Lagrange Equations for Top with One Fixed Point

We can **analyze the motion of a spinning top** using the Lagrange equations for the Euler angles. Let us assume that the top has its lowest point (tip) fixed on a surface. We will **use the fixed point as the origin**. The rotation about the origin will be described by the Euler angles so that **all the kinetic energy is contained in the rotation**.

$$T = T_{rot}$$

For a symmetric top, we can immediately write things in terms of the **rotation about principal axes of inertia**.

$$T = \frac{1}{2}\sum_i I^{(i)}\omega_i^2$$

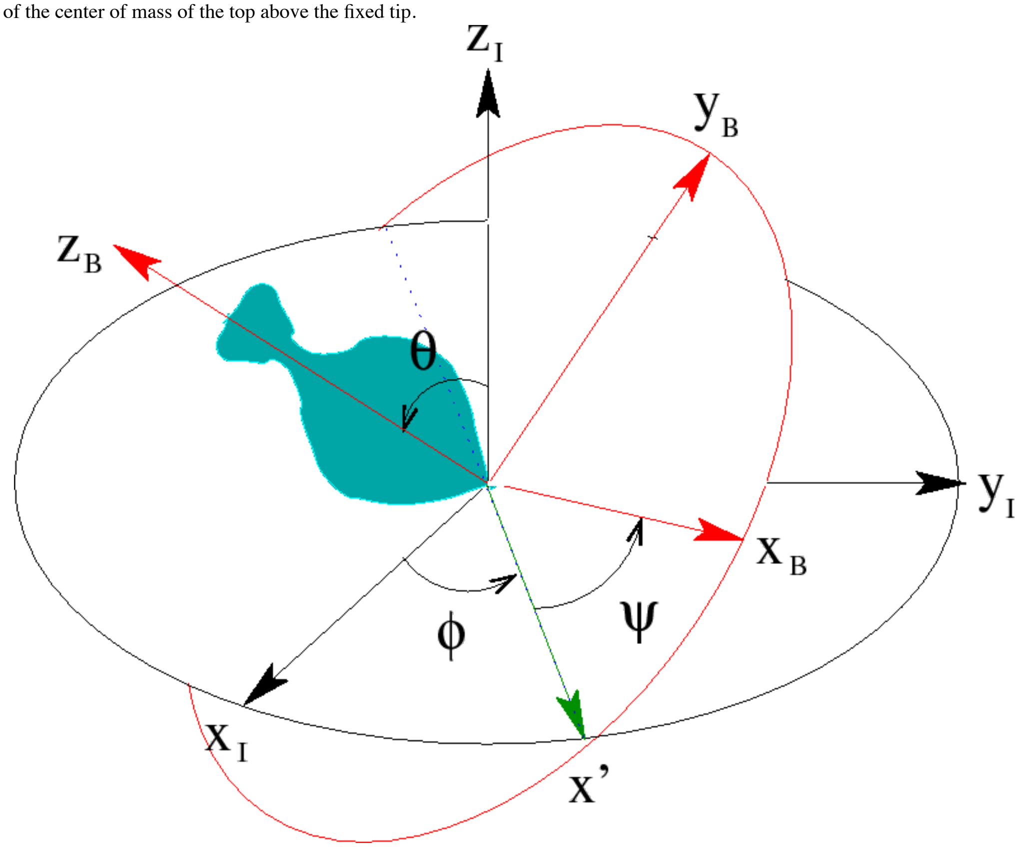
(Remember we have three principal moments of inertia but they don't make up a vector.) We have already written ω in terms of the Euler angles.

$$\vec{\omega} = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \dot{\phi}\cos\theta + \dot{\psi})$$

Recalling that θ is the angle between the Inertial z axis and the z axis in the rotating frame, the **potential energy can be written**,

$$U = mgh\cos\theta$$

where h is the height of the center of mass of the top above the fixed tip.



