Discrete Fourier Transform

In the design of a DSP system, two fundamental tasks are involved:

i) analysis of input signal

ii) design of a Processing system to give the desired output.

The Discrete Fourier Transform & Fast fourier Transform are very important mathematical tools for carrying out these tasks. The can be used to analyze two-dimensional signal.

- The FFT algorithms eliminate the redundant calculation & enable to analyse the spectral properties of a signal.
- They offer rapid frequency domain analysis & Processing of digital signals & investigation of digital systems.
- The FFT also allows time domain signal processing operations to be Performed equivalently in frequency-domain.
- In both domains FFT has considerable reduction in computation time.
- FFT algorithms are mainly useful in computing the DFT & IDFT, & also find applications in linear filtering, digital spectral analysis & correlation analysis.

Discrete Fourier Series :-

Any Periodic function can be expressed in a Fourier series represent^n. Consider a discrete time signal $x(n)$ that is periodic with period N defined by,

$x(n) = x(n + Nk)$ for any integer value of k.

Exponential form of Discrete Fourier series :-

A real periodic discrete-time signal $x(n)$ of period N can be expressed as a weighted sum of complex exponential sequences.

The exponential form of fourier series for a periodic discrete-time signal is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jk\omega_0 n}, \text{ for all } n.$$

where the coefficients $X(k)$ of fundamental digital frequency are expressed as,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-jk\omega_0 n}, \text{ for all } k$$

* W_N is called as twidelle factor that makes computation easier.

where, $\omega_0 = 2\pi/N$

above both equations are called Discrete fourier series synthesis & analysis Pair. Both $X(k)$ & $x(n)$ are Periodic sequences.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

where $W_N = e^{-j2\pi/N}$

Discrete - Time Fouries Transform :-

- Discrete - Time fourier Transform or fourier transform of a discrete time sequence $x(n)$ is represented by the complex exponential sequence $[e^{-j\omega n}]$ where ω is the real frequency variable.
- This transform is useful to map the time-domain sequence into a continous function of a frequency variable.
- DTFT is application to any arbitrary Sequences, whereas DFT can be applied only to finite length sequences.
- The discrete-time Fourier transform $X(e^{j\omega})$ of a seq. $x(n)$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

This equation represents the Fourier series representation of the periodic function $X(e^{j\omega})$. Hence Fourier coefficients $x(n)$ can be determined from $X(e^{j\omega})$ using fourier integral expressed by,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

called Inverse Discrete-Time Fourier Transform.

The above two equations are known as the Discrete-Time Fourier Transform Pair for the sequence $x(n)$, which relate time & frequency domain.

Discrete Fourier Transform (DFT)

The Discrete fourier Transform computes the values of z-Transform for evenly spaced points around unit circle for a given sequence.

If the sequence to be represented is of finite duration, ie. it has only a finite number of non-zero values, the transform used is Discrete Fourier Transform.

Let $x(n)$ be a finite duration sequence. The N-point DFT of the

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sequence $x(n)$ is expressed by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N},$$

$k = 0, 1, \ldots N-1$

The corresponding IDFT is,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi nk/N},$$

$n = 0, 1, \ldots, N-1$

Relationship bet^n DFT & DTFT :-

The DTFT is used for analyses of non-periodic discrete-time signals.

The DTFT is Discrete-Time Fourier transform & is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The range of ω is from $-\pi$ to π or 0 to 2π.

Now we know that discrete Fourier Transform is obtained by sampling one cycle of Fourier transform.

DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

Comparing above two equations, we can say that DFT is obtained from DTFT by putting $\omega = 2\pi k/N$

$$\therefore X(k) = X(\omega)\Big|_{\omega = 2\pi k/N}$$

By comparing DFT with DTFT we can write,

1) The continuous frequency spectrum $X(\omega)$ is replaced by discrete fourier spectrum $X(k)$.
2) Infinite summation in DTFT is replaced by finite summation in DFT.
3) The continuous frequency variable is replaced by finite number of frequencies located at $2\pi k/NT_s$, where T_s is sampling time.

Properties of DFT

Properties are useful in Practical techniques for Processing signals.

i) Periodicity:

If $X(k)$ is an N-point DFT of $x(n)$, then

$x(n+N) = x(n)$ for all n

$X(k+N) = X(k)$ for all k.

ii) Linearity:

If $X_1(k)$ & $X_2(k)$ are are N-point DFTs of $x_1(n)$ & $x_2(n)$ resp. & a & b are arbitrary constants either real or complex-valued, then

$$a x_1(n) + b x_2(n) \xleftrightarrow[N]{DFT} a X_1(k) + b X_2(k)$$

iii) Shifting Property:-

Let $x_p(n)$ is periodic seq. with Period N, which is obtained by extending $x(n)$ periodically ie.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Now shift $x_p(n)$ by k units to the r8. Then resultant signal as,

$$x_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

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The finite duration seq.

$$x'(n) = \begin{cases} x'_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

can be obtained from $x(n)$ by circular shift.

The circular shift can be represented by index modulo N.

$$x'(n) = x(n-k, (\text{mod}\, N))$$

4) Convolution Theorem

If $x_1(n) \xleftrightarrow{DFT} X_1(\omega)$

$x_2(n) \xleftrightarrow{DFT} X_2(\omega)$

then $x(n) = x_1(n) * x_2(n)$

$\xleftrightarrow{DFT} X(\omega) = X_1(\omega) X_2(\omega)$

5) Time Reversal of a sequence

If $x(n) \underset{N}{\xleftrightarrow{DFT}} X(k)$, then

$$x(-n, (\text{mod}\, N)) = x(N-n) \underset{N}{\xleftrightarrow{DFT}} X(-k, (\text{mod}\, N))$$

$$= X(N-k)$$

When N-point seq. in time is reversed, it is equnt to reversing the DFT values.

6. Circular Time Shift :-

If $x(n) \xleftrightarrow[N]{DFT} X(k)$ then

$$x(n-l, (\mathrm{mod}\, N)) \xleftrightarrow[N]{DFT} X(k)\, e^{-j2\pi kl/N}$$

Shifting of seq. by l units in the time-domain is equivalent to multiplication of $e^{-j2\pi kl/N}$ in freq. domain.

7) Circular Freq. Shift :-

$x(n) \xleftrightarrow[N]{DFT} X(k)$, then

$$x(n)\, e^{j2\pi ln/N} \xleftrightarrow[N]{DFT} X(k-l, (\mathrm{mod}\, N))$$

When seq. $x(n)$ is multiplied by complex exponential sequence $e^{j2\pi ln/N}$, it is equ^nt to circular shift of DFT by l units in freq. domain.

8) Complex Conjugate Property.

If $x(n) \xleftrightarrow[N]{DFT} X(k)$ then

$$x^*(n) \xleftrightarrow[N]{DFT} X^*(-k, (\mathrm{mod}\, N))$$

$$= X^*(N-k)$$

$$X^*(k) \xleftrightarrow{IDFT} \frac{1}{N} \sum_{k=0}^{N-1} X^*(k)\, e^{j2\pi kn/N}$$

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$$= \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N} \right]$$

Hence,

$$x^*(-n, (\text{mod } N)) = x^*(N-n) \xleftrightarrow[N]{DFT} X^*(k)$$

9) Circular Convolution:-

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ &

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then

$$x_1(n) \, N \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k) X_2(k)$$

10) Circular Correlation

For complex-valued sequences $x(n)$ & $y(n)$,

If $x(n) \xleftrightarrow[N]{DFT} X(k)$ &

$y(n) \xleftrightarrow[N]{DFT} Y(k)$ then

$$r_{xy}(l) \xleftrightarrow[N]{DFT} \frac{1}{N} R_{xy}(k)$$

$$= X(k) Y^*(k)$$

11) Multiplicar of two seq.

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then,

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$$x_1(n)\,x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \; (N) \; X_2(k)$$

12) Parseval's thm :-

For complex-valued seq. $x(n)$ & $y(n)$,

If, $x(n) \xleftrightarrow[N]{DFT} X(k)$ &

$y(n) \xleftrightarrow[N]{DFT} Y(k)$ then

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, Y^*(k)$$

If $y(n) = x(n)$, then above equa^n reduces to,

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

This expression relates the energy in the finite duration sequence $x(n)$ to the Power in freq. components $X_1(k)$

Circular convolution is equivalent to multiplications in DFT domain.

Steps for Circular Convolution using DFT & IDFT method as:-

i) for $x_1(n)$ & $x_2(n)$, obtain DFT $X_1(k)$ & $X_2(k)$.

ii) Multiply $X_1(k)$ & $X_2(k)$ to obtain $Y(k)$.

$$Y(k) = X_1(k) \cdot X_2(k)$$

iii) Obtain IDFT of $Y(k)$.

Linear convolution operation can be implemented by DFT & IDFT. Calculations are carried out in frequency domain.

Also, linear convolution opeⁿ can be implemented by Circular convolution → appending zeroes to length N. Calculations carried out in time domain.

In linear convolution DFT should be of length $N = L + M - 1$. So length of $x(n)$ & $h(n)$ should be N & for that pad zeroes.

Then N-point DFTs $X(k)$ & $H(k)$ are computed. The two DFTs $X(k)$ & $H(k)$ are multiplied to give $Y(k)$.

N-point IDFT of $Y(k)$ gives $y(n)$.

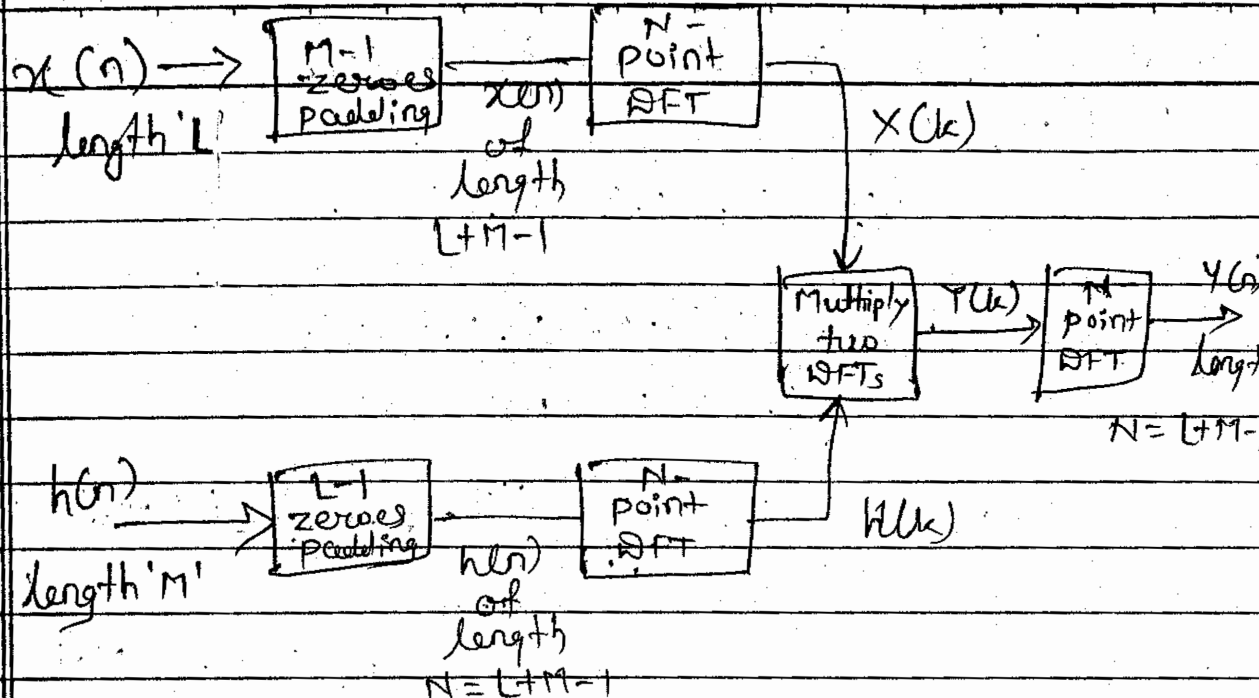

fig. Using DFT & IDFT Linear Convolution.

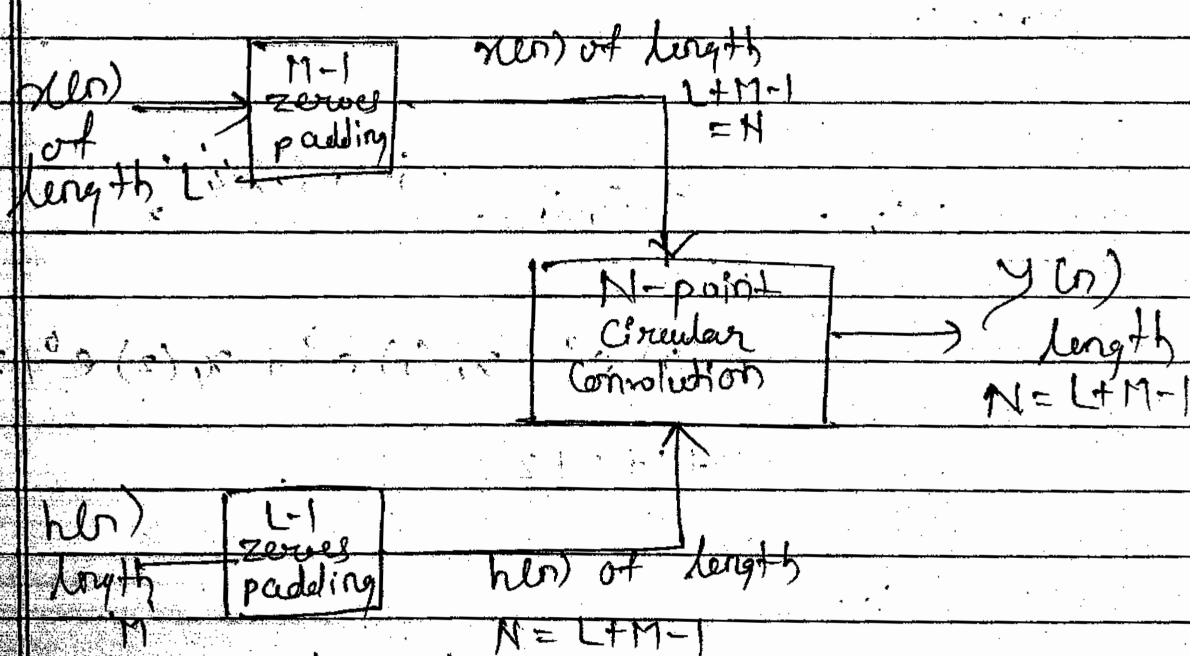

fig: Linear Convolution using Circular Convolution

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Eg. Find Circular Convolution using DFT & IDFT.

→ $$X_3(k) = X_1(k) \cdot X_2(k)$$

Seq. $x_1(n) = \{1, 1, 2, 2\}$
$x_2(n) = \{1, 2, 3, 4\}$

$$\therefore \quad x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j2\pi nk/N}$$

$n = 0, 1, \ldots, N-1$

i) $x_1(n) = \{1, 1, 2, 2\}$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots N-1$$

$N = 4$

$\therefore$ For $k = 0$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n(0)/4} =$$

$$= x_1(0) e^0 + x_1(1) e^0 + x_1(2) e^0 + x_1(3) e^0$$

$$= 1 + 1 + 2 + 2$$
$$= 6$$

For $k = 1$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n/4}$$

$$= x_1(0) e^{0} + x_1(1) e^{-j\pi/2} + x_1(2) e^{-j\pi}$$
$$+ x_1(3) e^{-j3\pi/2}$$

$$= 1 + e^{-j\pi/2} + 2e^{-j\pi} + 2e^{-j3\pi/2}$$

$$= 1 - j + 2(-1) + 2(j)$$

$$= -1 + j$$

For $k=2$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) e^{-jn\pi}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi}$$

$$= 1 - 1 + 2(1) + 2(-1) = 0$$

For $k=3$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) e^{-j3\pi n/2}$$

$$= 1 + e^{-j3\pi/2} + 2e^{-j3\pi} + 2e^{-j9\pi/2}$$

$$= 1 + j + 2(-1) - j(2)$$

$$= -1 - j$$

$$X_1(k) = \{6, -1+j, 0, -1-j\}$$

ii) when $x_2(n) = \{1, 2, 3, 4\}$

$\underline{N=4}$

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$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi nk/4}, \quad k = 0,1,2,3$$

For $k=0$,

$$X_2(0) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi n(0)/4} = 10$$

For $k=1$

$$X_2(1) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n/2}$$

$$= 1 + 2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2}$$

$$= 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2$$

For $k=2$

$$X_2(2) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n}$$

$$= 1 + 2e^{-j\pi} + 3e^{-j\pi} + 4e^{-j3\pi}$$

$$= 1 + 2(-1) + 3(1) + 4(-1) = -2$$

For $k=3$

$$X_2(3) = \sum_{n=0}^{3} x_2(n) e^{-j(3\pi/2)n}$$

$$= 1 + 2e^{-j3\pi/2} + 3e^{-j3\pi} + 4e^{-j9\pi/2}$$

$$= 1 + 2(j) + 3(-1) + 4(-j)$$

$$= -2 - 2j$$

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$X_2(k) = \{10, -2+2j, -2, -2-2j\}$

$X_3(k) = X_1(k) X_2(k)$

$= \{60, (-1+j)(-2+2j), 0, (-1-j)(-2-2j)\}$

$= \{60, -4j, 0, 4j\}$

we know that

$x_3(n) = IDFT\{X_3(k)\}$

$\therefore x_3(n) = \frac{1}{N} \sum_{k=0}^{3} X_3(k) e^{j2\pi nk/4}$, $n = 0, 1, 2, 3$

$= \frac{1}{4} \left[60 + (-4j) e^{j\pi n/2} + (4j) e^{j3\pi n/2}\right]$

$x_3(0) = \frac{1}{4}\left[60 + (-4j) + 4j\right] = 15$

$x_3(1) = \frac{1}{4}\left[60 - 4j e^{j\pi/2} + 4j e^{j3\pi/2}\right]$

$= \frac{1}{4}\left[60 - 4j(+j) + 4j(-j)\right]$

$= \frac{1}{4}\left[60 + 4 + 4\right] = 17$

$x_3(2) = \frac{1}{4}\left[60 - 4j e^{j\pi} + 4j e^{j3\pi}\right]$

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$$= \frac{1}{4}\left[60 - 4j(-1) + 4j(-1)\right] = 15$$

$$x_3(3) = \frac{1}{4}\left[60 + (-4j)e^{j3\pi/2} + 4je^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[60 + (-4j)(-j) + 4j(j)\right]$$

$$= \frac{1}{4}\left[60 - 4 - 4\right] = 13$$

$$\therefore x_3(n) = [15, 17, 15, 13]$$

Linear Convolution:-

Eg. An FIR filter has the impulse response of $h(n) = \{\underset{\uparrow}{1}, 2, 3\}$. Det. the response of the filter to the input seq. $x(n) = \{\underset{\uparrow}{1}, 2\}$. Use DFT & IDFT.

