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Euler–Rodrigues formula variations, quaternion conjugation and intrinsic connections

Jian S. Dai *

MOE key lab for mechanism theory and equipment design, International Centre for Advanced Mechanisms and Robotics, Tianjin University, PR China
Centre for Robotics Research, School of Natural Sciences and Mathematics, King's College London, University of London, United Kingdom

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ABSTRACT

This paper reviews the Euler–Rodrigues formula in the axis–angle representation of rotations, studies its variations and derivations in different mathematical forms as vectors, quaternions and Lie groups and investigates their intrinsic connections. The Euler–Rodrigues formula in the Taylor series expansion is presented and its use as an exponential map of Lie algebras is discussed particularly with a non-normalized vector. The connection between Euler–Rodrigues parameters and the Euler–Rodrigues formula is then demonstrated through quaternion conjugation and the equivalence between quaternion conjugation and an adjoint action of the Lie group is subsequently presented. The paper provides a rich reference for the Euler–Rodrigues formula, the variations and their connections and for their use in rigid body kinematics, dynamics and computer graphics.

1. Introduction

Euler–Rodrigues formula was first revealed in Euler's equations [1] published in 1775 in the way of change of direction cosines of a unit vector before and after a rotation. This was rediscovered independently by Rodrigues [2] in 1840 with Rodrigues parameters [3] of tangent of half the rotation angle attached with coordinates of the rotation axis, known as Rodrigues vector [4–6] sometimes called the vector–parameter [7], presenting a way for geometrically constructing a rotation matrix. The vector form of this formula was revealed by Gibbs [8], Bisshopp [9], and Bottema and Roth [10] in their presentation of the Rodrigues formulae in planar and spatial motion. In addition to Rodrigues parameters, Euler–Rodrigues parameters were revealed in the same paper [2] as the unit quaternion. It was illustrated by Cayley [11] that the rotation about an axis by an angle could be implemented by a quaternion transformation [12] that was again interpreted by Cayley [13] physically using Euler–Rodrigues parameters that we now know as the quaternion conjugation [14,15], this coincides with the result of using Rodrigues parameters in the Euler–Rodrigues formula, leading to Cayley transform [16] as a mapping between skew–symmetric matrices of Lie algebra elements and special orthogonal matrices of Lie group elements.

The Euler–Rodrigues formula for finite rotations [17,18] raised much interest in the second half of the 20th century. In 1969, Bisshop [9] studied the formula in vector form of the rotation tensor by presenting a derivation from rotating a vector about an axis by an angle. In 1979, Bottema and Roth [10] presented Rodrigues formulae for rigid body displacements of various motions and put forward vectorial representations. In 1980, Gray [4] reviewed Rodrigues' contribution to the combination of two rotations

* Tel.: +442078482321.
E-mail address: jian.dai@kcl.ac.uk.

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and presented a historical account of this development. In 1989, Cheng and Gupta [19] verified the original contribution of Euler and accounted a further contribution of Rodrigues based on Euler–Rodrigues parameters. Since 1980s, Euler–Rodrigues formula has been widely used in geometric algebra [20,21], theoretical kinematics [10,22,23], and robotics [24]. In modern mathematics, Euler–Rodrigues formula is used as an exponential map [25] that converts Lie algebra $so(3)$ into Lie group $SO(3)$, providing an algorithm for the exponential map without calculating the full matrix exponent [26–28] and for multi-body dynamics [29–32].

In the 21st century, Euler–Rodrigues formula continuously attracted broad interest. In 2003, Bauchau and Trainelli [33] developed an explicit expression of the rotation tensor in terms of vector parameterization based on the Euler–Rodrigues formula and in particular utilized tangent of half the angle of rotations [10,34]. In 2004 and 2006, Dai [3,35] reviewed Euler–Rodrigues parameters in the context of theoretical development of rigid body displacement historically. In 2007, Mebius [36] presented a way of obtaining the Euler–Rodrigues formula by substituting Euler–Rodrigues parameters in a 4×4 rotation matrix based on a quaternion representation. In 2008, Senan and O'Reilly [37] illustrated rotation tensors with a direct product of quaternions and examined the parameter constraint in Euler–Rodrigues parameters. In the same year, Norris [38] applied the Euler–Rodrigues formula to developing rotation of tensors in elasticity by projecting it onto the hexagonal symmetry defined by axes of rotations with Carton decomposition of rotation tensors [39]. In 2010, Müller [40] used a Cayley transformation to obtain a modified vector parameterization that represents an extension of the Rodrigues parameters, which reduces the computational complexity while increasing accuracy. In 2012, Kovács [41] gave a new derivation of the Euler–Rodrigues formula based on a matrix transformation of three continuous rotations. In the same year, Pujol [15,42] investigated the relation between the composition of rotations and the product of quaternions [43] and related the work to Cayley's early contribution [11,44] through Euler–Rodrigues parameters. Following various studies, the use of the Euler–Rodrigues formula and of the Euler–Rodrigues-parameters formulated unit-quaternion has been extended to a broad range of research topics including vector parameterization of rotations [33,40,45], rational motions [46–49], motion generation [50–52] and planning [53], kinematic mapping [54,55], orientation [56] and attitude estimation [57–59], mechanics [60], constraint analysis [61,62], reconfiguration [63,64], mechanism analysis [65–67] and synthesis [68,69], sensing [70] and computer graphics [71–73] and vision [74].

Though various studies were made, reconciliation of different versions of the Euler–Rodrigues formula and their derivations were vaguely known. This paper is to examine all Euler–Rodrigues formula variations, present their derivations and discuss their intrinsic connections to provide readers with a complete picture of variations and connections, leading to understanding of the Euler–Rodrigues formula in its variations and uses as an exponential map and a quaternion operator.

