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Stochastic Calculus Notes

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1 Itô's Formula.

In this section we state and prove the "fundamental theorem of calculus" for stochastic integrals. Recall that if $x(t)$ is continuously differentiable, then

$$f(x(t)) = f(x(0)) + \int_0^t f(x(s))x'(s)ds$$

However, for Brownian motion we have

$$f(B_t) = f(B_0) + \int_0^t f(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds$$

This is a special case of Itô's formula. If the process we are integrating possesses jumps, we will need to add more terms to the right-hand side.
In the proofs in this section will not be needed the right-continuity of the filtration $\{\mathcal{F}_t\}$.However, this may be necessary to construct the stochastic integral.In particular,this assures that the local martingale in the semimartingale decomposition whose local martingale is a local L^2-martingale.The problem only can arise when the local martingale has unbounded jumps(recall Lemma 1.26 from part 2).In particular, when the martingale is continuous or its jumps have an uniform bound, it can be localized into an L^2-martingale.

1.1 Itô's formula: proofs and special cases

We prove first Itô's formula for real-valued semimartingales and then consider the following special cases:continuous semimartingales, FV processes and Brownian motion.Next we consider vector-valued semimartingales.

If $F \subset \mathbb{R}, C^2(D)$ is the space of functions $f : D \to \mathbb{R}$ such that f', f'' exist on D and are continuous.For a real or vector-valued cadlag process X, the jump at time s is denoted by $\triangle X_s = X_s - X_{s-}$.Recall also the closure of the path over a time interval for a cadlag process, given by

$$\overline{X[0,t]} = \{X(s) : 0 \le s \le t\} \cup \{X(s-) : 0 < s \le t\}$$

Recall that a cadlag has at most countable many discontinuities, so in the theorem below the sum is over countable many terms.

Theorem 1.1. *Fix $0 < T < \infty$.Let D be a open subset of $\mathbb{R}$ and $f \in C^2(D)$.Let Y be a cadlag semimartingale with quadratic variation process $[Y]$.Assume that for all ω outside some event of probability zero,$\overline{Y}[0,T] \subset D$.Then*

$$\begin{aligned} f(Y_t) = f(Y_0) &+ \int_{(0,t]} f'(Y_{s-})dY_s + \frac{1}{2}\int_{(0,t]} f''(Y_{s-})d[Y]_s \\ &+ \sum_{s\in(0,t]} \left(f(Y_s) - f(Y_{s-}) - f'(Y_{s-})\triangle Y_s - \frac{1}{2} f''(Y_{s-})(\triangle Y_s)^2 \right) \end{aligned} \tag{1.1}$$

Part of the conclusion is that the last sum over $s \in (0,t]$ converges absolutely for almost every ω.Both sides of the equality above are cadlag processes, and the equality above means indistinguishability on $[0,T]$, that is, the equality (1.1) holds almost surely for $0 \le t \le T$.

Proof. Define the function γ on $D \times D$ by $\gamma(x,x) = 0$ for $x \in D$, and for $x \neq y$,

$$\gamma(x,y) = \frac{1}{(y-x)^2}\left(f(y) - f(x) - f'(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right) \tag{1.2}$$

Clearly, γ is continuous on $\{(x,y) \in D \times D : x \neq y\}$.We prove that γ is also continuous at diagonal points.To do this we use Taylor's theorem(Theorem A.1).Given $z \in D$, pick $r > 0$ small enough so that $(z-r, z+r) \subset D$.Then for $x, y \in D, x \neq y$,there exists $\theta_{x,y}$ between x and y such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2} f''(\theta_{x,y})(y-x)^2$$

So for these (x,y)

$$\gamma(x,y) = \frac{1}{2}(f''(\theta_{x,y}) - f''(x))$$

As $(x,y) \to (z,z), \theta_{x,y} \to z$, so by the continuity of f'',

$$\gamma(x,y) \to 0 = \gamma(z,z).$$

This proves the continuity of γ on $D \times D$.

Now we write

$$f(y) - f(x) = f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 + \gamma(x,y)(y-x)^2$$

Given a partition $\pi = \{t_i\}$ of $[0,\infty)$,we apply the identity above to obtain

$$f(Y_t) - f(Y_0) = \sum_i f'(Y_{t\wedge t_i})(Y_{t_{i+1}\wedge t} - Y_{t_i \wedge t}) \tag{1.3}$$

$$+ \frac{1}{2}\sum_i f''(Y_{t\wedge t_i})(Y_{t\wedge t_i} - Y_{t_i\wedge t})^2 \tag{1.4}$$

$$+ \sum_i \gamma(Y_{t_i}, Y_{t_{i+1}})(Y_{t\wedge t_i} - Y_{t_i\wedge t})^2 \tag{1.5}$$

By Proposition 1.37 and 1.59 from part 4 we can fix a sequence of partitions π^l such that. $mesh(\pi^l) \to 0$, and so the following limits happen almost surely, uniformly on $t \in [0,T]$, as $l \to \infty$.

i) The sum (1.3) converges to $\int_{(0,t]} f'(Y_{s-})dY_s$.

ii) The sum (1.4) converges to $\frac{1}{2}f''(Y_{s-})d[Y]_s$.

iii) $\sum_i (Y_{t\wedge t_{i+1}} - Y_{t\wedge t_i})^2 \to [Y]_t$.

Fix ω such that the limits in $i) - iii)$ hold.We apply Lemma A.2 in a simplified form so that the functions ϕ, γ in (A.3) have no time variables. We use this lemma with $d = 1$,the cadlag $s \to Y_s(\omega)$ on [0,t] an the sequence of partitions π^l chosen above.The closed set we take is $K = \overline{Y[0,T]}$.The continuous function is $\phi(x,y) = \gamma(x,y)(y-x)^2$.By hypothesis, $K \subset D$.Consequently, the function

$$\gamma(x,y) = \begin{cases} (x-y)^{-2}\phi(x,y), & x \neq y \\ 0, & x = y \end{cases}$$

is continuous on $K \times K$.Assumption (A.4) holds by $iii)$.Thus the hypotheses of Lemma A.2 are satisfied.The conclusion is that for this fixed ω and each $t \in [0,T]$, the sum on line (1.5) converges to

$$\begin{aligned}\sum_{s\in(0,t]}\phi(Y_{s-},Y_s)&=\sum_{s\in(0,t]}\gamma(Y_{s-},Y_s)(Y_s-Y_{s-})^2\\&=\sum_{s\in(0,t]}\left(f(Y_s)-f(Y_{s-})-f'(Y_{s-})\triangle Y_s-\frac{1}{2}f''(Y_{s-})(\triangle Y_s)^2\right)\end{aligned}$$

And this sum is absolutely convergent by Lemma A.2.