→ $x(n) = \{\underset{\uparrow}{1}, 2\}$ & $h(n) = \{\underset{\uparrow}{1}, 2, 3\}$

i) length N of DFT & IDFT

$$N = L + M - 1 = 3 + 2 - 1 = 4$$

$\therefore$ we should pad zeroes in $h(n)$ & $x(n)$ such that length $= 4$.

$$\therefore h(n) = \{1, 2, 3, 0\}$$

$$x(n) = \{1, 2, 0, 0\}$$

ii) Calculation of DFTs of $H(k)$ & $X(k)$

∴ we will calc. 4-point DFT of $h(n)$ & $x(n)$.

$$\therefore X_N = [W_N] x_N$$

$N = 4$

$$\therefore X_4 = [W_4] x_4$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$X_4 = \begin{bmatrix} 1+2+0+0 \\ 1-2j+0+0 \\ 1-2+0+0 \\ 1+j2+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-j2 \\ -1 \\ 1+j2 \end{bmatrix}$$

Now, DFT of $h(n)$

$$\therefore H_4 = [W_4] h_4$$

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$N=4$:

$$\therefore \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+0 \\ 1-j2+3+0 \\ 1-2+3+0 \\ 1+j2+3+0 \end{bmatrix} = \begin{bmatrix} 6 \\ -2-j2 \\ 2 \\ -2+j2 \end{bmatrix}$$

iii) Multiply DFTs :

$$\therefore Y(k) = H(k) \cdot X(k)$$

$$\therefore \begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \end{bmatrix} = \begin{bmatrix} 6 \times 3 \\ (-2-2j) \times (1-j2) \\ (2) \times (-1) \\ (-2+j2) \times (1+j2) \end{bmatrix}$$

$$Y(4) = \begin{bmatrix} 18 \\ -6+j2 \\ -2 \\ -6-j2 \end{bmatrix}$$

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This is 4-point DFT of $y(n)$.

iv) To obtain $y(n)$ from $Y(k)$ by IDFT

$$\therefore x_N = \frac{1}{N}[W_N^*]X_N$$

$$\therefore y_N = \frac{1}{N}[W_N^*]Y_N$$

N = 4

$$\therefore y_4 = \frac{1}{4}[W_4^*]Y_4$$

$$\therefore \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}^* \begin{bmatrix} 18 \\ -6+2j \\ -2 \\ -6-2j \end{bmatrix}$$

$$[W_4^*] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$\therefore y_4 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 18 \\ -6+j2 \\ -2 \\ -6-j2 \end{bmatrix}$$

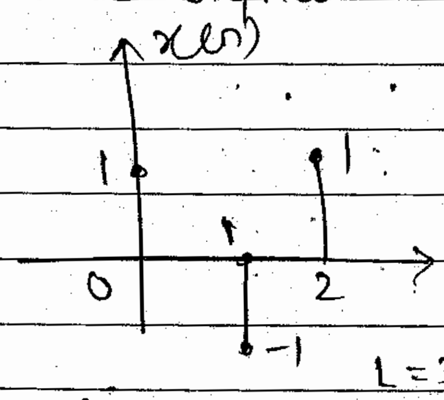
$$= \frac{1}{4}\begin{bmatrix} 18-6+j2-2-6-j2 \\ 18-j6-2+2+j6-2 \\ 18+6-j2-2+6+j2 \\ 18+j6+2+2-j6+2 \end{bmatrix}$$

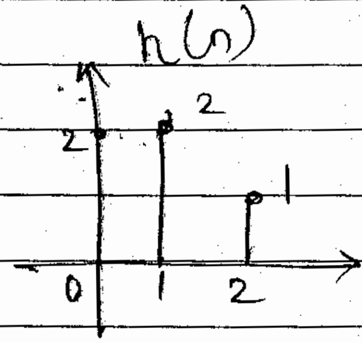
$$= \frac{1}{4}\begin{bmatrix} 4 \\ 16 \\ 28 \\ 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}$$

$\therefore y(n) = \{1, 4, 7, 6\}$

Eg. use DFT to compute linear Convolution of the signals.





$L = 3$

→ given seq. as

$x(n) = \{\underset{\uparrow}{1}, -1, 1\}$ &

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$h(n) = \{ \underset{\uparrow}{2}, 2, 1 \}$

i) $L = 3, \quad M = 3$

$\therefore N = L + M - 1 = 5 \Rightarrow k = 5 \ \& \ n = 5$
$0, 1, 2, 3, 4$

ii) $x(n) = \{ \underset{\uparrow}{1}, -1, 1 \}$

$\therefore x(0) = 1, \quad x(1) = -1, \quad x(2) = 1$

Calculation of $X(k)$:-
we have,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/5}$$

$$\therefore X(k) = \sum_{n=0}^{4} x(n) e^{-j2\pi kn/5}$$

for, $k = 0 \Rightarrow$

$$X(0) = \sum_{n=0}^{4} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) + x(4)$$

$\therefore X(0) = 1 - 1 + 1 = 1$

for $k = 1$

$$X(1) = \sum_{n=0}^{4} x(n) e^{-j2\pi n/5}$$

$$= x(0) e^{0} + x(1) e^{-j2\pi/5} + x(2) e^{-j4\pi/5} + 0 + 0$$

$$= 1 - 0.309 + j0.951 - 0.809 - j0.5878$$

$$= -0.118 + j0.3652$$

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for $k=2$

$$X(2) = \sum_{n=0}^{4} x(n) e^{-j4\pi n/5}$$

$$\therefore X(2) = x(0)e^{0} + x(1)e^{-j4\pi/5} + x(2)e^{-j8\pi/5} + 0 + 0$$

$$= 1 - 1[-0.809 - j0.5878] + 1[0.309 + j0.951]$$

$$\therefore X(2) = 2.118 + j1.5388$$

for $k=3$

$$X(3) = \sum_{n=0}^{4} x(n) e^{-j6\pi n/5}$$

$$= 1 - 1[-0.809 + j0.5878] + 1[0.309 - j0.951]$$

$$\therefore X(3) = 2.118 - j1.5388$$

for $k=4$

$$X(4) = \sum_{n=0}^{4} x(n) e^{-j8\pi n/5}$$

$$X(4) = x(0)e^{0} + x(1)e^{-j8\pi/5} + x(2)e^{-j16\pi/5} + 0 + 0$$

$$\therefore X(4) = 1 - 1[0.309 + j0.951] + 1[-0.809 + j0.5878]$$

$\therefore X(4) = -0.118 - j\,0.3632$

iii) Calcⁿ of $H(k)$

given,

$h(n) = \{\underset{\uparrow}{2}, 2, 1\} \quad \therefore h(0) = 2,$

$h(1) = 2, \; h(2) = 1$

we have, $H(k) = \sum_{n=0}^{4} h(n) e^{-j2\pi kn/5}$

For,

$k = 0$

$H(0) = \sum_{n=0}^{4} h(n) e^{0} = \sum_{n=0}^{4} h(n)$

$H(0) = h(0) + h(1) + h(2) + h(3) + h$

$\therefore H(0) = 2 + 2 + 1 + 0 + 0$

$\therefore H(0) = 5$

For,

$k = 1$

$H(1) = \sum_{n=0}^{4} h(n) e^{-j2\pi n/5}$

$\therefore H(1) = h(0) e^{0} + h(1) e^{-j2\pi/5} + h(2) e^{-j4\pi/5}$
$+ 0 + 0$

$H(1) = 2 + 2\,[0.309 - j\,0.951] +$
$[-0.809 - j\,0.5878]$

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$\therefore H[1] = 1.809 - j2.4898$

For $k=2$

$$H(2) = \sum_{n=0}^{4} h(n) e^{-j2\pi n/5}$$

$$\therefore H(2) = h(0)e^{0} + h(1)e^{-j4\pi/5} + h(2)e^{-j8\pi/5}$$

$$\therefore H(2) = 2 + 2[0.809 - j0.5878] + 1[0.309 + j0.951]$$

$$\therefore H(2) = 0.691 - j0.2246$$

For $k=3$

$$H(3) = \sum_{n=0}^{4} h(n) e^{-j6\pi n/5}$$

$$\therefore H(3) = h(0)e^{0} + h(1)e^{-j6\pi/5} + h(2)e^{-j12\pi/5} + 0 + 0$$

$$= 2 + 2[-0.809 + j0.5878] + 1[0.309 - j0.951]$$

$$\therefore H(3) = 0.691 + j0.2246$$

for $k=4$

$$H(4) = \sum_{n=0}^{4} h(n) e^{j8\pi n/5}$$

$$\therefore H(4) = h(0)e^{0} + h(1)e^{-j8\pi} + h(2)e^{-j16\pi/5} + 0 + 0$$

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$= 2+2\,[0.309+j0.951]+$
$1\,[-0.809+j0.5878]$

$\therefore H(4) = 1.809+j2.4898$

Step IV :-

Calculation of $Y(k) = X(k)\cdot H(k)$

$Y(0) = X(0)\cdot H(0) = (1)(5) = 5$

$Y(1) = X(1)\cdot H(1) = (-0.118+j0.3632)$
$(1.809-j2.4898)$
$= 0.6908+j0.9508$

$Y(2) = X(2)\cdot H(2)$
$= (2.118+j1.5388)(0.691-j0.2246)$
$= 1.8091+j0.5876$

$Y(3) = X(3)\cdot H(3)$
$= (2.118-j1.5388)(0.691+j0.2246)$
$= 1.8091-j0.5876$

$Y(4) = X(4)\cdot H(4)$
$= (-0.118-j0.3632)(1.809+j2.4898)$
$= 0.6908-j0.9508$

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v) Calcⁿ of y(n)

According to defⁿ of IDFT

$$y(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y(k) e^{-j2\pi kn/5}$$

$$y(n) = \frac{1}{5} \sum_{K=0}^{4} Y(k) e^{j2\pi 4n/5}$$

For $n=0$

$$y(0) = \frac{1}{5} \sum_{K=0}^{4} Y(k) e^{0}$$

$$y(0) = \frac{1}{5} [Y(0) + Y(1) + Y(2) + Y(3) + Y(4)]$$

$$\therefore y(0) = 2$$

For $n=1$

$$y(1) = \frac{1}{5} \sum_{K=0}^{4} Y(k) e^{j2\pi k/5}$$

$$\therefore y(1) = \frac{1}{5} \{ Y(0)e^{0} + Y(1) e^{j2\pi/5} + Y(2) e^{j4\pi/5} + Y(3) e^{j6\pi/5} + Y(4) e^{j8\pi/5} \}$$

$$\therefore y(1) = \frac{1}{5} \{ 5 + [0.6908 + j0.9508]$$

$$[0.309 + j0.951] + [1.8091 + j0.5876]$$

$$[-0.809 + j0.5878] + [1.8091 - j0.5876]$$

$$[-0.809 - j0.5878] + [0.6908 - j0.9508]$$

$$[0.309 - j0.951] \}$$

$$Y(1) = \frac{1}{5}\{5 + [0.691 + j0.951]$$

$$+ [-1.8089 + j0.588] + [-1.8089 - j0.588]$$
$$+ [-0.691 - j0.951]\}$$

$$\therefore y(1) = 0$$

For $n=2$

$$y(2) = \frac{1}{5}\sum_{K=0}^{4} Y(k)\, e^{j4\pi k/5}$$

$$\therefore y(2) = \frac{1}{5}\left\{ Y(0)e^{0} + Y(1)e^{j4\pi/5} + Y(2)e^{j8\pi/5} + Y(3)e^{j12\pi/5} + Y(4)e^{j16\pi/5} \right\}$$

$$\therefore y(2) = \frac{1}{5}\{5 + [0.6908 + j0.951] +$$
$$[1.8091 - j0.5876][0.309 + j0.951]$$

$$\therefore y(2) = \frac{1}{5}\{5 + [0.6908 + j0.9508][-0.809 + j0.5878]$$

$$+ [1.8091 + j0.5876][0.309 - j0.951] +$$
$$[1.8091 - j0.5876][0.309 + j0.951]$$
$$+ [0.6908 - j0.9808][-0.809 - j0.5878]\}$$

$$\therefore y(2) = 1$$

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~~PN → 70, 75, 80, 21~~

For $n=3$

$$y(3)=\frac{1}{5}\sum_{K=0}^{4} Y(k)\, e^{j6\pi k/5}$$

$$\therefore y(3)=\frac{1}{5}\left\{Y(0)+Y(1)e^{j6\pi/5}+Y(2)e^{j12\pi/5}\right.$$

$$\left.+Y(3)e^{j18\pi/5}+Y(4)e^{j24\pi/5}\right\}$$

$$\therefore y(3)=\frac{1}{5}\left\{5+[0.6908+j0.9508]\right.$$

$$[-0.809-j0.5878]+[1.8091+j0.5876][0.309+j0.951]$$

$$+[0.6908-j0.9508][-0.809+j0.5878]$$

$$\left.+[1.8091-j0.5876][0.309-j0.951]\right\}$$

$$\therefore y(3)=1$$

For $n=4$

$$y(4)=\frac{1}{5}\sum_{k=0}^{4} Y(k)\, e^{j8\pi k/5}$$

$$\therefore y(4)=\frac{1}{5}\left\{Y(0)+Y(1)e^{j8\pi/5}+Y(2)e^{j16\pi/5}\right.$$

$$\left.+Y(3)e^{j24\pi/5}+Y(4)e^{j32\pi/5}\right\}$$

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$$\therefore Y(4) = \frac{1}{5}\{5 + [0.6908 + j0.5878][0.309 - j0.951]$$

$$+ [1.8091 + j0.5876][-0.809 - j0.5878]$$

$$+ [1.8091 - j0.5876][-0.809 + j0.5878]$$

$$+ [0.6908 - j0.9608][0.309 + j0.951]\}$$

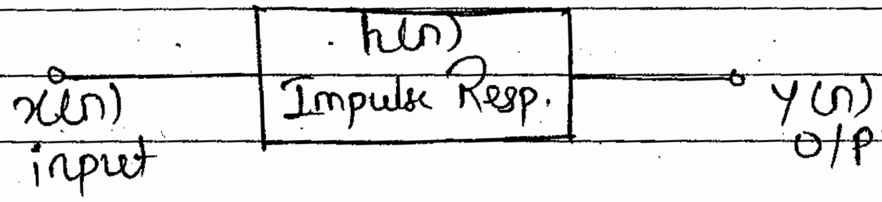
$\therefore y(4) = 1$

$\therefore$ O/P is

$y(n) = \{2, 0, 1, 1, 1\}$

#

Transfer Function of DT system in Frequency Domain using DFT :-



The output of DT system is given by equation of convolution.

$$y(n) = x(n) * h(n)$$

But the convolution is equivalent to multiplication in DFT domain.

$$\therefore Y(k) = X(k) \cdot H(k)$$

$$\therefore H(k) = Y(k) / X(k)$$

Here H(k) is called as system transfer function & it is the ratio of output to the input in DFT domain.

Response of FIR system Calculation in Frequency domain using DFT :-

- Linear filtering is same as linear convolution.
- In fourier transform, 'ω' is continuous function of frequency. So, the computation cannot be done on

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digital Computers. Because for DSP we want discrete signal & not continuous.

If we use DFT then computation will be more efficient because of the availability of Fast Fourier Transform algorithms.

So we must use DFT to obtain linear filtering opⁿ.

Steps to calculate linear filtering using DFT.

i) Calc. $N = L + M - 1$

ii) add zeroes to make length of $x(n)$ & $h(n) = N$.

iii) Calc. DFT of $x(n) \Rightarrow X(k)$

iv) Calc. DFT of $h(n) \Rightarrow H(k)$

v) Multiply $X(k)$ & $H(k)$ to get $Y(k) = X(k) \cdot H(k)$

vi) Obtain IDFT of $Y(k)$ means $y(n)$.

eg. Determine the response of FIR filter using DFT if :-

$x(n) = \{1, 2\}$ & $h(n) = \{2, 2\}$

$x(n) \Rightarrow L = 2$

$h(n) \Rightarrow M = 2$

$N = L + M - 1 = 2 + 2 - 1 = 3$

We have to calculate 3-point DFT.
We will compute Std. 4-point DFT.
$\therefore N = 4$

ii) $\therefore x(n) = \{\underset{\uparrow}{1}, 2, 0, 0\}$

& $h(n) = \{\underset{\uparrow}{2}, 2, 0, 0\}$

iii) Calc^n of $X(k)$

We will calculate DFT of $x(n)$, $X(k)$ using matrix method.

we have, $X[k] = W_N \cdot X_N$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} X[0] \\ X[1] \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+0+0 \\ 1-2j+0+0 \\ 1-2+0+0 \\ 1+2j+0+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

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$$X[k] = \{3, 1-2j, -1, 1+2j\}$$

IV) Calcⁿ of H(k) :

$$H(k) = W_N \cdot h_N$$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2+0+0 \\ 2-2j+0+0 \\ 2-2+0+0 \\ 2+2j+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2-2j \\ 0 \\ 2+2j \end{bmatrix}$$

$$\therefore H(k) = \{4, 2-2j, 0, 2+2j\}$$

V) $Y(k) = X(k) \cdot H(k)$

$$\therefore Y(k) = \{\underset{\uparrow}{3}, 1-2j, -1, 1+2j\} \cdot \{\underset{\uparrow}{4}, 2-2j, 0, 2+2j\}$$

$$Y(0) = X(0) \cdot H(0) = 3 \times 4 = 12$$

$$Y(1) = X(1) \cdot H(1) = (1-2j)(2-2j)$$

$$= 2-2j-4j+4j^2$$

$$= 2-6j-4 = 2-6j$$

$$Y(2) = X(2) \cdot H(2) = (-1)(0) = 0$$

$$Y(3) = X(3) \cdot H(3) = (1+2j)(2+2j)$$

$$= 2+2j+4j+4j^2$$

$$= 2+6j-4 = 2+6j$$

$$\therefore Y(k) = \{12, -2-6j, 0, -2+6j\}$$

VI) Now we will obtain $y(n)$ by taking IDFT $Y(k)$

$$y(n) = \frac{1}{N} W_N^{*} Y(k)$$

$$\therefore y(n) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ -2-6j \\ 0 \\ -2+6j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 12-2-6j+0-2+6j \\ 12-2j+6+0+2j+6 \\ 12+2+6j+0+2-6j \\ 12+2j-6+0-2j-6 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 \\ 24 \\ 16 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 0 \end{bmatrix}$$

$\therefore\ y(n) = \{\underset{\uparrow}{2}, 6, 4, 0\}$

Verification $\longrightarrow$

Linear Convolution

	1	2
2	2	4
2	2	4

$y(0) = 2\ ,\ y(1) = 6\ ,\ y(2) = 4$

$y(n) = \{2, 6, 4\}$

Eg. The first five Points of 8 point DFT of a real valued seq. are
{0.25, 0.125 - j0.3018, 0, 0.125 - j0.0518, 0}
Det. remaining three Points.