2. Geometrical interpretation of the Euler–Rodrigues formula

Rigid body rotation can be presented in the form of Rodrigues parameters [34,75] that integrate direction cosines of a rotation axis with tangent of half the rotation angle as three quantities in the form of

$$b_x = \tan\frac{1}{2}\theta s_x, \;\; b_y = \tan\frac{1}{2}\theta s_y, \;\; b_z = \tan\frac{1}{2}\theta s_z, \tag{1}$$

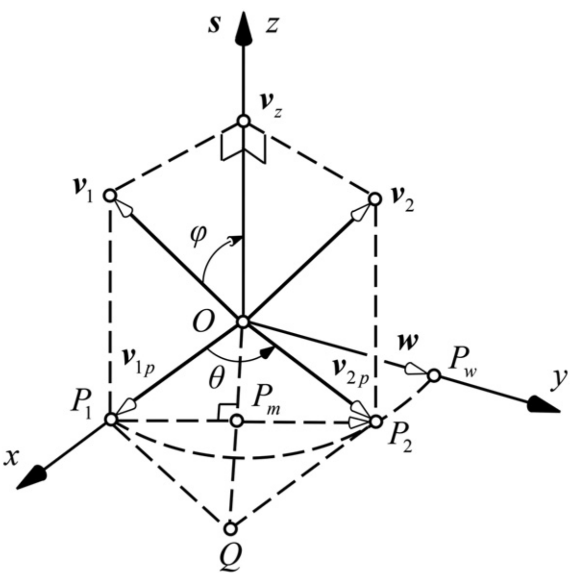

Fig. 1. Projected rhombus and the half-angle of rotation.

where $\boldsymbol{b} = (b_x, b_y, b_z)^{\mathrm{T}}$ is referred to as Rodrigues vector [4,6], the three quantities are known as Rodrigues parameters [10,14,19], where the axis of rotation is in the form of

$$\boldsymbol{s} = \begin{pmatrix} s_x, & s_y, & s_z \end{pmatrix}^{\mathrm{T}} \tag{2}$$

which is a unit vector.

The half-angle is an essential feature [3,10] of parameterization of rotations and of the measure of pure rotation in the most elegant representation of rotations in kinematics. This can be seen in Fig. 1.

In the figure, vector $\boldsymbol{v}_1$ with any magnitude rotates by angle θ about a unit axis $\boldsymbol{s}$ which is in line with axis $\boldsymbol{z}$ to form vector $\boldsymbol{v}_2$. Projection of vector $\boldsymbol{v}_1$ and that of its rotated vector $\boldsymbol{v}_2$ on xy-plane are presented as $\boldsymbol{v}_{1p}$ and $\boldsymbol{v}_{2p}$ with rotation angle θ. In this projection plane, a rhombus is formed by producing line P_1Q parallel to rotated vector projection $\boldsymbol{v}_{2p}$ and line P_2Q parallel to original vector projection $\boldsymbol{v}_{1p}$. Drawing diagonals P_1P_2 and OQ that are perpendicular to each other, intersection point P_m can be obtained. Tangent of half the rotation angle was then given as that by Rodrigues [2] and demonstrated again by Bottema and Roth [10] in the form of

$$\tan\frac{\theta}{2} = \frac{P_1P_m}{OP_m}. \tag{3}$$

This can further be replaced by diagonals of the rhombus in vector form of $\boldsymbol{v}_{2p} - \boldsymbol{v}_{1p}$ and $\boldsymbol{v}_{2p} + \boldsymbol{v}_{1p}$ as

$$\tan\frac{\theta}{2} = \frac{\left\|\boldsymbol{v}_{2p} - \boldsymbol{v}_{1p}\right\|}{\left\|\boldsymbol{v}_{2p} + \boldsymbol{v}_{1p}\right\|}. \tag{4}$$

The use of the half angle in the study of motions led to the Euler–Rodrigues formula and to the discovery of Euler–Rodrigues parameters. A geometrical interpretation of the Euler–Rodrigues formula is illustrated in Fig. 1, leading to rotated vector $\boldsymbol{v}_2$ as

$$\boldsymbol{v}_2 = \boldsymbol{v}_{2p} + \boldsymbol{v}_z \tag{5}$$

which is a combination of its projection on xy-plane and that on z-axis. Further from Fig. 1, vector $\boldsymbol{v}_2$ as a result of rotation from $\boldsymbol{v}_1$ can be given as

$$\boldsymbol{v}_2 = \boldsymbol{v}_{1p}\cos\theta + \boldsymbol{w}\sin\theta + \boldsymbol{v}_z \tag{6}$$

where

$$\boldsymbol{v}_{1p} = \boldsymbol{v}_1 - \boldsymbol{v}_z = \boldsymbol{v}_1 - (\boldsymbol{s}\cdot\boldsymbol{v}_1)\boldsymbol{s}, \tag{7}$$

and

$$\boldsymbol{w} = \boldsymbol{s}\times\boldsymbol{v}_1. \tag{8}$$

The above equation gives a skew-symmetric matrix $\boldsymbol{A}_s$ below having the effect of taking the vector cross product of $\boldsymbol{s}$ with a vector,

$$\boldsymbol{A}_s = [\boldsymbol{s}\times] = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix} \tag{9}$$

Here, matrix $\boldsymbol{A}_s$ gives Lie algebra $so(3)$ of $SO(3)$ in the form of a 3×3 skew–symmetric matrix [34]. Given the cross product by $\boldsymbol{s}$ as a linear operation on $\boldsymbol{w}\mapsto\boldsymbol{s}\times\boldsymbol{v}_1$ in the matrix form, it follows that

$$\boldsymbol{s}\times\boldsymbol{v}_1 = [\boldsymbol{s}\times]\boldsymbol{v}_1 = \boldsymbol{A}_s\boldsymbol{v}_1. \tag{10}$$

Substituting Eqs. (7) and (8) in Eq. (6) gives the following

$$\begin{aligned} \boldsymbol{v}_2 &= (\boldsymbol{v}_1 - (\boldsymbol{s}\cdot\boldsymbol{v}_1)\boldsymbol{s})\cos\theta + (\boldsymbol{s}\times\boldsymbol{v}_1)\sin\theta + (\boldsymbol{s}\cdot\boldsymbol{v}_1)\boldsymbol{s} \\ &= \boldsymbol{v}_1\cos\theta + (\boldsymbol{s}\times\boldsymbol{v}_1)\sin\theta + (\boldsymbol{s}\cdot\boldsymbol{v}_1)\boldsymbol{s}(1-\cos\theta) \end{aligned}. \tag{11}$$