We have proved that given $0<T<\infty$, for almost every ω (1.1) holds for all $0\le t\le T$. $\square$

Corollary 1.2. *Under the conditions of Theorem 1.1,$f(Y)$ is a semimartingale.*

Proof. Using the formula (1.1), we obtain $f(Y)=f(Y_0)+M+V$, with $M=\int f'(Y_-)dY$,and V the remaining sums. We prove that this is a FV process. Clearly $\int_{(0,t]}f''(Y_-)d[Y]$ is a FV process.Now we check that the sum over the jumps is also a FV process.Fix ω such that $Y_s(\omega)$ is a cadlag function and (1.1) holds.Let $\{s_i\}$ denote the (at most countable many) jumps of $s\to Y_s(\omega)$ in $[0,T]$.The theorem gives the absolute convergence

$$\sum_{s\in(0,t]}|f(Y_s)-f(Y_{s-})-f'(Y_{s-})\triangle Y_s-\frac{1}{2}f''(Y_{s-})(\triangle Y_s)^2|<\infty$$

Consequently for this fixed ω the sum in (1.1) defines a function in $BV[0,T]$. $\square$

We state some simplifications of Itô's formula.

Corollary 1.3. *under the hypotheses of Theorem 1.1 we have the following special cases.*

a) If Y is continuous on [0,T],then

$$f(Y_t)=f(Y_0)+\int_0^t f'(Y_s)dY_s+\frac{1}{2}\int_0^t f''(Y_s)d[Y]_s \tag{1.6}$$

b) If Y has bounded variation on $[0,T]$,then

$$f(Y_t)=f(Y_0)+\int_{(0,t]}f'(Y_{s-})dY_s+\sum_{s\in(0,t]}(f(Y_s)-f(Y_{s-})-f'(Y_{s-})\triangle Y_s) \tag{1.7}$$

c) if $Y_t=Y_0+B_t$, where B is a standard Brownian motion independent of Y_0, then

$$f(Y_t)=f(Y_0)+\int_0^t f'(Y_0+B_s)dB_s+\frac{1}{2}\int_0^t f''(Y_0+B_s)ds \tag{1.8}$$

Proof. a) If Y is continuous, the jumps are zero, so the sums over jumps disappear in (1.1).

b) By lemma 1.36 in part 2 applied to the cadlag BV path Y we have

$$[Y]_t = \sum_{s\in(0,t]} (\triangle Y_s)^2$$

Consequently

$$\frac{1}{2}f''(Y_{s-})d[Y]_s = \sum_{s\in(0,t]} \frac{1}{2}f''(Y_{s-})(\triangle Y_s)^2$$

and we get (1.7).

c) This is a special case of a),where $[B]_t = t$.

$\square$

Remark 1.4. *The hypothesis $\overline{Y[0,T]} \subset D$ is crucial in Theorem 1.1.The next example shows that $Y[0,T] \subset D$ is not enough.*

Example 1.5. *Let $D = (-\infty, 1) \cup (\frac{3}{2}, \infty)$, and define*

$$f(x) = \begin{cases} \sqrt{1-x}, & x < 1 \\ 0, & x > \frac{3}{2} \end{cases}$$

Then $f \in C^2(D)$.Define the deterministic process

$$Y_t = \begin{cases} t, & 0 \le t < 1 \\ 1+t, & t \ge 1. \end{cases}$$

Then $Y_t \in D$, but if $t < 1, \overline{Y}[0,t]$ is not contained in D because $1 \in \overline{Y[0,T]}$.In that case,

$$\int_{(0,t]} f'(Y_{s-})dY_s = \int_{(0,1)} f'(s)ds + f'(Y_{1-}) + \int_{(1,t]} f'(s)ds$$
$$= -1 + (-\infty) + 0$$

So the integral is not finite.

Now we proceed to prove Itô's formula for vector-valued semimartingales .For purposes of matrix multiplications we think of points $\mathbf{x} \in \mathbb{R}^d$ as columns vectors, so

$$\mathbf{x} = [x_1, x_2, \cdots, x_d]^T$$

Let $Y_1(t), Y_2(t), \cdots, Y_d(t)$ be cadlag semimartingales with respect to a common filtration $\{\mathcal{F}_t\}$.We write $Y(t) = [Y_1(t), \cdots, Y_d(t)]^T$ for the columns vector, and call Y an $\mathbb{R}^d-$valued semimartingale.Its jump is the vector of jumps in the coordinates,

$$\triangle Y(t) = [\triangle Y_1(t), \cdots, \triangle Y_d(t)]^T$$

For $0 < T < \infty$ and an open subset $D \subset \mathbb{R}^d, C^{1,2}([0,T], D)$ is the space of continuous functions $f : [0,T] \times D \to \mathbb{R}$ whose partial derivatives f_t, f_x,and f_{x_i,x_j} exists and are continuous in $(0,T) \times D$,and extend as continuous functions to $[0,T] \times D$.For $f \in C^{1,2}([0,T] \times D)$ and $(t, \mathbf{x}) \in [0,T] \times D$, the spatial gradient is

$$Df(t, \mathbf{x}) = [f_{x_1}(t, \mathbf{x}), f_{x_2}(t, \mathbf{x}), \cdots, f_{x_d}(t, \mathbf{x})]^T$$

and the Hessian matrix $D^2 f(t, \mathbf{x})$ is the $d \times d$ matrix of second-order spatial partial derivatives:

$$D^2 f(t, \mathbf{x}) = \begin{bmatrix} f_{x_1,x_1(t,\mathbf{x})} & f_{x_1,x_2}(t, \mathbf{x}) & \cdots & f_{x_1,x_d}(t, \mathbf{x}) \\ f_{x_2,x_1(t,\mathbf{x})} & f_{x_2,x_2}(t, \mathbf{x}) & \cdots & f_{x_2,x_d}(t, \mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_d,x_1(t,\mathbf{x})} & f_{x_d,x_2}(t, \mathbf{x}) & \cdots & f_{x_d,x_d}(t, \mathbf{x}) \end{bmatrix}$$

Theorem 1.6. *Fix $d \geq 1$, and $0 < T < \infty$.Let D be an open subset of $\mathbb{R}^d$ and $f \in C^{1,2}([0,T] \times D)$.Let Y be an $\mathbb{R}^d$-valued cadlag semimartingale such that outside some event of probability zero, $\overline{Y[0,T]} \subset D$.Then*

$$\begin{aligned} & f(t, Y(t)) \\ & = f(0, Y(0)) + \int f_t(s, Y(s))ds + \sum_{j=1}^{d} \int_{(0,t]} f_{x_j}(s, Y(s-))dY_j(s) \\ & + \frac{1}{2} \sum_{1 \leq j,k \leq d} \int_{(0,t]} f_{x_j,x_k}(s, Y(s-))d[Y_j, Y_k](s) \qquad (1.9) \\ & + \sum_{s \in (0,t]} \left(f(s, Y(s)) - f(s, Y(s-)) - Df(s, Y(s-))^T \triangle Y(s) - \frac{1}{2} \triangle Y(s)^T D^2 f(s, Y(s-)) \triangle Y(s) \right) \end{aligned}$$

Proof. The idea is similar as in the scalar case. We write $Y_t^k = Y_k(t)$.Define a function ϕ on

