Functional Type Cheats for Manifolds

Brian Beckman

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Abstract

The modern coordinate-free approach to manifolds comes with non-classical notation and vocabulary. These can be barriers to understanding and to calculational skill for people educated only in classical tensor calculus with indices. Short expressions in modern notation like $g\omega v$ entail multiple concepts like function application and linear transformation in abstract function spaces.

Classical tensor calculus assumes a coordinate system at the beginning. One carries the baggage all the way, writing everything in components in lengthy expressions full of indices. Modern notation is more concise, delaying choice of coordinate system, replacing intermediate steps with function composition. Function composition works without indices in any number of dimensions and on real or complex manifolds. Abstract indices that iterate over any set of coordinates, without choosing particular coordinates, are occasionally useful. They're abstractly well defined because all sensible coordinate systems over a given manifold have the same number of dimensions.

Function composition depends on *functional types*. Many classical results and new creative ideas follow easily just by checking the types. For example, co- and contravariance fall out naturally from pushforwards and pullbacks instead of relying on coordinate transformations. Checking types is the same activity as in functional programming, say with Haskell or OCaml. Type checking might not be enough to convince a mathematician, but it can usually convince a compiler.

Learning the new notation and vocabulary takes effort. As with sports, music, or human languages, the only pathway to fluency seems to be repetition and practice. Whether writing code, doing calculations, or writing proofs, one must eventually perceive and conceive types effortlessly in expressions like $g\omega v$.

Background

These are the types that have been most useful to me. They're loosely compatible with Baez and Muniain, *Gauge Theory, Knots, and Gravitation* (GKG). The exposition here roughly follows the order of that book.

Prerequisites

I presume here only that you know linear algebra, axioms of group theory, and vector and tensor calculus, at best in classical, index-heavy notation. Be comfortable with *injection* (1-to-1), *surjection* (onto), *bijection* (both), *equivalence class*, and the rest of naïve set theory.

A, B, …, X, Y, … refer to sets of elements a, b, …, x, y, …, respectively.

M, N, … refer to ***manifolds***, sets of abstract, featureless points p, q, …

One says that the point $p \in M$ is a member or ***element*** of the set M, defining the symbol $\in$. One also says p is ***in*** M, p ***inhabits*** M, p ***is an*** M, p ***has type*** M, $p : M$, all meaning the same thing.

The colon notation $p : M$ is a ***type signature***. The colon binds loosely, so $f : C^\infty M \subset (M \to \mathbb{R})$ means $f : (C^\infty M \subset (M \to \mathbb{R}))$ and, *en passant*, asserts a subset relation between the two types $C^\infty M$ and $M \to \mathbb{R}$. Likewise, $\phi^* f = f \circ \phi : C^\infty M$ means $(\phi^* f = f \circ \phi) : C^\infty M$ and, *en passant*, asserts equality of the two expressions $\phi^* f$ and $f \circ \phi$. Often, type notations appear in a lighter brown ink to set them off from other text.

For our purposes, there is no difference between being ***of a type*** and being ***in a set***. If $p : U$ and $U \subset M$, then $p : M$ by definition of *subset*. In general, there are big differences between types and sets, but they're not pertinent here.

The functional type signature $f : A \to B$ declares that function f takes elements of set A and produces elements of set B. One says that f ***maps*** A to B, (*maps* as a verb), f is an A-to-B, f ***has type*** $A \to B$, f ***inhabits*** the type or set $A \to B$, f ***is an*** $A \to B$, f ***sends*** $a \in A$ to $b \in B$, etc. ***Function***, ***mapping***, and ***map*** (as a noun) usually mean the same thing.

The notation $a \longmapsto b$ alone is an ***anonymous function***, meaning "a function of a that produces b." Typically, b is an expression that depends on a. Give an anonymous function a name as follows: $f : A \to B = a \longmapsto b,\ a \in A,\ b \in B$.

One also writes $f(\square) : A \to B$ to mean that f is an expression with a ***hole***, $\square$, waiting for an element of set A. Physicists write $f(a) : A \to B$, giving the hole a name, a, but we prefer to treat that notation as meaning the *value* of f at the input a, so we write $f(a) : B$ or $f\,a : B$ to emphasize that f has already received its argument of type A and the resulting ***function application*** produces a value of type B. Function application ***nests to the left***: $f\,a\,b = (f\,a)\,b$. If we need to name a hole, we'll write the name as a subscript, as in $f(\square_a) = a \longmapsto b$.

Write the types of functions of many parameters (***multiadic*** or ***multiary*** functions) as in Haskell: $f(\square, \square) : \mathbb{R} \to \mathbb{R} \to \mathbb{R} = \mathbb{R} \to (\mathbb{R} \to \mathbb{R})$, ***nesting to the right***. One might just as well write $f(\square, \square) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with a Cartesian product of parameters, but we prefer not to do so. Multiadic anonymous function notation also nests to the right, as in $f(\square, \square) : \mathbb{R} \to \mathbb{R} \to \mathbb{R} = x \longmapsto (y \longmapsto x + y) = x \longmapsto y \longmapsto x + y$.

Abstraction means three different things, often changing a type signature.

1. generalizing some realm of thought, leaving out details
2. pulling data out of an expression, making a function and parameters, e.g., $x + y \overset{\text{abst}}{\Longrightarrow} f(\square, \square) : \mathbb{R} \to \mathbb{R} \to \mathbb{R} = x \longmapsto y \longmapsto x + y$
3. taking away parts of a definition, e.g., abstracting $p : M$ from $v_p(f) : \mathbb{R} = (vf)(p)$ yields a new definition, $vf : M \to \mathbb{R}$.

This cheat sheet talks mostly about real manifolds. We can do likewise for complex manifolds. Just substitute $\mathbb{C}$ for $\mathbb{R}$.

Commutative Diagrams (closed up)

Concise Cheats

- ***Smooth Functions*** $f : C^\infty M \subset (M \to \mathbb{R})$
- ***Vector Field*** $v : C^\infty M \to C^\infty M$
- ***Set of all Vector Fields on M*** $VM = \{v : C^\infty M \to C^\infty M\}$
- ***Basis for Vector Fields*** ∂_μ
- ***Tangent Vector*** $v_p : C^\infty M \to \mathbb{R}$
- ***Tangent Space*** $T_p M = \{v_p : C^\infty M \to \mathbb{R} \mid p \in M\}$
- ***Curve*** $\gamma : \mathbb{R} \to M$
- ***Tangent to a Curve*** $\gamma'(t) : C^\infty M \to \mathbb{R}$
- ***Pullback of a Function*** $\phi^* : C^\infty N \to C^\infty M$
- ***Pushforward of a Tangent Vector*** $\phi_* : T_p M \to T_{\phi(p)} N$
- ***Lie Bracket*** $[v, w] = v \circ w - w \circ v : C^\infty M \to C^\infty M$
- ***Exterior Derivative*** $d f = v \longmapsto vf : VM \to C^\infty M$
- ***Basis for Differential Forms*** $d x^\mu$
- ***1-Form*** $\omega : VM \to C^\infty M$
- ***Set of all 1-Forms on M*** $\Omega^1 M$
- ***Cotangent Vector*** $\omega_p : T_p M \to \mathbb{R}$
- ***Cotangent Space*** $T_p^* M = \{\omega_p : T_p M \to \mathbb{R} \mid p \in M\}$
- ***Pullback of a Cotangent Vector*** $\phi^* : T_{\phi(p)}^* N \to T_p^* M$
- ***Pushforward of a Vector Field*** $\phi_* : VM \to VN$
- ***Pullback of 1-Form*** $\Omega^1 N \to \Omega^1 M$

- ***Pullback of a Differential*** $(\phi^* \mathbb{d} f) : \Omega^1 M$

Explanations and Demonstrations

Smooth Functions $f : C^\infty M \subset (M \to \mathbb{R})$

At the front gate of the Manifold Zoo is the set $C^\infty M$ of infinitely differentiable C^∞ (smooth) functions from the manifold M to the reals $\mathbb{R}$. That’s a subset of all functions from M to $\mathbb{R}$.

$C^\infty M$ is an indivisible symbol denoting the set of functions. It’s not multiplication of C^∞ times M or a function application of C^∞ to M.

After pullbacks, we can characterize smoothness on M precisely. Until then, take the term ***smooth*** intuitively.

See later that *curves* are certain functions going the other way, back from $\mathbb{R} \to M$.