This presents an action [34] of the Euler–Rodrigues formula [22,76] as an efficient algorithm for rotating a vector in space with a given axis and angle of the rotation. The above equation can be rewritten as

$$\begin{aligned}\boldsymbol{v}_2 &= \cos\theta\,\boldsymbol{v}_1 + \sin\theta[\boldsymbol{s}\times]\boldsymbol{v}_1 + (1-\cos\theta)\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{v}_1\\ &= \left(\cos\theta\boldsymbol{I} + \sin\theta[\boldsymbol{s}\times] + (1-\cos\theta)\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}}\right)\boldsymbol{v}_1\end{aligned} \tag{12}$$

where $\boldsymbol{I}$ is the 3×3 identify matrix and $\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}}$ is the outer product [77] or tensor product of two vectors. Factoring out $\boldsymbol{v}_1$ gives the Euler–Rodrigues formula [25,34] in matrix form as follows

$$\boldsymbol{R} = \cos\theta\boldsymbol{I} + \sin\theta[\boldsymbol{s}\times] + (1-\cos\theta)\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}}. \tag{13}$$

The variation of the Euler–Rodrigues rotation formula is hence presented. Since the skew–symmetric matrix $\boldsymbol{A}_s$ has the following property [34]

$$\boldsymbol{A}_s\boldsymbol{A}_s = \boldsymbol{s}\boldsymbol{s}^{\mathrm{T}} - \boldsymbol{I}, \tag{14}$$

the above Euler–Rodrigues formula can be rewritten in a standard form as

$$\boldsymbol{R} = \boldsymbol{I} + \sin\theta\boldsymbol{A}_s + (1-\cos\theta)\boldsymbol{A}_s\boldsymbol{A}_s. \tag{15}$$

This standard form of the formula has been used to construct a rotation matrix from the axis–angle representation of rotations. As such, the rotation matrix can be obtained as

$$\boldsymbol{R} = \begin{bmatrix} s_x^2 + \left(1-s_x^2\right)\mathrm{c}\theta & s_xs_y(1-\mathrm{c}\theta)-s_z\mathrm{s}\theta & s_xs_z(1-\mathrm{c}\theta)+s_y\mathrm{s}\theta \\ s_xs_y(1-\mathrm{c}\theta)+s_z\mathrm{s}\theta & s_y^2 + \left(1-s_y^2\right)\mathrm{c}\theta & s_ys_z(1-\mathrm{c}\theta)-s_x\mathrm{s}\theta \\ s_xs_z(1-\mathrm{c}\theta)-s_y\mathrm{s}\theta & s_ys_z(1-\mathrm{c}\theta)+s_x\mathrm{s}\theta & s_z^2 + \left(1-s_z^2\right)\mathrm{c}\theta \end{bmatrix}, \tag{16}$$

where $\mathrm{s}\theta = \sin\theta$ and $\mathrm{c}\theta = \cos\theta$. The matrix is used in projective geometry [78] and in most robotics literature [24]. In another way, the skew–symmetric matrix in Eq. (9) composed of Rodrigues parameters could be used in the Cayley transform [7,16,31,32] in place of the exponential map to generate a mapping between Lie algebra $so(3)$ elements and Lie group $SO(3)$ elements.

Considering the infinitesimal rotation $d\theta$, Eq. (12) can be rewritten while in the meantime taking into account of the matrix property expressed in Eq. (10), it follows that

$$\begin{aligned}\boldsymbol{v}_2 &= \left(\cos(d\theta)\boldsymbol{I} + \sin(d\theta)\boldsymbol{A}_s + (1-\cos(d\theta))\boldsymbol{A}_s^2\right)\boldsymbol{v}_1\\ &= (\boldsymbol{I} + d\theta\,\boldsymbol{A}_s)\boldsymbol{v}_1 = \boldsymbol{v}_1 + d\boldsymbol{v}_1\end{aligned}. \tag{17}$$

Thus, the following can be obtained

$$d\boldsymbol{v}_1 = \boldsymbol{A}_s\boldsymbol{v}_1 d\theta = \boldsymbol{s}\times\boldsymbol{v}_1 d\theta. \tag{18}$$

This infinitesimal rotation through the Euler–Rodrigues formula gives a linear velocity resulting from a rotation about rotation axis $\boldsymbol{s}$ with angular velocity $d\theta$ and presents the time derivative of a rotation matrix in terms of rotation axis $\boldsymbol{s}$ as

$$\boldsymbol{A}_s = \dot{\boldsymbol{R}}\boldsymbol{R}^{\mathrm{T}},$$

delivering Lie algebra $so(3)$ of rotation group $SO(3)$ as the infinitesimal generator of rotations and exhibiting the connection between the finite motion [45,55,79,80] and the infinitesimal motion [29,81,82]. This is to be further discussed in the exponential map presented in the following.

3. Algebraic interpretation and the exponential map

From the above, the relation between the finite and infinitesimal motion is presented by introducing the infinitesimal rotation to the Euler–Rodrigues formula. This can further be illustrated in the following by mapping a skew–symmetric matrix of a Lie algebra element into an orthogonal matrix of a Lie group using the Euler–Rodrigues formula.

3.1. Taylor series expansion and exponential map onto SO(3)

Algebraically, the Euler–Rodrigues formula can be derived by using the Taylor series expansion while considering the skew–symmetric matrix product property in the Appendix of [18] with the given axis $\boldsymbol{s} \in \mathbb{R}^3$ of a unit length and angle $\theta \in \mathbb{R}$. Using the standard matrix power series [83], it follows that

$$\begin{aligned} \boldsymbol{R} = e^{\theta \boldsymbol{A}_s} &= \sum_{k=0}^{\infty} \frac{(\theta \boldsymbol{A}_s)^k}{k!} = \boldsymbol{I} + \theta \boldsymbol{A}_s + \frac{1}{2}(\theta \boldsymbol{A}_s)^2 + \frac{1}{6}(\theta \boldsymbol{A}_s)^3 + \cdots \\ &= \boldsymbol{I} + \boldsymbol{A}_s\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + \boldsymbol{A}_s^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right) \\ &= \boldsymbol{I} + \sin\theta \boldsymbol{A}_s + (1 - \cos\theta)\boldsymbol{A}_s\boldsymbol{A}_s \end{aligned} \quad . \tag{19}$$

