Type Cheats for Manifolds

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January 2024

Abstract

The modern coordinate-free approach to manifolds comes with a terse, tricky, ambiguous, non-classical notation and vocabulary. These can be barriers to understanding. Short expressions like $g\,\omega\,v$ entail multiple concepts like function application and multiplication in abstract function spaces. Learning this notation takes as much effort as learning the traditional notations of multivariable and vector calculus, if not more, because the modern theory spans multiple levels of abstraction.

The only pathway I know to creative fluidity with this topic is repetition and practice. Whether writing code, doing calculations, or doing proofs, one must get to the point where the types appear effortlessly in the mind's eye when one looks at an expression like $g\,\omega\,v$. The zoo of types is big: vectors, covectors, vector and covector fields, tangent and cotangent spaces, bundles, sections, pullbacks, pushforwards, differential forms, exterior derivatives, Lie brackets, Lie derivatives, Lie groups, Lie algebras, adjoints, wedge products, curves, connections, metrics, geodesics, tensors, curvature, Hodge duals, De Rham cohomologies, and more. We must know these types intrinsically to read and write code, books, and papers. Starting out, we'll just be painfully explicit the way we must be when writing Java or Haskell code. Eventually, one gets "muscle memory" for the types.

We address the notation as one addresses functional programs: *analyze the types*. In many cases, type analysis alone yields easy proofs and creative new directions. We include several demonstrations empowered by *Mathematica*'s symbolical and numerical capabilities.

Background

The following cheats are more-or-less standard, though there is plenty of variation and, I daresay, sloppiness and ambiguity, in the literature.

These are the types that have been most useful to me. They're compatible with the notation in Baez and Muniain, *Gauge Theory, Knots, and Gravitation* (GKG), and the exposition here roughly follows the order of that book.

Prerequisites

I presume here only that you know multivariable calculus, linear algebra, and basics of group theory as commonly taught to undergraduates.

A, B, …, X, Y, … refer to sets of elements a, b, …, x, y, …, respectively.

M, N, … refer to ***manifolds***, sets of abstract, featureless points p, q, …

One says that the point $p \in M$ is a member or ***element*** of the set M, defining $\in$. One also says $p : M$, p is ***in*** M, p ***inhabits*** M, p ***is an*** M, p ***has type*** M, all meaning the same thing.

The colon notation $p : M$ is a ***type signature***. The colon binds tightly, so $f : C^\infty(M) \subset M \to \mathbb{R}$ means $f : (C^\infty(M) \subset (M \to \mathbb{R}))$ and, *en passant*, refers to a subset relation between the two types $C^\infty(M)$ and $M \to \mathbb{R}$. Likewise, $\phi^* f = f \circ \phi : C^\infty(M)$ means $(\phi^* f = f \circ \phi) : C^\infty(M)$ and, *en passant*, refers to the equality of the two expressions $\phi^* f$ and $f \circ \phi$. Sometimes, we color type notations in a lighter ink to set them off from other text.

For our purposes, there is no difference between being ***of a type*** and being ***in a set***. We could point out lots of differences, but it's not helpful to do so here.

The functional type signature $f : A \to B$ declares that function f takes elements of set A and produces elements of set B. One says that f ***maps*** A to B, (*map* as a verb), f ***has type*** $A \to B$, f ***inhabits*** the type or set $A \to B$, f ***is an*** $A \to B$, f ***sends*** $a \in A$ to $b \in B$, etc. ***Function***, ***mapping***, and ***map*** (as a noun) usually mean the same thing.

The notation $f = a \in A \longmapsto b \in B$ means "f is a function of the parameter a in A that produces the result b in B." b often is a complex expression depending on a. The notation $a \longmapsto b$ alone, with or without the sets, is an anonymous function, and it means "the function of a that produces b" or "a function of a that produces b."

Abstraction means multiple things, often changing a type signature.

1. generalizing some realm of thought, leaving out details
2. pulling data out of an expression into a function and parameters, e.g., $x + y \overset{\text{abst}}{\Longrightarrow} f(x, y) : \mathbb{R} \to \mathbb{R} \to \mathbb{R} = x + y$
3. taking away parts of a definition, e.g., abstracting $p : M$ from $v_p(f) : \mathbb{R} = (vf)(p)$ yields a new definition $vf : M \to \mathbb{R}$.

These three meanings are related but different.

This cheat sheet talks mostly about real numbers. We could do likewise for complex numbers, as well. Just substitute $\mathbb{C}$ for $\mathbb{R}$.

$f : C^\infty(M) \subset (M \to \mathbb{R})$

At the front door of the Manifold Zoo is the set of infinitely differentiable C^∞ functions from the manifold M to the reals $\mathbb{R}$.

