Barrier Options

This note is several years old and very preliminary. It has no references to the literature. Do not trust its accuracy! Note that there is a lot of more recent literature, especially on static hedging.

0.1 Introduction

In this note we discuss various kinds of barrier options. The four basic forms of these path-dependent options are **down-and-out**, **down-and-in**, **up-and-out** and **up-and-in**. That is, the right to exercise either appears ('in') or disappears ('out') on some **barrier** in (S, t) space, usually of the form $S =$ constant. The barrier is set above ('up') or below ('down') the asset price at the time the option is created. They are also often called **knock-out**, or **knock-in** options.

An example of a knock-out contract is a European-style option which immediately expires worthless if, at any time before expiry, the asset price falls to a lower barrier $S = B_-$, set below $S(0)$. If the barrier is not reached, the holder receives the payoff at expiry. When the payoff is the same as that for a vanilla call, the barrier option is termed a European **down-and-out call**. Figure 1 shows two realisations of the random walk, of which one ends in knock-out, while the other does not. The second walk in the figure leads to a payout of $S(T) - E$ at expiry, but if it had finished between B_- and E, the payout would have been zero. An **up-and-out** call has similar characteristics except that it becomes worthless if the asset price ever rises to an upper barrier $S = B^+$. (The barrier must be set above $S(0)$, because otherwise the options would be worthless.)

An 'in' option expires worthless *unless* the asset price reaches the barrier before expiry. If the asset value hits the line $S = B_-$ at some time prior to expiry then the option becomes a vanilla option with the appropriate payoff. If the payoff is that of a vanilla call, the option is a **down-and-in call**. Up-and-in options are defined in an analogous way.

Knock-out options can be further complicated in many ways. For example, the position of the knockout boundary may be a function of time; in particular it may only be active for part of the lifetime of the contract. Another complication for out options is to allow a **rebate**, whereby the holder of the option receives a specified amount R if the barrier is crossed; this can make the option more attractive to potential purchasers by compensating them for the loss of the option on knockout. Likewise it is common for in-type barrier options to give a rebate, usually a fixed amount, if the barrier is *not* hit, to compensate the holder for the loss of the option. A third possibility is to have more than one barrier, as in the **double knock-out option**, which has both upper and lower barriers where it expires lifeless. Any contract can in principle have barrier features added to it: as we shall see, this only changes the boundary conditions for the partial differential equation. The barrier idea, and our way of analysing it, can also be applied to interest-rate derivatives using spot rate

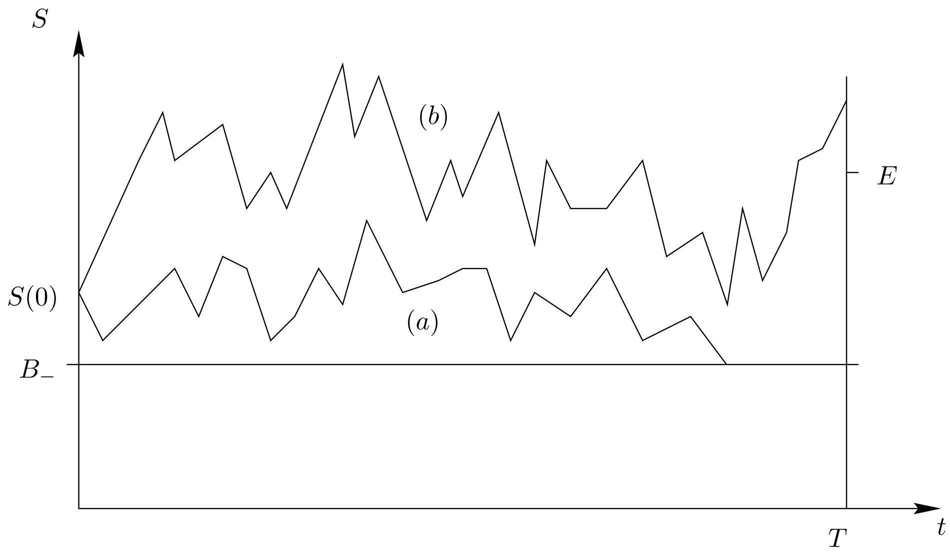

Figure 1: Random walks for a down-and-out call. Walk (a) results in knock-out, walk (b) does not.

models.

We only discuss European options in any detail; American barrier options are treated in Chapter 8. Although find a large number of formulæ for the values of various barrier options, our goal is not really to present a huge list of explicit solutions to the Black–Scholes equation. It is, rather, at least threefold.

First, I want to emphasise how easy it is to formulate these problems as boundary value problems for the Black–Scholes equation, which can then relatively easily be solved by numerical methods. Of course, the numerical methods can be used in many cases where exact solutions cannot be found, for example when using a 'deterministic volatility', or volatility surface, model. Secondly, the exact formulæ are undoubtedly useful in checking the accuracy of numerical solutions. Thirdly, they can be used as a quick guide to potential problems in hedging caused by inaccuracies in the model. This requires a brief discussion.

Barrier options are notoriously sensitive to misspecifications in the model, by which we mean two things. One is that even if the Black–Scholes model were a perfect description of reality, which it is not, errors in parameter estimation (of the volatility, in particular) can translate into large errors in pricing and especially hedging. These arise because many barrier options have discontinuities in their payoffs, and hence have large Gamma, and hence Vega, risks. (Gamma, Γ, and Vega, Ӂ, are closely related.) The other type of misspecification is that the difference between the real world and the Black–Scholes idealisation can also lead to errors that are particularly pronounced for barrier options. An example is that real price histories contain more large changes than the Brownian motion model allows, and these can lead to a greater risk of knock-out than allowed for in the Black–Scholes price. Another is that out barriers in particular are susceptible to market manipulation.

0.2 The down-and-out call

We begin with a European style down-and-out call option. At expiry it pays the usual call payoff $\max(S-E,0)$, provided that S has not fallen to B_- during the life of the option. If S ever reaches B_- then the option becomes worthless. Obviously the down-and-out call should cost less than the corresponding call, because of the additional risk of knock-out, with premature loss of the premium.

Who would want this option? One possible user would be a company that needs to buy a large quantity of, say, copper in three months' time. They are happy at the current price, but cannot afford for copper to become much more expensive. On the other hand, their gain if the copper price falls is less important to them (or, they may have a strong view that the price may well rise but is unlikely to fall). They could hedge the purchase with a forward contract, but then they would not benefit at all if the price falls, as they will be obliged to buy at the forward price. This might be termed an 'opportunity loss', the potential cost of peace of mind. They could buy an at-the-money vanilla call option, but although this limits the opportunity loss it is also more expensive. Compared with the vanilla call, the down-and-out call retains the upside protection but is cheaper. Its drawback is that does not protect the holder against a price that first falls below the barrier then rises sharply; the cost difference prices that risk.

Under the usual Black–Scholes assumptions, there is an explicit formula for the fair value of this option. We only consider in detail the case where the lower barrier is set below the option's strike price, $E > B_-$. In so doing, we see that there is a neat short cut which allows us to do many apparently more complicated cases with little effort.

Suppose that we are above the barrier, at asset value $S > B_-$ and time t, and we hold the down-and-out call. The next timestep, being infinitesimal, will not take us to the barrier. We can therefore apply the usual Black–Scholes hedging analysis, to show that the value of the option $C_{\mathrm{d/o}}(S,t)$ satisfies the Black–Scholes equation

$$\frac{\partial C_{\mathrm{d/o}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\mathrm{d/o}}}{\partial S^2} + (r-D)S\frac{\partial C_{\mathrm{d/o}}}{\partial S} - rC_{\mathrm{d/o}} = 0. \tag{1}$$

(We have included a constant dividend yield D.) Of course, this equation only holds for $B_- < S < \infty$. The option does not exist for $S < B_-$.

As before, the final condition for equation (1) is

$$C_{\mathrm{d/o}}(S,T) = \max(S-E,0),$$

but again only for $B_- < S < \infty$. As S becomes large the likelihood of the barrier being activated becomes negligible and so

$$C_{\mathrm{d/o}}(S,t) \sim Se^{-D(T-t)} - Ee^{-r(T-t)} \qquad \text{as} \quad S\to\infty.$$

We now see the most conspicuous way in which this valuation problem differs from that for a vanilla call. There, S runs from 0 to ∞. Here, the second 'spatial' boundary condition is applied at $S = B_-$ rather than at $S = 0$. If S ever reaches B_- then the option expires worthless; this financial condition translates into the mathematical condition that on $S = B_-$ the value of the option is zero:

$$C_{\mathrm{d/o}}(B_-,t) = 0.$$

This completes the formulation of the problem; we now find the explicit solution, using two methods. The first of these is, frankly, a bit plodding and will be eliminated on future rewritings. Skip straight to Section 0.3 and see if you don't agree.

Reduction to the heat equation

We use a slight variation[1] on the change of variables first introduced in Section 8. That is, we let

$$S = B_- e^x, \quad t = T - \tau/\tfrac{1}{2}\sigma^2, \quad C_{\mathrm{d/o}} = B_- e^{\alpha x + \beta\tau} u(x,\tau),$$

with $\alpha = \frac{1}{2}(1-k')$, $\beta = -\frac{1}{4}(k'-1)^2 - k$ and $k = r/\frac{1}{2}\sigma^2$, $k' = (r-D)/\frac{1}{2}\sigma^2$. (Without dividends, replace k' by k throughout.) In these new variables the barrier transforms to the point $x = 0$, and the barrier option problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

for $0 < x < \infty$, $\tau > 0$, with

$$u(x,0) = U(x) = \max\left(e^{\frac{1}{2}(k'+1)x} - (E/B_-)e^{\frac{1}{2}(k'-1)x}, 0\right), \qquad x \geq 0, \tag{3}$$

and

$$u(0,t) = 0. \tag{4}$$

The last boundary condition is new. We deal with it by the **method of images**.

We have several times related the problem of valuing simple call and put options to the flow of heat in an infinite bar. Boundary condition (4) is, however, imposed at a finite value of x: the analogy is now with heat flow in a *semi*-infinite bar whose end $x = 0$ is held at zero temperature.

It is clear that equation (2) is invariant under reflection, replacing x by $-x$. Thus, if $u(x,\tau)$ is a solution of (2), so is $u(-x,\tau)$. In the method of images we solve a semi-infinite problem by first solving an infinite problem made up of *two* semi-infinite problems with equal and opposite initial temperature distributions: one half is hot, the other cold. The net effect is cancellation at the join. The temperature there is guaranteed to be zero, since heat flowing in from one side is balanced by the equal and opposite flux of 'cold' from the other.

We can apply this method to the barrier option problem. We reflect the initial data about the point $x = 0$ (the 'join' of the two bars, corresponding to the barrier), at the same time changing its sign, thereby automatically satisfying (4). Thus, instead of solving (2)–(4) on the interval $0 < x < \infty$, we solve (2) for *all* x but subject to

$$u(x,0) = \begin{cases} U(x) & \text{for} \quad x > 0 \\ -U(-x) & \text{for} \quad x < 0, \end{cases} \tag{5}$$

that is,

$$u(x,0) = \begin{cases} \max\left(e^{\frac{1}{2}(k'+1)x} - (E/B_-)e^{\frac{1}{2}(k'-1)x}, 0\right) & \text{for} x > 0 \\ -\max\left(e^{-\frac{1}{2}(k'+1)x} - (E/B_-)e^{-\frac{1}{2}(k'-1)x}, 0\right) & \text{for} x < 0. \end{cases} \tag{6}$$

In this way we guarantee that $u(0,\tau) = 0$.

For a general payoff, we would have to write down the solution to this problem as an integral, as was done in Chapter 8. Here, though, there is a short cut. Consider a vanilla call, with the same expiry and exercise price but no barrier. Write its value as $C_{\mathrm{v}}(S,t;E)$,

[1] The only difference is that for convenience we have scaled with B_- rather than E.

and $U_v(x,\tau)$ for the corresponding solution of the heat equation. The payoff $C_{\mathrm{v}}(S,T;E)$ is zero for all S below the strike; this translates into $U_v(x,0) = 0$ for $x < \log(E/B_-)$. When we set the barrier below the strike, we ensure that $\log(E/B_-) > 0$. So, if we extend our down-and-out call payoff $U(x)$ into $x < 0$ by setting it equal to zero there, we have exactly the transformed payoff of the vanilla call, $U_v(x)$. This means that we can write equation (5) as

$$u(x,0) = U_v(x) - U_v(-x),$$

holding for *all* x. Now it clear that the solution to the problem for the transformed barrier option value is

$$u(x,\tau) = U_v(x,\tau) - U_v(-x,\tau).$$

After all, the right-hand side satisfies the heat equation and has the correct initial value. Furthermore, it always vanishes at $x = 0$, by antisymmetry.

We now need to return to the variables S and t. When we transform variables, we write

$$C_{\mathrm{v}}(S,t;E) = C_{\mathrm{v}}(B_- e^x, t(\tau); E) = B_- e^{\alpha x + \beta\tau} U_{\mathrm{v}}(x,\tau).$$

Thus, we have

$$U_{\mathrm{v}}(x,\tau) = e^{-\alpha x - \beta\tau} C_{\mathrm{v}}(B_- e^x, t(\tau); E)/B_-,$$

and so

$$U_{\mathrm{v}}(-x,\tau) = e^{\alpha x - \beta\tau} C_{\mathrm{v}}(B_- e^{-x}, t(\tau); E)/B_-.$$

We can now put the pieces together to show that the barrier option value is

$$\begin{aligned}
C_{\mathrm{d/o}}(S,t) &= B_- e^{\alpha x + \beta\tau} u(x,\tau) \\
&= B_- e^{\alpha x + \beta\tau} \left(U_{\mathrm{v}}(x,\tau) - U_{\mathrm{v}}(-x,\tau)\right) \\
&= C_{\mathrm{v}}(B_- e^x, t; E) - e^{2\alpha x} C_{\mathrm{v}}(B_- e^{-x}, t; E) \\
&= C_{\mathrm{v}}(S,t;E) - \left(\frac{S}{B_-}\right)^{2\alpha} C_{\mathrm{v}}(B_-^2/S, t; E).
\end{aligned}$$

Thus, when the strike is above the barrier, the barrier option value is revealed as the value of a vanilla call with the same payoff, $C_{\mathrm{v}}(S,t;E)$, less an 'image contribution' or 'reflection' which enables us to satisfy the barrier condition. Moreover, the image contribution has a particularly simple form; looking at it, we feel that we should have been able to get it by a less tortuous route. This is indeed the case, and in Section 8 we return to this idea, to value many other 'out' options. First, though, we complete our introductory study by dealing with the down-and-*in* option corresponding to our down-and-out call.

The down-and-in call

Let us turn to a down-and-in European call option. This contract expires worthless if the asset does not touch the barrier before expiry, while if the barrier *is* triggered, the down-and-in call is converted into a vanilla call. Although its name contains the word 'call', this contract looks more like a put! It becomes more valuable as the asset falls, because it is more likely to be converted into the underlying call. An investor who thinks that the market will fall and then rise again might buy this option in preference to, say, a calendar spread, because there is no need to specify when the fall and rise will occur, provided only that they do so in the lifetime of the option. This is precisely the situation that is *not* covered by a down-and-out call, suggesting that the cost of a down-and-in call should be the difference between a vanilla call and a down-and-out call.