→ Given DFT Points are,

$$X(0) = 0.25$$
$$X(1) = 0.125 - j0.3018$$
$$X(2) = 0$$
$$X(3) = 0.125 - j0.0518$$
$$X(4) = 0$$

Given seq. is real valued seq.
According to symmetry Property we have,

$$X^*(k) = X(N-k)$$
or $X(k) = X^*(N-k)$

This is N-point DFT.
$\therefore N = 8$

$$\therefore X(k) = X^*(8-k)$$

we want remaining three values $X(5)$, $X(6)$ & $X(7)$. Putting $k=7$ in equ^n (2).

$$X(5) = X^*(8-5) = X^*(3)$$
$$X(3) = 0.125 - j0.0518$$
$$X^*(3) = 0.125 + j0.0518$$

$\therefore X(5) = 0.125 + j0.0518$

For $k = 6$,

$X(6) = X^*(8-6) = X^*(2)$

$X^*(2) = 0$

$\therefore X^*(2) = 0$

$\therefore X(6) = 0$

For $k = 7$

$X(7) = X^*(8-7) = X^*(1)$

$X(1) = 0.125 - j0.3018$

$\therefore X(7) = 0.125 + j0.3018$

Eg. i) $x(n) = \{1, 2, 3, 4\}$ find DFT $X(k)$.

ii) Using results in Part (i) & not otherwise find DFT of foll. seq.

$x_1(n) = \{4, 1, 2, 3\}$ $\quad x_2(n) = \{2, 3, 4, 1\}$

$x_3(n) = \{3, 4, 1, 2\}$ $\quad x_4(n) = \{4, 6, 4, 6\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

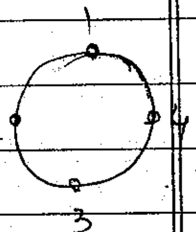
$\therefore X(k) = \{10, -2+2j, -2, -2-2j\}$

ii) $x_1(n) = \{4, 1, 2, 3\}$

$x(n) = \{1, 2, 3, 4\}$

Thus $x_1(n)$ is obtained by delaying $x(n)$ by one position.

$\therefore x_1(n) = x(n-1)$



According to circular time shift Property

$\therefore x(n-l) \xleftrightarrow{DFT} X(k)e^{-j2\pi kl/N}$

Here, $l = 1$

~~Here~~

$\therefore X_1(k) = X(k)e^{-j2\pi k/N}$

For

- $k=0 \Rightarrow X_1(0) = X(0)e^{0} = 10$
- $k=1 \Rightarrow X_1(1) = X(1)e^{-j2\pi/4}$

$$= (-2+2j)\left[\cos\frac{2\pi}{4} - j\sin 2\pi/4\right]$$

$\therefore X_1(1) = (-2+2j)(-j)$

$\therefore X_1(1) = 2j+2$

For $k=2 \Rightarrow$

$$X_1(2) = X(2)\, e^{-j4\pi/4} = X(2)\, e^{-j\pi}$$

$$\therefore X_1(2) = (-2)\,[\cos\pi - j\sin\pi] = (-2)(-1)$$

$$X_1(2) = 2$$

For $k=3 \Rightarrow$

$$X_1(3) = X(3)\, e^{-j6\pi/4}$$

$$= (-2-2j)\left[\cos 6\pi/4 - j\sin 6\pi/4\right]$$

$$= (-2-2j)(+j)$$

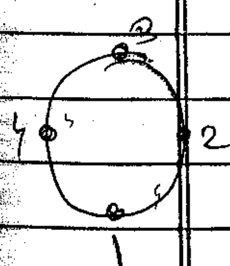
$$\therefore X_1(3) = -2j+2$$

$$\therefore X_1(k) = \{10,\ 2j+2,\ 2,\ -2j+2\}$$

b) $x_2(n) = \{2,3,4,1\}$

$x(n) = \{1,2,3,4\}$

Thus $x_2(n)$ is obtained by advancing $x(n)$ by 1 position.



$$\therefore x_2(n) = x(n+1) \Rightarrow \boxed{l=-1}$$

According to circular time shift property,

$$x(n-l) \xleftrightarrow{DFT} X(k)\, e^{-j2\pi kl/N}$$

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$$x(n+l) \xleftrightarrow{DFT} X(k)e^{-j2\pi lk/N}$$

$$X_2(k) = X(k)e^{-j2\pi lk/N}$$
$l = -1$

For $k=0 \Rightarrow X_2(0) = X(0)e^0 = 10$

$k=1 \Rightarrow X_2(1) = X(1)e^{+j2\pi/4}$

$= (-2+2j)\left[\cos 2\pi/4 + j\sin 2\pi/4\right]$

$\therefore X_2(1) = (-2+2j)(0+j)$

$\therefore X_2(1) = -2-2j$

For $k=2 \Rightarrow X_2(2) = X(2)e^{j4\pi/4}$

$= (-2)[\cos\pi + j\sin\pi]$

$= (-2)[-1]$

$\therefore X_2(2) = 2$

For $k=3 \Rightarrow X_2(3) = X(3)e^{j6\pi/4}$

$= (-2-2j)\left[\cos 6\pi/4 + j\sin 6\pi/4\right]$

$= (-2-2j)[-j]$

$\therefore X_2(3) = 2j-2$

$\therefore \cancel{X_2(k) = \{10, -2, -2j}$

$X_2(k) = \{10, -2-2j, 2, 2j-2\}$

c] $x_3(n) = \{3, 4, 1, 2\}$

$x(n) = \{1, 2, 3, 4\}$

$x_3(n)$ is obtained by delaying $x(n)$ by 2 positions.

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$\therefore x_3(n) = x(n-2)$

According to Circular time shifting Property

$x(n-l) \xleftrightarrow{DFT} X(k)\, e^{-j2\pi kl/N}$

$x(n-2) \xleftrightarrow{DFT} X(k)\, e^{-j4\pi k/4}$

$\therefore X_3(k) = X(k)\, e^{-j\pi k}$

$l = 2$

For $k=0 \Rightarrow X_3(0) = X(0)\, e^{0} = 10$

$k=1 \Rightarrow X_3(1) = X(1)\, e^{-j\pi} = (-2+2j)(\cos\pi - j\sin\pi)$

$= (-2+2j)(-1)$

$\therefore X_3(1) = 2-2j$

For $k=2 \Rightarrow X_3(2) = X(2)\, e^{-j2\pi}$

$= (-2)[\cos 2\pi - j\sin 2\pi]$

$= (-2)(1)$

$X_3(2) = -2$

For $k=3 \Rightarrow X_3(3) = X(3)\, e^{-j3\pi}$

$= (-2-j2)[\cos 3\pi - j\sin 3\pi]$

$= (-2-j2)(-1)$

$\therefore X_3(3) = 2+j2$

$\therefore X_3(k) = \{10, 2-2j, -2, 2+2j\}$

d) $x_4(n) = \{4, 6, 4, 6\}$

$x(n) = \{1, 2, 3, 4\}$

$x_4(n)$ & $x(n)$ are related as,

$x_4(n) = x(n) + x(n+2)$ $2 = N/2$

$= 4/2 = 2$

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Using half Period shift Property,

$X_4(k) = X(k) + (-1)^k X(k)$

$[\because x(n \mp 2) \longleftrightarrow (-1)^k X(k)]$

for $k=0 \Rightarrow X_4(0) = X(0) + (-1)^0 X(0)$

$= 10 + 10 = 20$

$k=1 \Rightarrow X_4(1) = X(1) + (-1)^1 X(1)$

$= -2+2j+2-2j$

$X_4(1) = 0$

$k=2 \Rightarrow X_4(2) = X(2) + (-1)^2 X(2)$

$= -2 + (-2)$

$\therefore X_4(2) = -4$

$k=3 \Rightarrow X_4(3) = X(3) + (-1)^3 X(3)$

$= -2-2j+2+2j$

$\therefore X_4(3) = 0$

$\therefore X_4(k) = \{20, 0, -4, 0\}$

Eg. Find DFT of the seq. $x[n] = \{1,2,3,4\}$ & using this result & not otherwise Find DFT of

i) $x_1[n] = \{1,0,2,0,3,0,4,0\}$

ii) $x_2[n] = \{1,2,3,4,0,0,0,0\}$

iii) $x_3[n] = \{1,2,3,4,1,2,3,4\}$

$\rightarrow$ $x(n) = \{1,2,3,4\}$

$X(k)$ is calculated as follows.

X(k) is calculated as,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2-3-4 \\ 1+2j-3-4j \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

i) $x_1(n) = \{1, 0, 2, 0, 3, 0, 4, 0\}$

If alternate zeroes are padded then then the DFT seq. repeats.

$\therefore X_1(k) = \{10, -2+2j, -2, -2-2j, 10, -2+2j, -2, -2-2j\}$

ii) $x_2[n] = \{1, 2, 3, 4, 0, 0, 0, 0\}$

If zeroes are padded at the end of the seq., then even point DFTs remains same, while odd point DFTs are different.

$\therefore X_2(k) = \{10, X_2(1), -2+2j, X_2(3), -2, X_2(5), -2-2j, X_2(7)\}$

iii) $x_3[n] = \{1, 2, 3, 4, 1, 2, 3, 4\}$

half period shifting property

$x_4(n) = \{1, 2, 3, 4, 0, 0, 0, 0\}$

Then $x_3(n) = x_4(n) + x_4\left[n - N/2\right]$

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$\therefore X_3(k) = X_1(k) + (-1)^k X_1(k)$

$$\therefore X_3(k) = \begin{cases} 2X_1(k), & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}$$

$\therefore \cancel{X_3(k) = \{20, 0, -4+4j, 0, -4, 0, -2-2j\}}$

$X_3(k) = \{20, 0, -4+4j, 0, -4, 0, -2-2j, 0\}$

Eg. Consider the length 8-seq. defined for $0 \leq n \leq 8$.

$x(n) = \{1, 2, -3, 0, 1, -1, 4, 2\}$

with a 8-point DFT. Evaluate the foll. function $X(k)$ w/o computing DFT.

i) $X(0)$ ii) $X(4)$ iii) $\sum_{k=0}^{7} X(k)$

iv) $\sum_{k=0}^{7} |X(k)|^2$

→ i)

$X(0) = \sum_{n=0}^{N-1} x(n)$ —— $k=0$

$N = 8$

$\therefore X(0) = x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$

$\therefore X(0) = 1 + 2 - 3 + 0 + 1 - 1 + 4 + 2$
$= 6$

ii) We have an identity,

$$X(N/2) = \sum_{n=0}^{N-1} (-1)^n x(n)$$

$N=8$

$$\therefore x(4) = \sum_{n=0}^{7} (-1)^n x(n)$$

$$\therefore x(4) = (-1)^0 x(0) + (-1)^1 x(1) + (-1)^2 x(2) + (-1)^3 x(3) + (-1)^4 x(4) + (-1)^5 x(5) + (-1)^6 x(6) + (-1)^7 x(7)$$

$$x(4) = 1 - 2 - 3 + 0 + 1 + 1 + 4 - 2$$

$$x(4) = 0$$

iii)

$$x(0) = \frac{1}{N} \sum_{n=0}^{N-1} X(k)$$

$N=8$

$$\therefore x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$\sum_{k=0}^{7} X(k) = 8 \times x(0)$$

$$\therefore \sum_{k=0}^{7} X(k) = 8 \times 1 = 8$$

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iv) According to Parseval's thm

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$N=8$

$$\therefore \sum_{n=0}^{7} |x(n)|^2 = \frac{1}{8} \sum_{k=0}^{7} |X(k)|^2$$

$$\therefore \sum_{k=0}^{7} |X(k)|^2 = 8 \sum_{k=0}^{7} |x(n)|^2$$

$$\therefore \sum_{k=0}^{7} |X(k)|^2 = 8 \left[|x(0)|^2 + |x(1)|^2 + \dots\dots + |x(7)|^2 \right]$$

$$= 8 \left[1+4+9+0+1+1+16+4 \right]$$

$$\therefore \sum_{k=0}^{7} |X(k)|^2 = 288$$

Fast Fourier Transform :-

The number N can be factored as,

$N = r_1, r_2, r_3, \dots r_v$

r is a Prime.

if $r_1 = r_2 = r_3 = \dots = r_v = r$

$\therefore N = r^v$

r is radix of FFT algorithm. & 'v' indicates number of stages in FFT alg.

Radix means base. & if its value is 2 then it is called as radix-2 FFT alg.

$\therefore$ when $r = 2$

$N = 2^v$

Suppose $N = 8$

$8 = 2^v$

$\therefore v = 3$

$\therefore$ For 8 point, there are three stages of FFT alg.

While computing FFT, divide number of input samples by 2, till you reach min^m two samples.

Based on this division there are two algos.

i) Radix-2 Decimation in Time alg.

ii) Radix-2 Decimation in frequency alg.

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Properties of twiddle factor :-

The twiddle factor W_N is given by

$$W_N = e^{-j2\pi/N}$$

1. $W_N^K = W_N^{K+N}$

It indicates that twiddle factor is periodic.

2. $W_N^{K+N/2} = -W_N^K$

It indicates twiddle factor is symmetric.

3. $W_N^2 = W_{N/2}$

Radix-2 Decimation In Time algorithm (DIT FFT)

Decimate means break into Parts. DIT indicates dividing seq. in Time domain.

First Stage of Decimation :-

- Let $x(n)$ be given input seq. containing 'N' samples.
- For decimation in time we will divide $x(n)$ into even & odd seq.

$\therefore x(n) = f_1(m) + f_2(m)$

$f_1(m)$ is an even seq.

& $f_2(m)$ is an odd seq.

$\therefore f_1(m) = x(2n)$, $m = 0, 1, 2, \ldots N/2 - 1$

~~$f_2(m) = x($~~

$f_2(m) = x(2n+1)$, $m = 0, 1, 2, \ldots N/2 - 1$

after decimation $f_1(m)$ & $f_2(m)$ will contain N/2 samples.

According to def^n of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

dividing x(n) into two Parts.

$$X(k) = \sum_{n\ \text{even}} x(n) W_N^{kn} + \sum_{n\ \text{odd}} x(n) W_N^{kn}$$

$\Downarrow$ $n = 2m$ $\qquad$ $\Downarrow$ $n = 2m+1$

Odd & even seq. contain N/2 samples each.

$$\therefore X(k) = \sum_{m=0}^{N/2-1} x(2m) W_N^{2km} + \sum_{m=0}^{N/2-1} x(2m+1) W_N^{k(2m+1)}$$

$x(2m) = f_1(m)$

& $x(2m+1) = f_2(m)$

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$$\therefore X(k) = \sum_{m=0}^{N/2-1} f_1(m) W_N^{2km} + \sum_{m=0}^{N/2-1} f_2(m) W_N^{2km} \cdot W_N^{k}$$

$$\therefore X(k) = \sum_{m=0}^{N/2-1} f_1(m) (W_N^2)^{km} + W_N^{k} \sum_{m=0}^{N/2-1} f_2(m) (W_N^2)^{km}$$

$$W_N^2 = W_{N/2}$$

$$\therefore X(k) = \sum_{m=0}^{N/2-1} f_1(m) W_{N/2}^{km} + W_N^{k} \sum_{m=0}^{N/2-1} f_2(m) W_{N/2}^{km}$$

Comparing with defn of DFT,

$$X(k) = F_1(k) + W_N^k F_2(k)$$

$$k = 0, 1, \dots N-1$$

Suppose $N=8$

$\therefore$ $F_1(k)$ is $N/2$ DFT of $f_1(m)$

$F_2(k)$ is $N/2$ DFT of $f_2(m)$

$\therefore$ $F_1(k)$ & $F_2(k)$ are 4-Point DFT's.

$F_2(k)$ is multiplied by ~~$W_{N/2}$~~ W_N^k & added with $F_1(k)$ to obtain 8-point DFT.

k varies from 0 to $N-1$.

$F_1(k)$ & $F_2(k)$ are 4-Point ($N/2$) DFT's

Using Periodicity Property,

of DFT, we can write,

$$F_1\left(k+N/2\right) = F_1(k)$$

& $F_2(k+N/2) = F_2(k)$

Replacing k by $(k+N/2)$

$$X(k+N/2) = F_1\left(k+N/2\right) + W_N^{k+N/2} F_2\left(k+N/2\right)$$

we here $W_N^{k+N/2} = -W_N^k$

$$X(k+N/2) = F_1\left(k+N/2\right) - W_N^k F_2(k+N/2)$$

X(k) is N-point DFT.