See later that *curves* are certain functions in $\mathbb{R} \to M$, so there is a way to get back.

What could it possibly mean to differentiate on M, a set of abstract, featureless points? One needs only a little topology.

Chart $\phi : (U \subset M) \to \mathbb{R}^n$

A ***chart*** ϕ is a continuous function with a continuous inverse from an ***open subset*** U of a manifold M to the n-dimensional Euclidean space of reals, $\mathbb{R}^n$. Continuity is defined precisely in topology, but we take it as understood.

Think of a chart of the World on flat paper, say in the Mercator projection. Go back and forth from numerical coordinates on the paper to points on the globe.

A chart implies a ***coordinate system***. Find the coordinates of any point or the components of a vector in the coordinate system. Intuit ***open subset*** as a "ball" of points in M without its boundary. This intuition is perfect if the manifold is itself $\mathbb{R}^n$, but is inadequate generally. The intuition is good enough for practical applications where one actually works in $\mathbb{R}^n$ as much as possible.

$\left(f \circ \phi^{-1}\right) : C^\infty(\mathbb{R}^n) \subset (\mathbb{R}^n \to \mathbb{R})$

(This gadget has no name of its own)

Any chart ϕ takes a point $p \in U \subset M$ and returns an $\mathbb{R}^n$. The inverse of the chart ϕ^{-1} goes the other way, from $\mathbb{R}^n$, the set of n-tuples of real numbers, to $p \in U \subset M$ (charts must be invertible). f picks up that p right away and returns an $\mathbb{R}$. So the whole composition $\left(f \circ \phi^{-1}\right)$ goes straight from $\mathbb{R}^n$ to $\mathbb{R}$ with M the forgotten middle of the sandwich.

Calculate derivatives by ordinary multivariable calculus through $\left(f \circ \phi^{-1}\right) : C^\infty(\mathbb{R}^n)$. Suddenly, $C^\infty(M)$ means something. Under practical circumstances, **any ol' chart will do** because all charts have the same dimension n.

$$f \text{ is } C^\infty(M) \text{ iff } \left(f \circ \phi^{-1}\right) : C^\infty(\mathbb{R}^n) \text{ for all charts.} \tag{1}$$

Point-Free Notation

Read $\left(f \circ \phi^{-1}\right)$ from right-to-left, as with all notations via $\circ$. The ***composition*** $\left(f \circ \phi^{-1}\right)$ is a new function in its own right. Apply it to arguments on its right, as in $\left(f \circ \phi^{-1}\right)(x)$. That feeds $x \in \mathbb{R}^n$ first to ϕ^{-1}, getting a point p in M, then p to f, like this $\left(f\left(\phi^{-1}(x)\right)\right)$. That's harder to read and write than $\left(f \circ \phi^{-1}\right)(x)$, so we go with the latter. Take off the (x) to get ***point-free notation*** $\left(f \circ \phi^{-1}\right)$. No longer bother to mention the "point" argument x. *Point* here is in a general sense, not to mean specifically a point p in M.

Arithmetic on Functions in $C^\infty(M)$

Let f and g be in $C^\infty(M) \subset M \to \mathbb{R}$. Let double-colon separate the name of a rule, like **C^∞ +**, from the definition of the rule.

$$\mathbf{C^\infty +} :: f + g : C^\infty(M) \quad (2)$$

is a new function in $C^\infty(M)$ that produces $f(p) + g(p) \in \mathbb{R}$, the sum of two real numbers $f(p)$ and $g(p)$, for any $p \in M$.

$$\mathbf{C^\infty *} :: f g : C^\infty(M) \quad (3)$$

is a new function in $C^\infty(M)$ that produces $f(p)\, g(p) \in \mathbb{R}$, the product of two real numbers $f(p)$ and $g(p)$, for any $p \in M$. $f g$ does not mean $f(g)$, f applied to g, even though $v f$ does mean $v(f)$, v applied to f. **The notation is ambiguous between multiplication and function application without the types!**

$$\mathbf{\mathbb{R} * C^\infty} :: \alpha f : C^\infty(M) \quad (4)$$

where $\alpha \in \mathbb{R}$, is a new function in $C^\infty(M)$ that produces $\alpha f(p) \in \mathbb{R}$ by multiplication of real numbers α and $f(p)$, for any point $p \in M$.

One says that functions producing values in $\mathbb{R}$ ***inherit*** ordinary, real-number arithmetic from $\mathbb{R}$.

