PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE
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DEPARTAMENTO DE MATEMÁTICA
PRIMER SEMESTRE DE 2018

Stochastic Calculus Notes

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July 1, 2018

1 Stochastic Integral with respect to Brownian motion

This section is a warm-up to the general theory of stochastic integrals. We consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$ and $B = \{B_t\}$ is a standard one-dimensional Brownian motion with respect to the filtration $\mathbb{F}_t$.As usual, we assume that $\mathcal{F}, \mathcal{F}_t$ contains all the subsets of events of probability zero.
The next lemma says that if we want to consider the integral $\int_0^t B_s dB_s$ as limit of Riemann sums, then the point of evaluation is crucial.

Lemma 1.1. *Fix a number $u \in [0,1]$.Given a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_{m(\pi)} = t\}$, let $s_i = (1-u)t_i + ut_{i+1}$, and define*

$$S(\pi) := \sum_{i=0}^{m(\pi)-1} B_{s_i}(B_{t_{i+1}} - B_{t_i})$$

Then

$$\lim_{mesh(\pi)\to 0} S(\pi) = \frac{1}{2}B_t^2 - \frac{1}{2}t + ut \quad in\ L^2(P)$$

Proof. For $a, b, c \in \mathbb{R}$ we have

$$b(a-c) = \frac{a^2}{2} - \frac{c^2}{2} - \frac{(a-c)^2}{2} + (b-c)^2 + (a-b)(b-c)$$

Apply this to $b = B_{s_i}, a = B_{t_{i+1}}, c = B_{t_i}$ to get

$$\begin{aligned} S(\pi) = &\sum_{i=0}^{m(\pi)-1} (B_{t+1}^2 - B_{t_i}^2) - \frac{1}{2}\sum_{i=0}^{m(\pi)-1} (B_{t_{i+1}} - B_{t_i})^2 + \sum_{i=0}^{m(\pi)-1} (B_{s_i} - B_{t_i})^2 + \sum_{i=0}^{m(\pi)-1} (B_{t_{i+1}} - B_{s_i})(B_{s_i} - B_{t_i}) \\ := &\frac{1}{2}B_t^2 - S_1(\pi) - S_2(\pi) - S_3(\pi) \end{aligned}$$

We now that

$$S_1(\pi) \overset{mesh(\pi)\to 0}{\longrightarrow} \frac{1}{2}t \ \ in\ L^2(P)$$

For the second sum,note that

$$E[S_2(\pi)] = \sum_{i=0}^{m(\pi)-1} u(t_{i+1}-t_i) = ut$$

because $B_{s_i} - B_{t_i} \sim N(0, s_i - t_i) = N(0, u(t_{i+1}-t_i))$.On the other hand,

$$Var(S_2(\pi)) = \sum_i Var((B_{s_i} - B_{t_i}))^2 = 2\sum_i (s_i - t_i)^2 \leq 2\sum_i (t_{i+1}-t_i)^2 \leq 2tmesh(\pi)$$

Here we used that if $X \sim N(0, s_i - t_i)$, then

$$Var(X^2) = E(X^4) - E(X^2)^2 = 3(s_i-t_i)^2 - (s_i-t_i)^2 = 2(s_i-t_i)^2$$

We conclude that $\lim_{mesh(\pi)\to} S_2(\pi) = ut$ in $L^2(P)$.

In the final sum,we check that $S_3(\pi) \to 0$ in $L^2(P)$.

$$\begin{aligned}
E[S_3(\pi)^2] &= E\left(\sum_i (B_{t_{i+1}-B_{s_i}})(B_{s_i} - B_{t_i})\right)^2 \\
&= E\left(\sum_i (B_{t_{i+1}-B_{s_i}})^2(B_{s_i} - B_{t_i})^2\right) \\
&+ E\left(\sum_{i\neq j} (B_{t_{i+1}-B_{s_i}})(B_{s_i} - B_{t_i})(B_{t_{j+1}-B_{s_j}})(B_{s_j} - B_{t_j})\right) \\
&= \sum_i (t_{i+1}-s_i)(s_i-t_i) \leq \sum_i (t_{i+1}-t_i)^2 \leq mesh(\pi)t
\end{aligned}$$

this goes to zero when $mesh(\pi) \to 0$. $\square$

Remark 1.2. *By the last lemma, and using that $B_t^2 - t$ is a martingale, then there exists a unique choice of u such that the limit of $S(\pi)$ is martingale, namely $u = 0$, so $s_i = t_i$.This election is used in the Itô integral.Under this integral we have*

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$$

The election $u = \frac{1}{2}$ leads to the Stratonovich integral.Here we have

$$\int_0^t B_s \circ dB_s = \frac{B_t^2}{2}$$

Under this integral the usual rules of calculus are satisfied.

Now we develop the Itô stochastic integral with respect to Brownian motion.now we describe the space of integrands.