$k = 0$ to $N/2 - 1$

$$\therefore X(k) = F_1(k) + W_N^k F_2(k) \quad , \; k = 0, 1 \ldots N/2 - 1 \qquad \text{I}$$

$$\& \; X(k+N/2) = F_1(k) - W_N^k F_2(k), \quad k = 0, 1, \cdots N/2 - 1 \qquad \text{II}$$

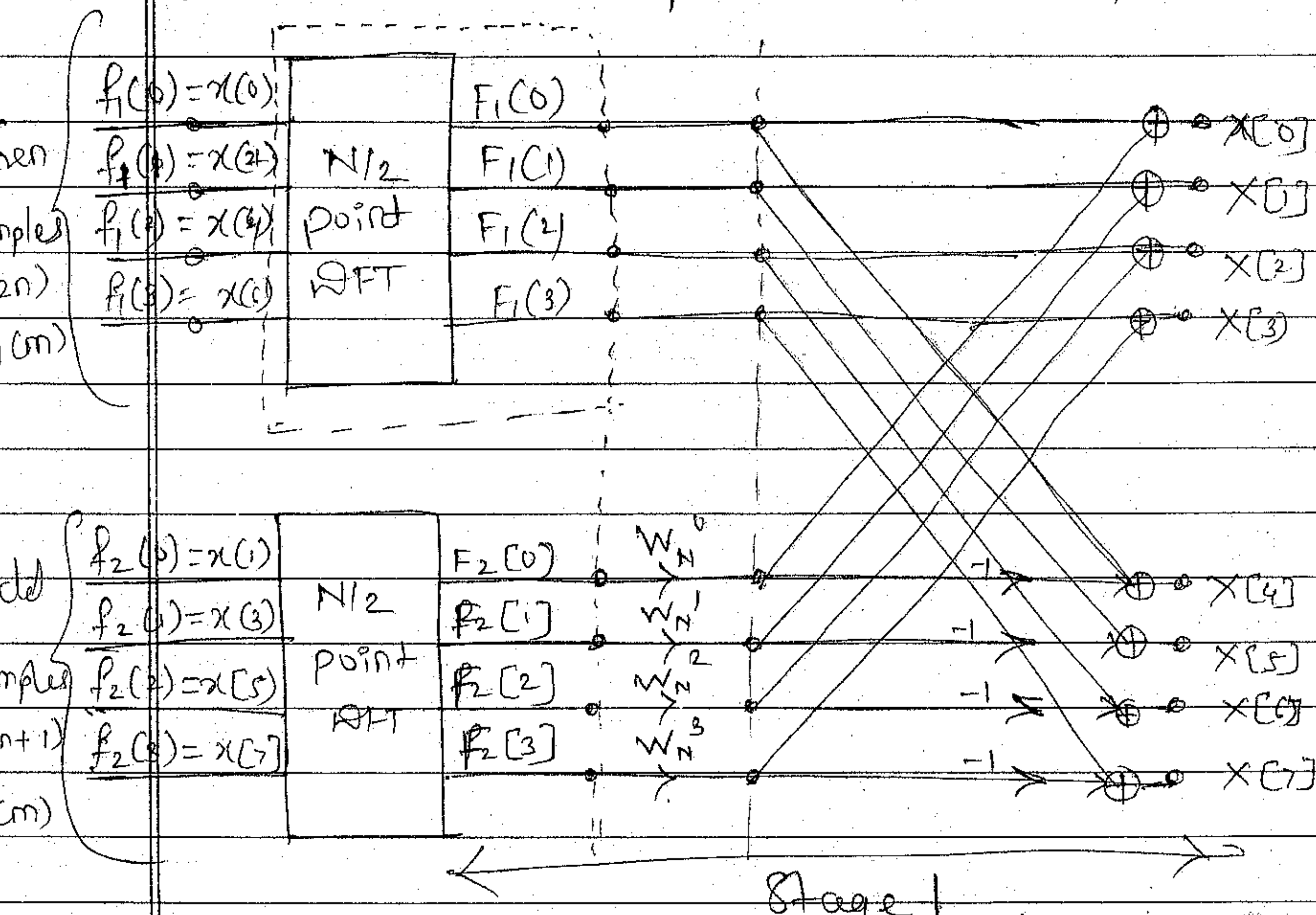
eg. of 8 point DFT ⟹

Putting $k = 0$ to 3 in above two equⁿ.

$$\left.\begin{aligned} X(0) &= F_1(0) + W_N^0 F_2(0) \\ X(1) &= F_1(1) + W_N^1 F_2(1) \\ X(2) &= F_1(2) + W_N^2 F_2(2) \\ X(3) &= F_1(3) + W_N^3 F_2(3) \end{aligned}\right\} \text{ I}^{st} \text{ equ}^n.$$

$$X(0+4) = X(4) = F_1(0) - W_N^0 F_2(0)$$
$$X(1+4) = X(5) = F_1(1) - W_N^1 F_2(1)$$
$$X(2+4) = X(6) = F_1(2) - W_N^2 F_2(2)$$
$$X(3+4) = X(7) = F_1(3) - W_N^3 F_2(3)$$

Representation of first stage of decimation for 8 point DFT is as,



First Stage of Decimation.

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Second Stage of Decimation :-

- In first stage we obtained sequences of length $N/2$.
- Length of each seq. is 4.
- We can further decimate $f_1(m)$ into even & odd samples.
- Let $g_{11}(n) = f_1(2m)$, which contains even samples & $g_{12}(n) = f_1(2m+1)$ contains odd samples of $f_1(m)$.

$\therefore$ range of n & m from ~~N/4~~ 0 to $N/4 - 1$

- We obtained seq. $X(k)$ & $X(k+N/2)$ from $F_1(k)$ & $F_2(k)$.
- Length of each seq. was $N/2$.
- In second stage we are further dividing seq. into even & odd.

$$\therefore F_1(k) = G_{11}(k) + W_{N/2}^{k} G_{12}(k)$$

$$k = 0, 1, \dots N/4 - 1$$

$$\& \quad F_1(k+N/4) = G_{11}(k) - W_{N/2}^{k} G_{12}(k)$$

$$k = 0, 1, \dots\dots N/4 - 1$$

Now range $k = 0$ to 1

$G_{11}(k)$ is DFT of $g_{11}(n)$ & $G_{12}(k)$ is DFT of $g_{12}(n)$.

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So putting values of k.

$$F_1(0) = G_{11}(0) + W_{N/2}^{0}\, G_{12}(0)$$

$$F_1(1) = G_{11}(1) + W_{N/2}^{1}\, G_{12}(1)$$

&

$$F_1\left(0 + 8/4\right) = F_1(2) = G_{11}(0) - W_{N/2}^{0}\, G_{12}(0)$$

$$F_1\left(1 + 8/4\right) = F_1(3) = G_{11}(1) - W_{N/2}^{1}\, G_{12}(1)$$

Here values of k are 0 & 1. So it is 2-point DFT.

We can obtain 4-point DFT by combining 2-point DFTs.

Here,

$$g_{11}(n) = f_1(2m) = x(4n) = \{x(0), x(4)\}$$

$$g_{12}(n) = f_1(2m+1) = x(4n+2) = \{x(2), x(6)\}$$

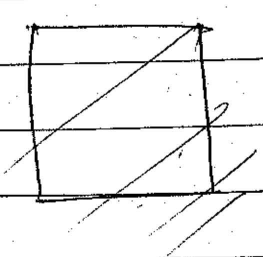
For $F_2(k)$ equa^ns as,

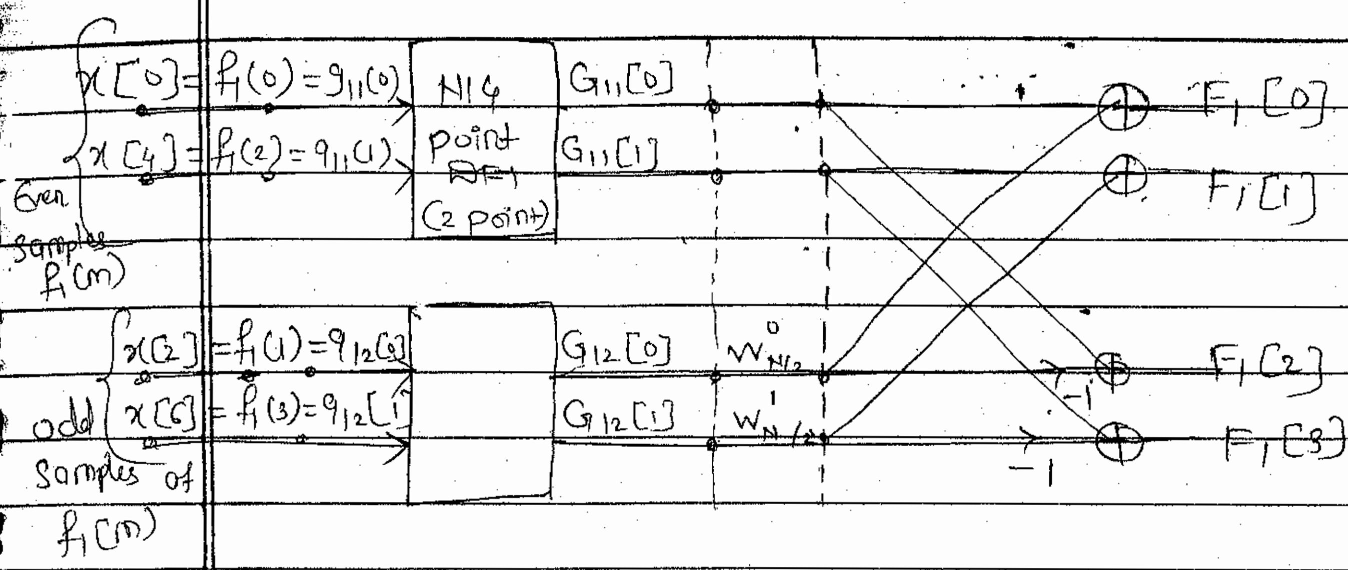
$$F_2(k) = G_{21}(k) + W_{N/2}^{k}\, G_{22}(k)$$

$$k = 0, 1, \ldots N/4 - 1$$

$$F_2(k + N/4) = G_{21}(k) - W_{N/2}^{k}\, G_{22}(k),$$

$$k = 0, 1, \ldots N/4 - 1$$





$F_1[k]$, N/2 point DFT.

Here $G_{12}(k)$ is DFT of $g_{21}(n)$ & $G_{22}(k)$ is DFT of $g_{22}(n)$. The values of k & n are 0 & 1. So Putting these values in above equa^ns.

$$F_2(0) = G_{21}(0) + W^0_{N/2}\, G_{22}(0)$$
$$F_2(1) = G_{21}(1) - W^1_{N/2}\, G_{22}(1)$$

& $F_2(0+8/4) = F_2(2) = G_{21}(0) - W^0_{N/2}\, G_{22}(0)$

$$F_2(1+8/4) = F_2(3) = G_{21}(1) - W^1_{N/2}\, G_{22}(1)$$

Here,

$$g_{21}(n) = f_2(2n) = x(4n+1) = \{x(1), x(5)\}$$
$$g_{22}(n) = f_2(2n+1) = x(4n+3) = \{x(3), x(7)\}$$

Graphical Representation

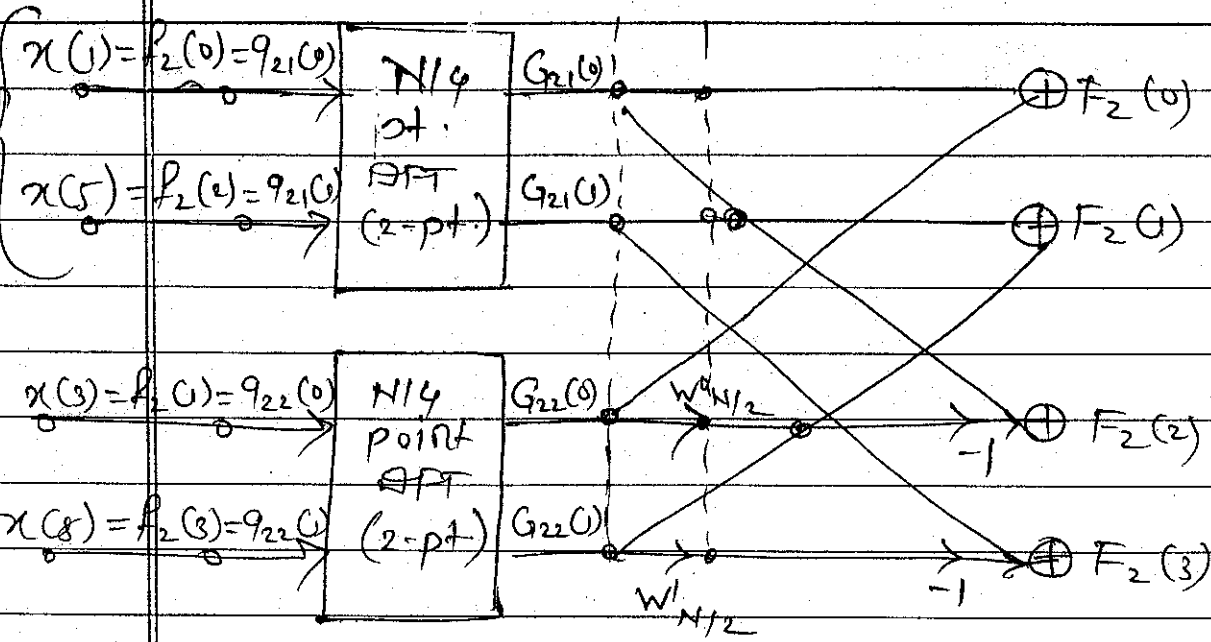


fig. $F_2(k)$, $N/2$ point DFT

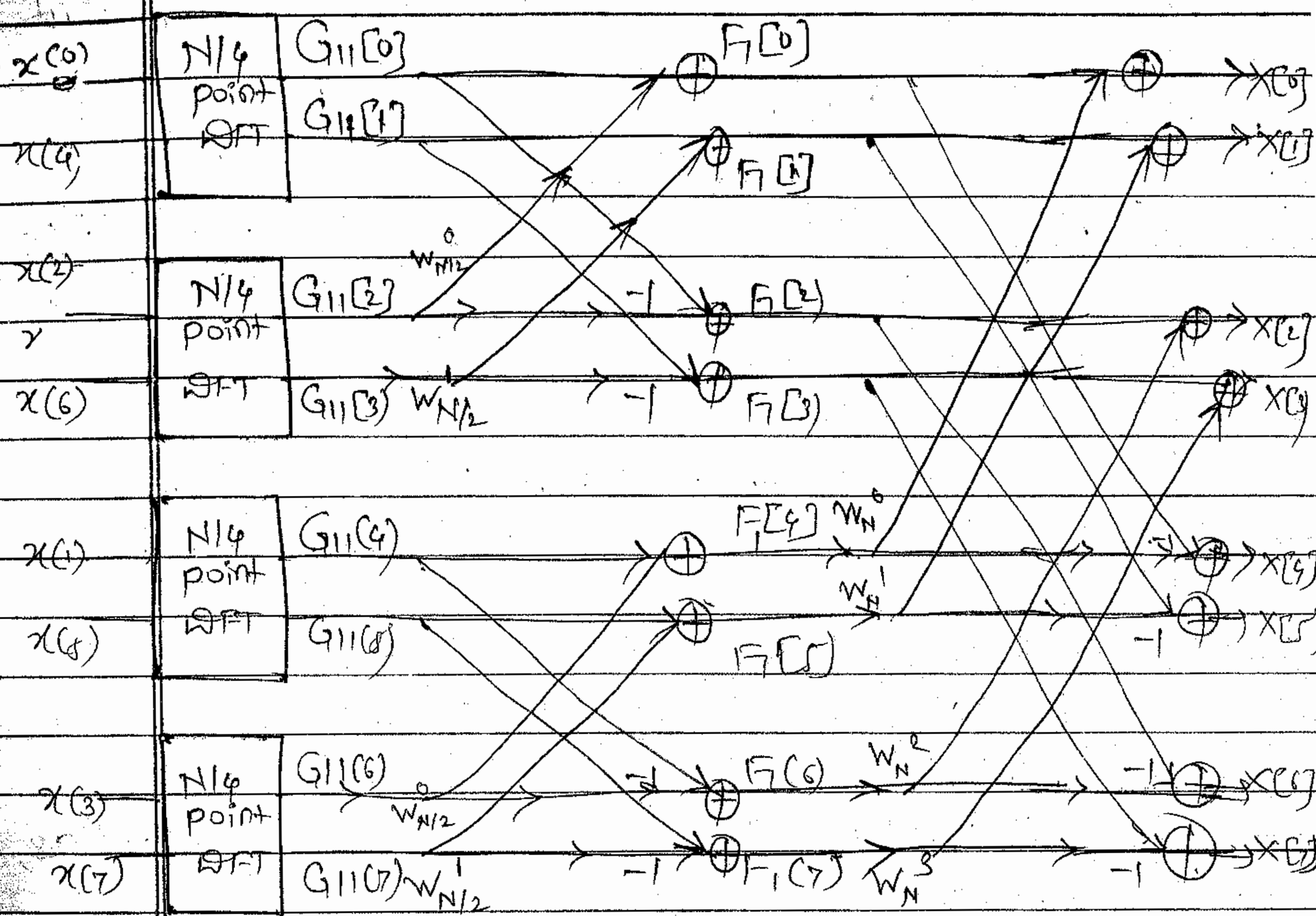
Combination of first & second stage combination :-

Combining first & second stage we get stages of decimation.

At this stage we have $N/4$ means 2 point sequences.

So further decimation is not possible.

We have to compute 2-point DFT.



Combination of first & second stage

Computation of 2 Point DFT :-

According to basic definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

Will use above equa^n to compute 2-Point DFT.

Consider the first block of 2-point DFT.

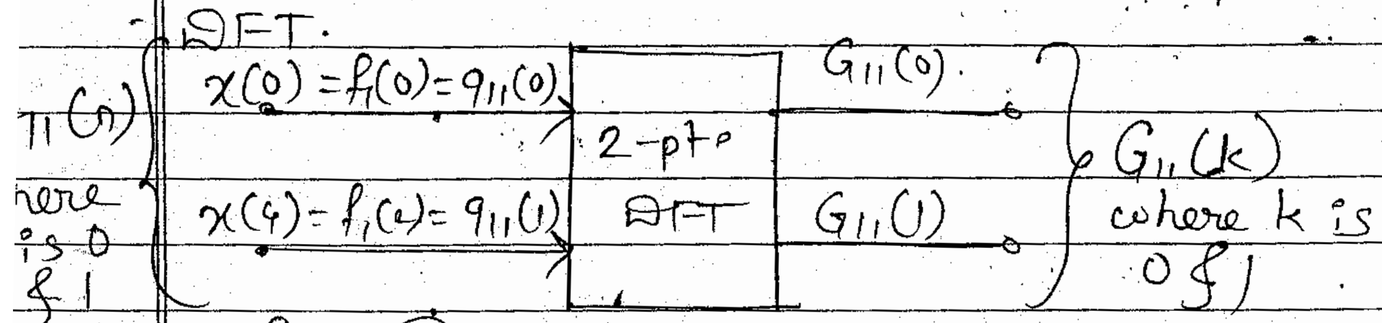


fig. Block of 2-pt. DFT.

- Here input seq. $g_{11}(0)$ & $g_{11}(1)$.
- We can denote input by $g_{11}(n)$ where $n = 0, 1$.
- O/P seq. are $G_{11}(0)$ & $G_{11}(1)$.
- $G_{11}(k)$ where $k = 0, 1$. $G_{11}(k)$ is DFT of $g_{11}(n)$.

Thus for $G_{11}(k)$ we can write equ^n as,

$$G_{11}(k) = \sum_{n=0}^{1} g_{11}(n) W_2^{kn}, \quad k = 0, 1$$

- This is 2 point DFT. So $N = 2$.
- Putting values of k,

for $k = 0$, $G_{11}(0) = \sum_{n=0}^{1} g_{11}(n) W_2^{0}$

- But $W_2^0 = 1$

$$\therefore G_{11}(0) = \sum_{n=0}^{1} g_{11}(n)$$

$$\therefore G_{11}(0) = g_{11}(0) + g_{11}(1)$$

for $k=1 \Rightarrow$

$$G_{11}(1) = \sum_{n=0}^{1} g_{11}(n) W_2^{n}$$

Expanding summation we get,

$$G_{11}(1) = g_{11}(0) W_2^{0} + g_{11}(1) W_2^{1}$$

$$W_N = e^{-j2\pi/N}$$

$$W_2^{1} = \left[e^{(-j2\pi/N)}\right]^{1} = e^{-j\pi}$$

$$= \cos\pi - j\sin\pi$$

$$= -1$$

So, $W_2^{0} = 1$ & $W_2^{1} = -1$

$$\therefore G_{11}(1) = g_{11}(0) - g_{11}(1)$$

We can represent computation of 2-point DFT as in fig. Structure looks like a butterfly. So it is FFT butterfly structure.