This presents the standard form of the Euler–Rodrigues formula in Eq. (15) again and has been widely used as an algorithm to compute the exponential map [25,28] converting an element of Lie algebra $so(3)$ into an element of Lie group $SO(3)$ as follows

$$\exp: \quad so(3) \rightarrow SO(3). \tag{20}$$

The exponential map from a skew–symmetric matrix gives an orthogonal matrix of determinant 1. In the standard form of the Euler–Rodrigues formula in Eq. (19), axis $\boldsymbol{s}$ of rotation and angle θ in the range $-\pi < \theta < \pi$ are thus defined in the case where matrix $\boldsymbol{R}$ is not equal to an identity matrix. When $\theta = 2k\pi$, it becomes that

$$e^{2k\pi \boldsymbol{A}_s} = \boldsymbol{I} \tag{21}$$

where $k \in \mathbb{Z}$, the exponential map is hence surjective and not commutative [27,84].

3.2. Exponential map of a non-normalized rotation axis

In the above, rotation axis $\boldsymbol{s}$ forming the skew–symmetric matrix is a unit vector and can be obtained from a rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where a_{ij} is an entry of the i-th row and j-th column. From this orthogonal matrix [18,49,85], non-normalized axis $\boldsymbol{u}$ of rotation can be obtained in the following as

$$\boldsymbol{A}_u = [\boldsymbol{u}\times] = \boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}} = \begin{bmatrix} 0 & a_{12} - a_{21} & a_{13} - a_{31} \\ a_{21} - a_{12} & 0 & a_{23} - a_{32} \\ a_{31} - a_{13} & a_{32} - a_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \tag{22}$$

and can be expressed in its vector form as

$$\boldsymbol{u} = \begin{pmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{pmatrix}. \tag{23}$$

This corresponds to the eigenvector of rotation matrix $\boldsymbol{R}$ associated with eigenvalue $\lambda = 1$. The magnitude of vector $\boldsymbol{u}$ is

$$\|\boldsymbol{u}\| = 2\sin\theta. \tag{24}$$

The unit vector $\boldsymbol{s}$ after normalizing the obtained rotation axis $\boldsymbol{u}$ is hence given

$$\boldsymbol{s} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = (\, l, \quad m, \quad n \,)^{\mathrm{T}}. \tag{25}$$

Thus, skew–symmetric matrix $\boldsymbol{A}_s$ in Eq. (9) can be obtained from the non-normalized rotation axis $\boldsymbol{u}$ as

$$\boldsymbol{A}_s = \frac{\boldsymbol{A}_u}{\|\boldsymbol{u}\|}, \tag{26}$$

where skew–symmetric matrix $\boldsymbol{A}_u$ is composed of components of vector $\boldsymbol{u}$ in the manner of Eq. (9). Considering the above in the Taylor series expansion (19), it follows that

$$e^{\theta \boldsymbol{A}_u} = \boldsymbol{I} + \sin(\|\boldsymbol{u}\|\theta)\frac{\boldsymbol{A}_u}{\|\boldsymbol{u}\|} + (1 - \cos(\|\boldsymbol{u}\|\theta))\frac{\boldsymbol{A}_u^2}{\|\boldsymbol{u}\|^2}. \tag{27}$$

This gives an exponential map [25] from a Lie algebra element to a Lie group element together with its normalization as another variation of the Euler–Rodrigues formula. The formula maps a non-normalized rotation axis into a rotation matrix.

4. Quaternion conjugation and equivalent Euler–Rodrigues formula operation

In addition to the geometrical and algebraic derivations, the Euler–Rodrigues formula can also be derived from conjugation of quaternions. With the Euler–Rodrigues parameters of integrating direction cosines of the rotation axis with the sine function of half the rotation angle, a quaternion [15,86,87] can be expressed in the form of

$$\boldsymbol{Q} = q_0 + \boldsymbol{q} = (\, q_0, \quad q_1, \quad q_2, \quad q_3\,) = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{s}, \tag{28}$$

where $\boldsymbol{s}$ is defined in Eq. (2), and four components are Euler–Rodrigues parameters forming the unit quaternion [2,71] in the following

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{29}$$

Its conjugate is

$$\boldsymbol{Q}^* = \boldsymbol{Q}^{-1} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\boldsymbol{s}. \tag{30}$$

The quaternions form a continuous group and are a Lie group. The group of unit quaternions is isomorphic to the group of three-dimensional rotations with a composition operation [49,55] of the rotation group. The unit quaternions form Lie group $Sp(1)$ in $\mathbb{H}$ field [78,83,88–90] that is isomorphic to topological group $SU(2)$ and a double cover of $SO(3)$ [27,91–93]. There exists a two-to-one surjective Lie group homomorphism from $SU(2)$ to $SO(3)$ sharing the same Lie algebra as that of Lie group $SO(3)$, leading to an extension and interpretation of the quaternion method [94,95] as the formalism in geometric algebra [6,7,96,97] and in use in computer graphics and mechanism analysis. Conjugation of unit quaternions can be written in the following form

$$\boldsymbol{V}' = \boldsymbol{Q}\boldsymbol{V}\boldsymbol{Q}^*, \tag{31}$$

where pure quaternion $\boldsymbol{V}$ is given in the form of

$$\boldsymbol{V} = 0 + \boldsymbol{v} \tag{32}$$

as a vector quaternion corresponding to $\boldsymbol{Q}(\pi\boldsymbol{s})$ [34]. The pure quaternions are Lie algebra elements with both well-defined addition and multiplication and meet two criteria as skew-commutative law and Jacobi identity. The pure quaternions constitute a subspace of $\mathbb{H}$ and are isomorphic to $\mathbb{R}^3$. Thus, the quaternion conjugation in Eq. (31) completes the Lie group action on its Lie algebra in pure quaternion form. It can be seen from the equation that quaternion $\boldsymbol{Q}$ and its conjugate $\boldsymbol{Q}^*$ are operators and pure quaternion $\boldsymbol{V}$ is an operand. The conjugation applies [34] when Lie algebra elements are in the form of pure quaternions or skew-symmetric matrices and displaces pure quaternion $\boldsymbol{V}$ into another pure quaternion $\boldsymbol{V}'$.