What could it possibly mean to differentiate functions on M, a set of abstract, featureless points? One needs a little topology.

Chart $\varphi : (U \subset M) \to \mathbb{R}^n$

A ***chart*** φ is a continuous function with a continuous inverse from an ***open subset*** U of a manifold M to the n-dimensional Euclidean space of reals, $\mathbb{R}^n$, the mother of all manifolds. We take continuity on $\mathbb{R}^n$ as understood. Continuity on $U \subset M$ takes more topology than we want to cover here, so take it intuitively.

Think of a chart of the World on flat paper, say in the Mercator projection. Go back and forth from numerical coordinates in $\mathbb{R}^n$ on the paper to points in $U \subset M$ on an physical spherical globe.

A chart implies a ***coordinate system*** on the destination set $\mathbb{R}^n$. Intuit the ***open subset*** U as a “ball” of points in M excluding the boundary of the “ball.” This intuition is good enough for practical applications where works in $\mathbb{R}^n$ as much as possible.

$\left(\boldsymbol{f \circ \varphi^{-1}}\right) : C^\infty \mathbb{R}^n \subset (\mathbb{R}^n \to \mathbb{R})$

(This gadget has no name of its own)

Point-Free Notation

Read $\left(f \circ \varphi^{-1}\right)$ from right-to-left. The ***function composition*** $\left(f \circ \varphi^{-1}\right)$ is a new function. Apply it to arguments on its right, as in $\left(f \circ \varphi^{-1}\right)(x)$, sending $x \in \mathbb{R}^n$ first to φ^{-1}, getting a point $p \in U \subset M$, then p to f, like this, $\left(f\left(\varphi^{-1}(x)\right)\right)$, to get a real number. That's harder to read and write than $\left(f \circ \varphi^{-1}\right)(x)$ or $\left(f \circ \varphi^{-1}\right) x$, so we go with the latter. Take off (abstract) the x to get ***point-free notation*** $\left(f \circ \varphi^{-1}\right): \mathbb{R}^n \to \mathbb{R}$, excluding the "point" x.

Function composition is associative (https://en.wikipedia.org/wiki/Function_composition), i.e., $(f \circ (g \circ h)) = ((f \circ g) \circ h)$, so the parentheses are not needed: $(f \circ (g \circ h)) = ((f \circ g) \circ h) = f \circ g \circ h$.

Any chart φ takes a point $p \in U \subset M$ and returns an $\mathbb{R}^n$. The inverse of the chart φ^{-1} goes the other way, from $\mathbb{R}^n$ to $p \in U \subset M$ (charts must be invertible). f picks up that p right away and returns an $\mathbb{R}$. So the whole composition $\left(f \circ \varphi^{-1}\right)$ goes straight from $\mathbb{R}^n$ to $\mathbb{R}$ with $U \subset M$ the forgotten middle of the sandwich.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions. The parameters or arguments of the functions go without names in such diagrams.

Out[7]=

$$\begin{tikzcd} U \subset M \arrow[dr, "f"'] & & \mathbb{R}^n \arrow[ll, "\varphi^{-1}"'] \arrow[dl, "f \circ \varphi^{-1}"] \\ & \mathbb{R} & \end{tikzcd}$$

Calculus

Suddenly, $C^\infty M$ means something: we have ordinary multivariable calculus in $\mathbb{R}^n$ through $\left(f \circ \varphi^{-1}\right): C^\infty \mathbb{R}^n$.

$\mathbb{R}^n$ is a manifold, so $C^\infty \mathbb{R}^n$ makes sense: *the set of functions in $\mathbb{R}^n \to \mathbb{R}$ with an infinite number of continuous derivatives. Such functions are plentiful*.

Under practical circumstances, **any ol' chart will do** because all charts map to the same $\mathbb{R}^n$ with the same dimension n, just through various coordinate systems. One assigns this same dimension, n, to U and to M. One can reason about φ without mentioning n more than once.

f is in $C^\infty M$ iff $\left(f \circ \varphi^{-1}\right): C^\infty \mathbb{R}^n$ for all charts. (1)

Arithmetic Rules for Functions in $C^\infty M$

Let double-colon separate the name of a rule, like $\boldsymbol{C^\infty +}$, from the definition of the rule.

Let f and g be in $C^\infty M \subset (M \to \mathbb{R})$.

$\mathbf{C^\infty +} :: f + g : C^\infty M$ (2)

$f + g$ is a new function in $C^\infty M$ that produces $f(p) + g(p) \in \mathbb{R}$, the sum of two real numbers $f(p)$ and $g(p)$, for any $p \in U \subset M$.

$\mathbf{C^\infty *} :: f\, g : C^\infty M$ (3)

$f g = f \times g$ (f *times* g) is a new function in $C^\infty M$ that produces $f(p)\, g(p) = f(p) \times g(p) \in \mathbb{R}$ for any $p \in U \subset M$. $f(p)\, g(p)$ denotes the product of two real numbers $f(p)$ and $g(p)$.

fg does not mean $f(g)$, f applied to g, even though vf does mean $v(f)$, v applied to f. **Juxtaposition is ambiguous without the types!**

$\mathbf{\mathbb{R} * C^\infty} :: \alpha f : C^\infty M$ (4)

αf, where $\alpha \in \mathbb{R}$, is a new function in $C^\infty M$ that produces $\alpha f(p) \in \mathbb{R}$ by multiplication of real numbers α and $f(p)$.

One says that functions producing values in $\mathbb{R}$ ***inherit*** ordinary, real-number arithmetic from $\mathbb{R}$.

Vector Field $v : C^\infty M \to C^\infty M$

A ***vector field*** v is a linear function that sends a function $f \in C^\infty M$ to another function $vf \in C^\infty M$. A vector field adheres to the Laws below.

It's weird to think of vector fields as functions from functions to functions, but it's just an abstraction of directional derivative (DD), which we already know about from classical vector calculus. v abstracts out the chart, necessary for DD, because everything works on any ol' chart.

Assume there is some chart φ so that $\frac{\partial f}{\partial x}$ really means $\frac{\partial (f \circ \varphi^{-1})}{\partial x}$, remembering that $\left(f \circ \varphi^{-1}\right) \in C^\infty \mathbb{R}^n \subset (\mathbb{R}^n \to \mathbb{R})$. Take partial derivatives with respect to coordinates $\left\{x^1, x^2, \ldots, x^n\right\} \in \mathbb{R}^n$. Indices are upstairs. Why? Later!

Define the notation vf to mean the vector field function v applied to the argument $\left(f \circ \varphi^{-1}\right) : C^\infty \mathbb{R}^n$ through any ol' chart φ.

vg does not mean $v \times g$, v times g, even though gf does mean $g \times f$, g times f. **Juxtaposition is ambiguous without the types!**

One might write $v(f)$ instead, but that's noisier. vf looks like multiplication. When v is linear, and it is, vf *is equal to* matrix multiplication, so function application and matrix multiplication look the same because they are the same in the all-important linear case.

In compounded function-applications notations like $v f p$, nest to the left, as $((v\, f)\, p) = ((v(f))\,(p))$.

Find components $\{v^1, v^2, \ldots, v^n\}$ of a classical, traditional vector, indices upstairs. Write

$$v f = v^1 \frac{\partial f}{\partial x^1} + v^2 \frac{\partial f}{\partial x^2} + \ldots + v^n \frac{\partial f}{\partial x^n} \overset{\text{trad}}{=} \nabla f \cdot v \tag{5}$$

with the traditional “grad f dot v” on the extreme right. Now shorten this notation.

$$v f = \left(v^1\, \partial_1 f + v^2\, \partial_2 f + \ldots + v^2\, \partial_n f\right) = \left(\sum_{i=1}^{n} v^i\, \partial_i f\right) \overset{\text{def}}{=} \left(v^i\, \partial_i f\right) = \left(v^i\, \partial_i\right) f \tag{6}$$

We’ve done three things:

- replaced $\frac{\partial}{\partial x^i}$ with ∂_i (indices *downstairs*)
- automatically summed over the repeated index i, one up, the other down—**always automatically sum over repeated up-down indices**
- parenthesized $\left(v^i\, \partial_i\right)$ in the expression $\left(v^i\, \partial_i\right) f$ to emphasize that we need not mention f

Let $v \in C^\infty M \to C^\infty M$ be a vector field. Give an $f : C^\infty M$ to v and get $v f : C^\infty M$. Give $v f$ a point $p : U \subset M$ to get an $\mathbb{R}$. $v f$ is a ***point-free notation***.