This is easy to show by a financial argument. The value of a portfolio consisting of one in-option and one out-option (with the same barrier, exercise price and expiry dates) is obviously equal to the value of a vanilla call (with the same exercise price and expiry dates). This is because only one of the two barrier options can be active at expiry. If the barrier is triggered, the out option is knocked out and the in option becomes the underlying vanilla call, while if the barrier is not triggered, the in option expires worthless and the out option pays the vanilla call payoff. (Again, see walks (a) and (b) in Figure 1.) Either way, the portfolio has the same outcome as the vanilla call. It follows that the value of the down-and-in call is

$$\left(\frac{S}{B_-}\right)^{2\alpha} C_{\mathrm{v}}(B_-^2/S, t; E).$$

There is a mathematical translation of this argument:

1. Formulate the boundary value problem for the down-and-in call (solve Black–Scholes for $S > B_-$, with zero final data and value equal to that of the vanilla call on the barrier);
2. Subtract off the value of the vanilla call, to get zero on the barrier;
3. Observe that the resulting problem is precisely that of a down-and-out call.

The details are left for the reader to fill in, with a degree of rigour appropriate to their temperament.

0.3 A reflection principle for the Black–Scholes equation

Before we go to value other barrier contracts, we state a very useful result. In fancy mathematical language it would be called a lemma, but it is just a restatement of the image principle in the variables S and t:

- Suppose we have a solution $V(S,t)$ of the Black–Scholes equation with a constant dividend yield D. That is,

  $$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0.$$

  Then for any constant B (representing a barrier),

  $$W(S,t) = S^{2\alpha} V(B^2/S, t)$$

  satisfies the same equation provided that $\alpha = \frac{1}{2}(1-(r-D)/\frac{1}{2}\sigma^2)$. This is the same α that we use when we transform the Black–Scholes equation to the diffusion equation as above.

The proof is simple. It can be done by direct differentiation, which is moderately arduous; or we can transform to the heat equation, observe that if $u(x,\tau)$ is a solution so is $u(-x,\tau)$, and translate the latter back again. Either way, ten minutes' work will do it.

So, we have a way of generating new solutions from old ones. The really useful thing about this trick is that S and B^2/S are on opposite sides of the barrier for $S \neq B$, and coincide when $S = B$. Let us see how to exploit this result to condense the calculations of Section 8.

When the underlying is a futures contract, the reflection takes a particularly simple form:

- Suppose that $V(F,t)$ is a solution of the Black–Scholes equation for derivatives contingent on a future,
$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0.$$
Then $W(S,t) = FV(B^2/F, t)$ is also a solution.

The proof is trivial, setting $D = r$ in the more general result above. We return to this result in Section **??**.

0.4 Single barrier contracts

We can now rattle through a large class of barrier options, using just three standard tools:

1. We use the idea of 'reflecting' solutions about a barrier to satisfy knockout conditions;
2. We use special option values, in particular digital options, to deal with rebates and payoff discontinuities;
3. Out-in parity gives values for in options from out options.

We start with options without a rebate. Before we can handle rebates, we need to value American digital call and put options. We therefore discuss rebates in general after these special options have been treated, in Section 0.5.

We shall make extensive use of the European vanilla call and put solutions, and of the standard European digital options, which we use as building blocks for more complex barrier structures. For convenience, we recall them here:

- The vanilla call has value
$$C_{\mathrm{v}}(S,t;E) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2),$$
where
$$d_1 = \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
and
$$d_2 = \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}};$$
- The vanilla put has value
$$P_{\mathrm{v}}(S,t;E) = Ee^{-r(T-t)}N(-d_2) - Se^{-D(T-t)}N(-d_1);$$
- The standard digital call, with payoff \$1 for $S > E$, 0 otherwise, has value
$$C_{\mathrm{d}}(S,t;E) = e^{-r(T-t)}N(d_2);$$
- The standard digital put has value
$$P_{\mathrm{d}}(S,t;E) = e^{-r(T-t)}N(-d_2).$$

We also recall the two put-call parity results,
$$C_{\mathrm{v}}(S,t;E) - P_{\mathrm{v}}(S,t;E) = Se^{-D(T-t)} - Ee^{-r(T-t)}$$
and
$$C_{\mathrm{d}}(S,t;E) + P_{\mathrm{d}}(S,t;E) = e^{-r(T-t)}.$$
These both follow from the result that $N(d) + N(-d) = 1$. They are useful for simplifying some formulæ.

Down-and-out calls: barrier below the strike (revisited)

Recall that the value of the down-and-out call option must be equal to the vanilla call payoff at expiry, and vanish on the barrier. We can satisfy the first of these conditions using the vanilla call value $C_{\mathrm{v}}(S,t;E)$. So, write

$$C_{\mathrm{d/i}}(S,t) = C_{\mathrm{v}}(S,t;E) - C_1(S,t).$$

(Note the minus sign; it is chosen to make the payoff for C_1 more attractive.) Then C_1 vanishes at expiry, but now the barrier condition $C_{\mathrm{d/o}}(B_-,t) = 0$ becomes $C_1(B_-,t) = C_{\mathrm{v}}(B_-,t;E)$. It looks as if we have exchanged one difficulty for another. But no: the reflection result above tells us to consider $(S/B_-)^{2\alpha}C_{\mathrm{v}}(B_-^2/S,t;E)$, which is also a solution of the Black-Scholes equation. Because the vanilla call value vanishes at expiry for $S < E$, and because the barrier is set below the strike, $(S/B_-)^{2\alpha}C_{\mathrm{v}}(B_-^2/S,t;E)$ vanishes at expiry for $S > B_-$: just what we want. Even better, when $S = B_-$,

$$\begin{aligned}(S/B_-)^{2\alpha}C_{\mathrm{v}}(B_-^2/S,t;E) &= (B_-/B_-)^{2\alpha}C_{\mathrm{v}}(B_-^2/B_-,t;E)\\ &= C_{\mathrm{v}}(B_-,t;E).\end{aligned}$$

So, we can satisfy *all* the boundary conditions by taking $C_1 = (S/B_-)^{2\alpha}C_{\mathrm{v}}(B_-/S,t;E)$. Thus,

$$C_{\mathrm{d/o}}(S,t) = C_{\mathrm{v}}(S,t;E) - (S/B_-)^{2\alpha}C_{\mathrm{v}}(B_-^2/S,t;E)$$

is the value of the down-and-out call.

Down-and-out calls: barrier above the strike

Suppose now that the barrier is above the strike. Our argument above fails: the reflected solution no longer vanishes above the barrier at expiry, because the vanilla call payoff does not vanish below it. But we can easily cure this defect. We simply truncate the payoff of the vanilla call at the barrier as shown in Figure 4 before extending it by zero for $S < B_-$.

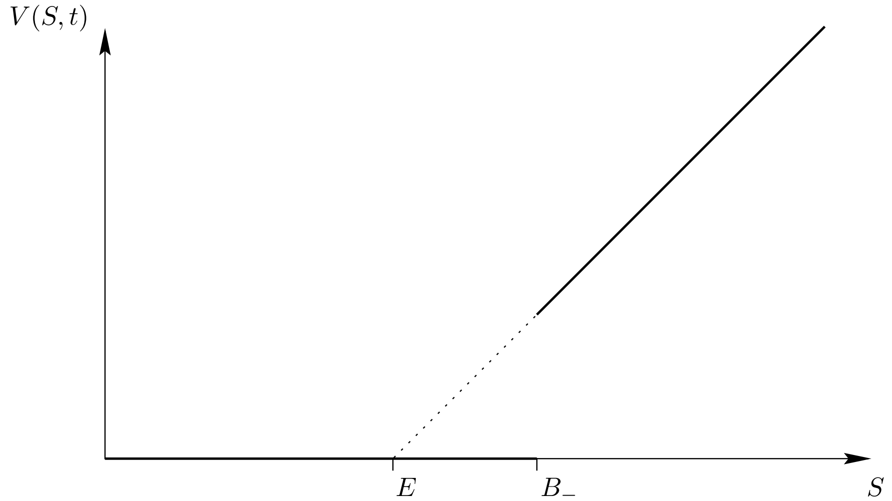

Figure 2: Truncated and extended call payoff.

Before we reflect, we need to value the option with the truncated payoff (for all S). This is easy to do. The payoff is the same as that of a vanilla call struck at B_- plus $(B_- - E)$ times a standard digital call, also struck at B_-, paying \$1. (Subtracting off the latter obviously leaves the former.) Then the down-and-out call has value

$$\begin{aligned} C_{\mathrm{d/o}}(S,t) = {} & C_{\mathrm{v}}(S,t;B_-) + (B_- - E)C_{\mathrm{d}}(S,t;B_-) \\ & - \left(\frac{S}{B_-}\right)^{2\alpha} \big(C_{\mathrm{v}}(B_-^2/S,t;B_-) + (B_- - E)C_{\mathrm{d}}(B_-^2/S,t;B_-)\big). \end{aligned}$$

Forward-start barrier options

Many barrier options are written with a forward-start condition. That is, at time $t = 0$, when the contract is initiated, it is agreed that at an intermediate time T_1 the holder will receive a barrier contract with later expiry $T > T_1$. The strike and barrier are not known at the outset, but are set by reference to the asset value at T_1, when the barrier option comes into being. A typical example might be a six-month down-and-out call, starting in three months, with the strike set at-the-money (i.e. at the spot price in three months from initiation) and the barrier set at 90% of this value. Here T_1 is three months and T is nine months. There is no barrier for the first three months of the option's life, and for the next six months the option is a regular down-and-out call.

These options are very little different from forward-start vanilla options. The method of valuation is the same: first work back to the intermediate date T_1, then use the values at that time as the intermediate payoff to work back to the present day.

It is not hard to obtain explicit formulæ. Consider as an example the forward-start down-and-out call above, in which the barrier option is at-the-money at time T_1 and the barrier is set equal to $cS(T_1)$ where $0 < c < 1$ (this ensures that the barrier is below the strike so the option is not instantaneously knocked out). The value of the option at time T_1 is obtained by setting $E = S$ and $B_- = cS$ in the formula for a down-and-out call given above. This gives

$$\begin{aligned} C^{\mathrm{f-s}}_{\mathrm{d/o}}(S,T_1) &= C_{\mathrm{d/o}}(S,T_1;E=S,B_-=cS) \\ &= C_{\mathrm{v}}(S,T_1;S,T) - (S/cS)^{2\alpha} C_{\mathrm{v}}\big((cS)^2/S,T_1;S,T\big) \\ &= A^* S, \end{aligned}$$

where A^* is easily worked out from the vanilla call formula. The value of the forward-start down-and-out option at earlier times t is then just

$$C^{\mathrm{f-s}}_{\mathrm{d/o}}(S,t) = A^* S e^{-D(T_1-t)}.$$

Note that this procedure works just as well if different constant volatilities are used for the two legs of the contract, as is sometimes done for forward-start vanilla options, even for time-varying $\sigma(t)$.

The digital down-and-out call

The digital down-and-out call pays \$1 if the asset reaches expiry without touching the lower barrier. By our reflection principle, its value is

$$C^{\mathrm{dig}}_{\mathrm{d/o}}(S,t) = C_{\mathrm{d}}(S,t;E) - (S/B_-)^{2\alpha} C_{\mathrm{d}}(B_-^2/S,t;E)$$

if the strike is above the barrier, and

$$C_{\mathrm{d/o}}\mathrm{dig}(S,t) = C_{\mathrm{d}}(S,t;B_-) - (S/B_-)^{2\alpha} C_{\mathrm{d}}(B_-^2/S,t;B_-)$$

if it is at or below the barrier.

Down-and-out puts

The down-and-out put is knocked out worthless if the asset falls to the barrier B_-, while if it does not, the holder receives the payoff of a vanilla put. Clearly, the vanilla put must be struck above, and preferably well above, the barrier: if it is struck below, the option does not pay out either at expiry or at the barrier. The holder has a good return if the asset stays above the barrier but finishes near to it. Thus the reward profile is rather like that of a butterfly spread, but with the ever-present risk of knock-out before expiry. It is easy to write down its value in terms of vanilla and digital options. The truncated payoff is shown in Figure 3. It is equivalent to a bear spread struck at B_- and E (that is, short one put struck at B_- and long one struck at E) together with a short position in a European cash-or-nothing put with payoff $E - B_-$ and strike B_-. The down-and-out put value can now easily be written down, using the formulæ given above. It is

$$\begin{aligned} P_{\mathrm{d/o}}(S,t) &= P_{\mathrm{v}}(S,t;E) - P_{\mathrm{v}}(S,t;B_-) - (E-B_-)P_{\mathrm{d}}(S,t;B_-) \\ &\quad - \left(\frac{S}{B_-}\right)^{2\alpha} \big(P_{\mathrm{v}}(B_-^2/S,t;E) - P_{\mathrm{v}}(B_-^2/S,t;B_-) + (E-B_-)P_{\mathrm{d}}(B_-^2/S,t;B_-)\big). \end{aligned}$$

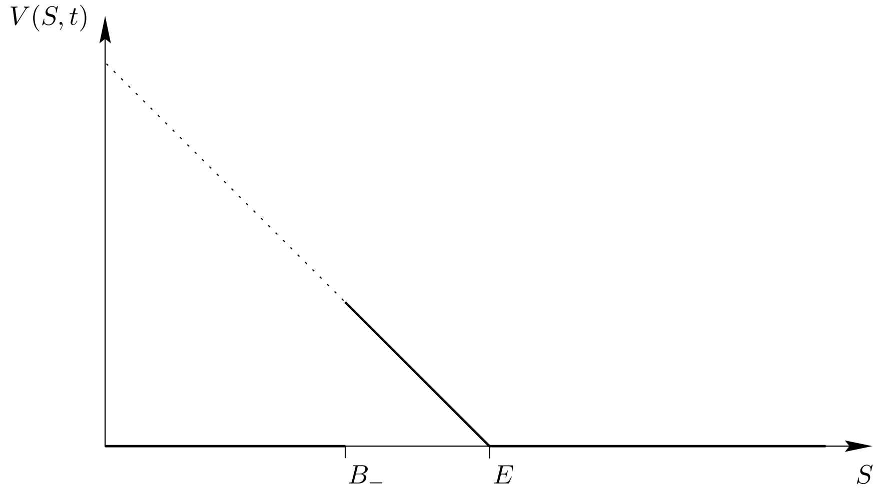

Figure 3: Truncated and extended payoff for a down-and-in put.

Down-and-in contracts

A moment's thought shows that, when there are no rebates,

$$\text{Down-and-out} + \text{Down-and-in} = \text{Vanilla},$$

for both calls and puts. Whatever the asset does, one or other of the contracts on the left-hand side is knocked out, while the other pays out the same as the vanilla contract. Which of the two survives depends on whether the asset reaches the barrier, but the payoff is the same whichever it is. Without rebates, then, down-and-in contracts are valued by first calculating the corresponding down-and-out value as above, and then subtracting it from the vanilla value. (Note that the down-and-in put is the same as the vanilla put if the barrier is above the strike, since the down-and-out put is then worthless.)

Up-and-out/in calls and puts

Up options are analysed in a way exactly analogous to their down counterparts. Assume that there are no rebates, and consider first the up-and-out call. This option expires worthless if the upper barrier $S = B_+$ is touched, and otherwise pays out the same as a vanilla call. Clearly, the barrier must be set above the strike; if it is below, the option is worthless because no asset path gives any payoff at all. In general terms it is very like a down-and-in put, except that it pays off on the upside not the downside. It is useful in the valuation of ladder options; see Section 0.9.2.