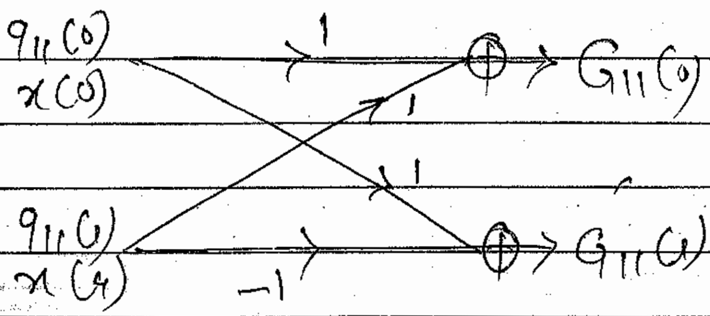


fig. FFT ~~fel~~ butterfly structure

$W_2^{0} = 1$. So we can modify equ^n as,

$$G_{11}(0) = g_{11}(0) + W_2^{0} g_{11}(1)$$

$$G_{11}(1) = g_{11}(0) - W_2^{0} g_{11}(1)$$

For other 2-point DFT, butterfly structure is same.

Total signal flow graph for 8-point DIT-FFT.

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Eg. Compute the eight-Point DFT of a sequence.

$x(n) = \{1/2, 1/2, 1/2, 1/2, 0, 0, 0, 0\}$

Using in-place radix-2 decimation in time FFT algorithm.

→ flow graph is as shown in fig. $S_1(n)$ represents output of stage 1 & $S_2(n)$ represents o/p of stage-2. The diff. values of twiddle factor are,

$W_8^0 = e^0 = 1 \qquad W_N^1 = e^{-j2\pi/N}$

$W_8^1 = e^{-j\pi/4} = 0.707 - j0.707$

$W_8^2 = e^{-j\pi/2} = -j$

$W_8^3 = -0.707 - j0.707$

O/P of stage-1

2-point DFT ⇒

$S_1(0) = x(0) + W_8^0 x(4) = \frac{1}{2} + 1(0) = \frac{1}{2}$

$S_1(1) = x(0) - W_8^0 x(4) = \frac{1}{2} - 1(0) = \frac{1}{2}$

$S_1(2) = x(2) + W_8^0 x(6) = \frac{1}{2} + 1(0) = \frac{1}{2}$

$S_1(3) = x(2) - W_8^0 x(6) = \frac{1}{2} - 1(0) = \frac{1}{2}$

$S_1(4) = x(1) + W_8^0 x(5) = \frac{1}{2} + 1(0) = \frac{1}{2}$

$S_1(5) = x(1) - W_8^0 x(5) = \frac{1}{2} - 1\cdot(0) = \frac{1}{2}$

$S_1(6) = x(3) + W_8^0 x(7) = \frac{1}{2} + 1(0) = \frac{1}{2}$

$S_1(7) = x(3) - W_8^0 x(7) = 1/2 - 1\cdot(0) = 1/2$

O/P of stage-2

$S_2(0) = S_1(0) + W_8^0 S_1(2) = \frac{1}{2} + 1\cdot(1/2) = 1$

$S_2(1) = S_1(1) + W_8^2 S_1(3) = \frac{1}{2} - j1/2$

$S_2(2) = S_1(0) - W_8^0 S_1(2) = \frac{1}{2} - 1/2 = 0$

$S_2(3) = S_1(1) - W_8^2 S_1(3) = \frac{1}{2} + j1/2$

$S_2(4) = S_1(4) + W_8^0 S_1(6) = \frac{1}{2} + \frac{1}{2} = 1$

$S_2(5) = S_1(5) + W_8^2 S_1(7) = \frac{1}{2} \cancel{+\frac{1}{2}} =$

$= \frac{1}{2} - j1/2$

$S_2(6) = S_1(4) - W_8^0 S_1(6) = 1/2 - 1/2 = 0$

$S_2(7) = S_1(5) - W_8^2 S_1(7) = 1/2 + j1/2$

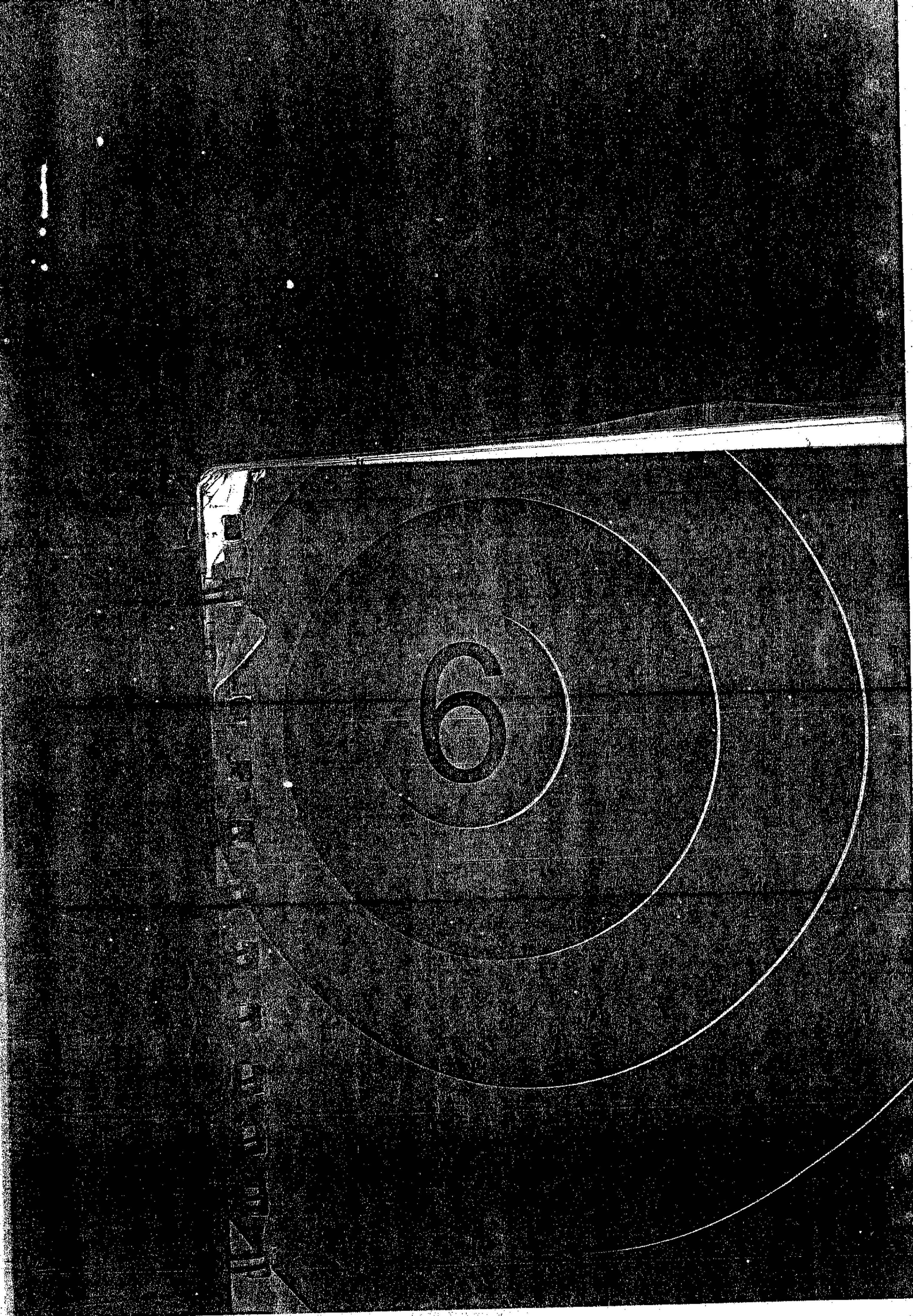
Final Output :-

$X(0) = S_2(0) + W_8^0 S_2(4) = 1+1 = 2$

$X(1) = S_2(1) + W_8^1 S_2(5) = (1/2 - j1/2)$

$+ (0.707 - j0.707)\left(\frac{1}{2} - j1/2\right)$

$= 0.5 - j1.207$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$$X(2) = S_2(2) + W_8^2 S_2(6) = 0 + (-j)(0) = 0$$

$$X(3) = S_2(3) + W_8^3 S_2(7) = (1/2 + j\,1/2)$$

$$+ (-0.707 - j0.707)\,(1/2 + j\,1/2)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + (0 - j\,0.707)$$

$$= 0.5 - j\,0.207$$

$$X(4) = S_2(0) - W_8^0 S_2(4) = 1 - 1 \cdot 1 = 0$$

$$X(5) = S_2(1) - W_8^1 S_2(5)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (0.707 - j\,0.707)\left(\frac{1}{2} - j\frac{1}{2}\right)$$

$$= \left(\frac{1}{2} - j\frac{1}{2}\right) - (-0.707j)$$

$$X(5) = 0.5 + j\,0.207$$

$$X(6) = S_2(2) - W_8^2 S_2(6) = 0 + j \cdot (0) = 0$$

$$X(7) = S_2(3) - W_8^3 S_2(7)$$

$$= \left(\frac{1}{2} + j\frac{1}{2}\right) - (-0.707 - j\,0.707)\left[\frac{1}{2} + j\frac{1}{2}\right]$$

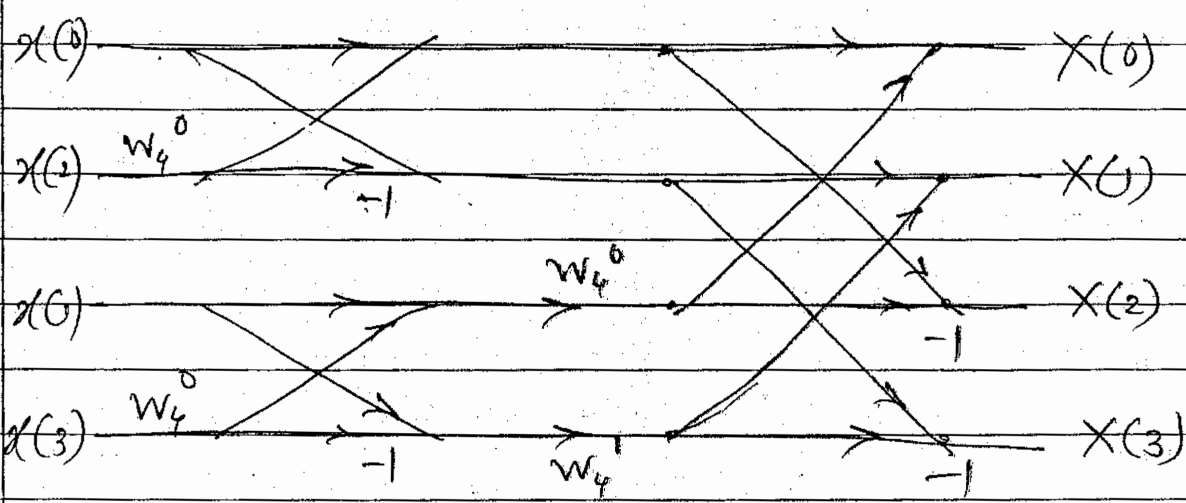
$$= \left(\frac{1}{2} + j\frac{1}{2}\right) + 0.707j$$

$$= 0.5 + j\,1.21$$

$$\therefore X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$

$$X(k) = \{2,\ 0.5 - j\,1.207,\ 0,\ 0.5 - j\,0.207,\ 0,$$
$$0.5 + j\,0.207,\ 0,\ 0.5 + j\,1.21\}$$

Eg. Derive DIT-FFT flowgraph for $N=4$.



Input seq. $x(0), x(1), x(2)$ & $x(3)$ & DFT in bit reversed order.

eg. Use 8-point radix-2 DIT FFT alg. to find DFT of the seq.

$x(n) = \{0.707, 1, 0.707, 0, -0.707, -1, -0.707, 0\}$

→ i) Computation of twiddle factors:-

$$W_N = e^{-j2\pi/N}$$

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{-j\pi/4} = \cos(\pi/4) - j\sin(\pi/4)$$

$$= 0.707 - j0.707$$

$$W_8^2 = -j$$

$$W_8^3 = -0.707 - j0.707$$

ii) Computation of 2-point DFT :-

As per given seq. values are,

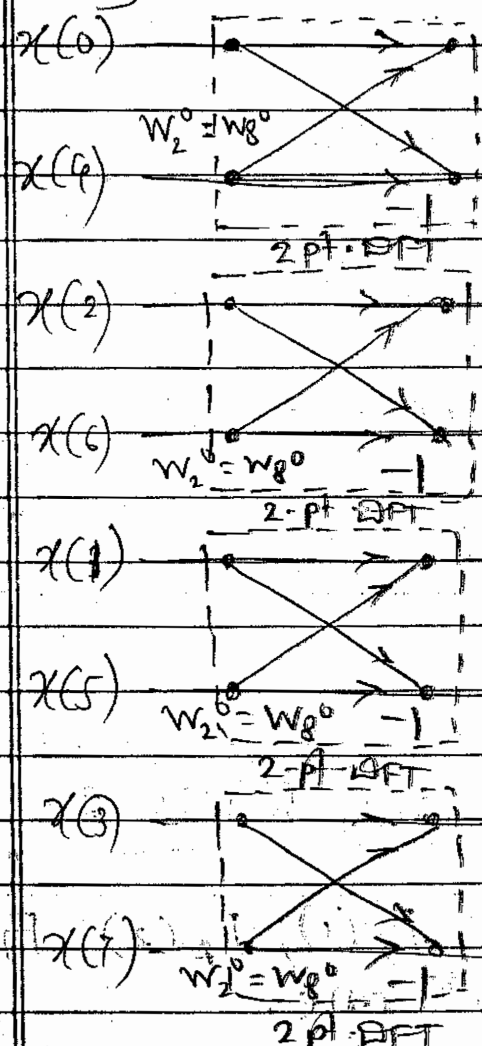
$x(0) = 0.707$ $\quad$ $x(4) = -0.707$

$x(1) = 1$ $\quad$ $x(5) = -1$

$x(2) = 0.707$ $\quad$ $x(6) = -0.707$

$x(3) = 0$ $\quad$ $x(7) = 0$

fig.



$V_{11}(0) = x(0) + W_8^0 x(4)$

$V_{11}(1) = x(0) - W_8^0 x(4)$

$V_{12}(0) = x(2) + W_8^0 x(6)$

$V_{12}(1) = x(2) - W_8^0 x(6)$

$V_{21}(0) = x(1) + W_8^0 x(5)$

$V_{21}(1) = x(1) - W_8^0 x(5)$

$V_{22}(0) = x(3) - W_8^0 x(7)$

$V_{22}(1) = x(3) - W_8^0 x(7)$

$\therefore V_{11}(0) = 0.707 + (1)(-0.707)$

$= 0$

$V_{11}(1) = 0.707 + (1) \times (-0.707)$

$= 1.414$

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$V_{12}(0) = x(2) + W_8^0 x(6)$
$= 0.707 + (1)(-0.707)$
$= 0$

$V_{12}(1) = x(2) - W_8^0 x(6)$
$= 0.707 - (1) \times (-0.707)$
$= 1.414$

$V_{21}(0) = x(1) + W_8^0 x(5)$
$= 1 + (1) \times (-1)$
$= 0$

$V_{21}(1) = x(1) - W_8^0 x(5)$
$= 1 - (+1) \times (-1)$
$= 2$

$V_{22}(0) = x(3) + W_8^0 x(7)$
$= 0 + 1(0) = 0$

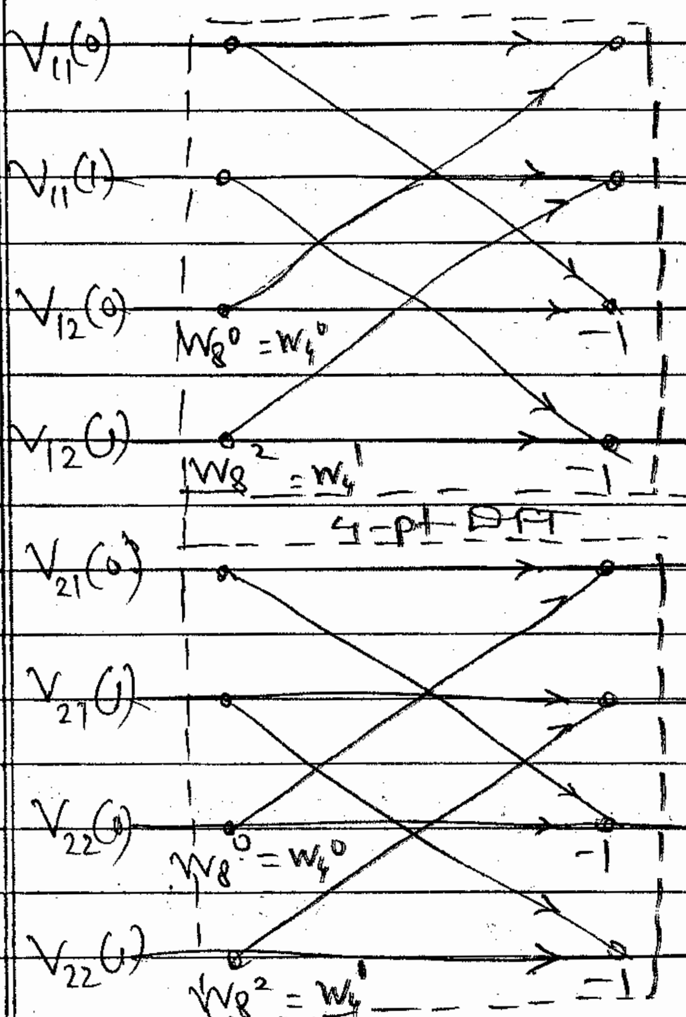
$V_{22}(1) = x(3) - W_8^0 x(7)$
$= 0 - 1 \times 0 = 0$

Combining two-point DFTs :-

Calcⁿ of $F_1(0), F_1(1), F_1(2), F_1(3)$ of 4-pt. DFT - Ist one.

