \triangle M(\tau)\mathbb{1}_{\{t\geq\tau\}}$.Then we have

$$\begin{aligned} (\mathbb{1}_{[0,\tau]}G) \cdot Y^{\tau-} &= (\mathbb{1}_{[0,\tau]}G) \cdot M^{\tau} + (\mathbb{1}_{[0,\tau]}G) \cdot U^{\tau-} - G(\tau)\triangle M_{\tau}\mathbb{1}_{[\tau,\infty)} \\ &\overset{(1.30)}{=} (G \cdot M)^{\tau} + (G \cdot U)^{\tau-} - G(\tau)\triangle M_{\tau}\mathbb{1}_{[\tau,\infty)} \end{aligned}$$

Similarly,

$$(G \cdot Y^{\tau-}) = (G \cdot M)^{\tau} + (G \cdot U)^{\tau-} - G(\tau)\triangle M_{\tau}\mathbb{1}_{[\tau,\infty)}$$

The relation $(\mathbb{1}_{[0,\tau]}G) \cdot U^{\tau-} = G \cdot U^{\tau-} = (G \cdot U)^{\tau-}$ comes from the path-by-path Lebesgue-Stieltjes integration: because U stops just before τ, we obtain $\Lambda_{U^{\tau-}}(A) = 0$ for any Borel set $A \subset [\tau, \infty)$, and so for any bounded Borel function g and any $t \geq \tau$,

$$\int_{[0,t} g d\Lambda_{U^{\tau-}} = \int_{[0,\tau)} g d\Lambda_{U^{\tau-}}$$

This gives us the first equality.Finally, using Theorem 1.43,

$$(G \cdot M)_t^{\tau} - G(\tau)\triangle M_{\tau}\mathbb{1}_{[\tau,\infty)}(t) = (G \cdot M)_t^{\tau} - \triangle(G \cdot M)_{\tau}\mathbb{1}_{[\tau,\infty)}(t) = (G \cdot M)_t^{\tau-}$$

Thus

$$(\mathbb{1}_{[0,\tau]}G) \cdot Y^{\tau-} = (G \cdot Y^{\tau-}) = (G \cdot M)_t^{\tau-} + (G \cdot U)_t^{\tau-} = (G \cdot Y)_t^{\tau-}$$

□

1.5 Quadratic variation

Recall that the quadratic variation $[X]$ of a process X(when it exists) is a nondecreasing process with $[X]_0 = 0$, whose value at time t is determined, up to a null event, by the limit in probability

$$[X]_t = \lim_{mesh(\pi)\to 0} \sum_i (X_{t_{i+1}} - X_{t_i})^2 \tag{1.56}$$

Here $\pi = \{0 = t_0 < t_1 < \cdots < t_{m(\pi)} = t\}$ is a partition of $[0, t]$.Quadratic covariation of two processes X, Y is defined as the FV process

$$[X, Y] = [(X + Y)/2] - [(X - Y)/2] \tag{1.57}$$

assuming the processes on the right exist.Again, there is a limit in probability

$$[X, Y] = \lim_{mesh(\pi)\to 0} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \tag{1.58}$$

For cadlag semimartingales X and Y, by corollary 1.37 in chapter 2, $[X], [Y], [X, Y]$ exist and have cadlag versions.

The limit (1.58) shows that, for processes X, Y and Z and reals α, β,

$$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z] \tag{1.59}$$

provided that these processes exist.The equality can be taken in the sense of indistinguishability for cadlag versions, provided such exist.Consequently, $[\cdot,\cdot]$ operates somewhat in the manner of an inner product.

Lemma 1.51. *Suppose M_n, M, N_n and N are L^2-martingales. Fix $0 \le T < \infty$.Assume $M_n(T) \to M(T)$ and $N_n(T) \to N(T)$ in L^2.Then*

$$E\left(\sup_{0\le t\le T} |[M_n, N_n]_t - [M, N]_t|\right) \to \text{ as } n \to \infty$$

Proof. By (1.57), it suffices to consider the case $M_n = N_n, M = N$.Applying

$$|[X]_t - [Y]_t| \le [X - Y]_t + 2[X - Y]_t^{1/2}[Y]_t^{1/2} \quad a.s \tag{1.60}$$

and using that $[X]_t$ is nondecreasing in t,

$$|[M_n]_t - [M]_t| \le [M_n - M]_t + 2[M_n - M]_t^{1/2}[M]_t^{1/2} \le [M_n - M]_T + 2[M_n - M]_T^{1/2}[M]_T^{1/2}$$