In valuing the option, the call payoff must be truncated at the barrier, and extended by zero above B_+. The payoff is therefore the same as that of a bull spread (long one call struck at E, short one call struck at B_+), minus a cash-or-nothing call struck at B_+ paying $B_+ - E$. Its value is therefore

$$\begin{aligned} C_{\mathrm{u/o}}(S,t) &= C_{\mathrm{v}}(S,t;E) - C_{\mathrm{v}}(S,t;B_+) - (B_+ - E)C_{\mathrm{d}}(S,t;B_+) \\ &\quad - \left(\frac{S}{B_+}\right)^{2\alpha} \big(C_{\mathrm{v}}(B_+^2/S,t;E) - C_{\mathrm{v}}(B_+^2/S,t;B_+) + (B_+ - E)C_{\mathrm{d}}(B_+^2/S,t;B_+)\big). \end{aligned}$$

The up-and-out put is very similar to the down-and-out call. If the barrier is set above the strike, we just need to reflect the vanilla put solution in the barrier. Then, the option value is

$$P_{\mathrm{u/o}}(S,t) = P_{\mathrm{v}}(S,t;E) - (S/B_+)^{2\alpha} P_{\mathrm{v}}(B_+^2/S,t;E),$$

If the barrier is below the strike, we need to truncate the put payoff as shown in Figure **??**. This payoff is equivalent to a vanilla put plus $E - B_+$ times a standard cash-or-nothing put paying \$1, both struck at B_+. The option value is then

$$\begin{aligned} P_{\mathrm{u/o}}(S,t) &= P_{\mathrm{v}}(S,t;B_+) + (E - B_+)P_{\mathrm{d}}(S,t;B_+) \\ &\quad - \left(\frac{S}{B_+}\right)^{2\alpha} \left(P_{\mathrm{v}}(B_+^2/S,t;B_+) + (E - B_+)P_{\mathrm{d}}(B_+^2/S,t;B_+)\right), \end{aligned}$$

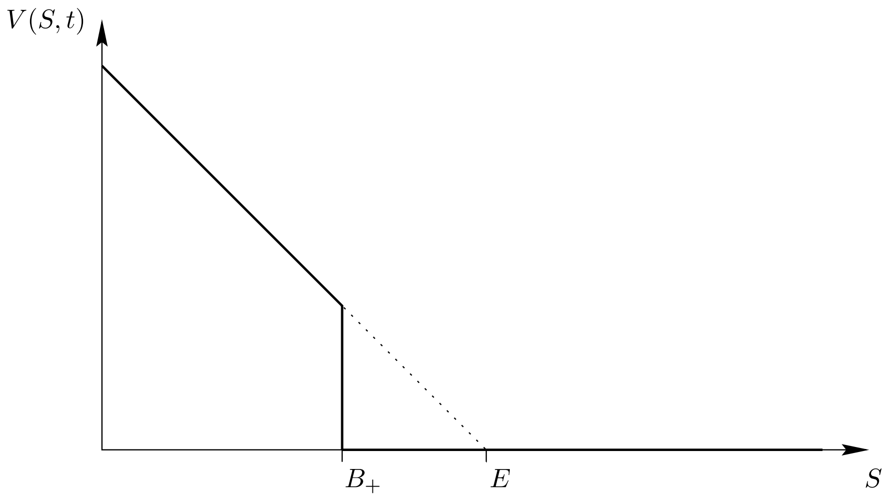

Figure 4: Truncated and extended payoff for an up-and-out put.

Once the up-and-out option values are known, their up-and-in counterparts are valued by out-in parity with the corresponding vanilla option. Note that the up-and-in call has the same value as the vanilla call when the barrier is below the strike, because the up-and-out call is then worthless.

0.5 American digital calls and puts

American digital options, also called 'one-touch' options, are really barrier options with a rebate but no other payoff. Having valued these options, we can easily use them as add-ons to barrier options to deal with rebates.

There are two fundamental digital options: a call and a put. The digital call pays \$1 if the asset rises to the strike E from below, and the put pays \$1 if it falls to the strike from above. (In the terminology of this chapter, the strike is an upper barrier for the call and a lower barrier for the put.)

These options are American in style, because they should be exercised as soon as the strike has been reached. There is no gain from holding the option longer: failure to exercise both loses the income from investing the payoff, and risks losing the payoff altogether if the asset moves the other way. It follows that the optimal exercise boundary is always at the strike.

Two sensible possibilities for receipt of the payoff are: the \$1 may be received immediately, or it may be paid at expiry. In the former case, the value of the option at the money, when $S = E$, is \$1; in the latter it is $\$1e^{-r(T-t)}$, the present value of \$1 at expiry. These become the boundary conditions for the Black–Scholes equation.

Our solution strategy is to satisfy the boundary condition by subtracting off an appropriate solution of the Black–Scholes equation. Then, we value the remainder as a conventional barrier contract, knocking out at the strike, and with a modified payoff. We begin with the more difficult case when the payment is immediate.

Digital options with immediate payouts

Write $C_{\mathrm{d}}^{\mathrm{Am}}(S,t;E)$ for the value of the American digital call with immediate payout. It satisfies the Black–Scholes equation

$$\frac{\partial C_{\mathrm{d}}^{\mathrm{Am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_{\mathrm{d}}^{\mathrm{Am}}}{\partial S^2} + (r-D)S\frac{\partial C_{\mathrm{d}}^{\mathrm{Am}}}{\partial S} - rC_{\mathrm{d}}^{\mathrm{Am}} = 0, \tag{7}$$

for $0 < S < E$. On $S = E$, we receive the payoff:

$$C_{\mathrm{d}}^{\mathrm{Am}}(E,t;E) = 1.$$

At expiry,

$$C_{\mathrm{d}}^{\mathrm{Am}}(S,T;E) = 0.$$

We need a time-independent solution of equation (7). Trying a solution of the form S^{λ} shows that two independent solutions are S^{λ_+} and S^{λ_-}, where λ_+ and λ_- are the positive and negative roots respectively of the quadratic

$$\lambda^2 + (k'-1)\lambda - k = 0.$$

Here as in Chapter 8, $k = r/\frac{1}{2}\sigma^2$ and $k' = (r-D)/\frac{1}{2}\sigma^2$. The roots are

$$\begin{aligned}\lambda_{\pm} &= \tfrac{1}{2}(1-k') \pm \sqrt{\tfrac{1}{4}(k'-1)^2 + k}\\ &= \alpha \pm \sqrt{-\beta},\end{aligned}$$

where $\alpha = \frac{1}{2}(1-k')$ and $\beta = -\left(\frac{1}{4}(1-k')^2 + k\right)$ are also as before. When $D = 0$, the roots are 1 and $-k$. (See Appendix 8 for more details of these exact solutions.)

The solution appropriate for the call is S^{λ_+}, because it is bounded at the origin. We therefore write

$$C_{\rm d}^{\rm Am}(S,t;E) = (S/E)^{\lambda_+} - C_1(S,t),$$

so that $C_1(E,t) = 0$ and $C_1(S,T) = (S/E)^{\lambda_+}$ for $0 < S < E$. Now we can value $C_1(S,t)$ as a barrier contract. We first find the value of the contract that has payoff

$$\begin{cases} (S/E)^{\lambda_+} & \text{for } 0 < S < E \\ 0 & \text{for } E \le S < \infty. \end{cases}$$

Then, we reflect about $S = E$. Note that the reflected payoff is

$$\begin{cases} (S/E)^{2\alpha}\left((E^2/S)/E\right)^{\lambda_+} = (S/E)^{\lambda_-} & \text{for } S > E \\ 0 & \text{for } S \le E, \end{cases}$$

and is just what we need for the American digital put.

The valuation part is best done by solving the appropriate heat equation problem, although it can in fact be obtained by inspection from the catalogue of solutions in Appendix 8. After a brief but painful struggle with the integral it yields

$$\left(\frac{S}{E}\right)^{\lambda_+} N(-d_+),$$

where

$$d_+ = \frac{\log(S/E) + \sigma^2\sqrt{-\beta}(T-t)}{\sigma\sqrt{T-t}}.$$

To obtain the reflected part, replace S by E^2/S in this formula and then multiply by $(S/E)^{2\alpha}$. This gives

$$\left(\frac{S}{E}\right)^{\lambda_-} N(d_-),$$

where

$$d_- = \frac{\log(S/E) - \sigma^2\sqrt{-\beta}(T-t)}{\sigma\sqrt{T-t}}.$$

Finally, putting the pieces together and using the relation $N(d_+) + N(-d_+) = 1$, we get

$$\begin{aligned} C_{\rm d}^{\rm Am}(S,t;E) &= \left(\frac{S}{E}\right)^{\lambda_+} - \left(\frac{S}{E}\right)^{\lambda_+} N(-d_+) + \left(\frac{S}{E}\right)^{\lambda_-} N(d_-) \\ &= \left(\frac{S}{E}\right)^{\lambda_+} N(d_+) + \left(\frac{S}{E}\right)^{\lambda_-} N(d_-). \end{aligned}$$

A similar argument for the American digital put shows that its value is

$$P_{\rm d}^{\rm Am}(S,t;E) = \left(\frac{S}{E}\right)^{\lambda_+} N(-d_+) + \left(\frac{S}{E}\right)^{\lambda_-} N(-d_-).$$

These formulæ simplify marginally when $D = 0$. Note that when $D > 0$, they cannot be obtained from the corresponding formulæ for $D = 0$ by replacing S by $Se^{-D(T-t)}$, because this amounts to a change of variable under which the strike does not remain fixed at E. Note too a neat relation for American digital calls with the same strike and expiry date:

$$C_{\mathrm{d}}^{\mathrm{Am}}(S,t;E) + P_{\mathrm{d}}^{\mathrm{Am}}(S,t;E) = \left(\frac{S}{E}\right)^{\lambda_+} + \left(\frac{S}{E}\right)^{\lambda_-}.$$

Of course, at a given time only one of the two options is alive, the other having been exercised.

American digital options with payout at expiry

It is much easier to deal with American digital options which pay out at expiry. The option is then worth the present value of the payoff (\$1) when the asset reaches the strike:

$$V(E,t) = e^{-r(T-t)}.$$

There is a neat way of finding $V(S,t)$. Suppose that it is a call, which must be valued for $0 < S < E$. Recall the put-call parity relation for European digital options with the same strike (here E):

$$e^{-r(T-t)} = C_{\mathrm{d}}(S,t;E) + P_{\mathrm{d}}(S,t;E).$$

The digital call C_{d} vanishes at expiry for $0 < S < E$, but the put P_{d} does not. However, its reflection in $S = E$ *does* vanish at expiry and is of course equal to P_{d} at $S = E$. So, the appropriate solution is, for $0 < S < E$,

$$V(S,t) = C_{\mathrm{d}}(S,t;E) + (S/E)^{2\alpha}\, P_{\mathrm{d}}(E^2/S,t;E).$$

Similarly, if $V(S,t)$ is a put, for $E < S < \infty$,

$$V(S,t) = P_{\mathrm{d}}(S,t;E) + (S/E)^{2\alpha}\, C_{\mathrm{d}}(E^2/S,t;E).$$

0.6 Barrier options with rebates

Knock-out options

Some knockout options specify that a fixed rebate R is to be paid to the holder if the barrier is breached and the option is knocked out. The payment may be at the time of knockout, or the holder may have to wait until expiry to receive it.

Suppose, for example, that the option is a down-and-out call, and that the payment is made immediately. Then, on the barrier the option value is equal to the rebate: $C_{\mathrm{d/o}}^{\mathrm{reb}}(B_-,t) = R$. We can deal with this variation on the usual boundary condition by splitting the contract into two parts. The first part has the appropriate payoff, but vanishes on the barrier; the second has zero payoff but is equal to R on the barrier. Obviously, whatever path the asset takes, the whole is equal to the sum of the two parts. But we know what the parts are. The first is just a down-and-out call with no rebate. The second is an *American* digital put, with payoff R; its value is

$$R\left((S/E)^{\lambda_+}\, N(-d_+) + (S/E)^{\lambda_-}\, N(-d_-)\right).$$

Adding the unrebated down-and-out call gives the desired value.

If, on the other hand, the payment is not made until expiry, the option value on the barrier is the present value of the rebate: $C_{\mathrm{d/o/r}}(B_-, t) = Re^{-r(T-t)}$. The value of this pay-at-expiry American digital put is (see Section 8)

$$R\big(P_{\mathrm{d}}(S,t;B_-) + (S/B_-)^{2\alpha}C_{\mathrm{d}}(B_-^2/S,t;B_-)\big).$$

Obviously the decomposition of an option with a rebate applies to down-and-out contracts in general:

- The value of any down-and-out (or up-and-out) contract with an immediate rebate R at a fixed barrier B_- (or B_+) is equal to its value with no rebate, plus the value of an American cash-or-nothing put (or call) with strike B_- (or B_+) and payoff R.

The necessary modification to allow for payment at expiry is as described above.

Knock-in options

A knock-in option contract may specify that a rebate R is paid if the barrier is not activated. This just increases the value by that of the corresponding digital knock-out option, struck at the barrier and paying R at expiry. This is a call for a down-and-in option, and a put for an up-and-in option.

0.7 Double knockout options

A **double knock-out** option is one that is knocked out the moment the asset reaches either the upper barrier $S = B_+$ or the lower barrier $S = B_-$. If it runs this gauntlet successfully, there is a payoff at expiry, for example that of a call, a put, a constant (a 'digital' double knock-out), or any other payoff the writer may offer. Likewise, a **double knock-in** option is converted to an underlying vanilla (or indeed any other) contract if either barrier is triggered. Failing this, it expires worthless. Either variety of option can have rebates attached.

As before, we first value the out option with no rebate. The in option value then follows from out-in parity and the corrections for rebates can be found separately. The complicating factor is the presence of *two* barriers. Our simple reflection technique no longer works: we can use it to satisfy the knock-out condition on one barrier or the other, but not on both simultaneously. All is not lost, though. The solution can be found in two quite different forms, each written as an infinite series. Of course, they give the same answer when all the terms have been added up, but as we see below, the rate of convergence of the sum to the solution can be quite different, depending on the time to expiry. The first solution is essentially an infinite series of reflections, alternately in the lower and upper barriers. As the number of reflections increases, the 'images' get further and further away from the original strip, and their contribution to the option value becomes correspondingly smaller. The second form is that of a Fourier series, an infinite sum of special solutions which individually satisfy the boundary conditions; the payoff condition is satisfied by taking the correct combination.[2] We return to the question of which to use later.

It is easiest to explain these solutions in the heat equation framework. For the first, it is easier to see how the reflection works, as reflection for Black–Scholes corresponds to translation by a constant for the heat equation. For the second, it is much easier to calculate the special solutions for the simpler equation.

[2]There is a sense in which this solution is also an image solution, but it is not central to the analysis.

The solution by images

To start with, consider the payoff. At this stage we do not need to pin ourselves down to a specific form; the reader can think of, say, the payoff of a vanilla call. We shall call it $\Lambda(S)$: it is of course defined only for $B_- < S < B_+$. As with single barrier options, we begin by extending the payoff to all other values of S by setting it equal to zero outside the interval $B_- < S < B_+$. (When the payoff is standard, as for a call option, this step can also be thought of as having a preliminary truncation.) Now transform the variables as in Section 8, to obtain the heat equation. When $x = \log(S/B_-)$, the interval $B_- < S < B_+$ becomes

$$0 < x < x_+ = \log(B_+/B_-).$$

Instead of solving just in this interval, though, we solve for $-\infty < x < \infty$, and we cook up an initial condition that will automatically make $u(x,\tau)$ vanish both on $x = 0$ and on $x = x_+$. Write $U_0(x)$ for the transformed version of the extended initial data, and $U(x,\tau)$ for the corresponding solution of the heat equation. As shown in Figure 5(a), $U_0(x)$ vanishes outside the interval $0 < x < x_+$, while in this interval it is equal to $e^{-\alpha x}\Lambda(B_- e^x)/B_-$. (The form shown in this figure is purely illustrative!)