$F_1(0) = V_{11}(0) + W_8^0 V_{12}(0)$
$= 0 + 1 \times (0) = 0$

$F_1(1) = V_{11}(1) + W_8^2 V_{12}(1)$
$= 1.414 + (-j) 1.414$



$F_1(0) = V_{11}(0) + W_8^0 V_{12}(0)$

$F_1(1) = V_{11}(1) + W_8^2 V_{12}(1)$

$F_1(2) = V_{11}(0) - W_8^0 V_{12}(0)$

$F_1(3) = V_{11}(1) - W_8^2 V_{12}(1)$

$F_2(0) = V_{21}(0) + W_8^0 V_{22}(0)$

$F_2(1) = V_{21}(1) + W_8^2 V_{22}(1)$

$F_2(2) = V_{21}(0) - W_8^0 V_{22}(0)$

$F_2(3) = V_{21}(1) - W_8^2 V_{22}(1)$

Combining two DFTs.

$$F_1(2) = V_{11}(0) - W_8^0 V_{12}(0)$$

$$F_1(2) = 0 - (1) \times 0 = 0$$

$$F_1(3) = V_{11}(1) - W_8^2 V_{12}(1)$$

$$F_1(3) = 1.414 - (-j) \times 1.414$$

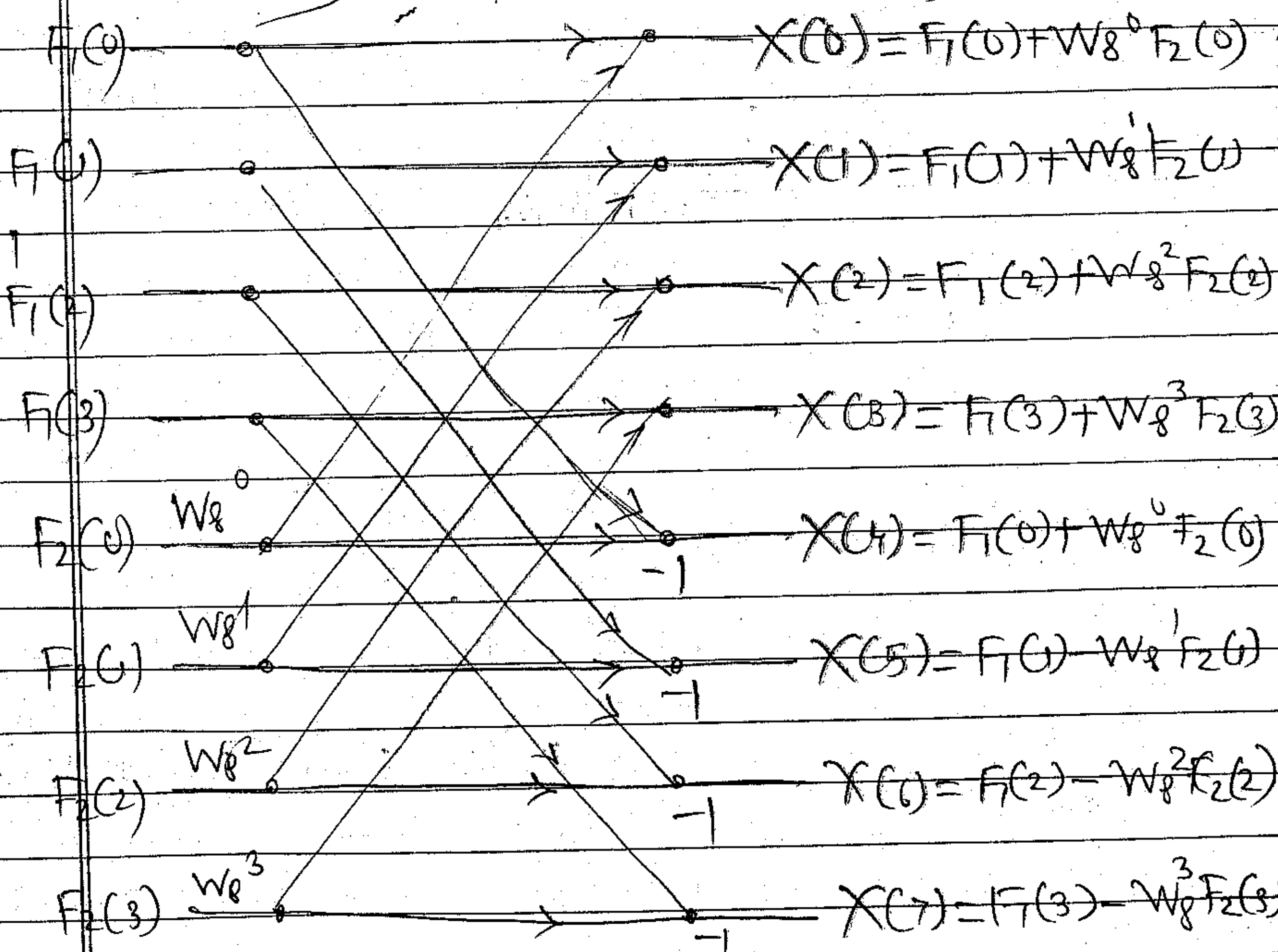
$$= 1.414 + j1.414$$

Now, Calc^n of $F_2(0), F_2(1), F_2(2), F_2(3)$

$F_2(0) = V_{21}(0) + W_8^0 V_{22}(0)$
$= 0 + 1 \times 0 = 0$
$F_2(1) = V_{21}(1) + W_8^2 V_{22}(1)$
$= 2 + (-j) \times 0$
$= 2$
$F_2(2) = V_{21}(0) - W_8^0 V_{22}(0)$
$= 0 - 1(0) = 0$

$F_2(3) = V_{21}(1) - W_8^2 V_{22}(1)$
$= 2 - (-j) \times 0$
$= 2$

Combining 4-point DFT $F_1(k)$ & $F_2(k)$



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$X(0) = 0 + 1 \times 0 = 0$

$X(1) = 1.414 - j1.414 + (0.707 - j0.707) \times 2$

$= 2.8284 - j2.8284$

$X(2) = F_1(2) + W_8^2 F_2(2)$

$= 0 + (-j) \times 0 = 0$

$X(3) = 1.414 + j1.414 + (-0.707 - j0.707) \times 2$

$= 0$

$X(4) = F_1(0) - W_8^0 F_2(0)$

$= 0 - 1(0) = 0$

$X(5) = 1.414 - j1.414 - (0.707 - j0.707) \times 2$

$= 1.414 - j1.414 - 1.414 + j1.414$

$= 0$

$X(6) = 0 - (-j) \times 0 = 0$

$X(7) = 1.414 + j1.414 - (-0.707 - j0.707) \times 2$

$= 2.82 + j2.82$

$\therefore X(k) = \{0, 2.8284 - j2.8284, 0, 0, 0, 0, 0,$
$2.8284 + j2.8284\}$

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Eg. Derive DIT-FFT flowgraph for $N=4$. Hence find DFT of $x(n)=\{1,2,3,4\}$

→ We have equations for first stage of decimation.

$$X(k) = F_1(k) + W_N^k F_2(k), \quad k=0,1,\ldots N/2-1$$

&

$$X(k+N/2) = F_1(k) - W_N^k F_2(k), \quad k=0,1,\ldots N/2-1$$

Here, $N=4$

$$\therefore X(k) = F_1(k) + W_4^k F_2(k), \quad k=0,1 \quad —③$$

$$\& \; X(k+2) = F_1(k) - W_4^k F_2(k), \quad k=0,1 \quad —④$$

Putting the values of k in equ^n ③.

$$X(0) = F_1(0) + W_4^0 F_2(0)$$

$$\& \; X(1) = F_1(1) + W_4^1 F_2(1)$$

Putting the values of k in equ^n ④.

$$X(2) = F_1(0) - W_4^0 F_2(0)$$

$$X(3) = F_1(1) - W_4^1 F_2(1)$$

So, signal flowgraph is as in fig.

Given seq. $x(n)=\{1,2,3,4\}$

Twiddle factor values are

$$W_4^0 = 1$$

$$W_4^1 = e^{-j2\pi \cdot 1/4} = e^{-j\pi/2} = -j$$

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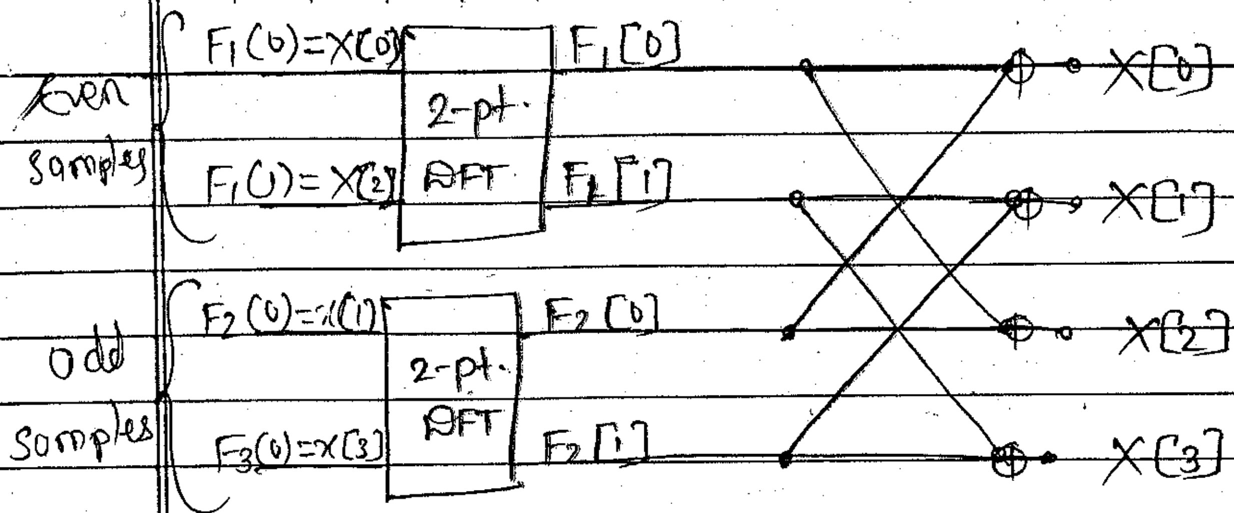
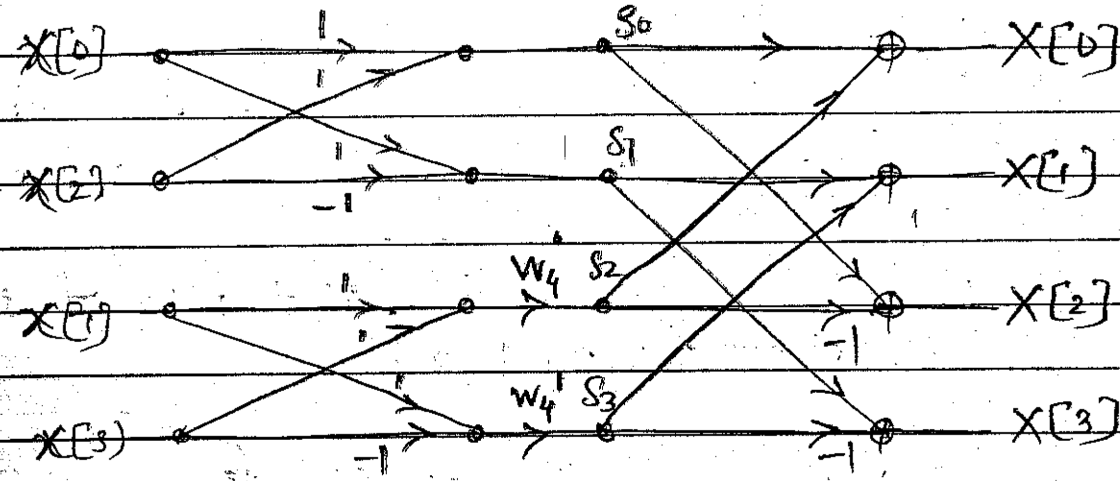


fig. flowgraph for 4-pt DFT

Replace 2pt. DFT by butterfly structure.



O/P $S(n)$ is,

$S_0 = x(0)+x(2) = 1+3=4$

$S_1 = x(0)-x(2) = 1-3=-2$

$S_2 = [x(1)+x(3)]W_4^0 = 2+4=6$

$S_3 = [x(1)-x(3)]W_4^1 = (2-4)(-j) = 2j$

Final O/P is,

$X(0) = S_0+S_2 = 4+6=10$

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$X(1) = S_1 + S_3 = -2+2j$

$X(2) = S_0 - S_2 = 4-6 = -2$

$X(3) = S_1 - S_3 = -2-2j$

$\therefore X(k) = \{X(0), X(1), X(2), X(3)\}$

$X(k) = \{10, -2+j2, -2, -2-j2\}$

Eg. $x(n)$ be a finite duration seq. of length 8 such that

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$

a) Find $X(k)$ using DIT FFT flowgraph.

b) Using the result in (a) & not otherwise find DFT of seq. $x_1(n) = \{-1, 2, -4, 2\}$. Justify your answer.

c) Using result in (b) find DFT of seq. $x_2(n) = \{-4, 2, -1, 2\}$

→ a) Flowgraph is as in fig.

Here $S_1(n)$ represents O/P of stage-1 & $S_2(n)$ → O/P of stage 2.

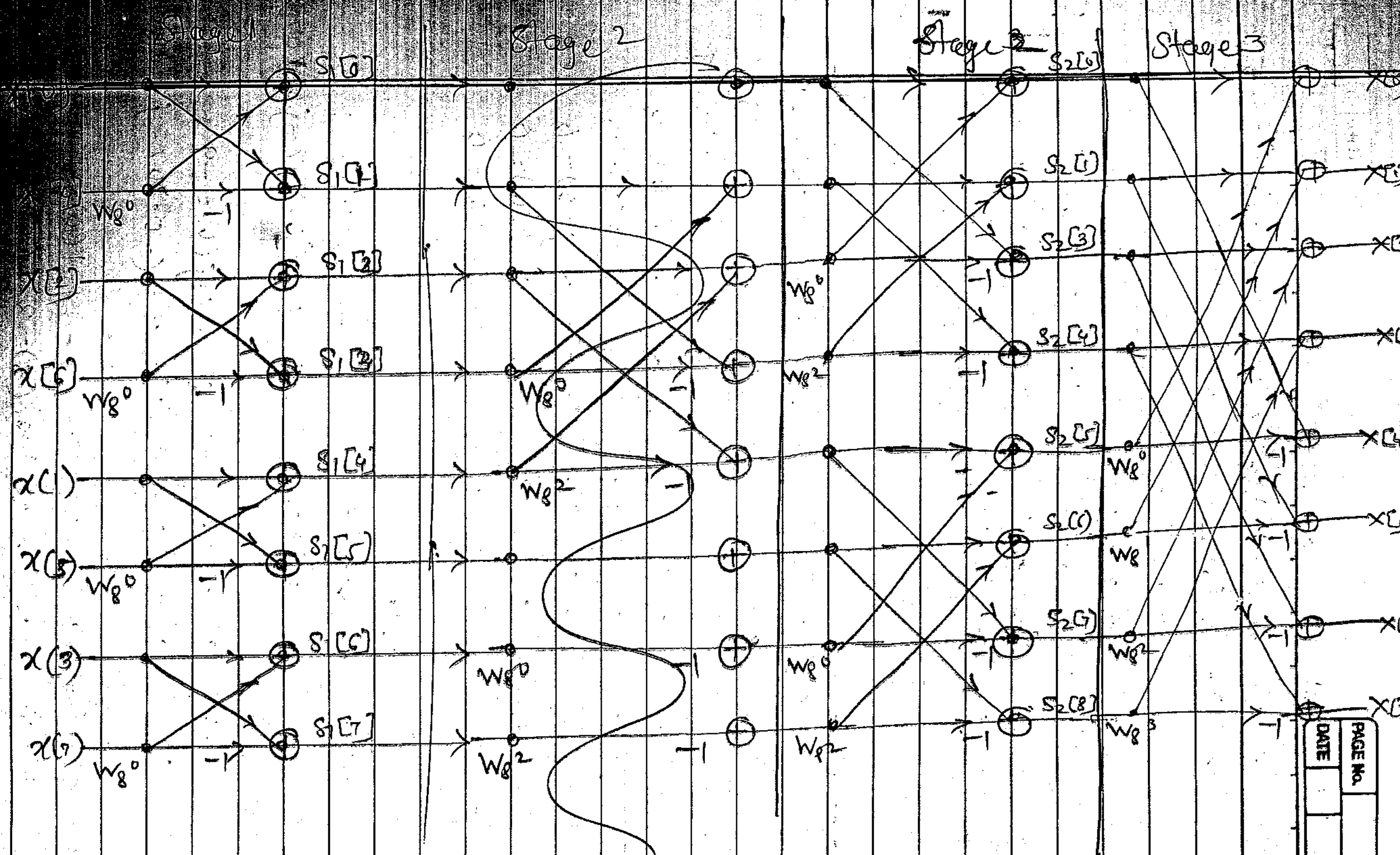
Values of twiddle factor are,

$W_8^0 = e^0 = 1$

$W_8^1 = e^{-j\pi/4} = 0.707 - j0.707$

$W_8^2 = e^{-j\pi/2} = -j$

$W_8^3 = e^{-j3\pi/4} = -0.707 - j0.707$



Flowgragh for 8-point

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Output of stage-1

$s_1(0) = x(0) + W_8^0 x(4) = -1 + 1(-4) = -5$

$s_1(1) = x(0) - W_8^0 x(4) = -1 - 1(-4) = 3$

$s_1(2) = x(2) + W_8^0 x(6) = 2 + 1(2) = 4$

$s_1(3) = x(2) - W_8^0 x(6) = 2 - 1(2) = 0$

$s_1(4) = x(1) + W_8^0 x(5) = 0 + 1(0) = 0$

$s_1(5) = x(1) - W_8^0 x(5) = 0 - 1(0) = 0$

$s_1(6) = x(3) + W_8^0 x(7) = 0$

$s_1(7) = x(3) - W_8^0 x(7) = 0 - 1(0) = 0$

Output of stage-2

$s_2(0) = s_1(0) + W_8^0 s_1(2) = -5 + 1(4) = -1$

$s_2(1) = s_1(1) + W_8^2 s_1(3) = 3 - j(0) = 3$

$s_2(2) = s_1(0) - W_8^0 s_1(2) = -5 + 1(4) = -9$

$s_2(3) = s_1(1) - W_8^2 s_1(3) = 3 + j(0) = 3$

$s_2(4) = s_1(4) + W_8^0 s_1(6) = 0 + 1(0) = 0$

$s_2(5) = s_1(5) + W_8^2 s_1(7) = 0 - j(0) = 0$

$s_2(6) = s_1(4) - W_8^0 s_1(6) = 0 - 1(0) = 0$

$s_2(7) = s_1(5) - W_8^2 s_1(7) = 0 + j(0) = 0$

Final O/P

$X(0) = s_2(0) + W_8^0 s_2(4) = -1 + 1(0) = -1$

$X(1) = s_2(1) + W_8^1 s_2(5) = 3 + (0.707 - j0.707) \cdot 0$

$= 3$

$X(2) = s_2(2) + W_8^2 s_2(6) = -9 - j(0) = -9$

$X(3) = s_2(3) + W_8^3 s_2(7) = 3 + (-0.707 - j0.707) \cdot 0$

$= 3$

$X(4) = s_2(0) - W_8^0 s_2(4) = -1 - 1(0) = -1$

$X(5) = s_2(1) - W_8^1 s_2(5) = 3 - 1(0) = 3$

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$X(6) = S_2(2) - W_8^2 S_2(6) = -9 + j(0) = -9$

$X(7) = S_2(3) - W_8^3 S_2(7) = 3 - (-0.707 - j0.707) \cdot 0$
$= 3$

$\therefore X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$.

b) Let $x(n) = \{a, b, c, d\}$ & lets its DFT seq. $X(k) = \{A, B, C, D\}$. If we add one zero after each sample in $x(n)$ the we will get seq.

$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$

This process is called unsampling. Process. Since in this seq. one zero is added after each sample, the entire DFT repeats one time. If two zeroes are added then DFT repeats two times.

