
Functional Type Cheats for Manifolds

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Abstract

The modern coordinate-free approach to manifolds comes with non-classical notation and vocabulary. These can be barriers to understanding for people educated only in classical tensor calculus with indices. Short expressions in modern notation like $g\omega\nu$ entail multiple concepts like function application and linear transformation in abstract function spaces.

Classical tensor calculus assumes dimensionality and a coordinate system chosen at the beginning. One carries that baggage all the way, writing everything in terms of components, with lengthy expressions full of indices. The modern notation delays choice of coordinate-system baggage to a calculation phase, replacing coordinates with function composition at intermediate stages. Function composition works, without concrete indices, in any number of dimensions and on real or complex manifolds.

The abstract forms depend on *functional types*. Many classical results and new creative ideas follow easily just by checking the types. For example, co- and contravariance fall out naturally from pushforwards and pullbacks. Checking types is the same activity as in functional programming, say with Haskell or OCaml. Mere type checking might not convince a mathematician, but it can usually convince a compiler.

Learning the new notation and vocabulary takes as much effort as learning the classical notation and vocabulary, if not more, because the modern theory spans multiple levels of abstraction. As with sports, music, or human languages, the only pathway to fluency seems to be repetition and practice. Whether writing code, doing calculations, or doing proofs, one must eventually perceive and conceive the types effortlessly in expressions like $g\omega\nu$.

The zoo of types is big: vector and covector fields; curves, tangent and cotangent vectors; tangent and cotangent spaces, bundles, and sections; p-forms, wedge products, and exterior derivatives; Lie brackets, Lie derivatives, Lie groups, and Lie algebras; representations and adjoints; metrics, connections, and geodesics; curvature and torsion; Hodge duals; De Rham cohomologies; and more. Many of us learned all this in index-heavy notation, but now must do it more tersely and generally without indices.

Background

These are the types that have been most useful to me. They're loosely compatible with the notation in Baez and Muniain, *Gauge Theory, Knots, and Gravitation* (GKG). The exposition here roughly follows the order of that book.

Prerequisites

I presume here only that you know linear algebra, axioms of group theory, and vector and tensor calculus in classical, index-heavy notation.

A, B, \dots, X, Y, \dots refer to sets of elements a, b, \dots, x, y, \dots , respectively.

M, N, \dots refer to **manifolds**, sets of abstract, featureless points p, q, \dots

One says that the point $p \in M$ is a member or **element** of the set M , defining \in . One also says p is **in** M , p **inhabits** M , p **is an** M , p **has type** M , $p : M$, all meaning the same thing.

The colon notation $p : M$ is a **type signature**. The colon binds loosely, so $f : C^\infty M \subset (M \rightarrow \mathbb{R})$ means $f : (C^\infty M \subset (M \rightarrow \mathbb{R}))$ and, *en passant*, refers to a subset relation between the two types $C^\infty M$ and $M \rightarrow \mathbb{R}$. Likewise, $\phi^* f = f \circ \phi : C^\infty M$ means $(\phi^* f = f \circ \phi) : C^\infty M$ and, *en passant*, refers to the equality of the two expressions $\phi^* f$ and $f \circ \phi$. Sometimes, type notations appear in a lighter ink to set them off from other text.

For our purposes, there is no difference between being **of a type** and being **in a set**. If $p : U$ and $U \subset M$, then $p : M$ by definition of *subset*.

The functional type signature $f : A \rightarrow B$ declares that function f takes elements of set A and produces elements of set B . One says that f **maps** A to B , (*maps* as a verb), f is an A -to- B , f **has type** $A \rightarrow B$, f **inhabits** the type or set $A \rightarrow B$, f **is an** $A \rightarrow B$, f **sends** $a \in A$ to $b \in B$, etc. **Function**, **mapping**, and **map** (as a noun) usually mean the same thing.

The notation $a \mapsto b$ alone is an **anonymous function**, meaning “a function of a that produces b .” Typically, b is an expression that depends on a . Give an anonymous function a name as follows: $f : A \rightarrow B = a \mapsto b, a \in A, b \in B$. One also writes $f(\square) : A \rightarrow B$ to mean that f is an expression with a **hole**, \square , waiting for an element of set A . Physicists will write $f(a) : A \rightarrow B$ but we will always write $f(a) : B$ or $fa : B$, as in Haskell, to emphasize that f has already received its argument of type A and the resulting **function application** produces a value of type B . The Haskell style of function application is indistinguishable from multiplication without knowledge of the types, but that's OK because function application is the same as multiplication in the case of linear functions. Function application **nests to the left**: $fab = (fa)b$.

We'll write the types of functions of many parameters (**multiadic** or **multiary** functions) as in Haskell: $f(x, y) : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} = \mathbb{R} \rightarrow (\mathbb{R} \rightarrow \mathbb{R})$, **nesting to the right**. One might just as well write $f(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ because it's isomorphic, but we prefer not to do so.

Abstraction means multiple things, often changing a type signature.

1. generalizing some realm of thought, leaving out details
2. pulling data out of an expression, making a function and parameters, e.g.,

$$x + y \xRightarrow{\text{abst}} f(x, y) : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} = x + y$$
3. taking away parts of a definition, e.g., abstracting $p : M$ from $v_p(f) : \mathbb{R} = (vf)(p)$ yields a new definition $vf : M \rightarrow \mathbb{R}$.

These three meanings are related but different.

This cheat sheet talks mostly about real manifolds. We can do likewise for complex manifolds. Just substitute \mathbb{C} for \mathbb{R} .

Commutative Diagrams (closed up)

Concise Cheats

- **Smooth Functions** $f : C^\infty M \subset (M \rightarrow \mathbb{R})$
- **Vector Field** $v : C^\infty M \rightarrow C^\infty M$
- **Set of all Vector Fields on M** $VM = \{v : C^\infty M \rightarrow C^\infty M\}$
- **Basis for Vector Fields** ∂_μ
- **Tangent Vector** $v_p : C^\infty M \rightarrow \mathbb{R}$
- **Tangent Space** $T_p M = \{v_p : C^\infty M \rightarrow \mathbb{R} \mid p \in M\}$
- **Curve** $\gamma : \mathbb{R} \rightarrow M$
- **Tangent to a Curve** $\gamma'(t) : C^\infty M \rightarrow \mathbb{R}$
- **Pullback of a Function** $\phi^* : C^\infty N \rightarrow C^\infty M$
- **Pushforward of a Tangent Vector** $\phi_* : T_p M \rightarrow T_{\phi(p)} N$
- **Lie Bracket** $[v, w] = v \circ w - w \circ v : C^\infty M \rightarrow C^\infty M$
- **Exterior Derivative** $df = v \mapsto vf : VM \rightarrow C^\infty M$
- **Basis for Differential Forms** dx^μ
- **1-Form** $\omega : VM \rightarrow C^\infty M$
- **Set of all 1-Forms on M** $\Omega^1 M$

- **Cotangent Vector** $\omega_p : T_p M \rightarrow \mathbb{R}$
- **Cotangent Space** $T_p^* M = \{\omega_p : T_p M \rightarrow \mathbb{R} \mid p \in M\}$
- **Pullback of a Cotangent Vector** $\phi^* : T_{\phi(p)}^* N \rightarrow T_p^* M$
- **Pushforward of a Vector Field** $\phi_* : VM \rightarrow VN$
- **Pullback of 1-Form** $\Omega^1 N \rightarrow \Omega^1 M$
- **Pullback of a Differential** $(\phi^* df) : \Omega^1 M$

Explanations and Demonstrations

Smooth Functions $f : C^\infty M \subset (M \rightarrow \mathbb{R})$

At the front gate of the Manifold Zoo is the set $C^\infty M$ of infinitely differentiable C^∞ (smooth) functions from the manifold M to the reals \mathbb{R} . That's a subset of all functions from M to \mathbb{R} .