Now we reflect in $x = 0$, considering the initial value

$$U_0(x) - U_0(-x) \tag{8}$$

(see Figure 5(b)). The corresponding solution of the heat equation is zero at $x = 0$, but not at $x = x_+$. We can satisfy this second boundary condition by reflecting in $x = x_+$, and it is easy to see that the new initial value created by this reflection is just that of (8) translated to the right by $2x_+$ (Figure 5(c)). This gives a term of the form

$$U_0(x - 2x_+) - U_0(2x_+ - x), \qquad x_+ < x < 3x_+.$$

Now, of course, we have messed up the boundary condition at $x = 0$, but we fix that up by adding to the initial condition (8) translated to the *left* by $2x_+$; and so on. In the end, we have an infinite series of copies of (8), as indicated in Figure 5(d). The initial data for the heat equation is therefore

$$u(x,0) = \sum_{n=-\infty}^{\infty} U_0(x - 2nx_+) - U_0(2nx_+ - x),$$

and the corresponding solution of the heat equation is

$$u(x,\tau) = \sum_{n=-\infty}^{\infty} U(x - 2nx_+,\tau) - U(2nx_+ - x,\tau).$$

The last step is to write the solution in terms of the original variables S and t. This entails a more elaborate version of the calculation in Section 8. Write $V_0(S,t)$ for the solution to the Black–Scholes equation with the payoff extended by zero, and no barriers. (In the down-and-out call example, this was the vanilla call solution, there called $C_{\mathrm{v}}(S,t;E)$.) Then

$$U(x,\tau) = e^{-\alpha x - \beta\tau} V_0(B_- e^x, t(\tau))/B_-.$$

It follows that

$$U(x - 2nx_+,\tau) = e^{-\alpha(x-2nx_+)-\beta\tau} V_0(B_- e^{x-2nx_+}, t(\tau))/B_-.$$

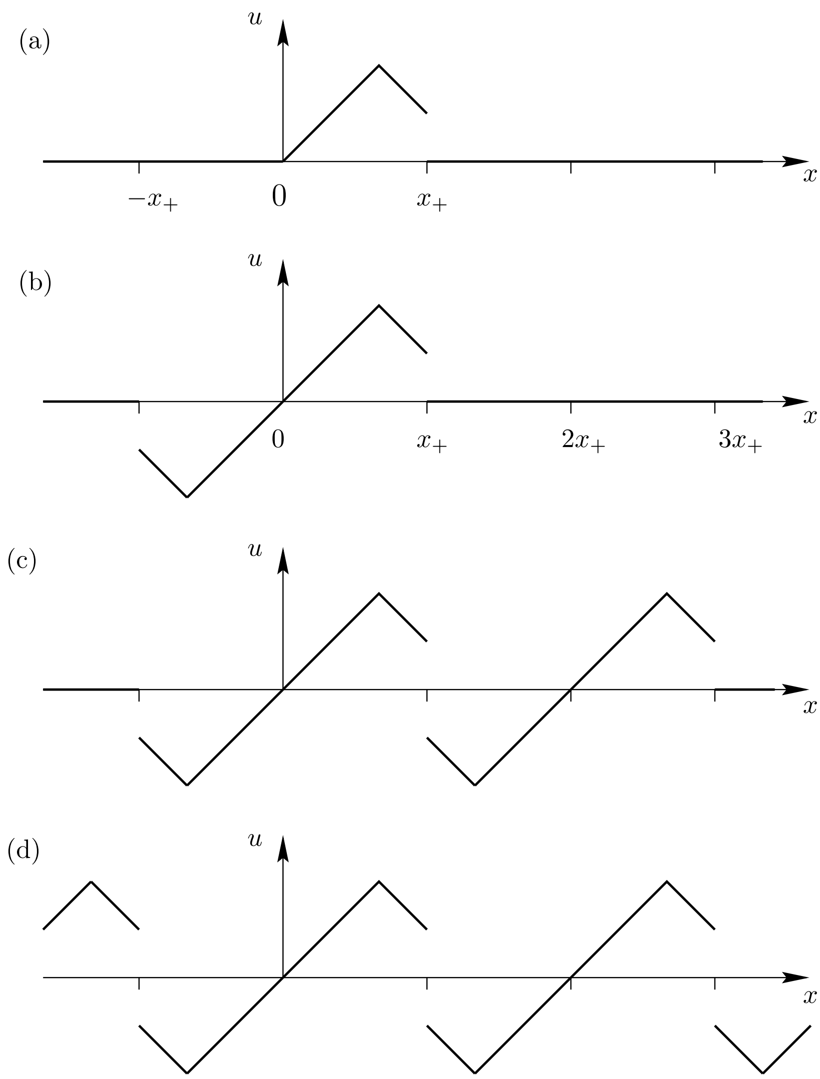

Figure 5: Truncated put payoff, and its extension to an odd periodic function.

We rearrange and substitute $B_- e^x = S$, $e^{x_+} = B_+/B_-$, to get

$$\begin{aligned} B_- e^{\alpha x + \beta\tau} U(x - 2nx_+, \tau) &= e^{2n\alpha x_+} V_0(B_- e^{-2nx_+} e^x, t(\tau)) \\ &= (B_+/B_-)^{2n\alpha} V_0((B_+/B_-)^{-2n} S, t). \end{aligned}$$

A similar calculation shows that

$$B_- e^{\alpha x + \beta\tau} U(2nx_+ - x, \tau) = (S/B_-)^{2\alpha} (B_+/B_-)^{-2n\alpha} V_0((B_+/B_-)^{2n} B_-^2/S, t).$$

We can now put the pieces together to get the double knock-out option value:

$$\begin{aligned} C_{\mathrm{dko}}(S,t) &= B_- e^{\alpha x + \beta\tau} u(x,\tau) \\ &= B_- e^{\alpha x + \beta\tau} \sum_{n=-\infty}^{\infty} U(x - 2nx_+, \tau) - U(2nx_+ - x, \tau) \\ &= \sum_{n=-\infty}^{\infty} (B_+/B_-)^{2n\alpha}\, V_0((B_+/B_-)^{-2n} S, t) \\ &\qquad - \sum_{n=\infty}^{\infty} (S/B_-)^{2\alpha} (B_+/B_-)^{-2n\alpha} V_0((B_+/B_-)^{2n} B_-^2/S, t). \end{aligned}$$

So, in this method we only have to solve the heat equation (or, equivalently, the Black–Scholes equation) once, with the extended payoff $U_0(x)$. In general, we can write this solution as an integral, but for most payoffs of interest we can write $\Lambda(S)$ as a sum of vanilla payoffs. Thus:

- A call with the strike between the barriers is the same as a bull spread struck at E and B_+, minus a digital call paying $B_+ - E$ struck at B_+:

$$C_{\mathrm{v}}(S,T;E) - C_{\mathrm{v}}(S,T;B_+) - (B_+ - E) C_{\mathrm{d}}(S,T;B_+);$$

- A call with the strike below the lower barrier is a bull spread struck at the two barriers, minus a digital call paying $B_+ - E$ struck at B_+, plus a digital call paying $B_- - E$, struck at B_-:

$$C_{\mathrm{v}}(S,T;B_-) - C_{\mathrm{v}}(S,T;B_+) - (B_+ - E) C_{\mathrm{d}}(S,T;B_+) + (B_- - E) C_{\mathrm{d}}(S,T;B_-);$$

- The corresponding puts are obtained by replacing 'bull' by 'bear', 'call' by 'put' and interchanging B_+ and B_-;
- A constant payoff ('receive \$1 if you are not KO'd') is obtained from the 'top hat' payoff, a bull spread in digital calls struck at the barriers:

$$C_{\mathrm{d}}(S,T;B_-) - C_{\mathrm{d}}(S,T;B_+).$$

Each term in the series is a translated and/or reflected version of this basic cell. Obviously, the further away the cell is translated, the smaller is its effect for $0 < x < x_+$, where we need to know $u(x,\tau)$. This effect is particularly marked for small τ, near expiry, and this series solution is clearly preferable to the Fourier series version at these times. For larger τ, the Fourier series converges much more rapidly since as we see below its later terms decay rapidly with increasing time to expiry.

The Fourier series solution

In the second approach, we only solve the heat equation for $0 < x < x_+$, and we choose a form of solution that is guaranteed to satisfy the boundary conditions. We have to solve

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for} \quad 0 < x < x_+,$$

with

$$u(0,\tau) = u(x_+,\tau) = 0$$

and

$$u(x,0) = U_0(x) = e^{-\alpha x}\Lambda(B_- e^x)/B_-.$$

The key observations are:

- Any 'reasonable' function $U_0(x)$ that vanishes at the barriers can be written as a Fourier sine series
$$U_0(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{x_+}\right),$$
where
$$c_n = \frac{1}{2x_+}\int_0^{x_+} U_0(x) \sin\left(\frac{n\pi x}{x_+}\right) dx;$$
- The function
$$\sin(n\pi x/x_+)e^{-n^2\pi^2\tau/x_+^2}$$
is a solution of the heat equation, initially equal to $\sin(n\pi x/x_+)$, which for $\tau \geq 0$ vanishes at $x = 0$ and $x = x_+$.

Put together, these give the solution of our heat equation problem, $u(x,\tau)$, as a sum of particular solutions, which by linearity is also a solution:

$$u(x,\tau) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/x_+)e^{-n^2\pi^2\tau/x_+^2}.$$

Putting $\tau = 0$ shows that the initial condition is satisfied.

In the original variables, we have

$$C_{\text{dko}}(S,t) = S^{\alpha} e^{\frac{1}{2}\sigma^2\beta(T-t)} \sum_{n=1}^{\infty} \tilde{c}_n \sin\left(\frac{n\pi \log(S/B_-)}{\log(B_+/B_-)}\right) e^{-\frac{1}{2}n^2\pi^2\sigma^2(T-t)/(\log(B_+/B_-))^2},$$

where

$$\begin{aligned}\tilde{c}_n &= B_-^{1-\alpha} c_n \\ &= \frac{2}{\log(B_+/B_-)} \int_{B_-}^{B_+} S^{-\alpha}\Lambda(S) \sin\left(\frac{n\pi \log(S/B_-)}{\log(B_+/B_-)}\right) \frac{dS}{S}.\end{aligned}$$

(We have cancelled some redundant factors of B_-.) These formulæ make it amply clear why it was preferable to solve the problem in terms of the heat equation, although with a lot of sweat it can be done in S and t. Even in terms of the heat equation the coefficients are not easy to evaluate, and it may be best to calculate them numerically using for example Simpson's rule (see page 30). Nevertheless, when the payoff is 1, we have

$$c_n =$$

In contrast to the reflection solution, the Fourier series solution converges *very badly* when t is close to T. *The reflection solution should always be used close to expiry,* because its terms are customised to the discontinuities of the payoff. Conversely, though, the Fourier series converges well for times long before expiry, because of the rapid time-decay of the exponential coefficients. It also converges better the larger σ is.

Technical Point: Fourier series and their convergence.
It is a well-known fact that a well-behaved periodic function can be written as an infinite sum of sines and cosines (themselves periodic). When we construct the reflected solution above, we are in effect extending $U_0(x)$ to a function with period $2x_+$. Moreover, the extension is an odd function of x, so the resulting Fourier series contains only sines, and no cosines.

The question of convergence of Fourier series is not simple. However, it can be shown that:

- *If a function $f(x)$ is analytic, then its Fourier series converges to $f(x)$ everywhere;*
- *If $f(x)$ is differentiable except at a finite number of jump discontinuities, at which it and its derivative have both left- and right-hand limits, then the Fourier series converges to $f(x)$ at points of continuity, and to*

$$\tfrac{1}{2}\left(f(x^+) + f(x^-)\right)$$

  at the jump points.

The first of these conditions applies to solutions of the heat equation for $\tau > 0$, and the second applies to any reasonable payoff. It says that the Fourier series converges to the average of the two values on either side of the jump. This in particular implies that the Fourier series of the periodic extension of $U_0(x)$ converges to 0 at $x = 0$ and $x = x_+$. Of course, the Fourier series for $u(x, \tau)$ satisfies the barrier conditions automatically for $\tau > 0$.

*The difficulty comes when we specify the notion of convergence. It is 'pointwise', meaning that if we fix a point x, we can make the Fourier series as close to $f(x)$ as we want by taking enough terms. The catch is that we may have to take many more terms at some points than others. When $f(x)$ is analytic, a value of N can be found that will guarantee getting close to the target $f(x)$ whatever x we choose (***uniform convergence***). But if $f(x)$ has jump discontinuities, the convergence is not at all uniform. Unfortunately, the partial sum to N terms overshoots near the jump by around 8.9%* however many terms we take*! This overshoot is known as the Gibbs phenomenon. Furthermore, the poor convergence carries over to the series solution of the heat equation for small τ. When τ is small, the x-derivative of $u(x, \tau)$ is large near the place where the jump in the initial data was, and although the convergence is uniform, we may have to take an unacceptably large number of terms to get an accurate approximation. However, this difficulty diminishes as τ increases. A similar, but less marked, effect is seen if $f(x)$ is continuous but its first (Delta) or second (Gamma) derivative has jumps. For this reason, the Fourier series solution should never be used for times close to expiry.*

(In case you had been wondering, the Gibbs overshoot is reconciled with pointwise convergence by noting that the point at which it occurs moves closer and closer to the jump point.

Away from the jump we get convergence, but only after we have waited for the overshoot to pass us.)

0.7.1 A digital double knock-out option

As an extension of the double knock-out option, consider an option which pays \$1 (and is then knocked out) if the asset touches either barrier, but otherwise pays nothing. If a double knock-out option pays a rebate of \$1 on knock-out, the value of this option must be added to the zero-rebate value computed as above. Note that this option is *not* equivalent to an American digital put struck at B_- plus an American digital call struck at B_+, because *both* barriers disappear on knock-out.

Its valuation is a combination of that for American digital options and double knock-out options. First, we subtract off the time-independent solution of the Black–Scholes equation that takes care of the barrier conditions, thereby changing the payoff. Then, we value the remaining part as a zero-rebate double-knockout option in the usual way.

We need the two time-independent solutions S^{λ_+} and S^{λ_-} of the Black–Scholes equation, found in Section 8. Now choose a combination $c_+S^{\lambda_+} + c_-S^{\lambda_-}$ to be equal to 1 on both barriers and subtract it from $V(S,t)$:

$$V(S,t) = \frac{1}{2}\left(\frac{(B_+)^{-\lambda_-} - (B_-)^{-\lambda_-}}{(B_+)^{\lambda_+-\lambda_-} - (B_-)^{\lambda_+-\lambda_-}}S^{\lambda_+} + \frac{(B_+)^{-\lambda_+} - (B_-)^{-\lambda_+}}{(B_+)^{\lambda_--\lambda_+} - (B_-)^{\lambda_--\lambda_+}}S^{\lambda_-}\right) - V_1(S,t).$$

Then $V_1(S,t)$ satisfies a zero-rebate double-knock-out problem with final value equal to $\left(c_+S^{\lambda_+} + c_-S^{\lambda_-}\right)$, and can be valued in the standard way.

0.8 Perpetual barrier options

There are two simple ‘perpetual’ barrier options, that is, options with no expiry date. One pays \$1 if the asset rises to B_+ (it does not matter when), and the other pays \$1 if it falls to B_-. They are thus a perpetual American digital call and put respectively. Both of these events have nonzero probability, and the values of the options can be thought of as the expected discounted first exit time. (Work out the probability distribution of the exit time, at which the holder receives \$1, discount to the present day, and take expectations, all of course under the risk-neutral random walk.) It is easier to solve directly, though. The option values are functions of S only, and must behave at $S = 0$ and $S = \infty$ respectively. The solutions are

$$C^{\mathrm{Am}}_{\mathrm{perp}}(S) = (S/B_+)^{\lambda_+}$$

and

$$P^{\mathrm{Am}}_{\mathrm{perp}}(S) = (S/B_-)^{\lambda_-}$$

for the up and down option respectively, where λ_+ and λ_- are as defined earlier (and in Appendix 8).

0.9 Up-down and ladder options

The main purpose of this section is to discuss ladder options, whose properties change if the asset price crosses a barrier. The basic idea is the same as that of an in option: if the barrier is triggered, the holder receives (exchanges into) a different contract. For example, in the case of a down-and-in call, the new contract is a vanilla call. It follows from no-arbitrage that the value of the original option on the barrier must be equal to that of the new contract. We begin with a simple example, which we call an up-down option, to illustrate the idea. Although to our knowledge this option is not traded, it could well be.