$\therefore DFT\{x_1(n)\} = X_1(k) = \{A, B, C, D, A, B, C, D\}$

In Part (a) for seq. $x(n)$

$x(n) = \{-1, 0, 2, 0, -4, 0, 2, 0\}$

We have obtained DFT,

$X(k) = \{-1, 3, -9, 3, -1, 3, -9, 3\}$

Observe that first four samples are repeated only once because only one zero is added after each sample.

given seq. $x_1(n) = \{-1, 2, -4, 2\}$

$\therefore X(k) = \{-1, 3, -9, 3\}$

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Justification of answer :-

We will prove property of DFT used in this eg. Let $x(n) = \{a, b, c, d\}$ & $X(k) = \{A, B, C, D\}$

Consider the seq.

$x_1(n) = \{a, 0, b, 0, c, 0, d, 0\}$

According to defn of DFT,

$$X_1(k) = \sum_{n=0}^{7} x_1(n) \cdot W_8^{kn}$$

We will divide seq. $x_1(n)$ into odd parts & even part. Let $x_1(2n)$ represent even & $x_1(2n+1)$ – odd part.

$$\therefore X_1(k) = \sum_{n=0}^{3} x_1(2n) W_8^{2kn} + \sum_{n=0}^{3} x_1(2n+1) W_8^{(2n+1) \cdot k}$$

In first summation 'n' is replaced by 2n & second summation 'n' by (2n+1). But in second summation $x_1(2n+1)$ represents odd samples of seq. $x_1(n)$ & all these are zero.

$$\therefore X_1(k) = \sum_{n=0}^{3} x_1(2n) W_8^{2kn}$$

Twiddle factor Property

$$W_N^{2kn} = W_{N/2}^{kn}$$

$$W_8^{2kn} = W_4^{kn}$$

$$\therefore X_1(k) = \sum_{n=0}^{3} x_1(2n) \cdot W_4^{kn}$$

But, $x_1(2n)$ represents even samples of $x_1(n)$.

means $x_1(2n) = x(n)$

$$\therefore X_1(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} = X(k)$$

But $x_1(n)$ is eight pt. seq.

$$\therefore X_1(k) = \{A, B, C, D, A, B, C, D\}$$

~~First Part of Notebook.~~

18/9/11C

Batch A4 8-10

58, 66, 69, 71, 74

$(3-5j)(2+2j)$

$= 6+6j-10j-10j^2$

$= 6-4j+10 = 16-4j$

$(3+5j)(2-2j)$

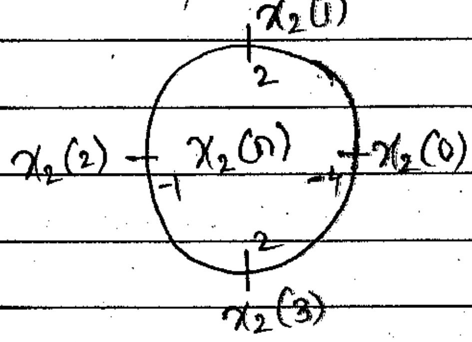
$= 6-6j+10j-10j^2$

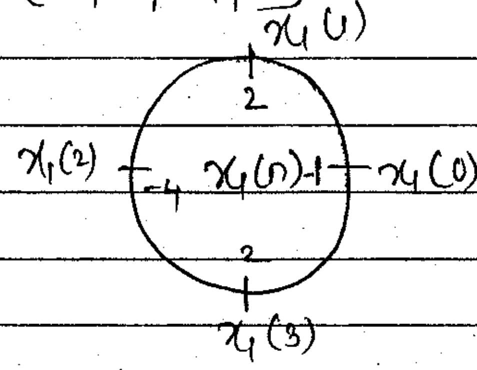
$= 6+4j+10$

$= 16+4j$

Eg. c] Here $x_2(n) = \{-4, 2, -1, 2\}$

we have, $x_1(n) = \{-1, 2, -4, 2\}$





Seq. $x_2(n)$

$x_2(n)$ is obtained by circularly rotating $x_1(n)$ by 2 positions in anticlockwise direction. That means $x_2(n)$ is obtained by delaying $x_1(n)$ by 2 positions.

$\therefore x_2(n) = x_1((n-2))$

according to circular time shift property.

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k) W_N^{kl}$$

We can write it as,

$$X_2(k) = X_1(k) \cdot W_4^{2k} = X_1(k) \cdot e^{-j2\pi k/4}$$

$$\therefore X_2(k) = e^{-j\pi/2 \cdot k} \cdot X_1(k)$$

We have $X_1(k) = \{-1, 3, \cancel{9}\, 9, 3\}$

We will calculate $X_2(k)$ for diff. values of k :-

for $k=0 \Rightarrow X_2(0) = e^0 \cdot X_1(0) = 1$

$k=1 \Rightarrow X_2(1) = e^{-j\pi} X_1(1) = (\cos\pi - j\sin\pi) \cdot 3$
$= -3$

$k=2 \Rightarrow X_2(2) = e^{-j\pi 2} X_1(2) = (\cos 2\pi - j\sin 2\pi) \cdot (-9) = -9$

For $k=3 \Rightarrow x_2(3) = e^{-j\pi 3} X_1(3) = (\cos 3\pi - j\sin 3\pi) \cdot 3$
$= -3$

$\therefore X_2(k) = \{-1, -3, -9, -3\}$

Computation of Inverse DFT using FFT algorithm :

DFT $\rightarrow$ $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn},$$

$$n = 0, 1, \dots N-1$$

IDFT $\rightarrow$ $$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \dots N-1$$

IDFT differs from DFT by,

a) Multiplication by $1/N$ factor

b) Negative sign of imaginary part (W_N).

Eg. Let $x(n)$ be 8 point seq. Its corresponding ~~D~~FT $X(k)$ is

$X(k) = \{(0.5), (2+j), (3+j2), (j), (3), (-j), (3-j2), (2-j)\}$

Find $x(n)$ by performing IFFT.

We will use inverse ~~DIT~~ FFT alg.

Flowgraph is as in fig.

Let $g(n)$ represent output of first stage & $h(n)$ represent output of second stage. By changing sign of imaginary part of twiddle factor the values of twiddle factor as follows:

For inverse $\Rightarrow W_N^{-kn} = e^{j2\pi nk/N}$

$W_8^0 = 1$

$W_8^{-1} = 0.707 + j0.707$

$W_8^{-2} = j$

$W_8^{3} = -0.707 + j0.707$

Output of stage-1

$g(0) = \frac{1}{8}X(0) + \frac{1}{8}X(4) = \frac{1}{8}(0.5) + \frac{1}{8}(3) = 0.44$

$g(1) = \frac{1}{8}X(1) + \frac{1}{8}X(5) = \frac{1}{8}(2+j) + \frac{1}{8}(-j) = 0.25$

$g(2) = \frac{1}{8}X(2) + \frac{1}{8}X(6) = \frac{1}{8}(3+j2) + \frac{1}{8}(3-j2) = 0.75$

$g(3) = \frac{1}{8}X(3) + \frac{1}{8}X(7) = \frac{1}{8}(j) + \frac{1}{8}(2-j) = 0.25$

$g(4) = \left[\frac{1}{8}X[0] + \frac{1}{8}X[4]\right] W_8^0 = \left[\frac{1}{8}(0.5) - \frac{1}{8}(3)\right] \cdot 1$

$= -0.31$

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$g(5) = \left[\frac{1}{8}X[1] - \frac{1}{8}X[5]\right] W_8^1$

$= \left[\frac{1}{8}(2+j) - \frac{1}{8}(-j)\right](0.707 + j0.707)$

$= (0.25 + j0.25)(0.707 + j0.707) = j0.35$

$g(6) = \left[\frac{1}{8}X(2) - \frac{1}{8}X(6)\right] W_8^2 = \left[\frac{1}{8}(3+j2) - \frac{1}{8}(3-j2)\right] j$

$= -0.5$

$g(7) = \left[\frac{1}{8}X(3) - \frac{1}{8}X(7)\right] W_8^3 = \left[\frac{1}{8}(j) - \frac{1}{8}(2-j)\right] \cdot [-0.707 + j0.707]$

$= \left(\frac{1}{4}j - \frac{1}{4}\right)(-0.707 + j0.707) = -j0.35$

Output of stage-II :-

$h(0) = g(0) + g(2) = 0.44 + 0.75 = 1.19$

$h(1) = g(1) + g(3) = 0.25 + 0.25 = 0.5$

$h(2) = [g(0) - g(2)] W_8^0 = 0.44 - 0.75 = -0.31$

$h(3) = [g(1) - g(3)] W_8^2 = 0$

$h(4) = g(4) + g(6) = -0.31 - 0.5 = 0.81$

$h(5) = g(5) + g(7) = j0.35 - j0.35 = 0$

$h(6) = [g(4) - g(6)] W_8^0 = -0.31 + 0.5 = \cancel{0.81}\ 0.91$

$h(7) = [g(5) - g(7)] W_8^2 = [j0.35 + j0.35] j = \cancel{0}\ -0.71$

Final Output :-

$X(0) = h(0) + h(1) = 1.19 + 0.5 = 1.69$

$X(1) = h(4) + h(5) = -0.81$

$X(2) = h(2) + h(3) = -0.31$

$X(3) = h(6) + h(7) = 0.19 - 0.7 = \cancel{0}\ -0.51$

$$x(4) = [h(0) - h(1)] \cdot W_8^0 = (1.19 - 0.5) \cdot 1 = 0.69$$

$$x(5) = [h(4) - h(5)]\, W_8^0 = -0.81$$

$$x(6) = [h(2) - h(3)]\, W_8^0 = -0.31$$

$$x(7) = [h(6) - h(7)]\, W_8^0 = (0.19 + 0.7) \cdot 1 = 0.89$$

$$\therefore\ x(n) = \{1.69, -0.81, -0.31, -0.51, 0.69, -0.81, -0.31, 0.89\}$$

Flowgraph :-

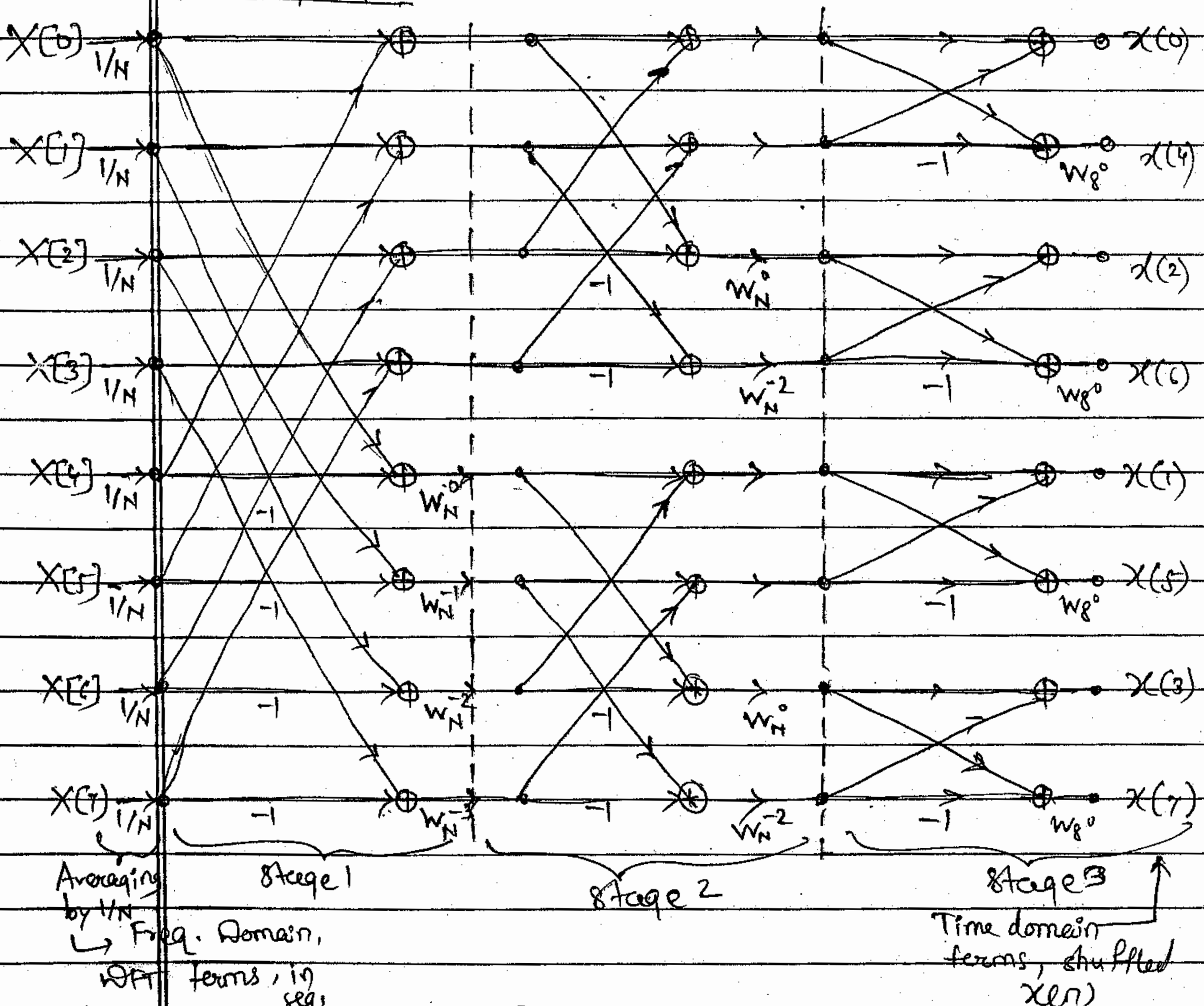

fig. IFFT

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Eg: i) If $x(n) = \{1+5j, 2+6j, 3+7j, 4+8j\}$ Find DFT $X(K)$ using DIFFFT.

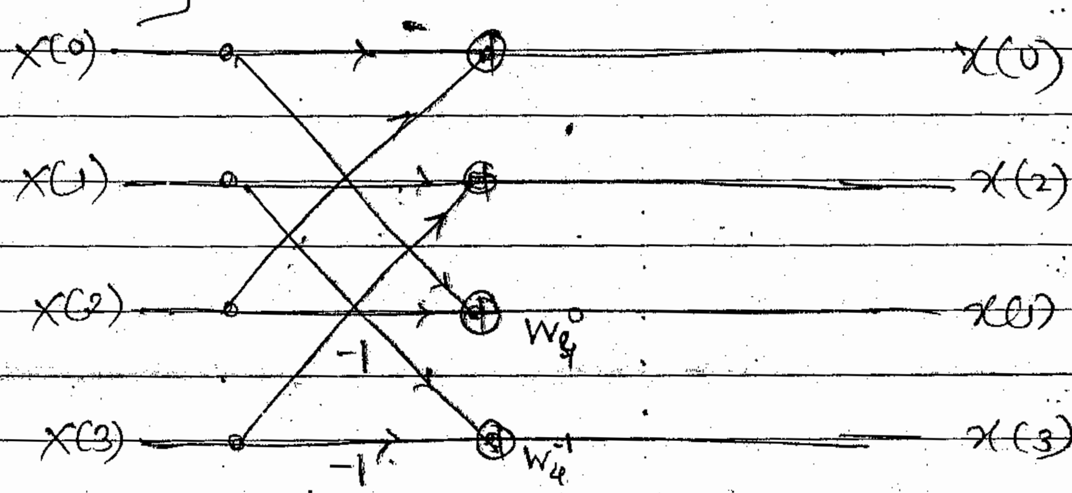
ii) Using the results obtained in (i) not otherwise, find DFT of foll. seq.

$x_1(n) = \{1,2,3,4\}$ & $x_2(n) = \{5,6,7,8\}$

→ i)

$x(n) = \{1+5j, 2+6j, 3+7j, 4+8j\}$

4-Point flowgraph DIF-FFT is as in fig.



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Eg. Using FFT & IFFT, find the output of the system if input $x(n)$ & impulse response $h(n)$ are given by,

$x(n) = \{2,2,4\}$

$h(n) = \{1,1\}$

→ $x(n) = \{2,2,4\}$ & $h(n) = \{1,1\}$

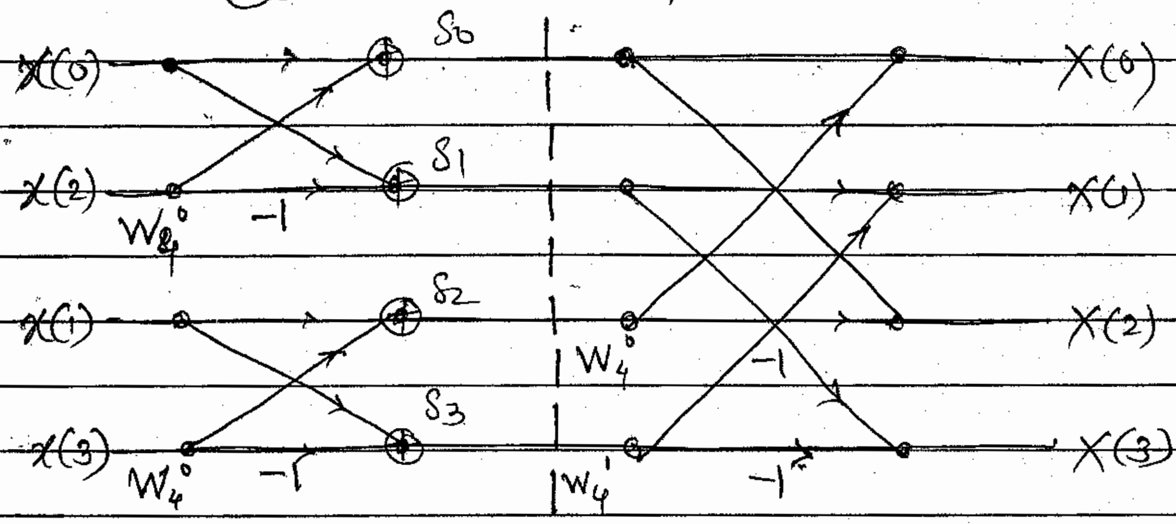
$\therefore L+M-1 = 3+2-1 = 4$

$\therefore x(n) = \{2,2,4,0\}$

$h(n) = \{1,1,0,0\}$

We will obtain ~~DIF~~ DFT of $x(n)$ using DIT FFT alg.

