Artificial Intelligence & Soft Computing

CSC 703

Subject In-charge

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Chapter 4

Fuzzy Logic

Based on CO5: Design fuzzy controller system.

Outline:

- **Introduction to Fuzzy Set**
  - Fuzzy set theory
  - Fuzzy set versus Crisp Set
  - Membership functions, Operations, Properties
  - Crisp relation & Fuzzy relations
- **Fuzzy Logic**
  - Fuzzy logic basics
  - Fuzzy rules and fuzzy Reasoning
- **Fuzzy Inference systems**
  - Fuzzification of input variables
  - De-fuzzification and fuzzy controllers

Introduction to Fuzzy Set

-Fuzzy set theory

Introduction- Fuzzy set theory

- The word "**Fuzzy**" means "**Vagueness**"(ambiguity)
- **Fuzziness** occurs when the boundary of a piece of information is not clear-cut.
- **Fuzzy set** theory provides a means for representing uncertainties.
- **Fuzzy set** theory uses Linguistic variables, rather than quantitative variables to represent imprecise concepts.

Introduction- Fuzzy set theory

- **Fuzzy sets-** In 1965, its developed by Lofti zadeh as an extension of classical notation set.
- **Classical set theory** allows the membership of the elements in the set in binary terms
- **Fuzzy set theory** permits membership function valued in the interval [0,1].

Fuzzy set Example:

Words like young, tall, good or high are fuzzy.

- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

Fuzzy set theory is an extension of classical set theory where elements have degree of membership.

Different types of set theory:

1. **Classical sets**, either an element belongs or it does not
   - ❑ **EXAMPLES:**
   - ✓ Set of integers – a real number is an integer or not
   - ✓ You are either in an airplane or not
   - ✓ A number is either odd or even
2. **Fuzzy sets** are sets that have gradations of belonging
   - ❑ **EXAMPLES:**
   - ✓ Green, Big, Near, Tall, Young etc.

Why Fuzzy Logic?

- The classifications of persons into males and females are easy, but it is problematic to classify them as being tall or not tall.
- Fuzzy logic is simply the extension of conventional logic to the case where partial set membership can exist, rule conditions can be satisfied partially and system outputs are calculated by interpolation

Example: **Tumblers full or empty ?**

- Consider the following five tumblers will be divided into **two** classes: ***full*** and ***empty***.
- 

- It is obvious: tumbler **1** belongs to the class ***full*** and tumbler **5** belongs to the class ***empty***.
- To which distinguishable unique class do the tumblers 2, 3 and 4 belong?
- These three tumblers are neither 100% full nor 100% empty.
- When we take a crisp decision then they belong neither to full nor to empty.

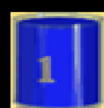
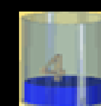
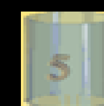
Example: Tumblers full or empty ?

- Consider the following five tumblers will be divided into **two** classes: ***full*** and ***empty***.
- 

- The following can be noticed:
  - **Tumbler 2 is 75% full**
  - **Tumbler 3 is 50% full**
  - **Tumbler 4 is 25% full**
- with other words:
  - **Tumbler 2 is 25% empty**
  - **Tumbler 3 is 50% empty**
  - **Tumbler 4 is 75% empty**

Definition of the set

- Let us define now two sets: F and E.
  - ❑ F is the set of all tumblers that belong to the class full
  - ❑ E is the set of all tumblers belonging to the class empty.
- These sets can be visualized by the following tabular graphical representation.

Tumbler					
Grade of membership to the set F	100%	75%	50%	25%	0%
Grade of membership to the set E	0%	25%	50%	75%	100

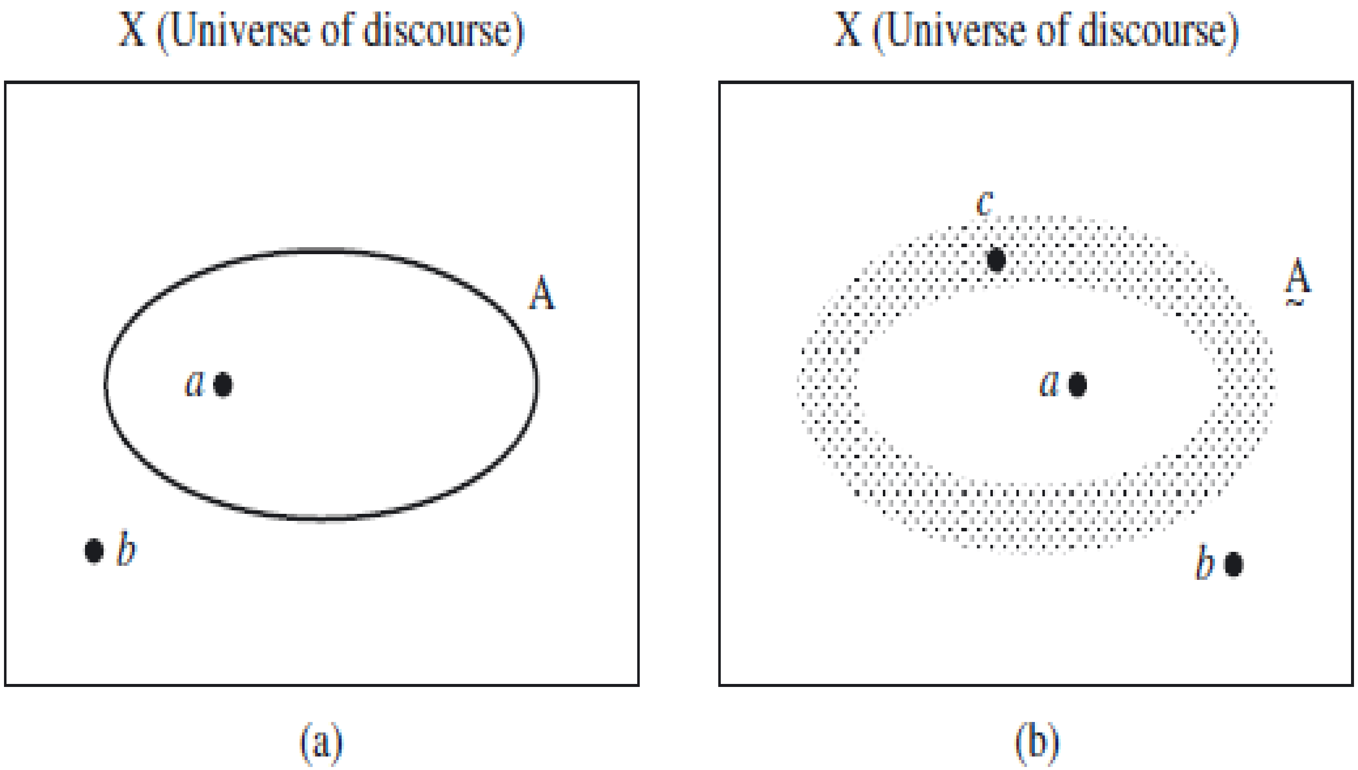
What is a fuzzy set ?

- The sets F and E have some elements, which have not the full, but a partial membership.
- Such kind of non-crisp sets are called fuzzy sets.

Introduction to Fuzzy Set

-Fuzzy set versus Crisp Set

Classical set theory versus Fuzzy set Theory: Boundary



Diagrams for (a) crisp set boundary and (b) fuzzy set boundary.

Classical set theory versus Fuzzy set Theory: Example

Crisp set has a unique membership function

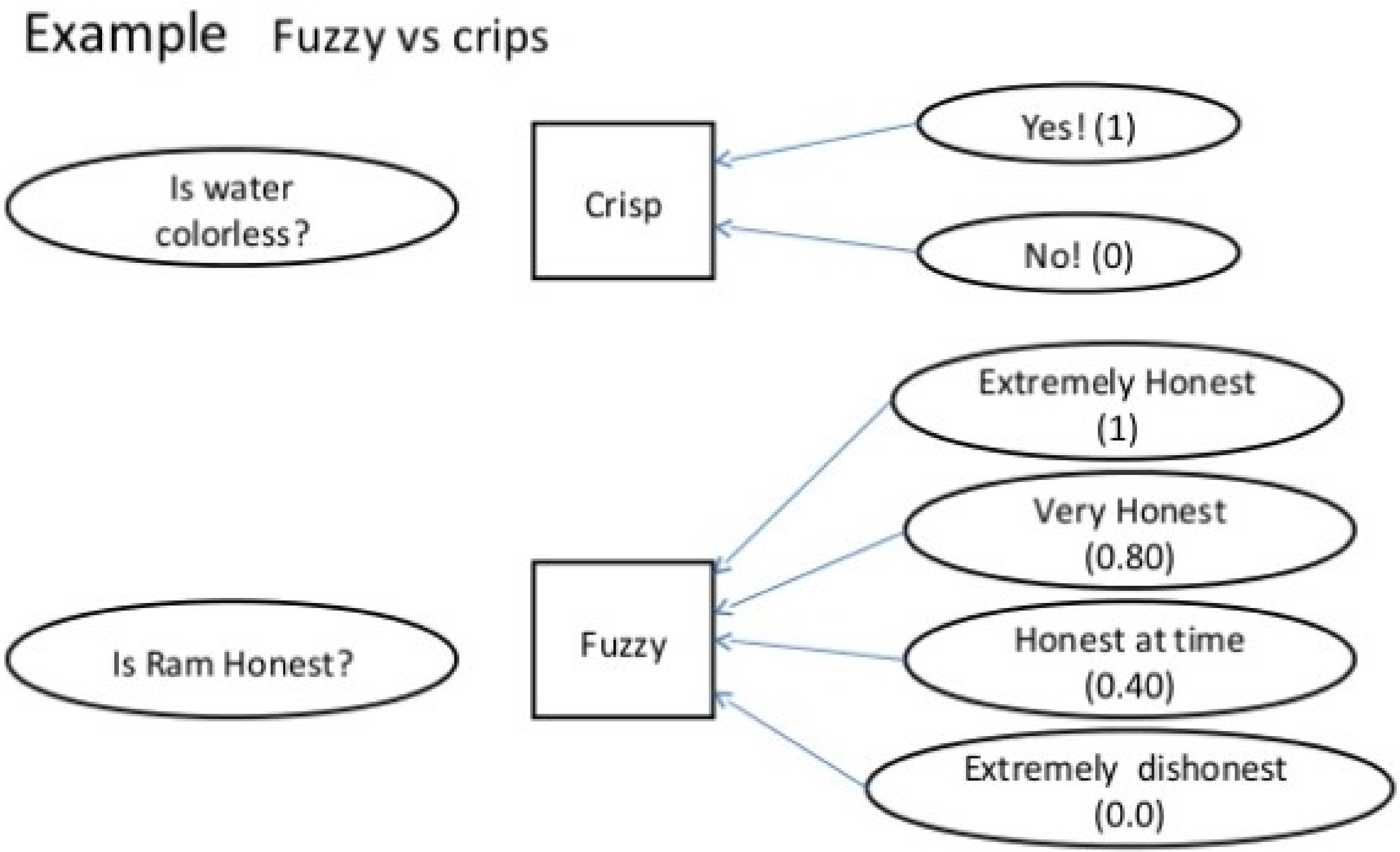
$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$$\mu_A(x) \in \{0, 1\}$$

Fuzzy Set can have an infinite number of membership functions

$$\mu_A \in [0,1]$$

Classical set theory versus Fuzzy set Theory: Example



Classical set theory versus Fuzzy set theory

Classical set theory	Fuzzy set theory
• Classes of objects with sharp boundaries.	• Classes of objects with un-sharp boundaries.
• A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.	• A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
• Widely used in digital system design	• Used in fuzzy controllers.

Classical set theory versus Fuzzy set theory

Crisp Set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu) \mid s \in X$ and $\mu(s)$ is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of **membership**.

Take quiz:

- https://docs.google.com/forms/d/e/1FAIpQLSe2erOxqP3iEtjjbgLIts1f3dLYRaRrRBkboiiX18zi-kHNsw/viewform

Quiz time:

- 1. What is the form of Fuzzy logic?
  a) Two-valued logic
  b) Crisp set logic
  c) Many-valued logic
  d) Binary set logic
- 2. Traditional set theory is also known as Crisp Set theory.
  a) True
  b) False
- 3. The truth values of traditional set theory is ____________ and that of fuzzy set is __________
  a) Either 0 or 1, between 0 & 1
  b) Between 0 & 1, either 0 or 1
  c) Between 0 & 1, between 0 & 1
  d) Either 0 or 1, either 0 or 1

Quiz time:

- 4. Fuzzy logic is extension of Crisp set with an extension of handling the concept of Partial Truth.
  a) True
  b) False
- 5. The room temperature is hot. Here the hot (use of linguistic variable is used) can be represented by _______
  a) Fuzzy Set
  b) Crisp Set
  c) Fuzzy & Crisp Set
  d) None of the mentioned
- 6. _______________ is/are the way/s to represent uncertainty.
  a) Fuzzy Logic
  b) Probability
  c) Entropy
  d) All of the mentioned

Introduction to Fuzzy Set

-Fuzzy set & Crisp Set

Membership, Operations, Properties

Classical set theory:

- Classical sets are also called *crisp* (sets).

  Lists: A = {apples, oranges, cherries, mangoes}

  $A = \{a_1, a_2, a_3\}$

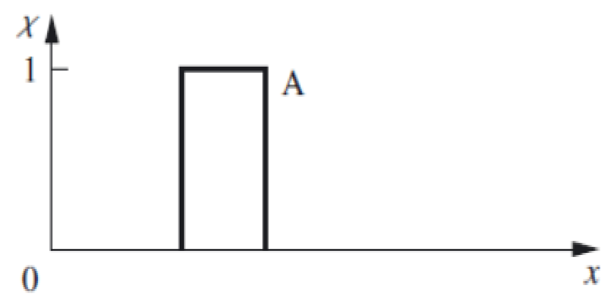
  $A = \{2, 4, 6, 8, \ldots\}$

- Formulas: $A = \{x \mid x \text{ is an even natural number}\}$

  $A = \{x \mid x = 2n, n \text{ is a natural number}\}$

- Membership or characteristic function

$$X_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

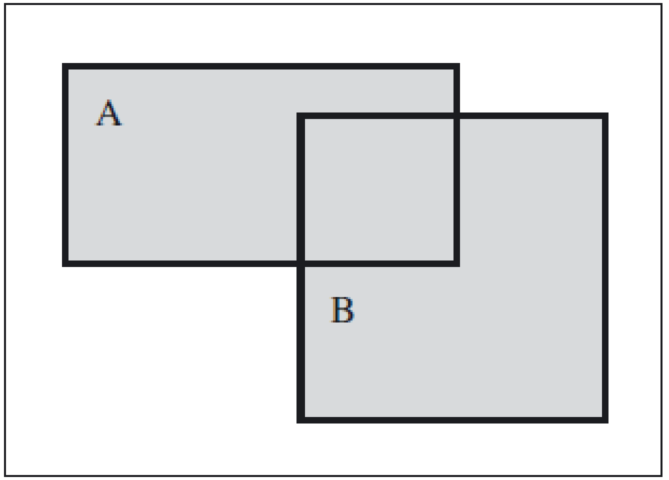


Classical set theory: Operations

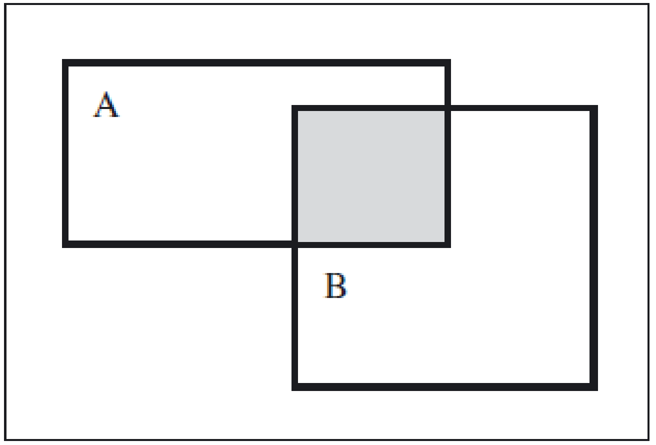
1. Union:
   $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
2. Intersection:
   $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
3. Complement:
   $A' = \{x \mid x \notin A, x \in X\}$; i.e. X – Universal Set
4. Set Difference:
   $A \mid B = \{x \mid x \in A \text{ and } x \notin B\}$
   Set difference is also denoted by A - B

Classical set theory: Operations

Union of sets A and B (logical or).

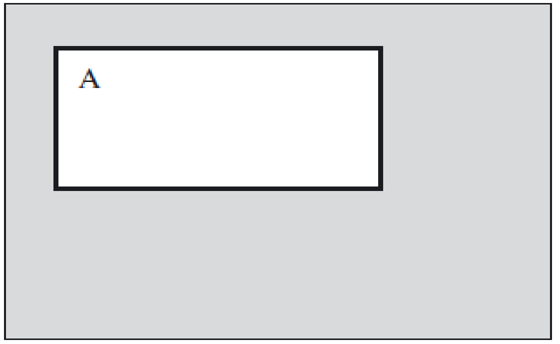


Intersection of sets A and B.

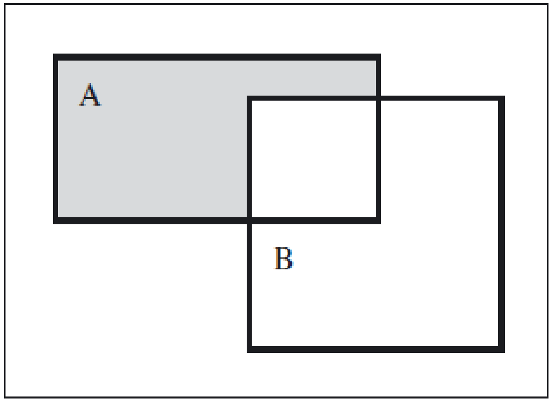


Classical set theory: Operations

Complement of set A.



Difference operation A|B.



Classical set theory: Properties

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup A = A$$
$$A \cap A = A$$

$$A \cup X = X$$
$$A \cap X = A$$
$$A \cup \varnothing = A$$
$$A \cap \varnothing = \varnothing$$

Fuzzy set theory:

- Fuzzy sets theory is an extension of classical set theory.
  - Elements have varying degree of membership. A logic based on two truth values.
  - True or false is sometimes insufficient when describing human reasoning.
- A **fuzzy set** is any set that allows its members to have different degree of membership, called **membership function**, having interval [0,1].
- Many degree of membership (between 0 to 1) are allowed.

Thus a membership function $\mu_{\tilde{A}}(x)$ is associated with a fuzzy sets $\tilde{A}$ such that the function maps every element of universe of discourse X to the interval [0,1].

The mapping is written as: $\mu_{\tilde{A}}(x): X \rightarrow [0,1]$.

Fuzzy set theory:

- **Fuzzy set** is defined as follows:
- If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs

$$A = \{(x, \mu_{A}(x)), x \in X\}$$

Here:

A: fuzzy set

***X*:** universe of discourse

$U_A(x)$: Membership function

Fuzzy set theory: Example

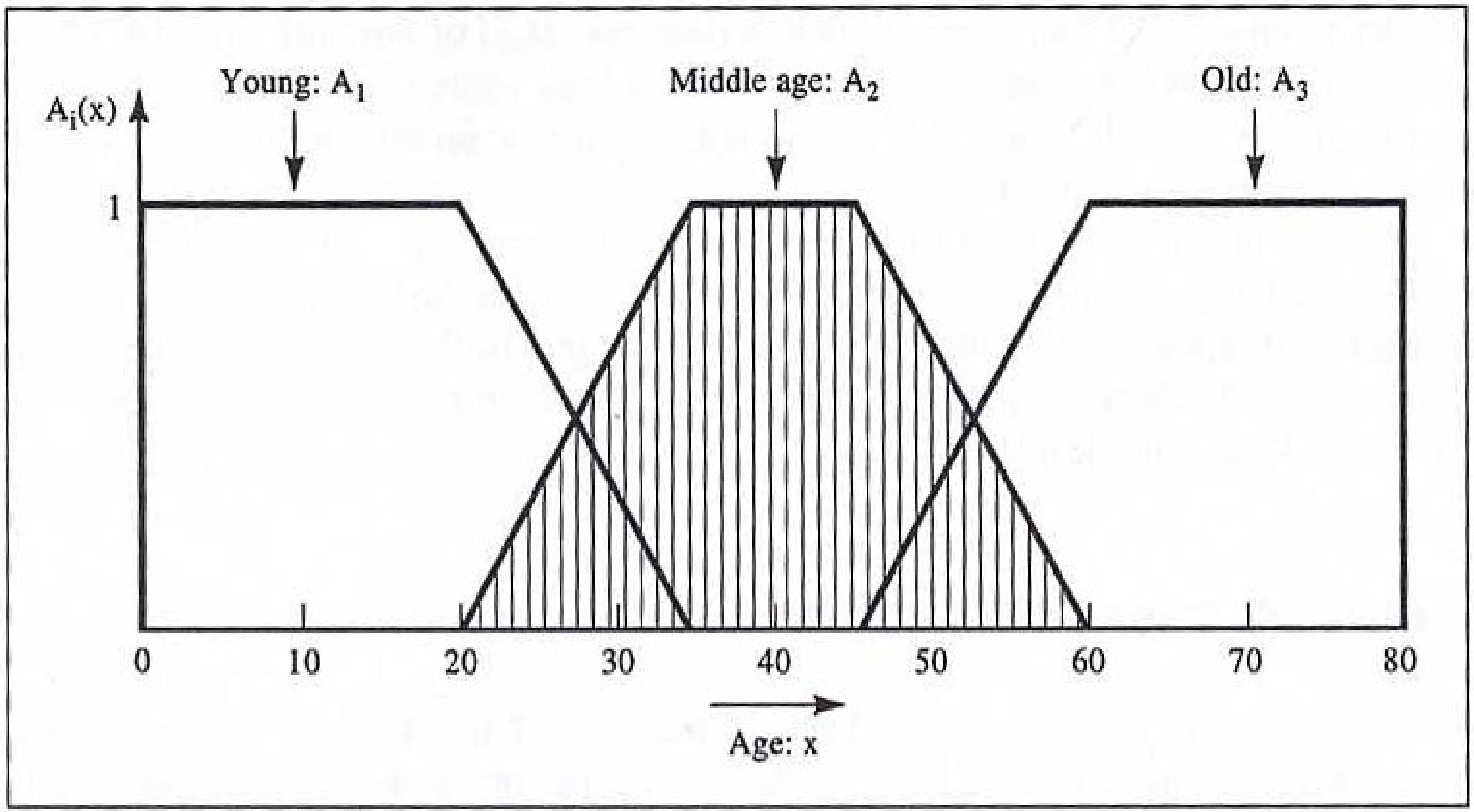
Example

- Let $X = \{g_1, g_2, g_3, g_4, g_5\}$ be the reference set of students.
- Let $\tilde{A}$ be the fuzzy set of "smart" students, where "smart" is fuzzy term.