Taking expectations, applying Cauchy-Schwarz and recalling that $E([X]_t) = E(X_t^2 - X_0^2) \le E(X_t^2)$ we get

$$\begin{aligned} E\left(\sup_{0\le t\le T} |[M_n, N_n]_t - [M, N]_t|\right) &\le E[(M_n(T) - M(T)^2] + 2E([M_n - M]_T)^{1/2}E([M]_T)^{1/2} \\ &= ||M_n(T) - M(T)||_{L^2}^2 + 2||M_n(T) - M(T)||_{L^2}||X(T)||_{L^2} \end{aligned}$$

Letting $n \to \infty$ we are done. □

Proposition 1.52. *Let $M, N \in \mathcal{M}_{2,loc}, G \in \mathcal{L}(M,\mathcal{P})$, and $H \in \mathcal{L}(N,\mathcal{P})$.Then*

$$[G \cdot M, H \cdot N] = \int_{(0,t]} G_s H_s d[M, N]_s \tag{1.61}$$

Proof. It's enough to show that for $L \in \mathcal{M}_{2,loc}$

$$[G \cdot M, L]_t = \int_{(0,t]} G_s d[M, L]_s \tag{1.62}$$

In that case, if we consider $L = H \cdot N$ we get

$$[M, H \cdot N]_t = [H \cdot N, M]_t = \int_{(0,t]} H_s d[M, N]_s$$

Thus $d[M, L]_t = H_t d[M, N]_t$, so that

$$[G \cdot M, H \cdot N] = [G \cdot M, L] \overset{(1.62)}{=} \int_{(0,t]} G_s d[M, L]_s = \int_{(0,t]} G_s H_s d[M, N]_s$$

Step 1 Suppose $M, L \in \mathcal{M}_2$.First consider $G = \xi \mathbb{1}_{(u,v]}$ for a bounded $\mathcal{F}_u$ random variable ξ.Then $G \cdot M = \xi(M^v - M^u)$.By (1.59) and lemma 1.34 in chapter 2

$$[G \cdot M, L]_t = \xi([M^v, L]_t - [M^u, L]_t) = \xi([M, L]_{t\wedge v} - [M, L]_{t\wedge u})$$
$$= \int_{(0,t]} \xi \mathbb{1}_{(u,v]}(s) d[M, L]_s = \int_{(0,t]} G(s) d[M, L]_s$$

Now take general $G \in \mathcal{L}_2(M, \mathcal{P})$.Pick simple predictable processes G_n such that $G_n \to G$ in $\mathcal{L}_2(M, \mathcal{P})$.Then $(G_n \cdot M)_t \to (G \cdot M)_t$ in $L^2(P)$.By Lemma 1.51, $[G_n \cdot M, L]_t \to [G \cdot M, L]_t$ in $L^1(P)$.By the previous step, we have

$$[G_n \cdot M, L] = \int_{(0,t]} G_n(s) d[M, L]_s$$

To get

$$[G \cdot M, L] = \int_{(0,t]} G_s d[M, L]_s$$

it's enough to prove the $L^1(P)$ convergence

$$\int_{(0,t]} G_n(s) d[M, L]_s \to \int_{(0,t]} G_s d[M, L]_s$$

Applying the Kunita-Watanabe inequality and Cauchy-Schwarz,

$$E\left(\left|\int_{(0,t]} (G_n(s) - G_s) d[M, L]_s\right|\right) \leq E\left(\int_{(0,t]} |G_n(s) - G_s|^2 d[M]_s\right) E([L]_t)^{1/2}$$
$$= \left(\int_{(0,t]\times\Omega} |G_n - G|^2 d\mu_M\right) E(L_t^2 - L_0^2)^{1/2} \to 0$$

when $n \to \infty$, because $G_n \to G$ in $\mathcal{L}_2(M, \mathcal{P})$ implies $G_n \to G$ in $L^2([0, t] \times \Omega, \mu_M)$.This finish the case $L, M \in \mathcal{M}_2, G \in \mathcal{L}_2(M, \mathcal{P})$.

Step 2 Now the case $L, M \in \mathcal{M}_{2,loc}, G \in \mathcal{L}(M, \mathcal{P})$.Pick stopping times $\{\tau_k\}$that localizes both L and (G, M).Write $G^k = \mathbb{1}_{(0,\tau_k]} G$.Then if $\tau_k(\omega) \geq t$,

$$[G \cdot M, L]_t = [G \cdot M, L]_{t\wedge\tau_k} = [(G \cdot M)^{\tau_k}, L^{\tau_k}]_t = [(G^k \cdot M^{\tau_k}), L^{\tau_k}]_t$$
$$= \int_{(0,t]} G_s^k d[M^{\tau_k}, L^{\tau_k}]_s = \int_{(0,t]} G_s d[M, L]_s$$

In the second and third equalities we used lemma 1.34 from chapter 2.In the last equality we used that $G^k = G$ if $t \leq \tau_k$ and again lemma 1.34 in chapter 2.

