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1 Stohastic Differential Equations.

We are interested in equations of the type

$$X(t,\omega) = H(t,\omega) + \int_{[0,t)} F(s,\omega,X(\omega))dY(s,\omega) \tag{1.1}$$

Where X is an unknown $\mathbb{R}^d$-valued process.Y is an $\mathbb{R}^m$-valued cadlag semimartingale and H is an $\mathbb{R}^d$-valued adapted cadlag process, both given.The coefficient $F(t,\omega,\eta)$ is a $d \times m$-matrix valued function of the time t, the sample point ω and the cadlag path η.
Underlying the equation is a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}$ where H, F, Y are defined.A solution is an $\mathbb{R}^d$-valued cadlag process X that is defined in the probability space, is $\mathcal{F}_t$- adapted, and satisfies (1.1) in the sense that both processes are indistinguishable. This is also called *strong solution*, in contrast to a weak solution, where the probability space and the filtration are part of the solution, that is, are not defined beforehand.
Equation (1.1) can be written in differential form

$$dX = dH + F(t,X)dY, \quad X(0) = H(0) \tag{1.2}$$

And these equations are called *stochastic differential equations* (SDEs).

1.1 Examples of stochastic equations and solutions.

1.1.1 Itô equations.

These are equations of the form

$$dX = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X(0) = \xi \tag{1.3}$$

or in integral form

$$X_t = \xi + \int_0^t b(x, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{1.4}$$

where B_t is a standard Brownian motion in $\mathbb{R}^m$ with respect to the filtration $\{\mathcal{F}_t\}$,ξ is an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable (in particular, ξ is independent of B_t) and the coefficients $b(t, x), \sigma(t, x)$ are Borel measurable functions of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. The *drift vector* $b(t, x)$ is $\mathbb{R}^d$-valued, and the *dispersion matrix* $\sigma(t, x)$ is $d \times m$-matrix valued. The $d \times d$-valued matrix $a(t, x) := \sigma(t, x)\sigma(t, x)^T$ is called the *diffusion matrix*.Of course, X is an unknown $\mathbb{R}^d$-valued process.
In that case, the cadlag semimartingale is $Y(t) := (t, B_t)^T$ and $F(s, x) := (b(s, x), \sigma(s, x)$,a $d \times (m + 1)$-matrix process.

Example 1.1. *(Orstein-Uhlenbeck process.)Let $\alpha > 0$ and $0 \leq \sigma < \infty$ be constants, and consider the SDE*

$$dX_t = -\alpha X_t dt + \sigma dB_t \tag{1.5}$$

with given initial value X_0 independent of the one-dimensional Brownian motion B_t.

Recalling the ODE case

$$x' + a(t)x = g(t) \tag{1.6}$$

when one multiply by the integrant factor $e^{\int_0^t a(s)ds}$, in (1.5) we consider the integrant factor $Z_t = e^{\alpha t}$.By Itô's formula applied to $f(t, x) = e^{\alpha t}x$, we get

$$d(ZX) = \alpha ZXdt + ZdX_t.$$

Assuming that X satisfies (1.5),

$$d(ZX) = \alpha ZXdt - \alpha XZdt + \sigma ZdB_t = \sigma ZdB_t$$

Integrating we obtain

$$Z_tX_t = X_0 + \sigma \int_0^t e^{as} dB_s$$

Therefore a solution is

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s \tag{1.7}$$

To check that (1.7) is a solution, we apply again Itô's formula to the function $f(t,x) = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} x$,where $Y(t) := \int_0^t e^{\alpha s} dB_s$,so that $dY_t = e^{\alpha t} dB_t$.Note that $f_t(t,x) = -\alpha f(t,x)$.We get

$$dX_t = df(t, Y_t) = -\alpha X_t dt + \sigma e^{-\alpha t} dY_t = -\alpha X_t dt + \sigma dB_t$$

and that is what we want. The solution process of (1.5) is called *Orstein-Uhlenbeck* process.

Example 1.2. *(Geometric Brownian motion).Let μ, σ be constants,B a one-dimensional Brownian motion, and consider the SDE*

$$dX = \mu X dt + \sigma X dB_t \tag{1.8}$$

In that case, the integrant factor will be

$$Z_t = e^{-\mu t - \sigma B_t + \frac{1}{2}\sigma^2 t}$$

Aplying Itô's formula we obtain

$$dZ = (-\mu + \frac{1}{2}\sigma^2) Z dt - \sigma Z dB + \frac{1}{2}\sigma^2 Z dt = (-\mu + \sigma^2) Z dt - \sigma Z dB$$

Now we use Theorem 1.56 in part 4 and (1.8) to deduce

$$\begin{aligned} d(XZ) &= XdZ + ZdX + d[X,Z] = (-\mu + \sigma^2) XZdt - \sigma XZdB + \mu XZdt + \sigma XZdB_t + d[X,Z] \\ &= \sigma^2 XZdt + d[X,Z] = 0 \end{aligned}$$

To verify the last equation, we compute

$$\begin{aligned} [Z,X]_t &= \left[(-\mu + \sigma^2) \int Zdt - \sigma \int ZdB, \mu \int Xdt + \sigma \int XdB \right]_t \\ &= -\sigma^2 \left[\int ZdB, \int XdB \right]_t = -\sigma^2 \int ZXd[B,B]_t = -\sigma^2 \int ZXd_t \end{aligned}$$

We have showed that $d(XZ) = 0$, so that

$$X_t = X_0 e^{-(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

And by Itô's formula again we check that the process satisfies the equation.This process is called *geometric Brownian motion*.

Example 1.3. *(Brownian Bridge).Fix* $0 < T < 1$*.The SDE is now*

$$dX = -\frac{X}{1-t}dt + dB. \ \ \textit{for } 0 \le t \le T, \ \textit{with } X_0 = 0 \tag{1.9}$$

The integrant factor here is $Z_t = e^{\int_0^t \frac{ds}{1-s}} = e^{-\log(1-s)} = \frac{1}{1-t}$.Then we obtain

$$\begin{aligned} d(ZX) = ZdX + XdZ + d[X,Z] &= -\frac{X}{(1-t)^2}dt + \frac{dB}{1-t} - \frac{X}{(1-t)^2}dt \\ &= \frac{dB}{1-t}. \end{aligned}$$

Integrating now we get

$$X_t = (1-t)\int_0^t \frac{dB_s}{1-s} \tag{1.10}$$

To check that (1.10) solves (1.9), we apply the product formula $d(UV) = UdV + VdU + [U,V]$ with $U = 1-t, V = \int (1-s)^{-1}dB_s$.The result is ([U,V]=0)

$$dX_t = dB_t - \int (1-s)^{-1}dBdt = dB_t - \frac{1}{1-t}Xdt$$

and this the result we want.

1.1.2 Stochastic exponential

The exponential function $g(t) = e^{ct}$ can be characterized as the unique function g that satisfies the equation

$$g(t) = 1 + \int_0^t g(s)ds$$

We want to generalize this changing the dt- integral to a semimartingale integral.

Theorem 1.4. *Let* Y *be a real-valued cadlag semimartingale such that* $Y_0 = 0$*.Define*

$$Z_t := e^{Y_t - \frac{1}{2}[Y]_t} \prod_{s\in(0,t]} (1 + \triangle Y_s)e^{-\triangle Y_s + \frac{1}{2}(\triangle Y_s)^2} \tag{1.11}$$

Then the process Z is a cadlag semimartingale, and it's the unique cadlag semimartingale Z that satisfies the equation

$$Z_t = 1 + \int_{(0,t]} Z_{s-} dY_s \tag{1.12}$$

Sketch of the proof. First we show that (1.11) defines a semimartingale. Rewrite Z_t as

$$Z_t = e^{Y_t - \frac{1}{2}[Y]_t + \frac{1}{2}\sum_{s\in(0,t]}(\triangle Y_s)^2} \prod_{s\in(0,t]} (1+\triangle Y_s)e^{-\triangle Y_s}$$

The continuous part

$$[Y_t]^c := [Y]_t - \sum_{s\in(0,t]} (\triangle Y_s)^2$$

of the quadratic variation is an increasing process, then $W_t := Y_t - \frac{1}{2}[Y]_t + \frac{1}{2}\sum_{s\in(0,t]}(\triangle Y_s)^2$ is a semimartingale.By Itô's formula, the process e^{W_t} is also a semimartingale.If we prove that the process

$$U_t := \prod_{s\in(0,t]} (1+\triangle Y_s)e^{-\triangle Y_s} \tag{1.13}$$

converges for a fixed ω and has paths of bounded variation.In that case, U is also a semimartingale, and by Theorem 1.57 in part 4, Z would be a semimartingale too.

Next we check that Z satisfies (1.12).Write $X_t = (e^{W_t}, U_t)$ and $f(w,u) := e^w u$.Clearly, $f_w = f, f_{w,w} = f, f_u = e^w, f_{u,u} = 0, f_{u,w} = e^w$.The gradient vector is $\nabla f = [f, e^w]$,and the Hessian is

$$D^2 f = \begin{bmatrix} f & e^w \\ e^w & 0 \end{bmatrix}$$

We apply Itô's formula to $Y_t = f(X_t)$ to obtain

$$\begin{aligned} Z_t = 1 &+ \int_{(0,t]} Z_{s-} dW_s + \int_{(0,t]} e^{W_{s-}} dU_s + \frac{1}{2}\int_{(0,t]} Z_{s-} d[W]_s + \frac{1}{2}\int_{(0,t]} e^{W_{s-}} d[U,W]_s \\ &+ \sum_{s\in(0,t]} \left\{ \triangle Z_s - Z_{s-}\triangle Y_s - e^{W_{s-}}\triangle U_s - \frac{1}{2} Z_{s-}(\triangle Y_s)^2 - e^{W_{s-}}\triangle Y_s \triangle U_s \right\} \end{aligned}$$

Where we used that $W_t = Y_t - \frac{1}{2}([Y]_t - \sum_{s\in(0,t]}(\triangle Y_s)^2)$ where the part in parentheses is FV and continuous, so its quadratic variation is zero .Thus $[W] = [Y]$ and $[W,U] = [Y,U] = \sum \triangle Y \triangle U$.Finally observe that most of the terms here cancels. For example, we have

$$\int_{(0,t]} e^{W_{s-}} dU_s - \sum_{(0,t]} e^{W_{s-}} \triangle U_s = 0$$

because U is an FV process whose paths are step functions, so the integral reduces to a sum over jumps.Next,

$$\int_{(0,t]} Z_{s-} dW_s + \frac{1}{2}\int_{(0,t]} Z_{s-} d[W]_s - \frac{1}{2}\sum_{s\in(0,t]} Z_{s-}(\triangle Y_s)^2 = \int_{(0,t]} Z_{s-} dY_s$$

By the definition of W_t and $[W] = [Y]$.Also,

$$\frac{1}{2}\int_{(0,t]} e^{W_{s-}} d[U,W]_s - \frac{1}{2}\sum_{s\in(0,t]} e^{W_{s-}} \triangle Y_s \triangle U_s = 0$$

Because $[U,W] = \sum \triangle Y \triangle U$.Finally, $\triangle Z_s - Z_{s-}\triangle Y_s = 0$ by definition of Z.The final result is

$$Z_t = 1 + \int_{(0,t]} Z_{s-} dY_s.$$

The uniqueness follows from the general existence and uniqueness theorem that will be proved lately. $\square$

The semimartingale Z defined before is called the *stochastic exponential* of Y,and denoted by $Z = \mathcal{E}(Y)$.

Example 1.5. *Let $Y_t = \lambda B_t$,where B is Brownian motion. The stochastic exponential $Z = \mathcal{E}(\lambda B)$ is given by $Z_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$,another instance of geometric Brownian motion. The equation*

$$Z_t = 1 + \lambda \int_0^t Z_s dB_s$$

and moment bounds show that geometric Brownian motion is a continuous L^2-martingale.

1.2 Itô equations

We present a existence and uniqueness theorem for SDEs.On $\mathbb{R}^d$ we use the Euclidean norm $|x| = (\sum x_i^2)^{1/2}$,and for a matrix $A = (a_{i,j})$ the norm is $|A| = (\sum |a_{i,j}|^2)^{1/2}$.