$$\tilde{A} = \{(g_1,0.4)(g_2,0.5)(g_3,1)(g_4,0.9)(g_5,0.8)\}$$

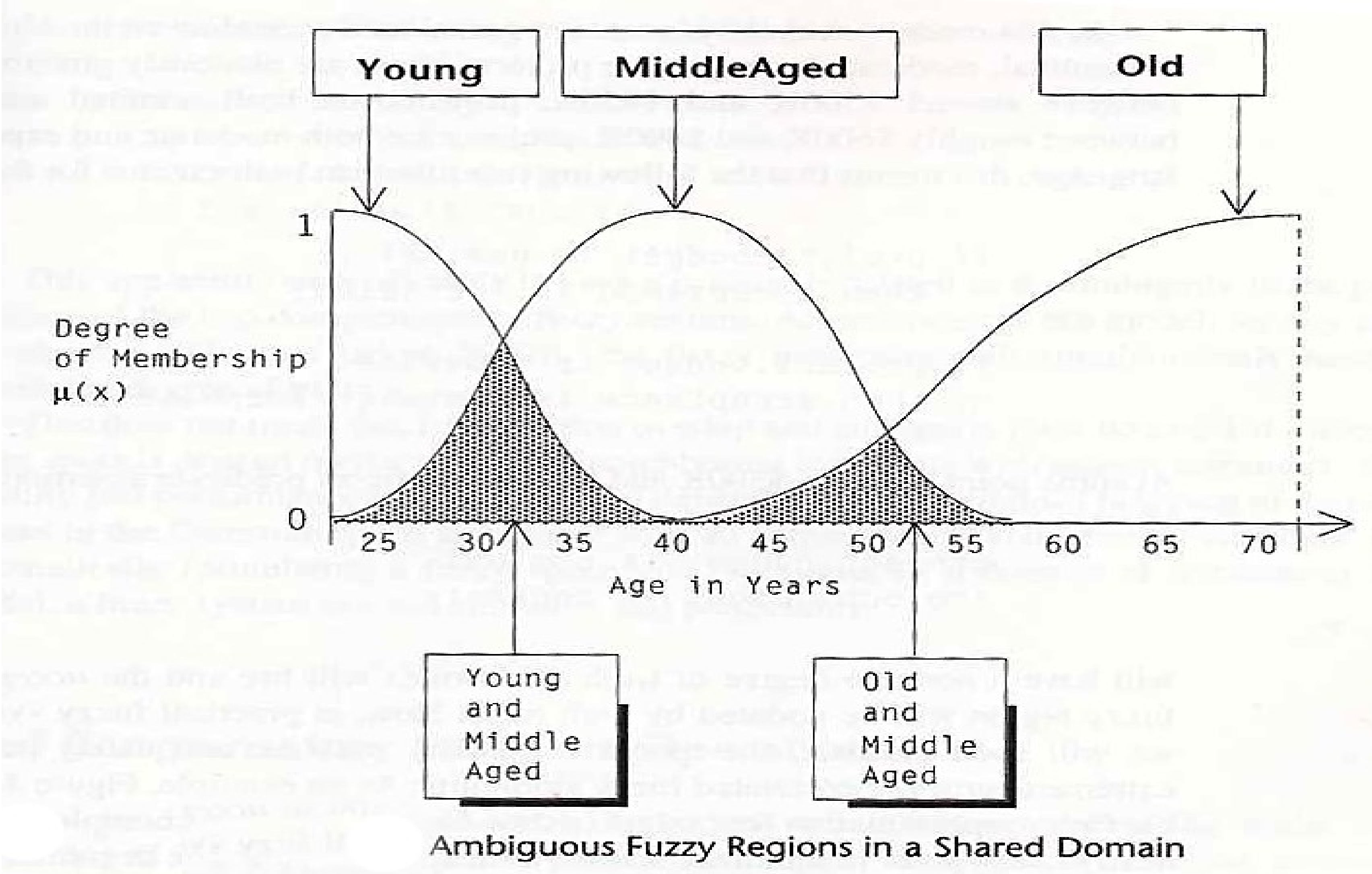
Here $\tilde{A}$ indicates that the smartness of g_1 is 0.4 and so on

Fuzzy set theory: Membership function example

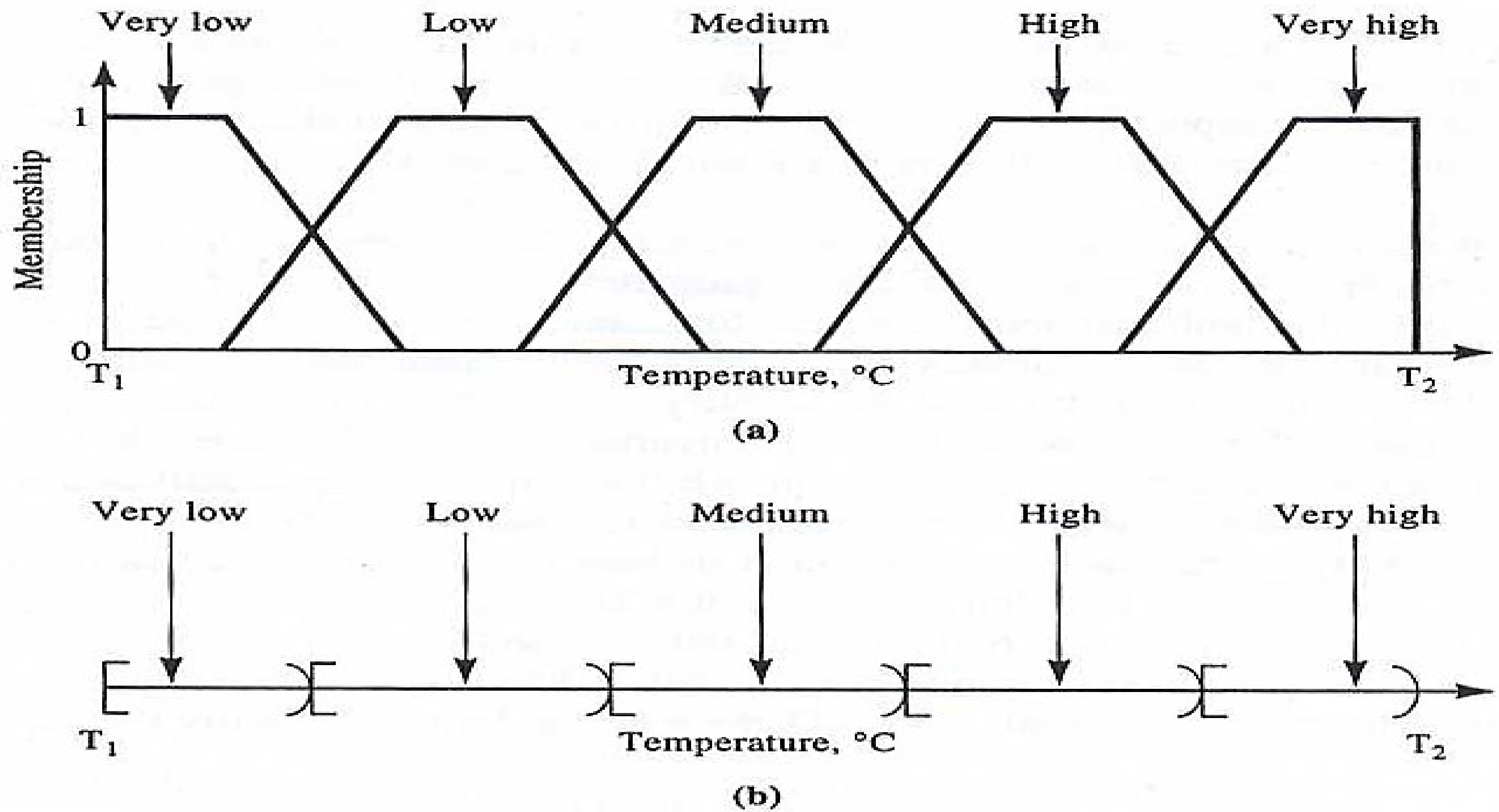


Membership functions representing the concepts of a young, middle-aged, and old person.

Fuzzy set theory: Membership function example



Fuzzy set theory: Membership function example



Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

Fuzzy set theory: Operations

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

Union:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Intersection:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Complement:

$$\mu_A(x) = 1 - \mu_A(x)$$

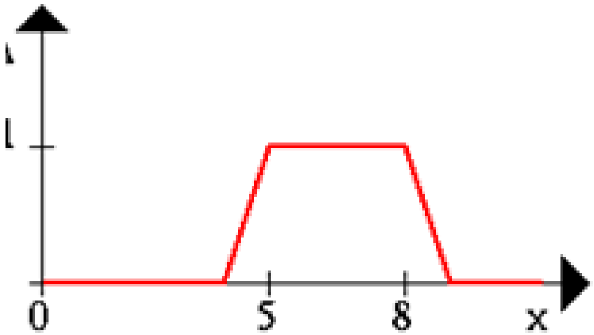
Fuzzy set theory: De Morgan's laws

- De Morgan's laws stated for classical sets also hold for fuzzy sets, as denoted by these expressions.

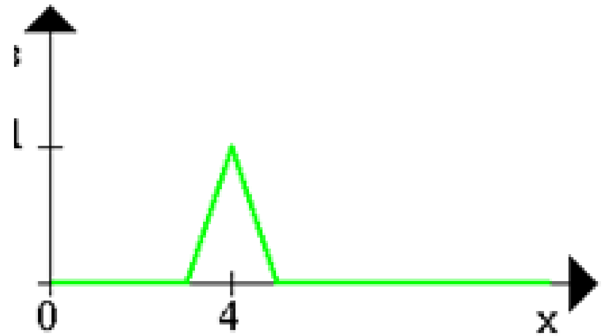
$$\overline{\tilde{A} \cup \tilde{B}} = \overline{\tilde{A}} \cap \overline{\tilde{B}}$$

$$\overline{\tilde{A} \cap \tilde{B}} = \overline{\tilde{A}} \cup \overline{\tilde{B}}$$

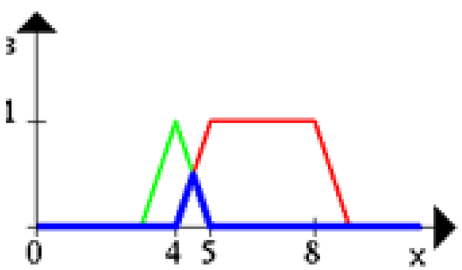
Fuzzy set theory: Operations



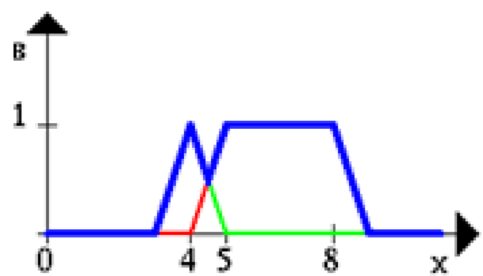
A



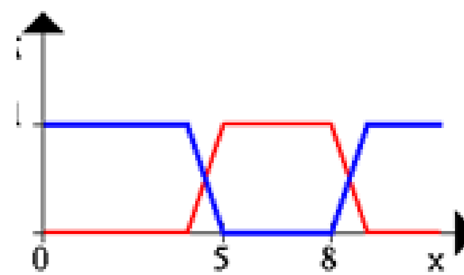
B



A ∧ B



A ∨ B



¬A

Fuzzy set theory: Operations- Union

- **Fuzzy union (∪):** the union of two fuzzy sets is the maximum (MAX) of each element from two sets.
- E.g.
  - A = {1.0, 0.20, 0.75}
  - B = {0.2, 0.45, 0.50}
  - A ∪ B = {MAX(1.0, 0.2), MAX(0.20, 0.45), MAX(0.75, 0.50)} = {1.0, 0.45, 0.75}

Fuzzy set theory: Operations-Intersection

- **Fuzzy intersection (∩):** the intersection of two fuzzy sets is just the MIN of each element from the two sets.
- E.g.
  - A = {1.0, 0.20, 0.75}
  - B = {0.2, 0.45, 0.50}
  - A ∩ B = {MIN(1.0, 0.2), MIN(0.20, 0.45), MIN(0.75, 0.50)} = {0.2, 0.20, 0.50}

Fuzzy set theory: Operations- Compliment

- **Fuzzy compliment (A’):** the remaining of each element from the fuzzy sets.
- $A' = 1 - U_A(x)$
- E.g.
  - A = {1.0, 0.20, 0.75}
  - A’ = {0.0, 0.80, 0.25}

Fuzzy set theory: Some more examples to solve

A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

1. Complement:

A' = {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

2. Union:

$A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

3. Intersection:

$A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

Fuzzy set theory: Some more examples to solve

1. Consider two fuzzy sets A and B . Find Complement, Union, Intersection and Difference.

x=2,3,4,5

A = { (2,1) , (3,0.5) , (4,0.3) , (5,0.2) }

B = { (2,0.5) , (3,0.7) , (4,0.2) , (5,0.4) }

Fuzzy set theory: Some more examples to solve

2. Consider two fuzzy sets A and B . Find Complement, Union, Intersection and Difference.

A = { (2,1) , (3,0.5) , (4,0.6) , (5,0.2) , (6,0.6) }

B = { (2,0.5) , (3,0.8) , (4,0.4) , (5,0.7) , (6,0.3) }

Fuzzy set theory: Properties

$A \cup B = B \cup A$
$A \cap B = B \cap A$

$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup A = A$ $A \cap A = A$
$A \cup X = X$ $A \cap X = A$
$A \cup \varnothing = A$ $A \cap \varnothing = \varnothing$

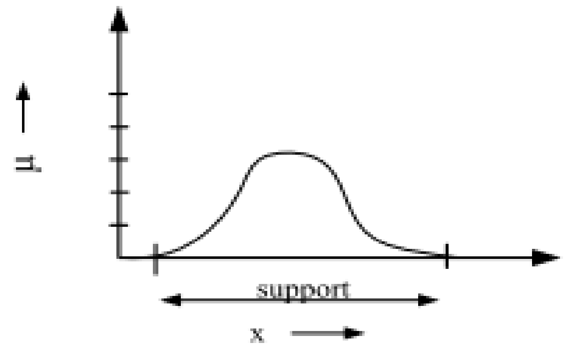
If $A \subseteq B \subseteq C$, then $A \subseteq C$

$A'' = A$

Fuzzy set theory Properties: Support of Fuzzy Set

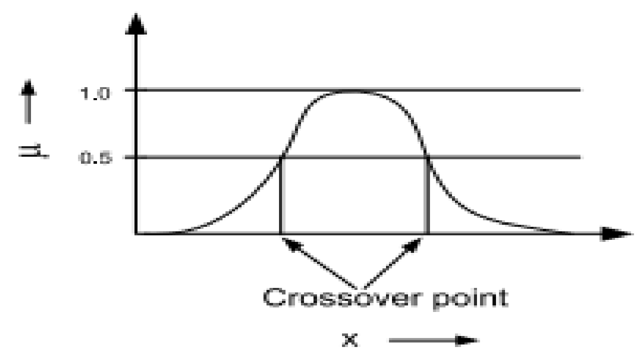
- The support of the fuzzy set A is S(A), which is a crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

$$\boxed{\text{Support }(A) = \{x \mid \mu_A(x) > 0\}}$$



- The element x in X at which $\mu_{\tilde{A}}(x) = 0.5$ is called crossover point.

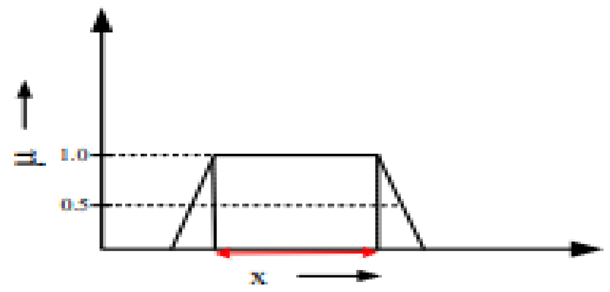
$$\boxed{\text{Crossover }(A) = \{x \mid \mu_A(x) = 0.5\}}$$



Fuzzy set theory Properties: Core of Fuzzy Set

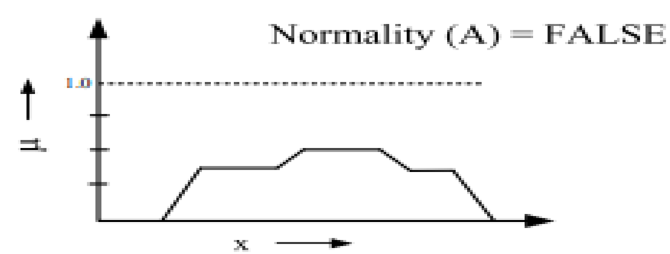
- The core of the fuzzy set A is C(A), which is a crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$.

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



Fuzzy set theory Properties: Normality

- A fuzzy set X is called normal, if there exist at least one element $x \in X$ such that $\mu\tilde{A}(x) = 1$
- A fuzzy set that is not normal is called subnormal

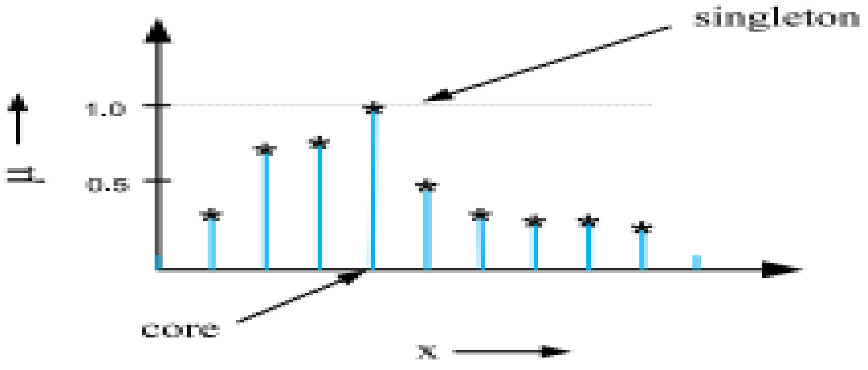


Fuzzy set theory Properties: Height

- The height of a fuzzy set A is the largest membership grade of an element in A
- Height (A) = max $\{\mu_{\tilde{A}}(x)\}$

Fuzzy set theory Properties: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{ x \mid \mu_A(x) = 1 \}$.



Fuzzy set theory Properties: α-cut and Strong α-cut

An α-**cut or** α-**level** set of a fuzzy set *A* is a crisp set of elements of *A* at least to degree α

$$A_\alpha = \{\, x \in X \;\; | \; \mu_A(x) \geq \alpha \,\}$$

Strong α-**cut or strong** α-**level** set are defined as:

$$A'_\alpha = \{\, x \in X \;\; | \; \mu_A(x) > \alpha \,\}$$

Fuzzy set theory Properties: α-cut and Strong α-cut

λ Cut /α-Cut

1. It is A Defuzzification Method
2. In this method a fuzzy set A is transformed into a crisp set Aλ for a given value of λ ($0 \le \lambda \le 1$)
3. In other-words, $A\lambda = \{x \mid \mu A(x) \ge \lambda\}$
4. That is, the value of Lambda-cut set Aλ is x, when the membership value corresponding to x is greater than or equal to the specified λ.
5. This Lambda-cut set Aλ is also called alpha-cut set.

Fuzzy set theory Properties: α-cut and Strong α-cut

Example

A1 = {(x1, 0.9),(x2, 0.5),(x3, 0.2),(x4, 0.3)}

Then

A0.6 = {(x1, 1),(x2, 0),(x3, 0),(x4, 0)} = {x1}

and

A2 = {(x1, 0.1),(x2, 0.5),(x3, 0.8),(x4, 0.7)}

A0.2 = {(x1, 0),(x2, 1),(x3, 1),(x4, 1)} = {x2, x3, x4}

MU asked question: Dec 2019 (5 m)

- Determine (alfa) α-level sets and strong α-level sets for the following fuzzy sets.

  A={(1,0.2), (2,0.5), (3, 0.8), (4,1), (5, 0.7), (6,0.3)}

  [hint: consider alfa-0.5]

Solution:

- A0.5={2,3,4,5}--- {0,1,1,1,1,0}
- A`0.5={3,4,5}---- {0,0,1,1,1,0}

Fuzzy set theory Properties: Open & closed

A fuzzy set A is

Open left

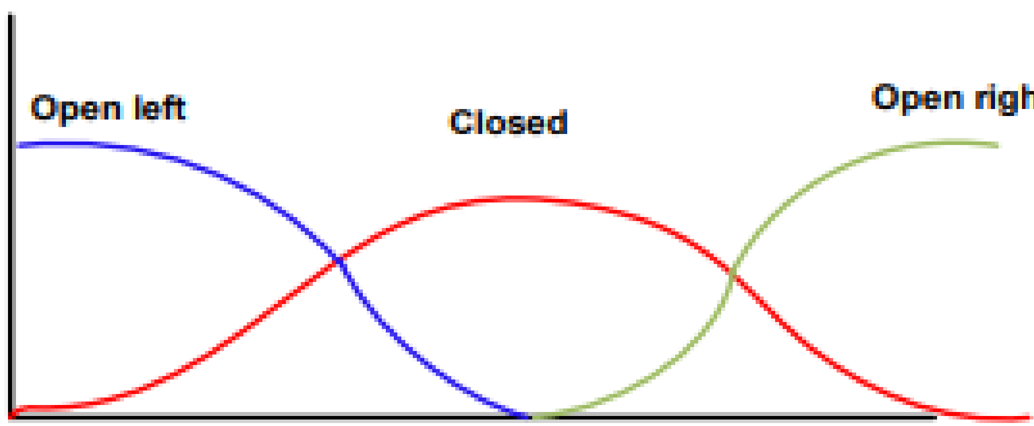
If $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

Open right:

If $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$

Closed

If : $\lim_{x \to -\infty} \mu_A(x) = \lim_{x \to +\infty} \mu_A(x) = 0$



Fuzzy set theory Properties: Cardinality

- Cardinality is defined as the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A
- $|A| = \sum \mu(x)$

Fuzzy set theory Properties: Relative Cardinality

- Relative Cardinality is defined as
- $||A|| = \sum \mu(x)$ / Number of non-zero elements

Fuzzy set theory Properties: Numerical 1

- Consider two fuzzy subsets of the set X, X = {a, b, c, d, e } referred to as A and B
- A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e} and
- B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

- Find Support of A,B
- Find Core for A, B
- Which among set A and B is Normal and sub normal?
- Find Height of Set A and B
- Which among set A and B is a fuzzy singleton set
- Find alpha cut and strong alpha cut for set A & Set B , for alpha=0.6
- Find Cardinality and relative Cardinality for set A and Set B.

Fuzzy set theory Properties: Numerical 2

- Find the support, core , height, cardinality and relative cardinality of A given below. Write all possible α cuts and strong α cuts.

X = { 1,2,3, 4, 5, 6}

A = { (1,0.2), (2,0.5), (3,0.8), (4,1),(5,0.7), (6,0.3) }

Support(A)=1,2,3,4,5,6
Core(A) = 4
Height(A)=1
Cardinality(A)=3.5
Relative Cardinality(A)=3.5/6
Alpha cut (0.5)=(0,1,1,1,1,0)=(2,3,4,5)
Strong Alpha cut(0.5)=(0,0,1,1,1,0)= (3,4,5)

Fuzzy set theory Properties: Numerical 3

- Find the support, core , height, cardinality and relative cardinality of A given below. Write all possible α cuts and strong α cuts.
- X = { 3, 4, 5, 6, 7, 8, 9 }
- A= { (3,0.1) , (4,0.2) , (5,0.3) , (6,0.4) , (7,1) ,(8,0.8) , (9,0.6) }

Support(A)=3,4,5,6,7,8,9
Core(A) = 7
Height(A)=1
Cardinality(A)= 3.4
Relative Cardinality(A)=3.4/7
Alpha cut (0.4)=(0,0,0,1,1,1,1)=(6,7,8,9)
Strong Alpha cut(0.4)=(0,0,0,0,1,1,1)=(7,8,9)

Fuzzy set theory: Representation

Note $\mu(x) \in [0,1]$

not {0,1} like Crisp set

$A = \{\mu_A(x1) / x1 + \mu_A(x2) / x2 + \ldots\}$

$= \{\sum \mu_A(xi) / xi\}$

Note: '+' $\neq$ add

'/ ' $\neq$ divide

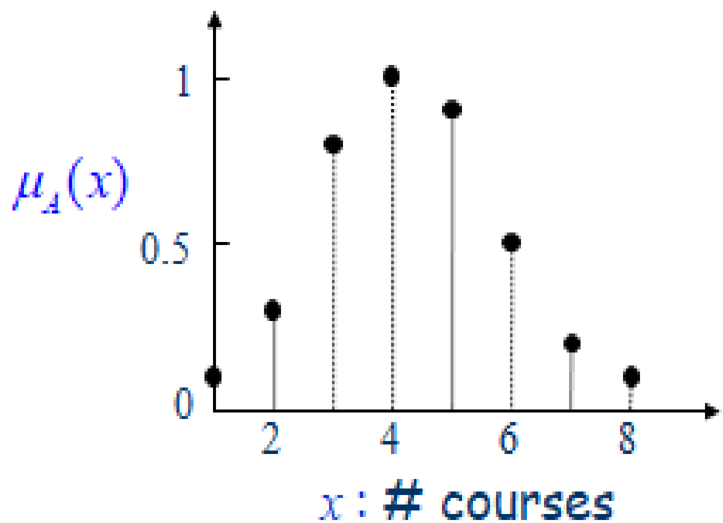
Only for representing element and its membership.