$C^\infty M$ is an indivisible symbol denoting the set of functions. It's not multiplication of C^∞ times M or a function application of C^∞ to M .

After defining pullbacks of functions, we can characterize smoothness precisely. Until then, take the term **smooth** intuitively.

See later that *curves* are certain functions going the other way, back from $\mathbb{R} \rightarrow M$.

What could it possibly mean to differentiate functions on M , a set of abstract, featureless points? One needs only a little topology.

Chart $\varphi : (U \subset M) \rightarrow \mathbb{R}^n$

A **chart** φ is a continuous function with a continuous inverse from an **open subset** U of a manifold M to the n -dimensional Euclidean space of reals, \mathbb{R}^n , the mother of all manifolds. We take continuity on \mathbb{R}^n as understood. Continuity on $U \subset M$ takes more topology than we want to cover here, so take it intuitively.

Think of a chart of a globe model of the World on flat paper, say in the Mercator projection. Go back and forth from numerical coordinates in \mathbb{R}^n on the paper to points in $U \subset M$ on the globe.

A chart implies a **coordinate system** on the destination set \mathbb{R}^n . Find coordinates of any point or components of any vector in the coordinate system. Intuit the **open subset** U as a “ball” of points in M excluding the boundary of the “ball.” This intuition is perfect in the manifold \mathbb{R}^n , but is inadequate generally. The intuition is good enough for practical applications where one does actual work in \mathbb{R}^n as much as

possible.

$$(f \circ \varphi^{-1}) : C^\infty \mathbb{R}^n \subset (\mathbb{R}^n \rightarrow \mathbb{R})$$

(This gadget has no name of its own)

Point-Free Notation

Read $(f \circ \varphi^{-1})$ from right-to-left. The **function composition** $(f \circ \varphi^{-1})$ is a new function. Apply it to arguments on its right, as in $(f \circ \varphi^{-1})(x)$, sending $x \in \mathbb{R}^n$ first to φ^{-1} , getting a point $p \in U \subset M$, then p to f , like this, $(f(\varphi^{-1}(x)))$, to get a real number. That's harder to read and write than $(f \circ \varphi^{-1})(x)$ or $(f \circ \varphi^{-1})x$, so we go with the latter. Take off the x to get **point-free notation** $(f \circ \varphi^{-1}) : \mathbb{R}^n \rightarrow \mathbb{R}$, excluding the “point” x .

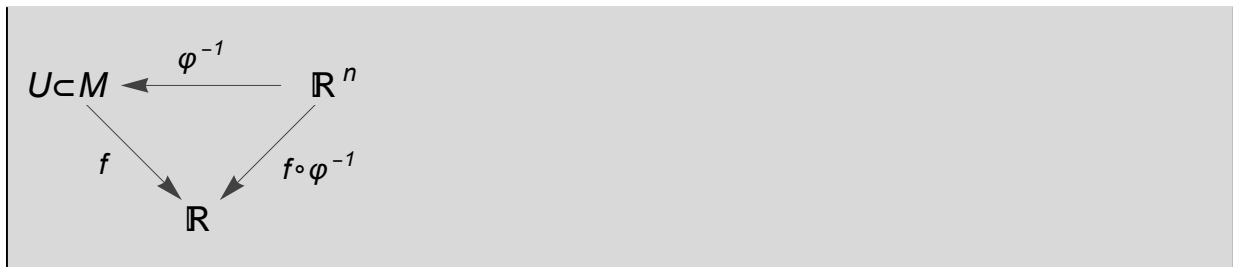
Function composition is associative (https://en.wikipedia.org/wiki/Function_composition), i.e., $(f \circ (g \circ h)) = ((f \circ g) \circ h)$, so the parentheses are not needed: $(f \circ (g \circ h)) = ((f \circ g) \circ h) = f \circ g \circ h$.

Any chart φ takes a point $p \in U \subset M$ and returns an \mathbb{R}^n . The inverse of the chart φ^{-1} goes the other way, from \mathbb{R}^n to $p \in U \subset M$ (charts must be invertible). f picks up that p right away and returns an \mathbb{R} . So the whole composition $(f \circ \varphi^{-1})$ goes straight from \mathbb{R}^n to \mathbb{R} with $U \subset M$ the forgotten middle of the sandwich.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions. The parameters or arguments of the functions go without names in such diagrams, and that's sometimes a bit of a weakness when relating such diagrams to notations.

Out[22]=



Calculus

Suddenly, $C^\infty M$ means something. Do ordinary linear algebra in \mathbb{R}^n . Calculate derivatives by ordinary multivariable calculus in \mathbb{R}^n through $(f \circ \varphi^{-1}) : C^\infty \mathbb{R}^n$.

\mathbb{R}^n is a manifold, so $C^\infty \mathbb{R}^n$ makes sense: the set of functions in $\mathbb{R}^n \rightarrow \mathbb{R}$ with an infinite number of

continuous derivatives. Such functions are plentiful.

Under practical circumstances, **any ol' chart will do** because all charts map to the same \mathbb{R}^n with the same dimension n . One assigns this same dimension n to U and M . One can reason about φ without mentioning n more than once.

f is in $C^\infty M$ iff $(f \circ \varphi^{-1}) : C^\infty \mathbb{R}^n$ for all charts.

(1)

Arithmetic Rules for Functions in $C^\infty M$

Let double-colon separate the name of a rule, like $C^\infty +$, from the definition of the rule.

Let f and g be in $C^\infty M \subset (M \rightarrow \mathbb{R})$.

$C^\infty + :: f + g : C^\infty M$

(2)

$f + g$ is a new function in $C^\infty M$ that produces $f(p) + g(p) \in \mathbb{R}$, the sum of two real numbers $f(p)$ and $g(p)$, for any $p \in U \subset M$.

$C^\infty * :: f g : C^\infty M$

(3)

$fg = f \times g$ (f times g) is a new function in $C^\infty M$ that produces $f(p)g(p) = f(p) \times g(p) \in \mathbb{R}$ for any $p \in U \subset M$. $f(p)g(p)$ denotes the product of two real numbers $f(p)$ and $g(p)$.

fg does not mean $f(g)$, f applied to g , even though vf does mean $v(f)$, v applied to f . **Juxtaposition is ambiguous without the types!**

$\mathbb{R} * C^\infty :: \alpha f : C^\infty M$

(4)

αf , where $\alpha \in \mathbb{R}$, is a new function in $C^\infty M$ that produces $\alpha f(p) \in \mathbb{R}$ by multiplication of real numbers α and $f(p)$.

Of all these rules, one says that functions producing values in \mathbb{R} **inherit** ordinary, real-number arithmetic from \mathbb{R} .

Vector Field $v : C^\infty M \rightarrow C^\infty M$

A **vector field** v is a linear function that sends a function $f \in C^\infty M$ to another function $vf \in C^\infty M$. A vector field adheres to the Laws below.

It's weird to think of vector fields as functions from functions to functions, but it's just an abstraction of directional derivative (DD), which we already know about from vector calculus. v abstracts out the chart, necessary for DD, because everything works on any ol' chart.