0.9.1 An up-down option

Suppose we have a contract that pays \$1 if, before expiry, the asset first crosses an upper barrier B_+ and subsequently crosses a lower barrier B_-; otherwise, it pays nothing. How do we value this? There are two possibilities: either the barrier is triggered, or it is not. Following the general principle of working backwards from expiry, the first task is to calculate the value of the contract we get after the asset reaches B_+. Then, we use this value as the boundary condition for the valuation of the option before the first barrier B_+ has been reached; of course, this includes the initial time. If the barrier is never reached, this second value holds right up to expiry. Note that this option is path-dependent. If S is below the upper barrier, the option has one value if the barrier has been crossed previously (although then S must be above B_-), and another if it has not. Of course, the two values agree on the barrier itself.

The answer to the first question, to find the value above the upper barrier, is simple. Once the upper barrier has been triggered, it disappears and the contract turns into one that pays \$1 if the asset falls to B_-. This is just an American digital put, struck at B_-, with value $P_{\mathrm{d}}^{\mathrm{Am}}(S,t;B_-)$. So, before the upper barrier has been reached, the value of the up-down option there is that of the put:

$$V_{\mathrm{u/d}}(B_+,t) = P_{\mathrm{d}}^{\mathrm{Am}}(B_+,t;B_-). \tag{9}$$

Note that this holds for all $B_- < S < \infty$, which includes part of the region below $S = B_+$.

When we value the option before the upper barrier is reached, we solve the Black–Scholes equation for $0 < S < B_+$, with (9) on $S = B_+$ and $V_{\mathrm{u/d}}(S,T) = 0$. We would like to subtract off the value of the American digital put, in order to get zero on $S = B_+$. But we cannot do this as it stands, because $P_{\mathrm{d}}^{\mathrm{Am}}(S,t;B_-)$ is only defined for $S > B_-$. However, if we reflect in $S = B_+$, we get a solution of the Black–Scholes equation that is defined for $0 < S < (B_+/B_-)B_+$, which includes the region we need. So, write

$$V_{\mathrm{u/d}}(S,t) = \left(\frac{S}{B_+}\right)^{2\alpha} P_{\mathrm{d}}^{\mathrm{Am}}(B_+^2/S,t;B_-) - V_1(S,t).$$

But $V_1(S,T) = 0$, because the American digital put is worthless at expiry; also $V_1(B_+,t) = 0$ by construction. So, $V_1(S,t) = 0$, and the up-down option option has the value $V_{\mathrm{u/d}}(S,t) = (S/B_+)^{2\alpha} P_{\mathrm{d}}^{\mathrm{Am}}(B_+^2/S,t;B_-)$.

It is straightforward to adapt this idea to cover the cases when the option pays \$1 if the asset rises to B_+ and is below (or above) B_- at expiry, rather than at any time before. The down-up option is likewise easily treated.

0.9.2 Ladder options

Imagine you hold a European call. The asset price soars: joy is unconfined. But then, just before expiry, it plummets. Gloom and doom all round. What could be more distressing? The **ladder option** insulates the holder from the worst plunges of the roller coaster. It works like this.

The underlying option is a European vanilla call with strike E (a similar put option can be constructed but we do not give the details). Above the strike, a number of 'rungs' of the ladder are set: $E < R_1 < R_2 < \cdots < R_N$. We assume that the initial value is below the first rung: $S(0) < R_1$. If before expiry the asset does *not* reach R_1, the payoff is the usual $\max(S - E, 0)$. But the moment the first rung is reached, the payoff is altered. The holder is *guaranteed* to receive $R_1 - E$, which is what he would get from a vanilla call if the asset were to be exactly on the rung R_1 at expiry. If the asset rises further, the call payoff stands, while if it falls, the amount $R_1 - E$ is locked in. Thus, the payoff alters to

$$\max(S - E, R_1 - E, 0) = R_1 - E + \max(S - R_1, 0),$$

showing the locked-in profit $R_1 - E$ and the remaining upside potential.[3] This payoff applies until the second rung is reached, at which point the payoff is again reset, to

$$\max(S - E, R_2 - E, 0) = R_2 - E + \max(S - R_2, 0);$$

and so on. In figure 6 we see three random walks with different outcomes, as explained in the caption.

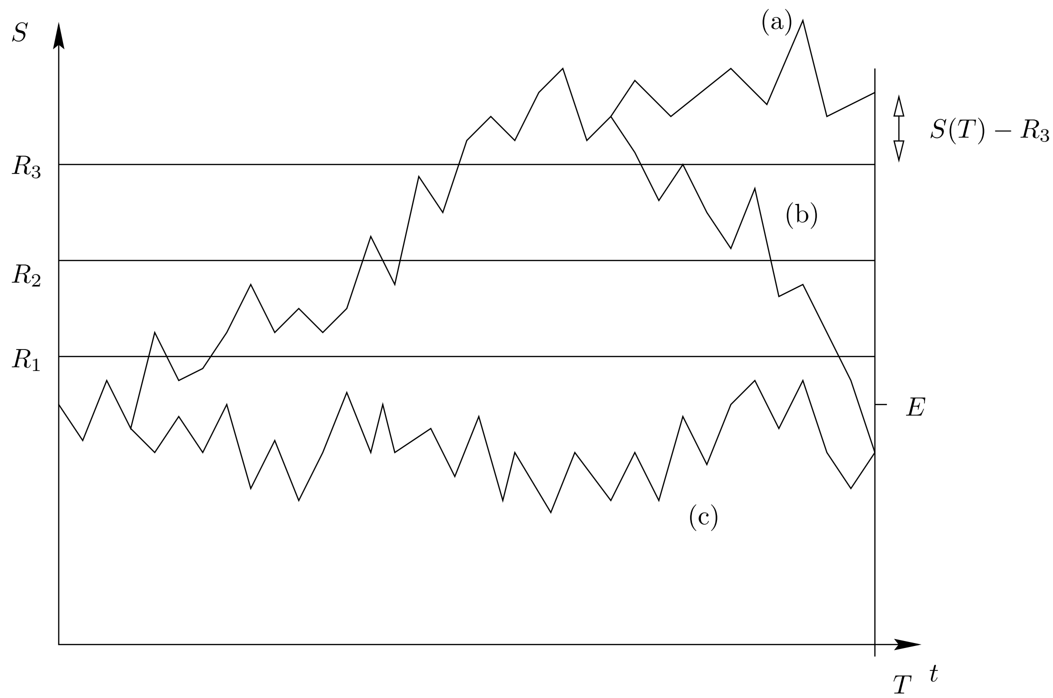

Figure 6: A three-rung ladder option. Random walk (a) pays out $(R_3 - E) + (S(T) - R_3) = S - E$; walk (b) pays $R_3 - E$ despite its low closing price; walk (c) pays nothing.

Like the up-down option, the ladder option is path-dependent. Its value depends on how many of the rungs have been crossed. It is therefore a kind of lookback option, in that its value depends on the position of the maximum asset price to date *vis à vis* the rungs of the ladder. Nevertheless, it is easier to treat it as a barrier-type option because the rungs are discrete.[4]

[3] A minor complication is if the locked-in profit is paid immediately, not at expiry.

[4] If you want to do some mathematical weight-lifting, try to recover the continuously-sampled lookback option as the limit of an appropriate ladder contract as the number of rungs tends to infinity and the rung spacing tends to zero...

For a given S, we can only calculate the option value if we also know how many rungs have been crossed. Of course, the rungs below S have already been crossed, so we just need to consider those above. As with lookback options, let us write

$$J = \max_{0 \le \tau \le t} S(\tau)$$

for the maximum asset value to date. Note that when $t = 0$, $J = S(0)$, and that $J \ge S$. The easiest case is when J is above the highest rung, R_N. Then, we know that S has crossed all the rungs, and the contract cannot change again. We have to solve for all $0 < S < \infty$, as the asset can fall again having risen above R_N. If $R_{N-1} < J < R_N$, we value the option for $0 < S < R_{N-1}$, with a boundary condition on $S = R_N$ which says that we exchange into the previous contract, in which all the rungs have been crossed. In this way, we proceed backwards down the ladder until the current value of J is reached.

Consider first the case $J > R_N$, so the top rung has been crossed. Then the payoff is

$$R_N - E + \max(S - R_N, 0).$$

The first of these is a cash sum and the second is a vanilla option struck at R_N. The value of the option in this case is

$$V_N(S,t) = (R_N - E)e^{-r(T-t)} + C_{\mathrm{v}}(S,t;R_N),$$

for all $0 < S < \infty$. Now suppose that the top rung has not been reached but the penultimate one has, so $R_{N-1} < J < R_N$. On reaching the top rung, the ladder option exchanges into the contract above, with value $V_N(R_N,t)$, while at expiry it receives the payoff

$$R_{N-1} - E + \max(S - R_{N-1}, 0),$$

but only for $0 < S < R_N$. That is,

- If we reach the top rung R_N, we receive a vanilla call struck at R_N and a cash sum $R_N - E$ (paid at expiry);
- If we do not, at expiry we receive a call payoff, struck at R_{N-1}, plus a cash amount $R_{N-1} - E$.

We can rearrange the possible outcomes into three independent contracts:

- A contract that pays $R_{N-1} - E$ at expiry.
- A contract that exchanges into a vanilla call (struck at R_N) if we reach the rung R_N, and pays nothing otherwise;
- A contract that knocks out at the rung, yielding a rebate $R_N - R_{N-1}$ (paid at expiry: this is the profit locked in), but if not knocked out, at expiry pays the vanilla call payoff (struck at R_{N-1}).

That is, the ladder option between rungs R_{N-1} and R_N consists of cash plus two options:

- The present value of $R_{N-1} - E$;
- A vanilla call option struck at R_N;
- An up-and-out call struck at R_{N-1}, paying (at expiry) a rebate $R_N - R_{N-1}$, with barrier at R_N.

The first of the two options could be thought of as an up-and-in call option, with both barrier and strike at R_N; but, as shown in Section 8, this is the same as the vanilla call that it exchanges into. The second option can be split further, into an up-and-out call with no rebate and an American pay-at-expiry digital call. Using the formulæ given earlier in this chapter, its value is

$$V_{N-1}(S,t) = (R_{N-1} - E)e^{-r(T-t)} + C_{\mathrm{v}}(S,t;R_{N-1})$$
$$+(S/R_N)^{2\alpha}\left(C_{\mathrm{v}}(R_N^2/S,t;R_N) - C_{\mathrm{v}}(R_N^2/S,t;R_{N-1}) + (R_N - R_{N-1})e^{-r(T-t)}\right),$$

and this holds for $0 < S < R_{N-1}$. (We have used the put-call parity result for digital options to simplify this formula. Several cancellations occurred in the process. It is easy to check that it satisfies the boundary and payoff values.)

Now suppose that $R_{N-2} < J < R_{N-1}$. We are guaranteed to receive $R_{N-2} - E$ whatever happens, and the payoff if the next rung R_{N-1} is not reached is

$$R_{N-2} - E + \max(S - R_{N-2}, 0).$$

On reaching the rung R_{N-1}, we exchange into the previous contract. In particular, we receive the present value of $R_{N-1} - R_{N-2}$, locking in the profit, plus the vanilla and up-and-out calls above. Now the latter two contracts have zero payoff for $0 < S < R_{N-1}$. We can just add them on, to give the value of $V_{N-2}(S,t)$, for $0 < S < R_{N-1}$, as the sum of:

- The present value of $R_{N-2} - E$;
- A vanilla call option struck at R_N, with barrier at R_N;
- An up-and-out call struck at R_{N-1}, paying (at expiry) a rebate $R_N - R_{N-1}$, with barrier at R_N;
- An up-and-out call struck at R_{N-2}, paying (at expiry) a rebate $R_{N-1} - R_{N-2}$, with barrier at R_{N-1}.

The pattern is now clear. As we work down the ladder we successively reduce the sum we are guaranteed to receive, at the same time adding a new up-and-out call. To go from V_{N-k+1} to V_{N-k}, i.e. to evaluate the option for $R_{N-k} \le J < R_{N-k+1}$, we must do the following to V_{N-k+1}:

- Replace the present value of $R_{N-k+1} - E$ by that of $R_{N-k} - E$;
- Add a new up-and-out call struck at R_{N-k}, paying (at expiry) a rebate $R_{N-k+1} - R_{N-k}$, with barrier at R_{N-k+1}.

Note that we have not made any special assumptions about the evolution of the asset price. If your market model is not standard geometric Brownian motion, you will have to value the components numerically, but the computational time only increases linearly with the number of rungs.

0.10 Shout options

A ladder option has fixed pre-specified rungs. A **shout** option is a ladder option with just one rung, but the twist is that the rung level is set, equal to the asset price, *at a time chosen by the holder*, provided only that the option is in-the-money at the time. The holder thus has to determine the 'optimal shouting frontier' (!) as part of the solution, and this option is therefore American in character. As with a ladder option, we first determine the value of the contract exchanged into.

When the option is in-the-money, $S > E$, the contract immediately after shouting is an at-the-money call, plus a guaranteed sum $S - E$, paid at expiry. The value of the at-the-money call is

$$SF(t) = S\left(e^{-D(T-t)}N\left(\frac{(r - D + \frac{1}{2}\sigma^2)\sqrt{T-t}}{\sigma}\right) - e^{-r(T-t)}N\left(\frac{(r - D - \frac{1}{2}\sigma^2)\sqrt{T-t}}{\sigma}\right)\right).$$

It is easy to see that $F(0)$=0 (an at-the-money call is worthless at expiry). Less easy to see whether dF/dt is negative, or what F does as $t \to -\infty$. The present value of $S - E$ paid at expiry is $(S - E)e^{-r(T-t)}$. It follows that on shouting, the holder effectively receives

$$\Lambda(S,t) = SF(t) + (S - E)e^{-r(T-t)},$$

for $E < S < \infty$ (the at-the-money call could in principle be sold for $SF(t)$). As with an American option, we must solve the Black–Scholes inequality

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV \leq 0$$

for $E < S < \infty$. It is supplemented by the constraint that

$$V(S,t) \geq \Lambda(S,t),$$

and the final condition

$$V(S,T) = \max(S - E, 0) \quad \text{for} \quad E < S < \infty.$$

(In practice, if the option is always shouted out before expiry or expires out-of-the money, so this condition is not used.)

When the option is out-of-the-money, $S < E$, the option to shout does not exist. Therefore we have the Black–Scholes *equation* to solve in this region, with the final condition

$$V(S,T) = 0 \quad \text{for} \quad 0 < S < E.$$

The option value for $S < E$ is of course affected by the potential for shouting: even if S is currently out-of-the-money, it may later rise above E. The connection is at $S = E$, where we impose the conditions that

$$V \quad \text{and} \quad \frac{\partial V}{\partial S} \quad \text{are continuous.}$$

So we see that the shout option is a European-American hybrid: European for $S < E$, American for $S > E$. Its value must be found numerically, Jeff to comment.

Another variation on the shout option idea is to have a call option whose strike is initially set at E, but which the holder can reset to be at-the-money by shouting at a time of his choice. The difference from the previous option is that no profit is locked in. Such an option would do well if the asset first fell then rose, and if the holder shouted near the bottom of the market. In this case the contract exercised into is an at-the-money call, whose value is $SF(t)$. We therefore solve the Black–Scholes inequality, with the constraint that the option value $V(S,t) \geq SF(t)$, and that its delta is continuous. In Fig 8 we see that Jeff has plotted out...

