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Primer Semestre de 2018

Directed Polymers in Random Environment

Chapter 4

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May 20, 2018

1 Semimartingale Approach and Localization Transition

In this section, we will study semimartingales to deduce a phase transition of the localization of the polymer.

1.1 Semimartingale Decomposition

If $(a_n)_{n\geq 0}$ is a sequence (random or not), we write $\triangle a_n := a_n - a_{n-1}$ for $n \geq 1$.We will use the following fact: **Doob's Decomposition:**Any $(\mathcal{F}_n)$ process $X = (X_n)_{n\geq 0} \subset L^1(\mathbb{P})$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), n \geq 1$$

where $M(X)$ is an $(\mathcal{F}_n)$ martingale and $A(X)$ is predictable,i.e, $A_n(X)$ is $\mathcal{F}_{n-1}$ measurable, and $A_0(X) = 0$. To obtain such processes, we compute their increments
$\triangle A_n = \mathbb{P}(\triangle X_n|\mathcal{F}_{n-1}), \quad \triangle M_n = \triangle X_n - \mathbb{P}(\triangle X_n|\mathcal{F}_{n-1})$.
Then $A_n = \sum_{t=1}^n \triangle A_t, \quad M_n = X_0 + \sum_{t=1}^n \triangle M_t$ The processes $M_n(X), A_n(X)$ are called the martingale part and the compensator of X respectively.

Remark 1.1. *If N is a square integrable martingale, then $N^2 \subset L^1(\mathbb{P})$ and the compensator $A(N^2)$,denoted by $\langle N_n\rangle$ is given by*

$$\triangle\langle N\rangle_n = \mathbb{P}[N_n^2 - N_{n-1}^2|\mathcal{F}_{n-1}] = \mathbb{P}[(N_n - N_{n-1})^2|\mathcal{F}_{n-1}] = \mathbb{P}[(\triangle N_n)^2|\mathcal{F}_{n-1}] \qquad (1.1.1)$$

We are interested in the process $X_n = -\log(W_n)$.Because W_n is a martingale, then X_n is a submartingale. In particular $\mathbb{P}(\triangle X_n|\mathcal{F}_{n-1}) \geq 0$.Therefore, A_n is an increasing process. To obtain M_n, A_n, we introduce

$$U_n := P_n^{\beta,\omega}[e^{\beta\omega(n,S_n)-\lambda(\beta)}] - 1$$

Recall the relation

$$Z_{n+m} = Z_n \times P_n^{\beta,\omega}(Z_m \circ \theta_{n,S_n}) \tag{1.1.2}$$

Using this, we have $W_n/W_{n-1} = 1 + U_n$.Then we write $W_n = \prod_{j=1}^n (1+U_t) \Rightarrow X_n = -\sum_{t=1}^n \log(1+U_t)$.Finally, we decompose each term $-\log(1+U_t) = \triangle M_t + \triangle A_t$, with $\triangle A_t = -\mathbb{P}(\log(1+U_t)|\mathcal{F}_{t-1}), \triangle M_t = -\log(1+U_t) + \mathbb{P}(\log(1+U_t)|\mathcal{F}_{t-1})$ As stated in [1], a key role in the asymptotics of the model is in the random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$

$$I_n := \sum_{x\in\mathbb{Z}^d} P_{n-1}^{\beta,\omega}(S_n = x)^2 \tag{1.1.3}$$

We can see this random variable as the probability that two independent random walks intersect. More precisely, on the product space Ω^2_{traj} we consider the probability measure $(P_n^{\beta,\omega})^{\otimes 2}$ and we will consider in this space two independent random walks $(S, \tilde{S})$, with $S, \tilde{S}$ have the same law $P_n^{\beta,\omega}$.The path $S, \tilde{S}$ are called replica.So, we can write I_n as

$$I_n = (P_n^{\beta\omega})^{\otimes 2}(S_n = \tilde{S}_n) \tag{1.1.4}$$

Hence, $\sum_{k=1}^n I_k$ is the expected amount of overlap up to time n of two independent polymers in the same (fixed) environment.The following theorem relates this expression with W_n:

Theorem 1.2. *Let $\beta \neq 0$.Then*

$$A := \{W_\infty = 0\} = B := \{\sum_{n=1}^\infty I_n = \infty\} \quad \mathbb{P}\ a.s \tag{1.1.5}$$

Moreover,if $\mathbb{P}(W_\infty = 0) = 1$,there exist $c_1, c_2 \in (0,\infty)$ depending on $\beta, \mathbb{P}$ such that $\mathbb{P}$ a.s

$$c_1 \sum_{k=1}^n I_k \leq -\log(W_n) \leq c_2 \sum_{k=1}^n I_k \tag{1.1.6}$$

and also

$$\lim_{n\to\infty} \frac{-\log(W_n)}{A_n} = 1 \quad \mathbb{P}\ a.s \tag{1.1.7}$$