Let B_t be a standard $\mathbb{R}^m$ Brownian motion with respect to the complete filtration $\{\mathcal{F}_t\}$ on the complete probability space $(\Omega, \mathcal{F}, P)$.The given data are functions $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and the $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable ξ that gives the initial position .In particular, ξ is independent of the Brownian motion B.

We want to show that in this context, there exists unique $\mathbb{R}^d$-valued process X,adapted to $\{\mathcal{F}_t\}$ such that

$$X_t = \xi + \int_0^t b(s, X_x)ds + \int_0^t \sigma(s, X_s)dB_s \tag{1.14}$$

in the sense of indistinguishability.Part of the requirement is that the integrals are well defined, for which is required that

$$P\left(\forall\ T < \infty : \int_0^T |b(s, X_s)|ds + \int_0^t |\sigma(s, X_s)|^2 ds < \infty\right) = 1. \tag{1.15}$$

The assumptions on the coefficients are the Lipschitz assumptions,

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y| \tag{1.16}$$

$$|b(t, x)| + |\sigma(t, x)| \leq L(1 + |x|) \tag{1.17}$$

for a constant $L < \infty$ and all $t, \in \mathbb{R}_+, x, y \in \mathbb{R}^d$.
We consider first the case $\xi \in L^2(P)$.

Theorem 1.6. *Assume* (1.16) *and* (1.17)*,and* $E(|\xi^2|) < \infty$*.*

a) There exists a continuous process X *adapted to* $\{\mathcal{F}_t\}$*that satisfies* (1.14) *and this moment bound: for each* $T < \infty$ *there exists a constant* $C = C(T, L) < \infty$ *such that*

$$E\left(\sup_{t\in[0,T]} |X_t|^2\right) \leq C(|1 + E(|\xi^2|)) \quad \textit{for } t \in [0, T] \tag{1.18}$$

In particular, the integrand $\sigma(s, X_x) \in \mathcal{L}_2(B)$ *and*

$$E\int_0^t |b(s, X_s)|^s ds < \infty \quad \textit{for each } T < \infty.$$

b) Let $\tilde{\xi} \in L^2(P)$ *be* $\mathcal{F}_\prime$*-measurable and let* $\tilde{X}$ *be the solution constructed in part a) with initial condition* $\tilde{\xi}$*.Then there exist a constant* $C < \infty$ *such that for all* $t < \infty$*,*

$$E\left(\sup_{0\leq s\leq t} |X_s - \tilde{X}_s|^2\right) \leq 9e^{Ct^2} E[|\xi - \tilde{\xi}|^2] \tag{1.19}$$

Sketch of the proof. a) The existence is by Picard iteration.Define $X_0(t) = \xi$ and for $n \geq 0$,

$$X_{n+1}(t) := \xi + \int_0^t b(s, X_n(s))ds + \int_0^t \sigma(s, X_n(s))dB_s \tag{1.20}$$

Step 1 By induction is proved that for $n \in \mathbb{N}$,

$$E\left(\sup_{s\in[0,t]} X_n(s)^2\right) < \infty \quad \forall\, t \in \mathbb{R}_+ \tag{1.21}$$

Useful inequality: $(a+b+c)^2 \leq 9(a^2+b^2+c^2)$.

Step 2 For each $T<\infty$ there exists a constant $A = A(T,L) < \infty$ such that

$$E\left(\sup_{s\in[0,t]} X_n(s)^2\right) \leq A(1+E(|\xi^2|)) \quad \forall\, n \in \mathbb{N}, \forall\, t \in [0,T]. \tag{1.22}$$

In fact, by the previous step can be deduced that

$$E\left(\sup_{s\in[0,t]} X_n(s)^2\right) \leq C(1+E(|\xi^2|)) + C\int_0^t E\left(\sup_{u\in[0,s]} X_n(u)^2\right) ds$$

If we write $y_n(t) := E(\sup_{s\in[0,t]} X_n(s)^2)$ and $B = C(1+E(|\xi^2|))$ we have

$$y_0(t) \leq B \quad \text{and} \quad y_{n+1}(t) \leq B + C\int_0^t y_n(s)ds$$

Inductively it is obtained that

$$y_n(t) \leq B\sum_{k=0}^{n} \frac{C^k t^k}{k!} \leq Be^{Ct} = \underbrace{Ce^{Ct}}_{=A}(1+E|\xi^2|)$$

Step 3 There exists a continuous, adapted process X such that $X_n \to X$ uniformly on compacts time intervals both almost surely and for each $T<\infty$,

$$\lim_{n\to\infty} E\left(\sup_{s\in[0,T]} |X(s)-X_n(x)|^2\right) = 0 \tag{1.23}$$

Furthermore, X satisfies (1.18).Part of this step is included in the following lemma

Lemma 1.7. *Assume (1.16),(1.17).Let X,Y be continuous adapted L^2-processes that satisfy (1.22),$\xi,\eta \in L^2(P)$ are random variables, and the processes $\overline{X},\overline{Y}$ defined by*

$$\overline{X} := \xi + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s$$
$$\overline{Y} := \eta + \int_0^t b(s,Y_s)ds + \int_0^t \sigma(x,Y_s)dB_s$$

Then there exists a constant $C = C(L) < \infty$ such that for all $0 \le t \le T < \infty$

$$E\left(\sup_{0\le s\le t}|\overline{X}_s - \overline{Y}_s|^2\right) \le 9E|\eta-\xi|^2 + CT\int_0^t E\left(\sup_{0\le u\le s}|X_u - Y_u|^2\right)ds. \quad (1.24)$$

The proof is similar to Step 1.

Applying this lemma we obtain

$$E\left(\sup_{0\le s\le t}|X_{n+1}(s) - X_n(s)|^2\right) \le C\int_0^t E\left(\sup_{0\le u\le s}|X_n(u) - X_{n-1}(u)|^2\right)ds \quad (1.25)$$

As before, if $y_n(t) := E\left(\sup_{0\le s\le t}|X_{n+1}(s) - X_n(s)|^2\right)$, then for some constants B, C and for $t \in [0, T]$,

$$y_0(t) \le B \quad \text{and} \quad y_{n+1}(t) \le C\int_0^t y_n(s)ds \quad (1.26)$$

And thus

$$y_n(t) \le B\frac{C^n t^n}{n!} \quad (1.27)$$

Applying Chebyshev's inequality and Borel-Cantelli, we deduce that if $T \in \mathbb{N}$, for almost every ω there exists some $N_T(\omega) < \infty$ such that if $n \ge N_T(\omega)$,

$$\sup_{0\le s\le T}|X_{n+1}(s) - X_n(s)| < 2^{-n}$$

Therefore the sequence $\{X_n\}$ is Cauchy in the space $C[0,T]$,so that there exists $\{X_s(\omega) : s \in \mathbb{R}_+\}$,defined for almost every ω,such that

$$\sup_{0\le s\le T}|X(s) - X_n(s)| \to 0 \quad as\ n \to \infty. \quad (1.28)$$

for all $T \in \mathbb{N}$ and almost every ω.This defines X and proves the almost surely convergence part of Step 3.Adaptedness of X comes from $X_n(t) \to X(t)$ a.s. and the completeness of $\mathcal{F}_{\sqcup}$.

From the uniform convergence we get

$$\sup_{0\le s\le t}|X(s) - X_n(s)| = \lim_{m\to\infty}\sup_{0\le s\le T}|X_m(s) - X_n(s)|.$$

Abbreviate $\eta_n := \sup_{0\le s\le T}|X_{n+1}(s) - X_n(s)|$.By Fatou's lemma we obtain

$$||\sup_{0\le s\le t}|X(s) - X_n(s)||_2 \le \liminf_{m\to\infty}||\sup_{0\le s\le T}|X_m(s) - X_n(s)|| \le \liminf_{m\to\infty}\sum_{k=n}^{m-1}||\eta_k||_2$$
$$= \sum_{k=n}^{\infty}||\eta_k||_2$$

Taking $n \to \infty$ and recalling (1.27) ,this converges to zero.Finally,(1.22) and Fatou's lemma implies (1.18).

b) We check that X satisfies (1.14).It's enough to check that each term of (1.20) converges in L^2 to the respective terms in (1.14).For example ,by (1.16),for $T < \infty$

$$E\int_0^T |\sigma(s, X_n(s)) - \sigma(s, X_s)|^2 ds \leq L^2 E\int_0^T |X_n(s) - X(s)|^2 \to 0$$

Consequently we have L^2- convergence os stochastic integrals :

$$\int_0^t \sigma(s, X_n(s))dB_s \to \int_0^t \sigma(s, X(s))dB_s$$

which also holds uniformly on compact time intervals. The other convergence is similar, and this proves part a) of the theorem.

c) Apply Lemma 1.7 to $\eta = \tilde{\xi}, \overline{X} = X, \overline{Y} = Y$ and then Gronwall's inequality(Lemma A.1)

□

Next we prove the uniqueness. To this end, let η be another initial condition and Y a process that satisfies

$$Y_t = \eta + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dB_s \tag{1.29}$$

Theorem 1.8. *Let ξ and η be $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variables without any integrability assumptions. Assume that coefficients and σ are Borel functions that satisfy Lipschitz condition (1.16).Let X and Y be two continuous process on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}$.Assume X satisfies integral equation (1.14) and condition (1.15).Assume Y satisfies the integral equation (1.29) and condition (1.15) with X replaced by Y. Then on the event $\{\xi = \eta\}$ processes X and Y are indistinguishable.*

Proof. Fix $n \in \mathbb{N}$,and define the stopping time

$$\nu := \inf\{t \geq 0 : |X_t - \xi| \geq n \text{ or } |Y_t - \eta| \geq n\}$$

So that the processes $X^\nu - \xi, X^\nu - \eta$ are bounded.We write

$$\begin{aligned}(X_t^\nu - Y_t^\nu) \cdot \mathbb{1}_{\{\eta=\xi\}} &= \int_0^{t\wedge\nu} [b(s, X_s^\nu) - b(s, Y_s^\nu)]\mathbb{1}_{\{\eta=\xi\}}ds + \int_0^{t\wedge\nu} [\sigma(s, X_s^\nu) - \sigma(s, Y_s^\nu)]\mathbb{1}_{\{\eta=\xi\}}dB_s \\ &:= G(t) + M(t).\end{aligned}$$

We take L^2 bound on each term. First consider $G(t)$.We have by Cauchy-Schwarz, bounding $\nu \wedge t$ by t, and Lipschitz condition (1.16)

$$
\begin{aligned}
E|G(t)|^2 &\leq E\left(\int_0^t |b(s,X_s^\nu) - b(s,Y_s^\nu)|^2 ds \int_0^t \mathbb{1}_{\{\eta=\xi\}} ds\right) \\
&= tE\left(\int_0^t |b(s,X_s^\nu) - b(s,Y_s^\nu)|^2 \mathbb{1}_{\{\eta=\xi\}} ds\right) \\
&\leq L^2 t \int_0^t E(|X_s^\nu - Y_s^\nu| \mathbb{1}_{\{\eta=\xi\}} ds)
\end{aligned}
$$

By the isometry of the stochastic integral,similarly we get

$$
E[|M_t|^2] \leq L^2 \int_0^t E[|X_s^\nu - Y_s^\nu|^2 \mathbb{1}_{\{\eta=\xi\}} ds]
$$

Combining both equations the result is

$$
E[|X_t^\nu - Y_t^\nu|^2 \mathbb{1}_{\{\eta=\xi\}}] = E[(G(t)+M(t))^2] \leq 2(E[G(t)^2]+E[M(t)^2] \leq 2L^2(t+1) \int_0^t E[|X_s^\nu - Y_s^\nu|^2 \mathbb{1}_{\{\eta=\xi\}} ds]
$$

If we restrict to $t \in [0,T]$ we can replace $2L^2(t+1)$ with $2L^2(T+1)$.Finally, by Gronwall's inequality (Lemma A.1) the result is

$$
E[|X_t^\nu - Y_t^\nu|^2 \mathbb{1}_{\{\eta=\xi\}}] = 0 \quad \text{for } t \in [0,T].
$$

If we repeat this argument for each $T \in \mathbb{N}$ we extend this result for all $t \in \mathbb{R}_+$.Thus on the event $\{\xi = \eta\}$ the processes X^ν, Y^ν are almost surely equal for each fixed time, and hence they are indistinguishable (by continuity).Consider this argument for each $n \in \mathbb{N}$, and by the continuity of the processes $X, Y, \nu \nearrow \infty$ when $n \nearrow \infty$.Therefore we conclude that X and Y are indistinguishable when $\xi = \eta$. □

Taking $\xi = \eta$, we obtain strong uniqueness.