For a measurable process X, the L^2-norm over the set $[0,T] \times \Omega$ is

$$||X||_{L^2([0,T]\times\Omega)} = \left(E \int_{[0,T]} |X(t,\omega)|^2 \right)^{1/2} \tag{1.1}$$

Let $\mathcal{L}_2(B)$ denote the collection of all measurable,adapted processes X such that

$$||X||_{L^2([0,T]\times\Omega)} < \infty$$

for all $T < \infty$.A metric on $\mathcal{L}_2(B)$ is given by $d_{L_2}(X,Y) := ||X-Y||_{\mathcal{L}_2(B)}$ where

$$||X||_{\mathcal{L}_2(B)} := \sum_{k=1}^{\infty} 2^{-k} (1 \wedge ||X||_{L^2([0,k]\times\Omega)}) \tag{1.2}$$

We call it a norm, although it's not a geniune norm, as in the martingale case. However, it's true that $||X+Y||_{\mathcal{L}_2(B)} \leq ||X||_{\mathcal{L}_2(B)} + ||Y||_{\mathcal{L}_2(B)}$, so we have the triangle inequality $d_{\mathcal{L}_2}(X,Y) \leq d_{\mathcal{L}_2}(X,Z) + d_{\mathcal{L}_2}(Z,Y)$.To conclude that $d_{\mathcal{L}_2}$ is a metric, we need the property $d_{\mathcal{L}_2}(X,Y) = 0 \leftrightarrow X = Y$.As in L^p spaces, we use the convention that two processes are considered as equal if the set of points (t,ω) such that $X(t,\omega) \neq Y(t,\omega)$ has $m \otimes P$-measure zero.Equivalently,

$$\int_0^{\infty} P(X(t) \neq Y(t)dt = 0 \tag{1.3}$$

In particular, processes that are indistinguishable, or modifications of each other have to be considered equal under this interpretation.

The symmetry property follows from the definition, so we conclude that $\mathcal{L}_2(B)$ is a metric space.Convergence $X_n \to X$ in $\mathcal{L}_2(B)$ is equivalent to $X_n \to X$ in $L^2([0,T]\times\Omega)$ for each $T < \infty$.

The class $\mathcal{L}_2(B)$ is quite restrictive, so we move to a wider class of function $\mathcal{L}(B)$ where the mean square requirement is satisfied only locally .More precisely, we define the class $\mathcal{L}(B)$ of all the measurable, adapted processes X such that

$$P\left(\omega : \int_0^T X(t,\omega)^2 dt < \infty \text{ for all T} < \infty \right) = 1 \tag{1.4}$$

Under this class we will define the process $(X \cdot B)_t := \int_0^t X_s dB_s$.

We start with a class of function such that the integral can be defined directly.

Definition 1.3. *A simple predictable process is a process of the form*

$$X(t,\omega) = \xi_0(\omega)(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t) \tag{1.5}$$

where n is finite, $0 = t_0 < t_1 < t_2 < \cdots < t_n$ are time points, and for $0 \le i \le n-1, \xi_i$ is a bounded $\mathcal{F}_{t_i}$-measurable random variable on $(\Omega, \mathcal{F}, P)$.
Call $\mathcal{S}_2$ the space of simple predictable processes.

Now we state a key aprroximation lemma.

Lemma 1.4. *Suppose X is a bounded,measurable,adapated process. Then there exists a sequence $\{X_n\}$ of simple predictable processes such that, for any $0 < T < \infty$,*

$$\lim_{n\to\infty} E\int_0^T |X_n(t) - X(t)|^2 dt = 0.$$

Proof. The first step is show that for each $0 < T < \infty$,there exists a simple predictable process $Y_k^{(T)}$ which vanish outside $[0,T]$ and satisfy

$$\lim_{k\to\infty} E\int_0^T |Y_k^{(T)}(t) - X(t)|^2 dt = 0 \tag{1.6}$$

Extend X to $\mathbb{R}\times\Omega$ by defining $X(t,\omega) = 0$ for $t < 0$.For each $n \in \mathbb{N}$ and $s \in [0,1]$,define

$$Z^{n,s}(t,\omega) := \sum_{j\in\mathbb{Z}} X(s + 2^{-n}j, \omega)\mathbb{1}_{(s+2^{-n}j, s+2^{-n}(j+1))}(t)\mathbb{1}_{[0,T]}(t)$$

$Z^{n,s}$ is a simple predictable process. It's jointly measurable as a function of the triple (s,t,ω), so it can be integrated over all the three variables, in any order due to Fubini's theorem. We claim that

$$\lim_{n\to\infty} E\int_0^T\int_0^1 |Z^{n,s}(t) - X(t)|^2 ds dt = 0 \tag{1.7}$$

To prove this, first consider a fixed ω, and write

$$\begin{aligned}
&\int_0^T\int_0^1 |Z^{n,s}(t) - X(t)|^2 ds dt \\
&= \int_0^T \sum_{j\in\mathbb{Z}} \int_0^1 |X(s+2^{-n}j,\omega) - X(t,\omega)|^2)\mathbb{1}_{(s+2^{-n}j, s+2^{-n}(j+1))}(t) ds dt \\
&= \int_0^T \sum_{j\in\mathbb{Z}} \int_0^1 |X(s+2^{-n}j,\omega) - X(t,\omega)|^2)\mathbb{1}_{(t-2^{-n}(j+1), t-2^{-n}j)}(s) ds dt
\end{aligned}$$