Proof. In order to prove (1.1.5),(1.1.6), it's enough to show the following:

$$\{W_\infty = 0\} \subset \{\sum_{n=1}^{\infty} I_n = \infty\} \quad \mathbb{P}\ a.s \tag{1.1.8}$$

and that there exists $c_1, c_2 \in (0,\infty)$ such that

$$\{\sum_{n=1}^{\infty} I_n = \infty\} \subset \{(1.1.6) \text{ holds}\} \quad \mathbb{P}\ a.s \tag{1.1.9}$$

To convince of that,note that using (1.1.8) we only need to show that $\mathbb{P}(B\backslash A) = 0$ .If $\mathbb{P}(A) = 1$ this is clear,and if $\mathbb{P}(A) = 0$,then $W_\infty > 0\ \ \mathbb{P}\ a.s$, and in that case, $\mathbb{P}(B) = 0$,otherwise, $\mathbb{P}(\{(1.1.6) \text{ holds}\}) > 0$,but this cannot happen if $-\log(W_\infty) < \infty$ because $\sum_{n\geq 1} I_n < \infty$ Recall that $\triangle M_n = -\log(1+U_n) + \mathbb{P}(\log(1+U_n)|\mathcal{F}_{n-1})$.Then

$$\mathbb{P}[(\triangle M_n)|\mathcal{F}_{n-1}] = \mathbb{P}[\log^2(1+U_n)|\mathcal{F}_{n-1}] - \mathbb{P}[\log(1+U_n)|\mathcal{F}_{n-1}]^2 \leq \mathbb{P}[\log^2(1+U_n)|\mathcal{F}_{n-1}]$$

Using (1.1.1),we deduce that

$$\langle M\rangle_n \leq \mathbb{P}[\log^2(1+U_n)|\mathcal{F}_{n-1}] \tag{1.1.10}$$

Now we claim that there exists a constant $c \in (0,\infty)$ such that

$$\frac{1}{c} I_n \leq \triangle A_n \leq c I_n, \quad \triangle\langle M\rangle_n \leq\ c I_n \tag{1.1.11}$$

The proof of the claim will be postponed.

Now we use the fact that a square integrable martingale converges a.s on the event $\{\langle M\rangle_\infty < \infty\}$ and (1.1.11) to conclude (1.1.8) $\mathbb{P}\ a.s$:

$$\{\sum_{n\geq 1} I_n < \infty\} \subset \{A_\infty < \infty, \langle M\rangle_\infty < \infty\} \subset \{A_\infty < \infty, \lim_{n\to\infty} M_n \text{ exists and is finite}\}$$
$$\subset \{\lim_{n\to\infty} \log W_n \text{ exists and is finite}\} = \{W_\infty > 0\}$$

Now we prove (1.1.9).By (1.1.11),it's enough to show that

$$\{A_\infty = \infty\} \subset \{\lim_{n\to\infty} -\frac{\log W_n}{A_n} = 1\}, \quad \mathbb{P}\ a.s \tag{1.1.12}$$

To show this,let's suppose that $A_\infty = \infty$.If $\langle M\rangle_\infty < \infty$, then $\lim_{n\to\infty} M_n$ exists and is finite a.s.Therefore, (1.1.12) holds.Now,if $\langle M\rangle_\infty = \infty$, using the Law of Large Numbers for martingales, we deduce that $M_n/\langle M\rangle_n \to 0$ a.s on the event $\{\langle M\rangle_\infty = \infty\}$. In this case, we see that

$$-\frac{-\log W_n}{A_n} = \frac{M_n + A_n}{A_n} = 1 + \frac{M_n}{\langle M\rangle_n}\frac{\langle M\rangle_n}{A_n} \to 1 \quad \mathbb{P}\ a.s$$

by (1.1.11), because $\langle M\rangle_n \le c^2 A_n$. $\square$

To prove the claim (1.1.11),we need the following lemma:

Lemma 1.3. *Let $e_i, 1 \le i \le m$ be positive, non constant i.i.d random variables on a probability space $(H, \mathcal{G}, \mathbb{P})$ such that*

$$\mathbb{P}(e_1) = 1, \quad \mathbb{P}(e_1^3 + \log^2 e_1) < \infty$$

For $\{\alpha_i\}_{1\le i\le m} \subset [0,\infty)$ such that $\sum_{1\le i\le m}\alpha_i = 1$, define a centered random variable $U > -1$ by $U = \sum_{1\le i\le m}\alpha_i e_i - 1$.Then there exists a constant $c \in (0,\infty)$, independent of m and $\{\alpha_i\}_{1\le i\le m}$ such that

$$\frac{1}{c}\sum_{1\le i\le m}\alpha_i^2 \le \mathbb{P}\left[\frac{U^2}{U+2}\right] \tag{1.1.13}$$

$$\frac{1}{c}\sum_{1\le i\le m}\alpha_i^2 \le -\mathbb{P}[\log(1+U)] \le c\sum_{1\le i\le m}\alpha_i^2 \tag{1.1.14}$$

$$\mathbb{P}[\log^2(1+U)] \le c\sum_{1\le i\le m}\alpha_i^2 \tag{1.1.15}$$

Proof. We will use constants $c_1, c_2, \cdots$ for constants independent of $\{\alpha_i\}_{1\le i\le m}$.We have

$$\mathbb{P}[U^2] = c_1\sum_{1\le i\le m}\alpha_i^2,\ \ \mathbb{P}[U^3] \le c_2\sum_{1\le i\le m}\alpha_i^2$$

Now we deduce the first inequality as follows:

$$\begin{aligned} c_1\sum_{1\le i\le m}\alpha_i^2 = \mathbb{P}\left[\frac{U}{\sqrt{2+U}}\sqrt{2+U}\cdot U\right] &\le \mathbb{P}\left[\frac{U^2}{2+U}\right]^{1/2}\mathbb{P}[2U^2+U^3]^{1/2} \\ &\le c_3\mathbb{P}\left[\frac{U^2}{2+U}\right]^{1/2}\left(\sum_{1\le i\le m}\alpha_i^2\right)^{1/2} \end{aligned}$$

To prove the other two inequalities, we define a function $\phi : (-1,\infty) \to [0,\infty)$ by $\phi(u) = u - \log(1+u)$, so that (recall that $\mathbb{P}(U) = 0$)

$$-\mathbb{P}[\log(1+U)] = \mathbb{P}[\phi(U)]$$

So, $-\mathbb{P}[\log(1+U)] = \mathbb{P}[\phi(U)] \geq \frac{1}{4}\mathbb{P}\left[\frac{U^2}{2+U}\right] \geq \frac{1}{c}\sum_{1\leq i\leq m}\alpha_i^2$
because if $u > -1, \phi(u) \geq \frac{1}{4}\frac{u^2}{2+u}$.This give us the first inequality in (1.1.14).To obtain the second one, note that for $\epsilon \in (0,1)$,

$$\begin{aligned}\mathbb{P}(\phi(U)) &= \mathbb{P}(\phi(U) : 1+U \geq \epsilon) + \mathbb{P}(\phi(U) : 1+U \leq \epsilon)\\ &\leq \mathbb{P}(\phi(U) : 1+U \geq \epsilon) - \mathbb{P}[\log(1+U) : 1+U \leq \epsilon]\end{aligned}$$

Note that if $1+u \geq \epsilon, \phi(u) \leq \frac{1}{2}u^2 \leq \frac{1}{2}(\frac{u}{\epsilon})^2$. Then we have

$$\mathbb{P}(\phi(U) : 1+U \geq \epsilon) \leq \frac{1}{2}\epsilon^{-2}\mathbb{P}(U^2) = \frac{1}{2}e^{-2}c_1\sum_{1\leq i\leq m}\alpha_i^2 \tag{1.1.16}$$

Now define $\gamma := -\mathbb{P}[\log e_1] \geq 0$ by Jensen, and take $\epsilon > 0$ small enough such that $\log(1/\epsilon) - \gamma \geq 1$. Define another centered random variable by $V := \sum_{1\leq i\leq m}\alpha_i(\log(e_i)+\gamma)$.Note that by Jensen inequality (-log is convex) we have that $\sum\alpha_i log(e_i) \leq \log(\sum\alpha_i e_i)$. Using that, we have

$$\begin{aligned}\{1+U\leq\epsilon\} &\subset \{V-\gamma \subset \log(1+U) \leq \log\epsilon\}\\ &\leq \{-\log(1+U) \leq -V+\gamma\} \cap \{1 \leq -V\}\end{aligned}$$