Abstract off the f and *define* the vector field as $v = v^i\, \partial_i$. **A vector field is a linear combination of partial derivative operators through any ol’ chart**. It turns out that when v is applied to $\left(f \circ \varphi^{-1}\right)$ on the same chart φ to get $v f : C^\infty M$, and then applied to any ol’ point $p : U \subset M$ to get $v f p : \mathbb{R}$, the result does not depend on the chart φ, so don’t mention it any more, just write $v f : C^\infty M$. But we need not mention f either.

We get a two-level point-free notation, just $v : C^\infty M \to C^\infty M$.

Why is v called a vector *field*? Classically, a *vector field* is a tuple of components defined at every point in, say, $\mathbb{R}^3$. $v f$ is defined at every point $p : U \subset M$, so v really is a field of objects defined at every point $p : U \subset M$, an object for calculating directional derivatives at p.

Don’t confuse *field*, here, with ***algebraic field***, which generalizes ordinary arithmetic via associative, commutative, and distributive laws.

Shorthand

Write $v_p\, f$ instead of $v f p$, which is, itself, shorthand for $(v(f))\,(p)$ and $(v f)\,(p)$.

Exercise

In another notebook, https://community.wolfram.com/groups/-/m/t/3098298, vectors are defined as equivalence classes of curves, there proved to satisfy all eight axioms of a ***vector space***. Prove likewise for $v = v^i\,\partial_i$.

Laws for Vector Fields

Vector fields in $C^\infty M \to C^\infty M$ must satisfy linearity laws and a Leibniz product rule so that vector fields act like derivative operators.

With $f : C^\infty M$ and $g : C^\infty M$:

Distributivity of function application:

$$\textbf{LINV}+ :: v\,(f+g) = vf + vg, \text{ via rule } \mathbf{C^\infty +}, \text{ both sides in } C^\infty M \tag{7}$$

Letting α be a real number:

$$\textbf{LINV}* :: v\,(\alpha g) = \alpha(vg), \text{ via rule } \mathbb{R} * \mathbf{C^\infty}, \text{ both sides in } C^\infty M \tag{8}$$

Leibniz’s Law, aka product rule of differentiation

$$\textbf{LEIBV} :: v\,(f g) = g(vf) + f(vg)\,, \text{ via rule } \mathbf{C^\infty} *, \text{ both sides in } C^\infty M \tag{9}$$

Arithmetic on Vector Fields

Let $v \in C^\infty M \to C^\infty M$ and $w \in C^\infty M \to C^\infty M$ be two vector fields, and let f and g each be in $C^\infty M$. $(v + w)$ is a new vector field. Apply it to a function $f : C^\infty M$ and invoke $\mathbf{C^\infty +}$.

$$\textbf{VF+} :: (v+w)\,f = vf + wf, \text{ right-hand side by } \mathbf{C^\infty +} \tag{10}$$

The next one defines gv, multiplication by function $g : C^\infty M$ on the left by a vector field $v : C^\infty M \to C^\infty M$ on the right.

$$\mathbf{C^\infty}\,\textbf{VF}* :: (gv)\,f = g(vf), \text{ right-hand side by } \mathbf{C^\infty} * \tag{11}$$

The rules above make the space of vector fields very much like a vector space. Technically, it’s a ***module*** over $C^\infty M$ because multiplication need not be commutative and inverses need not exist (https://math.stackexchange.com/questions/4142071).

Set of all Vector Fields on M $VM = \{v : C^\infty M \to C^\infty M\}$

VM, an indivisible symbol, is a type: the set of all vector fields on M.

$$VM = \{v : C^\infty M \to C^\infty M\} \tag{12}$$

Sets are types in this document, so we might just as well write $VM = C^\infty M \to C^\infty M$.

Basis for Vector Fields ∂_μ

Given components v^μ in some chart, the expression $v = v^\mu \partial_\mu$ suggests that ∂_μ be a basis in VM. Indices μ here are not concrete indices. They do not presume a particular coordinate system because any ol' chart will do. But all charts have in common the number of dimensions. The indices μ are ***abstract indices*** that range over the ordered dimensions of the manifold M. Once a coordinate system is chosen, the abstract indices identify with the concrete indices of the coordinate system. For example, in spherical coordinates in 3D, we have coordinates ρ, θ, and ϕ, with corresponding basis vectors ∂_ρ, ∂_θ, and ∂_ϕ defined at every point, so μ ranges over ρ, θ, ϕ. In Cartesian coordinates, we have coordinates x, y, and z, with corresponding basis vectors ∂_x, ∂_y, and ∂_z, and μ ranges over x, y, z. We abstract out the particular coordinate by saying that in any coordinate system, we will have coordinates x^1, x^2, x^3 and basis vectors ∂_{x^1}, ∂_{x^2}, and ∂_{x^3} or just ∂_1, ∂_2, ∂_3 for short.

Tangent Vector $v_p : C^\infty M \to \mathbb{R}$

Apply a vector field $v : C^\infty M \to C^\infty M$ to a function $f : C^\infty M$ to get a new function $v f \,\square_p : C^\infty M \subset M \to \mathbb{R}$ with a ***hole***, $\square_p$, waiting for a point $p : U \subset M$.

v, alone, has *two* holes, one for a function in $C^\infty M$ and one for a point in $U \subset M$. Via the shorthand $v_p f$,

$v = v_\square \square : C^\infty M \to C^\infty M$

The holes are invisible. No one writes them with $\square$ like that, but this is a cheat sheet, to get us comfortable with invisible things.

Give v an f in $C^\infty M$ as before and get an $C^\infty M \subset M \to \mathbb{R}$.

$v f : C^\infty M \subset M \to \mathbb{R}$

with an invisible hole waiting for a point $p : U \subset M$.

Start over with just v. Give v the point $p : U \subset M$, *first*. Get a new thing, v_p, with an invisible hole that waits for a $f : C^\infty M$.

$$v_p : C^\infty M \to \mathbb{R} \text{ such that } v_p f : \mathbb{R} \overset{\text{trad}}{=} \nabla f(p) \cdot v \tag{13}$$

Interpret the right-hand side traditionally. Assume any ol' chart. $\nabla f(p)$ is the gradient of f evaluated at p, and v is a tuple of components.

v_p is a ***tangent vector***. Given any ol' chart, the tangent vector lives in the ***tangent plane***, a hyperplane of type $\mathbb{R}^n$, at point p, where one does ordinary calculus.

Arithmetic and Laws for Tangent Vectors

Tangent vectors inherit arithmetic laws from vector fields, but the types change. In fact, *only* via type analysis can we develop **LEIBTV**. Type analysis reveals that one must apply the open functions f and g to p.

$$\textbf{LINTV}+ :: v_p\,(f+g) = v_p\,f + v_p\,g, \text{ both sides in } \mathbb{R} \tag{14}$$

$$\textbf{LINTV}* :: v_p\,(\alpha g) = \alpha(v_p\,g), \text{ both sides in } \mathbb{R} \tag{15}$$

$$\textbf{LEIBTV} :: v_p\,(f g) = g(p)\,(v_p\,f) + f(p)\,(v_p\,g), \text{ both sides in } \mathbb{R} \tag{16}$$

$$\textbf{TV}+ :: (v_p + w_p)\,f = v_p\,f + w_p\,f, \text{ both sides in } \mathbb{R} \tag{17}$$

$$\textbf{RTV}* :: \alpha v_p : C^\infty M \text{ such that } (\alpha v_p)\,f = \alpha(v_p\,f) \overset{\text{def}}{=} \alpha v_p\,f, \text{ both sides in } \mathbb{R} \tag{18}$$

for $\alpha \in \mathbb{R}$. **RTV*** defines the precedence of $v_p\,f : \mathbb{R}$. Function application $v_p\,f$ binds more tightly than multiplication by α.

Tangent Space $T_p\,M = \{v_p : C^\infty M \to \mathbb{R} \mid p \in U \subset M\}$

$T_p\,M$ is a type, the *set of tangent vectors at point* $p : U \subset M$. Tangent vectors send C^∞ functions to their directional derivatives in $\mathbb{R}$.

$$T_p\,M = \{v_p : C^\infty M \to \mathbb{R} \mid p \in U \subset M\} \tag{19}$$

$T_p\,M$ is an indivisible symbol denoting the set. The notation $T_p\,M$ is not a product or a function application.