For a fixed t,the s-integral vanishes unless

$$0 < t - 2^{-n}j \ and \ t - 2^{-n}(j+1) < 1$$

which is equivalent to $2^n(t-1) - 1 < j < 2^n t$.For each fixed t and j, change variables in the $s-$integral: let $h = t - s - 2^{-n}j$,and observe that $s \in [t - 2^{-n}(j+1), t - 2^{-n}j)$ iff $h \in (0, 2^{-n}]$.The last integral above is transformed into

$$\int_0^T \sum_{j\in\mathbb{Z}} \mathbb{1}_{2^n(t-1)-1<j<2^n t} \int_0^{2^{-n}} |X(t-h,\omega) - X(t,\omega)|^2 dh dt$$
$$\leq (2^n + 1) \int_0^{2^{-n}} \int_0^T |X(t-h,\omega) - X(t,\omega)|^2 dt dh$$

The last bound follows using the fact that there are at most $2^n + 1$ j-values such that $2^n(t-1) - 1 < j < 2^n t$.Now we take expectations to deduce

$$E \int_0^T \int_0^1 |Z^{n,s}(t) - X(t)|^2 ds dt \leq (2^n + 1) \int_0^{2^{-n}} \int_0^T |X(t-h,\omega) - X(t,\omega)|^2 dt dh \to 0$$

when $n \to \infty$.To show this, first note that for fixed ω,

$$\lim_{h\to 0} \int_0^T |X(t-h,\omega) - X(t,\omega)|^2 dt = 0$$

This is proved first for simple functions, and then by approximation to general $X(\cdot,\omega) \in L^2([0,T])$.Since X is bounded, by dominated convergence

$$\lim_{h\to 0} E \int_0^T |X(t-h,\omega) - X(t,\omega)|^2 dt = 0$$

Last,

$$\lim_{n\to\infty} (2^n + 1) \int_0^{2^{-n}} \left\{ \int_0^T |X(t-h,\omega) - X(t,\omega)|^2 dt = 0 \right\} dh = 0$$

This follows from this general fact: if $f(x) \to 0$ when $x \to 0$,then

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^\epsilon f(x) dx = 0$$

To check this, fix $\delta > 0$,and take ϵ_0 such that if $|x| < \epsilon_0$,then $|f(x)| \leq \delta$.Then, if $\epsilon < \epsilon_0$,

$$\frac{1}{\epsilon} \int_0^\epsilon f(x) dx \leq \delta$$

As $\delta > 0$ is arbitrary, we are done.This concludes the proof of (1.7).Observe that (1.7) is equivalent to say that the sequence $\phi_n(s) := E \int_0^T |Z^{n,s} - X(t)|^2 dt$ satisfies $\phi_n(s) \to 0$ in

$L^1[0,1]$.Then there exists a subsequence ϕ_{n_k} such that $\phi_{n_k}(s) \to 0$ for a.e $s \in [0,1]$.Pick any such s, and define $Y_k^{(T)} = Z^{n_k,s}$.By construction

$$\lim_{k\to\infty} \phi_{n_k}(s) = E\int_0^T |Y_k^{(T)} - X(t)|^2 dt = 0$$

This is (1.6).To complete the proof, for $m \in \mathbb{N}$, choose k_m such that

$$E\int_0^m |Y_{k_m}^{(m)} - X(t)|^2 dt < \frac{1}{m}$$

and define $X_m = Y_{k_m}^{(m)}$.Therefore for any $T > 0$, for any $m > T$ we have

$$E\int_0^T |X_m(t) - X(t)|^2 dt \leq E\int_0^m |X_m(t) - X(t)|^2 dt < \frac{1}{m} \overset{m\to\infty}{\longrightarrow} 0$$

□

Proposition 1.5. *Suppose $X \in \mathcal{L}_2(B)$-Then there exists a sequence of simple predictable processes $\{X_n\}$ such that $||X_n - X||_{\mathcal{L}_2(B)} \to 0$.*

Proof. Let $X^{(k)} := (X \wedge k) \vee (-k)$.Since $|X^{(k)} - X| \leq |X|$ and $|X^{(k)} - X| \to 0$ when $k \to \infty$, pointwise on $\mathbb{R}_+ \times \Omega$,

$$\lim_{k\to\infty} E\int_0^m |X^{(k)}(t) - X(t)|^2 dt = 0$$

for each $m \in \mathbb{N}$.This is equivalent to $||X - X^{(k)}||_{\mathcal{L}_2(B)} \to 0$.Given $\epsilon > 0$, pick k such that $||X - X^{(k)}||_{\mathcal{L}_2(B)} \leq \epsilon/2$.Since $X^{(k)}$ is a bounded process, we an use the last lemma to find a predictable process Y_ϵ such that $||X^k - Y_\epsilon||_{\mathcal{L}_2(B)} \leq \epsilon/2$.By the triangle inequality, $||X - Y_\epsilon||_{\mathcal{L}_2(B)} \leq \epsilon$.Repeat this argument for each $\epsilon = 1/n$, and call $X_n = Y_\epsilon$.This give the approximating sequence $\{X_n\}$.