□

Corollary 1.53. *Suppose $M, N \in \mathcal{M}_2, G \in \mathcal{L}_2(M, \mathcal{P}), H \in \mathcal{L}_2(N, \mathcal{P})$. Then*

$$(G \cdot M)_t (H \cdot N)_t - \int_{(0,t]} G_s H_s d[M, N]_s$$

is a martingale.If we weaken the hypotheses to $M, N \in \mathcal{M}_{2,loc}, G \in \mathcal{L}(M, \mathcal{P}), H \in \mathcal{L}(N, \mathcal{P})$,then the process above is a local martingale.

Proof. Use the last proposition and theorem 1.35 from chapter 2. □

Lemma 1.54. *Let $M \in \mathcal{M}_{2,loc}$ and assume $M_0 = 0$.*

a) For any $0 \le t < \infty, [M]_t = 0$ almost surely iff $\sup_{0 \le s \le t} |M_s| = 0$ almost surely.

b) Let $G \in \mathcal{L}(M, \mathcal{P})$. The stochastic integral $G \cdot M$ is the unique process $Y \in \mathcal{M}_{2,loc}$ that satisfies $Y_0 = 0$ and

$$[Y, L] = \int_{(0,t]} G_s d[M, L]_s \text{ almost surely}$$

for each $0 \le t < \infty$ and each $L \in \mathcal{M}_{2,loc}$.

Proof. a) Clearly, if $\sup_{0 \le s \le t} |M_s| = 0$, then $[M]_t = 0$ almost surely. Conversely, for a fixed t, let $\{\tau_k\}$ be a localizing sequence for M.Then for $s \le t, [M^{\tau_k}]_s \le [M^{\tau_k}]_t = [M]_{t \wedge \tau_k} \le [M]_t = 0$ by lemma 1.31 in chapter 2 and the monotonicity of $t \to [M]_t$.Consequently, $E[(M_s^{\tau_k})^2] = E([M^{\tau_k}]_s) = 0$, from which $M_s^{\tau_k} = 0$ almost surely. Consider k such that $\tau_k \ge s$, then $M_s = 0$ almost surely.As M has cadlag paths, there exists an event Ω_0 with $P(\Omega_0) = 1$ and $M_s(\omega) = 0$ for all $\omega \in \Omega_0, s \in [0, t]$(you can use that $M_s(\omega) = 0$ for the rationals in [0,t] and use the cadlag argument to extend this to all [0,t]).

b) We already proved that $G \cdot M$ satisfies the required property.Now suppose that $Y \in \mathcal{M}_{2,loc}$ satisfies the property.Then

$$[Y - G \cdot M, L]_t = [Y, L] - [G \cdot M, L]_t = 0$$

for any $L \in \mathcal{M}_{2,loc}$.Taking $L = Y - G \cdot M$ and applying part a), we obtain $Y = G \cdot M$ almost surely.

□

The last property shows that the quadratic covariation behaves as an inner product.

Proposition 1.55. *Let $M \in \mathcal{M}_{2,loc}, G \in \mathcal{L}(M,\mathcal{P})$ and let $N = G \cdot M$.Suppose $H \in \mathcal{L}(M,\mathcal{P})$.Then $HG \in \mathcal{L}(M,\mathcal{P})$ and $H \cdot N = (HG) \cdot M$.*

Proof. Let $\{\tau_k\}$ be a localizing sequence for (G, M) and (H, N).We have $N^{\tau_k} = (G \cdot M)^{\tau_k} = G \cdot M^{\tau_k}$.Using Proposition 1.52 we obtain

$$[N^{\tau_k}] = [G \cdot M^{\tau_k}] = [G \cdot M^{\tau_k}, G \cdot M^{\tau_k}] = \int_{(0,t]} G_s^2 d[M^{\tau_k}]_s$$

So $d[N^{\tau_k}]_s = G_s^2 d[M^{\tau_k}]_s$.Then for any $T < \infty$,

$$E \int_{[0,T]} \mathbb{1}_{[0,\tau_k]}(t) H_t^2 G_t^2 d[M^{\tau_k}] = E \int_{[0,T]} \mathbb{1}_{[0,\tau_k]} H_t^2 d[N^{\tau_k}] < \infty$$

because we assumed that $\{\tau_k\}$ localizes (H, N).This proves that $\{\tau_k\}$ localizes (HG, M).In particular,$HG \in \mathcal{L}(M,\mathcal{P})$.
Let $L \in \mathcal{M}_{2,loc}$.We obtain by (1.62) $d[N, L] = G_s d[M, L]$,and so by the same equation

$$[(HG) \cdot M, L]_t = \int_{(0,t]} G_s H_s d[M, L]_s = \int_{(0,t]} H_s d[N, L]_s = [H \cdot N, L]_t$$

Finally, by part b) in Lemma 1.54, $(HG) \cdot M$ must coincide with $H \cdot N$. □

As a corollary of this last result,we obtain a change of measure formula.