Also some books use $\mu(x)$ for Crisp Sets too.

Fuzzy set theory: Example (Discrete Universe)

$$U = \{1,2,3,4,5,6,7,8\}$$ — # courses a student may take in a semester.

$$A = \left\{ \begin{matrix} (1,0.1) & (2,0.3) & (3,0.8) & (4,1) \\ (5,0.9) & (6,0.5) & (7,0.2) & (8,0.1) \end{matrix} \right\}$$ — appropriate # courses taken



Fuzzy set theory: Example (Discrete Universe)

$$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$$ — # courses a student may take in a semester.

$$A = \begin{Bmatrix} (1, 0.1) & (2, 0.3) & (3, 0.8) & (4, 1) \\ (5, 0.9) & (6, 0.5) & (7, 0.2) & (8, 0.1) \end{Bmatrix}$$ — appropriate # courses taken

Alternative Representation:

$$A = 0.1/1 + 0.3/2 + 0.8/3 + 1.0/4 + 0.9/5 + 0.5/6 + 0.2/7 + 0.1/8$$

Fuzzy set theory: Example (Continuous Universe)

U : the set of positive real numbers — possible ages

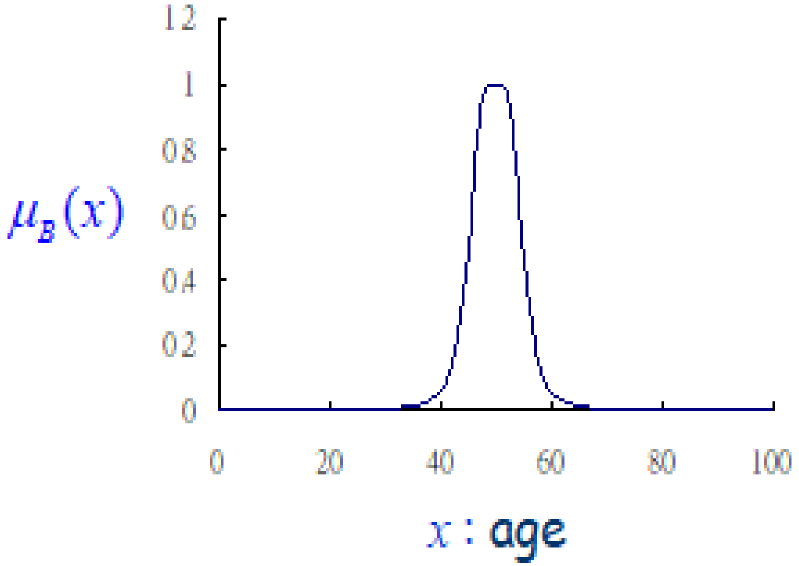
$$B = \{(x, \mu_B(x)) \mid x \in U\}$$

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{5}\right)^4}$$

about 50 years old

Alternative Representation:

$$B = \int_{R+} \frac{1}{1+\left(\frac{x-50}{5}\right)^4} \Big/ x$$



Fuzzy set theory: Fuzzy set Representation

$$A = \left\{ (x, \mu_A(x)) \middle| x \in U \right\}$$

U : discrete universe ➡ $A = \sum_{x_i \in U} \mu_A(x_i) / x_i$

U : continuous universe ➡ $A = \int_U \mu_A(x) / x$

Note that $\sum$ and integral signs stand for the union of membership grades; " / " stands for a marker and does not imply division.

Introduction to Fuzzy Set

-Fuzzy set & Crisp Set Relations

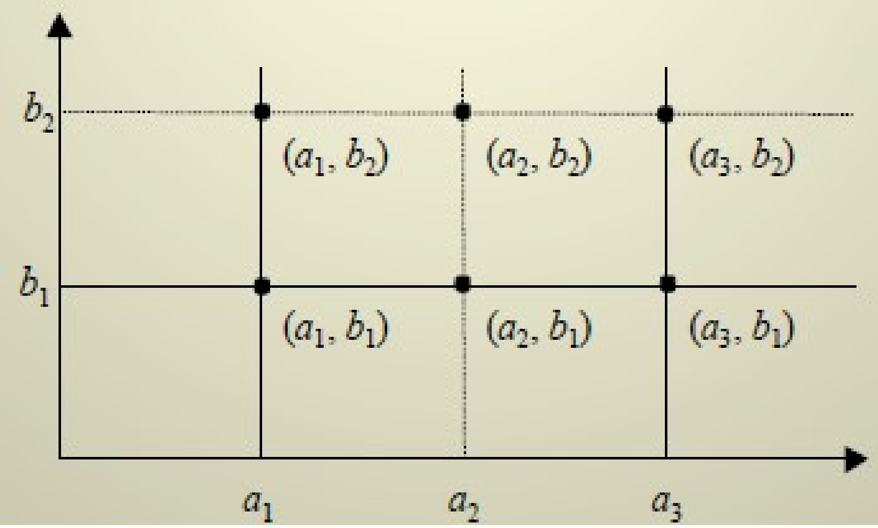
Classical set Relations:

- **Definition (Product set)** Let A and B be two non-empty sets, the product set or Cartesian product $A \times B$ is defined as follows,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Example $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$$



Classical set Relations: strength

- A relation among classical sets A and B is a subset of Cartesian product. It is denoted by R.
- **Strength of Relation:**
- It is measured by the characteristic function |R| given as
- $$|R| = \begin{cases} 1 & (a,b) \in R \\ 0 & (a,b) \notin R \end{cases}$$

Classical set Relations: Problem on strength

- Consider two sets A = { 1, 2 } and B = { a, b } and relation R = { (1,a) , (2,b) }
- Compute the strength of R
- **Solution:** $|R| = 1$

Classical set Relations: Operations

- For two relations R and S :

1. Union : (R U S) = max { |R| (x, y) , |S|(x, y) }
2. Intersection : (R Π S) = min { |R| (x, y) , |S|(x, y) }
3. Complement : $\bar{R}$ (x, y) = 1- |R| (x, y)

Classical set Relations: Operation-Composition

Let **R** be a relation that relates elements from universe X to universe Y (R: X→Y) and

S be a relation that relates elements from universe Y to universe Z (S: Y→Z) , then

a relation **T** that relates elements in X to elements in Z (T:X→Z)

can be formed by using an operation called **Composition**.

Max-min Composition

$$\mu_{R_1 \circ R_2}(x,z) = \bigvee_y [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]$$

Classical set Relations: Composition-problem

Max-min Composition

$$\mu_{R_1 \circ R_2}(x,z) = \bigvee_y [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]$$

1. Using max-min composition find the relation between R and S

- $R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

max{min(1,0), min(1, 1), min(0, 1)} = max{ 0, 1, 0} = 1
max{min(0,0), min(0, 1), min(1, 1)} = max{ 0, 0, 1} = 1
max{min(0,0), min(1, 1), min(0, 1)} = max{ 0, 1, 0} = 1
max{min(1,1), min(1, 0), min(0, 1)} = max{ 1, 0, 0} = 1
max{min(0,1), min(0, 0), min(1, 1)} = max{ 0, 0, 1} = 1
max{min(0,1), min(1, 0), min(0, 1)} = max{ 0, 0, 0} = 0

T(3*2)=

1	1
1	1
1	0

Fuzzy set Relations:

- A fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets A1, A2, ..., An where tuples (x1, x2, ..., xn) may have varying degrees of membership within the relation.
- The membership grade indicates the strength of the relation present between the elements of the tuple.

Fuzzy set Relations: Example

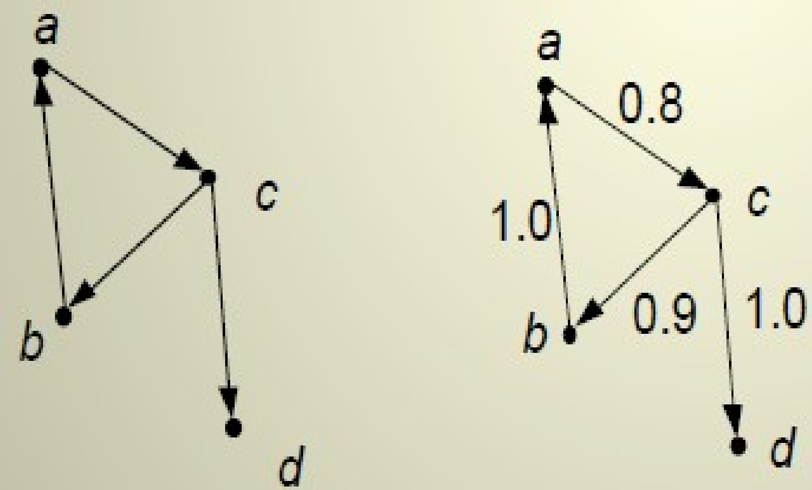
Example

Crisp relation R

$\mu_R(a, c) = 1,\ \mu_R(b, a) = 1,\ \mu_R(c, b) = 1$ and $\mu_R(c, d) = 1.$

Fuzzy relation R

$\mu_R(a, c) = 0.8,\ \mu_R(b, a) = 1.0,\ \mu_R(c, b) = 0.9,\ \mu_R(c, d) = 1.0$



(a) Crisp relation (b) Fuzzy relation

$A \backslash A$	a	b	c	d
a	0.0	0.0	0.8	0.0
b	1.0	0.0	0.0	0.0
c	0.0	0.9	0.0	1.0
d	0.0	0.0	0.0	0.0

corresponding fuzzy matrix

Fuzzy set Relations: Domain & Range

- Domain (set of all possible values of x)of a relation R (x, y) is given as

$$\mu_{dom(R)}(x) = \max_{y \in B} \mu_R(x,\ y)$$

- Range (set of all possible values of y) of a relation R (x,y) is given as

$$\mu_{ran(R)}(y) = \max_{x \in A} \mu_R(x,\ y)$$

Fuzzy set Relations: Domain & Range

Consider a universe $\underset{\sim}{X} = \{x_1, x_2, x_3, x_4\}$ & a binary fuzzy relation $\underset{\sim}{R}$ as:

$$\underset{\sim}{R}(\underset{\sim}{X}, \underset{\sim}{X}) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.7 & 0.8 \\ 0.1 & 0 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.1 \end{bmatrix} \end{array}$$

Solution:

Domain = {0.5, 0.8, 0.4, 0.6} (Take max on rows)
&

Range = {0.2, 0.6, 0.7, 0.8}
(Take max on columns)

Fuzzy set Relations: Domain & Range

Find the domain and range of the relation given below:

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} .9 & 1 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 \\ 0 & .5 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & .2 \end{bmatrix} \end{array}$$

Solutions:

Domain (x)= { 1,0.4,1,1,0.5,0.2}

Range (y)= {0.9,1,1,1,0.5}

Fuzzy set Relations: Cartesian Product

Let

- A be a fuzzy set on universe X, and
- B be a fuzzy set on universe Y,
- Then the Cartesian product between fuzzy sets A and B is defined as

$$\mu_{\tilde{A}x\tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

Fuzzy set Relations: Cartesian Product

2D Fuzzy Relation

- For example, for a fuzzy set (vector) $\underline{A}$ that has four elements, hence column vector of size 4×1, and for a fuzzy set (vector) $\underline{B}$ that has five elements, hence a row vector size of 1×5, the resulting fuzzy relation, $\underset{\sim}{R}$, will be represented by a matrix of size 4 × 5,

A= {x1, x2,x3,x4} 4*1
B= {y1,y2,y3,y4,y5} 1* 5

X1,y1	X1,y2	X1,y3	X1,y4	X1,y5
X2,y1	X2,y2	X2,y3	X2,y4	X2,y5
X3,y1	X3,y2	X3,y3	X3,y4	X3,y5
X4,y1	X4,y2	X4,y3	X4,y4	X4,y5

4*5

Fuzzy set Relations: Cartesian Product

- Consider two fuzzy sets A and B . Let A represent universe of three distinct temperatures x = $\{x_1, x_2, x_3\}$ and B represents universe of two discrete flow y = $\{y_1, y_2\}$. Find the Cartesian product between them given

$$\mu_{\tilde{A}x\tilde{B}}(x, y) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))$$

- A = { (x_1,0.4) , (x_2,0.7) (x_3, 0.1) } ----- (3*1) and
- B = { (y_1,0.5) , (y_2,0.8) } -----(1*2)

Solutions:

U (AXB)= { (x1,y1), (x1,y2), (x2,y1), (x2,y2), (x3,y1), (x3,y2)}
= { (0.4,0.5),(0.4,0.8) , (0.7,0.5),(0.7,0.8), (0.1,0.5),(0.1,0.8)}
= { 0.4,0.4,0.5,0.7,0.1,0.1 } -----(3*2)

0.4	0.4
0.5	0.7
0.1	0.1

Fuzzy set Relations: Operations

⌗ Operation of fuzzy relation

1) Union relation

$$\forall (x, y) \in A \times B$$

$$\mu_{R \cup S}(x, y) = \text{Max}\,[\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \vee \mu_S(x, y)$$

2) Intersection relation

$$\mu_{R \cap S}(x) = Min\,[\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \wedge \mu_S(x, y)$$

3) Complement relation

$$\forall (x, y) \in A \times B$$

$$\mu_R(x, y) = 1 - \mu_R(x, y)$$

4) Inverse relation

For all $(x, y) \subseteq A \times B$, $\mu_R^{-1}(y, x) = \mu_R(x, y)$

Fuzzy set Relations: Operations Example

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R\cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

$M_{R\cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

$M_{\bar{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Fuzzy set Relations: Operation-Composition

- Let R be a fuzzy relation on X x Y and S be a fuzzy relation on Y x Z , then a relation T that relates elements in X to elements in Z can be formed by using an operation called Composition.
- Fuzzy Max-min Composition

$$\mu_T(x,z) = \bigvee_{y\in Y} \left(\mu_{\underset{\sim}{R}}(x, y) \wedge \mu_{\underset{\sim}{S}}(y, z) \right).$$

- Fuzzy Max Product Composition

$$\mu_T(x,z) = \bigvee_{y\in Y} \left(\mu_{\underset{\sim}{R}}(x,y) \bullet \mu_{\underset{\sim}{S}}(y,z) \right)$$

Fuzzy set Relations: Composition- Max-Min

- Let the two relations R and S be, respectively:

R	y_1	y_2	y_3
x_1	0.4	0.6	0
x_2	0.9	1	0.1

S	z_1	z_2
y_1	0.5	0.8
y_2	0.1	1
y_1	0	0.6

- The goal is to compute R^oS using both Max-min and Max-product composition rules.

Fuzzy set Relations: Composition- Max-Min

$$R^oS = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.9 & 1 & 0.1 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.8 \\ 0.1 & 1 \\ 0 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 1 \end{bmatrix}$$

max{min(0.4,0.5), min(0.6, 0.1), min(0, 0)}

= max{ 0.4, 0.1, 0} = 0.4

max{min(0.4,0.8), min(0.6, 1), min(0, 0.6)}

= max{ 0.4, 0.6, 0} = 0.6

max{min(0.9,0.5), min(1, 0.1), min(0.1, 0)}

= max{ 0.5, 0.1, 0} = 0.5

max{min(0.9,0.8), min(1, 1), min(0.1, 0.6)}

= max{ 0.8, 1, 0.1} = 1

Fuzzy set Relations: Composition- Max product

$$RoS = \begin{bmatrix} 0.4 & 0.6 & 0 \\ 0.9 & 1 & 0.1 \end{bmatrix} \circ \begin{bmatrix} 0.5 & 0.8 \\ 0.1 & 1 \\ 0 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.06 & 0.6 \\ 0.45 & 1 \end{bmatrix}$$

max{(0.4,0.5), (0.6,0.1), (0,0)} = max{0.02,0.06,0} = 0.06

max{(0.4,0.8), (0.6,1), (0,0.6)} = max{0.32, 0.6, 0} = 0.6

max{(0.9,0.5), (1,0.1), (0.1,0)} = max{0.45, 0.1, 0} = 0.45

max{(0.9,0.8), (1,1,) ,(0.1,0.6)} = max{0.72, 1, 0.06} = 1

Fuzzy set Relations: Operation-Composition

Find the max-min and max product composition of the fuzzy relations given below:

$R = \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix}$ and $S = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}$

Fuzzy set Relations: Operation-Composition

Find the max-min and max product composition of the fuzzy relations given below:

$$R = \begin{bmatrix} 0.7 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{bmatrix}$$

Fuzzy set Relations: Composition- MU asked

- The formation of algal solutions in surface water is strongly dependent on Ph of water, temperature (T) and oxygen content(O). T = { 50, 55, 60 } and O = { 1, 2, 6 }. The fuzzy set of T is given by
- $T = \left\{\frac{0.7}{50} + \frac{0.8}{55} + \frac{0.9}{60}\right\}$
- and fuzzy set O is given by
- $O = \left\{\frac{0.1}{1} + \frac{0.6}{2} + \frac{0.8}{6}\right\} I = \left\{\frac{0.5}{50} + \frac{1}{55} + \frac{0.7}{60}\right\}$

i. Find R = T x O

ii. Find S = I o R using max product composition

iii. Find S = I o R using max min composition

Introduction to Fuzzy Set

-Fuzzy set & Crisp Set

Equivalence Relations

Crisp Equivalence Relation

- The relation R is said to be an equivalence relation if it has the following characteristics

1. **Reflexivity**

$$\mu_R(x_i, x_i) = 1$$

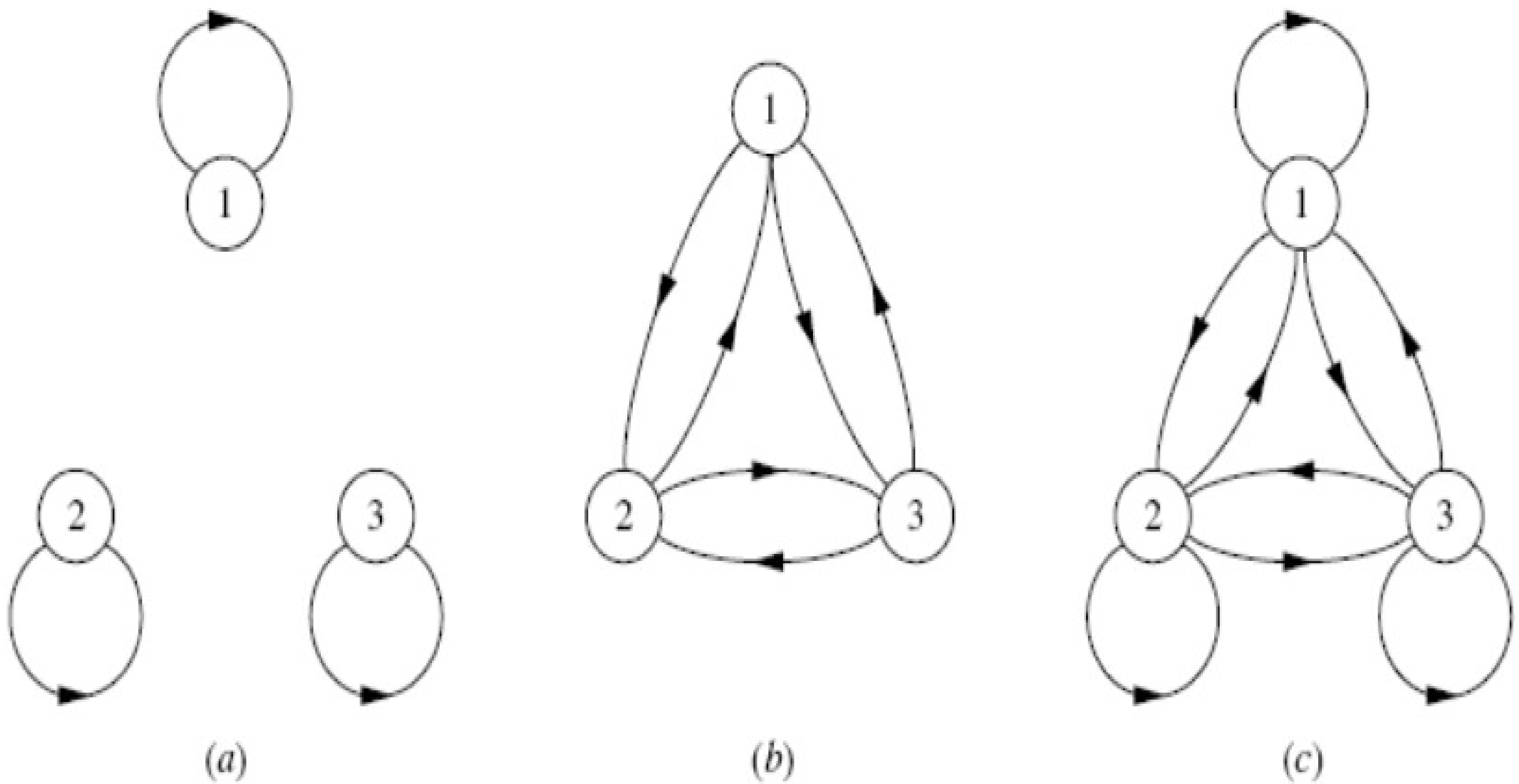
2. **Symmetry**

$$\mu_R(x_i, x_j) = \mu_R(x_j, x_i)$$

3. **Transitivity**

$$\mu_R(x_i, x_j) = \mu_R(x_j, x_k) = 1 \ , \ \mu_R(x_i, x_k) = 1$$

Crisp Equivalence Relation



FIGURE

Three-vertex graphs for properties of (*a*) reflexivity, (*b*) symmetry, (*c*) transitivity [Gill, 1976].

Crisp Tolerance Relation

- A relation is said to be a **tolerance relation** if it exhibits only the properties of reflexivity and symmetry.

