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Directed Polymers in Random Environment

Chapter 5

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1 The Localized Phase

We will try to understand better the polymer behavior in the localized phase.An important result in this section is that if $d = 1$ or 2,then the polymer is localized for every temperature.

1.1 Path Localization

In this subsection, we will need the integration by parts formula for Gaussian random variables. Therefore, we will assume that the environments are Gaussian,i.e., $\omega(t, y) \sim N(0, 1)$. Recall the definition of $N_n = N_n(S, \tilde{S}) := \sum_{t=1}^{n} 1_{S_t=\tilde{S}_t}$,where $S, \tilde{S}$ are two independent paths. We will consider a modified version of N_n, namely , we define for $y = (y_t)_t : \mathbb{N} \to \mathbb{Z}^d$, and S a path,

$$N_n(S, y) = \sum_{t=1}^{n} 1_{S_t=y_t}$$

the number of intersections of S and y up to time n.Define the parameter region

$$\mathscr{I} = \{\beta > 0 : p \text{ is differentiable at } \beta, p'(\beta) < \lambda'(\beta)\}$$

As p is convex, the set of points where p is not differentiable is at most countable.Note that $\mathscr{I} \subset (\beta_c, \infty)$. It's conjectured that this is an equality:

Conjecture:$\mathscr{I} = (\beta_c, \infty)$.

Now we state a path localization result for Gaussian environments:

Theorem 1.1. *Assume that the environment is Gaussian.There exists $y^n : [0,n] \to \mathbb{Z}^d$ such that*

$$\liminf_{n\to\infty} \mathbb{P}P_n^{\beta,\omega}\left[\frac{N_n(S,y^{(n)})}{n}\right] \geq 1 - \frac{p'}{\lambda'}(\beta) > 0 \tag{1.1.1}$$

for all $\beta \in \mathscr{I}$.Moreover,

$$\lim_{\beta\to\infty}\liminf_{n\to\infty} \mathbb{P}P_n^{\beta,\omega}\left[\frac{N_n(S,y^{(n)})}{n}\right] = 1 \tag{1.1.2}$$

Before the proof,we state a integration by parts formula for Gaussian. random variables.

Lemma 1.2. *If X is centered normal, and f is a smooth numerical function which doest not grow too fast at infinity,i.e.,*

$$\lim_{|x|\to\infty} f(x)exp\{-x^2/(2EX^2)\} = 0$$

then

$$E(Xf(X)) = E(X^2)E(f'(X)) \tag{1.1.3}$$

Proof. Let $\sigma^2 = E(X^2)$,and $g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}}exp\{-x^2/(2\sigma^2)\}$.Then using integration by parts and the grow of f, we deduce that

$$\int_{-\infty}^{\infty} xf(x)g_\sigma(x)dx = \sigma^2\int_{-\infty}^{\infty} f'(x)g_\sigma(x)dx$$

$\square$

Proof of Theorem 1.1.

Step 1: We use the integration by parts to compute

$$\begin{aligned}\frac{d}{d\beta}\mathbb{P}p_n(\omega,\beta) &= \mathbb{P}\frac{d}{d\beta}p_n(\omega,\beta) = \frac{1}{n}\mathbb{P}P_n^{\beta,\omega}[H_n] \\ &= \frac{1}{n}\mathbb{P}\sum_{t=1}^{n}P_n^{\beta,\omega}[\omega(t,S_t)] \\ &= \frac{1}{n}\sum_{t=1}^{n}\sum_{x}\mathbb{P}[P_n^{\beta,\omega}(S_t=x)\omega(t,x)] \\ &\overset{(1.1.3)*}{=} \frac{\beta}{n}\sum_{t=1}^{n}\sum_{x}\mathbb{P}[P_n^{\beta,\omega}(S_t=x) - P_n^{\beta,\omega}(S_t=x)^2] \\ &= \beta\left(1 - \mathbb{P}P_n^{\beta,\omega^{\otimes 2}}\left[\frac{N_n(S,\tilde{S})}{n}\right]\right)\end{aligned}$$

To verify that $*$ holds, we have to write $X = \omega(t,x)$ for t, x fixed.Then

$$F(X) = \frac{\sum_{\mathbf{x}:S_t=x} e^{\beta X} e^{\beta(H_n(\mathbf{x})-X)}(2d)^{-n}}{\sum_{\mathbf{x}:S_t=x} e^{\beta X} e^{\beta(H_n(\mathbf{x})-X)}(2d)^{-n} + \sum_{\mathbf{x}:S_t\neq x} e^{\beta H_n(\mathbf{x})}(2d)^{-n}}$$