□

Now we are ready to construct the stochastic integral. The main steps are:

i) Define an explicit formula $X \cdot B$ for a simple predictable process X. This integrall will be a continuous L^2- martingale.

ii) A general process $X \in \mathcal{L}_2(B)$ is approximated by simple proesses X_n.One shows that the integrals $X_n \cdot B$ converges to a uniquely defined continuous L^2-martingale, which we call $X \cdot B$.

iii) A localization argument is used to get from integrands in $\mathcal{L}_2(B)$ to integrands in $\mathcal{L}(B)$.The integral $X \cdot B$ is a continuous local L^2- martingale.

Let's start with step i).For a simple predictable process X like (1.5), the *stochastic integral* is the process $X \cdot B$ defined by

$$(X \cdot B)_t := \sum_{i=1}^{n-1} \xi_i(\omega)(B_{t_{i+1}\wedge t}(\omega) - B_{t_i \wedge t}(\omega). \tag{1.8}$$

We also write $I(X) = X \cdot B$ when we need a symbol for the maping $I : X \to X \cdot B$.
As always, we need to check that the integral does not depend of the representation as simple predictable process, and also that the integral map $I(X)$ is linear on $\mathcal{S}_2$.

Lemma 1.6.

a) Suppose the process X in (1.5) also satisfies

$$X_t(\omega) = \eta_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{j=1}^{m-1} \eta_j \mathbb{1}_{(s_j, s_{j+1}]}(t)$$

for all (t, ω), where $0 = s_0 = s_1 < s_2 < \cdots < s_m < \infty$ and $\eta_j \in \mathcal{F}_{s_j}$-measurable for $0 \leq j \leq m-1$.Then for each (t, ω),

$$\sum_{i=1}^{n-1} \xi_i(\omega)(B_{t_{i+1}\wedge t}(\omega) - B_{t_i \wedge t}(\omega) = \sum_{j=1}^{m-1} \eta_j(\omega)(B_{s_{j+1}\wedge t}(\omega) - B_{s_j \wedge t}(\omega). \tag{1.9}$$

b) $\mathcal{S}_2$ is a linear space, in other words, for $X, Y \in \mathcal{S}_2$ and reals $\alpha, \beta, \alpha X + \beta Y \in \mathcal{S}_2$.The integral satisfies

$$(\alpha X + \beta Y) \cdot B = \alpha(X \cdot B) + \beta(Y \cdot B).$$

Proof. a) Let $\{u_k\}_{k=1}^{l}$ be a common refinement of the partitions $\{t_i\}$ and $\{s_j\}$.Then X is equal to the process $\kappa_0(\omega)\mathbb{1}_{\{0\}}(\omega)(t) + \sum_{k=1}^{l} \kappa_k \mathbb{1}_{(u_k, u_{k+1}]}(t)$, where $\kappa_k = \xi_i$ if $(u_k, u_{k+1}] \subset$

$(t_i, t_{i+1}], \kappa_k = \eta_j$ if $(u_k, u_{k+1}] \subset (s_j, s_{j+1}]$.Therefore

$$\begin{aligned}\sum_{i=1}^{n-1} \xi_i(\omega)(B_{t_{i+1}\wedge t}(\omega) - B_{t_i\wedge t}(\omega)) &= \sum_{i=1}^{n-1} \xi_i(\omega) \sum_{k:(u_k,u_{k+1}]\subset(t_i,t_{i+1}]} (B_{u_{k+1}\wedge t}(\omega) - B_{u_k\wedge t}(\omega)) \\ &= \sum_{i=1}^{n-1} \sum_{k:(u_k,u_{k+1}]\subset(t_i,t_{i+1}]} \kappa_k(\omega)(B_{u_{k+1}\wedge t}(\omega) - B_{u_k\wedge t}(\omega)) \\ &= \sum_{k=1}^{l} \kappa_k(\omega)(B_{u_{k+1}\wedge t}(\omega) - B_{u_k\wedge t}(\omega))\end{aligned}$$

And we can do the same with the another representation.

b) If $X, Y \in \mathcal{S}_2$,by the argument in the first part we can assume that both X, Y have the same partition,namely

$$X(t,\omega) = \xi_0(\omega)(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t)$$
$$Y(t,\omega) = \eta_0(\omega)(t) + \sum_{i=1}^{n-1} \eta_i(\omega)\mathbb{1}_{(t_i,t_{i+1}]}(t)$$

Clearly,

$$\alpha X + \beta Y = X(t,\omega) = \alpha\xi_0(\omega)(t) + \beta\eta_0(\omega)(t) + \sum_{i=1}^{n-1}(\alpha\xi_i(\omega) + \beta\eta_i(\omega))\mathbb{1}_{(t_i,t_{i+1}]}(t)$$

so $\alpha X + \beta Y \in \mathcal{S}_2$, and the linearity follows from this representation.