0.11 Cliquet or ratchet options

The final variation on the ladder theme is the **cliquet** or **ratchet option**. With a ladder option, the profit is locked in when preset price levels are reached; with a shout option it is locked in at a time chosen by the holder. In the case of a ratchet option, the profit locked in is determined by the asset price on certain prespecified dates. As we showed in Chapter 8, a ratchet option can be decomposed into a series of forward start options, and it is discussed there. In Chapter 8 we discuss the lookback put option, which represents the ultimate in locked-in profits, as it allows the holder to sell for the maximum realised value of the asset price over its lifetime. Needless to say, it is expensive; ratchet, shout and ladder options are cheaper and less effective alternatives.

0.12 Accrual options

An **accrual** note (or option) is a knockout option with a rebate which is proportional to the time between initiation and knockout, and paid immediately. The barrier condition in general is therefore

$$V(B,t) = \bar{R}t,$$

where $\bar{R}$ is constant; B may represent an upper barrier, a lower barrier, or both. It is equally possible to specify that the rebate is proportional to the time remaining to expiry when the option is knocked out:

$$V(B,t) = \bar{R}(T-t).$$

This turns out to be slightly easier to manipulate; of course, the sum of the two forms of rebate is just the constant rebate $\bar{R}T$, so one can be obtained from the other using the results of Section 8.

The rebate is an add-on to the unrebated contract. It is therefore enough to value the knock-out contract that pays the rebate on the barrier, and nothing otherwise. Suppose the option is a down-and-out contract. We therefore need to solve the Black–Scholes equation for $S > B_-$, with $V(B_-,t) = \bar{R}(T-t)$, $V \to 0$ as $S \to \infty$, and $V(S,T) = 0$. Our strategy is again to find a particular solution of the Black–Scholes equation that is equal to the rebate on the barrier, subtract it off, and value the remainder as a conventional barrier contract with no rebate. It is easiest to do this using the heat equation. After setting $V(S,t) = e^{\alpha x + \beta\tau}u(x,\tau)$ as in Section 8, we have to solve

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

for $x > 0$, with

$$u(0,\tau) = R^*\tau e^{-\beta\tau},$$

where $R^* = R/\frac{1}{2}\sigma^2$. Now we know that

$$U(x,\tau;\beta) = e^{-\beta\tau - \sqrt{-\beta}\,x}$$

is a solution of the heat equation (recall that β is negative), equal to $e^{-\beta\tau}$ at $x = 0$. It follows that

$$\frac{\partial U}{\partial \beta} = \left(-\tau + \tfrac{1}{2}x/\sqrt{-\beta}\right) e^{-\beta\tau - \sqrt{-\beta}\,x}$$

is also a solution. It is equal to $-\tau e^{-\beta\tau}$ at $x=0$, and it decays as $x \to \infty$ (translating into good behaviour as $S \to \infty$). This gives us our particular solution: write

$$u(x,\tau) = R^* \left(\tau - \tfrac{1}{2}x/\sqrt{-\beta}\right) e^{-\beta\tau - \sqrt{-\beta}\,x} + u_1(x,\tau),$$

and then $u_1(x,\tau)$ is a conventional down-and-out contract with transformed payoff

$$u_1(x,0) = (\tfrac{1}{2}R^*/\sqrt{-\beta})xe^{-\sqrt{-\beta}x}$$

for $x > 0$. Extending the payoff by zero and evaluating the resulting integral (not much fun) eventually leads to the solution

$$V(S,t) = \frac{\bar{R}\sqrt{T-t}}{\sigma\sqrt{-\beta}} \left(\left(\frac{S}{B_-}\right)^{\lambda_-} (N'(d_-) - d_-N(d_-)) - \left(\frac{S}{B_-}\right)^{\lambda_+} (N'(d_+) - d_+N(d_+)) \right),$$

where $N'(d) = (1/\sqrt{2\pi})e^{-\frac{1}{2}d^2}$; as in Section 8

$$d_\pm = \frac{\log(S/B_-) \pm \sigma^2\sqrt{-\beta}(T-t)}{\sigma\sqrt{T-t}},$$

and $\lambda_\pm = \alpha \pm \sqrt{-\beta}$ are as defined in Section 8. The corresponding formula for an up-and-out option is obtained by reflection, and is obtained from the down-and-out formula by replacing $d_\pm$, $\lambda_\pm$ and B_- by $-d_\mp$. $\lambda_\mp$ and B_+ respectively.

These formulæ are complicated enough for us not to want to present many further generalisations! However it is easy enough in principle to deal with a double barrier option, using a combination of the down-and-out and the up-and-out solutions to find the particular integral. Finally, if the rebate is paid at expiry, we must take the present value of the rebate, $\bar{R}(T-t)e^{-r(T-t)}$. This results in a small modification of the method.

0.13 Corridor and switch options

A **switch option** is a generalisation of an accrual option. A switch call option with trigger level B_+ pays out an amount proportional to the time that the asset price spends above B_+: see Figure 7. Typically the contract is specified discretely, with a set payment for each day during which the asset is above B_+, but it is easier to price the continuous-time version. (We sidestep the issue of how exactly to measure the time spent above a barrier by a Brownian motion...)

Suppose that the option pays its holder at the rate \$1 per unit time spent above the trigger level, or barrier, and nothing otherwise. As usual, there are two modes of payment. In one, the payment is made 'in running', that is immediately; in the other it is made at expiry. If we want to formulate this option as a partial differential equation problem, we have to introduce a new, path-dependent, variable to measure the amount of time spent above the barrier. We look at the switch option from this point of view in Chapter 8. However, we can also value it by thinking of it as a continuous distribution of standard digital call options.

Suppose that the option is written today, at time t, and expires at time T, and consider first the case in which payments are made immediately. Take any time T' between t and T. In the small interval $(T', T'+\delta T')$, the holder will receive $\$1 \cdot \delta T'$ if $S > B_+$, and nothing otherwise. That is, he will receive the payoff of $\delta T'$ standard digital options struck at B_+

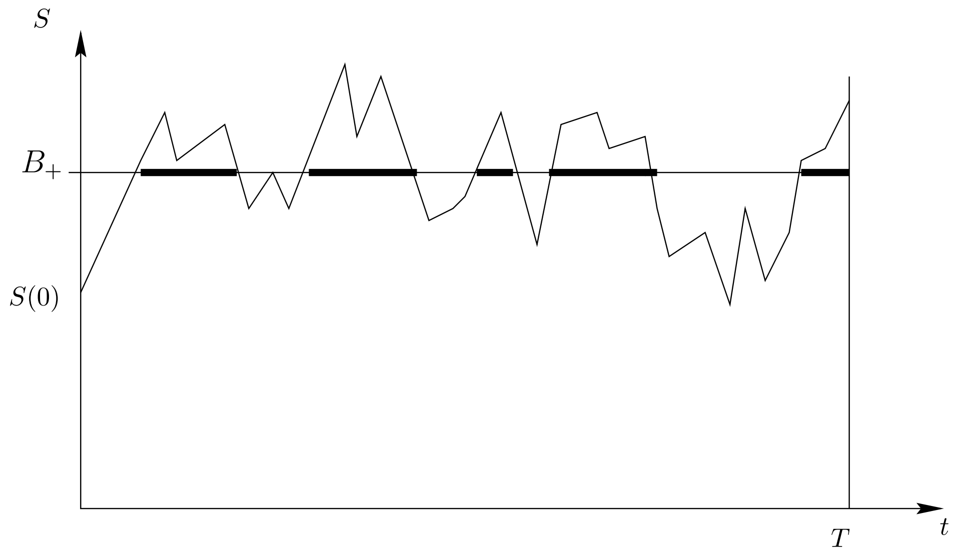

Figure 7: Random walk for a switch option. The holder receives \$1 per unit time that the asset is above the trigger level, B_+ as shown.

and expiring at T'. The value of the switch option is then just the superposition of all these digital calls as T' varies from t to T. That is,

$$\begin{aligned} C_{\mathrm{sw}}(S,t) &= \int_t^T C_{\mathrm{d}}(S,t;B_+,T')\,dT' \\ &= \int_t^T e^{-r(T'-t)} N\left(d_2(S,t;B_+,T')\right)\,dT'. \end{aligned}$$

The integral in this formula can be evaluated explicitly, after integration by parts and a lot of algebra, but it is probably best to evaluate it numerically.

When the payment is made at expiry, the payoffs of the constituent digital calls must be discounted to the final expiry date. The corresponding formula is slightly simpler:

$$\begin{aligned} C_{\mathrm{sw}}(S,t) &= \int_t^T e^{-r(T-T')} C_{\mathrm{d}}(S,t;B_+,T')\,dT' \\ &= e^{-r(T-t)} \int_t^T N\left(d_2(S,t;B_+,T')\right)\,dT'. \end{aligned}$$

A**corridor option** either pays out at a constant rate whenever the asset is outside the range $B_- < S < B_+$ and pays nothing when it is within this range, or vice versa. The former can clearly be valued as the sum of a switch call struck at the upper barrier, and a switch put struck at the lower barrier. To value the latter with immediate payment, note that a corridor option that pays when the asset is within the range and one that pays when it is outside add up to a security that pays \$1 per unit time whatever the value of S, and vanishes at expiry. Its value is $\left(1 - e^{-r(T-t)}\right)/r$. When payment is at expiry, the sum of the two is $Te^{-r(T-t)}$.[5]

[5]The units look funny because both are multiplied by a nominal \$1 per unit time.

0.14 Intermittent barrier options

All the barriers we have so far described have been simple and static: the barrier is always there. Sometimes, though, barrier options are sampled intermittently. That is, the option only knocks out or in if the asset price is beyond the barrier at specified times. For example, the basis may be the closing price each Friday. This scheme greatly reduces the potential for dispute whether the barrier has or has not been reached. On the other hand, the value, and hence the Greeks, can vary very rapidly near the barrier.

In Figure **??** we see a sketch of two random walks for an asset negotiating an intermittent barrier option with one sampling date T_1. Walk (a) passes the barrier successfully and receives (say) the call payoff $\max(S-E)$; walk (b) results in knock-out.

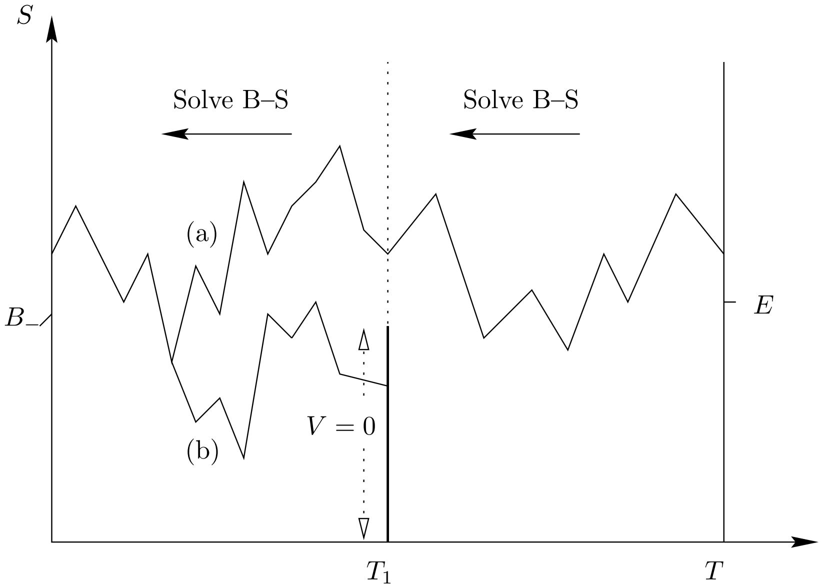

Figure 8: intermittent

The key to valuing this option is, as ever, to work backwards from expiry. For times from T back to T_1, the barrier can have no effect and the option has the vanilla value. What happens at $t = T_1$ depends on whether the asset is above or below the barrier. If $S > B_-$, no-arbitrage says that the option value should be continuous. But for $S \leq B_-$, the option value as $t \to T_1$ from below must be zero, to represent the knock-out condition, while as $t \to T_1$ from above it is the vanilla value. This jump does not represent an arbitrage opportunity, as there are no sample paths that go from one side of the barrier to the other. Thus, to value this option,

- Solve the Black–Scholes equation backwards from expiry to time T_1;
- To go from just after T_1 to just before,
  - Discard the values of V for $) < S \leq B_-$ and replace them by zero;
  - Retain the values of V for $S > E$.
- Use the new values as final data to solve backwards from $t = T_1$.

This procedure is repeated for each sampling date.

In practice, the reset interval is often a lot smaller than the life of the option. In this case, it is possible to find an excellent approximation to the solution based on a large number$N$ of resets. In a landmark paper in 1997, Broadie, Glasserman and Kou introduced the 'BGK barrier correction', whereby the value of a down-and-out call (other barrier options are similar) with barrier B_- and discrete sampling is approximately equal to that of the same *contin uously-sampled* option, but with the barrier replaced by

$$B_- \mathrm{e}^{-\beta\sigma\sqrt{T/N}}$$

where $\beta = -\zeta(\frac{1}{2})/2\pi \approx 0,5826$, $\zeta(\cdot)$ being Riemann's zeta function, is a universal constant of discrete sampling theory. This says that the discretely sampled option is slightly more valuable; the extra value comes from the paths that fall below the barrier between reset dates but rise above it at those dates. The approximation is in principle accurate[6] only up to $O(1/N)$ (ie the error is $O(N^{-\frac{3}{2}})$, but it is still very accurate even for small N. Although BGK only analysed the constant=parameter Black–Scholes case, the correction is widely applicable to other models and details can be found in two papers on my website.

0.15 Parisian and corridor options

A Parisian option is one which is not available in August.

0.16 Barrier options on futures

Many of the formulæ of this chapter become much simpler when the underlying is a futures contract. One reason is that the formulæ for the basic vanilla and digital options are simpler, as the "d's" are just

$$d_{1,2} = \frac{\log(F/E) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

Another reason is that the awkward power term $S^{2\alpha}$ is replaced by the simpler F, as $2\alpha = 1$. The futures formulæ can be obtained from the usual ones as follows:

- Set $D = r$, and so $2\alpha = 1$.
- Replace S by F.

Note also the simplicity of the reflection principle:

- If $V(F,t)$ is a solution of the Black–Scholes equation for derivatives contingent on a future,

  $$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0,$$

  then $W(F,t) = FV(B^2/F,t)$ is also a solution.

In Section 8, we showed that vanilla call and put prices on futures are related by

$$C_{\mathrm{v}}(F,t;E) = (F/E)P_{\mathrm{v}}(E^2/F,t;E)$$

[6] BGK only show up to $O(1/\sqrt{N})$, but it is a better approximation than they perhaps expected.

and, conversely,

$$P_{\mathrm{v}}(F,t;E) = (F/E)C_{\mathrm{v}}(E^2/F,t;E).$$

This result is now seen to be a consequence of the reflection principle. Take, for example the first relation. Both sides of the equation are solutions of the Black–Scholes equation. To show that they are equal, all we have to do is to show that their payoffs are the same. That is,

$$\begin{aligned}(F/E)P_{\mathrm{v}}(E^2/F,T;E) &= (F/E)\max\big(E - E^2/F, 0\big)\\ &= \max(F-E,0)\\ &= C_{\mathrm{v}}(F,T;E).\end{aligned}$$

We can use these results to simplify some barrier formulæ even further. Consider, for example, a down-and-in call when the barrier is at or below the strike, $B_- \le E$. Its value is

$$C_{\mathrm{d/i}}(S,t) = (F/B_-)C_{\mathrm{v}}(B_-^2/F,t;E).$$

We cannot use the put-call reflection result when B_- and E are different. However, at expiry, we have

$$\begin{aligned}(F/B_-)C_{\mathrm{v}}(B_-^2/F,T;E) &= (F/B_-)\max\big(B_-^2/F - E, 0\big)\\ &= (E/B_-)\max\big(B_-^2/E - F, 0\big)\\ &= (E/B_-)P_{\mathrm{v}}(F,T;B_-^2/E).\end{aligned}$$

It follows that this down-and-in call is equivalent to E/B_- puts struck at B_-^2/E:

$$C_{\mathrm{d/i}}(F,t) = (E/B_-)P_{\mathrm{v}}(F,t;B_-^2/E).$$

As a consequence, the down-and-out call formula is also simplified:

$$C_{\mathrm{d/o}}(F,t) = C_{\mathrm{v}}(F,t;E) - (E/B_-)P_{\mathrm{v}}(F,t;B_-^2/E).$$

It is worth noting again the ingredients of this recipe. First, we use the reflection idea to find a formula for the value of the barrier option. Although this formula involves a vanilla option price, here a call, it is not itself a vanilla product or combination of them. Then, we look at the final value of the reflected solution. This *can* be written in terms of vanilla option payoffs, here just that of a put. Now, two derivatives with the same payoff and cashflows have the same value. Hence, the original down-and-out call must have the same value as the vanilla put. Put another way, we have artificially extended the payoff into the 'irrelevant' region $F < B_-$, in such a way that the derivative with this artificial payoff automatically satisfies the barrier condition.