$[0,T]^2 \times D^2$ by the equality

$$\begin{aligned} f(t,\mathbf{y}) - f(s,\mathbf{x}) &= f_t(s,\mathbf{x})(t-s) + Df(s,\mathbf{x})^T(\mathbf{y}-\mathbf{x}) \\ &\quad + \frac{1}{2}(\mathbf{y}-\mathbf{x})^T D^2 f(s,\mathbf{x})(\mathbf{y}-\mathbf{x}) + \phi(s,t,\mathbf{x},\mathbf{y}) \end{aligned} \tag{1.10}$$

Apply this to the partition intervals to write

$$f(t,Y_t) - f(0,Y_0)$$

$$= \sum_i f_t(t \wedge t_i, Y_{t\wedge t_i})((t \wedge t_{i+1}) - (t \wedge t_i) \tag{1.11}$$

$$+ \sum_{k=1}^{d} \sum_i f_{x_k}(t \wedge t_i, Y_{t\wedge t_i})(Y^k_{t\wedge t_{i+1}} - Y^k_{t\wedge t_i}) \tag{1.12}$$

$$+ \frac{1}{2} \sum_{1\le j,k\le d} \sum_i f_{x_j,x_k}(t \wedge t_i, Y_{t\wedge t_i})(Y^j_{t\wedge t_{i+1}} - Y^j_{t\wedge t_i})(Y^k_{t\wedge t_{i+1}} - Y^k_{t\wedge t_i}) \tag{1.13}$$

$$+ \sum_i \phi(t \wedge t_i, t \wedge t_{i+1}, Y_{t\wedge t_i}, Y_{t\wedge t_{i+1}}). \tag{1.14}$$

By Propositions 1.37 and 1.59 in part 4, we an fix a sequence of partitions π^l such that $mesh(\pi^l) \to 0$ and so that the following limits happen almost surely,uniformly on $[0,T]$,as $l \to \infty$.

i) (1.11) converges to $\int_{(0,t]} f_t(s,Y_s)ds$.

ii) (1.12) converges to $\sum_{k=1}^d \int_{(0,t]} f_{x_k}(s,Y_{s-})dY^k_s$

iii) (1.13) converges to $\frac{1}{2}\sum_{1\le j,k\le d} \int_{(0,t]} f_{x_j,x_k}(s,Y_{s-})d[Y^j,Y^k]_s$

iv) $\sum_i (Y^k_{t\wedge t_{i+1}} - Y^k_{t\wedge t_i})^2 \to [Y^k]_t$ for $1 \le k \le d$.

in $i) - iii)$ the integrand is the left-limit process.For example in $ii)$

$$\lim_{r\searrow s} f_{x_k}(r,Y_r) = f_{x_k}(\lim_{r\searrow s}(r,Y_r)) = f_{x_k}(s,Y_{s-})$$

However in $i)$ the ds integral does not distinguish between Y_{s-} and Y_s because a cadlag path has at most countable many jumps.

Fix ω such that $\overline{Y[0,T]} \subset D$ and the limits in items $i) - iv)$ hold.These conditions hold for almost every ω.We need to study the sum (1.14).To do this,we apply Lemma A.2 to the $\mathbb{R}^d$-valued cadlag function $s \to Y_s(\omega)$ on $[0,T]$,with the function ϕ defined by (1.10),the closed set $K = \overline{Y[0,T]}$,and the sequence of partitions π^l chosen above.We need to check that ϕ and the set K satisfy the hypotheses of Lemma A.2.Continuity of ϕ follows from (1.10).Next we argue that if $(s_n, t_n, \mathbf{x}_n, \mathbf{y}_n) \to (u, u, \mathbf{z}, \mathbf{z})$ in $[0,T]^2 \times K^2$ while for each n,either $s_n \neq t_n$ or $\mathbf{x}_n \neq \mathbf{y}_n$, then

$$\frac{\phi(s_n, t_n, \mathbf{x}_n, \mathbf{y}_n)}{|t_n - s_n| + |\mathbf{y}_n - \mathbf{x}_n|^2} \to 0 \tag{1.15}$$

Given $\epsilon > 0$, let U be an interval around u in $[0,T]$ and let B be an open ball centered at $\mathbf{z}$ and contained in D such that

$$|f_t(v, \mathbf{w}) - f_t(u, \mathbf{z})| + |D^2 f(v, \mathbf{w}) - D^2 f(u, \mathbf{z})| \leq \epsilon \tag{1.16}$$

for all $v \in I, \mathbf{w} \in B$.Such an interval I and ball B exists by the openness of D and by the assumption of continuity of derivatives of f in $[0,T] \times D$. For large enough n, we have $s_n, t_n \in I, \mathbf{x}_n, \mathbf{y}_n \in B$.Since a ball is convex, by Taylor's formula (A.2) applied to $f(t_n, y_n)$ and the definition of ϕ, we can write

$$\begin{aligned}\phi(s_n, t_n, \mathbf{x}_n, \mathbf{y}_n) &= ((f_t(\tau_n, \mathbf{y}_n) - f_t(s_n, \mathbf{x_n}))(t_n - s_n) \\ &\quad + \frac{1}{2}(\mathbf{y}_n - \mathbf{x}_n)^T (D^2 f(s_n, \xi_n) - D^2 f(s_n, x_n))(\mathbf{y}_n - \mathbf{x}_n)\end{aligned}$$

where τ_n lies between s_n and t_n, and ξ_n is a point on the line segment connecting $\mathbf{x}_n$ and $\mathbf{y}_n$.In particular, $\tau_n \in I$ and $\xi_n \in B$.By the Schwarz inequality for vectors $|\mathbf{x}^t A \mathbf{y}| \leq |\mathbf{x}|A||\mathbf{y}|$(Recall that if $A = (a_{i,j})$,then $|A| = (\sum_{i,j} a_{i,j}^2)^{1/2}))$ we get

$$\begin{aligned}|\phi(s_n, t_n, \mathbf{x}_n, \mathbf{y}_n) &\leq |(f_t(\tau_n, \mathbf{y}_n) - f_t(s_n, \mathbf{x_n})| \cdot |t_n - s_n| \\ &\quad + |D^2 f(s_n, \xi_n) - D^2 f(s_n, x_n)| \cdot |\mathbf{y}_n - \mathbf{x}_n|^2 \\ &\overset{(1.16)}{\leq} 2\epsilon(|t_n - s_n| + |\mathbf{y}_n - \mathbf{x}_n|^2)\end{aligned}$$

Thus for n large enough,

$$\left| \frac{\phi(s_n, t_n, \mathbf{x}_n, \mathbf{y}_n)}{|t_n - s_n| + |\mathbf{y}_n - \mathbf{x}_n|^2} \right| \leq 2\epsilon$$