- A tolerance relation can be reformed into an equivalence relation by at most (n-1) compositions with itself. (where n is the cardinality of R)

Crisp Tolerance Relation: Example 1

Example 3.10. Suppose in an airline transportation system we have a universe composed of five elements: the cities Omaha, Chicago, Rome, London, and Detroit. The airline is studying locations of potential hubs in various countries and must consider air mileage between cities and takeoff and landing policies in the various countries. These cities can be enumerated as the elements of a set, i.e.,

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$
$$= \{\text{Omaha, Chicago, Rome, London, Detroit}\}$$

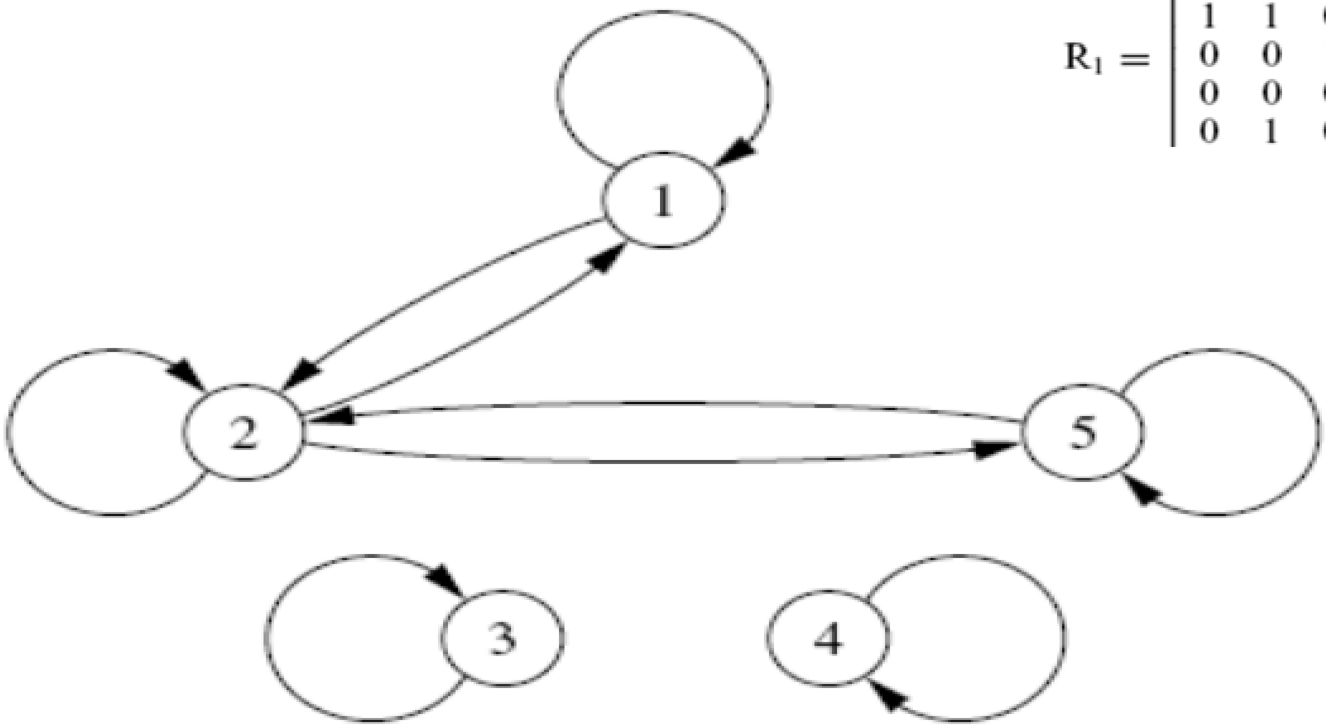
Further, suppose we have a tolerance relation, R_1, that expresses relationships among these cities:

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Crisp Tolerance Relation: Example 1

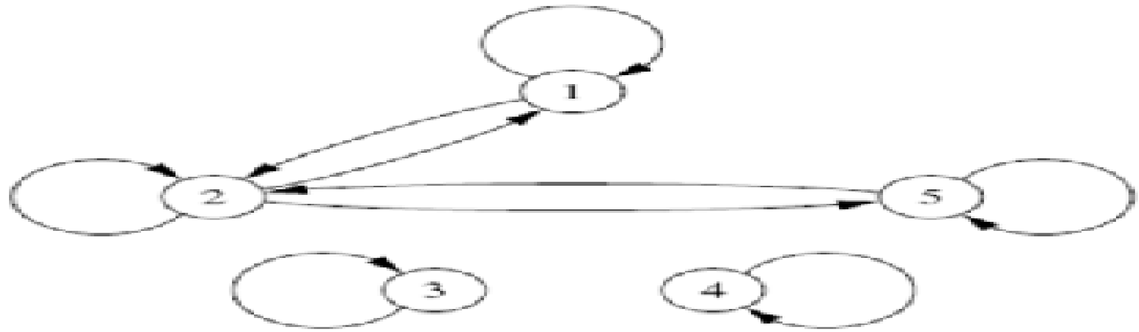
- This relation is reflexive and symmetric.
- but not transitivity

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



Crisp Tolerance Relation: Example 1

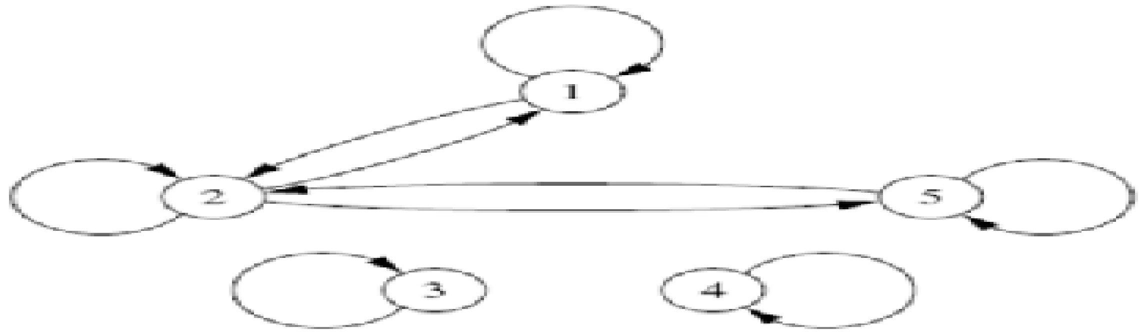
$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



- The property of reflexivity (diagonal elements equal unity) simply indicates that a city is totally related to itself.
- The property of symmetry might represent proximity: Omaha and Chicago (x_1 and x_2) are close (in a binary sense) geographically, and Chicago and Detroit (x_2 and x_5) are close geographically.
- This relation, R_1, does not have properties of transitivity, e.g.,$(x_1, x_2) \in R_1$ $(x_2, x_5) \in R_1$ but $(x_1, x_5) \notin R_1$

Crisp Tolerance Relation: Example 1

$$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



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- This relation, R_1, does not have properties of transitivity, e.g.,$(x_1, x_2) \in R_1$ $(x_2, x_5) \in R_1$ but $(x_1, x_5) \notin R_1$

Crisp Tolerance Relation: Example 1

R_1 can become an equivalence relation through one ($1 \le n$, where $n = 5$) composition. Using Eq. (3.20), we get

$$R_1 \circ R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = R$$

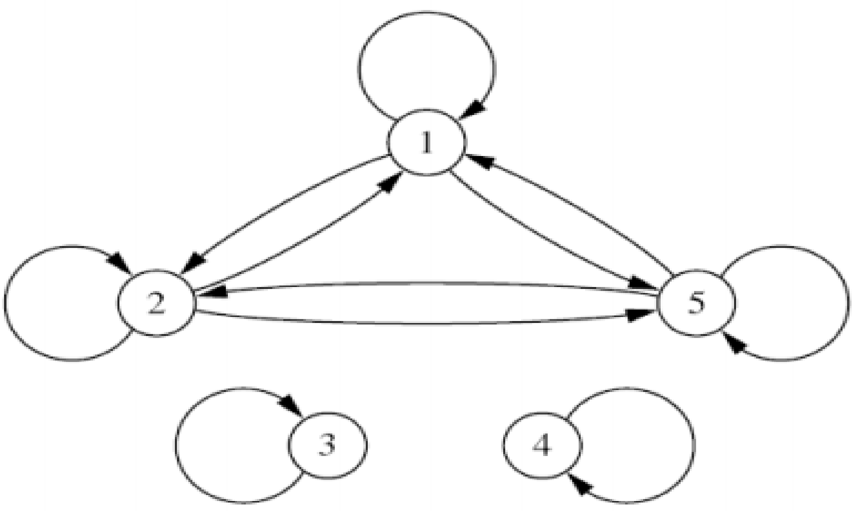

FIGURE 3.10
Five-vertex graph of equivalence relation (reflexive, symmetric, transitive) in Example 3.10.

Crisp Equivalence Relation

1. Find whether the given relation is an equivalence relation or not. If it is a tolerance relation, then convert it into equivalence relation by composition

	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	1	1	1	0	0	0	0	0
3	0	1	1	1	0	0	0	0
4	0	0	1	1	1	0	0	0
5	0	0	0	1	1	1	0	0
6	0	0	0	0	1	1	1	0
7	0	0	0	0	0	1	1	1
8	0	0	0	0	0	0	1	1

Fuzzy Equivalence Relation

The relation R is said to be an equivalence relation if it has the following characteristics

1. Reflexivity

   $\mu_R(x_i, x_i) = 1$

2. Symmetry

   $\mu_R(x_i, x_j) = \mu_R(x_j, x_i)$

3. Transitivity

   $\mu R(xi, xj) = \lambda 1$, $\mu R(xj, xk) = \lambda 2$ and $\mu R(xi, xk) = \lambda$ then $\lambda \geq \min(\lambda 1, \lambda 2)$

Fuzzy Tolerance Relation

- A relation is said to be a tolerance relation if it exhibits only the properties of reflexivity and symmetry.
- A tolerance relation can be reformed into an equivalence relation by at most (n-1) compositions with itself. (where n is the cardinality of R)

Fuzzy Tolerance Relation: Example 1

- **Example 3.11.** Suppose, in a biotechnology experiment, five potentially new strains of bacteria have been detected in the area around an anaerobic corrosion pit on a new aluminum–lithium alloy used in the fuel tanks of a new experimental aircraft. In order to propose methods to eliminate the biocorrosion caused by these bacteria, the five strains must first be categorized. One way to categorize them is to compare them to one another. In a pairwise comparison, the following "similarity" relation, $\underset{\sim}{R}_1$ is developed.

Fuzzy Tolerance Relation: Example 1

- For example, the first strain (column 1) has a strength of similarity to the second strain of 0.8.

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

is reflexive and symmetric. However, it is not transitive, e.g.,

$$\mu_{\underset{\sim}{R}}(x_1, x_2) = 0.8, \quad \mu_{\underset{\sim}{R}}(x_2, x_5) = 0.9 \geq 0.8$$

but

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = 0.2 \leq \min(0.8, 0.9)$$

Fuzzy Tolerance Relation: Example 1

One composition results in the following relation:

$$\underset{\sim}{R}_1^2 = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0 & 0.4 \\ 0.2 & 0.5 & 0 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

where transitivity still does not result; for example,

$$\mu_{\underset{\sim}{R}^2}(x_1, x_2) = 0.8 \geq 0.5 \qquad \text{and} \qquad \mu_{\underset{\sim}{R}^2}(x_2, x_4) = 0.5$$

but

$$\mu_{R^2}(x_1, x_4) = 0.2 \leq \min(0.8, 0.5)$$

Fuzzy Tolerance Relation: Example 1

Finally, after one or two more compositions, transitivity results:

$$\underset{\sim}{R}_1^3 = \underset{\sim}{R}_1^4 = \underset{\sim}{R} = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.5 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.4 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

$$\underset{\sim}{R}_1^3(x_1, x_2) = 0.8 \geq 0.5$$

$$\underset{\sim}{R}_1^3(x_2, x_4) = 0.5 \geq 0.5$$

$$\underset{\sim}{R}_1^3(x_1, x_4) = 0.5 \geq 0.5$$

Fuzzy Equivalence Relation

1. Find whether the given relation is an equivalence relation or not. If it is a tolerance relation, then convert it into equivalence relation by composition

$$R = \begin{bmatrix} 1 & 0.6 & 0 & 0.2 & 0.3 \\ 0.6 & 1 & 0.4 & 0 & 0.8 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.2 & 0 & 0 & 1 & 0.5 \\ 0.3 & 0.8 & 0 & 0.5 & 1 \end{bmatrix}$$

α-cut of fuzzy relation

Definition (α-cut relation) We can obtain α-cut relation from a fuzzy relation by taking the pairs which have membership degrees no less than α. Assume $R \subseteq A \times B$, and R_α is a α-cut relation. Then

$$R_\alpha = \{(x, y) \mid \mu_R(x, y) \geq \alpha, \quad x \in A, y \in B\}$$

Note that R_α is a crisp relation. □

Example: Write all possible α-cut for the fuzzy relation given below

$$M_R = \begin{matrix} 0.9 & 0.4 & 0.0 \\ 0.2 & 1.0 & 0.4 \\ 0.0 & 0.7 & 1.0 \\ 0.4 & 0.2 & 0.0 \end{matrix}$$

α-cut of fuzzy relation

Now the level set with degrees of membership function is,

$$\Lambda = \{0, 0.2, 0.4, 0.7, 0.9, 1.0\}$$

then we can have some α-cut relations in the following.

$M_{R\,0.4} =$

1	1	0
0	1	1
0	1	1
1	0	0

$M_{R\,0.7} =$

1	0	0
0	1	0
0	1	1
0	0	0

$M_{R\,0.9} =$

1	0	0
0	1	0
0	0	1
0	0	0

$M_{R\,1.0} =$

0	0	0
0	1	0
0	0	1
0	0	0

Fuzzy Set

-Fuzzy Extension Principle, Fuzzy Numbers

Fuzzy Extension Principle

Extension Principle

- Provides a general procedure for extending crisp domains of mathematical expressions to fuzzy domains.

- Generalizes a common point-to-point mapping of a function *f*(.) to a mapping between fuzzy sets.

Fuzzy Extension Principle

Extension Principle

Suppose that f is a function from X to Y and A is a fuzzy set on X defined as

$$A = \mu_A(x_1)/(x_1) + \mu_A(x_2)/(x_2) + ... + \mu_A(x_n)/(x_n)$$

Then the extension principle states that the image of fuzzy set A under the mapping $f(.)$ can be expressed as a fuzzy set B,

$$B = f(A) = \mu_A(y_1)/(y_1) + \mu_A(y_2)/(y_2) + ... + \mu_A(y_n)/(y_n)$$

Where $y_i = f(x_i)$, i=1,…,n. If f(.) is a many-to-one mapping then

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$

Fuzzy Extension Principle-Example

<u>example :</u>

Let both X and Y be the universe of natural numbers:

Function: $\mathbf{y = f(x) = x + 4}$;

- $A = 0.1/2 + 0.4/3 + 1/4 + 0.6/5$;
- $B = f(A) = 0.1/6 + 0.4/7 + 1/8 + 0.6/9$

Fuzzy Extension Principle-Numerical

- Given

$$A=0.1/-2+0.4/-1+0.8/0+0.9/1+0.3/2$$

$$f(x) = x^2-3$$

- Apply Fuzzy Extension Principle.

$$\begin{aligned} B &= 0.1/1+0.4/-2+0.8/-3+0.9/-2+0.3/1 \\ &= 0.8/-3+\max(0.4, 0.9)/-2+\max(0.1, 0.3)/1 \\ &= 0.8/-3+0.9/-2+0.3/1 \end{aligned}$$

Fuzzy Extension Principle

- ***Application of extension principle can result in two types of functions:***

1. **Monotonic Continuous Function:** Applying Fuzzy Extension principle, if the function f(x) on A results in continuous values of y , then f(x) is said to be a monotonic function
2. **Non-monotonic Continuous Function:** Applying Fuzzy Extension principle, if the function f(x) on A results in discrete values of y , then f(x) is said to be a non-monotonic function

Fuzzy Extension Principle

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2. **Non-monotonic Continuous Function:** Applying Fuzzy Extension principle, if the function f(x) on A results in discrete values of y , then f(x) is said to be a non-monotonic function

Fuzzy Extension Principle- Monotonic Continuous

- Function: $y=f(x)=0.6*x+4$
- Input: Fuzzy number - around-5
  - Around-5 = 0.3 / 3 + 1.0 / 5 + 0.3 / 7
- f(around-5) = 0.3/f(3) + 1/f(5) + 0.3/f(7)
- f(around-5) = 0.3/0.6*3+4 + 1/ 0.6*5+4 + 0.3/ 0.6*7+4
- f(around-5) = 0.3/5.8 + 1.0/7 + 0.3/8.2

Fuzzy Extension Principle- Non Monotonic Continuous

- Function: $y=f(x)=x^2-6x+11$
- Input: Fuzzy number - around-4

Around-4 = 0.3/2+0.6/3+1/4+0.6/5+0.3/6

y = 0.3/f(2)+0.6/f(3)+1/f(4)+0.6/f(5)+0.3/f(6)

y = 0.3/3+0.6/2+1/3+0.6/6+0.3/11

y = 0.6/2+(0.3 v 1)/3+0.6/6+0.3/11

y = 0.6/2 + 1/3 + 0.6/6 + 0.3/11

Fuzzy Numbers

- **Definition :**
- It is a fuzzy set with the following conditions :
  - convex fuzzy set
  - normalized fuzzy set
  - its membership function is piecewise continuous.
  - It is defined in the real number

Arithmetic Operation on Fuzzy Numbers

Applying the extension principle to arithmetic operations, we have

Fuzzy Addition: $$\mu_{A+B}(z) = \bigoplus_{\substack{x,y \\ x+y=z}} \mu_A(x) \otimes \mu_B(y)$$

Fuzzy Subtraction: $$\mu_{A-B}(z) = \bigoplus_{\substack{x,y \\ x-y=z}} \mu_A(x) \otimes \mu_B(y)$$

Fuzzy Multiplication: $$\mu_{A\times B}(z) = \bigoplus_{\substack{x,y \\ x\times y=z}} \mu_A(x) \otimes \mu_B(y)$$

Fuzzy Division: $$\mu_{A/B}(z) = \bigoplus_{\substack{x,y \\ x/y=z}} \mu_A(x) \otimes \mu_B(y)$$

Arithmetic Operation on Fuzzy Numbers-Addition

Given two fuzzy sets A and B, Compute F(A+B)

$A = 0.3/1 + 0.6/2 + 1/3 + 0.7/4 + 0.2/5$

$B = 0.5/10 + 1/11 + 0.5/12$

$$F(A+B) = 0.3/11 + 0.5/12 + 0.5/13 + 0.5/14 + 0.2/15 + 0.3/12 + 0.6/13 + 1/14 + 0.7/15 + 0.2/16 + 0.3/13 + 0.5/14 + 0.5/15 + 0.5/16 + 0.2/17$$

Get max of the duplicates,

$$F(A+B) = 0.3/11 + 0.5/12 + 0.6/13 + 1/14 + 0.7/15 + 0.5/16 + 0.2/17$$

Arithmetic Operation on Fuzzy Numbers-Addition

- Given two fuzzy sets A and B, Compute F(A+B)

$$A = \left\{\frac{0.7}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.6}{5}\right\} \qquad B = \left\{\frac{0.2}{6} + \frac{0.8}{7} + \frac{1}{8} + \frac{0.5}{9}\right\}$$

F(A+B)= 0.2/8 + 0.7/9 + 0.8/10 + 0.8/11 + 1/12 + 0.6/13 + 0.5/14

Arithmetic Operation on Fuzzy Numbers- Subtraction

Arithmetic Operations on Fuzzy Numbers through Extension Principle

Fuzzy Subtraction: $\mu_{A(-)B}(z) = \bigvee_{z=x-y} (\mu_A(x) \wedge \mu_B(y))$

Arithmetic Operation on Fuzzy Numbers- Subtraction

Given two fuzzy sets A and B, Compute F(A-B)

$$A = \left\{ \frac{0.5}{3} + \frac{1}{4} + \frac{0.6}{5} \right\} \qquad B = \left\{ \frac{0.5}{4} + \frac{1}{5} + \frac{0.3}{6} \right\}$$

F(A-B)= {0.3/-3 + 0.5/-2 + 1/-1 + 0.6/0 + 0.5/1}

Arithmetic Operation on Fuzzy Numbers- Multiplication & Division

Arithmetic Operations on Fuzzy Numbers through Extension Principle

Fuzzy Multiplication: $\mu_{A(*)B}(z) = \bigvee_{z=x*y} (\mu_A(x) \wedge \mu_B(y))$

Fuzzy Division: $\mu_{A(/)B}(z) = \bigvee_{z=x/y} (\mu_A(x) \wedge \mu_B(y))$

Membership function & its types

Give different membership functions of fuzzy logic?

MU- Dec 2019 5marks

Membership Function

- **Definition:**
- A membership function for a fuzzy set A on the universe of discourse X is defined as $\mu_A: X \rightarrow [0,1]$, where each element of X is mapped to a value between 0 and 1.
- This value, called membership value or degree of membership, quantifies the grade of membership of the element in X to the fuzzy set A.

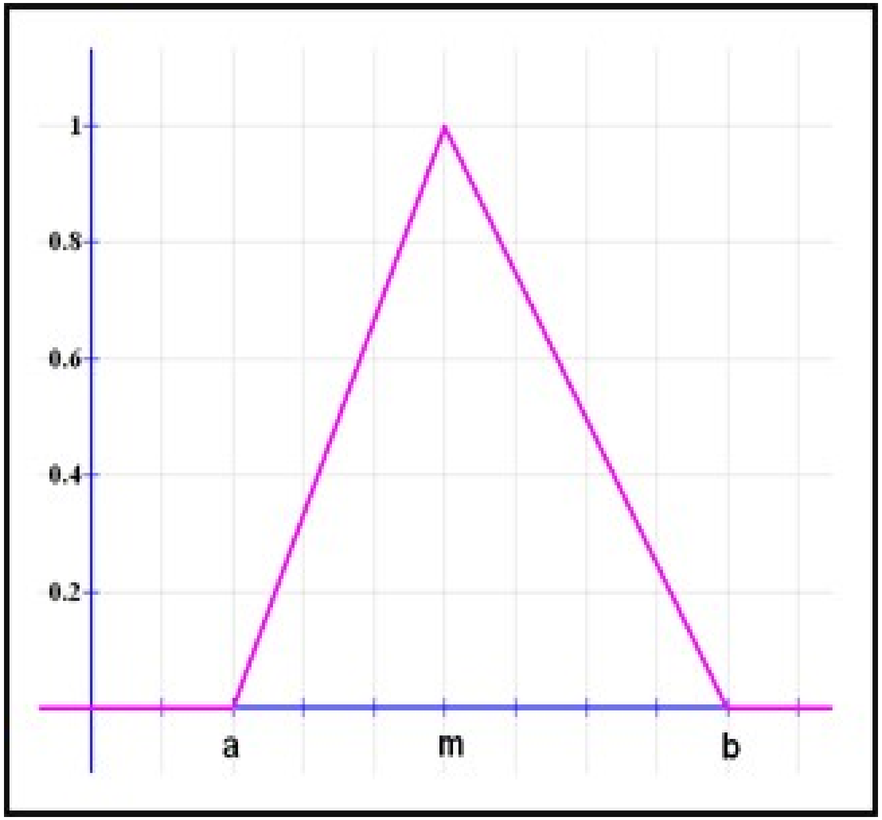
Membership Function

- Membership functions allow us to graphically represent a fuzzy set.
- The *x* axis represents the universe of discourse, whereas the *y* axis represents the degrees of membership in the [0,1] interval.
- Simple functions are used to build membership functions.

Membership Function- Triangular

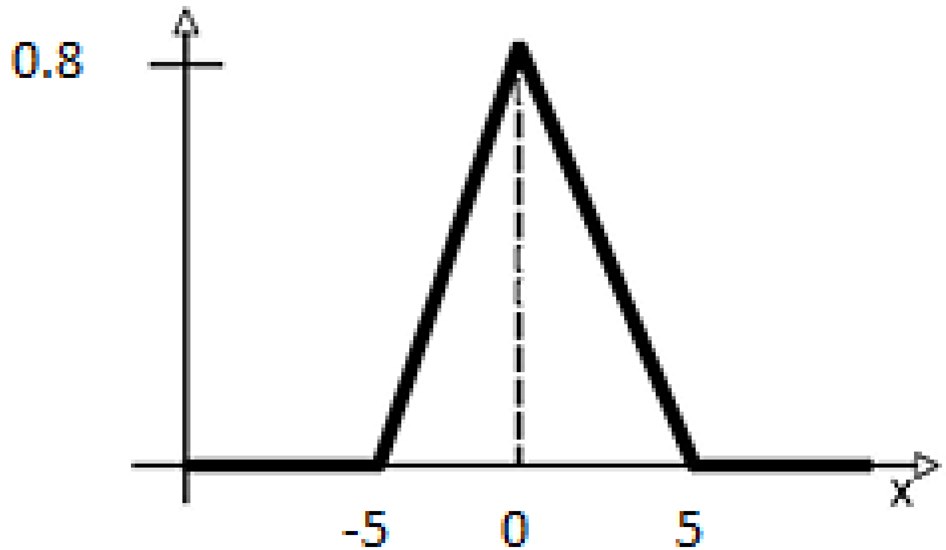
- **Triangular function**: defined by a lower limit **a**, an upper limit **b**, and a value **m**, where **a < m < b**.

$$\mu_A(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{m-a}, & a < x \le m \\ \dfrac{b-x}{b-m}, & m < x < b \\ 0, & x \ge b \end{cases}$$



Membership Function- Triangular

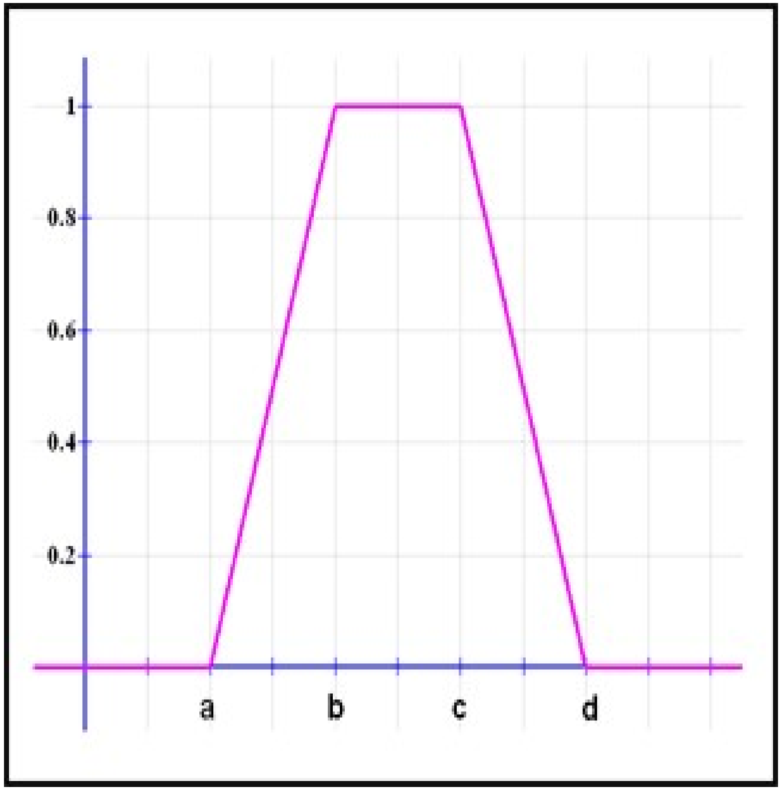
- Suppose the room temperature varies from -5°C to +5°C. If the membership function is defined as follows, find out the expression for it



Membership Function- Trapezoidal

- It is defined by a lower limit **a**, an upper limit **d**, a lower support limit **b**, and an upper support limit **c**,
- where **a < b < c < d**.