From rules **TV +** and **RTV** $*$, deduce that $T_p\,M$ is a vector space, i.e., satisfies all eight axioms of a vector space.

Curve $\gamma : \mathbb{R} \to M$

The type of a curve, $\mathbb{R} \to M$, is the reverse of the type $C^\infty M \subset (M \to \mathbb{R})$. Give γ a $t \in \mathbb{R}$, get a point in $U \subset M$, but not just any ol' point. Demand that γ be ***smooth***: nearby values of input yield nearby values of output. To define *smooth* properly below, we need first a vector tangent to the curve γ at the point that γ returns.

Tangent to a Curve $\gamma'(t) : C^\infty M \to \mathbb{R}$

Let $\gamma(t)$ be a point in $U \subset M$. In ordinary calculus, one learns that $\gamma'(t)$ is a number: the slope of the tangent to the curve. Picture the curve as a graph on graph paper from $\mathbb{R}$ to $\mathbb{R}$. Picture the tangent as a line of infinite length kissing the curve. Measure the slope of that line.

Generalize by defining the derivative $\gamma'(t)$ of a curve γ at $t \in \mathbb{R}$ to be **the tangent vector** of type $C^\infty M \to \mathbb{R}$ that produces the directional derivative, a real number, of any $f : C^\infty M$, but in the instantaneous direction of the classical tangent to $\gamma(t)$. Picture $\gamma'(t)$ an arrow in the tangent plane, kissing the curve, pointing in the direction of the tangent line of classical calculus, and of length somehow measuring the rate at which a point along the curve changes with respect to the parameter t. Because $\gamma'(t)$ is a tangent vector, we must feed it a function $f : C^\infty M$ to get a number.

$$\gamma'(t)(f) \overset{\text{trad}}{=} \frac{d}{dt} f(\gamma(t)) \tag{20}$$

To relate this to classical intuition in 2D, think of a curve as a pair of functions, $x(t)$ and $y(t)$, that trace out the curve on paper. The tangent vector to the curve has components $\frac{dx}{dt}(t)$ and $\frac{dy}{dt}(t)$, pointing in the direction of the tangent. Let's call it $\tau = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. The directional derivative of any function f in that direction is $\nabla f \cdot \tau = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$. Squinting at $\frac{d}{dt} f(\gamma(t))$, we see that it's equal to $\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ by the chain rule in this Cartesian chart. By thinking of x and y as abstract coordinates in any ol' chart—and there will always be just two coordinates—the definition of $\gamma'(t)(f)$ as a directional derivative at point $\gamma(t)$ seems plausible.

Check Types

$$\gamma'(t)(f) \overset{\text{trad}}{=} \frac{d}{dt}(f : C^\infty M \subset (M \to \mathbb{R}))(\gamma(t : \mathbb{R}) : M) : \mathbb{R}$$

"trad" means "traditional" and implies that one can only compute a numerical derivative given a chart, but any ol' chart will do. It turns out that the directional derivative has the same value in every chart / coordinate system.

Shorthand

Take away the f and write something that looks like traditional notation, but isn't. It's a trick of notation, a *trompe-l'oeil*! It's not the slope of a tangent, it's a function from $C^\infty M$ to $\mathbb{R}$.

$$\frac{d\gamma}{dt}(t) \overset{\text{def}}{=} \gamma'(t) : C^\infty M \to \mathbb{R} \tag{21}$$

Take away the t, leaving a hole □, which becomes invisible. This is a function from a time t in $\mathbb{R}$ and a function f in $C^\infty M \to \mathbb{R}$ to a real-number directional derivative.

$$\frac{d\gamma}{dt} \stackrel{\text{def}}{=} \gamma'(\square) \stackrel{\text{def}}{=} \gamma' : \mathbb{R} \to C^\infty M \to \mathbb{R} \tag{22}$$

Pullback of a Function $\phi^* : C^\infty N \to C^\infty M$

Let ϕ be a function from M to N, two manifolds. ϕ need not be a chart. Charts are usually denoted via curly phi, φ.

From this point on, for brevity, we won't continue to emphasize that we're always working in open subsets of the manifolds: $p \in M$ means $p \in U \subset M$.

Let $f : C^\infty N$ be a C^∞ function on N. We'd like a similar function $\phi^* f : C^\infty M$ on M. How? First, send a point $m \in M$ to $n \in N$ via ϕ, then from N to $\mathbb{R}$ via f. Write an ordinary function composition $f \circ \phi : C^\infty M \subset (M \to \mathbb{R} = (N \to \mathbb{R}) \circ (M \to N))$, remembering to read composition via $\circ$ from right to left, and *en passant*, defining the ***composition of types*** $N \to \mathbb{R}$ and $M \to N$ as $(N \to \mathbb{R}) \circ (M \to N) = M \to \mathbb{R}$.

Define the notation $\phi^* f = f \circ \phi$, the ***pullback of the function*** f, with star upstairs. Pushforward, introduced later, has star downstairs.

$$\phi^* f \stackrel{\text{def}}{=} f \circ \phi : C^\infty M \tag{23}$$

Take out f, leaving a hole □ for a function in $C^\infty N$.

$$\phi^* \stackrel{\text{def}}{=} \square \circ \phi : C^\infty N \to C^\infty M \tag{24}$$

Take out the ϕ, leaving two holes. Define the ***pullback infix operator*** *.

$$* \stackrel{\text{def}}{=} \square_{\text{left}}{}^* \square_{\text{right}} = \square_{\text{right}} \circ \square_{\text{left}} : ((\square_{\text{left}} : M \to N) \to (\square_{\text{right}} : C^\infty N \to C^\infty M)) \tag{25}$$

Less noisily:

$$* : (M \to N)_{\text{left}} \to C^\infty N_{\text{right}} \to C^\infty M \tag{26}$$

Diagram

Remember we're always working in open subsets in the manifolds M and N.

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[8]=

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "\phi^* f"'] & & N \arrow[dl, "f"] \\ & \mathbb{R} & \end{tikzcd}$$

Contravariance

Functions in $C^\infty N$ are called ***contravariant*** because they are pulled back from $N \to \mathbb{R}$ to $M \to \mathbb{R}$, upstream, against the grains, of the functions $f : C^\infty N$ and $\phi : M \to N$.

Example (Exercise 14 in GKG)

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by a constant angle θ, $\mathbb{R}$ and $\mathbb{R}^2$ being the two manifolds M and N.

In[9]:=

```
ClearAll[ϕ, x, y, xf, yf, pullback];
ϕ[{x_, y_}] := RotationMatrix[θ].{x, y};
ϕ[{x, y}]
```

Out[11]=

```
{x Cos[θ] - y Sin[θ], y Cos[θ] + x Sin[θ]}
```

Let $x_f : (M = \mathbb{R}^2) \to \mathbb{R}$ be a function that returns the first of a tuple of components in $\mathbb{R}^2$, and let y_f return the second. The subscript f serves only to remind that x_f and y_f are functions; f has no deeper meaning here.

In[12]:=

```
xf[tuple_List] := tuple〚1〛; yf[tuple_List] := tuple〚2〛;
```

Mathematica's composition operator is `@*`. Let's pull back both x_f and y_f.

In[13]:=

```
pullback[ϕ_, f_] := f@*ϕ;
pullback[ϕ, xf][{x, y}]
pullback[ϕ, yf][{x, y}]
```

Out[14]=

```
x Cos[θ] - y Sin[θ]
```

Out[15]=

```
y Cos[θ] + x Sin[θ]
```

Mathematica's Notation package serves us well, here.

In[16]:=
```
<< Notation`
```

In[17]:=
```
Notation[ ϕ_* f_  ⟺  pullback[ϕ_, f_] ]
```

In[18]:=
```
(ϕ* xf) [{x, y}]
(ϕ* yf) [{x, y}]
```

Out[18]=
```
x Cos[θ] - y Sin[θ]
```

Out[19]=
```
y Cos[θ] + x Sin[θ]
```

Smooth

A function g on the reals, i.e., in $C^\infty \mathbb{R}^n = \mathbb{R}^n \to \mathbb{R}$, is ***smooth*** because it has continuous derivatives in real-number spaces, an infinite number of derivatives for our purposes (https://mathworld.wolfram.-com/SmoothFunction.html). Lower orders of smoothness are appropriate in other applications.

How do we get from smooth in real-number spaces to smooth on manifolds?