This proves (1.15).Clearly, the function

$$(s, t, \mathbf{x}, \mathbf{y}) \to \frac{\phi(s, t, \mathbf{x}, \mathbf{y})}{|t - s| + |\mathbf{y} - \mathbf{x}|^2}$$

is continuous at points where either $s \neq t$ or $\mathbf{x} \neq \mathbf{y}$,because is the quotient of two continuous functions.This proves the continuity of γ defined in Lemma A.2 on $[0,T]^2 \times K^2$.The hypothesis in (A.4) is consequence of iv).This proves that the sum in (1.14) converges to

$$\begin{aligned}&\sum_{s\in(0,t]}\phi(s,s,Y_{s-},Y_s)\\&=\sum_{s\in(0,t]}\left(f(s,Y_s)-f(s,Y_{s-})-Df(s,Y_-)^T\triangle Y_s-\frac{1}{2}\triangle Y_s^TD^2f(s,Y_{s-})\triangle Y_s\right)\end{aligned}$$

This completes the proof. □

Remark 1.7. *If Y is a continuous $\mathbb{R}^d$-valued semimartingale, (1.6) can be expressed in differential notation as*

$$df(t,Y(t))=f_t(t,Y(t))dt+\sum_{j=1}^{d}f_{x_j}(t,Y(t))dY_j(t)+\frac{1}{2}\sum_{1\le j,k\le d}f_{x_j,x_k}(t,Y(t))d[Y_j,Y_k](t) \tag{1.17}$$

We state the Brownian motion case as a corollary.

Corollary 1.8. *Let $B(t)=(B_1(t),\cdots,B_d(t))$ be a Brownian motion in $\mathbb{R}^d$, with random initial point $B(0)$,and $f\in C^2(\mathbb{R}^d)$.Then*

$$f(B(t))=f(B(0))+\int_0^t Df(B(s))^TdB(s)+\frac{1}{2}\int_0^t\triangle f(B(s))ds \tag{1.18}$$

Suppose f is harmonic(i.e. $\triangle f=0$) in an open set $D\subset\mathbb{R}^d$.Let D_1 be an open subset of D such that $dist(D_1,D^c)>0$.Assume initially $B(0)=\mathbf{z}$ for some point $\mathbf{z}\in D_1$,and let

$$\tau=\inf\{t\ge 0: B(t)\in D_1^c\} \tag{1.19}$$

Then $f(B^\tau(t))$ is a local L^2-martingale.

Proof. Equality (1.18) comes directly from (1.6),noting that $[B_j,B_k]_t=t\delta_{i,j}$.
The process B^τ is a vector valued L^2-martingale that satisfies $\overline{B^\tau[0,T]}\subset D$ for all $T<\infty$.Thus Itô's formula applies.Note that $[B_i^\tau,B_j^\tau]_t=[B_i,B_j]_t^\tau=\delta_{i,j}(t\wedge\tau)$.$\triangle f=0$ implies that the second integral in (1.18) vanishes, so

$$f(B^\tau(t))=f(\mathbf{z})+\int_0^t Df(B^\tau(s))^TdB^\tau(s)$$

this shows that $f(B^\tau)$ is a local L^2-martingale. □

1.2 Applications of Itô's formula.

Example 1.9. *If we want to obtain $\int_0^t B_s^k dB_s$ for a standard Brownian motion and $k \geq 1$,consider $f(x) = \frac{1}{k+1}x^{k+1}$,so that $f'(x) = x^k, f''(x) = kx^{k-1}$.Now we apply Itô's formula to get*

$$\int_0^t B_s^k dB_s = \frac{1}{k+1}B_t^{k+1} - \frac{k}{2}\int_0^t B_s^{k-1} ds$$

the integral on the right is the Riemann integral of the (almost surely) continuous function $s \to B_s^{k-1}$.

The next lemma shows how find martingales using Brownian motion and the heat equation in one dimension.

Lemma 1.10. *Suppose $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and $f_t + \frac{1}{2}f_{xx} = 0$.Let B_t be a one-dimensional standard Brownian motion. Then $f(t, B_t)$ is a local L^2-martingale.If*

$$\int_0^T E[f_x(t, B_t)^2]dt < \infty \tag{1.20}$$

then $f(t, B_t)$ is an L^2-martingale on $[0, T]$.

Proof. Recall that $[B]_t = t$,so we apply (1.6) to get

$$\begin{aligned} f(t, B_t) &= f(0,0) + \int_0^t f_t(s, B_s)ds + \int_0^t f_x(s, B_s)dB_s + \frac{1}{2}\int_0^t f_{xx}(s, B_s)ds \\ &= f(0,0) + \int_0^t f_x(s, B_s)dB_s + \int_0^t \left(f_t(s, B_s) + \frac{1}{2}f_{xx}(s, B_s)\right) ds \\ &= f(0,0) + \int_0^t f_x(s, B_s)dB_s \end{aligned}$$

where the last line is a local L^2-martingale,because $f_x(s, B_s)$ is a continuous process,hence predictable, and is bounded on compact time intervals,so condition $i)$ in Proposition 1.28 from part 4 is satisfied.

The integrability condition (1.20) guarantees that $f_x(s, B_s)$ lies in the space $\mathcal{L}_2(B, \mathcal{P})$ of integrands on the interval $[0, T]$.Recall that our definition of stochastic integrals involves the time interval $[0, \infty)$.However, we can extend $f_x(s, B_s)$ by declaring it identicaaly zero on (T, ∞).This does not change the integral on $[0, T]$. $\square$

Example 1.11. *Let $\mu \in \mathbb{R}, \sigma \neq 0$ be constants.Let $a < 0 < b$,and let B_t be one-dimensional standard Brownian motion.Define $X_t := \mu t + \sigma B_t$,a Brownian motion with drift.We want to answer the following question: What is the probability that X_t exists the interval (a, b) throught the point b?*

To solve this,define the stopping time

$$\tau := \inf\{t > 0 : X_t = a \text{ or } X_t = b\}$$

We check first that $\tau < \infty$ almost surely.Consider the events

$$A_n := \{\sigma(B_{n+1} - B_n) > b - a + |\mu| + 1\}, n \in \mathbb{N}$$

Those events are independent, and have the same probability (because $B_{n+1}-B_n \sim N(0, 1)$).Let's say that $P(A_n) = z \in (0, 1)$.Note that

$$P\left(\bigcap_{n=1}^{N} A_n^c\right) = \prod_{n=1}^{n} P(A_n^c) = (1 - z)^N$$

As $z \in (0, 1), (1 - z)^N \to 0$ when $N \to \infty$.So $P(\cap_{n\in\mathbb{N}} A_n^c) = 0$.This is equivalent to say that $P(\cup_{n\in\mathbb{N}} A_n) = 1$.So almost surely some of the $A_n's$ must happen. In that case, note that

$$A_n \subset \{X_{n+1} - X_n > b - a + 1\}$$

This says that if $X_n \in (a, b)$,then $X_{n+1} \notin (a, b)$.In particular, X_n cannot remain in (a, b) for all n.We have checked that $\tau < \infty$ almost surely.
Next we are looking for a function h such that $h(X_t)$ is a martingale.In that case, if we could justify $Eh(X_\tau) = h(0)$, we could compute $P(X_\tau = b)$ from

$$h(0) = Eh(X_\tau) = h(a)P(X_\tau = a) + h(b)P(X_\tau = b).$$

Let $f(t, x) = h(\mu t + \sigma x)$.The condition from the last lemma $f_t + f_{xx} = 0$ is equivalent to

$$\mu h' + \frac{1}{2}\sigma^2 h'' = 0$$

Suppose $\mu \neq 0$.The general solution of this ode is

$$h(x) = C_1 e^{-2\mu\sigma^{-2}x} + C_2,\ C_1, C_2 \text{ are constants} \tag{1.21}$$