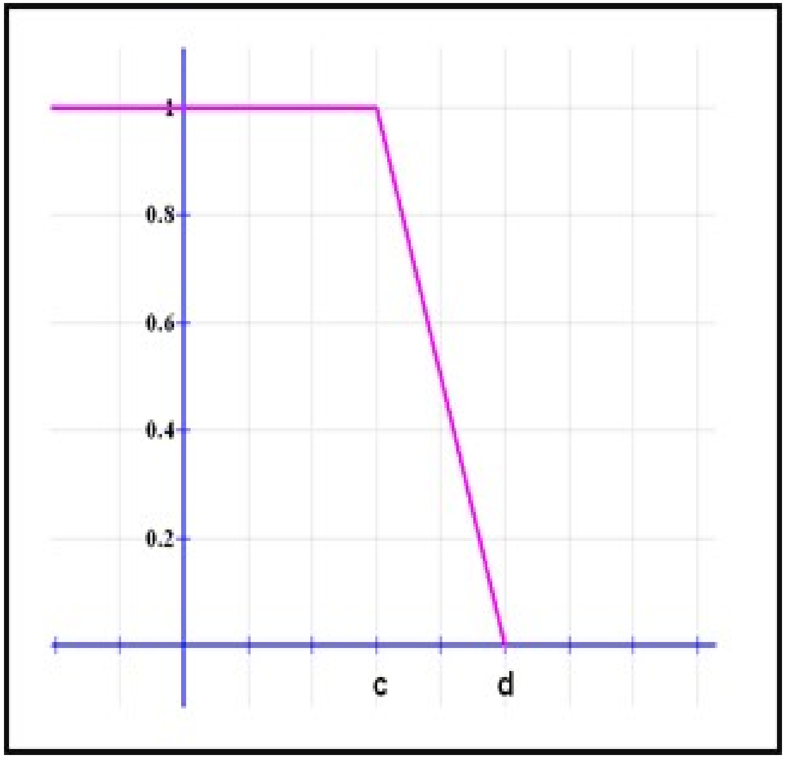
$$\mu_A(x) = \begin{cases} 0, & (x < a) \text{ or } (x > d) \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \end{cases}$$



Membership Function- Right

- It is the right trapezoidal function
- It is specified by two parameters c and d

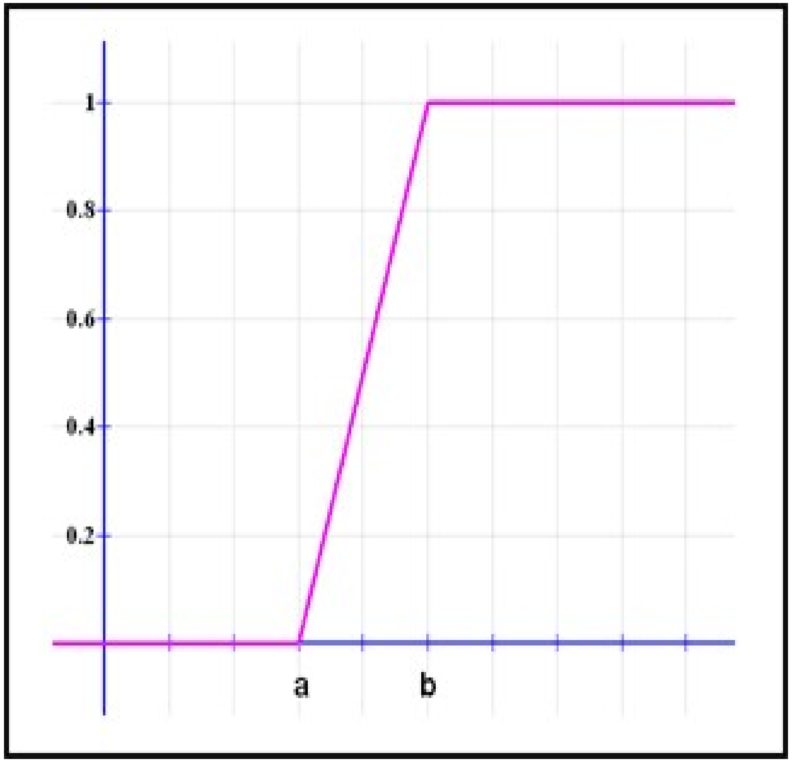
$$\mu_A(x) = \begin{cases} 0, & x > d \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 1, & x < c \end{cases}$$



Membership Function- Left

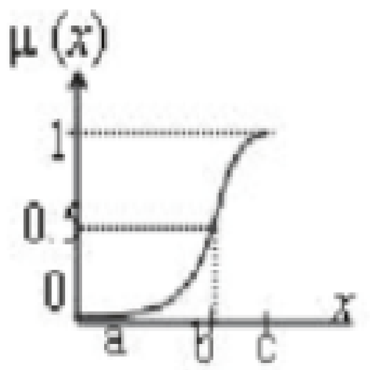
- It is the left trapezoidal function
- It is specified by two parameters a and b

$$\mu_A(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

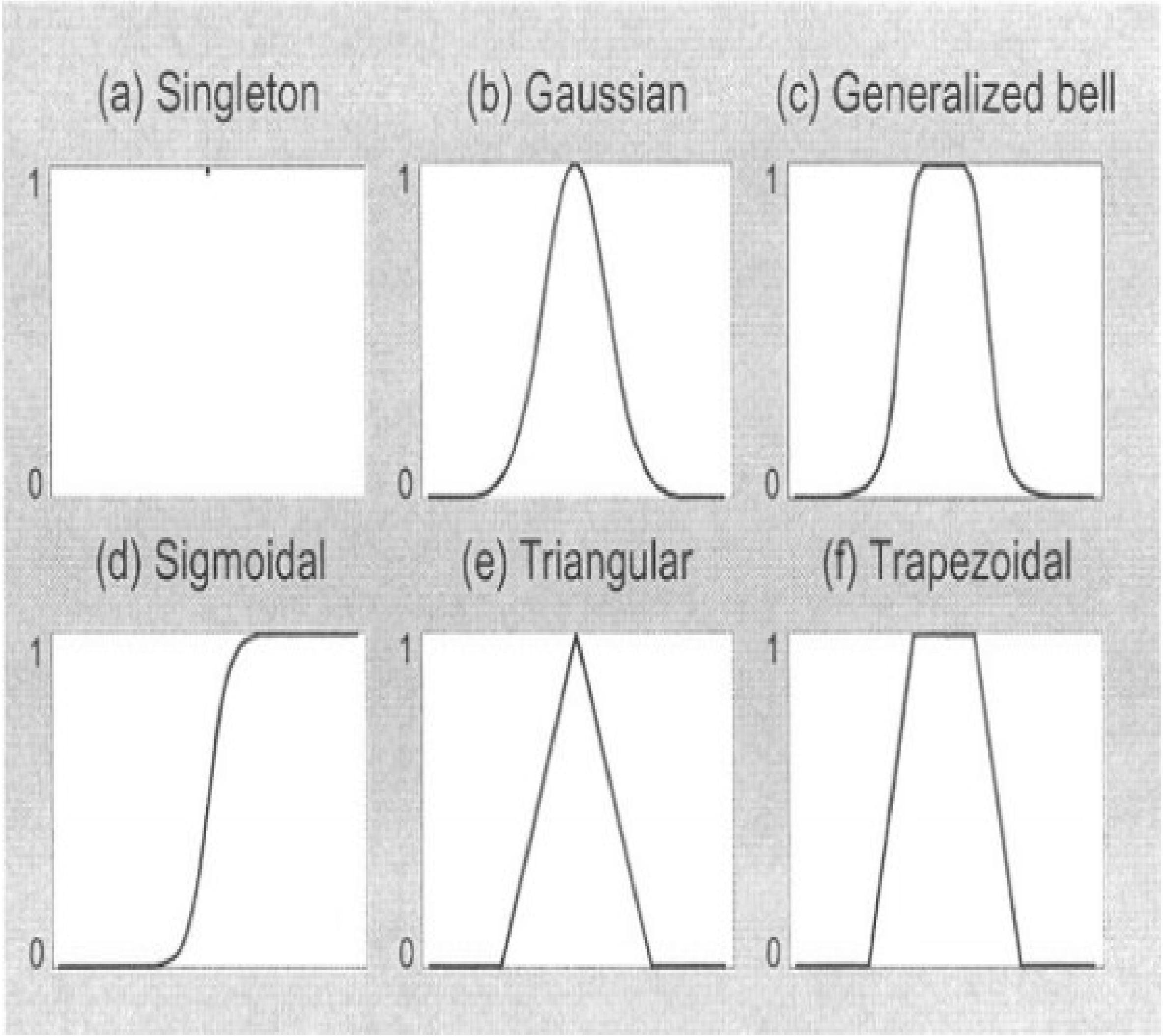


Membership Function- Other

- Guassian Function
- Bell shaped Function
- Sigmoidal Function
- S shaped Function
- Z shaped Function



S-shape membership function.



Fuzzy Logic

Basics of Fuzzy Logic

- **Definition of fuzzy**
  - Fuzzy – “not clear, distinct, or precise; blurred”
- **Definition of fuzzy logic**
  - A form of knowledge representation suitable for notions that cannot be defined precisely, but which depend upon their contexts.

Traditional representation of logic

Slow

Fast

Speed = 0

Speed = 1

```
bool speed;
get the speed
if ( speed == 0) {
//   speed is slow
}
else {
//   speed is fast
}
```

Fuzzy Logic representation

- For every problem must represent in terms of fuzzy sets.
- What are fuzzy sets?

Slowest

[0.0 – 0.25]

Slow

[0.25 – 0.50]

Fast

[0.50 – 0.75]

Fastest

[0.75 – 1.00]

Fuzzy Logic representation


```
float speed;
get the speed
if ((speed >= 0.0)&&(speed < 0.25)) {
//  speed is slowest
}
else if ((speed >= 0.25)&&(speed < 0.5))
{
//  speed is slow
}
else if ((speed >= 0.5)&&(speed < 0.75))
{
//  speed is fast
}
else // speed >= 0.75 && speed < 1.0
{
//  speed is fastest
}
```

Fuzzy Logic representation

- Fuzzy logic is conceptually easy to understand.
- Fuzzy logic is flexible.
- Fuzzy logic is tolerant of imprecise data.
- Fuzzy logic can model nonlinear functions of arbitrary complexity.
- Fuzzy logic can be built on top of the experience of experts.
- Fuzzy logic is based on natural language.
- Fuzzy logic can be blended with conventional control techniques.

Fuzzy Rules & Fuzzy Reasoning

Fuzzy Rules & Reasoning

- Fuzzy logic provides an alternative way to represent linguistic and subjective attributes of the real world in computing.
- It is able to be applied to control systems and other applications in order to improve the efficiency and simplicity of the design process.

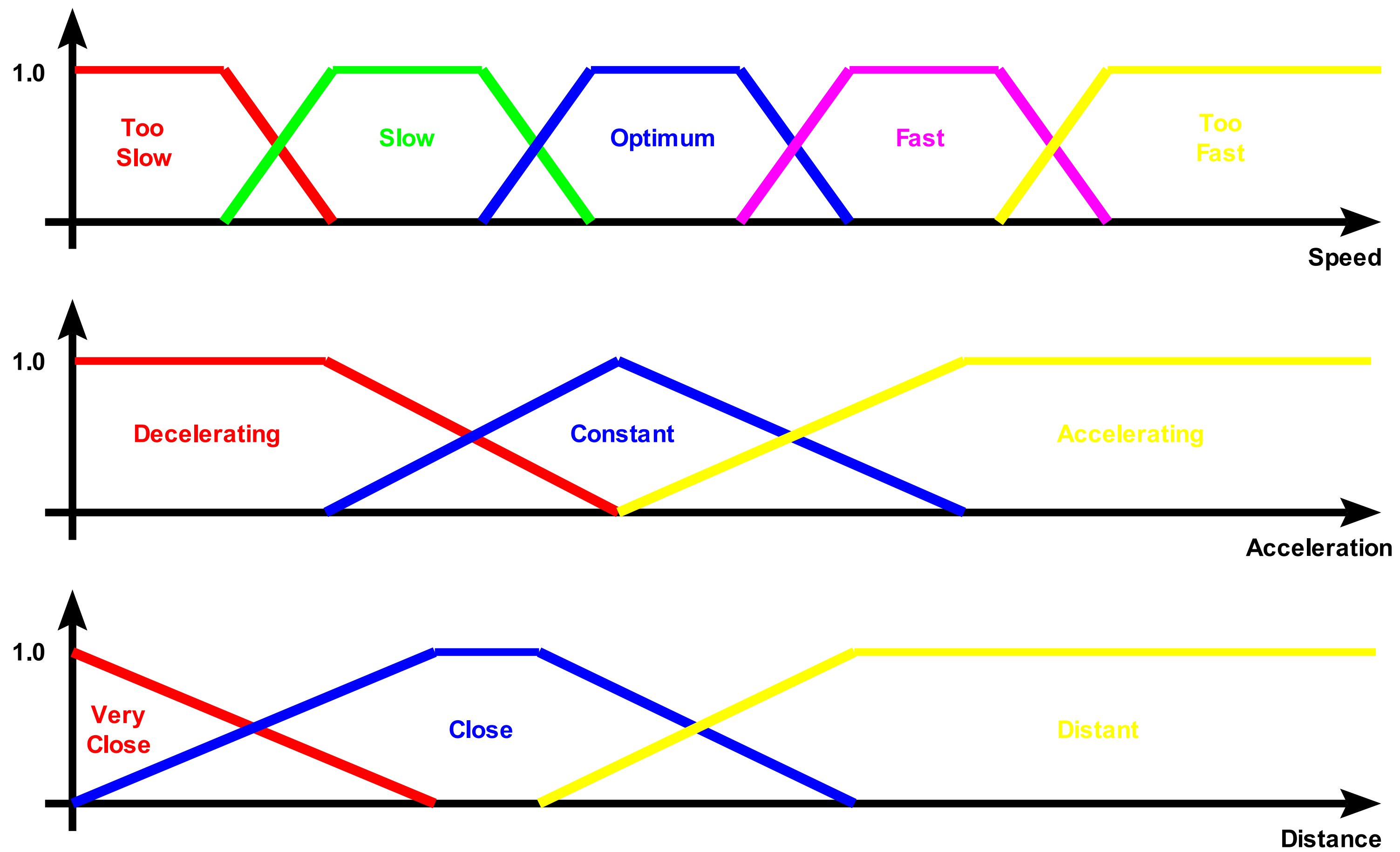
Fuzzy Rules & Reasoning -Example

- Fuzzy Logic provides a more efficient and resourceful way to solve Control Systems.
- **Some Examples**
  - Automotive Speed Controller
  - Temperature Controller
  - Anti – Lock Break System (ABS)

Fuzzy Rules & Reasoning- Example 1

- **The problem**
  - *Automotive Speed Controller*
  - **Input: Speed (5 levels), acceleration (3 levels), distance to destination (3 levels)**
- **Output:** power (fuel flow to engine)
- Using this we can define the fuzzy set rules to determine output based on input values

Fuzzy Rules & Reasoning -Example 1



Fuzzy Rules & Reasoning -Example 1

Example Rules

IF speed is TOO SLOW and acceleration is DECELERATING,
THEN INCREASE POWER GREATLY

IF speed is SLOW and acceleration is DECREASING,
THEN INCREASE POWER SLIGHTLY

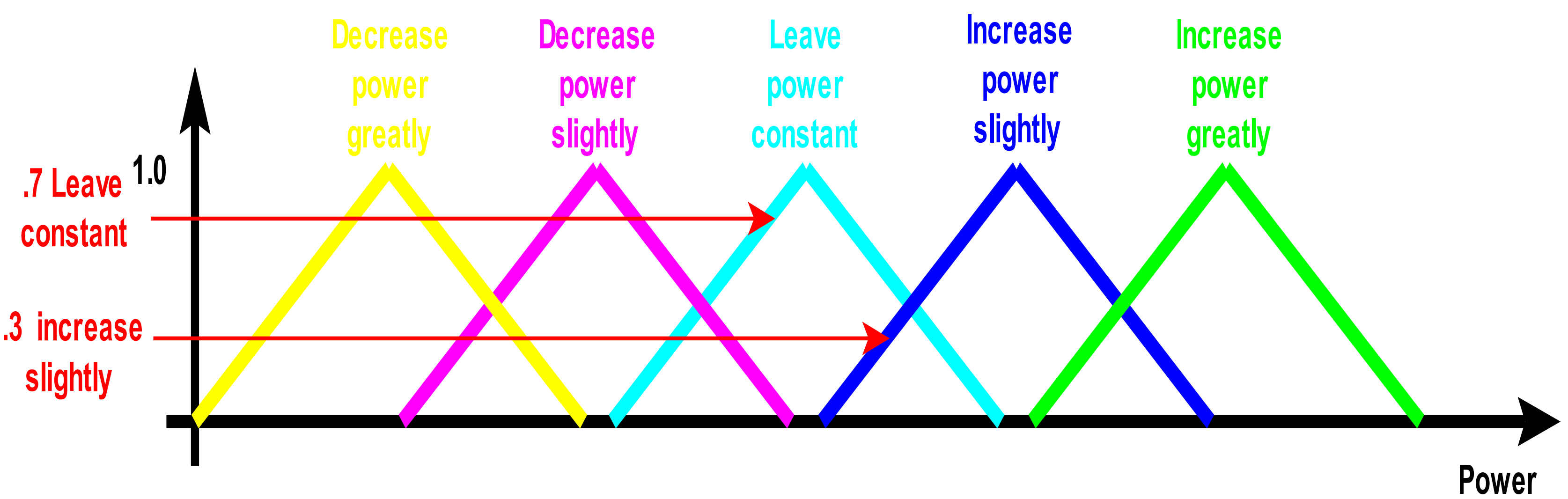
IF distance is CLOSE,
THEN DECREASE POWER SLIGHTLY

. . .

Output Determination

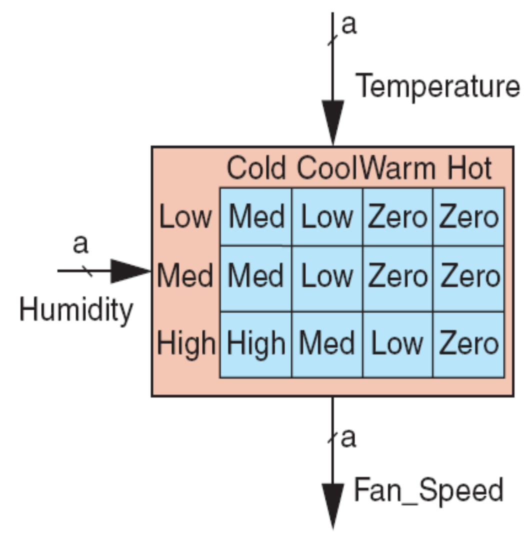
Degree of membership in an output fuzzy set now represents each fuzzy action.

Fuzzy actions are combined to form a system output.

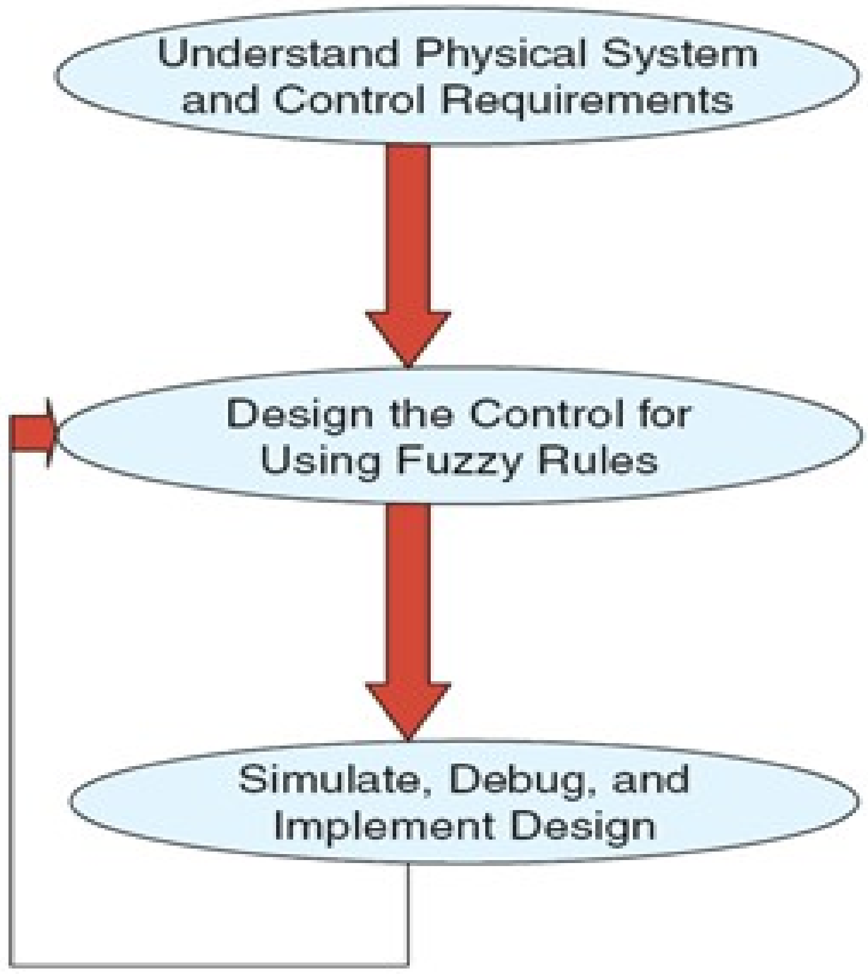


Fuzzy Rules & Reasoning- Example 2

- **The problem**
  - *Change the speed of a heater fan, based off the room temperature and humidity.*
- **Input: A temperature control system has four settings**
  - **Cold, Cool, Warm, and Hot**
- **Input: Humidity can be defined**
  - **Low, Medium, and High**
- **Output:** Fan Speed
- Using this we can define the fuzzy set rules to determine output based on input values as shown in Fig.