□

Now we check that the integral is a martingale, and we have an isommetry.

Lemma 1.7. *Let $X \in \mathcal{S}_2$.Then $X \cdot B$ is a continuous square-integrable martingale with respect to the original filtration $\{\mathcal{F}_t\}$.We have these isometries:*

$$E[(X\cdot B)_t^2] = E\int_0^t X_s^2 ds \text{ for all} t \geq 0, \tag{1.10}$$

and

$$||X \cdot B||_{\mathcal{M}_2} = ||X||_{\mathcal{L}_2(B)}. \tag{1.11}$$

Proof. To show that $X \cdot B$ is a martingale, it's enough to check the following: if $u < v$ and ξ is a bounded $\mathcal{F}_u$ measurable martingale, then $Z_t = \xi(B_{t\wedge v} - B_{t\wedge u})$ is a martingale.Suppose

$t > s$.If $s \leq u$, then $s \wedge u = s = s \wedge v$, so $Z_s = 0$.On the other hand , as $\mathcal{F}_s \subset \mathcal{F}_u$,

$$E[Z_t|\mathcal{F}_s] = E[E[Z_t|\mathcal{F}_u]|\mathcal{F}_s] = E[\underbrace{E[\xi(B_{t\wedge v} - B_{t\wedge u})|\mathcal{F}_u]}_{=0}|\mathcal{F}_s] = 0 = Z_s$$

If $s > u$,then ξ is $\mathcal{F}_s$-measurable, and we get

$$E[Z_t|\mathcal{F}_s] = \xi E[(B_{t\wedge v} - B_{t\wedge u})|\mathcal{F}_s] = Z_s$$

To prove (1.10), first square

$$(X \cdot B)_t^2 = \sum_{i=1}^{n-1} \xi_i^2 (B_{t\wedge t_{i+1}} - B_{t_i})^2 + 2\sum_{i<j} \xi_i \xi_j (B_{t\wedge t_{i+1}} - B_{t_i})(B_{t\wedge t_{j+1}} - B_{t_j})$$

Now we take and expectations,and by independence, the expectations of the second sum vanish(see Exercise 5).We have

$$E(X \cdot B)_t^2 = \sum_{i=1}^{n-1} E(\xi_i^2)(t \wedge t_{i+1} - w \wedge t_i)$$

On the other hand, we have

$$E\int_0^t X_2^2 ds = \int_0^t \sum_{i=1}^{n-1} E[\xi_i^2]\mathbb{1}_{(t_i,t_{i+1}])} = \sum_{i=1}^{n-1} E\xi_i^2 (t \wedge t_{i+1} - t \wedge t_i)$$

So (1.10) holds. Finally, to check (1.11),

$$||X \cdot B||_{\mathcal{M}_2} = \sum_{k=1}^{\infty} 2^{-k}(1 \wedge ||(X \cdot B)_k||_{L^2(P)}) \overset{(1.10)}{=} \sum_{k=1}^{\infty} 2^{-k}(1 \wedge ||X||_{L^2([0,k]\times\Omega)}) = ||X||_{\mathcal{L}_2(B)}$$

□

Lemma 1.8. *Let $X \in \mathcal{L}_2(B)$.Then there ir a unique continuous L^2-martingale Y such that for any sequence of simple predictable processes $\{X_n\}$ such that*

$$||X - X_n||_{\mathcal{L}_2(B)} \to 0$$

we have

$$||Y - X_n \cdot B||_{\mathcal{M}_\in} \to 0$$

Proof. Note that for any $X \in \mathcal{L}_2(B)$,there exists an approximating sequence by Proposition 1.5.By the last lemma, we have for $n, m \in \mathbb{N}$,

$$||X_n \cdot B - X_m \cdot B||_{\mathcal{M}_2} = ||X_n - X_m||_{\mathcal{L}_2(B)} \to 0$$

when $n,m\to\infty$.So $\{X_n\cdot B\}_n$ is a Cauchy sequence in $\mathcal{M}_2^c$.As this space is complete (Theorem 1.48 in Chapter 2), we deduce that there exists a limit Y. To show uniqueness, if $\tilde{Y}$ is another such limit,let's say that $\tilde{X}_n\cdot B\to\tilde{Y}$.We have

$$\begin{aligned}||Y-\tilde{Y}||_{\mathcal{M}_2} &\le ||Y-X_n\cdot B||_{\mathcal{M}_2}+||X_n\cdot B-\tilde{X}_n\cdot B||_{\mathcal{M}_2}+||\tilde{X}_n\cdot B-\tilde{Y}||_{\mathcal{M}_2}\\ &=||Y-X_n\cdot B||_{\mathcal{M}_2}+||X_n-\tilde{X}_n||_{\mathcal{L}_2(B)}+||\tilde{X}_n\cdot B-\tilde{Y}||_{\mathcal{M}_2}\to 0\end{aligned}$$

when $n\to\infty$,because in the middle term we can apply again triangle inequality.This shows the uniqueness. □

With this result, we can define the stochastic integral respect to Brownian motion.