A ***function*** $f : M \to \mathbb{R}$ is ***smooth*** if, for all charts $\varphi_\alpha : U \subset M \to \mathbb{R}^n$, $\left(f \circ \varphi_\alpha{}^{-1}\right) : \mathbb{R}^n \to \mathbb{R}$ is smooth.

A ***curve*** $\gamma : \mathbb{R} \to M$ is ***smooth*** if, for all functions $f : M \to \mathbb{R}$, $f\,\gamma : \mathbb{R} \to \mathbb{R}$ is smooth, i.e., in $C^\infty \mathbb{R}$.

A ***general function*** $\phi : M \to N$ from one manifold to another is ***smooth*** if f's being in $C^\infty N$ implies that the pullback $\phi^* f$ is in $C^\infty M$ (ϕ is not necessarily a chart, φ) .

Pushforward of a Tangent Vector $\phi_* : T_p M \to T_{\phi(p)} N$

Let $\phi : M \to N$, not necessarily a chart, φ. Recall the pullback for functions $\phi^* : C^\infty N \to C^\infty M$.

Define the ***pushforward*** $(\phi_* v_p) : T_{\phi(p)} N$ ***of the tangent vector*** $v_p : T_p M$ by its action on the pullback of a function:

$$\left(\phi_* v_p\right) f = v_p(\phi^* f) \text{ for any } f : C^*(N) \tag{27}$$

One might write $(\phi_* v_p) = (\phi_* v)_{\phi(p)}$ via sneak preview to pushforward of vector fields.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[20]=

$$
\begin{tikzcd}
T_p M \arrow[r, "\phi_*"] & T_{\phi(p)} N \\
C^\infty M \arrow[dr, "v_p"'] & C^\infty N \arrow[l, "\phi^*"'] \arrow[dl, "\phi_* v_p"] \\
& \mathbb{R}
\end{tikzcd}
$$

Check Types

The right-hand side of Equation 27, $v_p(\phi^* f)$, is a real number, a directional derivative in M.

$$v_p(\phi^* f) = [(v_p : C^\infty M \to \mathbb{R})\{[\phi^* : C^\infty N \to C^\infty M][f : C^\infty N] : C^\infty M\}] : \mathbb{R} \tag{28}$$

The left-hand side, $(\phi_* v_p) f$, must also be the same real number, this time, the directional derivative in N.

$$(\phi_* v_p) f = \{[\{\phi_* : (C^\infty M \to \mathbb{R}) \to (C^\infty N \to \mathbb{R})\}(v_p : C^\infty M \to \mathbb{R})] : C^\infty N \to \mathbb{R}\}[f : C^\infty N] : \mathbb{R} \tag{29}$$

The types $C^\infty M \to \mathbb{R}$ and $C^\infty N \to \mathbb{R}$, as sets, are just $T_p M$ and $T_{\phi(p)} N$, so, less noisily:

$$(\phi_* v_p) f = [\{\phi_* : T_p M \to T_{\phi(p)} N\}(v_p : T_p M)][f : C^\infty N] : \mathbb{R} \tag{30}$$

Covariance

Just as $\phi^* : C^\infty N \to C^\infty M$ pulls back a contravariant function $f : C^\infty N$ to a function $\phi^* f : C^\infty M$ backward through $\phi : M \to N$, $\phi_* : T_p M \to T_{\phi(p)} N$ pushes a tangent vector $v_p : C^\infty M \to \mathbb{R}$ forward through the same $\phi : M \to N$ to a tangent vector $\phi_* v_p : C^\infty N \to \mathbb{R}$.

Tangent vectors in $T_p M = \{v_p : C^\infty M \to \mathbb{R} \mid p \in M\}$ are ***covariant*** because they are pushed forward, upstream, along the grain, in the same direction as $\phi : M \to N$, to the tangent space $T_{\phi(p)} N = \{v_{\phi(p)} : C^\infty N \to \mathbb{R} \mid \phi(p) \in N\}$ of the destination manifold N.

Example (Exercise 16 in GKG)

A canonical example of a tangent vector is the tangent to a curve. Push it forward to another manifold and check that it meets expectations.

Prove $(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t))$.

Left-hand Side

$(\phi\circ\gamma)'(t)$ is a tangent vector. Apply it to a function f.

$(\phi\circ\gamma)'(t)\,f = \frac{d}{dt} f(\phi(\gamma(t))) = \frac{d}{dt}(f\circ\phi\circ\gamma)(t) \in \mathbb{R}$

Right-hand Side

$\phi_*(\gamma'(t)) = \gamma'(t)\,(\phi^* f) = \gamma'(t)\,(f\circ\phi) = \frac{d}{dt}(f\circ\phi)\,(\gamma(t)) = \frac{d}{dt}(f\circ\phi\circ\gamma)\,(t) \in \mathbb{R}$

by associativity of function composition.

Lie Bracket $[v, w] = v\circ w - w\circ v : C^\infty M \to C^\infty M$

$[v, w]$ is a vector field constructed from two other vector fields v and w, defined so that

$$[v, w]\,f \overset{\text{def}}{=} v\,(w f) - w\,(v f) \tag{31}$$

with v and w vector fields in $C^\infty M \to C^\infty M$, and all juxtapositions denoting function application, with and without parentheses.

Recall that $v f : C^\infty M$ is a directional derivative with an invisible hole, waiting for a point $p : M$. That has exactly the form of a $C^\infty M$, so it has the correct type for an input to another vector field, w. The notion and the notation make sense.

Get a point-free notation by sneaking f out of the right-hand side of Equation 31, being careful to notice that expressions vw and wv sneakily denote function composition and not multiplication of vector fields, *which is not defined*. Subtraction of vector fields *is* defined via rule **VF +** in *Arithmetic on Vector Fields*, above.

$$[v, w] = v\circ w - w\circ v \overset{\text{def}}{=} v w - w v \tag{32}$$

One sees that latter notation, $vw - wv$, frequently. I prefer the less ambiguous $v\circ w - w\circ v$. We already have juxtaposition ambiguously representing both function application and various kinds of multiplication and probably should not add function composition as a third ambiguity. Ambiguity slows down reading. It's the lazy writer's way to save ink, but it is a roadblock to readers.

Lie Bracket is a Vector Field

Linearity is almost obvious. The following proof of Leibniz's Law closely follows that on pg. 36 of GKG, with some of our own refinements.

- Invoke **C^∞ *** from *Arithmetic on Functions in C^∞ M* and **LEIBV** from *Laws for Vector Fields.*

- f, g, $v(f)$, $w(f)$, $v(g)$, $w(g)$, $(v \circ w)(f)$, $(v \circ w)(g)$, $(w \circ v)(f)$, $(w \circ v)(g)$, are all functions in $C^\infty M$, writing extra parentheses to avoid confusion with multiplication of functions.
- Multiplication of functions is commutative, so the blue cancels the red, below.
- Composition of vector fields is *not* commutative (else we wouldn't need the Lie bracket).

$(v \circ w - w \circ v)(f g)$

$= v(g w(f) + f w(g)) - w(g v(f) - f v(g))$

$= g((v \circ w)(f)) + v(g)\, w(f) + f((v \circ w)(g)) + v(f)\, w(g) - g((w \circ v)(f)) - w(g)\, v(f) - f((w \circ v)(g)) - w(f)\, v(g)$

$= g((v \circ w)(f)) + f((v \circ w)(g)) - g((w \circ v)(f)) - f((w \circ v)(g))$

$= g[v, w](f) + f[v, w](g)$

Example: Exercise 22 in GKG

Compute the Lie bracket of the vector fields defined below, writing the vector fields as functions of a named parameter, f.

In[21]:=
```
ClearAll[v, w, lieBracket];
v[f_] := (x ∂_x f + y ∂_y f) / Sqrt[x^2 + y^2];
w[f_] := (x ∂_y f - y ∂_x f) / Sqrt[x^2 + y^2];
lieBracket[v_, w_] := f ↦ (v@*w)@f - (w@*v)@f;
lieBracket[v, w]@f[x, y] // FullSimplify
```

Out[25]=

$$\frac{-x\, f^{(0,1)}[x, y] + y\, f^{(1,0)}[x, y]}{x^2 + y^2}$$

Symbolical Leibniz's Law

For these particular vector fields v and w, *Mathematica* verifies Leibniz's Law, with `@*` denoting function composition and `@` denoting function application.