Corollary 1.9. *Let ξ be an $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variable.Assume that b and σ are Borel functions that satisfy the Lipschitz condition (1.16).Then up to indistinguishability there is at most one continuous process X on $(\Omega, \mathcal{F}, P)$ adapted to $\{\mathcal{F}_t\}$ that satisfies (1.14)-(1.15).*

Now we prove the existence and uniqueness theorem without the L^2 condition on ξ.

Theorem 1.10. *Suppose* (1.16)-(1.17) *and* ξ *be an* $\mathbb{R}^d$*-valued* $\mathcal{F}_0$*-measurable random variable. Then there exists a continuous process* X *on* $(\Omega, \mathcal{F}, P)$ *adapted to* $\{\mathcal{F}_t\}$ *that satisfies* (1.14) *and* (1.15)*.The process* X *is unique up to indistinguishability.*

Proof. We have already proved the uniqueness. We need to remove the assumption $\xi \in L^2(p)$.For $m \in \mathbb{N}$,let X_m be the solution given by Theorem 1.6 with initial condition $\xi\{|\xi| \leq m\}$.From Theorem 1.8 is obtained that for $m < n, X_m$ and X_n are indistinguishable on the event $\{|\xi| \leq m\}$.Then we can define consistently

$$X(t) := X_m(t) \text{ on the event}\{|\xi| \leq m\}$$

The continuity of X is result of the same property for X_m, and similarly with adaptedness (because $\{|\xi| \leq m\} \in \mathcal{F}_0$).

To verify (1.14),we use Proposition 1.19 in part 4 to pass the $\mathcal{F}_0$-random variable $\mathbb{1}_{\{|\xi| \leq m\}}$ in and out of the stochastic integral. Additionally, we use the identities

$$b(s, X(s))\mathbb{1}_{\{|\xi| \leq m\}} = b(s, X_m(s))\mathbb{1}_{\{|\xi| \leq m\}}$$
$$\sigma(s, X(s))\mathbb{1}_{\{|\xi| \leq m\}} = \sigma(s, X_m(s))\mathbb{1}_{\{|\xi| \leq m\}}$$

To deduce

$$X(t)\mathbb{1}_{\{|\xi| \leq m\}} = X_m(t)\mathbb{1}_{\{|\xi| \leq m\}} = \mathbb{1}_{\{|\xi| \leq m\}}\left(\xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s\right)$$

Since the union of the events $\{|\xi| \leq m\}$ is almost surely Ω,we are done. $\square$

We address *weak uniqueness* or *uniqueness in distribution.*Suppose we have probability spaces $(\Omega, \mathcal{F}, P), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and assume that a Brownian motion B is adapted to the complete filtration $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, P)$ and ξ is an $\mathcal{F}_0$-measurable random variable. Let X be a continuous process $\{\mathcal{F}_t\}$-adapted process which satisfies (1.14)-(1.15).Similarly, suppose we have the same as above with $\tilde{B}, \tilde{\xi}, \{\tilde{\mathcal{F}}_t\}, \tilde{X}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Theorem 1.11. *Assume* (1.16)-(1.17)*and* b, σ *are continuous functions of* (t, x)*.Suppose* $\xi \overset{d}{=} \tilde{\xi}$*.Then the processes* $X, \tilde{X}$ *have the same probability distribution, that is, for* $A \in C_{\mathbb{R}^d}[0, \infty), P(X \in A) = \tilde{P}(\tilde{X} \in A)$*.*

Sketch of the proof. First assume $\xi, \tilde{\xi} \in L^2$.As in Step 1 in Theorem 1.6,consider sequences $X_n \in \Omega, \tilde{X}_n \in \tilde{\Omega}$ that converge in the sense of (1.23),(1.28).By strong uniqueness,$X_n \to X, \tilde{X}_n \to \tilde{X}$ in the sense mentioned.Thus it suffices to show that $X_n \overset{d}{=} \tilde{X}_n$.This is done by induction.

The case $(X_0, B) \overset{d}{=} (\tilde{X}_0, \tilde{B}_0)$ follows from the hypotheses.Now suppose that $(X_n, B) \overset{d}{=} (\tilde{X}_n, \tilde{B})$ for some $n \in \mathbb{N}$.The idea is to write the integrals as limits of Riemann sums.his can be done by continuity and uniform estimates as (1.22).If we fix partitions $s_i^k = i2^{-k}$,we have

$$X_{n+1}(t) = \xi + \lim_{k\to\infty} \left(\sum_{i\geq 0} b(s_i^k, X_n(s_i^k))(t \wedge s_{i+1}^k - t \wedge s_i^k) + \sum_{i \geq 0} \sigma(s_i^k, X_n(s_i^k))(B_{t\wedge s_{i+1}^k} - B_{t \wedge s_i^k}) \right)$$

with analogous result for $\tilde{X}_{n+1}$.The limits are uniform over bounded time intervals and L^2 or in probability over the probability space. By passing to a subsequence, the limit is also almost sure.Fix time points $0 \leq t_1 < \cdots < t_m$.The induction assumption and the limits above imply that the vectors

$$\begin{gathered}(X_{n+1}(t_1), \cdots X_{n+1}(t_m), B_{t_1}, \cdots, B_{t_m}) \\ (\tilde{X}_{n+1}(t_1), \cdots, \tilde{X}_{n+1}(t_m), \tilde{B}_{t_1}, \cdots, \tilde{B}_{t_m})\end{gathered}$$

have identical distributions. As finite-dimensional distributions determine the distribution of the entire process, this checks the case $n+1$ of the induction.This concludes the proof for the case of square integrable ξ and $\tilde{\xi}$.The general case comes by an approximation through $\xi\{|\xi| \leq m\}$ and $\tilde{\xi}\{|\tilde{\xi}| \leq m\}$. □

For $x \in \mathbb{R}^d$,let P^x be the distribution on $C = C_{\mathbb{R}^d}[0, \infty)$ of the process X that solves the SDE (1.14) with deterministic initial point $\xi = x$.Denote by E^x its expectation.The family of probability measures $\{P^x\}_{x \in \mathbb{R}^d}$ is well defined by the last theorem.An interesting result is that this family forms a Markov process, and satisfy the strong Markov property

Theorem 1.12. *Assume (1.16)-(1.17).Then the family $\{P^x\}_{x\in\mathbb{R}^d}$ of probability measures on C defined by the solutions of the Itô equation*

$$X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s$$

constitutes a Markov process that also satisfy the strong Markov property.

Proof. The strong Markov property is followed by the Feller property, that is,

$$x_j \to x \text{ implies } P^{x_j} \to P^x \text{ in the weak topology of } \mathcal{M}_1(C)$$

where $\mathcal{M}_1(C)$ is the space of probability measures on C.This convergence is result of (1.19). The Markov property is rather technical and is omitted. □

1.3 A semimartingale equation.

Fix a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$.We consider the equation

$$X(t,\omega) = H(t,\omega) + \int_{(0,t]} F(s,\omega,X(\omega))dY(s,\omega) \tag{1.30}$$

where Y is a given $\mathbb{R}^m$-valued cadlag semimartingale, H is a given $\mathbb{R}^d$-valued adapted cadlag process, and X is the unknown $\mathbb{R}^d$-valued process. The coefficient F is a $d\times m$-matrix valued function of its arguments.For the coefficient F we make these assumptions

Assumption 1.13. *The coefficient function $F(s,\omega,\eta)$ in (1.30) is a measurable function from the space $\mathbb{R}_+ \times \Omega \times D_{\mathbb{R}^d}[0,\infty)$ into the space $\mathbb{R}^{d\times m}$ of $d\times m$ matrices, and satisfies the following requirements:*

i) F satisfies a spatial Lipschitz condition uniformly in the other variables: there exits a finite constant L such that the following estimate holds for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and all $\eta, \zeta \in D_{\mathbb{R}^d}[0,\infty)$:

$$|F(t,\omega,\eta) - F(t,\omega,\zeta)| \leq L \sup_{s\in[0,t)} |\eta(s) - \zeta(s)|. \tag{1.31}$$

ii) Given any adapted $\mathbb{R}^d$-valued cadlag process X on Ω, the function $(t,\omega) \to F(t,\omega,X(\omega))$ is a predictable process.

iii) Given any adapted $\mathbb{R}^d$-valued cadlag process X on Ω, there exist stopping times $\nu_k \nearrow \infty$ such that $\mathbb{1}_{[0,\nu_k]}(t)F(t,X)$ is bounded for each k.

These conditions help to prove the following theorem, whose proof will be postponed to the next section.

Theorem 1.14. *Assume $\{\mathcal{F}_t\}$ is complete and right-continuous. Let H be an adapted $\mathbb{R}^d$-valued cadlag process and Y an $\mathbb{R}^m$-valued cadlag semimartingale. Assume F satisfies Assumption 1.13.Then there is a cadlag process $\{X(t) : 0 \leq t < \infty\}$ adapted to $\{\mathcal{F}_t\}$ that satisfies equation (1.30), and X is unique up to indistinguishability.*

Remark 1.15. *Observe that in (1.31) the supremum does not includes the endpoint t.If we define the stopped path*

$$\eta^{t-}(s) := \begin{cases} \eta(0), & t = 0, 0 \leq s < \infty \\ \eta(s), & 0 \leq s < t \\ \eta(t-), & s \geq t > 0 \end{cases}$$

By (1.31),we get

$$|F(t,\omega,\eta) - F(t,\omega,\eta^{t-})| \leq L \sup_{s\in[0,t)} |\eta(s) - \eta^{t-}(s)| = 0$$

Thus $F(t,\omega,\cdot)$ only depends on the path on the time interval $[0,t)$.
Parts $ii) - iii)$ guarantee that the stochastic integral $\int F(s,X)dY(s)$ exists for an arbitrary adapted cadlag process X and semimartingale Y.

The existence of the stopping times $\{\nu_k\}$ in part $iii)$ can be verified via this local boundedness condition.

Lemma 1.16. *Assume F satisfies part i)-ii) of Assumption 1.13.Suppose that there exists a path $\overline{\zeta} \in D_{\mathbb{R}^d}[0,\infty)$ such that for all $T < \infty$,*

$$c(T) := \sup_{t\in[0,T],\omega\in\Omega} |F(t,\omega,\overline{\zeta})| < \infty. \tag{1.32}$$

Then condition iii) is satisfied.

Sketch of the proof. Define the stopping times

$$\nu_k := \inf\{t \geq 0 : |X(t) \geq k|\} \wedge \inf\{t \geq 0 : |X(t-) \geq k|\} \wedge k.$$

Define the stopped process $X^{\nu_k -}$ as before.To check that $\mathbb{1}_{[0,\nu_k]}(s)F(s,X)$ is bounded, note first that the cases $s = 0$ of $\nu_k = 0$ are clear.Otherwise, X agrees with $X^{\nu_k -}$ on $[0,s)$, and then $F(s,X) = F(s, X^{\nu_k -})$.Finally, by (1.31) and the hypothesis,

$$
\begin{aligned}
|F(s,X)| &\le |F(s,\overline{\zeta})| + |F(s,X^{\nu_k -}) - F(s,\overline{\zeta})| \\
&\le c(k) + L \sup_{0\le t<\nu_k} |X(t) - \overline{\zeta}(t)| \\
&\le c(k) + L(k + \sup_{0\le t\le\nu_k} |\overline{\zeta}(t)|)
\end{aligned}
$$

where we used that $|X(s)| \le k$ if $0 \le s < \nu_k$.Finally, observe that the last quantity is finite because $\overline{\zeta}$ is locally bounded. □

The next corollary gives the solution for a processes defined on finite time.