Then we see that the derivative in the numerator only depends of the factor $e^{\beta X}$, and the same happens in the denominator, where the second sum in constant in X.Thus,

$$\begin{aligned} F'(X) &= \frac{\beta \sum_{\mathbf{x}:S_t=x} e^{\beta X} e^{\beta(H_n(\mathbf{x})-X)}(2d)^{-n} Z_n - \beta \left(\sum_{\mathbf{x}:S_t=x} e^{\beta X} e^{\beta(H_n(\mathbf{x})-X)}(2d)^{-n}\right)^2}{Z_n^2} \\ &= \beta P_n^{\beta,\omega}(S_t = x) - \beta (P_n^{\beta,\omega}(S_t = x))^2 \end{aligned}$$

We conclude that

$$\mathbb{P} P_n^{\beta,\omega\otimes 2}\left[\frac{N_n(S,\tilde{S})}{n}\right] = \left(1 - \frac{1}{\beta}\frac{d}{d\beta}\mathbb{P} p_n(\omega,\beta)\right)$$

Taking $n \to \infty$ and recalling that $\frac{d}{d\beta}\mathbb{P} p_n(\omega,\beta) \to p'(\beta)$, we have that

$$\lim_{n\to\infty} \mathbb{P} P_n^{\beta,\omega\otimes 2}\left[\frac{N_n(S,\tilde{S})}{n}\right] = 1 - \frac{1}{\beta} p'(\beta) = 1 - \frac{p'}{\lambda'}(\beta)$$

using that $\lambda(\beta) = \frac{\beta^2}{2}$ in the Gaussian case.

Step 2: For fixed n, β, ω, define

$$y^{(n)}(t) = arg\max_{x\in\mathbb{Z}^d} P_n^{\beta,\omega}(S_t = x), \quad t = 1, 2, \cdots, n. \tag{1.1.4}$$

By definition,

$$\begin{aligned} P_n^{\beta,\omega\otimes 2}(S_t = \tilde{S}_t) &= \sum_x P_n^{\beta,\omega}(S_t = x)^2 \le \max_{x\in\mathbb{Z}^d} P_n^{\beta,\omega}(S_t = x) \sum_x P_n^{\beta,\omega}(S_t = x) \\ &= \max_{x\in\mathbb{Z}^d} P_n^{\beta,\omega}(S_t = x) = P_n^{\beta,\omega}(S_y = y_t^{(n)}) \end{aligned}$$

Hence,

$$\mathbb{P} P_n^{\beta,\omega\otimes 2}\left[\frac{N_n(S,\tilde{S})}{n}\right] \le \mathbb{P} P_n^{\beta,\omega}\left[\frac{N_n(S, y^{(n)})}{n}\right]$$

And (1.1.1) is holds by the previous step.

Step 3: Recalling the improved annealed bound

$$p(\beta) \le \beta \inf_{b\in[0,\beta]} \frac{\lambda(b) + \log(2d)}{b} - \log(2d)$$

we see that p grows linearly in β,so there exists $C < \infty$ such that $p'(\beta) \leq C$ for all β by convexity.Thus, the left hand side of (1.1.1) is greater than $1 - \frac{C}{\beta}$.That proves (1.1.2)

$\square$

Remark 1.3. *1. We call $y^{(n)}$ the favorite path for the polymer, although it's not a random walk path (it has jumps). The first claim states that the polymer spends a positive proportion of the time in the favorite path.*

2. Because $N_n/n \leq 1$, the second claim is called complete localization, and it says that the preference is extreme for β large enough

1.2 Low dimensions

We will show that in dimensions 1 and 2,the strong disorder holds for all β.

Theorem 1.4. *Assume $d = 1$ of $d = 2$.For all $\beta \neq 0, p(\beta) < \lambda(\beta)$,and therefore $W_\infty = 0$*

We will derive the simpler result $W_\infty = 0$, and after that we state the general result. Note that for $z \in \mathbb{Z}^d$,

$$\begin{aligned}
[P_{t-1}^{\beta,\omega}]^{\otimes 2}(S_t = \tilde{S}_t + z) &= \sum_x P_{t-1}^{\beta,\omega}(S_t = x) P_{t-1}^{\beta,\omega}(S_t = x + z) \\
&\overset{Cauchy-Schwarz}{\leq} \left(\sum_x P_{t-1}^{\beta,\omega}(S_t = x)^2 \sum_x P_{t-1}^{\beta,\omega}(S_t = x + z)^2 \right)^{1/2} \\
&= \left(\sum_x P_{t-1}^{\beta,\omega}(S_t = x)^2 \sum_x P_{t-1}^{\beta,\omega}(S_t = x)^2 \right)^{1/2} \\
&= \sum_x P_{t-1}^{\beta,\omega}(S_t = x)^2 \\
&= [P_{t-1}^{\beta,\omega}]^{\otimes 2}(S_t = \tilde{S}_t) = I_t
\end{aligned}$$