Fuzzy Rules & Reasoning- Benefits



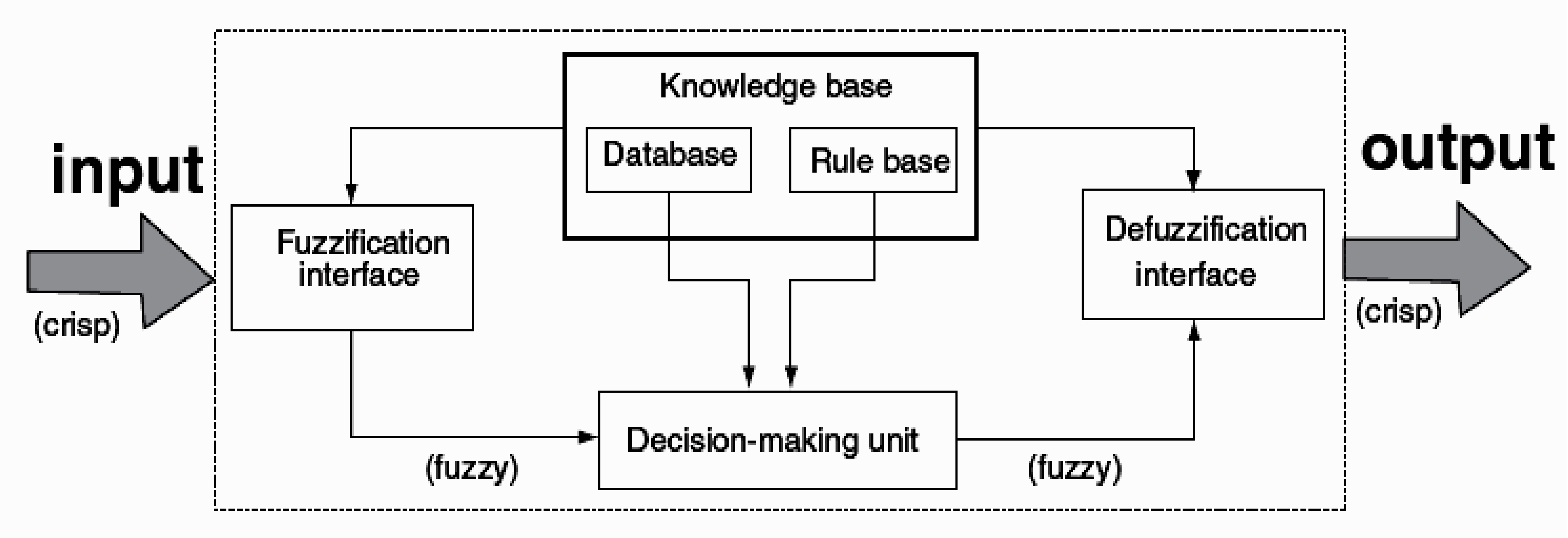
Fuzzy Inference Systems

- Fuzzification of input variables
- De-fuzzification

Fuzzy Inference

- Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic.
- FIS uses a collection of fuzzy membership functions and rules
- **If then rules**
  - *if temperature is cold then hot water valve is open and cold water valve is shut*
- **Rule Base**
  - If the distance to intersection (dti) is *far and the speed slow apply gentle breaks*
  - If dti is *near and the speed slow apply medium breaks*
  - If dti is far *and speed fast apply medium breaks*
  - If dti is *near and speed fast apply high breaks*

Fuzzy Inference- Block diagram



Fuzzy inference system consists of

1. a fuzzification interface
2. a rule base,
3. a database,
4. a decision-making unit and
5. a defuzzification interface

Fuzzy Inference- Block diagram working

1. **Rule base :** contains a number of fuzzy IF–THEN rules
2. **Database :** defines the membership functions of the fuzzy sets used in the fuzzy rules
3. **Decision-making unit :** performs the inference operations on the rules.
4. **Fuzzification interface :** transforms the crisp inputs into fuzzy values
5. **Defuzzification interface :** transforms the fuzzy results of the inference into a crisp output

Fuzzy Inference- Construction

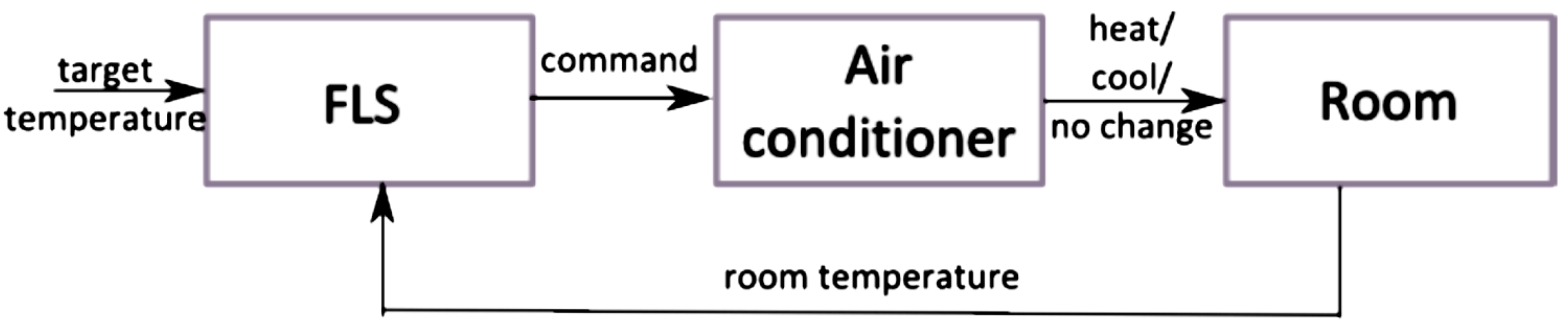
- Firstly, a crisp input is converted to fuzzy
- Then an inference is made based on a set of rules.
- Lastly, Fuzzy output is mapped to a crisp output

Steps in designing a Fuzzy Inference system

1. Define the linguistic variables and terms (initialization)
2. Construct the membership functions (initialization)
3. Construct the rule base (initialization)
4. Fuzzification
5. Inference (Evaluate the rules in the rule base and combine the results of each rule)
6. De-fuzzification

Fuzzy Inference- Example 1

- Let us consider an air conditioner system controlled by a FIS.
- **Input** :current temperature of the room and target temperature
- **Output :** command to heat or cool the room.

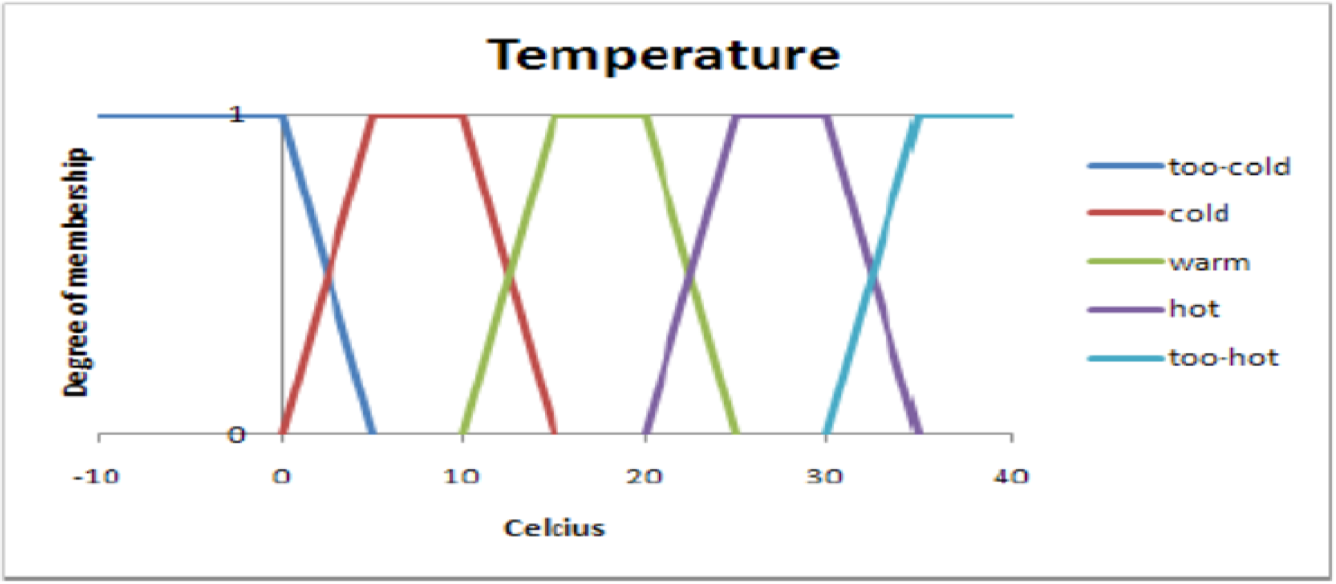


Fuzzy Inference- Step 1-Linguistic variables

- Consider the air conditioner FIS.
- E.g. Temperature (t) be the linguistic variable
- **Linguistic values : “hot"** and **“cold"**
- After decomposition,
- **T(t) = {too-cold, cold, warm, hot, too-hot}**
- Output : heat/cool/no-change

Fuzzy Inference- Step 2 Membership function

- Membership functions are used in the fuzzification and defuzzification
- A membership function is used to quantify a linguistic term.
- The membership functions for the linguistic terms of temperature variable are plotted as follows.



Fuzzy Inference- Step 3 Rule base

	Fuzzy Rules
1.	IF (temperature is *cold* OR *too-cold*) AND (target is *warm*) THEN command is *heat*
2.	IF (temperature is *hot* OR *too-hot*) AND (target is *warm*) THEN command is *cool*
3.	IF (temperature is *warm*) AND (target is *warm*) THEN command is *no-change*

Fuzzy Inference- Step 3 Rule base

Another way to represent rule base is the matrix representation of the fuzzy rules

temperature/target	too-cold	cold	warm	hot	too-hot
too-cold	no-change	heat	heat	heat	heat
cold	cool	no-change	heat	heat	heat
warm	cool	cool	no-change	heat	heat
hot	cool	cool	cool	no-change	heat
too-hot	cool	cool	cool	cool	no-change

For instance, the cell (2, 3) in the matrix can be read as follows: If temperature is cold and target is warm then command is heat.

Fuzzy Inference

Step 4 Fuzzification

Converts the crisp input to a linguistic variable using the membership functions stored in the fuzzy knowledge base

Step 5 Inference

- It evaluates the fuzzy rules and then combine the results of the individual rules
- Performed using fuzzy set operations.
- Mostly used are OR and AND
- The results can be combined in different ways.

Step 6 Defuzzification

- After the inference step, the overall result is a fuzzy value.
- This result is defuzzified to obtain a final crisp output.

Fuzzification of input variables

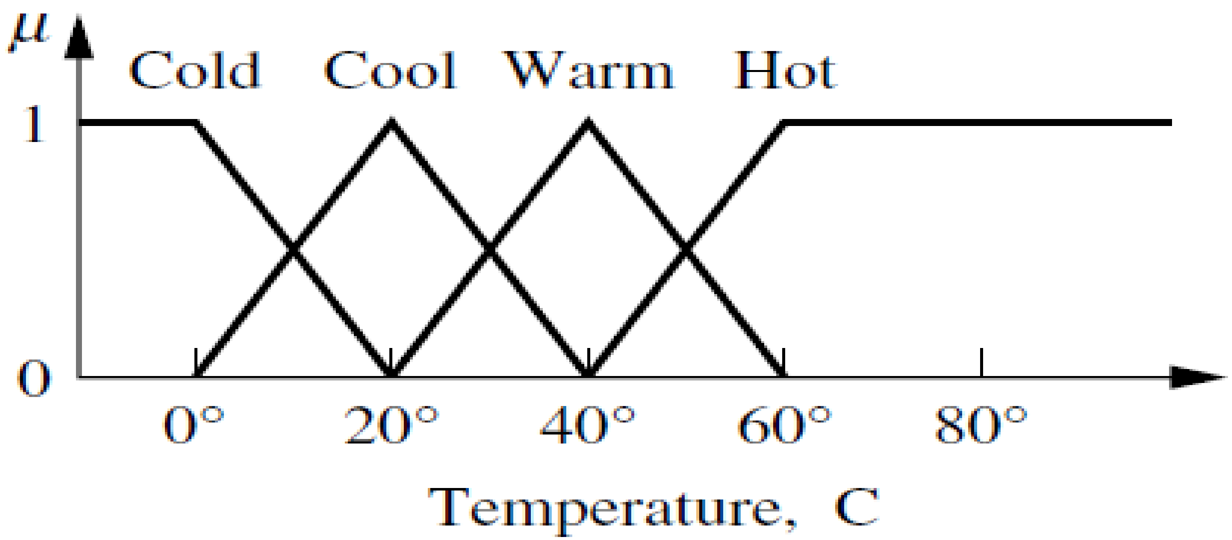
- It involves domain transformation where crisp inputs are converted into fuzzy inputs
- The conversion is represented by membership functions
- Fuzzification is assigning membership values for given crisp quantities

Different methods of Fuzzification

1. Intuition
2. Inference
3. Rank ordering
4. Neural networks
5. Genetic algorithms
6. Inductive reasoning

Different methods of Fuzzification- Intuition

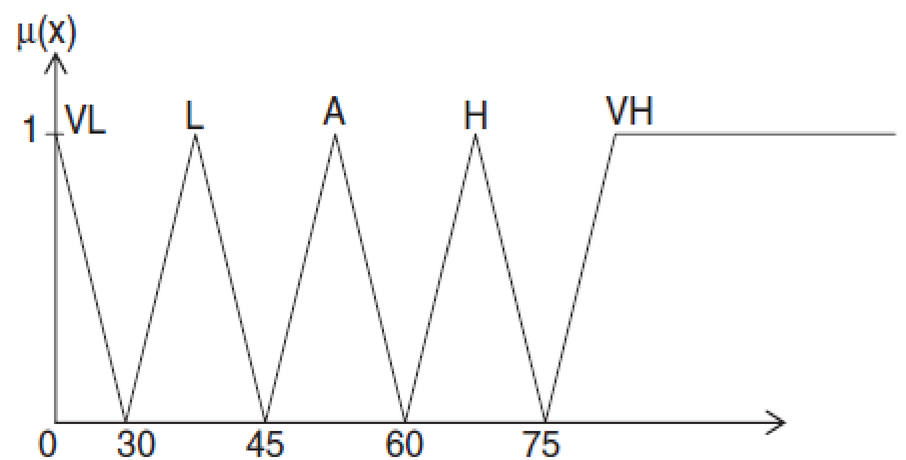
- It is simply derived from the capacity of humans to develop membership functions through their own innate intelligence and understanding.
- As an example, consider the membership functions for the fuzzy variable temperature.



Different methods of Fuzzification- Intuition

Using your own intuition and definitions of the universe of discourse, plot fuzzy membership functions for "**weight of people**."

- The universe of discourse is the weight of people. Let the weights be in "kg" – kilogram.
- **Let the linguistic variables are:**
  - Very light – $w \leq 30$
  - Light – $30 < w \leq 45$
  - Average – $45 < w \leq 60$
  - Heavy – $60 < w \leq 75$
  - Very heavy – $w > 75$



Different methods of Fuzzification- Intuition

Using your own intuition and definitions of the universe of discourse, plot fuzzy membership functions for "**Age of people**."

- Let A denotes age in years.
- **The linguistic variables are defined as,**

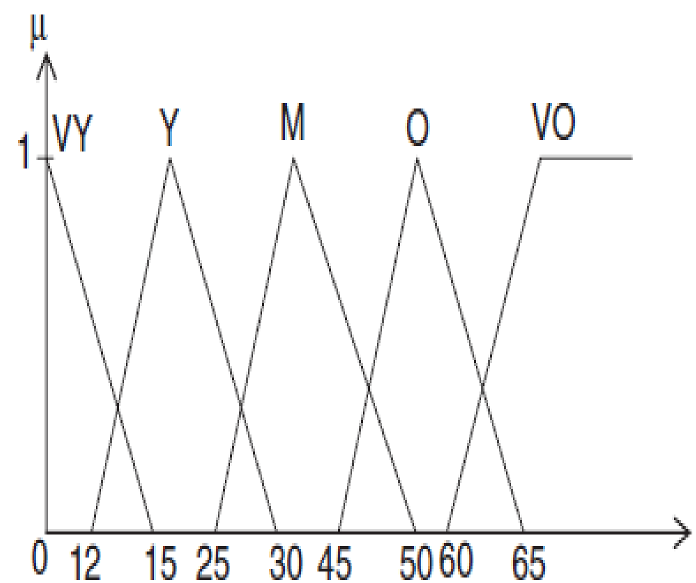
(1) Very Young (VY) – $A < 15$

(2) Young (Y) – $12 < A < 30$

(3) Middle aged (M) – $25 < A < 50$

(4) Old (O) – $45 < A < 65$

(5) Very Old (VO) – $A > 60$



Different methods of Fuzzification- Inference

- This method involves deductive reasoning.
- The membership function is formed from the facts known and knowledge.
- Let us use inference method for the identification of the triangle.
- Let U be universe of triangles and A,B, and C be the inner angles of the triangles. Also $A \geq B \geq C \geq 0$.
- Therefore the universe is given by:

$$U = \{(A, B, C), A \geq B \geq C \geq 0, A + B + C = 180^\circ\}.$$

Different methods of Fuzzification- Inference

- There are various types of triangles, for identifying, we define three types of triangles:
  - I = Approximate isosceles triangle
  - R = Approximate right triangle
  - O = Other triangles
- The membership for the approximate isosceles triangle is given as

$$\mu_{\underset{\sim}{I}}(A,B,C) = 1 - \frac{1}{60^\circ}\min(A-B, B-C).$$

Different methods of Fuzzification- Inference

- The membership for the approximate right triangle is

$$\mu_{\underset{\sim}{R}}(A,B,C) = 1 - \frac{1}{90^\circ}(A - 90^\circ)$$

- The membership for the other triangles can be given as the complement of the logical union of the two already defined membership functions

$$\mu_{\underset{\sim}{O}}(A,B,C) = \overline{\underset{\sim}{I} \cup \underset{\sim}{R}}$$

- using Demorgans law,

$$\mu_{\underset{\sim}{O}}(A,B,C) = \underset{\sim}{I} \cap \underset{\sim}{R} = \min\left\{\begin{matrix} 1 - \mu_{\underset{\sim}{I}}(A,B,C) \\ 1 - \mu_{\underset{\sim}{R}}(A,B,C) \end{matrix}\right\}$$

Different methods of Fuzzification- Rank Ordering

- Preferences by a single individual, a committee, a poll, and other opinion methods are used to assign membership values to a fuzzy variable.
- Preference is determined by pairwise comparisons, and these determine the ordering of the membership.

Different methods of Fuzzification- Rank Ordering

- Suppose 1000 people respond to a questionnaire about their pairwise preferences among five colors, X = {red, orange, yellow, green, blue}.
- Define a fuzzy set as A on the universe of colors "best color."

	Number who preferred				
	Red	Orange	Yellow	Green	Blue
Red	–	517	525	545	661
Orange	483	–	841	477	576
Yellow	475	159	–	534	614
Green	455	523	466	–	643
Blue	339	424	386	357	–
Total					

Different methods of Fuzzification- Rank Ordering

Problem:

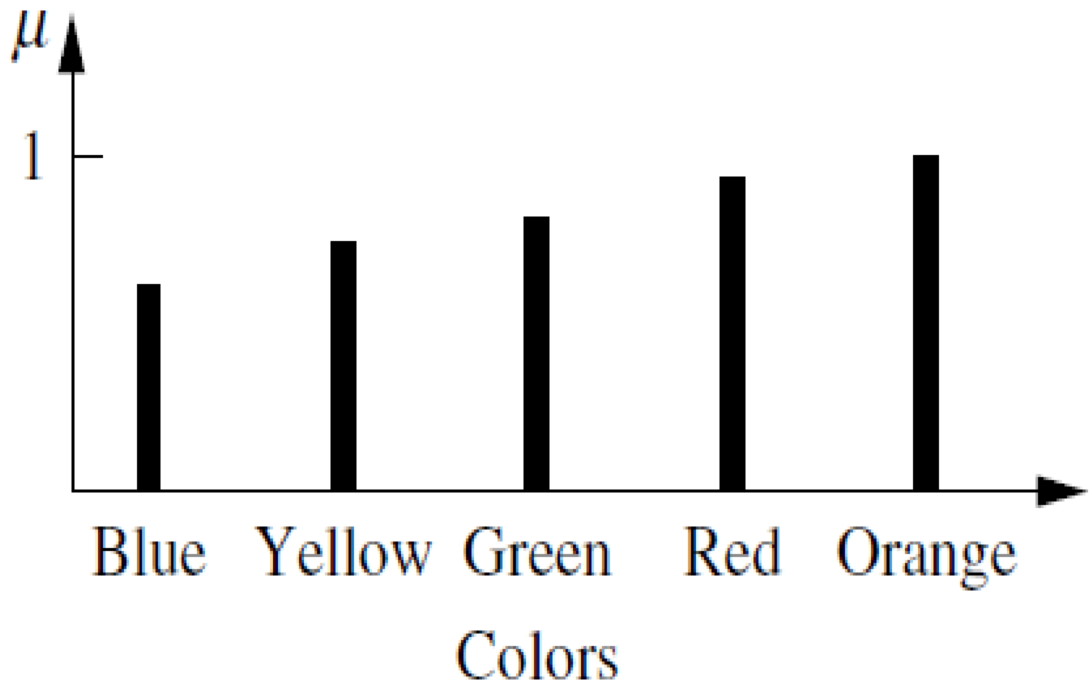
	Number who preferred				
	Red	Orange	Yellow	Green	Blue
Red	–	517	525	545	661
Orange	483	–	841	477	576
Yellow	475	159	–	534	614
Green	455	523	466	–	643
Blue	339	424	386	357	–
Total					

Solution:

	Number who preferred							
	Red	Orange	Yellow	Green	Blue	Total	Percentage	Rank order
Red	–	517	525	545	661	2248	22.5	2
Orange	483	–	841	477	576	2377	23.8	1
Yellow	475	159	–	534	614	1782	17.8	4
Green	455	523	466	–	643	2087	20.9	3
Blue	339	424	386	357	–	1506	15	5
Total						10,000		

Different methods of Fuzzification- Rank Ordering

- The normalized values are
- R – 0.945
- O - 1.0
- Y – 0.747
- G – 0.878
- B – 0.630



Different methods of Fuzzification- Neural Network

We have the data sets for inputs and outputs, the relationship between I/O may be highly nonlinear or not known.

We can classify them into different fuzzy classes.

Then, the output may not only be 0 or 1!

Different methods of Fuzzification- Neural Network

R1	0	0.2
R2	0	0.7
R3	0	0.1

memberships

Once the neural network is trained and tested, it can be used to find the membership of any other data points in the fuzzy classes (# of outputs)

Different methods of Fuzzification- Genetic Algorithms

Crossover

Mutation random selection

Reproduction

Chromosomes

Fitness Function

Stop (terminate conditions)

Converge

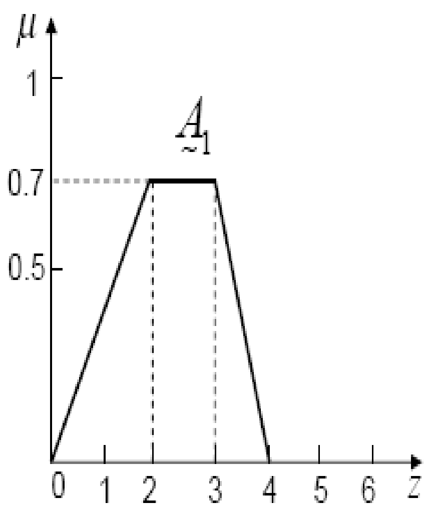
Reach the #limit

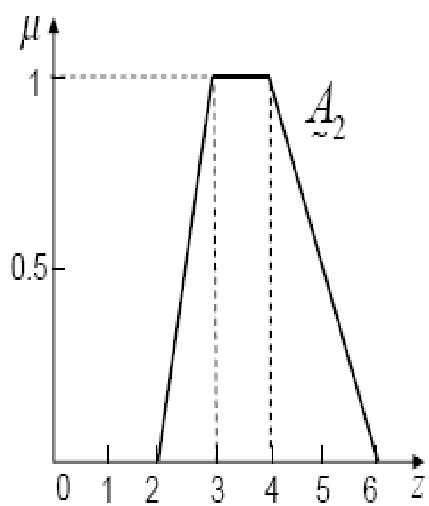
Different methods of Fuzzification- Inductive Reasoning

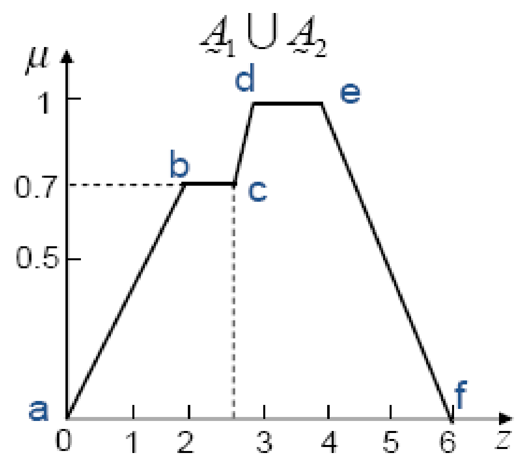
- Inductive reasoning is **a logical thinking process in which multiple premises that are believed to be true are combined to draw a conclusion**.
- It is a process that works in the opposite direction to deductive reasoning.
- **Example of Inductive Reasoning:**
    - **Most of our snowstorms come from the north**. It's starting to snow. This snowstorm must be coming from the north. Deductive Reasoning: All of our snowstorms come from the north.
- The induction is performed by the entropy minimization principle which clusters most optimally the parameters corresponding to the output classes.