Corollary 1.17. *Let $0 < T < \infty$.Assume $\{\mathcal{F}_t\}$ is right-continuous, Y is a cadlag semi-martingale and H is an adapted cadlag process, all defined for $0 \le t \le T$.Let F satisfy Assumption 1.13 for $(t,\omega) \in [0,T] \times \Omega$. In particular, part ii) takes this form: if X is a predictable process defined on $[0,T] \times \Omega$,then so is $F(t,X)$, and there is a nondecreasing sequence of stopping times $\{\sigma_k\}$ such that $\mathbb{1}_{(0,\sigma_k]}(t)F(t,X)$ is bounded for each k, and for almost every $\omega, \sigma_k = T$ for all large enough k.*
Then there exists a unique solution X to equation (1.30) on $[0,T]$.

Proof. The idea is extend all the terms to $[0,\infty)$ and apply Theorem 1.14.For $t \in (T,\infty)$ and $\omega \in \Omega$, define $\mathcal{F}_t = \mathcal{F}_T, H_t(\omega) = H_T(\omega), Y_t(\omega) = Y_T(\omega)$,and also extend the process $F(t,X)$ as a constant for $t > T$.Given any predictable process X given by the assumption, define

$$\nu_k = \sigma_k \mathbb{1}_{\sigma_k<T} + \infty_{\sigma_k=T}$$

These stopping times satisfy part iii) in Assumption 1.13 for $\mathbb{R}_+$.Theorem 1.14 gives a unique solution X for $t \in [0,\infty)$, and then X solves the equation on time $[0,T]$ with the original coefficients H, F, Y.
For the uniqueness, take a solution X of this equation on $[0,T]$, and extend it to $X_t = X_T$ for $t > T$.Then this is a solution of the extended equation on $[0,\infty)$.Then we apply the uniqueness theorem in this case. □

Now we consider the case when the Lipschitz constant in bounded at finite times.

Corollary 1.18. *Let the assumptions be as in Theorem 1.14,except that the Lipschitz assumption is weakened to this: for each* $0 < T < \infty$ *there exists a finite constant* $L(T)$ *such that this holds for all* $(t, \omega) \in [0, T] \times \Omega$ *and all* $\eta, \zeta \in D_{\mathbb{R}^d}[0, \infty)$*:*

$$|F(t, \omega, \eta) - F(t, \omega, \zeta)| \leq L(T) \sup_{s \in [0,t)} |\eta(s) - \zeta(s)| \tag{1.33}$$

Then equation (1.30) *has a unique solution* X *adapted to* $\{\mathcal{F}_t\}$.

Sketch of the proof. For $k \in \mathbb{N}$, the function $\mathbb{1}_{[0,k]}(t)F(t, \omega, \eta)$ satisfies the original hypotheses. By Theorem 1.14 there exists a process X_k that satisfies the equation

$$X_k(t) = H^k(t) + \int_{(0,t]} \mathbb{1}_{[0,k]}(s)F(s, X_n)dY^k(s) \tag{1.34}$$

where H^k, Y^k are the truncations up to time k.It's easy to see that for $k < m$,the stopped solution at time k X_m^k coincides with X_k.This permit us to define the X process unambiguously by $X(t) = X_k(t)$ if $t \in [0, k]$. Then the process X satisfies the equation

$$X(t) = H^k(t) + \int_{(0,k \wedge t]} F(s, X)dY(s), \quad 0 \leq t \leq k.$$

Since this holds for all k, X is a solution of the original SDE (1.30).
For the uniqueness, if $X, \tilde{X}$ are solutions of (1.30),then $X(t \wedge k), \tilde{X}(t \wedge k)$ solve (1.34).By the uniqueness theorem, $X(t \wedge k) = \tilde{X}(t \wedge k)$.As $k \in \mathbb{N}$ is arbitrary, $X = \tilde{X}$. □

Here we show how to recover the existence and uniqueness for Itô equations from the semimartingale equation

Corollary 1.19. *Let* B_t *be a standard Brownian motion in* $\mathbb{R}^m$ *with respect to a right-continuous filtration* $\{\mathcal{F}_t\}$ *and* $\xi \in \mathbb{R}^d$ *an* $\mathbb{R}^d$*-valued* $\mathcal{F}_0$*-measurable random variable. Fix* $0 < T < \infty$.*Assume the functions* $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ *satisfy the Lipschitz condition*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|$$

and the bound

$$|b(t, x)| + |\sigma(t, x)| \leq L(1 + |x|)$$

for a constant L and $0 \leq t \leq T, x, y \in \mathbb{R}^d$.

Then there exists a unique continuous process X on [0,T] that is adapted to $\{\mathcal{F}_t\}$ and satisfies

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{1.35}$$

for all $0 \leq t \leq T$.

Proof. Take $Y_t := [t, B_t]^t, H_t = \xi, F(t, \omega, \eta) := [b(t, \eta_{t-}), \sigma(t, \eta_{t-})]$. We write η_{t-} in order to obtain a predictable process.The hypotheses on b, σ guarantee that the assumptions for the semimartingale equation are satisfied.Therefore there is a cadlag solution X of (1.35).The continuity follows from the continuity on the right-hand side of this equation. □

1.4 Existence and uniqueness for a semimartingale equation.

In this section we prove Theorem 1.14.We will need Gronwall-type estimates, which will be proved after some technical lemmas.Observe that we can assume $L > 0$.If $L = 0$,then F does not depend on X,and (1.30) defined directly the process X,so there is nothing to prove.

1.4.1 Technical preliminaries

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Lemma 1.20. *Let X be a cadlag process and Y a cadlag semimartingale. Fix $t \geq 0$.Let γ be a nondecreasing continuous process on $[0, t]$ such that $\gamma(0) = 0$ and $\gamma(u)$ is a stopping time for each u.Then*

$$\int_{(0,\gamma(t)]} X(s-)dY(s) = \int_{(0,t]} X \circ \gamma(s-)d(Y \circ \gamma)(s) \tag{1.36}$$

Remark 1.21. *On the right the integrand is*

$$X \circ \gamma(s-) = \lim_{u \nearrow s, u < s} X(\gamma(u))$$

which is not the same as X evaluated on $\gamma(s-)$.To see this, suppose $\gamma(u) = u$ and X has a jump at s.

For the stochastic integral on the right of (1.36) the filtration changed to $\mathcal{F}_{\gamma(u)}$.

Proof. Simply write both sides of (1.36) as limit of Riemann sums, and these sums agree. □

Lemma 1.22. *Suppose A is a nondecreasing cadlag function such that $A(0) = 0$, and Z is a nondecreasing real-valued cadlag function. Then*

$$\gamma(u) := \inf\{t \geq 0 : A(t) \geq u\}$$

defines a nondecreasing cadlag function with $\gamma(0) = 0$ and

$$\int_{(0,u]} Z \circ \gamma(s-) d(A \circ \gamma)(s) \leq (A(\gamma(u)) - u) Z \circ \gamma(u-) + \int_{(0,u]} Z \circ \gamma(s-) ds \tag{1.37}$$

Proof. The nondecreasing property is clear. To prove right-continuity, let $\epsilon > 0$.By definition, $A(\gamma(u)+\epsilon) > u$.Pick $\delta > 0$ so that $A(\gamma(u)+\epsilon) > \delta+u$.Then for $v \in [u, u+\delta]$, $A(\gamma(u)+\epsilon) > v$, and thus $\gamma(v) \leq \gamma(u) + \epsilon$.

To prove (1.37), fix a partition $\{0 = s_0 < s_1 < \cdots < s_m = u\}$ be a partition of $[0, u]$.Since $Z \circ \gamma$ is cadlag, the integrals in (1.37) are limits of Riemann sums with integrand evaluated at the left endpoint of the partition. The following identity holds

$$\begin{aligned}\sum_{i=0}^{m-1} Z(\gamma(s_i))(A(\gamma(s_{i+1})) - A(\gamma(s_i))) = &\sum_{i=0}^{m-1} Z(\gamma(s_i))(s_{i+1} - s_i) + Z(\gamma(s_{m-1}))(A(\gamma(u)) - u) \\ &- \sum_{i=1}^{m-1} (Z(\gamma(s_i)) - Z(\gamma(s_{i-1}))(A(\gamma(s_i)) - s_i)\end{aligned}$$

Since $A(t) > u$ for $t > \gamma(u)$,by the cadlag property of A we have $A(\gamma(u)) \geq u$.Therefore the last sum in the right-hand side is positive, and then

$$\sum_{i=0}^{m-1} Z(\gamma(s_i))(A(\gamma(s_{i+1})) - A(\gamma(s_i))) \leq \sum_{i=0}^{m-1} Z(\gamma(s_i))(s_{i+1} - s_i) + Z(\gamma(s_{m-1}))(A(\gamma(u)) - u)$$

Letting the mesh of the partition to zero we obtain (1.37). ☐

Lemma 1.23. *Let X be an adapted cadlag process and $\alpha > 0$.Let $\tau_1 < \tau_2 < \cdots$ be the times of successive jumps in X of magnitude above α: with $\tau_0 = 0$,*

$$\tau_k := \inf\{t < \tau_{k-1} : |X(t) - X(t-)| > \alpha\}.$$

Then the $\{\tau_k\}$ are stopping times.

Sketch of the proof. Fix $k \in \mathbb{N}$.To prove that $\{\tau_l \leq t\} \in \mathcal{F}_t$,consider

$$A := \bigcup_{l\geq 1} \bigcup_{m\geq 1} \bigcup_{N\geq 1} \bigcap_{n\geq N} \{ \text{ there exis integers } 0 < u_1 < u_2 < \cdots < u_k \leq n \text{ such that } u_i - u_{i-1} \geq n/l$$
$$\text{and } |X(\frac{u_i t}{n}) - X(\frac{u_i t - t}{n})| > \alpha + \frac{1}{m}\} \in \mathcal{F}_t$$

Then is checked that $A = \{\tau_k \le t\}$. $\square$

Lemma 1.24. *Suppose F satisfies Assumption 1.13 and let*

$$\xi(t) := \int_{(0,t]} F(s, X) dY(s).$$

Let τ be a finite stopping time. Then

$$\xi^{\tau-}(t) = \int_{(0,t]} F(s, X^{\tau-}) dY^{\tau-}(s). \tag{1.38}$$

In particular, suppose X satisfies (1.30). Then the equation continues to hold when all the processes are stopped at τ^-, that is

$$X^{\tau-}(t) = H^{\tau-}(t) + \int_{(0,t]} F(s, X^{\tau-}) dY^{\tau-}(s). \tag{1.39}$$

Proof. We only need to check (1.38). We use part *b*) in Proposition 1.50 in part 4 with $G(s) = F(s, X^{\tau-}), J(s) = F(s, X)$. Observe that for $0 \le s \le \tau$,

$$|F(s, X^{\tau-}) - F(s, X^{\tau})| \le L \sup_{u \in (0,s)} |X_u^{\tau-} - X_u| \le L \sup_{u \in (0,\tau)} |X_u^{\tau-} - X_u| = 0.$$

Thus $G(s) = J(s)$ for $0 \le s \le \tau$. So by part *b*) in Proposition 1.50 in part 4,

$$\xi^{\tau-} = (J \cdot Y)^{\tau-} = (G \cdot Y)^{\tau-} = G \cdot Y^{\tau-}.$$

$\square$

Lemma 1.25. *Assume the filtration $\{\mathcal{F}_t\}$ is right-continuous. Let σ be a finite stopping time for $\{\mathcal{F}_t\}$ and $\overline{\mathcal{F}}_t = \mathcal{F}_{\sigma+t}$.*

a) Let ν be a stopping time for $\{\overline{\mathcal{F}}_t\}$. Then $\sigma+\nu$ is a stopping time for $\{\mathcal{F}_t\}$ and $\overline{\mathcal{F}}_\nu \subset \mathcal{F}_{\sigma+\nu}$.

b) Suppose τ is a stopping time for $\{\mathcal{F}_t\}$. Then $\nu := (\tau - \sigma)^+$ is an $\mathcal{F}_\tau$-measurable random variable, and an $\{\overline{\mathcal{F}}_t\}$-stopping time.

c) Let Z be a cadlag process adapted to $\{\mathcal{F}_t\}$ and $\overline{Z}$ a cadlag process adapted to $\{\overline{\mathcal{F}}_t\}$. Define

$$X(t) := \begin{cases} Z(t), & t < \sigma \\ Z(\sigma) + \overline{Z}(t - \sigma), & t \ge \sigma. \end{cases}$$

Then X is a cadlag process adapted to $\{\mathcal{F}_t\}$.