To verify (1.20), using (1.21) we obtain $f_x(t,x) = \sigma h'(\mu t + \sigma x) = -2C_1\mu\sigma^{-1}e^{-2\mu\sigma^{-2}(\mu t+\sigma x)}$.Therefore

$$\int_0^T E[f_x(t,B_t)^2]dt = -2C_1\mu\sigma^{-1}\int_0^T E\left(e^{-4\mu\sigma^{-1}(\mu t+\sigma B_t)}\right)dt = K(T)\int_0^T\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi t}}e^{\sigma x - x^2/2t}dxdt$$

$$= K(T)\int_0^T e^{\sigma^2 t/2}dt < \infty$$

For all $T < \infty$, and then (1.20) is satisfied.Then $M_t = f(t,B_t) = h(\mu t+\sigma B_t) = C_1 e^{-2(\frac{\mu}{\sigma})B_t - 2(\frac{\mu}{\sigma})^2 t} + C_2$ is a martingale.By optional stopping, $M_{t\wedge\tau}$ is also a martingale,and so $EM_{t\wedge\tau} = EM_0 = h(0)$.By path continuity and $\tau < \infty$ almost surely, $M_{t\wedge\tau} \to M_\tau$ almost surely as $t \to \infty$.Furthermore the process $M_{t\wedge\tau}$ is bounded, because up to time τ the process X_t remains in $[a,b]$, and so $|M_{t\wedge\tau}| \leq C := \sup_{a\leq x\leq b}|h(x)|$.Dominated convergence gives $EM_\tau = h(0)$. Finally, we can choose constants C_1, C_2 so that $h(b) = 1, h(a) = 0$.In that case, we obtain

$$P(X_\tau = b) = h(0) = \frac{e^{-2\mu a/\sigma^2} - 1}{e^{-2\mu a/\sigma^2} - e^{-2\mu b/\sigma^2}} \tag{1.22}$$

The case $\mu = 0$ is much simpler.In that case, $h'' = 0$, so $h(x) = C_1 x + C_2$ for some constants C_1, C_2.Then clearly $f(t,B_t) = C_1(\sigma B_t) + C_2$ is a martingale and using the same arguments as before,if we want $h(a) = 0, h(b) = 1$, then $C_1 = \frac{1}{b-a}, C_2 = \frac{-a}{b-a}$.So $P(X_\tau = b) = \frac{-a}{b-a}$.
The martingale found before is a special case of the following: $M_t = e^{X_t - \frac{1}{2}[X]_t}$ is a continuous local L^2-martingale whenever X is.To check this, we apply Itô's formula to the semimartingale $Y = X - \frac{1}{2}[X]$, $f(x) = e^x$ so that $f'(x) = f''(x) = f(x)$, and then

$$dM = MdY + \frac{1}{2}Md[Y,Y] = MdX - \frac{1}{2}Md[X] + \frac{1}{2}Md[X - [X]/2] = MdX$$

To obtain the last equality, we write $[X - [X]/2] = [X] + \frac{1}{4}[[X]] - 2[X,[X]/2] = [X]$ due to continuity and bounded variation (see Lemma 1.36 in part 2).We conclude that $M_t = M_0 + \int_0^t M_s dX_s$, a continuous local L^2-martingale by the construction of the stochastic integral.
Next we investigate transciense and recurrence in Brownian motion.

Proposition 1.12. *Let B_t be a Brownian motion on $\mathbb{R}$.Then $\limsup_{t\to\infty} B_t = \infty$ and $\liminf_{t\to\infty} B_t = -\infty$ almost surely.Consequently, almost every Brownian motion path visits every point infinitely often.*

Proof. Let $\tau_0 = 0$ and $\tau_{k+1} := \inf\{t > \tau_k : |B_t - B_{\tau_k}| = 4^{k+1}\}$.By the strong Markov property, for each k, $\{B_{t+\tau_k} - B_{\tau_k} : t \geq 0\}$ is a standard Brownian motion,independent of

$\mathcal{F}_{\tau_k}$.By symmetry,

$$P[B_{\tau_{k+1}} - B_{\tau_k} = 4^{k+1}] = P[B_{\tau_{k+1}} - B_{\tau_k} = -4^{k+1}] = \frac{1}{2}$$

And the random variables $\{B_{\tau_{k+1}} - B_{\tau_k} : k \geq 0\}$ are independent.Thus, for any $n \in \mathbb{N}$

$$P\left(\bigcap_{m \geq n} B_{\tau_{m+1}} - B_{\tau_m} = 4^{m+1}\right) = 0$$

$$P\left(\bigcap_{m \geq n} B_{\tau_{m+1}} - B_{\tau_m} = -4^{m+1}\right) = 0$$

This says that almost surely, for any $n \in \mathbb{N}$ there exists some indexes j, k such that $B_{\tau_{j+1}} - B_{\tau_j} = 4^{j+1}, B_{\tau_{k+1}} - B_{\tau_k} = -4^{k+1}$. But then since

$$|B_{\tau_j}| \leq \sum_{i=1}^{j} |B_{\tau_i} - B_{\tau_{i-1}}| \leq \sum_{i=1}^{j} 4^i = \frac{4^{j+1} - 1}{4 - 1} \leq \frac{4^{j+1}}{2}$$

$B_{\tau_{j+1}} \geq 4^j$.By the same argument, $B_{\tau_{k+1}} \leq -4^k$.Thus $\limsup_{t \to \infty} B_t = \infty$ and $\liminf_{t \to \infty} B_t = -\infty$ almost surely.Finally, note that as there are sequences $\{t_n\}, \{s_n\}$ with $t_n, s_n \nearrow \infty$ and $B_{t_n} \to \infty, B_{s_n} \to -\infty$,for any $x \in \mathbb{R}$ there are infinitely many n such that $B_{t_n} \geq x, B_{s_n} \leq x$ and by the continuity of Brownian motion, it has to cross the point x each oportunity,so it visits every point infinitely often. $\square$

Proposition 1.13. *Let B_t be a Brownian motion in $\mathbb{R}^d$, and let $P^{\mathbf{z}}$ be the probability measure when the process B_t started at point $\mathbf{z} \in \mathbb{R}^d$.Let $\tau_r := \inf\{t \geq 0 : |B_t| \leq r\}$ be the first time Brownian motion hits the ball of radius r around the origin*

a) If $d = 2, P^z(\tau_r < \infty) = 1$ for all $r > 0, z \in \mathbb{R}^d$.

b) If $d \geq 3$, then for $\mathbf{z}$ outside the ball or radius r,

$$P^{\mathbf{z}}(\tau_r < \infty) = \left(\frac{r}{|\mathbf{z}|}\right)^{d-2}$$

There will be an almost surely finite time T such that $|B_t| > r$ for all $t \geq T$.

c) For $d \geq 2$ and any $\mathbf{z}, \mathbf{y} \in \mathbb{R}^d$,

$$P^z[B_t \neq \mathbf{y} \textit{ for all } 0 < t < \infty] = 1$$

The case $\mathbf{y} = \mathbf{z}$ is allowed(that's why the time $t = 0$ is excluded).