De-Fuzzification

- Conversion of a fuzzy quantity to a precise quantity
- The output of a fuzzy process can be the logical union of 2 or more fuzzy membership function
- Suppose a fuzzy output comprises of two part A_1 and A_2. Then A_1 U A_2 involves max operator, which graphically is the outer envelope of the shape





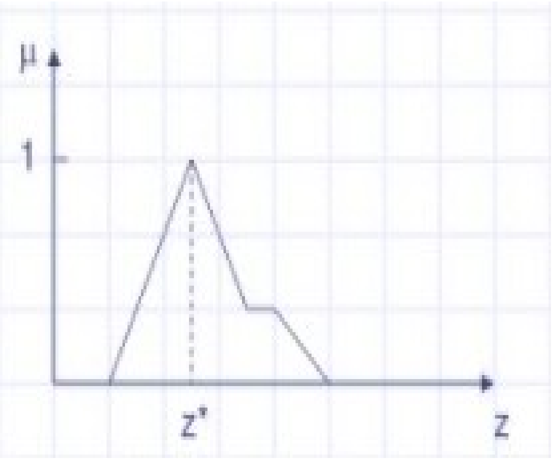


Different methods of De-Fuzzification

1. Max membership principle
2. Centroid method
3. Weighted Average Method
4. Mean Max membership
5. Center of sums
6. Centre of largest area
7. First of the Maxima
8. Last of the Maxima

Different methods of DeFuzzification- 1. Max membership principle

- Limited to peaked output functions
- Also referred as height method
- Applicable when height is unique
- It is given by the expression



$$\mu_A(z^*) \geq \mu_A(z) \qquad \forall z \in Z$$

where z* is the de-fuzzified value

Different methods of DeFuzzification- 1. Max membership principle

Example:

- Suppose, a fuzzy set Young is defined as follows:
- Young = {(15,0.5), (20,0.9), (25,0.8), (30,0.5), (35,0.3) }
- Then the crisp value of Young using Height method is 20
- Thus, a person of 20 years old is treated as young!

Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

- This method provides a crisp value based on the center of gravity of the fuzzy set.
- The total area of the membership function distribution used to represent the combined control action is divided into a number of sub-areas.
- The area and the center of gravity or centroid of each sub-area is calculated and then the summation of all these sub-areas is taken to find the de-fuzzified value for a discrete fuzzy set.

For discrete membership function, the defuzzified value denoted as x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum_{i=1}^{n} \mu(x_i)}$$

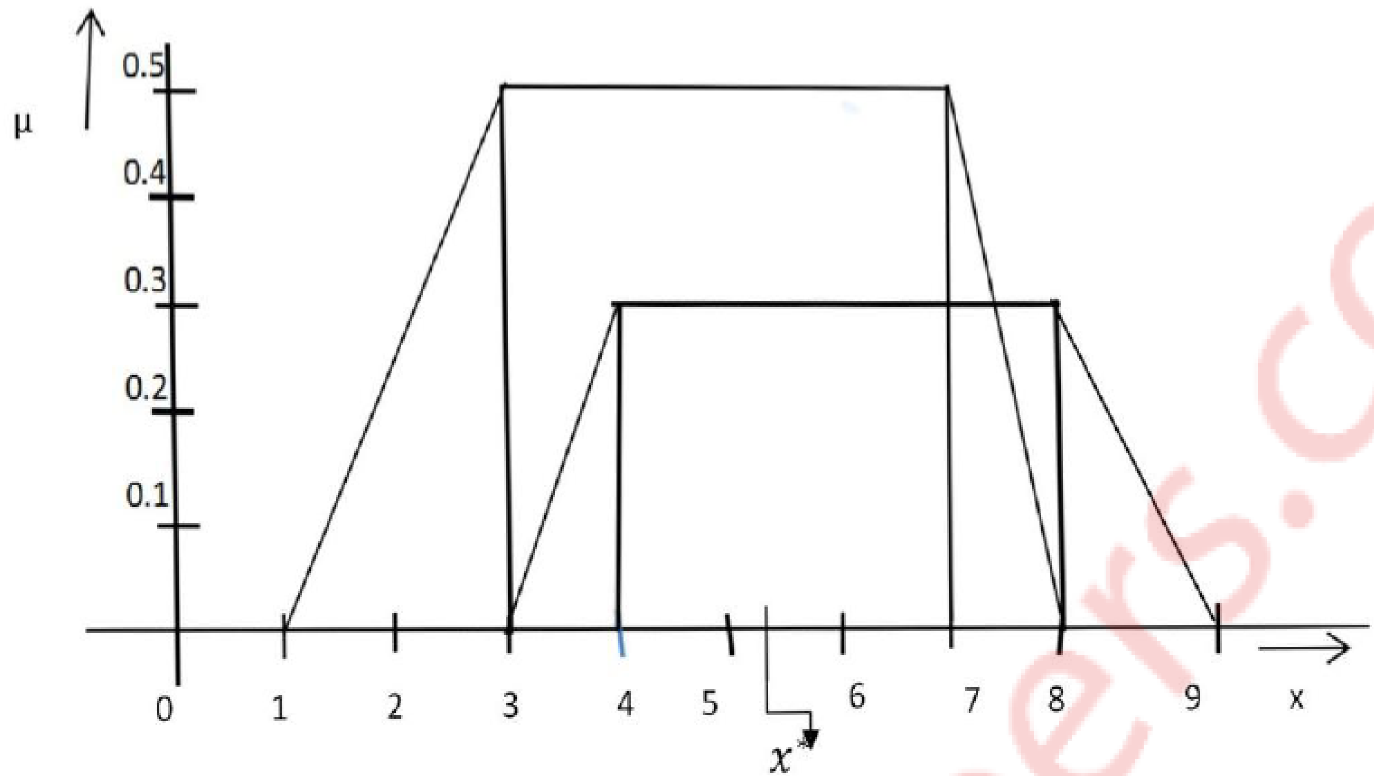
, Here x_i indicates the sample element, $\mu(x_i)$ is the membership function, and n represents the number of elements in the sample.

For continuous membership function, x^* is defined as :

$$x^* = \frac{\int x \, \mu_A(x) \, dx}{\int \mu_A(x) \, dx}$$

Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

(a) Explain defuzzification techniques. Apply defuzzification by using Center of Gravity (CoG) method on the following: [10]



MU asked Question Dec 2019

Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

Example:

The defuzzified value x^* using COG is defined as:

$$x^* = \frac{\sum_{i=1}^{N} A_i \times \bar{x_i}}{\sum_{i=1}^{N} A_i}$$

, Here N indicates the number of sub-areas, A_i and $\bar{x_i}$ represents the area and centroid of area, respectively, of i^{th} sub-area.

In the aggregated fuzzy set as shown in figure 2. , the total area is divided into six sub-areas.
For COG method, we have to calculate the area and centroid of area of each sub-area.
These can be calculated as below.
The total area of the sub-area 1 is ½ * 2 * 0.5 = 0.5
The total area of the sub-area 2 is (7-3) * 0.5 = 4 * 0.5 = 2
The total area of the sub-area 3 is ½ * (7.5-7) * 0.2 = 0.5 * 0.5 *0.2 =.05
The total area of the sub-area 4 is 0.5* 0.3 = .15
The total area of the sub-area 5 is 0.5* 0.3 = .15
The total area of the sub-area 6 is ½ *1* 0.3 = .15
Now the centroid or center of gravity of these sub-areas can be calculated as

Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

Centroid of sub-area1 will be (1+3+3)/3 = 7/3 =2.333
Centroid of sub-area2 will be (7+3)/2 = 10/2 = 5
Centroid of sub-area3 will be (7+7+7.5)/3 = 21.5/3 =7.166
Centroid of sub-area4 will be (7+7.5)/2 =14.5/2=7.25
Centroid of sub-are5 will be (7.5+8)/2 =15.5/2 = 7.75
Centroid of sub-area6 will be (8+8+9)/3 = 25/3 = 8.333
Now we can calculate $A_i \cdot \overline{x_i}$ and is shown in table 1.

Table 1

Sub-area number	Area(A_i)	Centroid of area($\overline{x_i}$)	$A_i \cdot \overline{x_i}$
1	0.5	2.333	1.1665
2	02	5	10
3	.05	7.166	0.3583
4	.15	7.25	1.0875
5	.15	7.75	1.1625
6	.15	8.333	1.2499

The defuzzified value x^* will be $\dfrac{\sum_{i=1}^{N} A_i \times \overline{x_i}}{\sum_{i=1}^{N} A_i}$

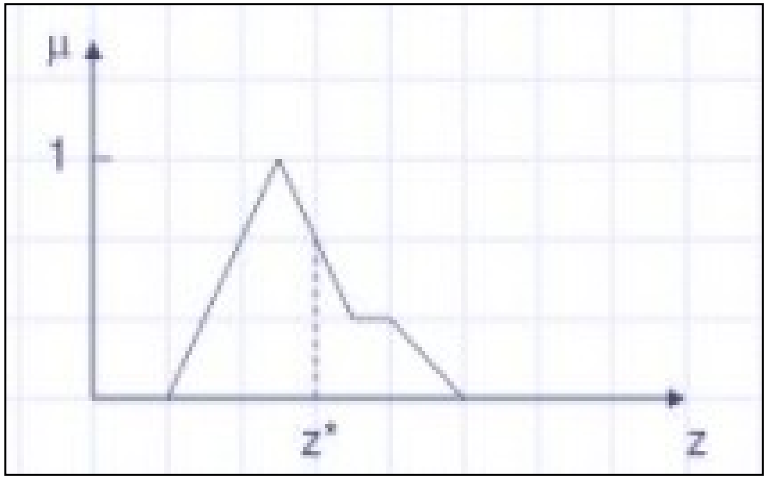
$$= \frac{(1.1665+10+0.3583+1.0875+1.1625+1.2499)}{(0.5+2+.05+.15+.15+.15)}$$

$$= (15.0247)/3 = 5.008$$

$$x^* = 5.008$$

Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

- It is the most prevalent and physically appealing of all the de-fuzzification method.
- It is given by the expression $z^* = \dfrac{\int \mu_A(z) z dz}{\int \mu_A(z) dz}$



Different methods of DeFuzzification- 2. Centroid Method/ Center of gravity/ Centroid of area method

$$z^* = \frac{\int \mu_B(z) \bullet z dz}{\int \mu_B(z) dz} =$$

$$\left[\int_0^1 (.3z)zdz + \int_1^{3.6} (.3z)dz + \int_{3.6}^4 \left(\frac{z-3}{2}\right)zdz + \int_4^{5.5} (.5)zdz + \int_{5.5}^6 (z-5)zdz + \int_6^7 zdz + \int_7^8 (8-z)zdz\right]$$

$$\div \left[\int_0^1 (.3z)dz + \int_1^{3.6} (.3)dz + \int_{3.6}^4 \left(\frac{z-3}{2}\right)dz + \int_4^{5.5} (.5)dz + \int_{5.5}^6 (z-5)dz + \int_6^7 dz + \int_7^8 (8-z)dz\right]$$

$$= 4.9 meters$$

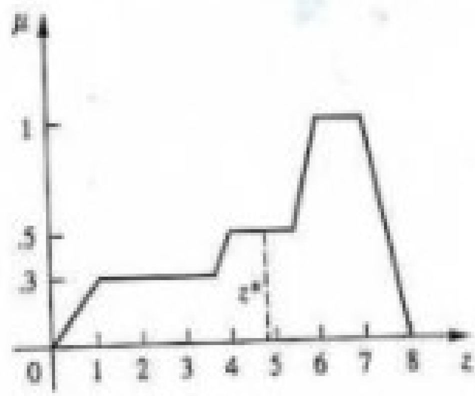
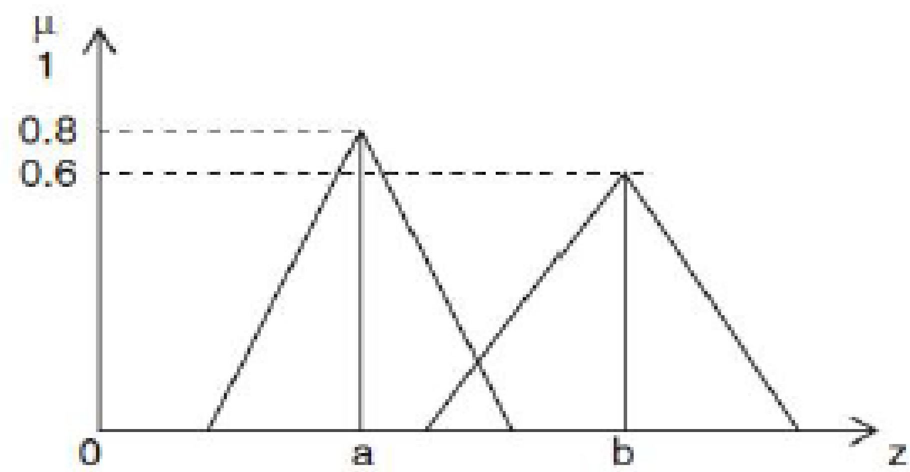


FIGURE 5.12
The centroid method for finding z^*.

Different methods of DeFuzzification- 3.Weighted Average method

- It is the most frequently used as it is most computationally efficient.
- It is restricted to symmetric output membership functions
- It is given by the expression $$z^* = \frac{\sum \mu_A(\bar{z})\bar{z}}{\sum \mu_A(\bar{z})}$$
- where $\bar{z}$ is the centroid of each symmetric membership function

$$z^* = \frac{a(0.8) + b(0.6)}{0.8 + 0.6}.$$



Different methods of DeFuzzification- 3.Weighted Average method

$$z^* = \frac{\sum \mu_A(\bar{z})\,\bar{z}}{\sum \mu_A(\bar{z})}$$

Weighted-Average Method:

$$z^* = \frac{(.3\times 2.5)+(.5\times 5)+(1\times 6.5)}{.3+.5+1} = 5.41 meters$$

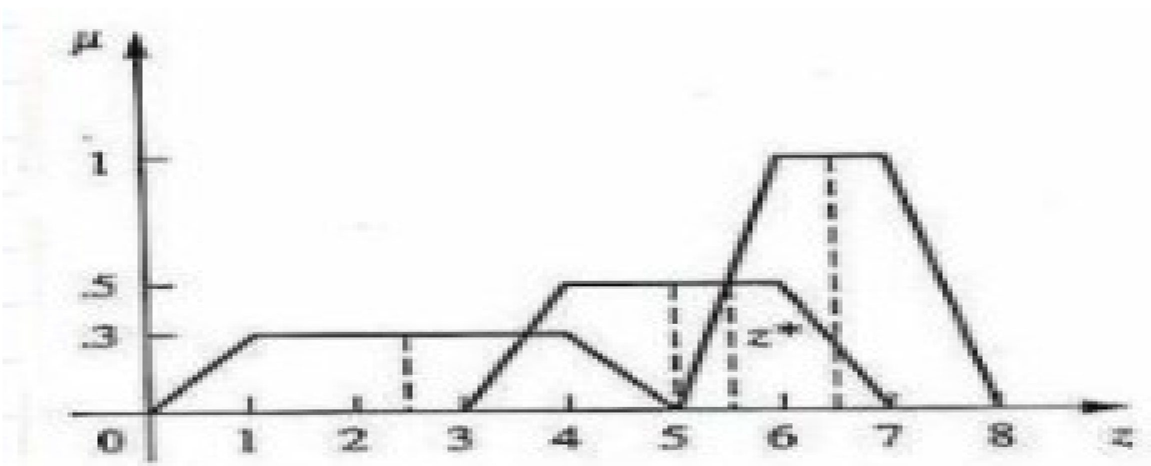
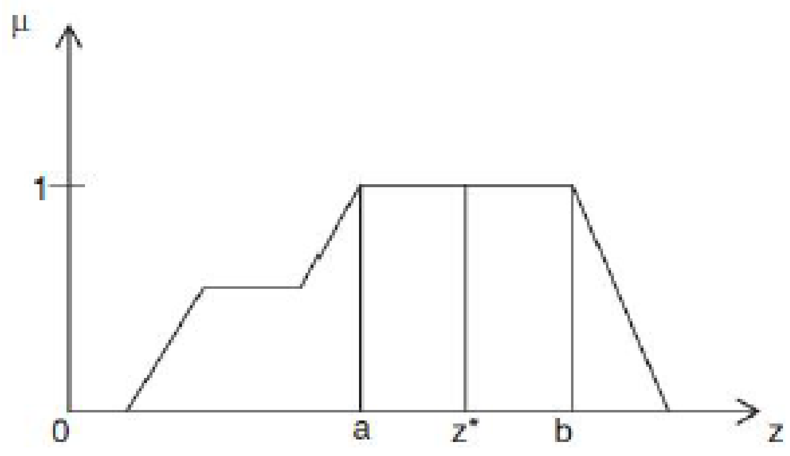


FIGURE 5.13
The weighted average method for finding z*.

Different methods of DeFuzzification- 4.Mean Max membership

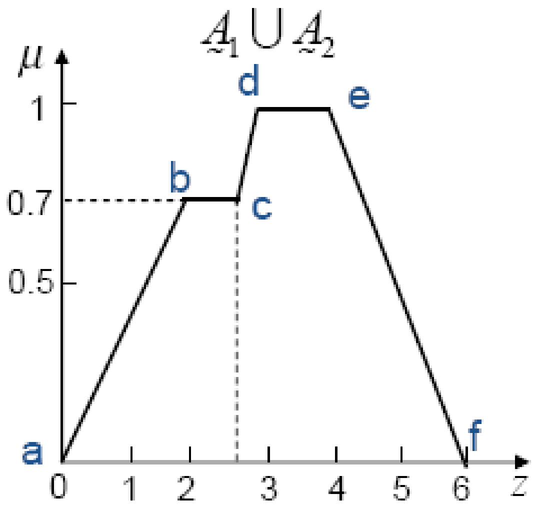
- It is very similar to maximum principle method, except that the locations of the maximum membership can be non-unique. (It can be a plateau rather than a single point)



$$z^* = \frac{a+b}{2}$$

Different methods of DeFuzzification- 4.Mean Max membership

- Another example:



$$z^* = \frac{3+4}{2} = 3.5$$

Different methods of DeFuzzification- 4.Mean Max membership

- Another example:

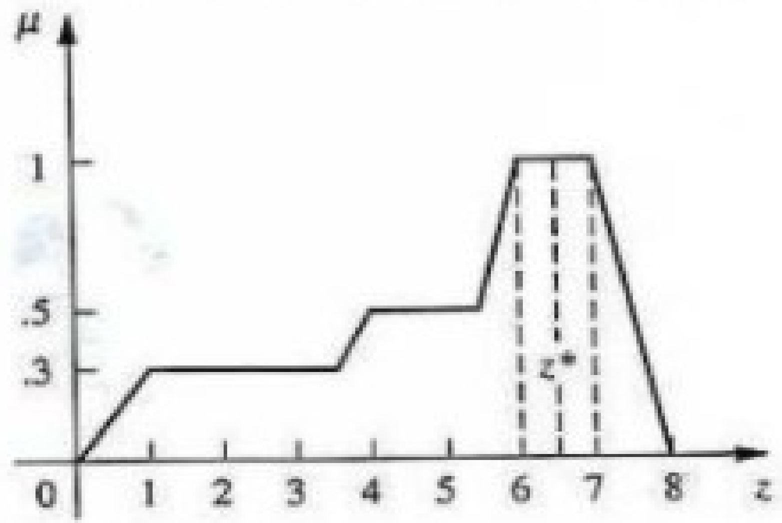


FIGURE 5.14
The mean-max membership method for finding z^*.

Mean-Max Method: $(6+7)/2 = 6.5 meters$

Different methods of DeFuzzification- 5. Centre of Sum

- This method is faster and not restricted to symmetric membership functions
- This process involves the algebraic sum of individual output fuzzy sets, instead of their union.
- **Two drawbacks:**
  - the intersecting areas are added twice
  - the method involves finding the centroids of the individual membership functions.
- This method is similar to weighted average method, except that the weights are the areas of respective membership function

$$z^* = \frac{\sum_{i=1}^{m} A_i \times c_i}{\sum_{i=1}^{m} A_i}$$

Where $A_1, A_2 \ldots A_m$ corresponds to area of individual output fuzzy sets and $c_1, c_2, \ldots c_m$ are there centroids respectively

Different methods of DeFuzzification- 5. Centre of Sum

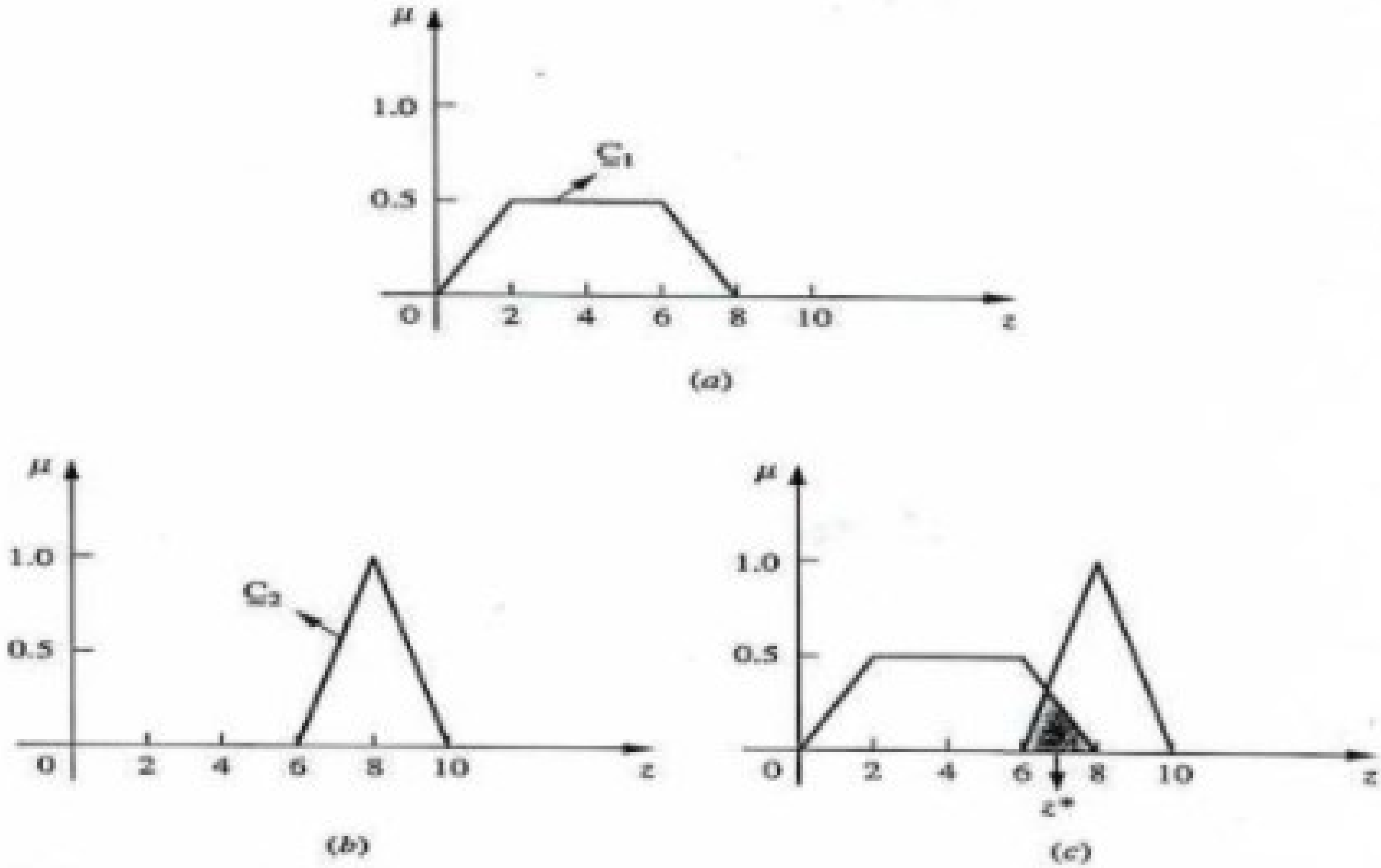


FIGURE 5.21
Center of sums method: (*a*) first membership function; (*b*) second membership function; and (*c*) defuzzification step.