Note that we are assuming that the **symmetry axis of the top is the z axis** so that $I^{(2)} = I^{(1)} \equiv I^{(12)}$. We can now write the kinetic energy in terms of the Euler angles.

$$T = \frac{1}{2} I^{(12)} \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I^{(3)} \omega_3^2$$

$$\omega_1^2 = \left(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi\right)^2$$

$$\omega_1^2 = \dot\phi^2 \sin^2\theta \sin^2\psi + 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \cos^2\psi$$

$$\omega_2^2 = \left(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi\right)^2$$

$$\omega_2^2 = \dot\phi^2 \sin^2\theta \cos^2\psi - 2\dot\phi\dot\theta \sin\theta \sin\psi \cos\psi + \dot\theta^2 \sin^2\psi$$

$$\omega_1^2 + \omega_2^2 = \dot\phi^2 \sin^2\theta + \dot\theta^2$$

$$\omega_3^2 = \left(\dot\phi \cos\theta + \dot\psi\right)^2$$

$$T = \frac{1}{2} I^{(12)} \left(\dot\phi^2 \sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I^{(3)} \left(\dot\phi \cos\theta + \dot\psi\right)^2$$

$$U = mgh\cos\theta$$

$$L = \frac{1}{2} I^{(12)} \left(\dot\phi^2 \sin^2\theta + \dot\theta^2\right) + \frac{1}{2} I^{(3)} \left(\dot\phi \cos\theta + \dot\psi\right)^2 - mgh\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Since the Lagrangian does not depend on ϕ or ψ (cyclic), so the momenta are conserved.

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = \left(I^{(12)} \sin^2\theta + I^{(3)} \cos^2\theta\right) \dot\phi + I^{(3)} \cos\theta \dot\psi = const.$$

This is the angular momentum about the $\hat z_I$ axis.

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I^{(3)} \left(\dot\psi + \dot\phi \cos\theta\right) = const.$$

This is the angular momentum about the $\hat z_B$ axis. This is reasonable since one can see that the torque is along the line of nodes. The actual values of p_ϕ and p_ψ are set by **initial conditions** in the problem.

So p_ϕ and p_ψ are constants of the motion and we can solve the equations for $\dot\phi$ and $\dot\psi$.

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{I^{(12)}\sin^2\theta}$$

$$\dot\psi = \frac{p_\psi}{I^{(3)}} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I^{(12)}\sin^2\theta}$$

There is a **third** Lagrange equation but it will be easier to understand the motion of the top by using the **total energy equation**, along with the two conserved momenta.

$$E = \frac{1}{2}I^{(12)}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}I^{(3)}\left(\dot\phi\cos\theta + \dot\psi\right)^2 + mgh\cos\theta$$

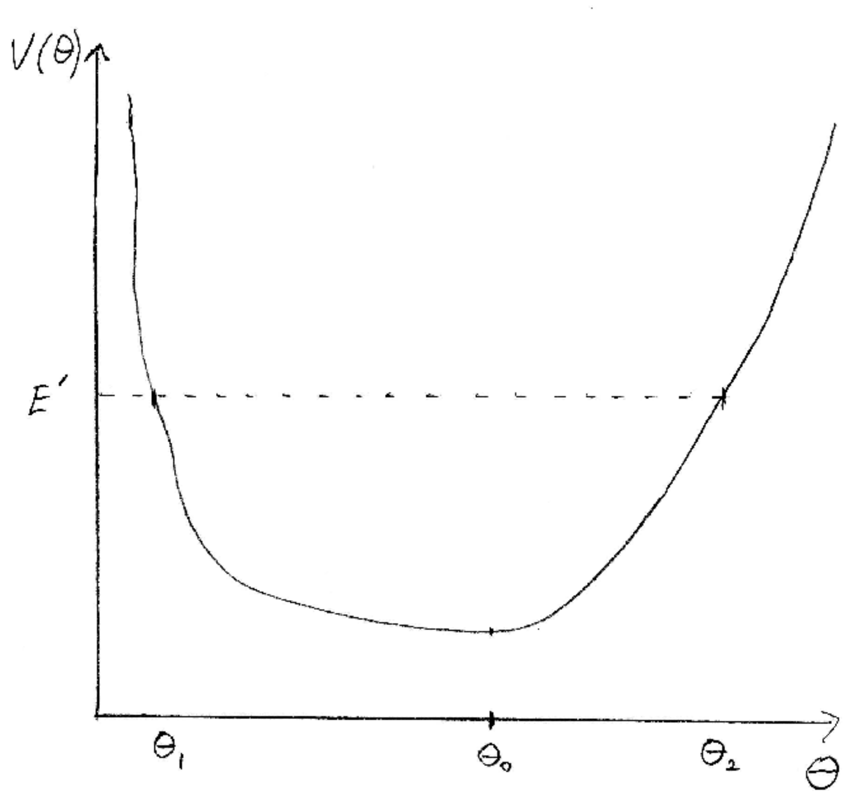
$$p_\psi = I^{(3)}\left(\dot\psi + \dot\phi\cos\theta\right)$$

$$E = \frac{1}{2}I^{(12)}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}\frac{p_\psi^2}{I^{(3)}} + mgh\cos\theta$$