Vector Field $v : C^\infty(M) \to C^\infty(M)$

Tangent Vector $v_p : C^\infty(M) \to \mathbb{R}$

Apply a vector field $v : C^\infty(M) \to C^\infty(M)$ to a function $f : C^\infty(M)$ to get a new function $v\, f\, \square : C^\infty(M) \subset M \to \mathbb{R}$ that has a ***hole***, $\square$, waiting for a point $p \in M$.

v, alone, has *two* holes, one for a function in $C^\infty(M)$ and one for a point in M.

$v = v_\square\, \square : (C^\infty(M) \to C^\infty(M)) \subset (C^\infty(M) \to M \to \mathbb{R})$

The holes are supposed to be invisible. No one writes them in like that, but this is a cheat sheet, designed to get you comfortable with invisible things.

Give v an f in $C^\infty(M)$ and get an $M \to \mathbb{R}$.

$v\, f : M \to \mathbb{R}$

Start over with just v. Give v the point $p : M$, *first*, and leave a hole for f. Get a new thing, v_p that waits for a $C^\infty(M)$ like f. v_p is a refinement of the $v_p\, f$ notation introduced above.

$v_p : C^\infty(M) \to \mathbb{R}$ such that $v_p\, f : \mathbb{R} \overset{\text{trad}}{=} \nabla f(p) \cdot v$ (13)

Interpret the right-hand side traditionally. Assume any ol' chart: $\nabla f(p)$ is the gradient of f evaluated at p, and v is a tuple of components.

v_p is a ***tangent vector***. Given any ol' chart, the tangent vector lives in the ***tangent plane***, a hyper-plane of type $\mathbb{R}^n$, at point p. One does ordinary calculus in the tangent plane.

Arithmetic and Laws for Tangent Vectors

Tangent vectors inherit arithmetic laws from vector fields, but the types change. In fact, *only* type analysis shows that in **LEIBTV** one must apply the open functions f and g to p.

LINTV +:: $v_p\,(f + g) = v_p\, f + v_p\, g$, both sides in $\mathbb{R}$ (14)

LINTV$\ast$:: $v_p\,(\alpha g) = \alpha\left(v_p g\right)$, both sides in $\mathbb{R}$ (15)

LEIBTV :: $v_p\,(f g) = \left(v_p\, f\right) g(p) + f(p)\left(v_p\, g\right)$, both sides in $\mathbb{R}$ (16)

TV +:: $\left(v_p + w_p\right) f = v_p\, f + w_p\, f$, both sides in $\mathbb{R}$ (17)

RTV$\ast$:: $\alpha v_p : C^\infty(M)$ such that $\left(\alpha v_p\right) f = \alpha\left(v_p\, f\right) \overset{\text{def}}{=} \alpha v_p\, f$, both sides in $\mathbb{R}$ (18)

for $\alpha \in \mathbb{R}$. **RTV**$\ast$ defines the precedence of $v_p\, f : \mathbb{R}$. Function application $v_p\, f$ binds more tightly than multiplication by α.

Tangent Space $T_p\, M = \{v_p : C^\infty(M) \to \mathbb{R} \mid p \in M\}$

$T_p\, M$ is a type, the *set of tangent vectors at point* $p : M$.

$T_p\, M = \{v_p : C^\infty(M) \to \mathbb{R} \mid p \in M\}$ (19)

The notation $T_p\, M$ looks like a product or a function application, but it's neither. Think of it as an indivisible symbol denoting the set. It's a disruptive and troubling bit of notation.

From rules **TV +** and **RTV**$\ast$, deduce that $T_p\, M$ is a vector space, i.e., satisfies all eight axioms of a vector space.

Curve $\gamma : \mathbb{R} \to M$

Tangent to a Curve $\gamma'(t) : C^\infty(M) \to \mathbb{R}$

Pullback $\phi^* f : C^\infty(M)$

Smooth

Pushforward $\phi_* v_p : T_p(M) \in T_{\phi(p)}(N)$

Let $\phi : M \to N$. Recall the pullback $\phi^* : C^\infty(N) \to C^\infty(M)$.

Let $v_p \in T_p M : C^\infty(M) \to \mathbb{R}$ be a tangent vector at $p \in M$. Define the ***pushforward of the tangent vector*** in terms of a pullback:

$$\phi_* v_p \in T_{\phi(p)} N \text{ such that } (\phi_* v_p) f = v_p(\phi^* f) \tag{27}$$

Unpack the types as follows. The right-hand side, $v_p(\phi^* f)$, is a real number, a directional derivative in M.

$$v_p(\phi^* f) = \left[(v_p : C^\infty(M) \to \mathbb{R})\{[\phi^* : C^\infty(N) \to C^\infty(M)][f : C^\infty(N)] : C^\infty(M)\}\right] : \mathbb{R} \tag{28}$$

So the left-hand side, $(\phi_* v_p) f$, must also be a real number, the same directional derivative.