Assume there is some chart φ so that $\frac{\partial f}{\partial x}$ really means $\frac{\partial(f \circ \varphi^{-1})}{\partial x}$, remembering that $(f \circ \varphi^{-1}) \in C^\infty \mathbb{R}^n \subset (\mathbb{R}^n \rightarrow \mathbb{R})$. Take partial derivatives with respect to coordinates $\{x^1, x^2, \dots, x^n\} \in \mathbb{R}^n$. Indices are upstairs. Why? Later!

Define the notation vf to mean the vector field function v applied to the argument $(f \circ \varphi^{-1}) : C^\infty \mathbb{R}^n$ through any ol' chart φ .

vg does not mean $v \times g$, v times g , even though gf does mean $g \times f$, g times f . **Juxtaposition is ambiguous without the types!**

One might write $v(f)$ instead, but that's noisier. vf looks like multiplication. When v is linear, and it is, vf is equal to multiplication. Applying linear functions acts just like multiplying matrices.

In compounded function-applications notations like vp , nest to the left, as $((vf)p) = ((v(f))(p))$. If functions are carefully designed, then function application, like function composition, can be associative, so the order of bracketing does not matter. For example, $\sin(\text{sqrt}(x)) = (\sin(\text{sqrt}))(x)$ if $\sin(\text{sqrt})$, the sin of the function sqrt is properly defined.

Find components $\{v^1, v^2, \dots, v^n\}$ of a classical, traditional vector, indices upstairs. Write

$$vf = v^1 \frac{\partial f}{\partial x^1} + v^2 \frac{\partial f}{\partial x^2} + \dots + v^n \frac{\partial f}{\partial x^n} \stackrel{\text{trad}}{=} \nabla f \cdot v \quad (5)$$

with the traditional “grad f dot v ” on the extreme right. Now shorten this notation.

$$vf = (v^1 \partial_1 f + v^2 \partial_2 f + \dots + v^n \partial_n f) = \left(\sum_{i=1}^n v^i \partial_i f \right) \stackrel{\text{def}}{=} (v^i \partial_i f) = (v^i \partial_i) f \quad (6)$$

We've done three things:

- replaced $\frac{\partial}{\partial x^i}$ with ∂_i (indices *downstairs*)
- automatically summed over the repeated index i , one up, the other down—**always automatically sum over repeated up-down indices**
- parenthesized $(v^i \partial_i)$ in the expression $(v^i \partial_i) f$ to emphasize that we need not mention f

Let $v \in C^\infty M \rightarrow C^\infty M$ be a vector field. Give to v an $f : C^\infty M$ to get $vf : C^\infty M$. Why is vf of type $C^\infty M \subset (M \rightarrow \mathbb{R})$? Give vf a point $p : U \subset M$ to get an \mathbb{R} .
 vf is a point-free notation.

Abstract off the f and *define* the vector field as $v = v^i \partial_i$. **A vector field is a linear combination of partial derivative operators through any ol' chart.** It turns out that when v is applied to $(f \circ \varphi^{-1})$ on the same chart φ to get $vf : C^\infty M$, and then applied to any ol' point $p : U \subset M$ to get $vp : \mathbb{R}$, the result does not depend on the chart φ , so don't mention it any more, just write $vf : C^\infty M$. But we need not mention f either.

We get a two-level point-free notation, just $v : C^\infty M \rightarrow C^\infty M$.

Why is v called a vector *field*? Classically, a *vector field* is a tuple of components defined at every point in, say, \mathbb{R}^3 . vf is defined at every point $p : U \subset M$, so v really is a field of objects defined at every point $p : U \subset M$, an object for calculating directional derivatives at p .

Don't confuse *field*, here, with **algebraic field**, which generalizes ordinary arithmetic via associative, commutative, and distributive laws.

Shorthand

Write $v_p f$ instead of $vf p$, which is, itself, shorthand for $(v(f))(p)$ and $(vf)(p)$.

Exercise

In another notebook, <https://community.wolfram.com/groups/-/m/t/3098298>, vectors are defined as equivalence classes of curves, there proved to satisfy all eight axioms of a **vector space**. Prove likewise for $v = v^i \partial_i$.

Laws for Vector Fields

Vector fields in $C^\infty M \rightarrow C^\infty M$ must satisfy linearity laws and a Leibniz product rule so that vector fields act like derivative operators.

With $f : C^\infty M$ and $g : C^\infty M$:

Distributivity of function application:

$$\text{LINV+} :: v(f + g) = vf + vg, \text{ via rule } C^\infty +, \text{ both sides in } C^\infty M \quad (7)$$

Letting α be a real number:

$$\text{LINV*} :: v(\alpha g) = \alpha(vg), \text{ via rule } \mathbb{R} * C^\infty, \text{ both sides in } C^\infty M \quad (8)$$

Leibniz's Law, aka product rule of differentiation

$$\text{LEIBV} :: v(fg) = g(vf) + f(vg), \text{ via rule } C^\infty *, \text{ both sides in } C^\infty M \quad (9)$$

Arithmetic on Vector Fields

Let $v \in C^\infty M \rightarrow C^\infty M$ and $w \in C^\infty M \rightarrow C^\infty M$ be two vector fields, and let f and g each be in $C^\infty M$. $(v + w)$ is a new vector field. Apply it to a function $f : C^\infty M$ and invoke $C^\infty +$.

$$\mathbf{VF}+:: (v + w) f = v f + w f, \text{ right-hand side by } \mathbf{C}^\infty+ \quad (10)$$

The next one defines gv , multiplication by function $g : \mathbf{C}^\infty M$ on the left by a vector field $v : \mathbf{C}^\infty M \rightarrow \mathbf{C}^\infty M$ on the right.

$$\mathbf{C}^\infty \mathbf{VF} *:: (gv) f = g(vf), \text{ right-hand side by } \mathbf{C}^\infty * \quad (11)$$

The rules above make the space of vector fields very much like a vector space. Technically, it's a **module** over $\mathbf{C}^\infty M$ because multiplication need not be commutative and inverses need not exist (<https://math.stackexchange.com/questions/4142071>).

Set of all Vector Fields on M $VM = \{v : \mathbf{C}^\infty M \rightarrow \mathbf{C}^\infty M\}$

VM , an indivisible symbol, is a type: the set of all vector fields on M .

$$VM = \{v : \mathbf{C}^\infty M \rightarrow \mathbf{C}^\infty M\} \quad (12)$$

Sets are types in this document, so we might just as well write $VM = \mathbf{C}^\infty M \rightarrow \mathbf{C}^\infty M$.

Basis for Vector Fields ∂_μ

Given components v^μ in some chart, the expression $v = v^\mu \partial_\mu$ suggests that ∂_μ be a basis in VM . Indices μ here are not concrete indices. They do not presume a particular coordinate system because any ol' chart will do. But all charts have in common the number of dimensions. The indices μ are **abstract indices** that range over the ordered dimensions of the manifold M . Once a coordinate system is chosen, the abstract indices identify with the concrete indices of the coordinate system. For example, in spherical coordinates in 3D, we have coordinates ρ , θ , and ϕ , with corresponding basis vectors ∂_ρ , ∂_θ , and ∂_ϕ defined at every point, so μ ranges over ρ , θ , ϕ . In Cartesian coordinates, we have coordinates x , y , and z , with corresponding basis vectors ∂_x , ∂_y , and ∂_z , and μ ranges over x , y , z . We abstract out the particular coordinate by saying that in any coordinate system, we will have coordinates x^1 , x^2 , x^3 and basis vectors ∂_{x^1} , ∂_{x^2} , and ∂_{x^3} or just ∂_1 , ∂_2 , ∂_3 for short. Now we can see the notation can get messy when we go to any

Tangent Vector $v_p : \mathbf{C}^\infty M \rightarrow \mathbb{R}$

Apply a vector field $v : \mathbf{C}^\infty M \rightarrow \mathbf{C}^\infty M$ to a function $f : \mathbf{C}^\infty M$ to get a new function $vf : \mathbf{C}^\infty M \subset M \rightarrow \mathbb{R}$ with a **hole**, \square , waiting for a point $p : U \subset M$.

v , alone, has two holes, one for a function in $\mathbf{C}^\infty M$ and one for a point in $U \subset M$. Via the shorthand $v_p f$,

$$v = v_{\square} : C^{\infty} M \rightarrow C^{\infty} M$$

The holes are invisible. No one writes them with \square like that, but this is a cheat sheet, to get us comfortable with invisible things.