Sketch of the proof. a) Given $A \in \overline{\mathcal{F}}_\nu$,we have

$$A \cap \{\sigma + \nu < t\} = \bigcup_{r \in (0,t) \cap \mathbb{Q}} A \cap \{\nu < r\} \cap \{\sigma \leq t - r\}$$

Check that $A \cap \{\nu < r\} \in \mathcal{F}_{\sigma+r}$,and then

$$A \cap \{\nu < r\} \cap \{\sigma + r \leq t\} \in \mathcal{F}_t$$

Taking $A = \Omega$,this shows that $\{\sigma + \nu < t\} \in \mathcal{F}_t$.As the filtration is right-continuous, by Lemma 1.16 in part 1 we conclude that $\sigma + \nu$ is $\mathcal{F}_t$-measurable.
Now if $A \in \overline{\mathcal{F}}_t$ is arbitrary, $A \cap \{\sigma + \nu \leq t\} = \cap_{m \geq n} A \cap \{\sigma + \nu < t + \frac{1}{m}\} \in \mathcal{F}_{t+\frac{1}{n}}$.As $n \in \mathbb{N}$ is arbitrary and the right-continuity of $\mathcal{F}_t$, we deduce that $A \cap \{\sigma + \nu \leq t\} \in \mathcal{F}_t$,so $A \in \mathcal{F}_{\sigma+\nu}$.Therefore $\overline{\mathcal{F}}_t \subset \mathcal{F}_{\sigma+\nu}$.

b) For $0 \leq t < \infty$, $\{(\tau - \sigma)^+ \leq t\} = \{\sigma + t \geq \tau\}$.By part *ii*) in Lemma 1.11 from part 1, this last set lie both in $\mathcal{F}_\tau$ and $\mathcal{F}_{\sigma+t}$.In particular, this shows that $(\tau - \sigma)^+$ is $\mathcal{F}_{\sigma+t} = \overline{\mathcal{F}}_t$-measurable.

c) Fix $0 \leq t < \infty$, and a Borel set B on the state space of these processes. We have

$$\{X(t) \in B\} = \{\sigma > t, Z(t) \in B\} \cup \{\sigma \leq t, Z(\sigma) + \overline{Z}(t - \sigma) \in B\}$$

Clearly, the first term in the union is $\mathcal{F}_t$-measurable. For the second term,note that

$$\{\sigma \leq t, Z(\sigma) + \overline{Z}(t - \sigma) \in B\} = \{\sigma \leq t, Z(\sigma) + \overline{Z}((t - \sigma)^+) \in B\}$$

By part *iii*) in Lemma 1.11 from part 1,since cadlag processes are progressively measurable, the process $Z(\sigma)$ is $\mathcal{F}_\sigma$-measurable and $\overline{Z}((t-\sigma)^+)$ is $\overline{\mathcal{F}}_{(t-\sigma)^+}$-measurable.Since $\mathcal{F}_\sigma \subset \mathcal{F}_{\sigma+\nu}$ and by part a) in this lemma, $\overline{\mathcal{F}}_{(t-\sigma)^+} \subset \mathcal{F}_{\sigma+(t-\sigma)^+}$,then $Z(\sigma) + \overline{Z}((t-\sigma)^+)$ is $\mathcal{F}_{\sigma+(t-\sigma)^+}$-measurable.Since $\sigma \leq t$ is equivalent to $\sigma + (t - \sigma)^+ \leq t$,we rewrite once more

$$\{\sigma \leq t, Z(\sigma) + \overline{Z}(t - \sigma)^+\} = \{\sigma + (t - \sigma)^+ \leq t\} \cap \{Z(\sigma) + \overline{Z}(t - \sigma)^+\} \in \mathcal{F}_t.$$

Completing the proof.

□

1.4.2 A Gronwall estimate for semimartingale equations.

In this section, the processes are defined for $0 \leq t < \infty, F$ satisfies Assumption 1.13.The semimartingales will satisfy the following definition.Recall that if S is a cadlag BV function on $[0,T]$ and Λ_S id its Lebesgue-Stieltjes measure,then the total variation function of S is denoted by $V_S(t)$.It satisfies $|\Lambda_S| = \Lambda_{V_S}$.For Lebesgue-Stieltjes integrals we have the inequality

$$\left| \int_{(0,T]} g(s) dS(s) \right| \leq \int_{(0,T]} |g(s)| dV_S(s). \tag{1.40}$$

Definition 1.26. *Let $0 < \delta, K < \infty$ be constants. We say that a $\mathbb{R}^m$-valued cadlag semimartingale $Y = (Y_1, \cdots, Y_m)^t$ is of type (δ, K) if Y has a decomposition $Y = Y(0) + M + S$ where $M = (M_1, \cdots, M_m)^t$ is an m-vector of L^2-martingales ,$S = (S_1, \cdots, S_m)^t$ is an m-vector of FV processes, and*

$$|\triangle M_j(t)| \leq \delta, |\triangle S_j(t)| \leq \delta, \text{ and } V_{S_j}(t) \leq K \tag{1.41}$$

for all $0 \leq t < \infty$ and $1 \leq j \leq m$,almost surely.

Define the increasing process

$$A(t) := 16L^2 dm \sum_{j=1}^{m} [M_j]_t + 4KL^2 dm \sum_{j=1}^{m} V_{S_j}(t) + t, \tag{1.42}$$

the stopping times

$$\gamma(u) := \inf\{t \geq 0 : A(t) \geq u\}, \tag{1.43}$$

and the constant

$$c = c(\delta, K, L) = 16\delta^2 L^2 dm^2 + 4\delta K L^2 dm^2 \tag{1.44}$$

Remark 1.27. *Some observations about $A(t), \gamma(t)$.*

i) $A(t) \geq t$, and $A(t)$ is strictly increasing.

ii) As $A(t + \epsilon) \geq t + \epsilon > t$,then $\gamma(u) \leq u$.

iii) *$\gamma(u)$ is a stopping time. In fact, $\{\gamma(u) \leq t\} = \{A(t) \geq u\} \in \mathcal{F}_t$ by the adaptedness of A.*

iv) *Because A is strictly increasing, γ is continuous.*

v) *Note that for any given ω and T,once $u > A(t), \gamma(u) \geq T$.Thus $\gamma(u) \to \infty$ when $u \to \infty$.*

vi) *If Y is of type (δ_0, K_0) for some $\delta_0 \leq \delta, K_0 \leq K$,then $\triangle A(t) \leq c$.This is because $\triangle[M_j]_t = (\triangle M_j(t))^2, \triangle V_{S_j}(t) = |\triangle S_j(t)|$.*

For $l = 1, 2$,let H_l, Z_l and X_l be adapted $\mathbb{R}^d$-valued cadlag processes.Assume they satisfy the equations

$$Z_l(t) = H_l(t) + \int_{(0,t]} F(s, X_l) dY(s), \quad l = 1, 2. \tag{1.45}$$

for all $0 \leq t < \infty$.Let

$$D_X(t) := \sup_{0 \leq s \leq t} |X_1(s) - X_2(s)|^2 \tag{1.46}$$

and

$$\phi_X(u) := E[D_X \circ \gamma(u)] = E\left[\sup_{0 \leq s \leq \gamma(u)} |X_1(s) - X_2(s)|^2\right] \tag{1.47}$$

Make the last two definitions with X replaced by Z and H.D_X, D_Z and D_H are nonnegative, nondecreasing cadlag processes, and ϕ_X, ϕ_Z and ϕ_H are nonnegative nondecreasing functions, and cadlag at least on any interval on which they are finite. We assume that

$$\phi_H(u) = E\left[\sup_{0 \leq s \leq \gamma(u)} |H_1(s) - H_2(s)|^2\right] < \infty. \tag{1.48}$$

for all $0 \leq u < \infty$.

The proposition below is the key tool for the proof of Theorem 1.14.Part b) is a Gronwall-type estimate for SDE's.

Proposition 1.28. *Suppose F satisfies Assumption 1.13 and Y is a semimartingale of type $(\delta, K - \delta)$.Furthermore,assume (1.47), and let the pairs (X_l, Z_l) satisfy (1.45).*

a) *For $0 \leq u < \infty$,*

$$\phi_Z(u) \leq 2\phi_H(u) + c\phi_X(u) + \int_0^u \phi_X(s) ds. \tag{1.49}$$

b) Suppose $Z_l = X_l$,that is, X_1 and X_2 satisfy the equations

$$X_l(t) = H_l(t) + \int_{(0,t]} F(s, X_l)dY(s), \quad l = 1, 2. \tag{1.50}$$

Then $\phi_X(u) < \infty$.If δ is small enough relative to K, L so that $c < 1$,then for all $u > 0$,

$$\phi_X(u) \leq \frac{2\phi_H(u)}{1-c} e^{\frac{u}{1-c}} \tag{1.51}$$

Before the proof we need an auxiliary lemma.

Lemma 1.29. *Let $0 < \delta, K < \infty$ and suppose Y is a semimartingale of type (δ, K).Let $G = (G_{i,j})$ be a bounded predictable $d \times m$-matrix valued process. Then for $0 \leq u < \infty$ we have the bounds*

$$E\left[\sup_{0\leq t\leq\gamma(u)} \left|\int_{(0,t]} G(s)dY(s)\right|^2\right] \leq \frac{L^{-2}}{2} E \int_{0,\gamma(u)]} |G(s)|^2 dA(s) \leq \frac{L^{-2}}{2}(u+c)||G||_\infty^2. \tag{1.52}$$

Sketch of the proof of Lemma 1.29. From the inequality $(x+y)^2 \leq 2(x^2+y^2)$,

$$\left|\int_{(0,t]} G(s)dY(s)\right|^2 \leq 2\sum_{i=1}^{d}\left(\sum_{j=1}^{m}\int_{(0,t]} G_{i,j}(s)dM_j(s)\right)^2 + 2\sum_{i=1}^{d}\left(\sum_{j=1}^{m}\int_{(0,t]} G_{i,j}(s)dS_j(s)\right)^2 \tag{1.53}$$

Each sum is bounded by separately.

For the first sum, we apply Doob's inequality, the inequality $(\sum_i a_i)^2 \leq m\sum_i a_i^2$,and the isometry of stochastic integrals to obtain

$$E\left[\sup_{0\leq t\leq\gamma(u)}\left(\sum_{j=1}^{m}\int_{(0,t]} G_{i,j}(s)dM_j(s)\right)^2\right] \leq 4m\sum_{j=1}^{m} E\int_{(0,\gamma(u)]} G_{i,j}^2 d[M_j](s) \tag{1.54}$$

Similarly, in the second sum we apply the inequality $(\sum_i a_i)^2 \leq m\sum_i a_i^2$,(1.40) and Cauchy-Schwarz to obtain

$$E\left[\sup_{0\leq t\leq\gamma(u)}\left(\sum_{j=1}^{m}\int_{(0,t]} G_{i,j}dS_j(s)\right)^2\right] \leq m\sum_{j=1}^{m} E\left[V_{s_j}(\gamma(u))\int_{(0,\gamma(u)]} G_{i,j}(s)^2 dV_{S_k}(s)\right] \tag{1.55}$$

To prove (1.52),we use (1.53),(1.54) , $V_{S_j} \leq K$ and the definition of A to deduce

$$
\begin{aligned}
E\left[\sup_{0\leq t\leq \gamma(u)}\left|\int_{(0,t]} G(s)dY(s)\right|^2\right] &\leq 8dm\sum_{j=1}^{m} E\int_{(0,\gamma(u)]}|G(s)|^2 d[M_j](s) + 2Km\sum_{j=1}^{m}\int_{(0,\gamma(u)]}|G(s)|^2 dV_{S_j}(s) \\
&\leq \frac{L^{-2}}{2}\int_{(0,\gamma(u)]}|G(s)|^2 dA(s) \\
&\leq \frac{L^{-2}}{2}\sup_{0\leq s\leq \gamma(u)}|G(s)|^2 A(\gamma(u)) \leq \frac{L^{-2}}{2}(u+c)||G||_{\infty}^2
\end{aligned}
$$

In the last inequality we use that $A(\gamma(u)-) \leq u$ and $\triangle A(s) \leq c$,so that

$$A(\gamma(u)) \leq A(\gamma(u)-) + \triangle A(\gamma(u)) \leq u + c.$$

□

Sketch of proof of Proposition 1.28.