Substituting the quaternion in Eq. (28) and its conjugate in Eq. (30) in the conjugation in Eq. (31), it follows that

$$\begin{aligned}\boldsymbol{V}' &= \left(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)\boldsymbol{V}\left(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)\\ &= \left(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)(0 + \boldsymbol{v})\left(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)\\ &= \left(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)\left(\sin\tfrac{1}{2}\theta\boldsymbol{v}\cdot\boldsymbol{s} + \cos\tfrac{1}{2}\theta\boldsymbol{v} - \sin\tfrac{1}{2}\theta(\boldsymbol{v}\times\boldsymbol{s})\right).\end{aligned} \tag{33}$$

This forms a new quaternion element with the real part of the above obtained as zero in the form of

$$\mathrm{Re}(\boldsymbol{V}') = \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\boldsymbol{v}\cdot\boldsymbol{s} - \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\boldsymbol{s}\cdot\boldsymbol{v} + \sin^2\tfrac{1}{2}\theta\,\boldsymbol{s}\cdot(\boldsymbol{s}\times\boldsymbol{v}) = 0. \tag{34}$$

This hence illustrates that the quaternion triple product preserves Lie algebra property. Further, the imaginary part of Eq. (33) can be obtained as

$$\begin{aligned}\mathrm{Im}(\boldsymbol{V}') &= \cos^2\tfrac{1}{2}\theta\boldsymbol{v} - \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta(\boldsymbol{v}\times\boldsymbol{s}) + \sin^2\tfrac{1}{2}\theta(\boldsymbol{v}\cdot\boldsymbol{s})\boldsymbol{s} + \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\boldsymbol{s}\times\boldsymbol{v} - \sin^2\tfrac{1}{2}\theta\boldsymbol{s}\times(\boldsymbol{v}\times\boldsymbol{s}) \\ &= \cos^2\tfrac{1}{2}\theta\boldsymbol{v} + \sin\theta(\boldsymbol{s}\times\boldsymbol{v}) + \sin^2\tfrac{1}{2}\theta(\boldsymbol{v}\cdot\boldsymbol{s})\boldsymbol{s} - \sin^2\tfrac{1}{2}\theta\boldsymbol{s}\times(\boldsymbol{v}\times\boldsymbol{s}).\end{aligned} \tag{35}$$

From the vector triple product identity $\boldsymbol{s}\times(\boldsymbol{v}\times\boldsymbol{s}) = (\boldsymbol{s}\cdot\boldsymbol{s})\boldsymbol{v} - (\boldsymbol{s}\cdot\boldsymbol{v})\boldsymbol{s}$, it follows that

$$\begin{aligned}\mathrm{Im}(\boldsymbol{V}') &= \cos^2\tfrac{1}{2}\theta\boldsymbol{v} + \sin\theta\boldsymbol{s}\times\boldsymbol{v} + \sin^2\tfrac{1}{2}\theta(\boldsymbol{v}\cdot\boldsymbol{s})\boldsymbol{s} - \sin^2\tfrac{1}{2}\theta(\boldsymbol{v}(\boldsymbol{s}\cdot\boldsymbol{s}) - \boldsymbol{s}(\boldsymbol{s}\cdot\boldsymbol{v})) \\ &= \cos\theta\boldsymbol{v} + \sin\theta(\boldsymbol{s}\times\boldsymbol{v}) + 2\sin^2\tfrac{1}{2}\theta(\boldsymbol{s}\cdot\boldsymbol{v})\boldsymbol{s}\end{aligned} \tag{36}$$

Combining both real and imaginary parts of the resultant conjugation in Eq. (33) results in pure quaternion V' as

$$\begin{aligned}\boldsymbol{V}' &= \left(\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\boldsymbol{s}\right)\boldsymbol{V}\left(\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\boldsymbol{s}\right) \\ &= (0 + \cos\theta\boldsymbol{v} + \sin\theta(\boldsymbol{s}\times\boldsymbol{v}) + (1-\cos\theta)(\boldsymbol{s}\cdot\boldsymbol{v})\boldsymbol{s}).\end{aligned} \tag{37}$$

If expressing the obtained pure quaternion in vector form as

$$\boldsymbol{V} = \begin{pmatrix} 0 \\ \boldsymbol{v} \end{pmatrix},$$

conjugation in Eq. (37) can be rewritten in vector form as follows

$$\boldsymbol{V}' = \begin{pmatrix} 0 \\ (\cos\theta\boldsymbol{I} + \sin\theta[\boldsymbol{s}\times] + (1-\cos\theta)\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}})\boldsymbol{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{R}\boldsymbol{v} \end{pmatrix}, \tag{38}$$

where the outer product or tensor product $\boldsymbol{s}\boldsymbol{s}^{\mathrm{T}}$ of two vectors is used. It can be seen that the quaternion operator in the form of conjugation preserves both length and orientation of an operand and is equivalent to exerting the Euler–Rodrigues formula on the operand.

Hence a rotation can be completed by quaternion conjugation in Eq. (31) which exerts a linear transformation on a Lie algebra element. The Euler–Rodrigues formula of rotations in Eq. (13) is in this case again obtained but this time from the quaternion operator. In the above derivation of conjugation, it can be seen that the conjugation is a way of executing the Euler–Rodrigues formula and subsequently applies the formula to a Lie algebra element. This quaternion conjugation implements an operation by the Euler–Rodrigues formula and the equivalence [34] is confirmed between the conjugation applied to a Lie algebra element in pure quaternion form and the left action [27,34] applied to a Lie algebra element in vector form.

The derivation also presents the intrinsic connection between Euler–Rodrigues parameters and Euler–Rodrigues formula and provides a way of implementing the Euler–Rodrigues formula through quaternion conjugation which utilizes the half angle of rotation in kinematics [7], that geometrically describes a rotation and makes the conjugation possible in the form of quaternion triple product [95].

5. Conclusions

This paper presented variations of the Euler–Rodrigues formula in different mathematical forms and revealed connections between these forms geometrically and algebraically and in particular the connection with Euler–Rodrigues parameters through quaternion conjugation.