Corollary 1.56. *Let Y be a cadlag semimartingale,G and H predictable processes that satisfies (1.35),and $X = \int GHdY$, also a cadlag semimartingale. Then*

$$\int GdX = \int GHdY$$

Proof. Let $Y = Y_0 + M + U$ be a semimartingale decomposition of Y.Let $V_t := \int_{(0,t]} H_s dU_s$,another FV process.Then, for fixed ω, H_s is the Radon-Nikodym derivative $d\Lambda_V / d\Lambda_U$ of the Lebesgue-Stieltjes measures on the real line.Clearly, $X = H \cdot M + V$ is a semimartingale decomposition of X.We use Proposition 1.55to compute

$$\int_{(0,t]} G_s dX_s = (G \cdot (H \cdot M))_t + \int_{(0,t]} G_s dV_s = ((GH) \cdot M)_t + \int_{(0,t]} G_s H_s dU_s$$
$$= \int_{(0,t]} G_s H_s dY_s$$

□

Recall that for a cadlag process X, we defined the caglad process X_- by $X_-(0) = X(0), X_-(t) = X(t-)$ if $t > 0$.The next theorem is a integration-by-parts formula for stochastic integrals.

Theorem 1.57. *Let Y and Z be cadlag semimartingales. Then $[Y,Z]$ exists as a cadlag,adapted FV process and satisfies*

$$[Y,Z]_t = Y_t Z_t - Y_0 Z_0 + \int_{(0,t]} Y_{s-} dZ_s - \int_{(0,t]} Z_{s-} dY_s \tag{1.63}$$

The product YZ is a semimartingale, and for a predictable process H that satisfies (1.35),

$$\int_{(0,t]} H_s d(YZ)_s = \int_{(0,t]} H_s Y_{s-} dZ_s + \int_{(0,t]} H_s Z_{s-} dY_s + \int_{(0,t]} H_s d[Y,Z]_s \tag{1.64}$$

Proof. We know that $[Y,Z]$ exists by 1.37 from chapter 2.However, we give another proof here that will be useful.Take a countable infinite partition $\pi = \{0 = t_1 < t_2 < \cdots\}$ of $\mathbb{R}_+$ such that $t_i \nearrow \infty$, and in fact, we consider a sequence of such partitions, with $mesh(\pi) \to 0$.We will use that for $i \in \mathbb{N}, Y_{t_i}(Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}) = Y_{t_i \wedge t}(Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t})$ and the same if we exchange Y with Z.For $t \in \mathbb{R}_+$, consider

$$\begin{aligned} S_\pi(t) &= \sum_i (Y_{t_{i+1}\wedge t} - Y_{t_i \wedge t})(Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}) \\ &= \sum_i (Y_{t_{i+1}\wedge t} Z_{t_{i+1}\wedge t} - Y_{t_i \wedge t} Z_{t_i \wedge t}) - \sum_i Y_{t_i}(Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}) - \sum_i Z_{t_i}(Y_{t_{i+1}\wedge t} - Y_{t_i \wedge t}) \end{aligned} \tag{1.65}$$

$$\overrightarrow{mesh(\pi) \to 0} \; Y_t Z_t - Y_0 Z_0 - \int_{(0,t]} Y_{s-} dZ_s - \int_{(0,t]} Z_{s-} dY_s \tag{1.66}$$

The convergence is due to Proposition 1.37, in probability on compact time sets.In particular, there exists a subsequence where the convergence is almost surely, uniformly on $[0,T]$.The limit process is cadlag.

