JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0894-0347(XX)0000-0

On Equation 16

1. Definitions

First, a few definitions.

T is the 6DOF of a rigid body, which lives in SE3. V is the velocity of the rigid body, which is also 6 dimensional, for linear and rotation velocity, and it lives in se3.

The Lagrangian is

$$L(T,V) = \frac{1}{2}V^T G V - P(T) \tag{1.1}$$

This is all discretized, of course, where T^k is the discrete version of position, and now the average velocity V^k can be defined as

$$V^k = \frac{1}{\triangle t}\log(\triangle T^k) \tag{1.2}$$

where $\triangle T^k = T^{-k}T^{k+1}$ is the displacement between discrete times.

2. Some Calculus

Equation 16 comes from equations 9, 10 and 14 in that paper. I am just going to repeat them here for consistency.

$$L_d(T^k, T^{k+1}) = \frac{\triangle t}{2}L(T^k, V^k) + \frac{\triangle t}{2}L(T^{k+1}, V^k) \tag{2.1}$$

The equation that really explains what is going on is the variation of V^k.

$$\delta V^k = \frac{1}{\triangle t} d\log_{\triangle t V^k}(-T^{-k})\delta T^k + \frac{1}{\triangle t} d\log_{\triangle t V^k} Ad_{exp(\triangle t[V^k])}(T^{-k-1})\delta T^{k+1} \tag{2.2}$$

You should think of the quantities before each variational term as though they were partial derivatives. Thus, you can think of the partial derivative of V^k in terms of T^k as the stuff before δT^k. More description here [2]

From 2.1, we get that

$$L_d(T^{k-1}, T^k) = \frac{\triangle t}{2}L(T^{k-1}, V^{k-1}) + \frac{\triangle t}{2}L(T^k, V^{k-1})$$

Expanding from 1.1, we simplify

$$L_d(T^{k-1}, T^k) = \frac{\triangle t}{2}(V^{k-1})^T G V^{k-1} - \frac{\triangle t}{2}P(T^{k-1}) - \frac{\triangle t}{2}P(T^k)$$

2020 *Mathematics Subject Classification.* Primary .

We can now compute 16b, and using the definition of directional derivative from [2] we get
(2.3)

$$D_2 L_d(T^{k-1}, T^k) = T^{k^*} L_d(T^{k-1}, T^k) = \frac{\triangle t}{2} T^{k^*}[(V^{k-1})^T G V^{k-1}] - \frac{\triangle t}{2} T^{k^*} P(T^{k-1}) - \frac{\triangle t}{2} T^{k^*} P(T^k)$$

Since T^{k-1} has no dependence on T^k, that derivative vanishes, leaving us with

$$D_2 L_d(T^{k-1}, T^k) = \frac{\triangle t}{2} T^{k^*}[(V^{k-1})^T G V^{k-1}] - \frac{\triangle t}{2} T^{k^*} P(T^k) \tag{2.4}$$

Now we can use the 2.2 to compute that first term, in addition to using equation 93 from [1],and using the fact the G is symmetric, we get

$$T^{k^*}[(V^{k-1})^T G V^{k-1}] = 2[T^{k^*} V^{k-1}]^T G V^{k-1} = \frac{2}{\triangle t}[T^{k^*}(d \log_{\triangle t V^k} Ad_{exp(\triangle t[V^k])}(T^{-k})]^T) G V^{k-1}$$

Now, using the fact that transpose reverses the order of linear operators (thats equation 5 from [1]) and the chain rule, we get

$$\begin{aligned} T^{k^*}[(V^{k-1})^T G V^{k-1}] &= \frac{2}{\triangle t}[(d \log_{\triangle t V^k} Ad_{exp(\triangle t[V^k])}(T^{-k} T^k)]^T) G V^{k-1} \\ &= \frac{2}{\triangle t}[Ad_{exp(\triangle t[V^k])}]^T [(d \log_{\triangle t V^k})]^T G V^{k-1} \end{aligned}$$

Plugging this back into 2.4, we get

$$D_2 L_d(T^{k-1}, T^k) = [Ad_{exp(\triangle t[V^k])}]^T [(d \log_{\triangle t V^k})]^T G V^{k-1} - \frac{\triangle t}{2} T^{k^*} P(T^k)$$

Which is equation 16b from the paper...up to a minus sign!! Which is what I got last time, and you were ok with that. Equation 16c is the same type mathematical gymnastics.

References

1. https://www.math.uwaterloo.ca/ hwolkowi/matrixcookbook.pdf
2. Frank W. Warner ,Foundations of Differentiable Manifolds and Lie Groups, Springer Verlag, new york NY 1983