In[26]:=
```
((v@*w)@(f[x, y] × g[x, y]) - (w@*v)@(f[x, y] × g[x, y])) // FullSimplify
```

Out[26]=

$$\frac{g[x, y]\left(-x\, f^{(0,1)}[x, y] + y\, f^{(1,0)}[x, y]\right) + f[x, y]\left(-x\, g^{(0,1)}[x, y] + y\, g^{(1,0)}[x, y]\right)}{x^2 + y^2}$$

In[27]:=
```
f[x, y] ((v@*w)@g[x, y] - (w@*v)@g[x, y]) +
  g[x, y] ((v@*w)@f[x, y] - (w@*v)@f[x, y]) // FullSimplify
```

Out[27]=

$$\frac{g[x, y] \left(-x f^{(0,1)}[x, y] + y f^{(1,0)}[x, y]\right) + f[x, y] \left(-x g^{(0,1)}[x, y] + y g^{(1,0)}[x, y]\right)}{x^2 + y^2}$$

More general cases are not so straightforward due to subtleties of *Mathematica*, but we have a proof in prose, so we're done for now.

Numerical Example

Pick a function from the *Mathematica* documentation for `Plot3D`. Apply the Lie bracket above to it and plot the results.

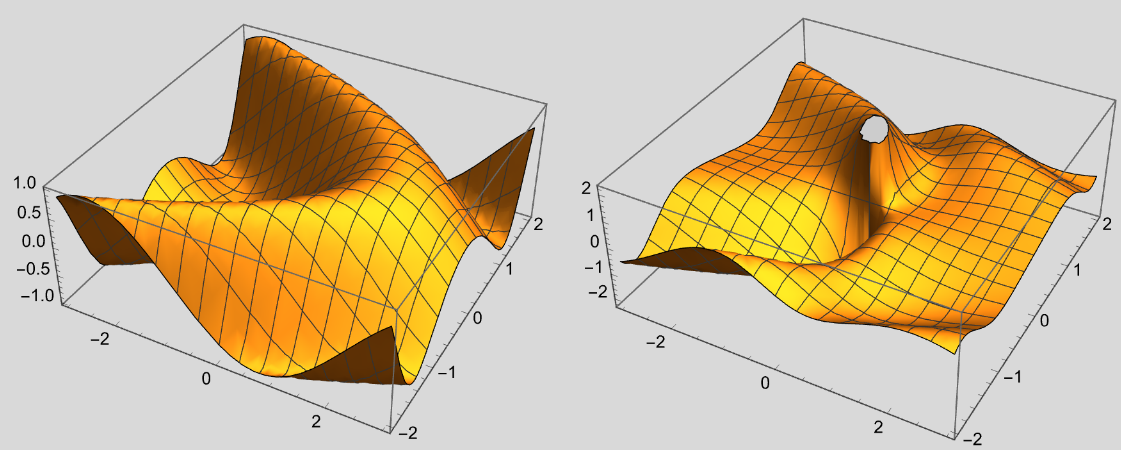
In[28]:=
```
lieBracket[v, w]@Sin[x + y^2] // FullSimplify
```

Out[28]=

$$\frac{(1 - 2 x) y \operatorname{Cos}\left[x + y^2\right]}{x^2 + y^2}$$

In[29]:=
```
GraphicsRow[{Plot3D[Sin[x + y^2], {x, -3, 3}, {y, -2, 2}],
  Plot3D[Evaluate@lieBracket[v, w]@Sin[x + y^2], {x, -3, 3}, {y, -2, 2}]}]
```

Out[29]=



Jacobi Identity (Exercise 24 in GKG)

The easiest way to prove the Jacobi Identity in *Mathematica* is to spell it out by hand because *Mathematica* does not automatically define addition of functions, **VF +**. Function composition is associative, so we don't need a lot of parentheses.

In[30]:=
```
Module[{u, v, w, f},
 u@*v@*w@f - v@*w@*u@f - (u@*w@*v@f - w@*v@*u@f) + v@*w@*u@f - w@*u@*v@f -
  (v@*u@*w@f - u@*w@*v@f) + w@*u@*v@f - u@*v@*w@f - (w@*v@*u@f - v@*u@*w@f)]
```
Out[30]=
```
0
```

Exterior Derivative $d\!\!d f = v \longmapsto vf : \text{VM} \to C^\infty M$

Applying a vector field $v : C^\infty M \to C^\infty M = VM$ to a function $f : C^\infty M$ yields $vf : C^\infty M$, a directional derivative (DD) with an invisible hole waiting for a point $p \in M$. Apply $vf : C^\infty M$ to a $p \in M$ to get an $\mathbb{R}$, the DD of f at p in the direction of v.

$vf = (vf)(\square) : C^\infty M \subset (M \to \mathbb{R})$

$vfp \overset{\text{def}}{=} v_p f : \mathbb{R} \overset{\text{trad}}{=} \nabla f(p) \cdot v$

The last formula on the right is in traditional "grad f dot v" notation in $\mathbb{R}^n$, assuming any ol' chart.

Start over. Take out $p \in M$ and put back the hole. Abstract out instead the vector field $v : VM \subset C^\infty M \to C^\infty M$ into a function parameter of type VM, and define ***the differential 1-form*** $d\!\!d f$ or ***exterior derivative of*** f as **the function of type** $VM \to C^\infty M$ **that sends a vector field** $v : VM$ **to** $vf : C^\infty M$, that is, such that $d\!\!d f v = (d\!\!d f)\, v = vf : C^\infty M$ with a hole, $\square$, for $p \in M$. The type of $d\!\!d f$, manifestly, is $VM \to C^\infty M$, sending a vector field in VM to a function in $C^\infty M$ that's waiting for a $p \in M$.

$$d\!\!d f \overset{\text{def}}{=} v \longmapsto (vf : C^\infty M) \overset{\text{trad}}{=} v \longmapsto \nabla f(\square) \cdot v \qquad (33)$$

Turn this around and fill the hole for $p \in M$ first, leaving a hole for $v \in VM$.

Take $d\!\!d f$ as an indivisible unit of notation so that $d\!\!d f_p$ means $(d\!\!d f)_p = (d\!\!d f)(p)$.

$$d\!\!d f_p \overset{\text{def}}{=} v \longmapsto (vfp : \mathbb{R}) \overset{\text{trad}}{=} v \longmapsto \nabla f(p) \cdot v \qquad (34)$$

Check Types

$$d\!\!d f_p \overset{\text{def}}{=} [v \longmapsto (vfp : \mathbb{R})] : VM \to \mathbb{R} \qquad (35)$$

Linearity

$d\!\!d f$ is manifestly linear in vector fields v and w, with the **scalars of linearity being functions** $\boldsymbol{g}$ **in** $\boldsymbol{C^\infty M}$. This is easies to see in some ol' chart:

- Dot product distributes over vector addition.

$$\text{đ} f(v+w) \overset{\text{trad}}{=} \nabla f(\square)\cdot(v+w) \overset{\text{trad}}{=} \nabla f(\square)\cdot v + \nabla f(\square)\cdot w = \text{đ} f v + \text{đ} f w \tag{36}$$

- Dot product commutes with multiplication by scalar function $g(\square)$.

$$\text{đ} f(gv) \overset{\text{trad}}{=} \nabla f(\square)\cdot(g(\square)\, v) \overset{\text{trad}}{=} g(\square)\,(\nabla f(\square)\cdot v) = g\, \text{đ} f v \tag{37}$$

Laws for Exterior Derivative

Because $\text{đ} f$ just abstracts a vector field v from expressions like $v f$, the following are easy to derive from *Arithmetic on Vector Fields* and *Laws for Vector Fields* above.

Abstract the function part f of $\text{đ} f$ and take the Laws above as defining the naked $\text{đ} : C^\infty M \to (VM \to C^\infty M)$ operator.

$$\textbf{LIND+} :: \text{đ}(f+g) = \text{đ} f + \text{đ} g \tag{38}$$

$$\textbf{LINR}{*} :: \text{đ}(\alpha f) = \alpha\, \text{đ} f \tag{39}$$

$$\textbf{LIND}{*} :: (f+g)\, \text{đ} h = f\, \text{đ} h + g\, \text{đ} h \tag{40}$$

$$\textbf{LEIBD} :: \text{đ}(f g) = f\, \text{đ} g + g\, \text{đ} f \tag{41}$$

Basis for Differential 1-Forms $\text{đ} x^\mu$

Given a chart (any ol' chart will do) and a function $f : C^*(M)$, define a basis set, $\text{đ} x^\mu$, for differential 1-forms (μ is an abstract index):

$$\text{đ} f \overset{\text{def}}{=} (\partial_\mu f)\, \text{đ} x^\mu \tag{42}$$

Remember to sum on the down-up index pair μ. Again, μ is an ***abstract index***, valid in and necessary for any ol' chart, as opposed to a concrete index that presupposes a particular chart. This works because the manifold itself has the same dimension n as the target $\mathbb{R}^n$ of any ol' chart.