The paper started by reviewing the Euler–Rodrigues formula from its geometrical content exerting a rotation on a vector and visited the connection between two various representations, relating the formula to the finite and infinitesimal motion in the realm of Lie groups and Lie algebras. The paper further reviewed the use of the Euler–Rodrigues formula for the exponential map and a way of generating the Euler–Rodrigues formula from a Taylor series expansion, presenting a derivation of the Euler–Rodrigues formula algebraically. This then extended through derivation the exponential map to another form of the Euler–Rodrigues formula with respect to a non-normalized vector and established their connections.

Through quaternion conjugation, the paper presented a way of obtaining the Euler–Rodrigues formula and demonstrated that the quaternion conjugation is an left action of the Lie group on a Lie algebra and equivalent to an operation using the Euler–Rodrigues formula as an operator. The connection between Euler–Rodrigues parameters in the form of the unit quaternion and the Euler–Rodrigues formula is simultaneously presented.

The paper hence provided a foundation of the Euler–Rodrigues formula and presented an aspect of use of the Euler–Rodrigues formula in different forms in the study of rigid body kinematics, dynamics, computer graphics and mechanism analysis and synthesis.

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References

[1] L. Euler, Nova Methodus Motum Corporum Rigidorum Determinandi, Novi Commentari Acad. Imp. Petrop., vol. 20 1775, pp. 208–238 (also Leonhardi Euleri Opera Omnia, Series Secunda, Opera mechanica Et Astronomica, Basileae MCMLXVIII, vol. **9**, pp. 99–125.).
[2] O. Rodrigues, Des lois géométriques qui régissent les déplacements d'un systéme solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire, J. Math. 5 (1840) 380–440.
[3] J.S. Dai, An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist, Mech. Mach. Theory 41 (1) (2006) 41–52.
[4] J.J. Gray, Olinde Rodrigues' paper of 1840 on transformation groups, Arch. Hist. Exact Sci. 21 (1980) 375–385.
[5] J. Marsden, Introduction to Mechanics and Symmetry, Springer, New York, 1999.
[6] F. Frank, Orientation mapping, Metall. Trans. 19A (1988) 403–408.
[7] C.D. Mladenova, I.M. Mladenov, Vector decomposition of finite rotations, Rep. Math. Phys. 69 (1) (2011) 107–117.
[8] J.W. Gibbs, in: E.B. Wilson (Ed.), Vector Analysis, Scribner, New York, 1901.
[9] K.E. Bisshopp, Rodrigues' formula and the screw matrix, J. Eng. Ind. Trans. ASME 91 (1969) 179–185.
[10] O. Bottema, B. Roth, Theoretical kinematics, North-Holland Series in Applied Mathematics and Mechanics, North-Holland, Amsterdam1979.
[11] A. Cayley, On certain results relating to quaternions, Phil. Mag. 26 (1845) 141–145 (also The Collected Mathematical Papers, **1**, 123–126 (paper No. 20) and Note 20, 586, Cambridge University Press, 1889).
[12] W.R. Hamilton, On quaternions; or on a new system of imaginaries in algebra, Philos. Mag. (1844) 489–495 (3rd ser. 25).
[13] A. Cayley, On the motion of rotation of solid body, Camb. Math. J. III (1843) 224–232 (also The Collected Mathematical Papers, vol **1**, 28–35, (Paper No. 6), Cambridge University Press, 1889).
[14] S.L. Altmann, Hamilton, Rodrigues, and the Quaternion Scandal, Math. Mag. 62 (5) (1989) 291–308.
[15] J. Pujol, Hamilton, Rodrigues, Gauss, Quaternions, and rotations: a historical reassessment, Commun. Math. Anal. 13 (2) (2012) 1–14.
[16] A. Cayley, On the rotation of a solid body round a fixed point, Camb. Dublin Math J. I (1846) 167–173 (also the Collected Mathematical Papers, vol **I**, 237-252, (Paper No. 37), Cambridge University Press, 1889).
[17] G.S. Chirikjian, A.B. Kyatkin, Engineering applications of noncommutative harmonic analysis: with emphasis on rotation and motion groups, Stochastic Models, Information Theory, and Lie Groups, Volume 2: Analytic Methods and Modern Applications, CRC Press LLC, Birkhäuser, Boston, 2001.
[18] J.S. Dai, Finite displacement screw operators with embedded Chasles' motion, J. Mech. Robot. Trans. ASME 4 (4) (2012) 041002.
[19] H. Cheng, K.C. Gupta, An historical note on finite rotations, Trans. ASME J. Appl. Mech. 56 (1989) 139–145.
[20] C.J.L. Doran, A.N. Lasenby, Geometric Algebra for Physicists, Cambridge University Press, 2003.
[21] J. Rooney, Aspects of Clifford algebra for screw theory, in: A. Kecskeméthy, A. Müller (Eds.),*Computational Kinematics*, Proc. 5th Int Workshop on Computational Kinematics, Springer, Berlin Heidelberg 2009, pp. 191–200.