$\therefore$ Calculations as,



$$W_4^0 = e^{-j\frac{2\pi(0)}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{2\pi(1)}{4}} = e^{-j\pi/2} = -j$$

So Stage-I

$S_0 = x(0) + x(2) W_4^0 = 2+4 = 6$

$S_1 = x(0) - x(2) W_4^0 = 2-4 = -2$

$S_2 = [x(1) + W_4^0 x(3)] = 2 + 0 = 2$

$S_3 = x(1) - W_4^0 x(3) = 2 - 0 = 2$

Final O/P :-

$X(0) = S_0 + S_2 (W_4^0) = 6 + 2 = 8$

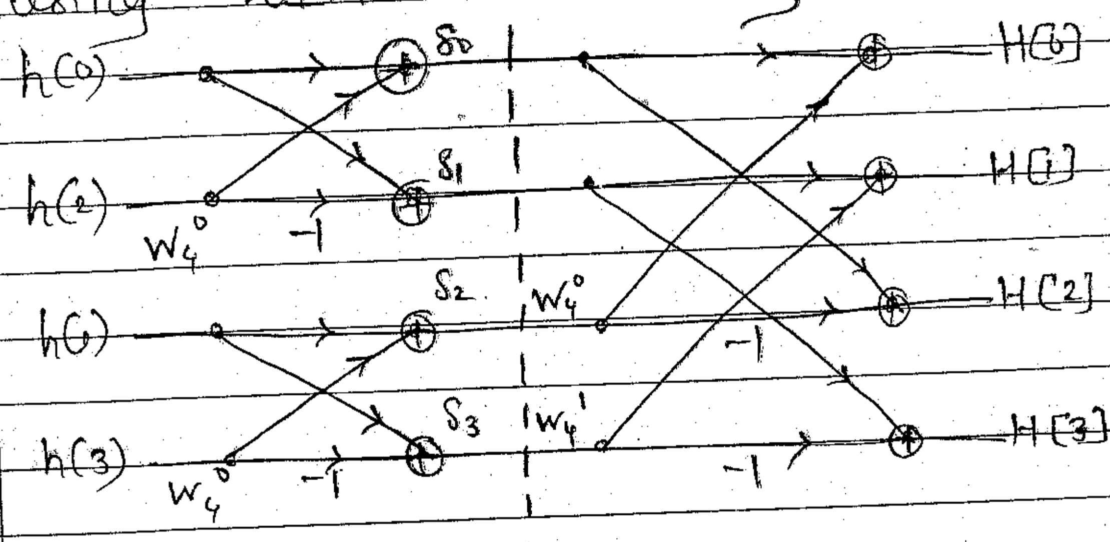
$X(1) = S_1 + W_4^1 S_3 = -2 + (-j) 2 = -2 - 2j$

$X(2) = S_0 - W_4^0 S_2 = 6 - 2 = 4$

$X(3) = S_1 - W_4^1 S_3 = -2 - (-j) 2 = -2 + j2$

$\therefore X(k) = \{8, -2-2j, 4, -2+j2\}$

Now we will obtain DFT of h(n) using ~~DIT~~FFT. as in fig.



Stage-1

$S_0 = h(0) + W_4^0 h(2) = 1 + (1)(0) = 1$

$S_1 = h(0) - W_4^0 h(2) = 1 - (1)(0) = 1$

$S_2 = h(1) + W_4^0 h(3) = (1) + (1)(0) = 1$

$S_3 = h(1) - W_4^0 h(3) = (1) - (0) = 1$

Find O/P :-

$H(0) = S_0 + W_4^0 S_2 = 1 + (1)(1) = 2$

$H(1) = S_1 + W_4^1 S_3 = 1 + (-j) = 1 - j$

$H(2) = S_0 - W_4^0 S_2 = 1 - 1 = 0$

$H(3) = S_1 - W_4^1 S_3 = 1 - (-j) = 1 + j$

$\therefore H(k) = \{2, 1-j, 0, 1+j\}$

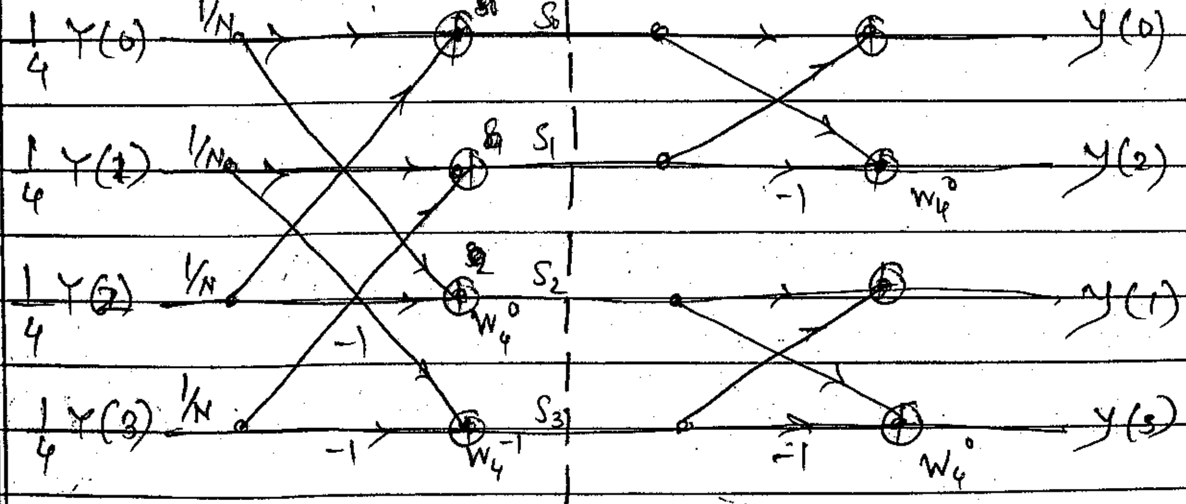
Now we will multiply, H(k) & X(k)

$Y(k) = X(k) \cdot H(k)$

$\therefore Y(k) = \{8, -2-2j, 4, -2+2j\} \cdot \{2, 1-j, 0, 1+j\}$

$= \{16, -4, 0, -4\}$

Now we will Perform IFFT to obtain seq. y(n). For this we have to multiply each input by 1/N means 1/4 & we have to change sign of imaginary Part of twiddle factor.



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Stage - I

$S_0 = \left[\frac{1}{4}Y(0) + \frac{1}{4}Y(2)\right] = \frac{1}{4}(16) + \frac{1}{4}(0)$

$= 4$

$S_1 = \left[\frac{1}{4}Y(1) + \frac{1}{4}Y(3)\right] = \frac{1}{4}(-4) + \frac{1}{4}(-4)$

$= -2$

$S_2 = \left[\frac{1}{4}Y(0) - \frac{1}{4}Y(2)\right] W_4^0 = \left[\frac{1}{4}(16) - \frac{1}{4}(0)\right](1)$

$= 4$

$S_3 = \left[\frac{1}{4}Y(1) - \frac{1}{4}Y(3)\right] W_4^{+1} = \left[\frac{1}{4}(-4) - \frac{1}{4}(-4)\right] j$

$= \cancel{2j}\,(-1+1)j = 0$

Inverse $\rightarrow$ $W_4^{+1} = e^{j2\pi nk/4} = e^{j\pi/2}$

$= \cos\pi/2 + j\sin\pi/2$

$= j$

$\therefore$ Final O/P is,

$y(0) = S_0 + S_1 = 4 + (-2) = 2$

$y(1) = S_2 + S_3 = 4 - \cancel{0} = 4$

$y(2) = (S_0 - S_1) W_4^0 = 4 - (-2) = 6$

$y(3) = (S_2 - S_3) W_4^0 = (4 - 0)(1) = 4$

$= \cancel{4j - 2j^2}$

$= \cancel{4j - 2(-1)}$

$= \cancel{4j + 2}$

$\therefore y(n) = \{2, 4, 6, 4\}$

Comparison of Complex & Real multiplication & Additions of DFT & FFT.

First we will calculate the computational complexity for direct DFT calculation.

A) For Direct computation:-

- According to definition of DFT we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

Equation indicates that we have to take multiplication of $x(n)$ & twiddle factor.

Then we have to add all the terms.

Since twiddle factor is complex, we need to Perform complex multiplication & complex addition.

Complex multiplications:

- for one value of 'k', multiplication should be Performed for all values of 'n', as given in above equaⁿ.
- The range of 'n' is from 0 to N-1. So for one value of 'k'; N complex multiplications are required.
- Now the range of k is also from k=0 to k=N-1. The total complex

multiplications are,

Complex multiplications = $N \times N = N^2$

Complex additions :

According to equation of DFT for each value of k we need to add the Product terms of $x(n) W_N^{kn}$.

for eg. Let us say $N=4$

For

$$k=0 \Rightarrow X(0) = \sum_{n=0}^{3} x(n) W_4^{0 \times n} = \sum_{n=0}^{3} x(n) W_4^0$$

$$\therefore X(0) = x(0) W_4^0 + x(1) W_4^0 + x(2) W_4^0 + x(3) W_4^0$$

Four complex multiplication & three complex additions are required.

Here we have considered $N=4$.

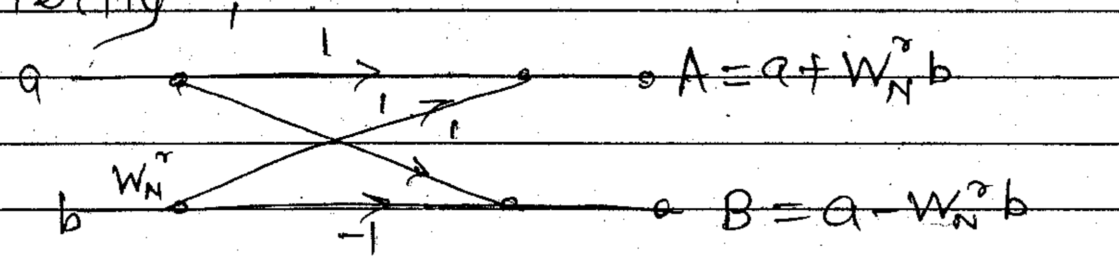
Thus for each value of 'k', N complex multiplications & N-1 complex additions are required.

Now the total value of k & N are N.

$\therefore$ Complex Additions $= N(N-1) = N^2 - N$

Computational Complexity using FFT algorithm :-

- First we will calculate the computation complexity required for one butterfly.
- Consider the general structure of butterfly,



- Here a & b are inputs & A & B are outputs of butterfly. The outputs are given by,

$$A = a + W_N^r b$$
$$B = b - W_N^r b$$

1. To calculate any output, we need to multiply input 'b' by twiddle factor W_N^r. So one complex multiplication is required for one butterfly diagram.
2. ~~To calculate any output (A or B), we need to multiply input 'b' by twiddle factor W_N^r. So one complex multiplication is required for one butter~~
2. To calculate output 'A', one complex addition is required, while to calculate 'B', one complex subtraction is required. But the computational complexity of addition &

subtraction is same. So we can say that for one butterfly two complex additions are required.

3. For 8 point DFT, 4 butterflies are there at each stage. So for 'N'-point DFT, at each stage $N/2$ butterflies are required.

4. Three stages are required to compute 8-point DFT. In general, for 'N' point DFT, $\log_2 N$ stages are required.

Complex multiplications:

At each stage there are $N/2$ butterflies. Total number of stages are $\log_2 N$. & for each butterfly one complex multiplication is required.

$\therefore$ Total complex multiplications $= \frac{N}{2} \log_2 N$

Complex additions:

Total number of stages are $\log_2 N$. At each stage $N/2$ butterflies are required.

$\therefore$ Total Complex additions $= 2 \times \frac{N}{2} \log_2 N$

Total Complex additions $= N \log_2 N$

Comparison of direct DFT computation & computation using FFT algorithms.
→ Table

Number of pts N	Direct Computation Complex multipl^n N^2	Complex Additions $(N^2 - N)$	Using FFT Complex multipl^n $(N/2 \log_2 N)$	Complex Addition $(N \log_2 N)$
4	16	12	2	4
8	64	56	12	24
16	256	240	32	64
32	1024	992	40	80

The table shows that, by use of FFT algorithms the number of complex multiplications & complex additions are reduced.
So tremendous improvement in speed.

In-place computation to reduce memory size :

- Firstly, we will discuss the memory requirement of each butterfly.
- Butterfly calculates values of A & B for inputs a & b.
- Remember a & b are complex inputs.
- So two memory locations are required to store any one of input a or b.

- One memory location is required to store real part & other to store imaginary part.
- Now to store both inputs a & b, 2+2=4 memory locations are required.
- Now outputs are computed as,

$$A = a + W_N^r b$$

$$\& \quad B = a - W_N^r b$$

- Thus output A & B are calculated by using the values of a & b stored in memory.
- Now A & B are also complex numbers.
  ∴ 2+2=4 memory locations are required to store both outputs A & B.
- Once the computation of A & B is done, then values of a & b are not required.
- So instead of storing A & B at other memory locations, these values are stored at the same place where a & b were stored.
- That means A & B are stored in place of a & b.
- This is called as in-place computation.
- In place computation reduces memory size.

Memory Requirement

- Four memory locations are required for every butterfly to store input & output
- Now there are $N/2$ butterflies per stage.
- So for each stage,

memory locations required to store inputs & outputs $= 4 \times N/2 = 2N$.

- One value of twiddle factor is required to compute A & B.
- Now there are $N/2$ butterflies at each stage. So for each stage,

Memory locations required to store twiddle factor $= N/2$

- Thus combined memory required per stage is $2N + N/2$.
- These many number of memory locations are required to store input values, o/p values & twiddle factor per stage.
- Actual computation of N-point FFT is done stage wise.
- That means computation of one stage is done at a time.
- So these memory locations can be used for other stages also.
- This will again reduce memory size.

$\therefore$ Total memory location $= 2N + N/2$

Eg. Show & compare computational complexity is reduced if 32 point DFT is computed using Radix-2 DIT FFT alg.

→ 32 point DFT using direct computation

Given ⇒ no. of pt. $= 32$

Number of complex multiplicⁿ $= N^2 = 32^2 = 1024$

Number of complex additions $= N^2 - N = 32^2 - 32$

$= 1024 - 32 = 992$

In case of Radix 2 DIT FFT alg.

Number of complex multiplicⁿ $= \frac{N}{2} \log_2 N$

$= \frac{32}{2} \log_2 32$

$= 16 \log_2 32 = 16 \times 5 = 80$

Number of complex addⁿ $= N \log_2 N = 32 \log_2 32$

$= 32 \times 5 = 160$

Comparing both answers, it is proved that computational complexity is reduced if 32 point DFT is computed using Radix-2 DIT FFT alg.

Eg. Find the number of complex additions & complex multiplicⁿ required to find DFT for 16 point signal. Compare them with no. of computation required if FFT alg. is used.

→ $N = 16$

Using DFT:

no. of complex multiplicⁿ $= N^2 = 16^2 = 256$

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Number of complex additions $= N^2 - N$
$= 16^2 - 16 = 240$

Using FFT :-

Number of Complex multiplications $= \frac{N}{2}\log_2 N = \frac{16}{2}\log_2 N$
$= 8\log_2 16 = 32$

Number of Complex additions $= N\log_2 N = 16\log_2 16 = 64$

Spectral Analysis :-

- Frequency Analysis of Discrete time signals:

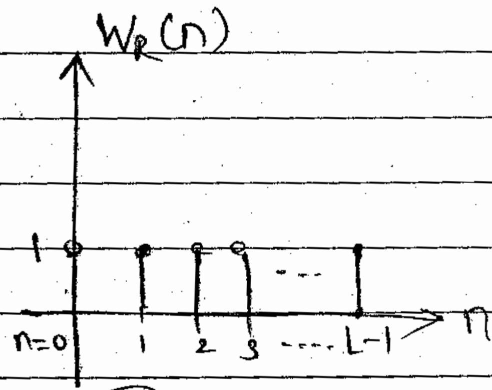
The freq. analysis is also called as spectrum analysis. To do the frequency analysis, time domain signal should be first converted into frequency domain.

- If input signal is analog then first it is passed thro antialiasing filter. Then this signal is sampled at the rate Fs; where Fs is sampling frequency. To avoid aliasing, Fs should be greater than or equals to 2W. Here W is the maximum freq. of input signal.
- In Practical cases, the time interval of signal is maintained at T_0 & $T_0 = LT$. Here L = sample number & $T \Rightarrow$ Sampling interval.

- Suppose we want to analyze the long i/p sequence $x(n)$. Now to limit the time interval of seq. means this input seq. is multiplied by a rectangular window. A rectangular window is denoted by $W_R(n)$ & it is defined as,

$$W_R(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

- This rectangular window is as in fig.
- Since output of $W_R(n)$ is zero after the interval $L-1$; multiplying $x(n)$ by $W_R(n)$ produces signal $\hat{x}(n)$ only in range 0 to $L-1$.



Rectangular window

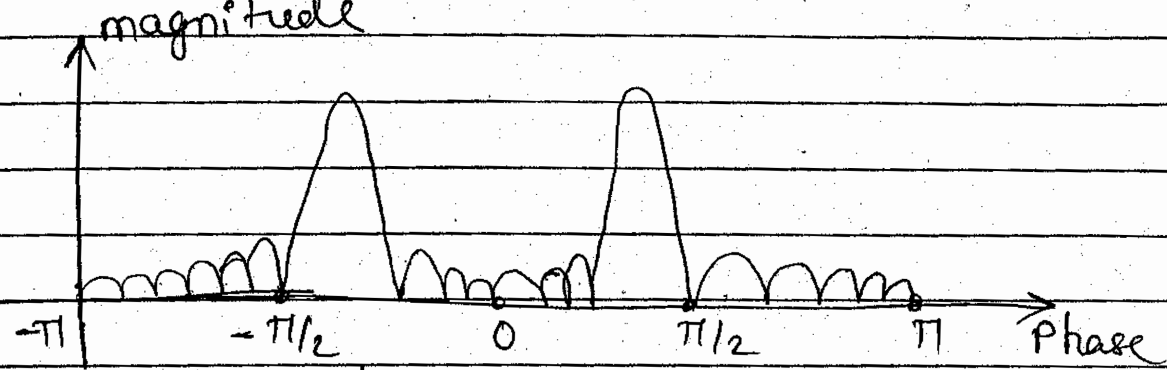
$\therefore \hat{x}(n) = x(n) W_R(n)$

- Then by taking DFT of equ^n. we get,

$$\hat{X}(k) = \sum_{n=0}^{L-1} [x(n) W_R(n)] e^{-j2\pi nk/N}$$

- Here we are considering only 'L' samples of input signal & not the complete input signal. The spectrum will be more accurate if the value of 'L' is large.

Spectral Leakage:-



Magnitude Spectrum

- Magnitude spectrum is not localized to a single frequency, but it is spreadout over the entire frequency range. That means the Power of a signal is spreadout in entire frequency range.
- The leakage of Power is called spectral leakage which is taking place because of windowing of input seq.

Advantages & Limitations of spectral analysis using DFT.

Advantages:

1) Fast processing of DFT can be done using FFT algorithms.
2) Estimation of power spectrum can be done.
3) Calculation of harmonics can also be done.
4) The resolution can be improved by increasing numbers of samples in the calculation of DFT.

Limitations:-

1. The frequency spectrum of entire input signal is not obtained because of windowing.
2. The leakage of power takes place.
3. If we increase number of samples to obtain the better accuracy then, the processing time is increased.