Definition 1.9. *Let B be a Brownian motion on a probability space $(\Omega,\mathcal{F},P)$ with respect to the filtration $\{\mathcal{F}_t\}$.For any measurable adapted process $X\in\mathcal{L}_2(B)$, the stochastic integral $I(X)=X\cdot B$ is the squuare-integrable continuous martingale that satisfies*

$$\lim_{n\to\infty}||X\cdot B-X_n\cdot B||_{\mathcal{M}_2}=0$$

for any sequence $X_n\in\mathcal{S}_2$ of simple predictable processes such that

$$||X-X_n||_{\mathcal{L}_2(B)}\to 0$$

The process $I(X)$ is unique up to indistinguishability. Alternative notation for the stochastic integral is

$$\int_0^t X_s dB_s=(X\cdot B)_t$$

Example 1.10. *We derive the integral for the process*

$$X(t)=\sum_{i=1}^{m-1}\eta_i\mathbb{1}_{(s_i,s_{i+1}]}(t)$$

where $0\le s_1<\cdots<s_m$ and each $\eta_i\in L^2(P)$ is $\mathcal{F}_{s_i}-$measurable.It can be proved that (see the exercises)

$$X_k(t)=\sum_{i=1}^{m-1}\eta_i^{(k)}\mathbb{1}_{(s_i,s_{i+1}]}(t)$$

is an approximating sequence for X, where $\eta_i^{(k)}=(\eta_i\wedge k)\vee(-k)$.Then

$$\int_0^t X(s)dB_s=\sum_{i=1}^{m-1}\eta_i(B_{t\wedge s_{i+1}}-B_{t\wedge s_i})$$

It's also needed to check that the right-side is square integrable. More details in the exercises.

Example 1.11. *One can check that the Brownian motion itself is an element of $\mathcal{L}_2(B)$.Let $t_i^n = i2^{-n}$ and*

$$X_n(t) = \sum_{i=0}^{2^n n-1} B_{t_i^n} \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t)$$

$X_n \notin\in \mathcal{S}_2$,but it can be used to approximate B.By Example 1.10,

$$\int_0^t X_n(s) dB_s = \sum_{i=0}^{2^n n-1} B_{t_i^n}(B_{t\wedge t_{i+1}^n} - B_{t\wedge t_i^n})$$

For any $T < n$,we use that $X_n(t) = B_{t_i^n}(t)$ in $(t_i^n, t_{i+1}^n]$ to get

$$\begin{aligned} E\int_0^T |X_n(s) - B_s|^2 dB_s &\leq \sum_{i=0}^{2^n n-1} \int_{t_i^n}^{t_{i+1}^n} E|X_n(s) - B_s|^2 ds \\ &= \sum_{i=0}^{2^n n-1} \int_{t_i^n}^{t_{i+1}^n} E|B_{t_i^n}(s) - B_s|^2 ds \\ &= \sum_{i=0}^{2^n n-1} \int_{t_i^n}^{t_{i+1}^n} (t_i^n - s) ds = \sum_{i=0}^{2^n n-1} \frac{1}{2}(t_{i+1}^n - t_i^n)^2 = \frac{1}{2} n 2^{-n} \end{aligned}$$

Thus $X_n \to B$ in $\mathcal{L}_2(B)$ as $n \to \infty$.By the isometry (1.12) in the next proposition, this integral converges to $\int_0^t B_s dB_s$ in L^2 as $n \to \infty$,so by Lemma 1.1,

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$

Proposition 1.12. *Let $X, Y \in \mathcal{L}_2(B)$.*

a) Linearity: $(\alpha X + \beta Y) \cdot B = \alpha(X \cdot B) + \beta(Y \cdot B)$.

b) The isometry holds:

$$E[(X \cdot B)_t^2] = E\int_0^t X_s^2 ds \quad \text{for all } t \geq 0 \tag{1.12}$$

and

$$||X \cdot B||_{\mathcal{M}_2} = ||X||_{\mathcal{L}_2(B)} \tag{1.13}$$

In particular, if $X, Y \in \mathcal{L}_2(B)$ are $m \otimes P$-equivalent in the sense of (1.3), then $X \cdot B$ and $Y \cdot B$ are indistinguishable.

c) Suppose τ is a stopping time such that $X(t,\omega)=Y(t,\omega)$ for $t\le\tau(\omega)$. Then for almost every ω, $(X\cdot B)_t(\omega)=(Y\cdot B)_t(\omega)$ for $t\le\tau(\omega)$

Proof. Parts $a)-b)$ are inherited by approximation. So we only prove $c)$.Taking $Z=X-Y$, it suffices to prove that if $Z\in\mathcal{L}_2(B)$ satisfies $Z(t,\omega)=0$ for all $t\le\tau(\omega)$, then $(Z\cdot B)_t(\omega)=0$ for $t\le\tau(\omega)$.