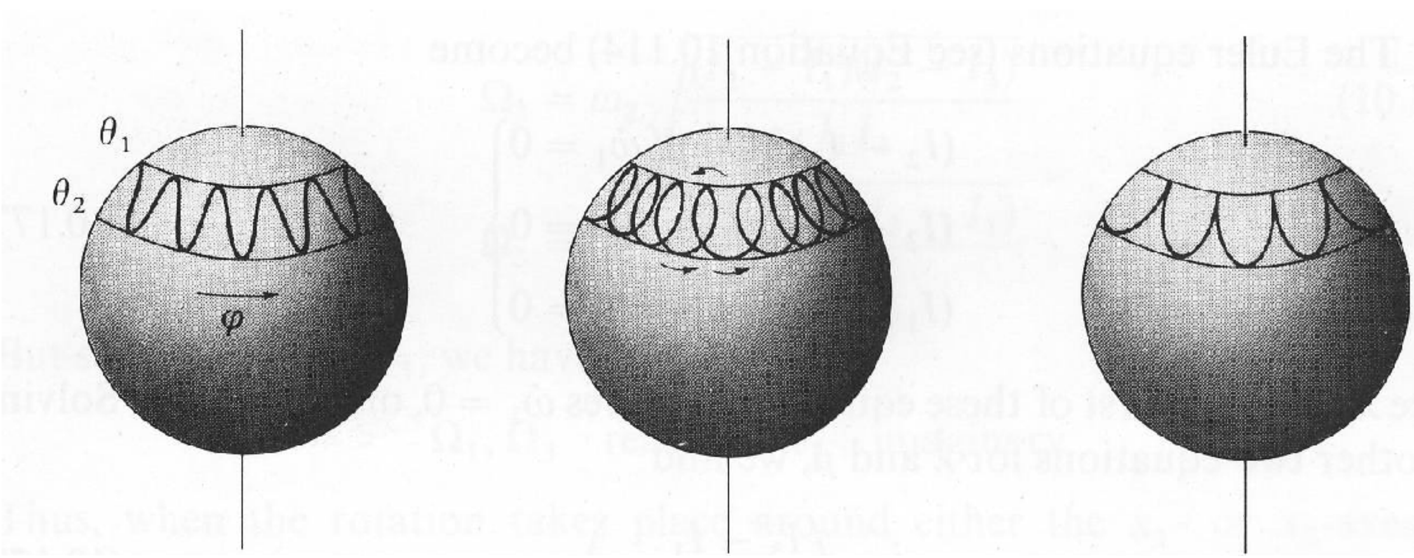
$$E - \frac{1}{2}\frac{p_\psi^2}{I^{(3)}} = \frac{1}{2}I^{(12)}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + mgh\cos\theta$$

$$E - \frac{1}{2}\frac{p_\psi^2}{I^{(3)}} = \frac{1}{2}I^{(12)}\dot\theta^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I^{(12)}\sin^2\theta} + mgh\cos\theta$$

$$E - \frac{1}{2}\frac{p_\psi^2}{I^{(3)}} = \frac{1}{2}I^{(12)}\dot\theta^2 + V(\theta) = const.$$



This is very much **like a central force problem** with the mass oscillating back and forth in the potential. $\dot{\theta}$ goes to zero at the limits. The motion will be limited between the some angles θ_1 and θ_2 at which $E - \frac{1}{2}\frac{p_\psi^2}{I^{(3)}} = V(\theta)$. This oscillation of θ as the angular momentum precesses is called **nutation**.



Next Up Previous Contents

Next: Homework **Up:** Dynamics of Rigid Bodies **Previous:** Stability of Rigid Body **Contents**
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