We can apply this idea in the more complex case when the barrier is above the strike, $B_- > E$. Here the down-and-out call value is

$$\begin{aligned}C_{\mathrm{d/o}}(F,t) &= C_{\mathrm{v}}(F,t;B_-) + (B_- - E)C_{\mathrm{d}}(F,t;B_-)\\ &\qquad - \left(\frac{F}{B_-}\right)\big(C_{\mathrm{v}}(B_-^2/F,t;B_-) + (B_- - E)C_{\mathrm{d}}(B_-^2/F,t;B_-)\big).\end{aligned}$$

The only new term is the reflected digital call, $(F/B_-)C_{\mathrm{d}}(B_-^2/F,t;B_-)$. At expiry, its value is

$$(F/B_-)C_{\mathrm{d}}(B_-^2/F,T;B_-) = \begin{cases} F/B_- & \text{if } 0 < F \le B_-,\\ 0 & \text{if } F > B_-.\end{cases}$$

This is the same as

$$\frac{1}{B_-}\Big(F - C_\mathrm{v}(F,T;B_-) - B_- C_\mathrm{d}(F,T;B_-)\Big),$$

equivalent to $1/B_-$ times a position long one asset and short one asset-or-nothing call (see Figure 9). (Incidentally, the reflected digital put is equivalent to $1/B_-$ asset-or-nothing options.) We can therefore replace the reflected digital call solution by the above combination

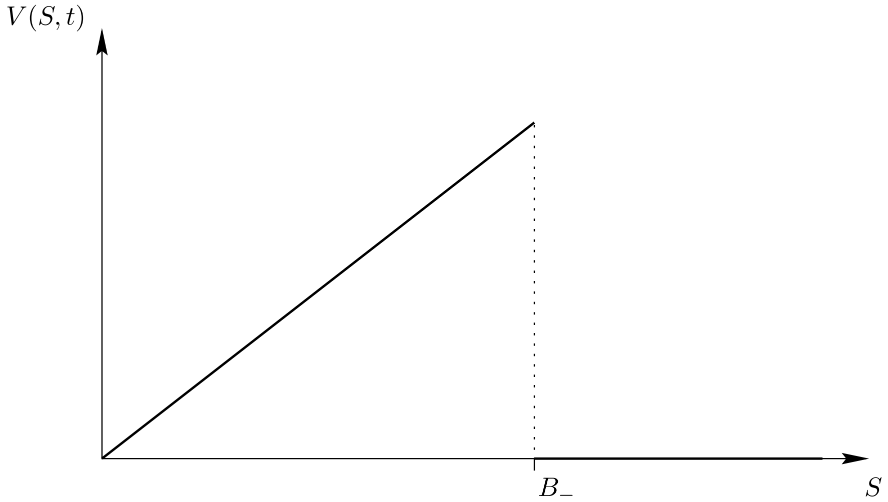

Figure 9: Payoff for the reflected digital call.

of standard contracts. After some arithmetic, using put-call parity for both the vanilla and the digital options, we obtain

$$C_{\mathrm{d/o}}(F,t) = C_\mathrm{v}(F,t;B_-) - (E/B_-)P_\mathrm{v}(F,t;B_-) \tag{10}$$

$$+ (B_- - E)\big(P_\mathrm{d}(F,t;B_-) - C_\mathrm{d}(F,t;B_-)\big). \tag{11}$$

for the down-and-in call when the barrier is above the strike. (Put-call parity can be used to write this in several different ways; we have chosen one which does not involve the asset or cash.) It is straightforward to see that this formula gives the correct payoff and barrier conditions. The corresponding down-and-in call value can then be written down using out-in parity.

0.17 Hedging barrier options

Note: the material in this section is both out of date and mathematically rather naive! There are better approaches available in the literature.

Barrier options can be a nightmare to hedge. The main reason is that they often have a very large gamma, so they are especially sensitive to volatility misspecification. For example, consider a down-and-out call when the barrier is above the strike. If the asset is near the barrier as expiry approaches, the option value is influenced by two quite different possible outcomes. On one hand, the barrier may not be touched before expiry, in which case the payoff will be substantial (approximately $B_- - E$). On the other hand, the probability of knockout, with complete loss of the option's value, is not small.

As with digital options, delta-hedging is not a very appealing strategy. One possible remedy is to set up a 'static' hedge which will mollify the worst features of the barrier. We saw an example of this in the previous section. A down-and-in call on a futures contract is exactly equivalent to a long position in E/B_- puts struck at B_-^2/E. The writer of this option can therefore set up a once-and-for-all hedge by buying the puts. If the barrier is activated, the hedge is liquidated and the proceeds used to pay for the underlying call option. If not, the hedge expires worthless, as does the down-and-in call. The down-and-out call is similarly hedged with the underlying vanilla call as well as with the puts. If the barrier is above the strike, the hedge must also contain digital options, again as outlined above.[7] Similar static hedges can be constructed for the other barrier options in this chapter, as long as they are written on a futures contract.

One of the attractions of static hedges is that they incur very small transaction costs and require little maintenance. Once the hedge is set up, it should in principle cover the barrier option perfectly for all its lifetime. Of course, this assertion assumes a Black–Scholes world, but another advantage of a static hedge is that it may cover the barrier option quite well even if the Black–Scholes assumptions do not hold. For example, it is common market practice to price exotic options using a 'volatility surface' inferred from vanilla prices. That is, the Black–Scholes equation is assumed to be valid, but with a volatility $\sigma(S,t)$ whose values are calculated to be consistent with the market prices of liquid vanilla options. This process is designed to cope with the implied volatility smile, or may be thought of as a measure taken to accommodate the market's 'view' of future volatility as expressed in vanilla prices (see Chapter 8). It is reasonable to suppose that a static hedge in terms of these same vanilla instruments is likely to have the same overall characteristics as the option it is hedging. It may not be a perfect hedge, but it should remove much of the risk. (If there is any flexibility in the construction of the static hedge, for example via put-call parity, it would appear sensible to make the 'derivative' part of it mirror the barrier as closely as possible, as illustrated below.)

Example: hedging a down-and-in call with variable volatility

We illustrate this procedure in Figure 8. It shows numerical calculations (see Chapter 8) of the values of a down-and-out call and its static hedge given by equation (10). The volatility was (arbitrarily) set to decrease linearly with F, from .... The barrier is at $B_- = 10$ and the strike at $E = 9$ (the bad case, with a payoff discontinuity). Of course, the individual components of the static hedge had to be found numerically as well. We see ....

0.17.1 Static hedges for options on the spot price

It is possible to set up static hedges when the underlying is the spot price rather the futures price. Because the reflected solutions are not so simple, the static hedge usually involves a continuous distribution of vanilla contracts. Consider for example setting up a static hedge for a down-and-in call, with the barrier below the strike. The value of the down-and-in call is

$$C_{\mathrm{d/i}}(S,t) = \left(\frac{S}{B_-}\right)^{2\alpha} C_{\mathrm{v}}(B_-^2/S, t; E).$$

[7] Smaller transaction costs may be incurred if some of the options in the static hedge are replaced by the asset and cash, using put-call parity. On the other hand, if the hedger is also a market maker, he may prefer to hedge by writing options in order to benefit from the bid-offer spread.

At expiry, the right-hand side is equal to

$$\begin{aligned}
\left(\frac{S}{B_-}\right)^{2\alpha} \max\left(B_-^2/S - E, 0\right) &= \left(\frac{E}{B_-}\right)\left(\frac{S}{B_-}\right)^{2\alpha-1} \max\left(B_-^2/E - S, 0\right) \\
&= \left(\frac{E}{B_-}\right)\left[\left(\left(\frac{S}{B_-}\right)^{2\alpha-1} - \left(\frac{B_-}{E}\right)^{2\alpha-1}\right) \max\left(B_-^2/E - S, 0\right)\right. \\
&\qquad \left. + \left(\frac{B_-}{E}\right)^{2\alpha-1} P_{\mathrm{v}}(S, T; B_-^2/E)\right] \\
&= \left(\frac{E}{B_-}\right)\left[\Lambda(S) + \left(\frac{B_-}{E}\right)^{2\alpha-1} P_{\mathrm{v}}(S, T; B_-^2/E)\right].
\end{aligned}$$

In the last line, we have written $\Lambda(S)$ for the difference between the final value of the static hedge and the vanilla puts in the last term. If 2α were equal to 1, as for options on futures, this difference would vanish. We have separated out the vanilla put in order to make $\Lambda'(S)$ continuous (and equal to 0) at the strike of the vanilla put, $S - B_-^2/E$. Now all we have to do is to synthesise the derivative whose payoff is $\Lambda(S)$, using vanilla puts (preferable to calls because $\Lambda(S)$ vanishes for large S. This is easy to do: the density of vanilla puts with strike E^* that we need to hold is $\Lambda''(E^*)$ (see Section 8). Then, the static hedge is

$$\left(\frac{B_-}{E}\right)^{2\alpha-1} P_{\mathrm{v}}(S, T; B_-^2/E) + \int_0^{B_-} \Lambda''(E^*) P_{\mathrm{v}}(S, t; E^*) dE^*.$$

Although this is a perfect hedge in theory, in practice puts with a continuum of strikes are not available. Even if the static hedge consists of a single put, as for options on futures, the put with that precise strike may not be exchange-traded (sometimes an advantage as it reduces the costs). We now consider how to deal with this difficulty.

0.17.2 Approximate static hedges

If 'continuous static hedging' is impossible, the next best thing is to set up a static hedge to minimise an appropriate measure of the exposure to the barrier option. The remaining, statically unhedged, part of the barrier option can then be hedged dynamically in the usual way. It, and therefore its risk, should be much smaller than before. Again, if a static hedge is optimal in this sense in a Black–Scholes world, one hopes that it will perform well in the non-Black–Scholes (real) world too.

In a typical situation, then, there may be N vanilla products, say call and put options, available to hedge the barrier. We can also include cash and the asset; one or other of the calls and puts can then eliminated[8] by put-call parity, so we suppose that only puts are used. Thus, the hedge is made up of $N + 2$ vanilla products, and has the form

$$V_{\mathrm{app}}(S, t) = \sum_{n=1}^{N} b_n P_{\mathrm{v}}(S, t; E_n) + b_{N+1} e^{-r(T-t)} + b_{N+2} S,$$

or, for short,

$$V_{\mathrm{app}}(S, t) = \sum_{n=1}^{N+2} b_n V_n(S, t).$$

[8] This is only true when the vanilla options are European: American options demand both calls and puts.

There are many possible ways to measure how much the approximate static hedge differs from the ideal one. For example, we may choose to minimise the error at expiry only, i.e. try to make the payoff of the approximate hedge, $V_{\text{app}}(S,T)$, as close as possible to the payoff of the exact hedge, $V_{\text{ex}}(S,T)$. However, this involves values of S beyond the barrier, where we do not care what the option value is. More to the point, this idea does not give as good results as the one we describe below. It is better to make the two close over a range of time values, as well as some or all asset values. A convenient method is the least squares approach. If the option has a lower barrier B_-, we may, for example, find the minimum of

$$\int_0^T \int_{B_-}^{S_{\max}} \left(V_{\text{ex}}(S,t) - V_{\text{app}}(S,t)\right)^2 W(S,t)\, dSdt.$$

The integration is taken only up to the (large) value $S_{\max}$ because the difference may not decay as $S \to \infty$ (i.e., we may not care what happens for very large S.). The function $W(S,t)$ is a 'weight' function, which allows us to give more importance to some values of S and t than others. Usually, though, it is not necessary to calculate a double integral. It is often good enough to calculate a time-averaged error such as

$$\int_0^T \left(V_{\text{ex}}(S_0,t) - V_{\text{app}}(S_0,t)\right)^2 W(t)\, dt$$

for a suitably chosen S_0 and weighting function $W(t)$. We illustrate this in some detail below for a down-and-in call option.

The minimisation is a straightforward application of calculus, and leads to the equations

$$\mathbf{A}\mathbf{b} = \mathbf{c},$$

in which the $(N+2)\times(N+2)$ matrix $\mathbf{A}$ has entries

$$A_{ij} = \int_0^T V_i(S_0,t)V_j(S_0,t)W(t)\, dt$$

(the integral may be a double one if preferred). The vector $\mathbf{b}$ contains the $N+2$ unknown coefficients in the hedge, and the components of the vector $\mathbf{c}$ are

$$c_i = \int_0^T V_i(S_0,t)V_{\text{ex}}(S_0,t)W(t)\, dt.$$

Example: approximate hedging of down-and-in calls

We illustrate the ideas of the previous section by constructing static hedges for down-and-in calls on a futures contract. The corresponding down-and-out call hedge just involves an extra position in the underlying vanilla option. In all the illustrations we took $\sigma = 0.15$, $r = 0.08$ and $T = 0.25$, for a three-month option. In each case we minimised the mean-square error on the barrier, as above, taking $W(t) = 1$ for simplicity. We carried out the integrations using the computer package Maple.

Suppose first that we are hedging a down-and-in call with barrier $B_- = \$10.00$ and strike $E = \$10.50$ (the 'nice' case, where digital options are not needed). The exact hedge here consists of E/B_- puts struck at $E_{\text{ex}} = B_-^2/E$, that is, 1.05 puts struck at \$9.52. Suppose also that puts are available with strikes $E_1 = \$9.00$, $E_2 = \$9.50$ and $E_3 = \$10.00$. Let us also look for a hedge consisting only of these puts, which is reasonable here as the payoff of

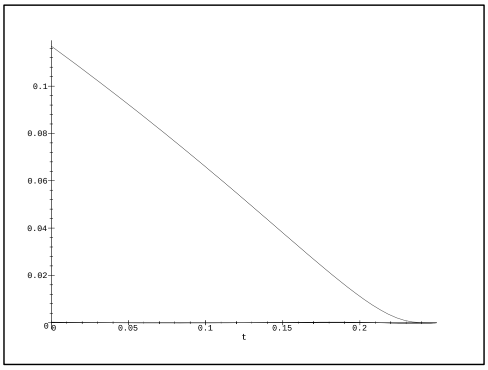

Figure 10: 'Good' down-and-in call at the barrier (upper curve), and the error in its optimal static hedge using three vanilla puts (lower curve).

the down-and-in call vanishes for $S > B_-$. That is, we make the payoff of the down-and-in call and the hedge agree exactly for $S > B_-$; it may sometimes be better not to impose this restriction, sacrificing accuracy at expiry for better hedging earlier on. Thus, the hedge has the form $b_1 P(S,t;E_1) + b_2 P(S,t;E_2) + b_3 P(S,t;E_3)$.