Give v an f in $C^{\infty} M$ as before and get an $C^{\infty} M \subset M \rightarrow \mathbb{R}$.

$$vf : C^{\infty} M \subset M \rightarrow \mathbb{R}$$

with an invisible hole waiting for a point $p : U \subset M$.

Start over with just v . Give v the point $p : U \subset M$, *first*. Get a new thing, v_p , with an invisible hole that waits for a $f : C^{\infty} M$.

$$v_p : C^{\infty} M \rightarrow \mathbb{R} \text{ such that } v_p f : \mathbb{R} \stackrel{\text{trad}}{=} \nabla f(p) \cdot v \quad (13)$$

Interpret the right-hand side traditionally. Assume any ol' chart. $\nabla f(p)$ is the gradient of f evaluated at p , and v is a tuple of components.

v_p is a **tangent vector**. Given any ol' chart, the tangent vector lives in the **tangent plane**, a hyperplane of type \mathbb{R}^n , at point p , where one does ordinary calculus.

Arithmetic and Laws for Tangent Vectors

Tangent vectors inherit arithmetic laws from vector fields, but the types change. In fact, *only* via type analysis can we develop **LEIBTV**. Type analysis reveals that one must apply the open functions f and g to p .

$$\mathbf{LINTV}+ :: v_p (f + g) = v_p f + v_p g, \text{ both sides in } \mathbb{R} \quad (14)$$

$$\mathbf{LINTV}* :: v_p (\alpha g) = \alpha (v_p g), \text{ both sides in } \mathbb{R} \quad (15)$$

$$\mathbf{LEIBTV} :: v_p (fg) = g(p) (v_p f) + f(p) (v_p g), \text{ both sides in } \mathbb{R} \quad (16)$$

$$\mathbf{TV}+ :: (v_p + w_p) f = v_p f + w_p f, \text{ both sides in } \mathbb{R} \quad (17)$$

$$\mathbf{RTV}* :: \alpha v_p : C^{\infty} M \text{ such that } (\alpha v_p) f = \alpha (v_p f) \stackrel{\text{def}}{=} \alpha v_p f, \text{ both sides in } \mathbb{R} \quad (18)$$

for $\alpha \in \mathbb{R}$. **RTV*** defines the precedence of $v_p f : \mathbb{R}$. Function application $v_p f$ binds more tightly than multiplication by α . As before, with proper understanding of types, this observation hardly matters.

$$\textbf{Tangent Space } T_p M = \{v_p : C^{\infty} M \rightarrow \mathbb{R} \mid p \in U \subset M\}$$

$T_p M$ is a type, the set of *tangent vectors at point $p : U \subset M$* . Tangent vectors send C^∞ functions to their directional derivatives in \mathbb{R} .

$$T_p M = \{v_p : C^\infty M \rightarrow \mathbb{R} \mid p \in U \subset M\} \quad (19)$$

$T_p M$ is an indivisible symbol denoting the set. The notation $T_p M$ is not a product or a function application.

From rules **TV+** and **RTV***, deduce that $T_p M$ is a vector space, i.e., satisfies all eight axioms of a vector space.

Curve $\gamma : \mathbb{R} \rightarrow M$

The type of a curve, $\mathbb{R} \rightarrow M$, is the reverse of the type $C^\infty M \subset (M \rightarrow \mathbb{R})$. Give γ a $t \in \mathbb{R}$, get a point in $U \subset M$, but not just any ol' point. Demand that γ be **smooth**: nearby values of input yield nearby values of output. To define *smooth* properly below, we need first a tangent vector to γ based at the point that γ returns.

Tangent to a Curve $\gamma'(t) : C^\infty M \rightarrow \mathbb{R}$

Let $\gamma(t)$ be a point in $U \subset M$. In ordinary calculus, one learns that $\gamma'(t)$ is a real number: the slope of the tangent to the curve. Picture the curve as a graph on graph paper from \mathbb{R} to \mathbb{R} . Picture the tangent as a line kissing the curve. Measure the slope of that line.

Generalize by defining the derivative $\gamma'(t)$ of a curve γ at $t \in \mathbb{R}$ to be the tangent vector of type $C^\infty M \rightarrow \mathbb{R}$ that produces the directional derivative, a real number, of any $f : C^\infty M$, but in the instantaneous direction of the tangent to $\gamma(t)$. Picture $\gamma'(t)$ an arrow in the tangent plane.

$$\gamma'(t)(f) \stackrel{\text{trad}}{=} \frac{d}{dt} f(\gamma(t)) \quad (20)$$

Check Types

$$\gamma'(t)(f) \stackrel{\text{trad}}{=} \frac{d}{dt} (f : C^\infty M \subset (M \rightarrow \mathbb{R})) (\gamma(t) : \mathbb{R} : M) : \mathbb{R}$$

“trad” means “traditional” and implies that one can only compute a numerical derivative given a chart, but any ol' chart will do. It turns out that the directional derivative has the same value in every chart / coordinate system.

Shorthand

Take away the f and write something that looks like traditional notation, but isn't. It's a trick of notation, a *trompe-l'oeil*!

$$\frac{d\gamma}{dt}(t) \stackrel{\text{def}}{=} \gamma'(t) : C^\infty M \rightarrow \mathbb{R} \quad (21)$$

Take away the t , leaving a hole \square , which becomes invisible.

$$\frac{d\gamma}{dt} \stackrel{\text{def}}{=} \gamma'(\square) \stackrel{\text{def}}{=} \gamma' : \mathbb{R} \rightarrow C^\infty M \rightarrow \mathbb{R} \quad (22)$$

Pullback of a Function $\phi^* : C^\infty N \rightarrow C^\infty M$

Let ϕ be a function from M to N , two manifolds. ϕ need not be a chart. Charts are usually denoted via curly phi, ϕ . From this point on, for brevity, we won't continue to emphasize that we're always working in open subsets of the manifolds: $p \in M$ means $p \in U \subset M$.

Let $f : C^\infty N$ be a C^∞ function on N . We'd like a similar function $\phi^* f : C^\infty M$ on M . How? First, send a point $m \in M$ to $n \in N$ via ϕ , then from N to \mathbb{R} via f . Write an ordinary function composition $f \circ \phi : C^\infty M \subset (M \rightarrow \mathbb{R} = (N \rightarrow \mathbb{R}) \circ (M \rightarrow N))$, remembering to read composition via \circ from right to left, and *en passant*, defining the **composition of types** $N \rightarrow \mathbb{R}$ and $M \rightarrow N$ as $(N \rightarrow \mathbb{R}) \circ (M \rightarrow N) = M \rightarrow \mathbb{R}$.