Therefore,

$$[P_{t-1}^{\beta,\omega}]^{\otimes 2}(S_t = \tilde{S}_t + z) \leq I_t \tag{1.2.1}$$

Proof of the claim $W_\infty = 0$ in Theorem 1.4:

Dimension 1: Note that

$$1 = \sum_{z:z\equiv 0 \pmod 2, |z|\le 2t} [P_{t-1}^{\beta,\omega}]^{\otimes 2}(S_t = \tilde{S}_t + z) \overset{(1.2.1)}{\le} (2t+1) I_t$$

Hence, $I_t \ge \frac{1}{2t+1}$, and $\sum_t I_t = \infty$. So, $W_\infty = 0$ when $d = 1$

Dimension 2: Assume by contradiction that $W_\infty > 0$ almost surely.Consider the event

$$A_n = \{|S_n^{(1)}| \le K\sqrt{n\log n}, |S_n^2| \le K\sqrt{n\log n}\}$$

where both coordinates of S_n are smaller in absolute value than $K\sqrt{n\log n}$. Let

$$X_n := P(e^{\beta H_{n-1} - (n-1)\lambda(\beta)}; A_n^c)$$

By Markov inequality, for large n we have

$$\begin{aligned}\mathbb{P}\left(X_n \ge e^{-\frac{K^2}{4}\log n}\right) &\le e^{\frac{K^2}{4}\log n}\mathbb{P}(X_n)\\ &= e^{\frac{K^2}{4}\log n} P(A_n^c)\\ &\le 4e^{\frac{-K^2}{4}\log n}\end{aligned}$$

In the last equation we used the bound

$$P(\pm S_n^{(1)} > K\sqrt{n\log n}) \le e^{-n\gamma^*(K\sqrt{n\log n})}$$

where γ^* is the convex conjugate of γ,

$$\gamma(u) := \log P(e^{uS_n^{(1)}}) = \log\frac{1+cosh(u)}{2} \le \log\frac{1+e^{u^2/2}}{2} \le \frac{u^2}{2}$$

implying that $\gamma^*(v) = \sup_u\{uv - \gamma(u)\} \ge v^2/2$. If $K > 2$,then $\sum_{n\ge 1} P(X_n) < \infty$, so by Borel-Cantelli lemma, $X_n \to 0$ $\mathbb{P}-$ almost surely .Then

$$Y_n = P_{n-1}^{\beta,\omega}(A_n^c) \to \frac{\lim_{n\to\infty} X_n}{W_\infty} = 0 \;\; \mathbb{P} - a.s$$

If we denote by $\mathcal{C}(n,K) := [-K\sqrt{n\log n}, K\sqrt{n\log n}]^2$,

$$\begin{aligned}(1-Y_n)^2 &= \sum_{x,y\in\mathcal{C}(n,k)} [P_{n-1}^{\beta,\omega}]^{\otimes 2}(S_n = x, \tilde{S}_n = y) \le \sum_{z\in\mathcal{C}(n,2K)} [P_{n-1}^{\beta,\omega}]^{\otimes 2}(S_n = \tilde{S}_n + z)\\ &\overset{(1.2.1)}{\le} (4K\sqrt{n\log n})^2 I_n\end{aligned}$$

Therefore, $\mathbb{P}-$ a.s., we have $I_n \ge \frac{1}{17K^2 n\log n}$, so $\sum_n I_n = \infty$,contradicting the fact that $W_\infty > 0$.That concludes the proof

$\square$

The next result gives asymptotics for $\lambda(\beta) - p(\beta)$ for dimension 1 and 2.In particular, it proves Theorem 1.4.

Theorem 1.5. *([1],[2],[3]) Assume $\omega(t, x)$ has mean 0 and variance 1.*

i) For $d = 1$,as $\beta \searrow 0$,

$$\lambda(\beta-) - p(\beta) \asymp \beta^4$$

ii) For $d = 2$, as $\beta \searrow 0$,

$$\lambda(\beta) - p(\beta) = e^{-\pi\beta^2(1+o(1))}$$

The conjecture in dimension $d = 1$ is that as $\beta \searrow 0$,

$$\lambda(\beta) - p(\beta) \sim \frac{1}{24}\beta^4$$

References

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