Step 1) First assume that $|F| \leq C_0$,and Y is of type (δ, K).From the inequality $(x+y)^2 \leq 2(x^2+y^2)$,

$$|Z_1(t) - Z_2(t)|^2 \leq 2|H_1(t) - H_2(t)|^2 + 2\left|\int_{(0,t]}(F(s,X_1) - F(s,X_2)dY(s)\right|^2 \tag{1.56}$$

It's easy to check that $\phi_Z(u) < \infty$ for all $0 \leq u < \infty$ applying (1.52) and $|F| \leq C_0$.

By the Lipschitz assumption,

$$|F(s,X_1) - F(s,X_2)|^2 \leq L^2 D_X(s-)$$

Applying this bound and (1.52) is obtained that

$$E\left[\sup_{0\leq t\leq \gamma(u)}\left|\int_{(0,t]}(F(s,X_1) - F(s,X_2)dY(s)\right|^2\right] \leq \frac{1}{2}E\int_{(0,\gamma(u)]} D_X(s-)dA(s). \tag{1.57}$$

By (1.56) and (1.57) we deduce

$$\phi_Z(u) \leq 2\phi_H(u) + E\int_{(0,\gamma(u)]} D_X(s-)dA(s). \tag{1.58}$$

By the change of variable formula in Lemma 1.20 and then (1.37), the result is

$$\phi_Z(u) \leq 2\phi_H(u) + E[A(\gamma(u) - u)D_X \circ (\gamma(u-))] + E\int_{(0,u]} D_X \circ \gamma(s-)ds \tag{1.59}$$

Finally,noting that $A(\gamma(u)) - u \leq c$ we conclude that

$$\phi_Z(u) \leq 2\phi_H(u) + c\phi_X(u) + \int_0^u \phi_X(s)ds. \tag{1.60}$$

This proves part $a)$ in this particular case.

To check part $b)$, note that if $Z_l = X_l$ and $c < 1$,then $\phi_X = \phi_Z < \infty$.Equation (1.60) transform into

$$\phi_X(u) \leq \frac{2}{1-c}\phi_H(u) + \frac{1}{1-c}\int_0^u \phi_X(s)ds.$$

Finally,apply Gronwall's inequality Lemma A.1 to deduce the desired bound.

Step 2) By part $iii)$ in Assumption 1.13,pick stopping times $\sigma_k \nearrow \infty$ and finite constants B_k such that

$$|\mathbb{1}_{(0,\sigma_k)]}(s)F(s,X_k)| \leq B_k, \quad l = 1,2.$$

Define truncated functions by

$$F_{B_k}(s,\omega,\eta) := [F(s,\omega,\eta) \wedge B_k] \vee (-B_k). \tag{1.61}$$

By (1.45) and Lemma 1.29 we have

$$Z_l^{\sigma_k-}(t) = H_l^{\sigma_k-}(t) + \int_{(0,t]} F(s, X_l^{\sigma_k-})dY^{\sigma_k-}(s), \quad l = 1,2.$$

Since $X_l = X_l^{\sigma_k-}$ on $[0,\sigma_k)$, $F(s,X_l^{\sigma_k-}) = F(s,X_l)$ on $[0,\sigma_k]$(because on $[0,\sigma_k]$,the integrand only depends on $\{X_l(s) : 0 \leq s < \sigma_k\}$).The truncation has no effect on $F(s,X_l)$ if $0 \leq s \leq \sigma_k$, and so also

$$F(s,X_l^{\sigma_k-}) = F_{B_k}(s,X_l^{\sigma_k}-) \text{ on } [0,\sigma_k].$$

Then we can substitute this in the equation above to deduce

$$Z_l^{\sigma_k-}(t) = H_l^{\sigma_k-}(t) + \int_{(0,t]} F_{B_k}(s, X_l^{\sigma_k-})dY^{\sigma_k-}(s), \quad l = 1,2.$$

And the advantage is that the integrand is bounded, so we are in position to apply Step 1. We only need to check that Y^{σ_k-} is of type (δ,K).This is true because

$$Y^{\sigma_k-} = Y_0 + M^{\sigma_k}(t) + S^{\sigma_k}(t) - \triangle M(\sigma_k)\mathbb{1}_{t\geq\sigma_k}. \tag{1.62}$$

M^{σ_k} is a L^2-martingale with jumps bounded by δ.The jth component of the new FV part is $G_j(t) := S_j^{\sigma_k -}(t) - \triangle M_j(\sigma_k)\mathbb{1}_{t\geq\sigma_k}$.Note that if $t<\sigma_k, |\triangle G_j(t)| = |\triangle S_j(t)| \leq \delta$,and if $t=\sigma_k, \triangle G_j(\sigma_k) = |\triangle S_j^{\sigma_k -}(\sigma_k)| = |\triangle M_j(\sigma_k)| \leq \delta$.Also, $V_{G_j} \leq V_{S_j} + |\triangle M_j(\sigma_k)| \leq K - \delta + \delta = K$.

Therefore $Y^{\sigma_k -}$ is of type (δ, K).The conclusion is that Step 1 holds for $Z_l^{\sigma_k -}, H_l^{\sigma_k -}, X_l^{\sigma_k -}$.Additionally, the following inequality is true

$$E\left[\sup_{0\leq s\leq\gamma(u)} |X_1^{\sigma_k -}(s) - X_2^{\sigma_k -}(s)|^2\right] \leq \phi_X(u)$$

while

$$\lim_{k\to\infty} E\left[\sup_{0\leq s\leq\gamma(u)} |X_1^{\sigma_k -}(s) - X_2^{\sigma_k -}(s)|^2\right] = \phi_X(u)$$

by monotone convergence because $\sigma_k \nearrow \infty$.The same results hold for H, Z.Finally,apply part $a)$ and the previous inequalities to obtain

$$E\left[\sup_{0\leq s\leq\gamma(u)} |Z_1^{\sigma_k -}(s) - Z_2^{\sigma_k -}(s)|^2\right] \leq 2\phi_H(u) + c\phi_X(u) + \int_0^u \phi_X(s)ds.$$

Taking $k\to\infty$ part $a)$ is complete.Now we apply part $b)$ in the Step 1 to conclude

$$E\left[\sup_{0\leq s\leq\gamma(u)} |X_1^{\sigma_k -}(s) - X_2^{\sigma_k -}(s)|^2\right] \leq 2\frac{\phi_H(u)}{1-c}e^{\frac{1}{1-c}}$$

and again take $k\to\infty$ to conclude the proof of the proposition.

$\square$

1.4.3 The uniqueness theorem.

Before state the uniqueness theorem, we prove an useful lemma that will be used in the posterior proof.

Lemma 1.30. *Assume F satiasfies Assumption 1.13 and X satisfies (1.30) on $[0,\infty)$.Let σ be a bounded stopping time.Define a new filtration, new processes, and a new coefficient by $\overline{\mathcal{F}}_t := \mathcal{F}_{t+\sigma}, \overline{X}(t) := X(t+\sigma) - X(\sigma), \overline{H}(t) := H(t+\sigma) - H(\sigma), \overline{Y}(t) := Y(t+\sigma) - Y(\sigma)$ and*

$$\overline{F}(t,\omega,\eta) := F(\sigma + t, \omega, \zeta^{\omega,\eta})$$

where the cadlag path $\zeta^{\omega,\eta} \in D_{\mathbb{R}^d}[0,\infty)$ is defined by

$$\zeta^{\omega,\eta}(s) := \begin{cases} X(s), & 0 \leq s < \sigma \\ X(\sigma) + \eta(s-\sigma), & s \geq \sigma \end{cases}$$

Then under $\{\overline{\mathcal{F}}_t\}$, $\overline{X}, \overline{H}$ are adapted cadlag processes,$\overline{Y}$ is a semimartingale and $\overline{F}$ satisfies Assumption 1.13.$\overline{X}$ is a solution of the equation

$$\overline{X}(t) = \overline{H}(t) + \int_{(0,t]} \overline{F}(s,\overline{X})d\overline{Y}(s)$$

Proof. First we check that $\overline{F}$ satisfies the hypotheses. Is clear that is Lipschitz.Let $\overline{Z}$ be a cadlag process adapted to $\{\overline{\mathcal{F}}_t\}$.Define the process Z by

$$Z(t) := \begin{cases} X(t), & t < \sigma \\ X(\sigma+t) - X(\sigma), & t \geq \sigma \end{cases}$$

Then Z is a cadlag adapted process to $\{\mathcal{F}_t\}$ due to part $c)$ in Lemma 1.25.Observe that by construction,$\overline{F}(t,\overline{Z}) = F(\sigma+t, Z)$ is predicable under $\{\overline{\mathcal{F}}_t\}$ by Lemma 1.46 from part 4.Pick stopping times $\nu_k \nearrow \infty$ such that $\mathbb{1}_{[0,\nu_k]}(s)F(s,Z)$ is bounded for each k.Define $\rho_k := (\nu_k - \sigma)^+$.Then $\rho_k \nearrow \infty$.By part $b)$ in Lemma 1.25 ρ_k is a stopping time under $\{\overline{\mathcal{F}}_t\}$,and $\mathbb{1}_{(0,\rho_k]}(s)\overline{F}(s,\overline{Z}) = \mathbb{1}_{(0,\nu_k]}(s+\sigma)F(s+\sigma,Z)$,whic is bounded.We have checked the hypotheses for $\overline{Z}$.

By Theorem 1.45 from part 4, $\overline{Y}$ is a semimartingale .From part $iii)$ in Lemma 1.11 from part 1,$\overline{X}$ and $\overline{H}$ are adapted to $\{\overline{\mathcal{F}}_t\}$.

To finish the proof, we check that $\overline{X}$ satisfies the equation:

$$\begin{aligned} \overline{X}(t) = X(t+\sigma) - X(\sigma) &= H(t+\sigma) - H(\sigma) + \int_{(\sigma,\sigma+t]} F(s,X)dY(s) \\ &= overline H(t) + \int_{(0,t]} F(\sigma+s,X)d\overline{Y}(s) \\ &= \overline{H}(t) + \int_{(0,t]} \overline{F}(s,\overline{X})d\overline{Y}(s) \end{aligned}$$

In the last equality we used that $\zeta^{\omega,\overline{X}} = X$.That finishes the proof of the lemma. □

Next we procede to the proof ot the uniqueness theorem.

Theorem 1.31. *Let H be an $\mathbb{R}^d$-valued adalted cadlag process and Y an $\mathbb{R}^m$-valued cadlag semimartingale. Assume F satisfies Assumption 1.13.Suppose X_1, X_2 are two adapted $\mathbb{R}^d$-valued cadlag processes, and both are solutions to the equation*

$$X(t) = H(t) + \int_{(0,t]} F(s, X)dY(s), \quad 0 \le t < \infty. \tag{1.63}$$

Then for almost every ω, $X_1(t,\omega) = X_2(t,\omega)$ for all $0 \le t < \infty$.

Sketch of the proof. Clearly $X_1(0) = X_2(0) = H(0)$.

Step 1) First we show that there exists a stopping time σ,defined in terms of Y, such that $P(\sigma > 0) = 1$, and for all choices of H, F, and for all solutions X_1, X_2 we have $X_1(t) = X_2(t)$ for $0 \le t \le \sigma$.