Define the notation $\phi^* f = f \circ \phi$, the **pullback of the function** f , with star upstairs. Pushforward, introduced later, has star downstairs.

$$\phi^* f \stackrel{\text{def}}{=} f \circ \phi : C^\infty M \quad (23)$$

Take out f , leaving a hole \square for a function in $C^\infty N$.

$$\phi^* \stackrel{\text{def}}{=} \square \circ \phi : C^\infty N \rightarrow C^\infty M \quad (24)$$

Take out the ϕ , leaving two holes. Define the **pullback infix operator** $*$.

$$* \stackrel{\text{def}}{=} \square_{\text{left}} * \square_{\text{right}} = \square_{\text{right}} \circ \square_{\text{left}} : ((\square_{\text{left}} : M \rightarrow N) \rightarrow (\square_{\text{right}} : C^\infty N \rightarrow C^\infty M)) \quad (25)$$

Less noisily:

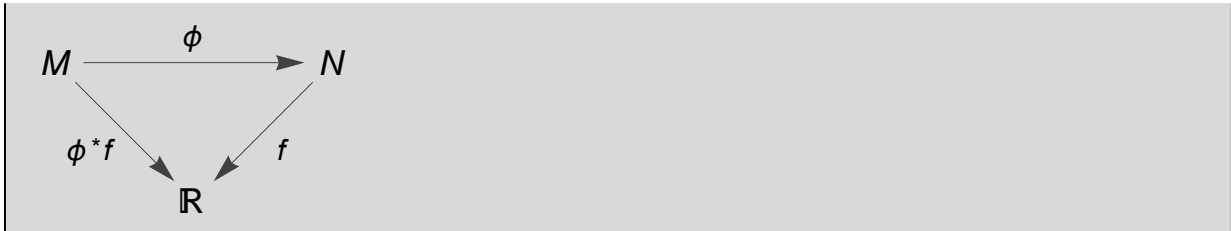
$$* : (M \rightarrow N)_{\text{left}} \rightarrow C^\infty N_{\text{right}} \rightarrow C^\infty M \quad (26)$$

Diagram

Remember we're always working in open subsets in the manifolds M and N .

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[23]=



Contravariance

Functions in $C^\infty N$ are called **contravariant** because they are pulled back from $N \rightarrow \mathbb{R}$ to $M \rightarrow \mathbb{R}$, upstream against the grains of $f : C^\infty N$ and $\phi : M \rightarrow N$.

Example (Exercise 14 in GKG)

Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be counterclockwise rotation by a constant angle θ , \mathbb{R} and \mathbb{R}^2 being the two manifolds M and N .

In[24]:=

```
ClearAll[phi, x, y, xf, yf, pullback];
phi[{x_, y_}] := RotationMatrix[theta].{x, y};
phi[{x, y}]
```

Out[26]=

```
{x Cos[theta] - y Sin[theta], y Cos[theta] + x Sin[theta]}
```

Let $x_f : (M = \mathbb{R}^2) \rightarrow \mathbb{R}$ be a function that returns the first of a tuple of components in \mathbb{R}^2 , and let y_f return the second. The subscript f serves only to remind that x_f and y_f are functions; f has no deeper meaning here.

In[27]:=

```
xf[tuple_List] := tuple[[1]]; yf[tuple_List] := tuple[[2]];
```

Mathematica's composition operator is `@*`. Let's pull back both x_f and y_f .

In[28]:=

```
pullback[phi_, f_] := f@*phi;
pullback[phi, xf][{x, y}]
pullback[phi, yf][{x, y}]
```

Out[29]=

```
x Cos[theta] - y Sin[theta]
```

Out[30]=

```
y Cos[theta] + x Sin[theta]
```

Mathematica's Notation package serves us well, here.

```
In[31]:= << Notation`

In[32]:= Notation[  $\phi_* f \Leftrightarrow \text{pullback}[\phi_*, f]$  ]

In[33]:= ( $\phi_* x f$ ) [{x, y}]
( $\phi_* y f$ ) [{x, y}]

Out[33]= x Cos[ $\theta$ ] - y Sin[ $\theta$ ]

Out[34]= y Cos[ $\theta$ ] + x Sin[ $\theta$ ]
```

Smooth

A function g in $C^\infty \mathbb{R}^n = \mathbb{R}^n \rightarrow \mathbb{R}$ is **smooth** because it has continuous derivatives in real-number spaces, an infinite number of derivatives for our purposes (<https://mathworld.wolfram.com/SmoothFunction.html>). Lower orders of smoothness are appropriate in other applications.

How do we get from smooth in real-number spaces to smooth on manifolds?

A **function** $f : M \rightarrow \mathbb{R}$ is **smooth** if, for all charts $\phi_\alpha : U \subset M \rightarrow \mathbb{R}^n$, $(f \circ \phi_\alpha^{-1}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth.

A **curve** $\gamma : \mathbb{R} \rightarrow M$ is **smooth** if, for all functions $f : M \rightarrow \mathbb{R}$, $f \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, i.e., in $C^\infty \mathbb{R}$.

A **general function** $\phi : M \rightarrow N$ from one manifold to another is **smooth** if f 's being in $C^\infty N$ implies that the pullback $\phi^* f$ is in $C^\infty M$ (ϕ is not necessarily a chart, ϕ).

Pushforward of a Tangent Vector $\phi_* : T_p M \rightarrow T_{\phi(p)} N$

Let $\phi : M \rightarrow N$, not necessarily a chart, ϕ . Recall the pullback for functions $\phi^* : C^\infty N \rightarrow C^\infty M$.

Define the **pushforward** $(\phi_* v_p) : T_{\phi(p)} N$ of the **tangent vector** $v_p : T_p M$ by its action on the pullback of a function:

$$(\phi_* v_p) f = v_p(\phi^* f) \text{ for any } f : C^*(N) \quad (27)$$

One might write $(\phi_* v_p) = (\phi_* v)_{\phi(p)}$ via sneak preview to pushforward of vector fields.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[35]=



Check Types

The right-hand side of Equation 27, $v_p(\phi^* f)$, is a real number, a directional derivative in M .

$$v_p(\phi^* f) = [(v_p : C^\infty M \rightarrow \mathbb{R}) \{ [\phi^* : C^\infty N \rightarrow C^\infty M] [f : C^\infty N] : C^\infty M \}] : \mathbb{R} \quad (28)$$

The left-hand side, $(\phi_* v_p) f$, must also be the same real number, this time, the directional derivative in N .

$$(\phi_* v_p) f = [\{ [\phi_* : (C^\infty M \rightarrow \mathbb{R}) \rightarrow (C^\infty N \rightarrow \mathbb{R})] \{ v_p : C^\infty M \rightarrow \mathbb{R} \} : C^\infty N \rightarrow \mathbb{R} \} [f : C^\infty N] : \mathbb{R} \quad (29)$$

The types $C^\infty M \rightarrow \mathbb{R}$ and $C^\infty N \rightarrow \mathbb{R}$, as sets, are just $T_p M$ and $T_{\phi(p)} N$, so, less noisily:

$$(\phi_* v_p) f = [\{ [\phi_* : T_p M \rightarrow T_{\phi(p)} N] \{ v_p : T_p M \} : T_{\phi(p)} N \} [f : C^\infty N] : \mathbb{R} \quad (30)$$

Covariance

Just as $\phi^* : C^\infty N \rightarrow C^\infty M$ pulls back a contravariant function $f : C^\infty N$ to a function $\phi^* f : C^\infty M$ backward through $\phi : M \rightarrow N$, $\phi_* : T_p M \rightarrow T_{\phi(p)} N$ pushes a tangent vector $v_p : C^\infty M \rightarrow \mathbb{R}$ forward through the same $\phi : M \rightarrow N$ to a tangent vector $\phi_* v_p : C^\infty N \rightarrow \mathbb{R}$.