The only interesting property of this basis is

$$\text{đ} x^\mu (\partial_\nu) = \delta^\mu_\nu \tag{43}$$

(no sum). The Kronecker δ^μ_ν is 1 when $\mu = \nu$ and zero otherwise (ν is Greek "nu," despite looking like v, English/Latin "vee").

Take the time to unpack the types on these. See GKG pg. 43 ff for much more.

1-Form $\omega : VM \to C^\infty M$

1-form has the same type as an exterior derivative. It is a generalization of that idea, a generalization that does not specify the function $f : C^\infty M$.

Leave the comfort zone of $\mathbb{R}^n$ for M. Assume dimension n and forget about the particular chart (any ol' chart will do). *Define* a ***1-form*** $\omega : VM \to C^\infty M$, abstractly, as any any ol' linear function ω that sends a vector field $v : VM$ to a function $\omega v : C^\infty M$ that, in turn, has an invisible hole for a point $p \in M$, and with the scalars of linearity being functions in $C^\infty M$.

Vector fields inhabit a vector space. 1-forms inhabit the dual space in the sense of linear algebra.

Linearity

1-forms live in a linear space, meaning that sums of 1-forms are defined and multiplying a 1-form by a scalar object is defined. The scalar objects for 1-forms are functions in $C^\infty M$.

- Addition via function application and $\mathbf{C^\infty +}$:

$$\omega(v + w) = \omega v + \omega w \tag{44}$$

- Multiplication by a scalar object, a function $g : C^\infty M$. Let $g v$ denote g times a vector field $v : \mathrm{VM}$, with and without $*$ for multiplication and parentheses:

$$\omega(g v) = g * \omega(v) = g * (\omega v) = g \omega v \tag{45}$$

Check Types

$\omega\,[\text{applied to}]\,(g\,[\mathbf{C^\infty\ VF} *]\,v) =$

$g\,[\mathbf{C^\infty} *,\ \text{commuted}] * \omega\,[\text{applied to}]\,(v) =$

$(g : C^\infty M) * ((\omega v) : C^\infty M) = g \omega v : C^\infty M$

$g \omega v : C^\infty M$ is similar to $g\, df\, v : C^\infty M$, the $\mathbf{C^\infty} *$ product of (on the left) a function $g : C^\infty M$, which has an invisible hole for a point $p : M$, and (on the right) a directional derivative $df\, v : C^\infty M$, which also has an invisible hole for the same point $p : M$. Fill the holes and get a real number. That's why the entire expression $g \omega v$ has type $C^\infty M \subset (M \to \mathbb{R})$.

$$g \omega v \overset{\text{def}}{=} g(\square)[\mathbf{C^\infty} *]\,((\omega v)\,(\square)) \tag{46}$$

Exterior Derivative df is a 1-Form

When you see any ol' 1-form ω, think of an exterior derivative df. They have the same types. With any ol' 1-form ω, we just don't specify a function $f : C^\infty M$. But every df is a 1-form. Though linearity of df was demonstrated on some ol' chart above, let's demonstrate linearity forgetting the chart.

$d\!\!/ f(v+w) = d\!\!/ f\,v + d\!\!/ f\,w = (d\!\!/ f)(v) + (d\!\!/ f)(w)$, by distributivity of function application, with and without noisy parentheses. $d\!\!/$ binds very tightly.

$$(gv)\,f = g(vf) \tag{47}$$

Check Types

$$((g : C^\infty M)\, v : (C^\infty M \to C^\infty M))\,(f : C^\infty M) = g(vf) \tag{48}$$

$$(g\,[\mathbf{C^\infty\ VF}*]\,v)\,[\text{applied to}]\,f = g\,[\mathbf{C^\infty\ VF}*]\,(v\,[\text{applied to}]f) \tag{49}$$

Arithmetic on 1-Forms

$$\mathbf{1\,F+} :: (\omega + \upsilon)\,v \overset{\text{def}}{=} \omega v + \upsilon\,v \tag{50}$$

υ, a 1-form, is Greek upsilon, despite looking like English/Latin u.

Check Types

$(\omega : (VM \to C^\infty M) + \upsilon : (VM \to C^\infty M))\,(v : VM) \overset{\text{def}}{=}$
$(\omega v + \upsilon v) : C^\infty M$

$(\omega + \upsilon)\,[\text{applied to}]\,v \overset{\text{def}}{=}$
$\omega\,[\text{applied to}]\,v + [\mathbf{C^\infty +}]\,\upsilon\,[\text{applied to}]\,v$

$$\mathbf{C^\infty\,1\,F*} :: (f\,\omega)\,v \overset{\text{def}}{=} f(\omega v) = f\,\omega v = f\,[\mathbf{C^\infty\ VF}*]\omega\,[\text{applied to}]\,v \tag{51}$$

Set of all 1-Forms on M $: \Omega^1 M = \{\omega : VM \to C^\infty M\}$

Just a definition of the indivisible symbol $\Omega^1 M = \{\omega : VM \to C^\infty M\}$, like the symbol $VM = C^\infty M \to C^\infty M$.

Cotangent Vector $\omega_p : T_p M \to \mathbb{R}$

A 1-form $\omega : VM \to C^\infty M$ sends a vector field $v : C^\infty M \to C^\infty M = VM$ to something with the same type as a directional-derivative-with-a-hole-for-a-point $p : M$, $\omega v \,\square : C^\infty M \subset M \to \mathbb{R}$. $\omega v p$ is a real number that has the same type as a directional derivative.

Define a ***cotangent vector*** $\omega_p : (C^\infty M \to R)$ as a linear gadget that applies a 1-form $\omega : VM \to C^\infty M$ to a vector field $v : VM$ at the point $p : M$. *En passant*, it entails a tangent vector, $v_p \,\square : C^\infty M \to \mathbb{R}$, having thrown away the hole for $f : C^\infty M$ because the 1-form ω doesn't specify a function. All together, get a number:

$$\omega_p\, v_p \overset{\text{def}}{=} \omega v p = \omega(v)\,(p) : \mathbb{R} \tag{52}$$

Cotangent Space $T_p^* M = \{\omega_p : T_p M \to \mathbb{R} \mid p \in M\}$

Just as the tangent space $T_p M = \{v_p : C^\infty M \to \mathbb{R} \mid p \in M\}$ is defined as the set of all tangent vectors $v_p : C^\infty M \to \mathbb{R}$ at the point $p : M$, define the ***cotangent space*** $T_p^*(M)$ as the set of all cotangent vectors at p. This is the dual space to $T_p M$ in the sense of linear algebra.

Pullback of a Cotangent Vector $\phi^* : T^*_{\phi(p)} N \to T_p^* M$

Just as are functions in $C^\infty N$, cotangent vectors are contravariant w.r.t. functions ϕ between manifolds M and N.

Recall the pushforward $\phi_* v_p : T_{\phi(p)} N$ of a tangent vector $v_p : T_p M$, defined via the function pullback $\phi^* : C^* N \to C^* M$

$$(\phi_* v_p)\, f = v_p(\phi^* f) \tag{53}$$

$$(\phi_* v_p : C^\infty N \to \mathbb{R})\,(f : C^\infty N) : \mathbb{R} = (v_p : C^\infty M \to \mathbb{R})\,(\phi^* f : C^\infty M) : \mathbb{R}) \tag{54}$$

with $\phi : M \to N$, not necessarily a chart, φ.

Likewise, define the **pullback** $(\phi^* \omega_{\phi(p)})$ **of a cotangent vector** $\omega_{\phi(p)} : T^*_{\phi(p)} N$ to a new cotangent vector $(\phi^* \omega)_p : T_p^* M$, via the action of $\omega_{\phi(p)}$ on the pushforward $v_{\phi(p)} = \phi_* v_p : T_{\phi(p)} N$ of a tangent vector $v_p : T_P M$. Let $q = \phi(p)$ to shorten things.

$$\omega_q\, v_q = \omega_q(\phi_* v_p) \overset{\text{def}}{=} (\phi^* \omega_q)\, v_p \overset{\text{def}}{=} (\phi^* \omega)_p\, v_p : \mathbb{R} \tag{55}$$

All these quantities are real numbers.