Assume first that Z is bounded, so $|Z(t,\omega)|\le C$. Pick a sequence $\{Z_n\}$ of simple predictable processes that converge to Z in $\mathcal{L}_2(B)$.Let Z_n be of the type

$$Z_n(t,\omega)=\sum_{i=1}^{m(\pi)-1}\xi_i^n\mathbb{1}_{(t_i^n,t_{i+1}^n]}(t)$$

We may assume $|\xi_i^n|\le C$,otherwise replace ξ_i^n by $(\xi_i^n\wedge k)\vee(-k)$(this also approximates Z).Define another sequence of predictable processes by

$$\tilde{Z}_n(t):=\sum_{i=1}^{m(\pi)-1}\xi_i^n\mathbb{1}_{\{\tau\le t_i^n\}}\mathbb{1}_{(t_i^n,t_{i+1}^n]}(t)$$

We claim that

$$\tilde{Z}_n\to Z\ in\ \mathcal{L}_2(B)\tag{1.14}$$

To prove (1.14), first note that $Z_n\mathbb{1}_{\tau<t}\to Z\mathbb{1}_{\tau<t}=Z$ in $\mathcal{L}_2(B)$.So it suffices to show that

$$Z_n\mathbb{1}_{\tau<t}-\tilde{Z}_n\to 0\ in\ \mathcal{L}_2(B).\tag{1.15}$$

We estimate

$$|Z_n(t)\mathbb{1}_{\tau<t}-\tilde{Z}_n(t)|\le C\sum_i|\mathbb{1}_{\tau<t}-\mathbb{1}_{\tau\le t_i^n}|\mathbb{1}_{(t_i^n,t_{i+1}^n]}\le C\sum_i\mathbb{1}_{\{t_i^n<\tau<t_{i+1}^n\}}\mathbb{1}_{(t_i^n,t_{i+1}^n]}(t)$$

The last inequality can be justified noting that $|\mathbb{1}_{\tau<t}-\mathbb{1}_{\tau\le t_i^n}|\mathbb{1}_{(t_i^n,t_{i+1}^n]}=1$ if $\tau<t,\tau>t_i^n$ and $t<t_{i+1}^n$, so $t_i^n<\tau<t_{i+1}^n$.The another case, namely $\tau\le t_i^n$ and $t\le\tau$ and $t_i^n<t\le t_{i+1}^n$ cannot happen. Now we integrate over $[0,T]\times\Omega$ to get

$$\begin{aligned}E\int_0^T|Z_n(t)\mathbb{1}_{\tau<t}-\tilde{Z}_n(t)|^2dt&\le C^2\sum_i P(t_i^n<\tau<t_{i+1}^n)\int_0^T\mathbb{1}_{(t_i^n,t_{i+1}^n]}(t)dt\\&\le C^2\max\{T\wedge t_{i+1}^n-T\wedge t_i^n:1\le i\le m(n)-1\}\end{aligned}$$

We can add as many points t_i^n to the partition of each Z_n, so the right hand side converges to zero when $n \to \infty$,for each fixed T.This proves (1.15) and also (1.14).
The integral of $\tilde{Z}_n$ is given explicitly by

$$(\tilde{Z}_n \cdot B)_t = \sum_{i=1}^{m(\pi)-1} \xi_i^n \mathbb{1}_{\tau \leq t_i^n} (B_{t \wedge t_{i+1}^n} - B_{t \wedge t_i^n})$$

By construction , we see that $(\tilde{Z}_n \cdot B)_t = 0$ if $t \leq \tau$.Byc (1.14), $\tilde{Z}_n \cdot B \to Z \cdot B$ in $\mathcal{M}_2^c$.By Lemma 1.49 in Chapter 1, there exists a subsequence $\tilde{Z}_{n_k} \cdot B$ and an event Ω_0 with $P(\Omega_0) = 1$ such that, for eacth $\omega \in \Omega_0$ and $T < \infty$,

$$(\tilde{Z}_{n_k} \cdot B)_t(\omega) \to (Z \cdot B)_t(\omega) \ \text{ uniformly for } 0 \leq t \leq T$$

For any $\omega \in \Omega_0$, in the limit $(Z \cdot B)_t(\omega) = 0$ for $t \leq \tau(\omega)$.This concludes the proof for bounded processes.
The general case comes from the standard approximation: if $Z \in \mathcal{L}_2(B)$, take $Z^{(k)} = (Z \wedge k) \vee (-k)$ satisfies property c), and the same holds in the limit. □

Next we extend the intgral to integrands in $\mathcal{L}(B)$.Given a process $X \in \mathcal{L}(B)$,define the stopping times

$$\tau_n(\omega) := \inf\{t \geq 0 : \int_0^t X(s, \omega^2) ds \geq n\} \tag{1.16}$$

These are stopping times by Lemma 1.20 in Chapter 1 because the function

$$t \to \int_0^t X(s, \omega)^2 ds$$

is continuous for each ω in th event in the definition of (1.4).By this same continuity, if $\tau_n(\omega) < \infty$,

$$\int_0^\infty X(s, \omega)^2 \mathbb{1}_{s \leq \tau_n(\omega)} ds = \int_0^{\tau_n(\omega)} X(s, \omega)^2 ds = n$$