Plugging in the numbers and solving the equations, we find that

$$b_1 = -0.189, \quad b_2 = 1.138, \quad b_3 = 0.006.$$

In figure 10 we show the values on the barrier of the down-and-in call, together with the difference between the exact value and the static hedge, i.e. the residual portion of the portfolio. In fact the latter is almost impossible to distinguish from the t-axis: this is a very good hedge. (Now variable sigma)

For our second example, we try to hedge a down-and-in call with the barrier at \$10.00 but now with the strike at \$9.50, again using the same three puts. The optimal portfolio has 24.62 puts struck at \$9, −13.58 puts struck at \$9.50, and 4,79 puts struck at \$10.00. Figure 11 shows the exact value and the error as before, evaluated just above the barrier. As expected, the hedge is markedly worse than the previous example, because of the singular behaviour of the down-and-in call near the barrier and near expiry.

Our third example shows how to improve the hedging of a 'bad' down-and-in call option by including a digital put option in the hedge. The barrier is again \$10.00, and strike again \$9.50. Now, though, we hedge with b_1 vanilla puts struck at \$9.50, b_2 vanilla puts struck at \$10.00, and b_3 standard digital puts struck at \$10.00. The optimal hedge has $b_1 = 2.075$, $b_2 = 0.192$, $b_3 = 0.960$. It is shown in Figure 12, and is clearly a great improvement on the hedge using only vanilla puts. Note the sudden loss of value in the exact solution as expiry approaches.

To round off this example, we note that the static hedge for a written down-and-in call performs well if the asset suddenly falls to well below the barrier, thereby knocking the option out.[9] If the contract specifies the delivery of the underlying call itself, this can be purchased more cheaply if the asset is well below the barrier; if it specifies merely that the holder receives the payoff of the call, the liability after knockout can likewise be hedged by buying the call, presumably more cheaply. The writer's position is strengthened by the fact

[9] these words are being written in October 1997!

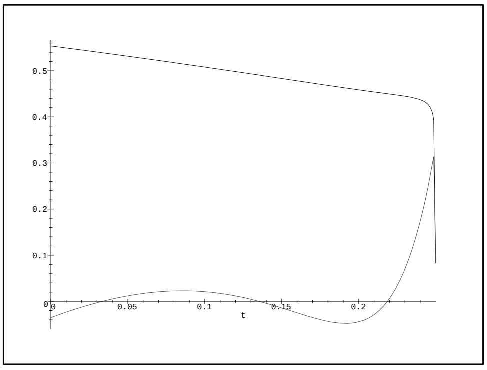

Figure 11: 'Bad' down-and-in call at the barrier (upper curve), and the error in its optimal static hedge using three vanilla puts (lower curve).

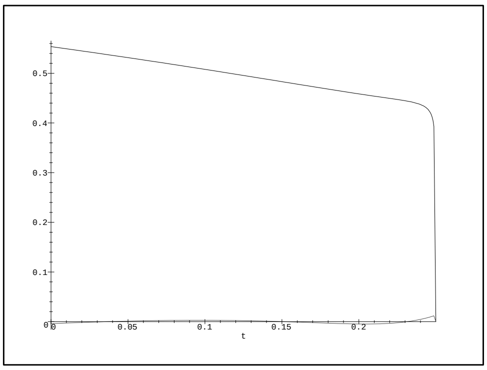

Figure 12: 'Bad' down-and-in call at the barrier (upper curve), and the error in its optimal static hedge using two vanilla puts and one digital put (lower curve).

that the puts used for hedging become *more* valuable as the asset falls. Barrier options are risky in volatile markets!

Appendix: Exact solutions of the Black–Scholes equation

Introduction

In this short appendix, we give a list of useful exact solutions of the Black–Scholes equation. These may be used as building blocks for many derivative contracts. This is not a complete catalogue (there are infinitely many solutions!) but it does contain most of those encountered in practice.

Any of these solutions can represent the value of a traded (or tradable) derivative. All one has to do is to choose an expiry date and work out the payoff. We indicate where this gives a potentially reasonable contract.

The Black–Scholes equation with a constant dividend yield D is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0. \tag{12}$$

We shall use the following notation, used throughout. We write

$$k = r/\tfrac{1}{2}\sigma^2 \quad \text{and} \quad k' = (r-D)/\tfrac{1}{2}\sigma^2,$$

then define

$$\alpha = \tfrac{1}{2}(1-k') \quad \text{and} \quad \beta = -\tfrac{1}{4}(1-k')^2 - k = -\alpha^2 - k.$$

We write λ_+ and λ_- for the roots of the quadratic equation

$$\lambda^2 + (k'-1)\lambda - k = 0.$$

They are

$$\begin{aligned}\lambda_\pm &= \tfrac{1}{2}(1-k') \pm \sqrt{\tfrac{1}{4}(k'-1)^2 + k} \\ &= \alpha \pm \sqrt{-\beta}.\end{aligned}$$

When $D = 0$, $k' = k$, $\beta = -\frac{1}{4}(k+1)^2$, and the roots of the quadratic are 1 and $-k$.

We shall use T for a general time constant, to be thought of as an expiry time. Likewise K is a general price constant or scale, usually a strike price (E) or barrier ($B_\pm$), but occasionally the maximum or minimum realised value (J).

It is occasionally useful to remember that equation (12) can be turned into the heat equation by the substitutions

$$x = \log(S/K), \quad \tau = \tfrac{1}{2}\sigma^2(T-t), \quad V(S,t) = e^{\alpha x + \beta\tau}u(x,\tau).$$

We also recall the 'reflection' principle:

> If $V(S,t)$ is a solution of the Black–Scholes equation (12), then for any constant K,
>
> $$S^{2\alpha}V(K^2/S,t)$$
>
> satisfies the same equation.

The fundamental solution

Corresponding to the fundamental solution of the heat equation,

$$\frac{1}{2\sqrt{\pi\tau}}e^{-x^2/4\tau},$$

we have the solution

$$\frac{e^{-(r-D)(T-t)}}{\sigma\sqrt{2\pi(T-t)}}e^{-\left(\log(S/K)+(r-D-\frac{1}{2}\sigma^2)(T-t)\right)^2/\,2\sigma^2(T-t)},$$

Apart from the discounting term $e^{-(r-D)(T-t)}$, this is the fundamental transition density function (replace T bt t' and K by S', and divide by S'). Note that we can obtain the transition density function for a random walk with an absorbing barrier at $S = K$ by subtracting the reflected solution from the density with no barrier.

Note also that the frequently occurring combination $r - D - \frac{1}{2}\sigma^2$ is the drift of $\log S$ under the risk-neutral random walk $dS = \sigma\, dX + (r-D)dt$.

Two basic solutions and their reflections

Whenever you devise a partial differential equation model for a financial derivative, two special solutions must be checked. If the underlying asset is traded, it must satisfy the equation (after discounting for dividends), and so must cash. Thus two solutions of equation (12) are

$$Se^{-D(T-t)} \quad \text{and} \quad Ke^{-r(T-t)}.$$

The latter is the only asset-independent solution.

Reflection yields two more solutions:

$$S^{2\alpha-1}e^{-D(T-t)} = S^{-(r-D)/\frac{1}{2}\sigma^2}e^{-D(T-t)}$$

and

$$S^{2\alpha}e^{-r(T-t)} = S^{-(r-D-\frac{1}{2}\sigma^2)/\frac{1}{2}\sigma^2}e^{-r(T-t)}.$$

Time-independent solutions

The time-independent solutions of (12) are

$$S^{\lambda_+} \quad \text{and} \quad S^{\lambda_-},$$

where $\lambda_\pm$ are defined above. They are each other's reflections. The value of a perpetual American put option is proportional to S^{λ_-} (see above).

Solutions with $N(\cdot)$

The formulæ for calls, puts etc all involve the ubiquitous

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2}\,ds.$$

In this section we find some solutions of (12) of the form

$$e^{\delta t}S^{\gamma}N(d(S,t))$$

for suitable δ, γ and d. The idea is just to substitute this into (12) and look for simplifications. The resulting mess has one sum of terms proportional to $N(d)$ and one proportional to $N'(d)$. We equate these two sums to zero *separately*, thereby cutting down on the number of possible solutions that we can find (but making life much much easier). The result is that we must have

$$\delta + \tfrac{1}{2}\sigma^2\gamma(\gamma-1) + (r-D)\gamma - r = 0 \tag{13}$$

and

$$\frac{\partial d}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\left(\frac{\partial^2 d}{\partial S^2} - d\left(\frac{\partial d}{\partial S}\right)^2\right) + (r-D+\gamma\sigma^2)S\frac{\partial d}{\partial S} = 0.$$

The partial equation for $d(S,t)$ is much worse that the Black–Scholes equation. However it does have some simple solutions.[10] It turns out (to nobody's surprise) that there is a solution of the form

$$d(S,t) = a(t)\log(S/K) + b(t);$$

[10]Other solutions might lead to some interesting new derivative products...

$a(t)$ and $b(t)$ can be found quite easily to give

$$d(S,t) = \pm \frac{\log(S/K) + \left(r - D + (\gamma - \frac{1}{2})\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

where of course γ satisfies (13).

We now list some special cases of this formula. There is little point in taking general values of δ. The only ones that are needed in practice are $\delta = 0$, $\delta = r$ and $\delta = D$.

Solutions with $\delta = 0$

When $\delta = 0$, we have $\gamma = \lambda_+$ or $\gamma = \lambda_-$. The corresponding solutions of (12) are

$$S^{\lambda_+} N(\pm d_+)$$

where

$$d_+ = \frac{\log(S/K) + \sigma^2\sqrt{-\beta}(T-t)}{\sigma\sqrt{T-t}},$$

and

$$S^{\lambda_-} N(\pm d_-)$$

where

$$d_- = \frac{\log(S/K) - \sigma^2\sqrt{-\beta}(T-t)}{\sigma\sqrt{T-t}}.$$

(Note that when $D = 0$, $\sigma^2\sqrt{-\beta} = \frac{1}{2}\sigma^2(k+1) = r + \frac{1}{2}\sigma^2$, and $\lambda_+ = 1$, $\lambda_- = -k = -r/\frac{1}{2}\sigma^2$.) The solutions with $N(\pm d_+)$ are the reflections of those with $N(\mp d_-)$ and vice versa.

Solutions with $\delta = r$

When $\delta = r$, we have $\gamma = 0$ or $\gamma = 2\alpha$. The former gives the solutions

$$e^{-r(T-t)} N(\pm d_{20})$$

where

$$d_{20} = \frac{\log(S/K) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

These are the only solutions of the form $e^{\delta t}N(d)$. (We have written d_{20} in consonance with the usage elsewhere. When $D = 0$, $d_{20} = d_2$.) These solutions are relevant to digital options.

When $\gamma = 2\alpha$, we have the solutions

$$e^{-r(T-t)} S^{2\alpha} N(\pm d) = e^{-r(T-t)} S^{-(r-D-\frac{1}{2}\sigma^2)/\frac{1}{2}\sigma^2} N(\pm d),$$

where

$$d = \frac{\log(S/K) - (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The solutions for $\gamma = 0$ and $\gamma = 2\alpha$ are each other's reflections after changing the sign of d.

Solutions with $\delta = D$

Lastly we look at solutions when $\delta = D$. Then, $\gamma = 1$ or $\gamma = 2\alpha - 1 = -(r-D)/\frac{1}{2}\sigma^2$. The former gives the solutions

$$e^{-D(T-t)} S N(\pm d_{10}),$$

where

$$d_{10} = \frac{\log(S/K) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

The latter choice for γ gives the solutions

$$e^{-D(T-t)} S^{-(r-D)/\frac{1}{2}\sigma^2} N(\pm d),$$

where

$$d = \frac{\log(S/K) - (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Again the solutions for the two values of γ are each other's reflections after changing the sign of d.

Power and log solutions

Solutions proportional to S^n have the form

$$e^{\delta t} S^n,$$

where substitution into (12) shows that

$$\delta = -\left(\tfrac{1}{2}\sigma^2 n(n-1) + (r-D)n - r\right).$$

Note the special cases

$$\delta = 0, \qquad n = \lambda_{\pm}$$

(the time-independent solutions),

$$\delta = r, \qquad n = 0, 2\alpha,$$

and

$$\delta = D, \qquad n = 1, -(r-D)/\tfrac{1}{2}\sigma^2.$$

There is also a solution corresponding to $n = 0$; it is

$$a(t)\log(S/K) + b(t),$$

where

$$a(t) = e^{-r(T-t)} \quad \text{and} \quad b(t) = (r - D - \tfrac{1}{2}\sigma^2)(T-t)e^{-r(T-t)}.$$

(Note that the two halves of this solution must be used together; however, any multiple of $e^{-r(T-t)}$ can be added to $b(t)$. This corresponds to changing K or T.) Reflection generates the further solution

$$S^{2\alpha}\left(a(t)\log(K/S) + b(t)\right).$$

Separable solutions

Separable solutions of (12) have the form

$$F(S)G(t).$$

Substitution into (12) shows that

$$\frac{G'}{G} = -\frac{\frac{1}{2}\sigma^2 S^2 F'' + (r-D)SF' - rF}{F}.$$

The left-hand side is a function of t only, the right-hand side a function of S only, so both must be a constant, δ. This gives

$$G(t) = e^{-\delta(T-t)},$$

and

$$\tfrac{1}{2}\sigma^2 S^2 F'' + (r-D)SF' - (r-\delta)F = 0.$$

The solutions of this Bernoulli equation have the form S^γ, where

$$\tfrac{1}{2}\sigma^2\gamma(\gamma-1) + (r-D)\gamma - (r-\delta) = 0,$$

so that

$$\begin{aligned}\gamma_\pm &= \tfrac{1}{2}(1-k') \pm \sqrt{\tfrac{1}{4}(k'-1)^2 + k - (\delta/\tfrac{1}{2}\sigma^2)} \\ &= \alpha \pm \sqrt{-\beta - (\delta/\tfrac{1}{2}\sigma^2)},\end{aligned}$$

There are three cases: real distinct roots, coincident roots, and complex conjugate roots. They occur when

$$\delta <, =, > -\tfrac{1}{2}\sigma^2\beta$$

respectively. The appropriate solutions are

$$F(S) = S^{\gamma_\pm}$$

when the roots are real, and

$$F(S) = S^\gamma \log(S/K)$$

when the double root is γ. When the roots are complex,

$$\gamma_\pm = \gamma_1 \pm i\gamma_2,$$

where $i^2 = -1$. Now the solutions are

$$F(S) = S^{\gamma_1}\sin\left(\gamma_2\log(S/K)\right) \quad \text{and} \quad F(S) = S^{\gamma_1}\cos\left(\gamma_2\log(S/K)\right).$$

(Of course, one can be changed into the other by suitable choice of K.)

Solutions *via* the heat equation

One can generate endless solutions of the Black–Scholes equation by transforming it to the heat equation and thinking of simple solutions of the latter. We have

$$\begin{aligned} V(S,t) &= e^{\alpha x+\beta\tau}u(x,\tau) \\ &= S^{\alpha}e^{\frac{1}{2}\sigma^2\beta(T-t)}u\left(\log(S/K),\frac{1}{2}\sigma^2(T-t)\right), \end{aligned}$$

where

$$\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial x^2}.$$

As an example of what can be done, the solution

$$u(x,\tau)=c_1+c_2x+c_3(\tfrac{1}{2}x^2+\tau)$$

of the heat equation, where c_1, c_2 and c_3 are independent constants, gives the solution

$$V(S,t)=S^{\alpha}e^{\frac{1}{2}\sigma^2\beta(T-t)}\left(c_1+c_2\log(S/K)+c_3\left(\frac{1}{2}\left(\log(S/K)\right)^2+\frac{1}{2}\sigma^2(T-t)\right)\right).$$

It is clear that this process can be continued indefinitely. It is also clear that we have had enough of this.

Further reading

- Rubinstein (1992) contains a catalogue and explicit formulæ for a large number of barrier options.