Tangent vectors in $T_p M = \{v_p : C^\infty M \rightarrow \mathbb{R} \mid p \in M\}$ are **covariant** because they are pushed forward, upstream, along the grain, in the same direction as $\phi : M \rightarrow N$, to the tangent space $T_{\phi(p)} N = \{v_{\phi(p)} : C^\infty N \rightarrow \mathbb{R} \mid \phi(p) \in N\}$ of the destination manifold N .

Example (Exercise 16 in GKG)

Prove $(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t))$.

Left-hand Side

$(\phi \circ \gamma)'(t)$ is a tangent vector. Apply it to a function f .

$$(\phi \circ \gamma)'(t) f = \frac{d}{dt} f(\phi(\gamma(t))) = \frac{d}{dt} (f \circ \phi \circ \gamma)(t) \in \mathbb{R}$$

Right-hand Side

$$\phi_*(\gamma'(t)) = \gamma'(t) (\phi^* f) = \gamma'(t) (f \circ \phi) = \frac{d}{dt} (f \circ \phi)(\gamma(t)) = \frac{d}{dt} (f \circ \phi \circ \gamma)(t) \in \mathbb{R}$$

by associativity of function composition.

Lie Bracket $[v, w] = v \circ w - w \circ v : C^\infty M \rightarrow C^\infty M$

$[v, w]$ is a vector field defined so that

$$[v, w] f \stackrel{\text{def}}{=} v(wf) - w(vf) \quad (31)$$

with v and w vector fields in $C^\infty M \rightarrow C^\infty M$, and all juxtapositions denoting function application, with and without parentheses.

Recall that $vf : C^\infty M$ is a directional derivative with an invisible hole, waiting for a point $p : M$. That has exactly the form of a $C^\infty M$, so it has the correct type for an input to another vector field, w . The notion and the notation make sense.

Get a point-free notation by sneaking f out of the right-hand side of Equation 31, being careful to notice that expressions vw and wv sneakily denote function composition and not multiplication of vector fields, *which is not defined*. Subtraction of vector fields is defined via rule **VF +** in *Arithmetic on Vector Fields*, above.

$$[v, w] = v \circ w - w \circ v \stackrel{\text{def}}{=} vw - wv \quad (32)$$

One sees that latter notation, $vw - wv$, frequently. I prefer the less ambiguous $v \circ w - w \circ v$. We already have juxtaposition ambiguously representing both function application and various kinds of multiplication and probably should not add function composition as a third ambiguity. Ambiguity slows down reading. It's the lazy writer's way to save ink, but it is a major roadblock to readers.

Lie Bracket is a Vector Field

Linearity is almost obvious. The following proof of Leibniz's Law closely follows that on pg. 36 of GKG, with some of our own refinements.

- Invoke **$C^\infty *$** from *Arithmetic on Functions in $C^\infty M$* and **LEIBV** from *Laws for Vector Fields*.
- $f, g, v(f), w(f), v(g), w(g), (v \circ w)(f), (v \circ w)(g), (w \circ v)(f), (w \circ v)(g)$, are all functions in $C^\infty M$, writing extra parentheses to avoid confusion with multiplication of functions.
- Multiplication of functions is commutative, so the blue cancels the red, below.

- Composition of vector fields is *not* commutative (else we wouldn't need the Lie bracket).

$$\begin{aligned}
& (v \circ w - w \circ v)(fg) \\
&= v(gw(f) + fw(g)) - w(gv(f) + fv(g)) \\
&= g((v \circ w)(f)) + v(g)w(f) + f((v \circ w)(g)) + v(f)w(g) - g((w \circ v)(f)) - w(g)v(f) - f((w \circ v)(g)) - w(f)v(g) \\
&= g((v \circ w)(f)) + f((v \circ w)(g)) - g((w \circ v)(f)) - f((w \circ v)(g)) \\
&= g[v, w](f) + f[v, w](g)
\end{aligned}$$

Example: Exercise 22 in GKG

Compute the Lie bracket of the vector fields defined below, writing the vector fields as functions of a named parameter, f .

```

In[36]:= ClearAll[v, w, lieBracket];
v[f_] := (x ∂x f + y ∂y f) / Sqrt[x2 + y2];
w[f_] := (x ∂y f - y ∂x f) / Sqrt[x2 + y2];
lieBracket[v_, w_] := f ↦ (v@*w)@f - (w@*v)@f;
lieBracket[v, w]@f[x, y] // FullSimplify

```

Out[40]=

$$\frac{-x f^{(0,1)}[x, y] + y f^{(1,0)}[x, y]}{x^2 + y^2}$$

Symbolical Leibniz's Law

For these particular vector fields v and w , *Mathematica* verifies Leibniz's Law, with $@*$ denoting function composition and $@$ denoting function application.

```

In[41]:= ((v@*w)@(f[x, y] × g[x, y]) - (w@*v)@(f[x, y] × g[x, y])) // FullSimplify

```

Out[41]=

$$\frac{g[x, y] (-x f^{(0,1)}[x, y] + y f^{(1,0)}[x, y]) + f[x, y] (-x g^{(0,1)}[x, y] + y g^{(1,0)}[x, y])}{x^2 + y^2}$$

```

In[42]:= f[x, y] ((v@*w)@g[x, y] - (w@*v)@g[x, y]) +
g[x, y] ((v@*w)@f[x, y] - (w@*v)@f[x, y]) // FullSimplify

```

Out[42]=

$$\frac{g[x, y] (-x f^{(0,1)}[x, y] + y f^{(1,0)}[x, y]) + f[x, y] (-x g^{(0,1)}[x, y] + y g^{(1,0)}[x, y])}{x^2 + y^2}$$

More general cases are not so straightforward due to subtleties of *Mathematica*, but we have a proof in prose, so we're done for now.

Numerical Example

Pick a function from the *Mathematica* documentation for `Plot3D`. Apply the Lie bracket above to it and plot the results.

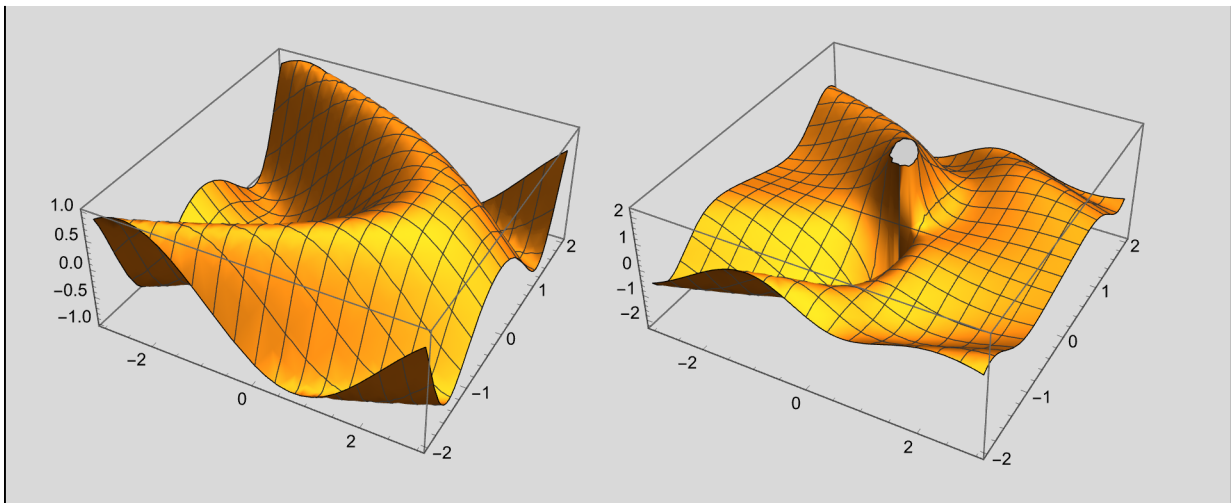
```
In[43]:= lieBracket[v, w]@Sin[x + y^2] // FullSimplify
```

```
Out[43]=
```

$$\frac{(1 - 2x) y \cos[x + y^2]}{x^2 + y^2}$$

```
In[44]:= GraphicsRow[{Plot3D[Sin[x + y^2], {x, -3, 3}, {y, -2, 2}],  
Plot3D[Evaluate@lieBracket[v, w]@Sin[x + y^2], {x, -3, 3}, {y, -2, 2}]]
```

```
Out[44]=
```