Take $Y = Z$, and suppose $s < t$.Once the mesh is smaller than $t - s$, there exist indices $k < l$ such that $t_k < s \leq t_{k+1}$ and $t_l < t \leq t_{l+1}$.Then

$$\begin{aligned} S_\pi(t) - S_\pi(s) &= (Y_t - Y_{t_l})^2 + \sum_{i=k}^{l-1} (Y_{t_{i+1}} - Y_{t_i})^2 - (Y_s - Y_{t_k})^2 \\ &\geq (Y_{t_{k+1}} - Y_{t_k})^2 - (Y_s - Y_{t_k})^2 \end{aligned}$$

When $l \to \infty$, the right side converges to $(\triangle Y_s)^2 - (\triangle Y_s)^2 = 0$.Since this limit holds almost surely simultaneously for all $s < t \in [0, T]$, we can conclude that the limit process is nondecreasing.This process that every semimartingale has a nondecreasing, cadlag quadratic variation $[Y]$.

By definition of quadratic covariation, we know that the limit process in (1.65) is $[Y, Z]_t$,so coincides with the limit process in (1.65) almost surely, at each fixed time. Then they are indistinguishable.This checks (1.63).Using this equation, we deduce that

$$YZ = Y_0 Z_0 + \int Y_- dZ + \int Z_- dY + [Y, Z]$$

this is a semimartingale representation ot YZ, so it is a semimartingale.The formula (1.64) is a direct application of the last corollary and the semimartingale representation of YZ. $\square$

Now we extend Proposition 1.52 to semimartingales.

Theorem 1.58. *Let Y and Z be cadlag semimartingales and G,H predictable processes that satisfy* (1.35)*.Then*

$$[G \cdot Y, H \cdot Z]_t = \int_{(0,t]} G_s H_s d[Y, Z]_s$$

Proof. For the same reason as in Proposition 1.52,it suffices to show

$$[G \cdot Y, Z]_t = \int_{(0,t]} G_s d[Y, Z]_s$$

Let $Y = Y_0 + M + A$ and $Z = Z_0 + N + B$ be decompositions into local L^2- martingales M and N and FV processes A and B.By the bilinearity of the quadratic covariation,

$$[G \cdot Y, Z] = [G \cdot M, N] + [G \cdot M, B] + [G \cdot A, Z]$$

by Proposition 1.52, $[G \cdot M, N] = \int G_s d[M, N]$.To analyze the other two terms, we consider ω such that the paths of the integrals and the semimartingales are cadlag, and Theorem 1.43 is valid for each integral that appear here.Since B and $G \cdot A$ are FV processes,lemma 1.36 in chapter 2 applies.Combining this lemma with Theorem 1.43 and the definition of a Lebesgue-

Stieltjes integral with respect to a step function gives

$$\begin{aligned}[G\cdot M,B]_t+[G\cdot A,Z]_t &= \sum_{s\in(0,t]}\triangle(G\cdot M)_s\triangle B_s+\sum_{s\in(0,t]}\triangle(G\cdot A)_s\triangle Z_s\\ &= \sum_{s\in(0,t]}\triangle G_s\triangle M_s\triangle B_s+\sum_{s\in(0,t]}\triangle G_s\triangle A_s\triangle Z_s\\ &= \int_{(0,t]}G_s d[M,B]_s+\int_{(0,t}G_s d[A,Z]_s\end{aligned}$$

Combining the three sums, we are done. □

The last result of this chapter will be used in the proof of Itô's formula.

Proposition 1.59. *Let Y and Z be a cadlag semimartingales,and G an adapted cadlag process. Given a partition $\pi=\{0=t_1<t_2<\cdots\}$ with $t_i\nearrow\infty$ of $[0,\infty)$,define*

$$R_t(\pi)=\sum_{i=1}^{\infty}G_{t_i}(Y_{t_{i+1}\wedge t}-Y_{t_i\wedge t})(Z_{t_{i+1}\wedge t}-Z_{t_i\wedge t}) \tag{1.67}$$

Then as $mesh(\pi)\to 0, R_t(\pi)$ converges to $\int G_{t-}d[Y,Z]$ in probability, uniformly on compact time intervals.

Proof. We write

$$R_t(\pi)=\sum_i G_{t_i}(Y_{t_{i+1}\wedge t}Z_{t_{i+1}\wedge t}-Y_{t_i\wedge t}Z_{t_i\wedge t})-\sum_i G_{t_i}Y_{t_i}(Z_{t_{i+1}\wedge t}-Z_{t_i\wedge t})-\sum_i G_{t_i}Z_{t_i}(Y_{t_{i+1}\wedge t}-Y_{t_i\wedge t})$$

We know that YZ is a semimartingale.We apply Proposition 1.37 to each sum gives the convergence to the limit

$$\int_{(0,t]}G_{s-}d(YZ)_s-\int_{(0,t]}G_{s-}Y_{s-}dZ_s-\int_{(0,t]}G_{s-}Z_{s-}dY_s=\int_{(0,t]}G_s d[Y,Z]_s$$

in the last equality we used (1.64). □