$$(\phi_* v_p) f = \{[\{\phi_* : (C^\infty(M) \to \mathbb{R}) \to (C^\infty(N) \to \mathbb{R})\}(v_p : C^\infty(M) \to \mathbb{R})] : C^\infty(N)\}[f : C^\infty(N)] : \mathbb{R} \tag{29}$$

$C^\infty(M) \to \mathbb{R}$ and $C^\infty(N) \to \mathbb{R}$, as sets, are just $T_p(M)$ and $T_{\phi(p)}(N)$, so

$$(\phi_* v_p) f = [\{\phi_* : T_p(M) \to T_{\phi(p)}(N)\}(v_p : C^\infty(M) \to \mathbb{R})][f : C^\infty(N)] : \mathbb{R} \tag{30}$$

Just as $\phi^* : C^\infty(N) \to C^\infty(M)$ pulls back a function $f : C^\infty(N)$ to a function $\phi^* f : C^\infty(M)$ through $\phi : M \to N$, $\phi_* : T_p(M) \to T_{\phi(p)}(N)$ pushes a tangent vector $v_p : C^\infty(M) \to \mathbb{R}$ forward to a tangent vector $\phi_* v_p : C^\infty(N)$ through the same $\phi : M \to N$.

Covariance

Tangent vectors in $T_p M = \{v_p : C^\infty(M) \to \mathbb{R} \mid p \in M\}$ are ***covariant*** because they are pushed forward in the same direction as $\phi : M \to N$ to the tangent space $T_{\phi(p)} N = \{v_{\phi(p)} : C^\infty(N) \to \mathbb{R} \mid \phi(p) \in N\}$ of the destination manifold N.

Example (Exercise 16 in GKG)

Prove $(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t))$.

Left-hand Side

$(\phi \circ \gamma)$'(t) is a tangent vector. Apply it to a function f

$(\phi \circ \gamma)'(t)\, f = \frac{d}{dt} f(\phi(\gamma(t))) = \frac{d}{dt} (f \circ \phi \circ \gamma)(t) \in \mathbb{R}$

Right-hand Side

$\phi_*(\gamma'(t)) = \gamma'(t)\,(\phi^* f) = \gamma'(t)\,(f \circ \phi) = \frac{d}{dt} (f \circ \phi)(\gamma(t)) = \frac{d}{dt} (f \circ \phi \circ \gamma)(t) \in \mathbb{R}$

by associativity of function composition.

Lie Bracket $[v, w] = v \circ w - w \circ v : C^\infty(M) \to C^\infty(M)$

Exterior Derivative $d f = v \longmapsto v\, f : \mathrm{Vect}(M) \to C^\infty(M)$

1-Form $\mathrm{Vect}(M) \to C^\infty(M)$

Cotangent Vector $\omega_p : T_p\, M \to \mathbb{R}$

A 1-form $\omega : \mathrm{Vect}(M) \to C^\infty(M)$ takes a vector field $v : C^\infty(M) \to C^\infty(M) \in \mathrm{Vect}(M)$ to a directional derivative with a hole for a point $p : M$, $\omega\, v\, \square : C^\infty(M) \subset M \to \mathbb{R}$ and produces a number that’s similar to a directional derivative.

Define a ***cotangent vector*** $\omega_p : (C^\infty(M) \to R)$ as a linear gadget that applies a 1-form $\omega : \mathrm{Vect}(M) \to C^\infty(M)$ to a vector field $v : \mathrm{Vect}(M)$ and fills the hole with a point $p : M$. *En passant*, it entails a tangent vector, $v_p\, \square : C^\infty(M) \to \mathbb{R}$, but throws away the hole by assuming a 1-form ω, which doesn’t specify a function. The whole thing produces a number:

$$\omega_p\, v_p \overset{\text{def}}{=} \omega\, v\, p = \omega(v)\,(p) : \mathbb{R} \tag{50}$$

Cotangent Space $T_p^*\, M = \{\omega_p : T_p(M) \to \mathbb{R} \mid p \in M\}$

Just as we defined the tangent space $T_p\, M = \{v_p : C^\infty(M) \to \mathbb{R} \mid p \in M\}$ as the set of all tangent vectors $v_p : C^\infty(M) \to \mathbb{R}$ at the point $p : M$, we define the ***cotangent space*** $T_p^*(M)$ as the set of all cotangent vectors at p. You’ll recognize this as the dual space to $T_p(M)$ in the sense of linear algebra.

Clean Up

The following cell is marked “not Evaluatable” during development.

In[ ]:=
```
RemoveNotation[ ϕ_* f_ ⟺ pullback[ϕ_, f_] ];
ClearAll[ϕ, x, y, xf, yf, pullback, lieBracket];
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