[22] J.M. McCarthy, An Introduction to Theoretical Kinematics, The MIT Press, London, 1990.
[23] F.C. Park, B. Ravani, Bezier curves on Riemannian manifolds and lie groups with kinematics applications, J. Mech. Des. ASME 117 (1) (1995) 36–40.
[24] J.J. Craig, Introduction to Robotics, 2nd edn. Addison-Wesley, Reading, MA, 1989.
[25] R.M. Murray, Z. Li, S.S. Sastry, A Mathematical Introduction to Robotic Manipulation, CRC Press, New York, 1994.
[26] G.S. Chirikjian, Stochastic Models, Information Theory, and Lie Groups, Volume 2: Analytic Methods and Modern Applications, Birkhäuser, Boston, 2011.
[27] J.M. Selig, Geometric Fundamentals of Robotics, Springer, New York, 2005.
[28] J. Gallier, D. Xu, Computing exponentials of skew–symmetric matrices and logarithms of orthogonal matrices, Int. J. Robot. Autom. 17 (4) (2002) 1–11.
[29] F.C. Park, J.E. Eobrow, S.R. Ploen, A lie group formulation of robot dynamics, Int. J. Robot. Res. 14 (6) (1995) 609–618.
[30] G.A. Sohl, J.E. Bobrow, A recursive multibody dynamics and sensitivity algorithm for branched kinematic chains, J. Dyn. Syst. Meas. Control. 123 (2001) 391–399.
[31] D. Lewis, J. Simo, Conserving Algorithms for the Dynamics of Hamilton Systems on Lie Groups, J. Nonlinear Sci. 4 (1994) 253–299.
[32] O. Bauchau, G. Damilano, N. Theron, Numerical integration of nonlinear elastic multi-body systems, Int. J. Numer. Methods Eng. 38 (1995) 272–2751.
[33] O.A. Bauchan, L. Trainnelli, The vectorial parameterization of rotation, Nonlinear Dyn. 32 (1) (2003) 71–92.
[34] J.S. Dai, Geometrical foundations and screw algebra for mechanisms and robotics (Translated From *Screw Algebra and Kinematic Approaches for Mechanisms and Robotics, 2016*), Higher Education Press, Beijing, 2014, ISBN 9787040334838.
[35] J.S. Dai, From Rodrigues' parameters to the finite twisting motion of a rigid body, Proc of 10th IFToMM World Congress, Tianjin, China Machine Press 2004, pp. 944–947.
[36] J.E. Mebius, Derivation of the Euler–Rodrigues formula for three-dimensional rotations from the general formula for four-dimensional rotations, Xiv General Mathematics2007.
[37] N.A.F. Senan, O.M. O'Reilly, On the use of quaternions and Euler–Rodrigues symmetric parameters with moments and moment potentials, Int. J. Eng. Sci. 47 (4) (2009)http://dx.doi.org/10.1016/j.ijengsci.2008.12.008.
[38] A.N. Norris, Euler–Rodrigues and Cayley formulae for rotation of elasticity tensors, Math. Mech. Solids 13 (2008) 465–498.
[39] S. Forte, M. Vianello, Functional bases for transversely isotropic and transversely hemitropic invariants of elasticity tensors, Q. J. Mech. Appl. Math. 51 (1998) 543–552.
[40] A. Müller, Group theoretical approaches to vector parameterization of rotations, J. Geom. Symm. Phys. 19 (2010) 43–72.
[41] E. Kovács, Rotation about an arbitrary axis and reflection through an arbitrary plane, Ann. Math. Inform. 40 (2012) 175–186.
[42] J. Pujol, Tutorial on rotations in the theories of finite deformation and micropolar (Cosserat) elasticity, Bull. Seism. Soc. Am. 99 (2B) (2009) 1011–1027.
[43] J. Synge, Classical Dynamics, McGraw-Hill, New York, 1960.
[44] A. Cayley, On the application of quaternions to the theory of rotation, Philos. Mag. 33 (1848) 196–200 (Reprinted in: The Collected Mathematical Papers of Arthur Cayley, vol. I, No. 68, pp 405–409, 1889).
[45] K. Wohlhart, Decomposition of a finite rotation into three consecutive rotations about given axes, Proc. VI-th Int Conf. on Theory of Machines and Mechanisms, IFToMM, Liberec, 1992.
[46] Q.J. Ge, P.M. Larochelle, Algebraic motion approximation with NURBS motions and its application to spherical mechanism synthesis, J. Mech. Des. ASME 121 (1999) 529–532.
[47] A. Purwar, Q.J. Ge, On the effect of dual weights in computer aided design of rotational motions, J. Mech. Des. 127 (5) (2005) 967–972.
[48] S. Bai, J. Angeles, A robust solution of the spatial Burmester problem, ASME J. Mech. Robot. 4 (3) (2012) 0311003.
[49] G.S. Chirikjian, A.B. Kyatkin, Engineering Applications of Noncommutative Harmonic Analysis: With Emphasis on Rotation and Motion Groups, LLC CRC Press, 2000.
[50] C. Belta, V. Kumar, Euclidean metrics for motion generation on SE(3), J. Mech. Eng. Sci. 216 (C1) (2002) 47–61.
[51] L. Cui, J.S. Dai, A Darboux-frame-based formulation of spin-rolling motion of rigid objects with point contact, IEEE Trans. Robot. 26 (2) (2010) 383–388.
[52] Q.J. Ge, B. Ravani, Computer-aided geometric design of motion interpolants, ASME J. Mech. Des. 116 (3) (1994) 756–762.