Proof. Note first that in c) it's enough to consider $\mathbf{y} = 0$,because

$$P^{\mathbf{z}}[B_t \neq \mathbf{y} \text{ for all } 0 < t < \infty] = P^{\mathbf{z}-\mathbf{y}}[B_t \neq 0 \text{ for all } 0 < t < t\infty]$$

Also, it's suffices to consider $\mathbf{z} \neq 0$,because if we know the result for $\mathbf{z} \neq 0$, then the case $\mathbf{z} = 0$ comes from the Markov property:

$$\begin{aligned} P^0[B_t \neq 0 \text{ for all } 0 < t < \infty] &= \lim_{s \searrow 0} P^0[B_t \neq 0 \text{ for all } s < t < \infty] \\ &= \lim_{s \searrow 0} P^{B(s)}[B_t \neq 0 \text{ for all } 0 < t < \infty] = 1 \end{aligned}$$

because $P(B(s) = 0) = \lim_{\epsilon \to 0} P(B_s \in (-\epsilon, \epsilon)) = 0$,thus $B_s \neq 0$ with probability one.
We can assume $\mathbf{z} \neq 0$ and $r < |\mathbf{z}|$.
First we consider the case $d = 2$.The function $g(\mathbf{x}) = \log(|\mathbf{x}|)$ is harmonic in $D = \mathbb{R}^2 \setminus \{0\}$.Let $\sigma_R := \inf\{t \geq 0 : |B_t| \geq R\}$.Pick $r < |\mathbf{z}| < R$,and define the annulus $A = \{\mathbf{x} : r < |\mathbf{x}| < R\}$.The time to exit the annulus is $\zeta = \tau_r \wedge \sigma_R$.As we proved before, each coordinate of B_t has $\limsup\ \infty$ almost surelt, the exit time of σ_R is finite, and hence ζ.We apply (1.8) to the harmonic function

$$f(\mathbf{x}) = \frac{logR - log|\mathbf{x}|}{logR - logr}$$

and the annulus A.We get that $f(B_{\zeta \wedge t})$ is a local L^2-martingale,and since f is bounded on the closure of A,by dominated convergence, $f(B_{\zeta \wedge t})$ is an L^2-martingale.If we repeat the optional stopping argument from the last example, we obtain that

$$E^{\mathbf{z}} f(B_{\zeta \wedge t}) = E^z(f(B_0)) = f(\mathbf{z})$$

Taking $t \to \infty$ and using that $f(r) = 1, f(R) = 0$,

$$f(\mathbf{z}) = E^z f(B_\zeta) = P^{\mathbf{z}}(|B_\zeta| = r) f(r) + P^{\mathbf{z}}(|B_\zeta| = R) f(R) = P^{\mathbf{z}}(|B|_\zeta = r) = P^{\mathbf{z}}(\tau_r < \sigma_R)$$

We conclude that

$$P^{\mathbf{z}}(\tau_r < \sigma_R) = \frac{logR - log|z|}{logR - logr} \tag{1.23}$$

From this we can obtain $a), c)$ for $d = 2$.To check a), note that $\sigma_R \nearrow \infty$ as $R \nearrow \infty$,because for a fixed path Brownian motion is bounded on bounded time intervals. We deduce that

$$P^{\mathbf{z}}(\tau_r < \infty) = \lim_{R \to \infty} P^{\mathbf{z}}(\tau_r < \sigma_R) = 1$$

To prove $c)$ in case $d = 2$, consider $r = r(k) = (1/k)^k$and $R = R(k) = k$.Then we get

$$P^{\mathbf{z}}(\tau_{r(k)} < \sigma_{R(k)}) = \frac{logk - log|z|}{(k+1)logk} \xrightarrow{k\to\infty} 0$$

Let $\tau = \inf\{t \geq 0 : B_t = 0\}$.For $0 < r < |z|, \tau_r \leq \tau$ because if has to enter to the ball of radius r before hitting 0.Since $\sigma_{R(k)} \nearrow \infty$ as $k \nearrow \infty$,

$$P^{\mathbf{z}}(\tau < \infty) = \lim_{k\to\infty} P^{\mathbf{z}}(\tau < \sigma_{R(k)}) \leq \lim_{k\to\infty} P^{\mathbf{z}}(\tau_{r(k)} < \sigma_{R(k)}) = 0$$

Now we proceed to the case $d \geq 3$.Consider the harmonic function $g(\mathbf{z}) = |\mathbf{x}|^{2-d}$, and apply Itô's formula to the function

$$f(\mathbf{x}) = \frac{R^{2-d} - |\mathbf{z}|^{2-d}}{R^{2-d} - r^{2-d}}$$

The annulus A and the stopping times σ_R, ζ are defined as before.Applying the same reasoning as above the result is

$$P^{\mathbf{z}}(\tau_r < \sigma_R) = \frac{R^{2-d} - |z|^{2-d}}{R^{2-d} - r^{2-d}} \tag{1.24}$$

Taking $R \to \infty$,

$$P^{\mathbf{z}}(\tau_r < \infty) = \frac{|z|^{2-d}}{r^{2-d}} = \left(\frac{r}{|z|}\right)^{d-2}$$

this shows part $b)$ for $d \geq 3$.Part $c)$ follows because the quantify above converges to zero when $r \to 0$.

It's remains to show that there will be an almost surely finite time T such that $|B_t| \geq r$ for all $t \geq T$.Let $r < R$.Define $\sigma_R^1 = \sigma_R$ and for $n \geq 2$,

$$\tau_r^n := \inf\{t > \sigma_R^{n-1} : |B_t| \leq r\}$$
$$\sigma_R^n := \inf\{t > \tau_r^n : |B_t| \geq R\}$$

So that $\sigma_R^1 < \tau_r^2 < \sigma_R^2 < \tau_r^3 < \cdots$ are the successive visits to radius R and back to radius r.Let $\alpha = (r/R)^{d-2} < 1$.We claim that for $n \geq 2, P^{\mathbf{z}}(\tau_r^n < \infty) = \alpha^{n-1}$.In that case,

$$\sum_n P^{\mathbf{z}}(\tau_r^n < \infty) < \infty$$

and Borel-Cantelli tell us that $\tau_r^n < \infty$ only can happen finitely many times.To prove the claim, the case $n = 2$ followd from the strong Markov property,

$$P^{\mathbf{z}}(\tau_r^2 < \infty) = P^{\mathbf{z}}(\sigma_R^1 < \tau_r^2 < \infty) = P^{B(\sigma_R^1)}(\tau_r < \infty) = \left(\frac{r}{|B_{\sigma_R^1}|}\right)^{d-2} = \alpha$$