Jacobi Identity (Exercise 24 in GKG)

The easiest way to prove the Jacobi Identity in *Mathematica* is to spell it out by hand because *Mathematica* does not automatically define addition of functions, **VF** +. Function composition is associative, so we don't need a lot of parentheses.

```
In[45]:= Module[{u, v, w, f},  
u@*v@*w@f - v@*w@*u@f - (u@*w@*v@f - w@*v@*u@f) + v@*w@*u@f - w@*u@*v@f -  
(v@*u@*w@f - u@*w@*v@f) + w@*u@*v@f - u@*v@*w@f - (w@*v@*u@f - v@*u@*w@f) ]
```

```
Out[45]=
```

0

Exterior Derivative $df = v \mapsto vf : VM \rightarrow C^\infty M$

Applying a vector field $v : C^\infty M \rightarrow C^\infty M = VM$ to a function $f : C^\infty M$ yields vf , a directional derivative (DD) with an invisible hole waiting for a point $p \in M$. That DD with a hole is of type $C^\infty M$. Apply it to a $p \in M$ to get an \mathbb{R} , the DD of f at p in the direction of v .

$$vf = (vf)(\square) : C^\infty M \subset (M \rightarrow \mathbb{R})$$

$$vf p \stackrel{\text{def}}{=} v_p f : \mathbb{R} \stackrel{\text{trad}}{=} \nabla f(p) \cdot v$$

The last formula on the right is in traditional “grad f dot v ” notation in \mathbb{R}^n , assuming any ol’ chart.

Start over. Take out $p \in M$ and put back the hole. Abstract instead the vector field $v : VM \subset C^\infty M \rightarrow C^\infty M$ into a function parameter, and define **the differential 1-form df** or **exterior derivative of f** as **the function of type $VM \rightarrow C^\infty M$ that sends the vector field $v : VM$ to $vf : C^\infty M$** , that is, such that $df v = (df) v = vf : C^\infty M$

$$df \stackrel{\text{def}}{=} v \mapsto (vf : C^\infty M) \stackrel{\text{trad}}{=} v \mapsto \nabla f(\square) \cdot v \quad (33)$$

with a hole, \square , for $p \in M$. The type of df , manifestly, is $VM \rightarrow C^\infty M$, sending a vector field in VM to a function in $C^\infty M$ that’s waiting for a $p \in M$.

Turn this around and fill the hole for $p \in M$ first, leaving a hole for $v \in VM$.

Take df as an indivisible unit of notation so that df_p means $(df)_p = (df)(p)$.

$$df_p \stackrel{\text{def}}{=} v \mapsto (vf p : \mathbb{R}) \stackrel{\text{trad}}{=} v \mapsto \nabla f(p) \cdot v \quad (34)$$

Check Types

$$df_p \stackrel{\text{def}}{=} [v \mapsto (vf p : \mathbb{R})] : VM \rightarrow \mathbb{R} \quad (35)$$

Linearity

df is manifestly linear in vector fields v and w , with the scalars of linearity being functions g in $C^\infty M$. This is easy to see in some ol’ chart:

- Dot product distributes over vector addition.

$$\nabla f(\square) \cdot (v + w) \stackrel{\text{trad}}{=} \nabla f(\square) \cdot v + \nabla f(\square) \cdot w \quad (36)$$

- Dot product commutes with multiplication by scalar value $g(\square)$.

$$\nabla f(\square) \cdot (g(\square) v) = g(\square) (\nabla f(\square) \cdot v) \quad (37)$$

Laws for Exterior Derivative

Because df just abstracts a vector field v from expressions like vf , the following are easy to derive from *Arithmetic on Vector Fields* and *Laws for Vector Fields* above.

Abstract the function part f of df and take the Laws above as defining the naked $d : C^\infty M \rightarrow (VM \rightarrow C^\infty M)$ operator.

$$\text{LIND+} :: d(f + g) = df + dg \quad (38)$$

$$\text{LINR*} :: d(\alpha f) = \alpha df \quad (39)$$

$$\text{LIND*} :: (f + g)dh = f dh + g dg \quad (40)$$

$$\text{LEIBD} :: d(fg) = f dg + g df \quad (41)$$

Basis for Differential 1-Forms dx^μ

Given a chart (any ol' chart will do) and a function $f : C^*(M)$, define a basis set, dx^μ , for differential 1-forms:

$$df \stackrel{\text{def}}{=} (\partial_\mu f) dx^\mu \quad (42)$$

Remember to sum on the down-up index pair μ . Again, μ is an **abstract index**, valid in and necessary for any ol' chart, as opposed to a concrete index that presupposes a particular chart. This works because the manifold itself has the same dimension n as the target \mathbb{R}^n of any ol' chart.

The only interesting property of this basis is

$$dx^\mu(\partial_\nu) = \delta_\nu^\mu \quad (43)$$

(no sum). The Kronecker δ_ν^μ is 1 when $\mu = \nu$ and zero otherwise (ν is Greek “nu,” despite looking like ν , English/Latin “vee”).

Take the time to unpack the types on these. See GKG pg. 43 ff for much more.

1-Form $\omega : VM \rightarrow C^\infty M$

1-form has the same type as an exterior derivative. It is a generalization of that idea, a generalization that does not specify the function $f : C^\infty M$.

Leave the comfort zone of \mathbb{R}^n for M . Assume dimension n and forget about the particular chart (any ol' chart will do). Define a **1-form** $\omega : VM \rightarrow C^\infty M$, abstractly, as any any ol' linear function ω that

sends a vector field $v : VM$ to a function $\omega v : C^\infty M$ that, in turn, has an invisible hole for a point $p \in M$, and with the scalars of linearity being functions in $C^\infty M$.

Vector fields inhabit a vector space. 1-forms inhabit the dual space. It's a good idea to think of 1-forms as a field, just as one thinks of a vector field as a field.

Linearity

Check Types

Exterior Derivative df is a 1-Form

Check Types

Arithmetic on 1-Forms

Set of all 1-Forms on M : $\Omega^1 M = \{\omega : VM \rightarrow C^\infty M\}$

Cotangent Vector $\omega_p : T_p M \rightarrow \mathbb{R}$

A 1-form $\omega : VM \rightarrow C^\infty M$ sends a vector field $v : C^\infty M \rightarrow C^\infty M = VM$ to something with the same type as a directional derivative-with-a-hole for a point $p : M$, $\omega v \square : C^\infty M \subset M \rightarrow \mathbb{R}$. $\omega v p$ is a real number that has the same type as a directional derivative.