[53] R. Campa, K. Camarillo, Unit quaternions: a mathematical tool for modeling, path planning and control of robot manipulators, in: M. Ceccarelli (Ed.), Robot Manipulators, InTech Europe, Croatia 2008, pp. 21–48.
[54] A. Müller, On the manifold property of the set of singularities of kinematic mappings: genericity conditions, ASME J. Mech. Robot. 4 (1) (2012) 011006.
[55] J.S. Dai, N. Holland, D.R. Kerr, Finite twist mapping and its application to planar serial manipulators with revolute joints, J. Mech. Eng. Sci. Proc IMechE 209 (3) (1995) 263–271.
[56] B.K.P. Horn, Closed-form solution of absolute orientation using unit quaternions, J. Opt. Soc. Am. A 4 (4) (1987) 620–642.
[57] J.L. Crassidis, F.L. Markley, Attitude estimation using modified Rodrigues parameters, Proceedings of the Flight Mechanics/Estimation Theory Symposium, (NASA/CP-1996-3333) NASA-Goddard Space Flight Center, Greenbelt, MD 1996, pp. 71–83.
[58] J. Diebel, Representing attitude: Euler angles, quaternions, and rotation vectors, Tech Rep., Stanford University, Palo alto, CA, 2006.
[59] Y. Yu, X. Ding, J.J. Zhu, Attitude tracking control of a Quadrotor UAV in the exponential coordinates, J. Franklin Inst. 350 (8) (2013) 2044–2068.
[60] N.A.F. Senan, O.M. O'Reilly, On the use of quaternions and Euler–Rodrigues symmetric parameters with moments and moment potentials, Int. J. Eng. Sci. 47 (4) (2008)http://dx.doi.org/10.016/j.ijengsci.2008.12.008.
[61] A. Perez, J.M. McCarthy, Dual quaternion synthesis of constrained robotic systems, ASME J. Mech. Des. 126 (3) (2004) 425–435.
[62] K. Zhang, J.S. Dai, Y. Fang, Topology and constraint analysis of phase change in the metamorphic chain and its evolved mechanism, J. Mech. Des. Trans. ASME 132 (12) (2010) 121001.
[63] X. Ding, S. Lu, Fundamental reconfiguration theory of chain-type modular reconfigurable mechanisms, Mech. Mach. Theory 70 (2013) 487–507.
[64] X. Kong, Reconfiguration analysis of a 3-DOF parallel mechanism using Euler parameter quaternions and algebraic geometry method, Mech. Mach. Theory 74 (2014) 188–201.
[65] K. Zhang, J.S. Dai, A kirigami-inspired 8R linkage and its evolved overconstrained 6R linkages with the rotational symmetry of order two, J. Mech. Robot. Trans. ASME 6 (2) (2014) 021008.
[66] D. Gan, Q. Liao, S. Wei, J.S. Dai, S. Qiao, Dual quaternion-based inverse kinematics of the general spatial 7R mechanism, J. Mech. Eng. Sci. 222 (8) (2008) 1593–1598.
[67] S. Qiao, Q. Liao, S. Wei, H.J. Su, Inverse kinematic analysis of the general 6R serial manipulators based on double quaternions, Mech. Mach. Theory 45 (2) (2010) 193–199.
[68] J.M. McCarthy, G.S. Soh, Geometric Design of Linkage, Springer, London, 2010.
[69] G. Wei, Y. Chen, J.S. Dai, Synthesis, mobility and multifurcation of deployable polyhedral mechanisms with radially reciprocating motion, J. Mech. Robot. Trans. ASME 136 (9) (2014) 091003.
[70] P. Corke, J. Lobo, J. Dias, An introduction to inertial and visual sensing, Int. J. Robot. Res. 26 (6) (2007) 519–535.
[71] J. Vince, Quaternions for Computer Graphics, Springer-Verlag, London, 2011.
[72] K. Shoemake, Animating rotation with quaternion curves, Comput. Graph. 19 (3) (1985) 245–254.
[73] D.F. Rogers, J.A. Adams, Mathematical Elements for Computer Graphics, 2nd ed. McGraw-Hill, Inc. New York, 1989.
[74] B.K.P. Horn, Robot Vision, MIT/McGraw-Hill, New York, 1986.
[75] J. Davidson, K.H. Hunt, Robots and Screw Theory, Applications of Kinematics and Statics to Robotics, Oxford University Press, New York, 2004.
[76] H. Hickey, More on vector representation of rigid body rotation, Math. Mag. 43 (1970) 38–39.
[77] N. Bourbaki, Elements of Mathematics, Algebra I, Springer-Verlag, 1989.
[78] O. Veblen, J.W. Young, Projective Geometry, vol. I, Ginn and Co., Boston, 1910.
[79] C. Huang, B. Roth, Analytic expressions for the finite screw systems, Mech. Mach. Theory 29 (2) (1994) 207–222.
[80] J.S. Dai, J. Rees Jones, Interrelationship between screw systems and corresponding reciprocal systems and applications, Mech. Mach. Theory 36 (5) (2001) 633–651.
[81] E. Pennestri, P.P. Valentini, Linear dual algebra algorithms and their application to kinematics, in: G.L. Bottasso (Ed.), Multibody Dynamics, Computational Methods and Applications, Series of Computational Methods in Applied Sciences, vol. 12, Netherlands, Springer 2009, pp. 207–229.
[82] J.S. Dai, J. Rees Jones, Null-space construction using cofactors from a screw algebra context, Proc. Roy. Soc. Lond. A Math. Phys. Eng. Sci. 458 (2024) (2002) 1845–1866.
[83] I.M. Sheffer, A note of matrix power series, Am. Math. Mon. 37 (5) (1930) 228–231.
[84] J. Stillwell, *Naive Lie Theory*, undergraduate texts in Mathematics, Springer, in: R.A. Wilson (Ed.), The Finite Simple Groups, Graduate Texts in Mathematics, 251, Springer, London 2008, pp. 69–75 (2009).
[85] R.A. Wilson, The finite simple groups, Graduate Texts in Mathematics, 251, Springer, London, 2009. 69–75.
[86] Klein, F., 1884, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Tubner, Leipzig. Translated as: *Lectures on the Icosahedron and the Solutions of Equations of the Fifth*, 2nd ed., (Translated by Morrice, G.G.), Ballantyne, Hanson Co., 1914; Dover Publications, New York, 1956.
[87] D. Simonovitch, Rotations in NMR: part I. Euler–Rodrigues parameters and quaternions, Concepts Magn. Reson. 9 (1997) 149–171.
[88] D. Widdows, Quaternionic algebra described by Sp(1) representations, Q. J. Math. 54 (2003) 463–481.
[89] M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa 15 (1961) 179–246.
[90] P.R. Girat, The quaternion group and modern physics, Eur. J. Phys. 5 (1984) 25–32.
[91] A.J. Hanson, *Visualizing Quaternions*, Series in Interactive 3D Technology, Morgan Kaufmann Publishing, San Francisco, 2006.
[92] W. Fulton, J. Harris, Representation Theory: A First Course, Springer-Verlag, New York, 1991.
[93] M.L. Curtis, Matrix Groups, Springer-Verlag, Berlin, 1979.
[94] F. Thomas, Approaching dual quaternions from matrix algebra, IEEE Trans. Robot. 30 (5) (2014) 1037–1048.
[95] Kuipers, J.B., 1999, "Quaternions and Rotation Sequences, Geometry", *Integrability and Quantization*, 1–10 Sept. 1999, Varna, Bulgaria, eds Mladenov, I.M. and Naber, G.L. *Coral Press*, Sofia 2000, pp 127–143.
[96] M.L. Husty, M. Pfurmer, H.-P. Schröcker, K. Brunnthaler, Algebraic methods in mechanism analysis and synthesis, Robotica 25 (6) (2007) 661–675.
[97] T. Arponen, A. Müller, S. Püpponent, J. Tuomela, Computational algorithm algebraic geometry and global analysis of regional manipulators, Appl. Math. Comput. 232 (2014) 820–835.