$(\phi^* \omega)_p : T_p^* M$ is a sneak preview of the pullback of a 1-form.

Diagram

In the following diagram and others like it, the vertices are labeled with sets and the edges are labeled with functions.

Out[31]=

$$\begin{tikzcd}
T_p^* M \arrow[rr, "\phi_*"] \arrow[dr, "\phi^* \omega_{\phi(p)}"'] & & T_{\phi(p)}^* N \arrow[dl, "\omega_{\phi(p)}"] \\
& \mathbb{R} &
\end{tikzcd}$$

Pushforward of a Vector Field $\phi_* : VM \to VN$

Vector fields are covariant w.r.t. functions ϕ between manifolds, just as are tangent vectors.

Recall that the pushforward $(\phi_* v_p) : T_{\phi(p)} N$ of a tangent vector $v_p : T_p M$ is a tangent vector in $T_{\phi(p)} N$ that sends functions in $C^\infty N$ to reals.

Likewise, the pushforward $(\phi_* v) : VN$ of a vector field $v : VM$ is a vector field in VN that sends functions in $C^\infty N$ to functions in $C^\infty N$, functions that are waiting for a point $q : N$.

Pullback of 1-Form $\phi^* : \Omega^1 N \to \Omega^1 M$

1-forms are contravariant w.r.t. functions ϕ between manifolds, just as are cotangent vectors.

Let $\phi : M \to N$ be a mapping from manifold M to manifold N (not necessarily a chart, φ). Let $q : N = \phi(p : M)$.

Recall the pullback $(\phi^* \omega)_p : T_p^* M$ of a cotangent vector $\omega_q : T_q^* N$ defined by its action on the pushforward $\phi_* v_p : T_q N$ of a tangent vector $v_p : T_P M$.

$$\omega_q(\phi_* v_p) \overset{\text{def}}{=} (\phi^* \omega_q)\, v_p \overset{\text{def}}{=} (\phi^* \omega)_p\, v_p$$

Abstract away the points p and q, define the **pullback** $(\phi^* \omega)$ **of a 1-form** $\boldsymbol{\omega}$ by its action on the pushforward $(\phi_* v)$ of a vector *field* v.

$$(\phi^* \omega)\, v p = \omega(\phi_* v)\, \phi(p) = \omega(\phi_* v)\, q \tag{56}$$

Check Types

$$\{[(\phi^* \omega) : \Omega^1 M = VM \to C^\infty M][v : VM] : C^\infty M\}\{p : M\} : \mathbb{R}$$
$$= \{[\omega : \Omega^1 N = VN \to C^\infty N](\phi_* v : VN) : C^\infty N\}\{\phi(p) : N\} : \mathbb{R}$$

Pullback of a Differential $(\phi^* \mathbb{d} f) : \Omega^1 M$

The pullback of a differential is the differential of the pullback.

$$\phi^* \mathbb{d} f = \mathbb{d}(\phi^* f) \tag{57}$$

Check Types

Let $f : C^\infty N$ and remember that $\mathbb{d} : C^\infty M \to (VM \to C^\infty M)$

$$[(\phi^* (\mathbb{d} f : VN \to C^\infty N)) : VM \to C^\infty M] = [\mathbb{d}((\phi^* (f : C^\infty N)) : C^\infty M) : VM \to C^\infty M] \tag{58}$$

Less noisily:

$$[(\phi^* (\mathbb{d} f : \Omega^1 N)) : \Omega^1 M] = [\mathbb{d}((\phi^* (f : C^\infty N)) : C^\infty M) : \Omega^1 M] \tag{59}$$

remembering that $\Omega^1 N = VN \to C^\infty N$ and $\Omega^1 M = VM \to C^\infty M$.

Fill the Holes

$\phi^* \mathbb{d} f = \mathbb{d}(\phi^* f)$ has two invisible holes, one for a vector field in VM and then another for a point $p : M$. Fill the hole for p first and get something that waits for a vector field.

$$(\phi^* \mathbb{d} f)_p = (\mathbb{d}(\phi^* f))_p \tag{60}$$

Both sides of this equality have type $VM \to \mathbb{R}$. Given a vector field in VM, they yield the directional derivative of $(\phi^* f)$ in the direction of v at p.

Now, fill the hole for the vector field $v : VM$. Recall that the pullback of a 1-form is defined via the pushforward of a vector field, $(\phi^* \omega)\, v p = \omega(\phi_* v)\, \phi(p)$ or $(\phi^* \omega)_p = \omega_{\phi(p)}(\phi_* v)$. Let $\omega = \mathbb{d} f$ to get directional derivatives in both M and N, which must be equal real numbers:

$$(\phi^* \mathbb{d} f)_p\, v : \mathbb{R} = \mathbb{d} f_{\phi(p)}(\phi_* v) : \mathbb{R} \tag{61}$$

Check Types

$$[(\phi^* \mathbb{d} f)_p : VM \to \mathbb{R}][v : VM] : \mathbb{R} = [\mathbb{d} f_{\phi(p)} : VN \to \mathbb{R}][(\phi_* v) : VN] : \mathbb{R} \tag{62}$$

Exterior Algebra: ΛV (UNDONE)

The science of the wedge product; most things follow from the definition $\omega \wedge \mu = -\mu \wedge \omega$, or, more generally, $\omega \wedge \mu = (-1)^{pq}\, \mu \wedge \omega$ if ω is a p-form and μ is a q-form.

ΛV is the name of the algebra generated by the wedge product over vectors (usually cotangent vectors).

P-Forms (UNDONE)

pullback extends to p-forms, so p-forms are contravariant (Ex 47 GKG).

$\Lambda^p V$ is the subspace of ΛV of p-fold wedge products.

ΩM is the algebra generated by $\Omega^1 M$ (set of 1-forms on M) and the wedge product.

Let $\Omega^0 M$ be the set of functions $C^\infty M$, the set of coefficients of polynomials with wedge products.

2-forms and vectors can be identified only in $\mathbb{R}^3$; it’s an accident. Hodge-$\star$ relates them.

3-forms and 0-forms can be identified in $\mathbb{R}^3$.

Direct Sum and Tensor Product

https://www.math3ma.com/blog/the-tensor-product-demystified

Exterior Derivative in $\Omega^p M$ (UNDONE)

The exterior derivative of a function, $d f$, is the exterior derivative applied to members of $\Omega^0 M$, yielding 1-forms, members of $\Omega^1 M$.

The set of maps $d : \Omega^p M \to \Omega^{p+1} M$ that satisfy the following properties is a unique set.

$d : \Omega^0 M \to \Omega^1 M$ is as above (63)

LINΩ1 :: linearity with respect to $\mathbb{R}$, *i.e.*, $d(\omega + \mu) = d\omega + d\mu$ and $d(r\omega) = r\, d\omega$ for $r \in \mathbb{R}$ (64)

LEIBΩp :: $d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p\, \omega \wedge \mu$, where $\omega \in \Omega^p M$ and $\mu \in \Omega^1 M$ (65)

$d(d\omega) = 0$ for $\omega \in \Omega^1 M$ (66)

Gradient: $d : \Omega^0 \mathbb{R}^3 \to d : \Omega^1 \mathbb{R}^3$

Curl: $d : \Omega^1 \mathbb{R}^3 \to d : \Omega^2 \mathbb{R}^3$

Divergence: $d : \Omega^2 \mathbb{R}^3 \to d : \Omega^3 \mathbb{R}^3$

Maxwell’s Equations (UNDONE)

Metric (UNDONE)

Hodge Star (UNDONE)

Stoke’s Theorem (UNDONE)

DeRham Cohomology (UNDONE)

Lie Groups (UNDONE)

Lie Algebras (UNDONE)

Maxwell’s Equations Again (UNDONE)

Bundles and Connections

Bianchi Identity (UNDONE)

Yang-Mills Equation (UNDONE)

Link Invariants (UNDONE)

Chern-Simon Theory (UNDONE)

Semi-Riemannian Geometry (UNDONE)

Levi-Civita Connection (UNDONE)

Geodesics (UNDONE)

Clean Up

The following cell is marked “not Evaluatable” during development.

```
In[ ]:= RemoveNotation[ ϕ_* f_ ⟺ pullback[ϕ_, f_] ];
ClearAll[ϕ, x, y, xf, yf, pullback, lieBracket];
```