Let $X_n(t, \omega) := X(t, \omega) \mathbb{1}_{t \leq \tau_n}(\omega)$.Adapteness of X_n follows from $\{t \leq \tau_n\} = \{\tau_n < t\}^c \in \mathcal{F}_t$.The function $(t, \omega) \to \mathbb{1}_{t \leq \tau_n(\omega)}$ is $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$-measurable (see the exercises),hence X_n is a measurable process. Together these properties say that $X_n \in \mathcal{L}_2(B)$, and the stochastic integrals $X_n \cdot B$ are well defined.
Now we want to show that there exists a unique limit when $n \to \infty$ of $X_n \cdot B$, and this will be our definition of $X \cdot B$.

Lemma 1.13. *For almost every ω, $(X_m \cdot B)_t(\omega) = (X_n \cdot B)_t(\omega)$ for all $t \leq \tau_m(\omega) \wedge \tau_n(\omega)$.*

Proof. This follows from Proposition 1.12,part c). □

Observe that by (1.4) we have that $\tau_n(\omega) \nearrow \infty$ for almost every ω.Now we extend the stochastic integral to $\mathcal{L}(B)$.

Definition 1.14. *Let B be a Brownian motion on a probability space $(\Omega, \mathbb{F}, P)$ with respect to the filtration $\mathcal{F}_t$,and $X \in \mathcal{L}(B)$.Let Ω_0 be the event of full probability on which $\tau_n \nearrow \infty$ and the conclusion of Lemma 1.13 holds for all pairs m, n.The stochastic integral $X \cdot B$ is defined for $\omega \in \Omega_0$ by*

$$(X \cdot B)_t(\omega) = (X_n \cdot B)_t(\omega) \quad \text{for any } n \text{ such that } \tau_n(\omega) \geq t \tag{1.17}$$

For $\omega \notin \Omega_0$,define $(X \cdot B)_t(\omega) \equiv 0$.The process $X \cdot B$ is a continuous local L^2-martingale.

To justify the claim that $X \cdot B$ is a local L^2-martingale, just note that $\{\tau_n\}$ serves as a localizing sequence:

$$(X \cdot B)_t^{\tau_n} = (X \cdot B)_{t\wedge\tau_n} = (X_n \cdot B)_{t\wedge\tau_n} = (X_n \cdot B)_t^{\tau_n}$$

so $(X \cdot B)_t^{\tau_n} = (X_n \cdot B)_t^{\tau_n}$, which is an L^2- martingale (by Corollary 1.12 in Chapter 2).The above equality also implies that $(X \cdot B)_t(\omega)$ is continuous for $t \in [0, \tau_n(\omega)]$,which contains any given interval $[0, T]$ if n is taken large enough.

Definition 1.15. *Let X be an adapted measurable process. A nondecreasing sequence of stopping times $\{\sigma_n\}$ is a localizing sequence for X if $X(t)\mathbb{1}_{t\leq\sigma_n}$ is in $\mathcal{L}_2(B)$ for all n, and $\sigma_n \nearrow \infty$ with probability one.*

Remark 1.16. *In the exercises we prove that $X \in \mathcal{L}(B)$ iff X has a localizing sequence $\{\sigma_n$.In fact, the definition of stochastic integral works equally well if we replace the stopping times $\{\tau_n\}$ by a localizing sequence $\{\sigma_n\}$.To see this, fix a localizing sequence $\{\sigma_n\}$ and define $\tilde{X}_n(t) = \mathbb{1}_{t\leq\sigma_n} X(t)$.Let Ω_1 the event of full probability on which $\sigma_n \nearrow \infty$ and forl all pairs m, n, $(\tilde{X}_m \cdot B)_t = (\tilde{X}_n \cdot B)_t$ for $t \leq \sigma_m \wedge \sigma_n$.Let Y be the process defined by*

$$Y_t(\omega) = (\tilde{X}_n \cdot B)_t(\omega) \quad \text{for any } n \text{ such that } \sigma_n \geq t \tag{1.18}$$

for $\omega \in \Omega_1$, and zero outside.

Lemma 1.17. *$Y = X \cdot B$ in the sense of indistinguishability.*

Proof. Let $\Omega_2 = \Omega_0 \cap \Omega_1$.This is a full probability event.Applying Proposition 1.12(c) to the stopping time $\sigma_n \wedge \tau_n$ and the processes $X_n, \tilde{X}_n$,we conclude that for almost every $\omega \in \Omega_2$, if $t \leq \sigma_n(\omega) \wedge \tau_n(\omega)$,

$$Y_t(\omega) = (\tilde{X}_n \cdot B)_t(\omega) = (X_n \cdot B)_t(\omega) = (X \cdot B)_t(\omega)$$

since $\sigma_n \wedge \tau_n \nearrow \infty$, the above equality holds almost surely for all $0 \leq t < \infty$.

$\square$

This lemma says that if $X \in \mathcal{L}(B)$,the stochastic integral $X \cdot B$ can be defined for any localizing sequence.