Now we procede by induction

$$P^{\mathbf{z}}(\tau_r^n < \infty) = P^{\mathbf{z}}(\tau_r^{n-1} < \sigma_R^{n-1} < \tau_r^n < \infty) = P^{\mathbf{z}}(\tau_r^{n-1} < \sigma_R^{n-1} < \infty, \sigma_R^{n-1} < \tau_r^n < \infty)$$
$$\overset{indep}{=} P^{\mathbf{z}}(\tau_r^{n-1} < \sigma_R^{n-1} < \infty) P^{\mathbf{z}}(\sigma_R^{n-1} < \tau_r^n < \infty) = P^z(\tau_r^{n-1} < \infty) \cdot \alpha = \alpha^{n-1}$$

In the last step,we used that if $\tau_r^{n-1} < \infty$,then σ_R^{n-1} has to be finite, because each coordinate of B_t has $\limsup\ \infty$.This concludes the claim, and also the proof. $\square$

Now comes an important theorem of characterization of Brownian motion. We will need this lemma.

Lemma 1.14. *Let X be a random d-vector and $\mathcal{A}$ a sub-σ-algebra on $(\Omega, \mathcal{F}, P)$.Let*

$$\phi(\theta) := \int_{\mathbb{R}^d} e^{i\theta^T \mathbf{x}} \mu(d\mathbf{x}) \ (\theta \in \mathbb{R}^d)$$

be the characteristic function of a probability distribution μ on $\mathbb{R}^d$.Assume

$$E\left(e^{i\theta^T X} \mathbb{1}_A\right) = \phi(\theta) P(A)$$

for all $\theta \in \mathbb{R}^d, A \in \mathcal{A}$.Then X has distribution $\mu$and is independent of $\mathcal{A}$.

Proof. Taking $A = \Omega$, we get the first claim.Fix A such that $P(A) > 0$, and define the probability measure ν_A on $\mathbb{R}^d$ by

$$\nu_A(B) := \frac{E(\mathbb{1}_B(X)\mathbb{1}_A)}{P(A)}, \quad B \in \mathcal{B}(\mathbb{R}^d)$$

By definition, we have

$$\int_{\mathbb{R}^d} \mathbb{1}_B(x) \nu(dx) = \frac{1}{P(A)} \int_\Omega \mathbb{1}_A \mathbb{1}_B(X) dP$$

Thus for any $f : \mathbb{R}^d \to \mathbb{C}$ continuous and bounded,

$$\int_{\mathbb{R}^d} f(x) \nu(dx) = \frac{1}{P(A)} \int_\Omega \mathbb{1}_A f(X) dP$$

Taking $f = e^{i\theta^T x}$ for some fixed θ,

$$\int_{\mathbb{R}^d} e^{i\theta^T x} \nu(dx) = \frac{1}{P(A)} \int_\Omega \mathbb{1}_A e^{i\theta^T X} dP = \phi(\theta)$$

So ϕ is the characteristic function of ν, and so $\nu = \mu$.In particular,

$$P(\{X \in B\} \cap A) = \nu_A(B) P(A) = \mu(B) P(A) = P(X \in B) P(A)$$

As $A \in \mathcal{A}$ is arbitrary, the independence of $X, \mathcal{A}$ follows. $\square$

Theorem 1.15 (Lévy's Characterization of Brownian Motion.)**.** *Let* $M = [M_1, \cdots, M_d]^T$ *be a continuous* $\mathbb{R}^d$*-valued local martingale and* $X(t) = M(t) - M(0)$*.Then* X *is a standard Brownian motion relative to* $\{\mathcal{F}_t\}$ *iff* $[X_i, X_j]_t = \delta_{i,j}t$*.In particular, in this case the process* X *is independent of* $\mathcal{F}_0$*.*

Proof. We have already seen that d-dimensional standard Brownian motion satisfies $[B_i, B_j]_t = \delta_{i,j}t$.Now we proceed to prove the converse. First observe that a continuous local martingale is also a local L^2-martingale.Fix a vector $\theta = (\theta_1, \cdots, \theta_d)^T \in \mathbb{R}^d$,and define

$$f(t, x) := e^{i\theta^T \mathbf{x} + \frac{1}{2}|\theta|^2 t}$$

Let $Z_t := f(t, X(t))$,and apply Itô's formula(it also applies to complex.valued functions) to this function,

$$Z_t = 1 + \frac{1}{2}|\theta|^2 \int_0^t Z_s ds + \sum_{j=1}^d i\theta_j \int_0^t Z_s dX_j(s) - \frac{1}{2}\int_0^t Z_s ds \underbrace{\sum_{j=1}^d \theta_j^2}_{=|\theta|^2}$$

$$= 1 + i\sum_{j=1}^d \theta_j \int_0^t Z_s dX_j(s)$$

This shows that Z is a local L^2-martinagale.However, on any bounded time interval, Z is bounded, so that Z is an L^2-martingale.The martingale property $E[Z_t|\mathcal{F}_s] = Z_s$ for $s < t$ can be rewritten as

$$E\left(e^{i\theta^T(X(t)-X(s))}|\mathcal{F}_s\right) = e^{-\frac{1}{2}|\theta^2|(t-s)} \tag{1.25}$$

The right-hand side is $\phi(\theta)$,where ϕ is the characteristic function of a $N(0, t-s)$ random variable.Thus $X(t) - X(s) \sim N(0, t-s)$ by the previous lemma and $X(t) - X(s)$ is independent of $\mathcal{F}_s$.Therefore X satisfies all the properties of a standard Brownian motion relative to $\{\mathcal{F}_t\}$. □

Here is an application of Lévy's critetion.

Example 1.16. *(Bessel processes.)Let* $d \geq 2$ *and* $B(t) = [B_1(t), \cdots, B_d(t)]^T$ *a d-dimensional Brownian motion. Set*

$$R_t := |B(t)| = (B_1(t)^2 + \cdots + B_d(t)^2)^{1/2} \tag{1.26}$$

We will find the semimartingale decomposition of R_t.