Hence, we have

$$\begin{aligned}-\mathbb{P}[\log(1+U) : 1+U \leq \epsilon] &\leq \mathbb{P}[-V+\gamma : 1 \leq -V] \leq \mathbb{P}[-V : 1 \leq -V] + \gamma\mathbb{P}[1 \leq -V]\\ &\leq (1+\gamma)\mathbb{P}[V^2] = c_4\sum_{1\leq i\leq m}\alpha_i^2\end{aligned}$$

where the last inequality comes from on the event $-V \geq 1, V^2 \geq -V \geq 1$, and this implies that $\mathbb{P}[-V : -V \geq 1] \leq \mathbb{P}[V^2], \mathbb{P}[1 \leq -V] \leq \mathbb{P}[V^2]$
This result,together with (1.1.16),deduce the second part in (1.1.14).To prove the last inequality, the argument is similar. We use the fact that $|\log(1+u)| \leq \epsilon^{-1}\log(\epsilon^{-1})|u|$ if $\epsilon < 1+u$, we have that

$$\mathbb{P}[\log^2(1+U) : \epsilon \leq 1+U] \leq \epsilon^{-2}\log^2(\epsilon^{-1})\mathbb{P}[U^2]$$

On the other hand, the following holds:

$$\{1+U\leq\epsilon\} = \{V-\gamma \leq \log(1+U) \leq \log(\epsilon)\} \subset \{\log^2(1+U) \leq 2V^2+2\gamma^2\} \cap \{1 \leq -V\}$$

(Because $0 \leq -\log(1+U) \leq \gamma - V \to \log^2(1+U) \leq (V-\gamma)^2 \leq 2V^2 + 2\gamma^2$)

Therefore, we have

$$\mathbb{P}[\log^2(1+U) : 1+U \leq \epsilon] \leq 2\mathbb{P}[V^2] + 2\gamma^2 \mathbb{P}[1 \leq -V] \leq c_5 \sum_{1 \leq i \leq m} \alpha_i^2$$

□

Proof of the claim. Recall that $\triangle A_n = -\mathbb{P}(\log(1+U_n)|\mathcal{F}_{n-1})$ and(1.1.10). We apply the lemma with $\{e_i\}, \{\alpha_i\}$ and $\mathbb{P}$ as $\{e^{\beta\omega(n,z)-\lambda(\beta)}\}_{|z|_1 \leq n}, \{P_{n-1}^{\beta,\omega}(S_n = z)\}_{|z|_1 \leq n}$ and $\mathbb{P}[\cdot|\mathcal{G}_{n-1}]$.More precisely, we use the last two equation in the lemma, and recall that in our case, $\alpha_i^2 = I_i$ by (1.1.4) □

1.2 Localization versus Delocalization

The strong disorder regime present a strange behavior. On one hand, the polymer has a tendency to diffuse. On the other hand, it tends to localize in certain regions (the regions which maximizes H_n).To study this phenomenon, we will consider the random variable J_n, which is the probability of the favorite endpoint for the polymer of size n:

$$J_n := \max_{x \in \mathbb{Z}^d} P_{n-1}^{\beta,\omega}(S_n = x) \tag{1.2.1}$$

This random variable measures how much spread is the polymer. For example, if $\beta = 0$, we have $J_n = O(n^{-d/2})$, but it should be larger if the polymer is localized in certain regions. Recalling the definition of $I_n = \sum_{x \in \mathbb{Z}^d} P_{n-1}^{\beta,\omega}(S_n = x)^2$, we have the following inequalities

$$J_n^2 \leq I_n \leq J_n \tag{1.2.2}$$

Definition 1.4.

*We say the polymer is **localized** if*

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} J_t > 0 \quad \mathbb{P}\ a.s \tag{1.2.3}$$

*and the polymer is **delocalized** if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} J_t = 0 \quad \mathbb{P}\ a.s \tag{1.2.4}$$

Now we show that a criterion to distinguish each case, and also a phase transition:

Theorem 1.5 (Localization Transition). *Let $\beta \neq 0$. Then*

- *localized if and only if $p < \lambda$*
- *delocalized if and only if $p = \lambda$*

Localization occurs for all $\beta > \beta_c$, and delocalization for $\beta \leq \beta_c$

Proof. Note first that $\frac{1}{n}\sum_n I_n \approx \frac{1}{n}\sum_n J_n$, so they have the same limits.
If the polymer is localized, then $\sum_n I_n = \infty \Rightarrow W_\infty = 0 \Rightarrow p < \lambda$. If $p < \lambda$, then $W_\infty = 0$. Then $\sum_n I_n/n \approx -\log(W_n)/n$. But $\log(W_n)/n = -\log(Z_n)/n\lambda \to \lambda - p > 0$. Thus, the polymer is localized.
If $p = \lambda$, then $\sum_n I_n < \infty$, so the polymer is delocalized. If the polymer is delocalized, then must happen that $p = \lambda$, because we proved that $p < \lambda$ implies that the polymer is localized. That concludes the proof. □

References

[1] Francis Comets. Directed polymers in random environments. *Lecture Notes in Mathematics*, 2175, 2017.