Define a **cotangent vector** $\omega_p : (C^\infty M \rightarrow \mathbb{R})$ as a linear gadget that applies a 1-form $\omega : VM \rightarrow C^\infty M$ to a vector field $v : VM$ and fills the hole with a point $p : M$. *En passant*, it entails a tangent vector, $v_p \square : C^\infty M \rightarrow \mathbb{R}$, having thrown away the hole for $f : C^\infty M$ because the 1-form ω doesn't specify a function. All together, get a number:

$$\omega_p v_p \stackrel{\text{def}}{=} \omega v p = \omega(v)(p) : \mathbb{R}$$

(52)

Cotangent Space $T_p^* M = \{\omega_p : T_p M \rightarrow \mathbb{R} \mid p \in M\}$

Just as the tangent space $T_p M = \{v_p : C^\infty M \rightarrow \mathbb{R} \mid p \in M\}$ is defined as the set of all tangent vectors $v_p : C^\infty M \rightarrow \mathbb{R}$ at the point $p : M$, define the **cotangent space** $T_p^*(M)$ as the set of all cotangent vectors at p . This is the dual space to $T_p M$ in the sense of linear algebra.

Pullback of a Cotangent Vector $\phi^* : T_{\phi(p)}^* N \rightarrow T_p^* M$

Just as are functions in $C^\infty N$, cotangent vectors are contravariant w.r.t. functions ϕ between manifolds M and N .

Recall the pushforward $\phi_* v_p : T_{\phi(p)} N$ of a tangent vector $v_p : T_p M$, defined via the function pullback $\phi^* : C^* N \rightarrow C^* M$

$$(\phi_* v_p) f = v_p(\phi^* f) \quad (53)$$

$$(\phi_* v_p : C^\infty N \rightarrow \mathbb{R})(f : C^\infty N) : \mathbb{R} = (v_p : C^\infty M \rightarrow \mathbb{R})(\phi^* f : C^\infty M) : \mathbb{R} \quad (54)$$

with $\phi : M \rightarrow N$, not necessarily a chart, ϕ .

Likewise, define the **pullback** $(\phi^* \omega_{\phi(p)})$ of a cotangent vector $\omega_{\phi(p)} : T_{\phi(p)}^* N$ to a new cotangent vector $(\phi^* \omega)_p : T_p^* M$, via the action of $\omega_{\phi(p)}$ on the pushforward $v_{\phi(p)} = \phi_* v_p : T_{\phi(p)} N$ of a tangent vector $v_p : T_p M$. Let $q = \phi(p)$ to shorten things.

$$\omega_q v_q = \omega_q(\phi_* v_p) \stackrel{\text{def}}{=} (\phi^* \omega_q) v_p \stackrel{\text{def}}{=} (\phi^* \omega)_p v_p : \mathbb{R} \quad (55)$$

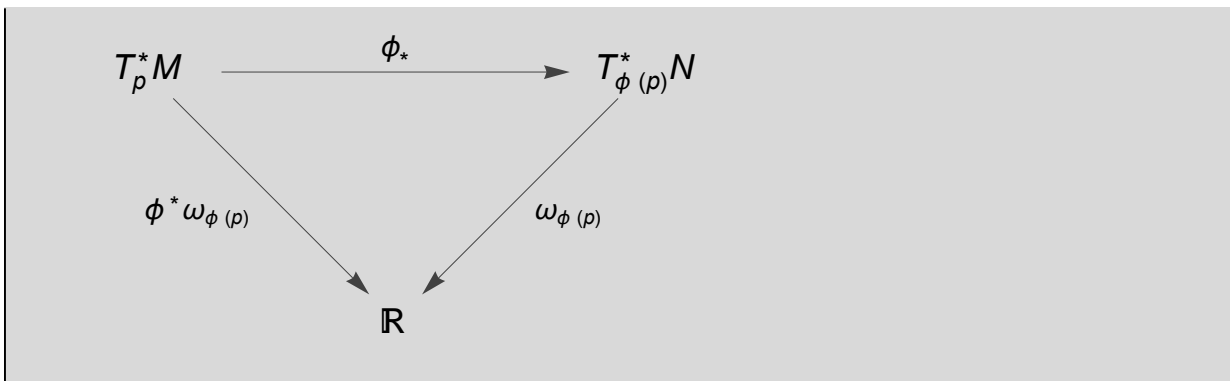
All these quantities are real numbers.

$(\phi^* \omega)_p : T_p^* M$ is a sneak preview of the pullback of a 1-form.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[46]=



Pushforward of a Vector Field $\phi_* : VM \rightarrow VN$

Vector fields are covariant w.r.t. functions ϕ between manifolds, just as are tangent vectors.

Recall that the pushforward $(\phi_* v_p) : T_{\phi(p)} N$ of a tangent vector $v_p : T_p M$ is a tangent vector in $T_{\phi(p)} N$ that sends functions in $C^\infty N$ to reals.

Likewise, the pushforward $(\phi_* v) : VN$ of a vector field $v : VM$ is a vector field in VN that sends functions in $C^\infty N$ to functions in $C^\infty N$, functions that are waiting for a point $q : N$.

Pullback of 1-Form $\phi^* : \Omega^1 N \rightarrow \Omega^1 M$

1-forms are contravariant w.r.t. functions ϕ between manifolds, just as are cotangent vectors.

Let $\phi : M \rightarrow N$ be a mapping from manifold M to manifold N (not necessarily a chart, ϕ). Let $q : N = \phi(p : M)$.

Recall the pullback $(\phi^* \omega)_p : T_p^* M$ of a cotangent vector $\omega_q : T_q^* N$ defined by its action on the pushforward $\phi_* v_p : T_q N$ of a tangent vector $v_p : T_p M$.

$$\omega_q(\phi_* v_p) \stackrel{\text{def}}{=} (\phi^* \omega)_p v_p \stackrel{\text{def}}{=} (\phi^* \omega)_p v_p$$

Abstract away the points p and q , define the **pullback** $(\phi^* \omega)$ of a **1-form** ω by its action on the pushforward $(\phi_* v)$ of a vector field v .

$$(\phi^* \omega) v p = \omega(\phi_* v) \phi(p) = \omega(\phi_* v) q \quad (56)$$

Check Types

$$\begin{aligned} & \{[(\phi^* \omega) : \Omega^1 M = VM \rightarrow C^\infty M][v : VM] : C^\infty M\} \{p : M\} : \mathbb{R} \\ &= \{[\omega : \Omega^1 N = VN \rightarrow C^\infty N](\phi_* v : VN) : C^\infty N\} \{\phi(p) : N\} : \mathbb{R} \end{aligned}$$

Pullback of a Differential $(\phi^* df) : \Omega^1 M$

Exterior Algebra: $\wedge V$

This is a bit abstract, but it's the appropriate generalization of *cross-product* to any number of dimensions.

P-Forms (UNDONE)

Direct Sum and Tensor Product

<https://www.math3ma.com/blog/the-tensor-product-demystified>

Metric (UNDONE)

Hodge Star (UNDONE)

Stoke's Theorem (UNDONE)

DeRham Cohomology (UNDONE)

Lie Groups (UNDONE)

Lie Algebras (UNDONE)

Maxwell's Equations (UNDONE)

Bundles and Connections

Bianchi Identity (UNDONE)

Yang-Mills Equation (UNDONE)

Link Invariants (UNDONE)

Chern-Simon Theory (UNDONE)

Semi-Riemannian Geometry (UNDONE)

Levi-Civita Connection (UNDONE)

Geodesics (UNDONE)

Clean Up

The following cell is marked “not Evaluatable” during development.

```
In[ ]:= RemoveNotation[  $\phi_* f_*$   $\Leftrightarrow$  pullback[ $\phi_*$ ,  $f_*$ ] ];  
ClearAll[ $\phi$ , x, y, xf, yf, pullback, lieBracket];
```