By Theorem 1.27 from part 2, we can choose a martingale decomposition of Y as $Y = Y(0) + M + G$,so that the local L^2-martingale M has bounded jumps. More precisely, pick $0 < \delta < K/3 < \infty$ so that $c < 1$,where c is defined as in (1.44), and if assume that the jumps of M are bounded by $\delta/2$.Fix a constant $0 < C < \infty$.Define the following stopping times:

$$\tau_1 := \inf\{t > 0 : |Y(t) - Y(0)| \ge C \text{ or } |Y(t-) - Y(0)| \ge C\},$$
$$\tau_2 := \inf\{t > 0 : |\triangle Y(t)| \ge \delta/2\},$$
$$\tau_3 := \inf\{t > 0 : V_{G_j} \ge K - 2\delta \text{ for some } 1 \le j \le m\}.$$

By Lemma 1.20 from part 1 and Lemma 1.23, τ_1, τ_2 are stopping times.τ_3 is also a stopping time since $V_{G_j}(t)$ is nondecreasing and cadlag,

$$\{\tau_3 \le t\} = \bigcup_{j=1}^{m} \{V_{G_j} \ge K - 2\delta\} \in \mathcal{F}_t$$

It's not difficult to see that the three stopping times are strictly positive. Let T be an arbitrary positive number and

$$\sigma = \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge T. \tag{1.64}$$

so that $P(\sigma > 0) = 1$.We claim that $Y^{\sigma-}$ satisfies the hypotheses of Proposition 1.28.To see this,decompose

$$Y^{\sigma-}(t) = Y(0) + M^{\sigma}(t) + G^{\sigma-}(t) - \triangle M(\sigma)\mathbb{1}_{t\ge\sigma}.$$

Arguin as in Step 2 of Proposition 1.28,we deduce that $Y^{\sigma-}$ is of type $(\delta, K-\delta)$ and M^{σ} is bounded, and hence an L^2-martingale.

By assumption both X_1, X_2 satisfy (1.63), and by Lemma 1.24 both $X_1^{\sigma-}, X_2^{\sigma-}$ satisfy the equation

$$X(t) = H^{\sigma-}(t) + \int_{(0,t]} F(s, X) dY^{\sigma-}(s), \quad 0 \le t < \infty. \tag{1.65}$$

To this equation we apply Proposition 1.28 with $H_1 = H_2 = H^{\sigma-}$, so $\phi_H(u) = 0$, and $\phi_H(u) < \infty$.By part $b)$ of this proposition, we get

$$E\left[\sup_{0\le t\le\gamma(u)} |X_1^{\sigma-}(t) - X_2^{\sigma-}|^2\right] = 0$$

For any $u > 0$.Taking $u \nearrow \infty$ we conclude that $X_1^{\sigma-}(t) = X_2^{\sigma-}(t)$ for all $0 \le t < \infty$.In particular, $X_1(t) = X_2(t)$ for $0 \le t < \sigma$.

The case $t = \sigma$ is result of

$$\begin{aligned} X_1(\sigma) &= H(\sigma) + \int_{(0,\sigma]} F(s, X_1) dY(s) \\ &= H(\sigma) + \int_{(0,\sigma]} F(s, X_2) dY(s) = X_2(\sigma) \end{aligned} \tag{1.66}$$

where we used that $F(s, X_1)$ only depends on $\{X_1(s) : 0 \le s < \sigma\}$.So we have proves that $X_1(t) = X_2(t)$ for $0 \le t \le \sigma$.

Step 2) Let $0 < T < \infty$.By Lemma 1.30 we can restart the equations for X_1, X_2 at time $\sigma = \tau \wedge T$.Since $X_1 = X_2$ on $[0, \sigma]$,both X_1, X_2 lead to the same new coefficient $\overline{F}(t, \omega, \eta)$ for the restarted equation. Consequently we have

$$\overline{X}_l(t) = \overline{H}(t) + \int_{(0,t]} \overline{F}(s, \overline{X}_l) d\overline{Y}(s), \quad l = 1, 2.$$

We apply Step 1 to this equation to find a stopping time $\overline{\sigma} > 0$ in the filtration $\{\mathcal{F}_{(\tau\wedge T)+t}\}$ such that $\overline{X}_1 = \overline{X}_2$ on $[0, \overline{\sigma}]$.This implies that $X_1 = X_2$ on $[0, \tau \wedge T + \overline{\sigma}]$.Therefore $\tau \ge \tau \wedge T + \overline{\sigma}$.If $\tau \wedge T = \tau$ we would have $\overline{\sigma} \le 0$,a contradiction. So $\tau \wedge T = T$, and this is equivalent to $\tau \ge T$.As τ is arbitrary, $\tau = \infty$, and so $X_1 = X_2$ at all time.

□

1.4.4 Existence theorem.

We start the existence theorem with stringent assumptions on Y.These hypotheses are subsequently relaxed with a localization argument.

Proposition 1.32. *A solution to* (1.30) *on* $[0,\infty)$ *exists under the following assumptions:*

i) F satisfies Assumption 1.13,

ii) There are constants $0<\delta<K<\infty$ such that c defined by (1.44) *satisfies $c<1$, and Y is of type $(\delta, K-\delta)$.*

Sketch of the proof. We define the Picard iteration as follows.Let $X_0(t) := H(t)$,and for $n\geq 0$,

$$X_{n+1}(t) := H(t) + \int_{(0,t]} F(s, X_n) dY(s) \tag{1.67}$$

By part *iii*) of Assumption 1.13, fix stopping times $\nu_k \nearrow \infty$ and constants B_k such that $\mathbb{1}_{(0,\nu_k]}(s)|F(s,H)| \leq B_k$ for all k.By Lemma 1.24,(1.67) continues to hold for stopped processes:

$$X_{n+1}^{\nu_k-}(t) = H^{\nu_k-}(t) + \int_{(0,t]} F(s, X_n^{\nu_k-}) dY^{\nu_k-}(s). \tag{1.68}$$

Fix k for a while.For $n\geq 0$,let

$$D_n(t) := \sup_{0\leq s\leq t} |X_{n+1}^{\nu_k-}(s) - X_n^{\nu_k-}(s)|^2$$

And for $0\leq u<\infty$,let

$$\phi_n(u) := E[D_n \circ \gamma(u)].$$

Lemma 1.33. *The function ϕ_0 is bounded on bounded intervals.*

Proof of Lemma 1.33. Because ϕ_0 is nondecreasing it suffices to show that $\phi_0(u)$ is finite for any u.First, define the truncation

$$F_{B_k}(s,\omega,\eta) := [F(s,\omega,\eta)\wedge B_k] \vee (-B_k)$$

Note that on $[0,\nu_k]$,

$$F(s, H^{\nu_k}) = F(s,H) = F_{B_k}(s,H) = F_{B_k}(s, H^{\nu_k-}).$$

Thus

$$|X_1^{\nu_k -}(t) - X_0^{\nu_k -}(t)|^2 = \left| \int_{(0,t]} F_{B_k}(s, H^{\nu_k -}) dY^{\nu_k -}(s) \right|^2$$

Now we apply (1.52)(Recall that $Y^{\nu_k -}$ is of type (δ, K)) and this last bound to deduce

$$\phi_0(u) \le E\left[\sup_{0\le t\le \gamma(u)} \left| \int_{(0,t]} F_{B_k}(s, H^{\nu_k -}) dY^{\nu_k -}(s) \right|^2 \right] \overset{(1.52)}{\le} \frac{L^{-2}}{2}(u+c)B_k^2$$

This shows that ϕ_0 is bounded on bounded intervals. $\square$

Inductively, we check that ϕ_n is bounded on bounded intervals.Apply part $a)$ in Proposition 1.28 with $(Z_1, Z_2) = (X_n^{\nu_k -}, X_{n+1}^{\nu_k -})$ and $(H_1, H_2) = (H^{\nu_k -}, H^{\nu_k -})$ to conclude

$$\phi_{n+1}(u) \le c\phi_n(u) + \int_0^u \phi_n(s)ds \tag{1.69}$$

By inductive hypothesis we deduce the boundedness condition.

Now we want to check that $\sum_n \phi_n < \infty$.The following lemma is proved easily by induction.

Lemma 1.34. *Fix $0 < T < \infty$.Let $\{\phi_n\}$ be nonnegative measurable functions on $[0,T]$ such that $\phi_0 \le B$ for some constant B,and (1.69) holds for all $n \ge 0, 0 \le u \le T$.Then for all $n \in$ and $0 \le u \le T$,*

$$\phi_n(u) \le B \sum_{k=0}^{n} \frac{1}{k!}\binom{n}{k} u^k c^{n-k} \tag{1.70}$$

Lemma 1.35. *For any $0 < \delta < 1$ and $0 < u < \infty$,*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!}\binom{n}{k} u^k \delta^{n-k} = \frac{1}{1-\delta} e^{\frac{u}{1-\delta}}.$$

Sketch of the proof of Lemma 1.35. Use the identity for $0 < x < 1, k \ge 0$,

$$\sum_{m=0}^{\infty} (m+1)(m+1)\cdots(m+k)x^m = \frac{k!}{(1-x)^{k+1}}. \tag{1.71}$$

and change the order of summation. $\square$

These two lemmas implies that

$$\sum_{n=0}^{\infty} \phi_n < \infty \tag{1.72}$$

By Chebyshev's inequality and (1.70) is obtained for $\alpha \in (c, 1)$

$$\sum_{n=0}^{\infty} P\left(\sup_{0\le t\le\gamma(u)} |X_{n+1}^{\nu_k -}(t) - X_n^{\nu_k -}(t)| > \alpha^{n/2} \right) \le \sum_{n=0}^{\infty} \alpha^{-n}\phi_n(u) \le B \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!}\binom{n}{k}\left(\frac{u}{\alpha}\right)^k \left(\frac{c}{\alpha}\right)^{n-k}$$

and the last sum converges by Lemma 1.35.By the Borel-Cantelli lemma there exists an almost surely finite time $N(\omega)$ such that for $n \geq N(\omega)$,

$$\sup_{0\leq t\leq \gamma(u)} |X_{n+1}^{\nu_k-}(t) - X_n^{\nu_k-}(t)| < \alpha^{n/2}.$$

Consequently for $p > m \geq N(\omega)$,

$$\sup_{0\leq t\leq \gamma(u)} |X_m^{\nu_k-}(t) - X_p^{\nu_k-}(t)| \leq \sum_{n=m}^{p-1} \alpha^{n/2} \leq \sum_{n=m}^{\infty} \alpha^{p/2} = \frac{\alpha^{m/2}}{1-\alpha^{1/2}}.$$

The last expression vanishes when $m \to \infty$.This give us the Cauchy property.By the completeness of cadlag functions under uniform distance,for almost every ω there exists a cadlag function $t \to \tilde{X}_k(t)$ on the interval $[0, \gamma(u)]$ such that

$$\lim_{n\to\infty} \sup_{0\leq t\leq \gamma(u)} |\tilde{X}_k(t) - X_n^{\nu_k-}(t)| = 0. \tag{1.73}$$

By considering a sequence of u-values increasing to infinity, we get a single event of probability one on which (1.73) holds for all $0 \leq u < \infty$.For any fixed ω and $T < \infty, \gamma(u) \geq T$ for all large enough u.We conclude that,with probability one, there exists a cadlag function $\tilde{X}_k$ on the interval $[0, \infty)$ such that

$$\lim_{n\to\infty} \sup_{0\leq t<T} |\tilde{X}_k(t) - X_n^{\nu_k-}(t)| = 0 \text{ for all } T < \infty. \tag{1.74}$$

Next we show that , for fixed k, $\tilde{X}_k$ is a solution of

$$\tilde{X}_k(t) = H^{\nu_k-}(t) + \int_{(0,t]} F(s, \tilde{X}_k) dY^{\nu_k-}(s). \tag{1.75}$$

Take $n \to \infty$ in (1.68).The left hand converges to the left hand of (1.75).To show the convergence of the stochastic integrals we recall the dominated convergence for stochastic integrals (Theorem 1.44 in from part 4). From the Lipschitz property of F,

$$|F(t, \tilde{X}_k) - F(t, X_n^{\nu_k-})| \leq L \sup_{0\leq s<t} |\tilde{X}_k(s) - X_n^{\nu_k-}(s)|.$$

Using this result,take $H_n(t) := F(t, \tilde{X}_k) - F(t, X_n^{\nu_k-})$ and the cadlag bound $G_n(t) := L \cdot \sup_{0\leq s\leq t} |\tilde{X}_k(s) - X_n^{\nu_k-}(s)|$,so that $|H_n(t)| \leq G_n(t-)$.The convergence (1.74) gives us the hypotheses so that we can apply dominated convergence.In conclusion, the right-hand side of (1.68) converges in probability, uniformly over t in compact time intervals to the right-hand side of (1.75).Along a subsequence, we can obtain almost sure convergence.This check (1.75).