Suppose that $B(0) = \mathbf{z} \neq 0$ so that $R_0 = |\mathbf{z}| > 0$.Let $D = R^d \setminus \{0\}$ and $f(\mathbf{x}) = |\mathbf{x}|$.Then $f \in C^2(D), f_{x_i} = x_i|\mathbf{x}|^{-1}, \triangle f = (d-1)|\mathbf{x}|^{-1}$.By Proposition 1.13 c), with probability 1 the path $B[0,T]$ is a closed subset of D,for any $T < \infty$.We can apply Itô's formula and gives

$$R_t = |z| + \sum_{j=1}^{d} \int_0^t \frac{B_j(s)}{|B_s|} dB_j(s) + \frac{d-1}{2} \int_0^t \frac{ds}{R_s} \tag{1.27}$$

We prove that the stochastic integral is a 1-dimensional Brownian motion using the Lévy's criterion.Let

$$W_t := \sum_{j=1}^{d} \int_0^t \frac{B_j(s)}{|B_s|} dB_j(s)$$

and compute their quadratic variation

$$\begin{aligned}
[W]_t &= \left[\sum_{j=i}^{d} \int_0^t \frac{B_i(s)}{|B_s|} dB_i(s), \sum_{j=1}^{d} \int_0^t \frac{B_j(s)}{|B_s|} dB_j(s) \right]_t \\
&= \sum_{i,j} \left[\int_0^t \frac{B_i(s)}{|B_s|} dB_i(s), \int_0^t \frac{B_j(s)}{|B_s|} dB_j(s) \right] \\
&= \sum_{i,j} \int_0^t \frac{B_i(s) B_j(s)}{|B_s|^2} d[B_i, B_j]_s \\
&= \sum_{i} \int_0^t \frac{B_i(s)^2}{|B_s|^2} ds = \int_0^t ds = t
\end{aligned}$$

Thus W is a standard Bownian motion.We rewrite (1.27) as

$$R_t = |z| + \frac{d-1}{2} \int_0^t R_s^{-1} ds + W_t$$

Process R_t is the *Bessel process* with dimension d,or with parameter $\frac{d-1}{2}$.
As a final application, we prove one case of the Burkholder-Davis-Gundy inequalities. Recall the notation $M_t^* = sup_{0 \leq s \leq t} |M_s|$.

Proposition 1.17. *Let $p \in [2, \infty)$,and $C_p = (p(p-1)e)^{p/2}$.Then for all continuous local martingales M with $M_0 = 0$ and all $0 < t < \infty$,*

$$E[(M_t^*)^p] \leq C_p E([M_t]^{p/2}) \tag{1.28}$$

Proof. Let $f(x) = |x|^p$.We have $f \in C^2(\mathbb{R})$,with $f'(x) = sign(x)p|x|^{p-1}, f''(x) = p(p-1)|x|^{p-2}$,where $sign(x) = \frac{x}{|x|}$ for $x \neq 0$ (and the convention at $x = 0$ is immaterial).We apply Itô's formula for continuous local martingales to get

$$|M_t|^p = p\int_0^t sign(M_s)|M_s|^{p-1}dM_s + \frac{p(p-1)}{2}\int_0^t |M_s|^{p-2}d[M]_s$$

Suppose first M is bounded.Then M is an L^2-martingale and $M \in \mathcal{L}_2(M, \mathcal{P})$.Consequently the term $p\int_0^t sign(M_s)|M_s|^{p-1}dM_s$ is a mean zero L^2-martingale.Take expectations and apply Holder inequality with exponents $\frac{p-2}{p}, \frac{p}{2}$:

$$\begin{aligned} E(|M_t|^p) &= \frac{p(p-1)}{2}E\int_0^t |M_s|^{p-2}d[M]_s \leq \frac{p(p-1)}{2}E((M_t^*)^{p-2}[M]_t) \\ &\leq \frac{p(p-1)}{2}(E((M_t^*)^p))^{1-2/p}E([M]_t^{p/2})^{2/p} \end{aligned}$$

On the other hand, by Doob's inequality and the fact that $(\frac{p}{p-1})^p \leq 2e$ for $p \geq 2$,

$$E[(M_t^*)^p] \leq \left(\frac{p}{p-1}\right)^p E(|M_t|^p) \leq ep(p-1)(E((M_t^*)^p))^{1-2/p}E([M]_t^{p/2})^{2/p}$$

Rearranging the above inequality gives the conclusion for a bounded martingale.
The general case comes from localization.Let $\tau_k = \inf\{t \geq 0 : |M|_t \geq k\}$.By continuity, M^{τ_k} is bounded, so we can apply the previous result to M^{τ_k} and obtain

$$E[(M_{\tau_k \wedge t})^p] \leq C_p E([M]_{\tau_k \wedge t}^{p/2})$$

Finally,take $k \nearrow \infty$ and apply motonone convergence and deduce (1.28). $\square$

A Some auxilliary results.

We state the Taylor's theorem for real and vector-valued functions.This is used in the proof of Itô's formula.

Theorem A.1. *a) Let (a, b) be an open interval in $\mathbb{R}$, $f \in C^{1,2}([0,T] \times (a,b))$, $s, t \in [0,T]$ and $x, y \in (a,b)$.Then there exists a point τ between s and t,θ between x and y such that*

$$f(t,y) = f(s,x) + f_x(s,x)(y-x) + f_t(\tau,y)(t-s) + \frac{1}{2}f_{xx}(s,\theta)(y-x)^2 \tag{A.1}$$

b) Let G be an open convex set in $\mathbb{R}^d$, $f \in C^{1,2}([0,T] \times G), s, t \in [0,T]$ and $\mathbf{x}, \mathbf{y} \in G$.Then there exists a point τ between s and t,$\theta \in [0,1]$ such that ,with $\xi = \theta\mathbf{x} + (1-\theta)\mathbf{y}$,

$$f(t,\mathbf{y}) = f(s,\mathbf{x}) + Df(s,\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + f_t(\tau,\mathbf{y})(t-s) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^T D^2 f(s,\xi)(\mathbf{y}-\mathbf{x}) \quad \text{(A.2)}$$

The following technical lemma is also used in the proof of Itô's formula.

Lemma A.2. *Let $g_1, \cdots, g_d$ be cadlag functions on $[0,T]$,and form the $\mathbb{R}^d$-valued function $\mathbf{g} = (g_1, \cdots, g_d)^T$.A cadlag function on a bounded set is bounded, so there exists a closed set $K \subset \mathbb{R}^d$ such that $g(s) \in K$ for all $s \in [0,T]$.*

Let ϕ be a continuous function on $[0,T]^2 \times K^2$ such that the function

$$\gamma(s,t,\mathbf{x},\mathbf{y}) = \begin{cases} \frac{\phi(s,t,\mathbf{x},\mathbf{y})}{|t-s|+|y-x|^2}, & s \neq t \text{ or } \mathbf{x} \neq y \\ 0, & s = t \text{ and } \mathbf{x} = \mathbf{y} \end{cases} \quad \text{(A.3)}$$

is also continuous on $[0,T]^2 \times K^2$.Let $\pi^l = \{0 = t_0^l < t_1^l < \cdots < t_{m(l)}^l = T\}$ be a sequence of partitions on $[0,T]$ such that $mesh(\pi^l) \to 0$ as $l \to \infty$, and

$$C_0 = \sup_l \sum_{i=0}^{m(l)-1} |\mathbf{g}(t_{i+1}^l - \mathbf{g}(t_i^l))|^2 < \infty \quad \text{(A.4)}$$

Then

$$\lim_{l\to\infty} \sum_{i=0}^{m(l)-1} \phi(t_i^l, t_{i+1}^l, \mathbf{g}(t_i^l), \mathbf{g}(t_{i+1}^l)) = \sum_{s\in(0,T]} \phi(s,s,\mathbf{g}(s-),\mathbf{g}(s)) \quad \text{(A.5)}$$

The limit on the right is a finite, absolutely convergent sum.