This argument can be repeated for all values of k.The limit (1.74) implies that if $k < m, \tilde{X}_k(t) = \tilde{X}_m(t)$ on $[0, \nu_k)$ (because $X_n^{\nu_k} = X_n^{\nu}$ on $[0, \nu_k)$).Since $\nu_k \nearrow \infty$,we conclude that there is a single-well defined cadlag process X on $[0, \infty)$ such that $X = \tilde{X}_k$ on $[0, \nu_k)$.In this interval, (1.75) agrees term by term with the equation

$$X(t) = H(t) + \int_{(0,t]} F(s, X)dY(s) \tag{1.76}$$

This is true since $X = \tilde{X}_k$ on $[0, \nu_k)$, so that $F(s, X) = F(s, \tilde{X}_k)$ on $[0, \nu_k]$.Finally,apply part $b)$ in Proposition 1.50 from part 4.We have finished the proof of Proposition 1.32. $\square$

Now we finish the proof of the uniqueness without restrictions.

Theorem 1.36. *A solution of* (1.30) *on* $[0, \infty)$ *exists under Assumption 1.13 on F, or an arbitrary semimartingale* Y *and cadlag process* H*-*

Proof. Given an arbitrary semimartingale Y and a cadlag process H,fix constants $0 < \delta < K/3 < \infty$ so that c defined by (1.44) satisfies $c < 1$.Pick a decomposition $Y = Y_0 + M + G$ such that the local L^2-martingale M has jumps bounded by $\delta/2$.Fix a constant $0 < C < \infty$.Define the following stopping times ρ_k, σ_k and τ_k^i for $1 \leq i \leq 3$ and $k \in \mathbb{Z}_+$.First $\rho_0 = \sigma_0 = \tau_0^i = 0$ for $1 \leq i \leq 3$.For $k \geq 1$,

$$\begin{aligned}
\tau_k^1 &:= \inf\{t > 0 : |Y(\rho_{k-1} + t) - Y(\rho_{k-1})| \geq C \text{ or } |Y((\rho_{k-1} + t)-) - Y(\rho_{k-1})| \geq C\}, \\
\tau_k^2 &:= \inf\{t > 0 : |\triangle Y(\rho_{k-1} + t)| > \delta/2\}, \\
\tau_k^3 &:= \inf\{t > 0 : V_{G_j}(\rho_{k-1} + t) - V_{G_j}(\rho_{k-1}) \geq K - 2\delta \text{ for some } 1 \leq j \leq m\} \\
\sigma_k &:= \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3 \wedge 1 \\
\tau_k &:= \sigma_1 + \cdots + \sigma_k.
\end{aligned}$$

Each $\rho_k > 0$ by the same reasons that $\sigma > 0$,where σ was defined in (1.64).Consequently, $0 = \rho_0 < \rho_1 < \cdots$.Also, it can be showed that $\rho_k \nearrow \infty$. Intuitively, these stopping times are defined so that we can apply Proposition 1.28 with the stopping times σ_k.

Iterating part a) of Lemma 1.25 we deduce that ρ_k is a $\mathbb{F}_t$-stopping time for each k.Because $\rho_{k+1} = \rho_{k+1} - \rho_k$, applying now part b) in the same lemma we obtain that σ_{k+1} is a stopping time for $\{\mathcal{F}_{\rho_k+t} : t \geq 0\}$.

Now we state the core lemma that will help us to prove the theorem.

Lemma 1.37. *For each k,there exists an adapted cadlag process $X_k(t)$ such that the equation*

$$X_k(t) = H^{\rho_k}(t) + \int_{(0,t]} F(s, X_k) dY^{\rho_k}(s) \tag{1.77}$$

is satisfied.

Assuming for a moment this result, we proceed to finish the proof of the theorem. If $k < m$, stopping the processes of the equation

$$X_m(t) = H^{\rho_m}(t) + \int_{(0,t]} F(s, X_m) dY^{\rho_m}(s)$$

at ρ_k gives the equation

$$X_m^{\rho_k}(t) = H^{\rho_k}(t) + \int_{(0,t]} F(s, X_m^{\rho_k}) dY^{\rho_k}(s)$$

By the uniqueness theorem, $X_k = X_m^{\rho_k}$ for $k < m$.Thus we can define the process X in an unique way, so that $X = X_k$ on $[0, \rho_k]$ for each k.Then for $0 \leq t \leq \rho_k$, equation (1.77) agrees term by term with the desired equation

$$X(t) = H(t) + \int_{(0,t]} F(s, X) dY(s).$$

As $\rho_k \nearrow \infty$,this proves that X solves the equation for all $t \in [0, \infty)$,concluding the proof of the existence theorem. □

Proof of Lemma 1.37. The proof is by induction on k.For $k = 1, \rho_1 = \sigma_1$.The semimartingale $Y^{\sigma_1 -}$ satisfies the hypotheses of Proposition 1.32 by the same arguments as in (1.62).Consequently this proposition applies, and there exists a solution $\tilde{X}$ of the equation

$$\tilde{X}(t) = H^{\rho_1 -}(t) + \int_{(0,t]} F(s, \tilde{X}) dY^{\rho_1 -}(s). \tag{1.78}$$

Define

$$X_1(t) := \begin{cases} \tilde{X}(t), & 0 \leq t < \rho_1. \\ X_1(\rho_1 -) + \triangle H(\rho_1) + F(\rho_1, \tilde{X}) \triangle Y(\rho_1), & t \geq \rho_1. \end{cases}$$

We want to check that X_1 solves (1.77). Note that $H^{\rho_1-} = H^{\rho_1}$ on $[0,\rho_1)$ and and $F(t,\tilde{X}) = F(t,X_1)$ on $[0,\rho_1]$.Thus X_1 is a solution of (1.77) if $0 \le t < \rho_1$.For $t \ge \rho_1$, the following is true:

$$\begin{aligned}
X_1(t) &= X_1(\rho_1-) + \triangle H(\rho_1) + F(\rho_1,\tilde{X})\triangle Y(\rho_1) \\
&= H(\rho_1-) + \int_{(0,\rho_1)} F(s,\tilde{X})dY(s) + \triangle H(\rho_1) + F(\rho_1)\triangle Y(\rho_1) \\
&= H(\rho_1) + \int_{(0,\rho_1]} F(s,\tilde{X})dY(s). \\
&= H^{\rho_1}(t) + \int_{(0,t]} F(s,\tilde{X})dY^{\rho_1}(s).
\end{aligned}$$

Recalling that $H^{\rho_1}(t) = H(\rho_1)$ for $t \ge \rho_1$, and the general fact $G\cdot M^\tau = (G\cdot M)^\tau$.This checks the case $k=1$ in the lemma.

Assume a process $X_k(t)$ solves (1.77).Define $\overline{\mathcal{F}}_t := \mathcal{F}_{t+\rho_k}$,

$$\begin{aligned}
\overline{H}(t) &:= H(\rho_k + t) - H(\rho_k) \\
\overline{Y}(t) &:= Y(\rho_k + t) - Y(\rho_k) \\
\overline{F}(t,\omega,\eta) &:= F(\rho_k + t, \omega, \zeta^{\omega,\eta})
\end{aligned}$$

where the cadlag path $\zeta^{\omega,\eta}$ is defined by

$$\zeta^{\omega,\eta}(s) := \begin{cases} X_k(s), & 0 \le s < \rho_k \\ X_k(\rho_k) + \eta(s-\rho_k), & s \ge \rho_k. \end{cases}$$

We want a solution of the equation

$$\overline{X}(t) = \overline{H}^{\sigma_{k+1}-}(t) + \int_{(0,t]} \overline{F}(s,\overline{X})d\overline{Y}^{\sigma_{k+1}-}(s) \tag{1.79}$$

under the filtration $\{\overline{\mathcal{F}}_t\}$.Observe that the semimartingale $\overline{Y}^{\sigma_{k+1}-}$ satisfies the hypotheses of Proposition 1.32 (again,by the arguments in (1.62)).Also $\overline{F}$ satisfies Assumption 1.13,so exists a such process $\overline{X}$.Note that $\overline{X}(0) = \overline{H}(0) = 0$.Define

$$X_{k+1}(t) := \begin{cases} X_k(t), & t < \rho_k \\ X_k(\rho_k) - \overline{X}(t-\rho_k), & \rho_k \le t < \rho_{k+1} \\ X_{k+1}(\rho_{k+1}-) + \triangle H(\rho_{k+1}) + F(\rho_{k+1}, X_{k+1})\triangle Y(\rho_{k+1}), & t \ge \rho_{k+1}. \end{cases}$$

We proceed to verify that X_{k+1} solves (1.77).

By induction hypothesis, X_{k+1} satisfies the equation (1.77) for $k+1$ on $[0,\rho_k]$ (because $H^{\rho_{k+1}} = H^{\rho_k}$ in that interval).Note that for $0 \leq s \leq \rho_{k+1}, \overline{F}(s,\overline{X}) = F(\rho_k + s, X_{k+1})$. To check this, in the interval $[0,\rho_k)$, $X_{k+1} = X_k = \zeta^{\omega,\overline{X}}$.In the interval $[\rho_k,\rho_{k+1}), \zeta^{\omega,\overline{X}} = X_k(\rho_k) + \overline{X}(s-\rho_k) = X_{k+1}(s)$.Then for $\rho_k < t < \rho_{k+1}$ we have $t-\rho_k < \sigma_{k+1}$,so $\overline{H}(t-\rho_k) = \overline{H}^{\sigma_{k+1}-}(t-\rho_k)$, and the same with $\overline{Y}$.As $\overline{X}(t-\rho_k)$ satisfies (1.79),we deduce

$$\begin{aligned}
X_{k+1}(t) &= X_k(\rho_k) + \overline{X}(t-\rho_k) \\
&= X_k(\rho_k) + \overline{H}(t-\rho_k) + \int_{(0,t-\rho_k]} \overline{F}(s,\overline{X})d\overline{Y}(s) \\
&= X_k(\rho_k) + H(t) - H(\rho_k) + \int_{(\rho_k,t]} F(s,X_{k+1})dY(s) \\
&\overset{(1.77)}{=} H(t) + \int_{(0,t]} F(s,X_{k+1})dY(s) \\
&= H^{\rho_{k+1}}(t) + \int_{(0,t]} F(s,X_{k+1})dY^{\rho_{k+1}}(s).
\end{aligned}$$

We have proved that if $\rho_k < t < \rho_{k+1}$,

$$X_{k+1}(t) = H^{\rho_{k+1}}(t) + \int_{(0,t]} F(s,X_{k+1})dY^{\rho_{k+1}}(s). \tag{1.80}$$

To conclude, we need to check the case $t \geq \rho_{k+1}$:

$$\begin{aligned}
X_{k+1}(t) &= X_{k+1}(\rho_{k+1}-) + \triangle H(\rho_{k+1}) + F(\rho_{k+1},X_{k+1})\triangle Y(\rho_{k+1}) \\
&\overset{(1.80)}{=} H(\rho_{k+1}-) + \int_{(0,\rho_{k+1})} F(s,X_{k+1})dY(s) + \triangle H(\rho_{k+1}) + F(\rho_{k+1},X_{k+1})\triangle Y(\rho_{k+1}) \\
&= H(\rho_{k+1}) + \int_{(0,\rho_{k+1}]} F(s,X_{k+1})dY(s) \\
&= H^{\rho_{k+1}}(t) + \int_{(0,t]} F(s,X_{k+1})dY^{\rho_{k+1}}(s)
\end{aligned}$$

In the last equation we used the general fact $(G\cdot M)^\tau = G\cdot M^\tau$.By induction, the lemma is proved. □

A Gronwall's inequality.

We state the Gronwall's lemma that it is used in this chapter.

Lemma A.1 (Gronwall's inequality)**.** *Let g be an integrable Borel function on $[a,b]$,f a nondecreasing function on $[a,b]$, and assume that there exists a constant B such that*

$$g(t) \leq f(t) + B \int_a^t g(s)ds, \quad a \leq t \leq b.$$

Then

$$g(t) \leq f(t)e^{B(t-a)}, \quad a \leq t \leq b.$$