Different methods of DeFuzzification- 5.Centre of Sum

Using Center of sums:

S1 = 0.5 * 0.5(8+4) = 3

S2 = 0.5 * 1 * 4 = 2

Note: area of trapezoid= ((a+b)/2) * h

Area of Triangle= (h*b)/2

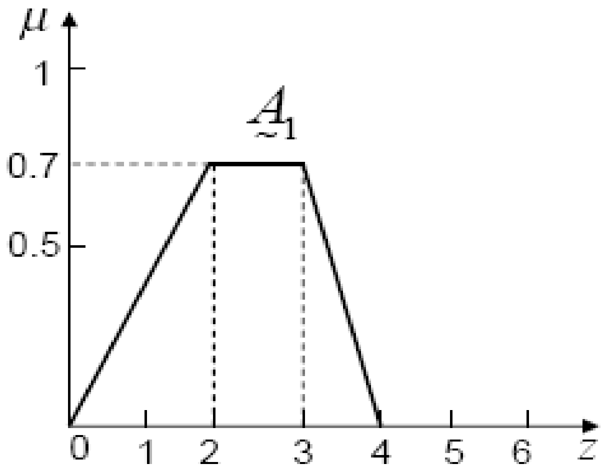
$z_1 = 4 \qquad z_2 = 8$

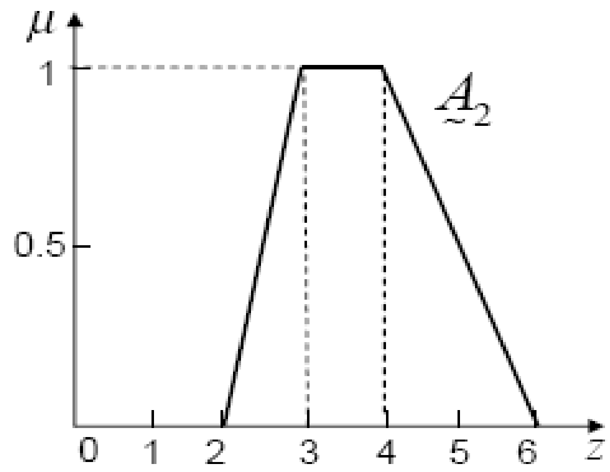
$$\frac{z_1 s_1 + z_2 s_2}{s_1 + s_2} = \frac{4\times3+8\times2}{3+2} = \frac{28}{5} = 5.6$$

So, z*= 5.6

Different methods of DeFuzzification- 5.Centre of Sum

Find z * using Centre of sum method?





Different methods of DeFuzzification- 6.Centre of largest area

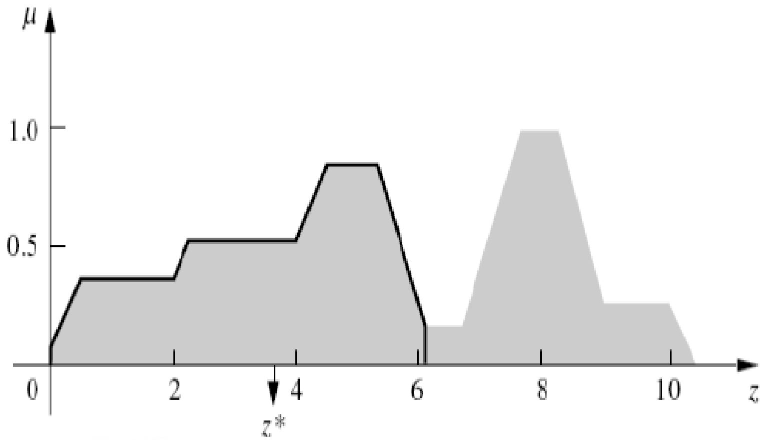
- If the output fuzzy set has at least two convex sub regions, then the centroid of the largest area is used to obtain the de-fuzzified value z*of the output.

$$z^* = \frac{\int \mu_{\underset{\sim}{C}_m}(z) z\,dz}{\int \mu_{\underset{\sim}{C}_m}(z)\,dz}$$

Where $\underline{C}_m$ is the convex sub-region that has the largest area making up $\underline{C}_k$. (see figure)

Different methods of DeFuzzification- 6.Centre of largest area

- **Example**



Centre of largest area is 3.188 same as centroid) , since the complete output fuzzy set is convex

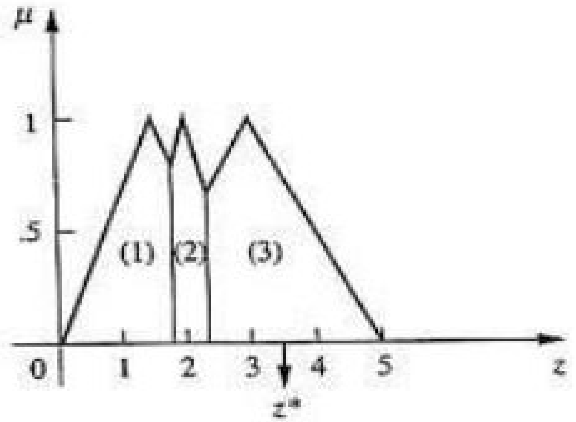


FIGURE 5.28
Center of largest area method for Example 5.6.

Centre of largest area is 3.5 same as centroid of 3rd triangle) , since the complete output fuzzy set is convex

Different methods of DeFuzzification- 7.First and last of the Maxima

First of Maxima

- This method uses the overall output (union) of all individual fuzzy sets to determine the smallest value of the domain with maximum membership degree

$$z^* = \inf_{z \in Z} \left\{ z \in Z \mid \mu_{C_k}(z) = hgt\left(C_k\right) \right\}$$

Infimum (Inf): the greatest lower bound

Last of Maxima

- This method uses the overall output (union) of all individual fuzzy sets to determine the largest value of the domain with maximum membership degree

$$z^* = \sup_{z \in Z} \left\{ z \in Z \mid \mu_{C_k}(z) = hgt\left(C_k\right) \right\}$$

Supremum (Sup): the least upper bound

Different methods of DeFuzzification- First and last of the Maxima

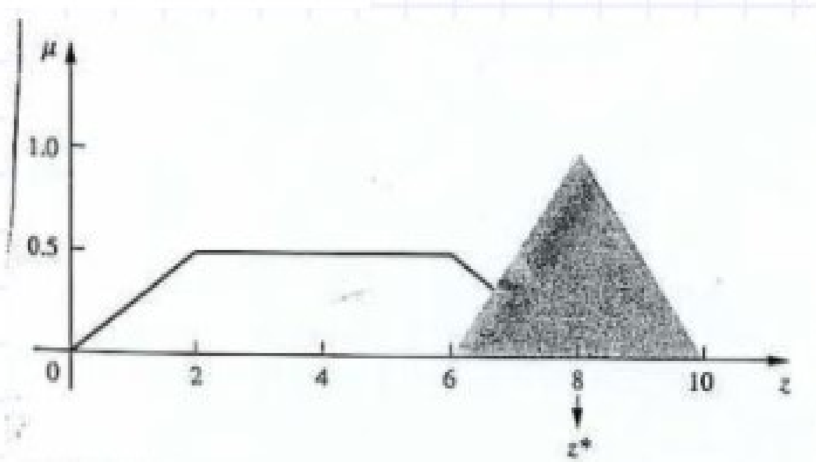


FIGURE 5.23
First of max (and last of max) method.

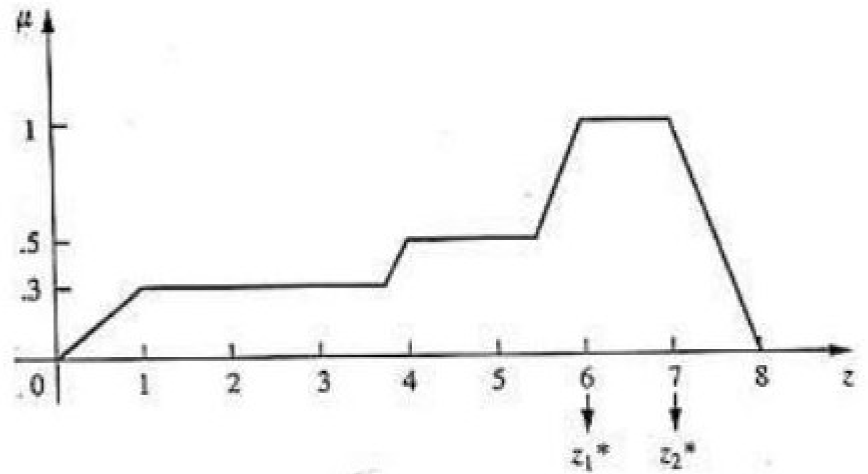


FIGURE 5.26
First of maxima solution ($z_1^* = 6$) and last of maxima solution ($z_2^* = 7$).

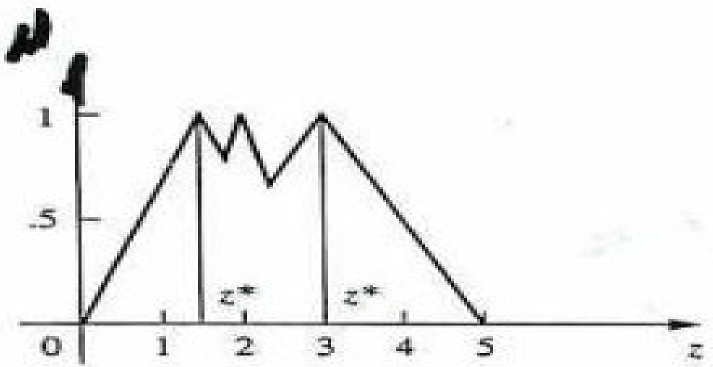
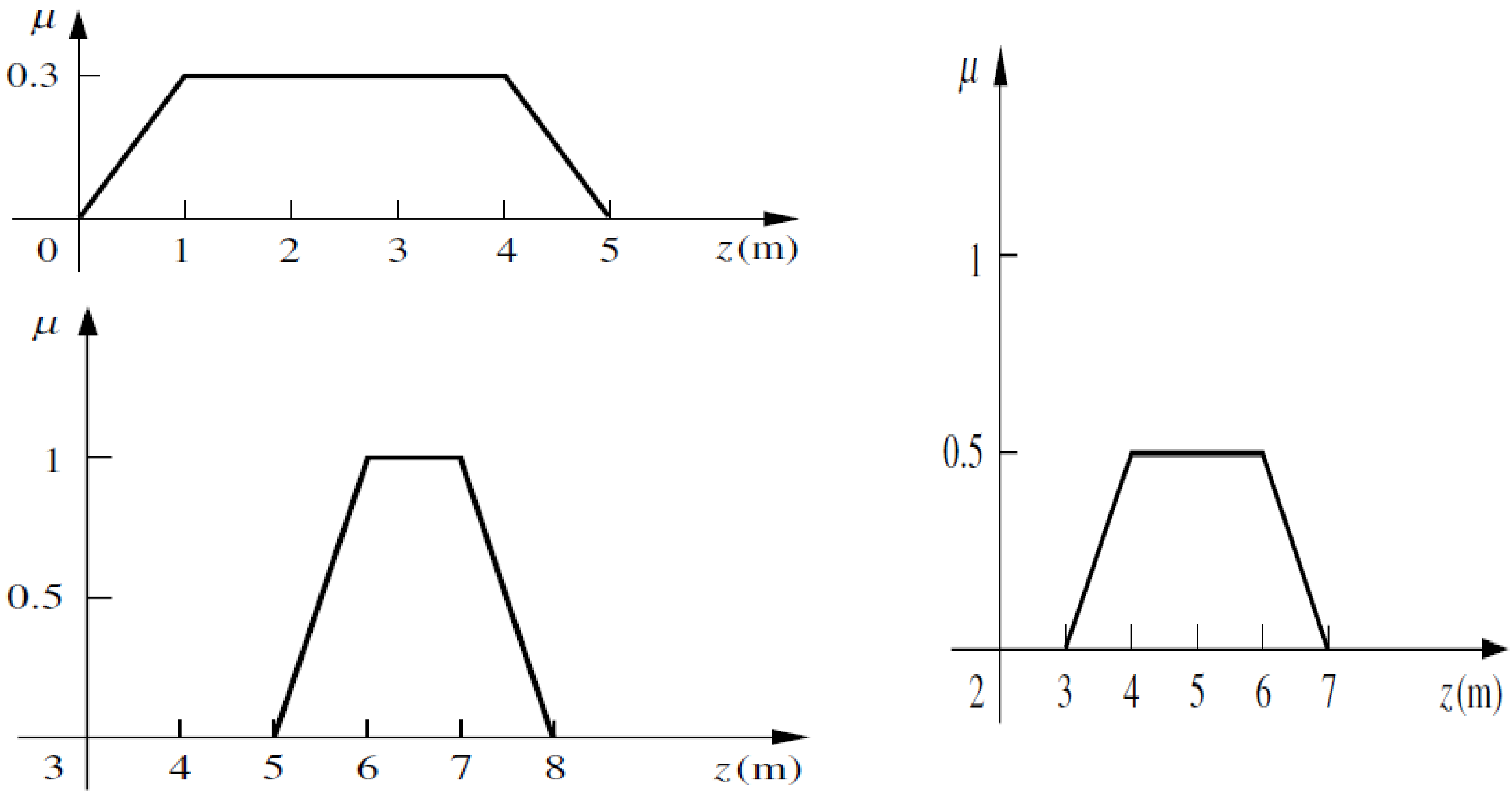


FIGURE 5.29
First of maxima gives $z^* = 1.5$ and last of maxima gives $z^* = 3$.

Different methods of DeFuzzification- Numerical

- For the given membership functions determine the defuzzified output by seven methods.



Different methods of DeFuzzification- Numerical

- Output of fuzzy set.

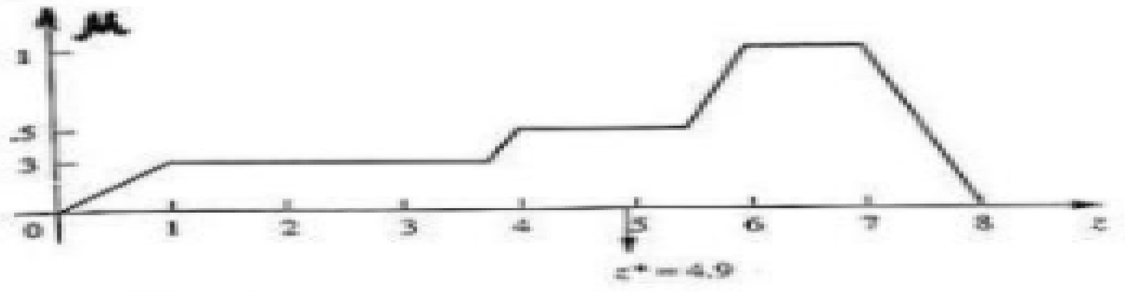


FIGURE 5.25
Output fuzzy set for Example 5.5 is convex.

- Center of sum methods.

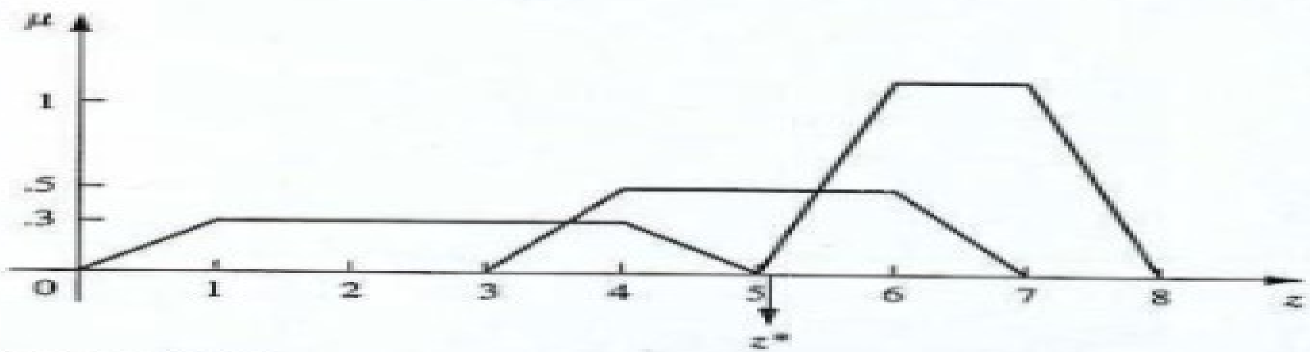


FIGURE 5.24
Center of sums result for Example 5.5.

Different methods of DeFuzzification- Numerical

- First and last maxima

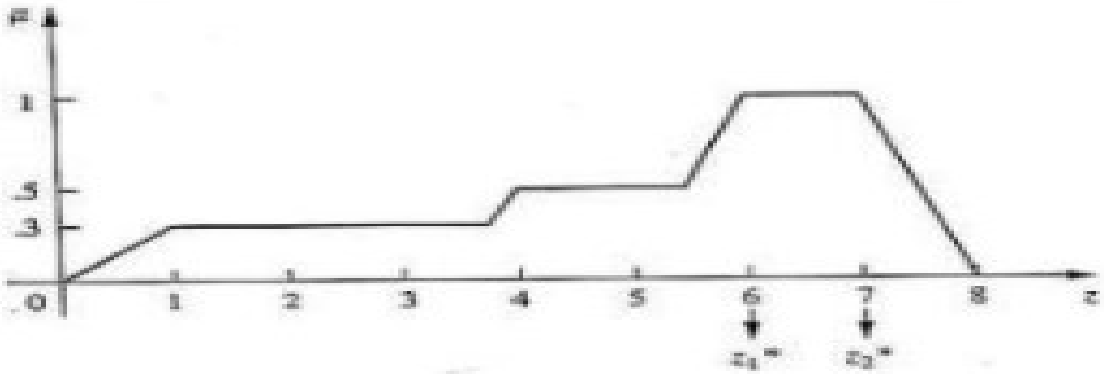
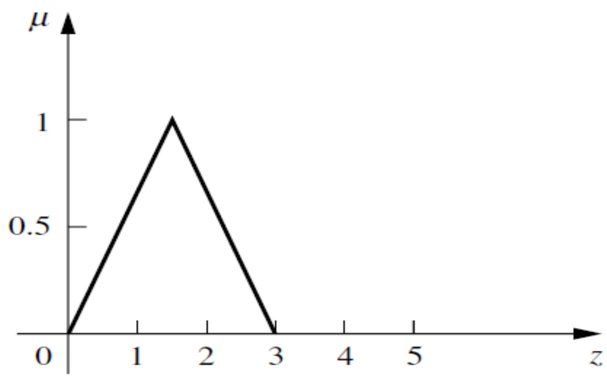
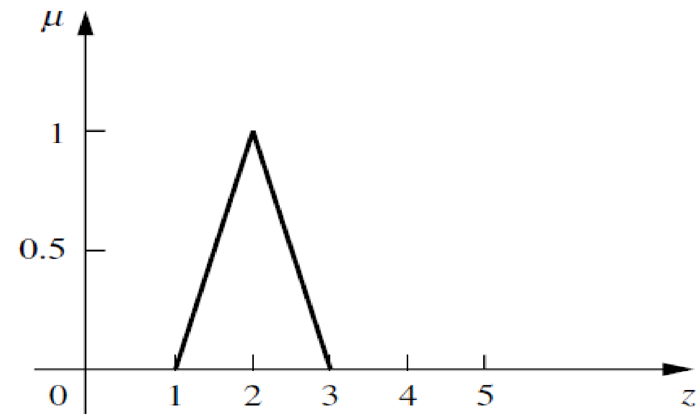


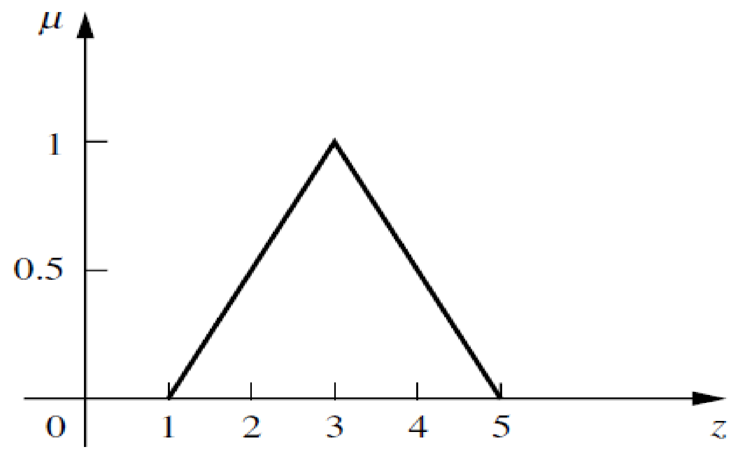
FIGURE 5.26
First of maxima solution ($z_1^* = 6$) and last of maxima solution ($z_2^* = 7$).

Different methods of DeFuzzification- Numerical

- For the given membership functions determine the defuzzified output by seven methods.







Different methods of DeFuzzification- Numerical

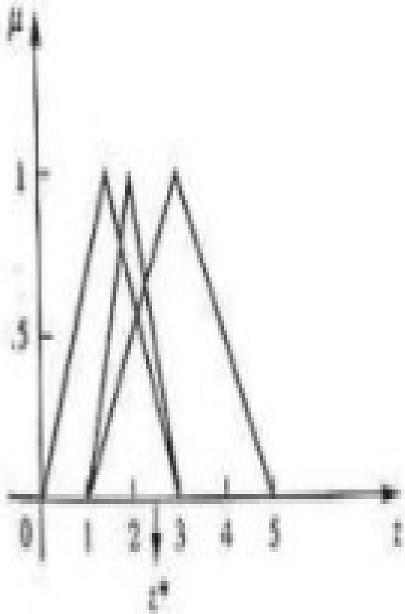


FIGURE 5.27
Center of sums solution for Example 5.6.

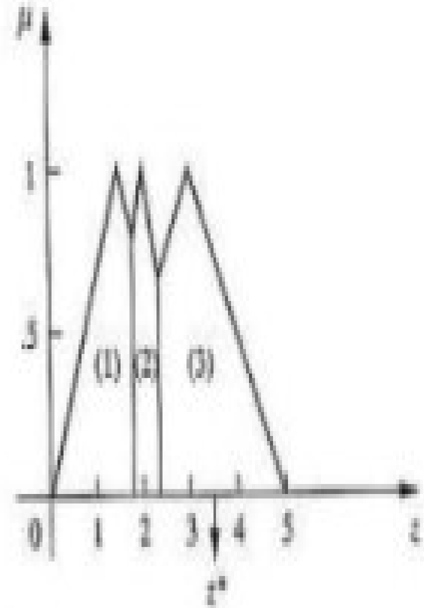


FIGURE 5.28
Center of largest area method for Example 5.6.

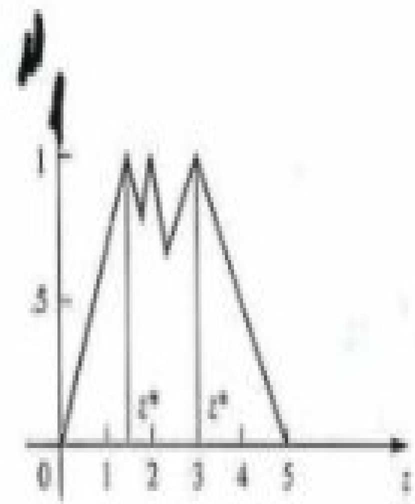


FIGURE 5.29
First of maxima gives $z^* = 1.5$ and last of maxima gives $z^* = 3$.

Fuzzy Inference Systems

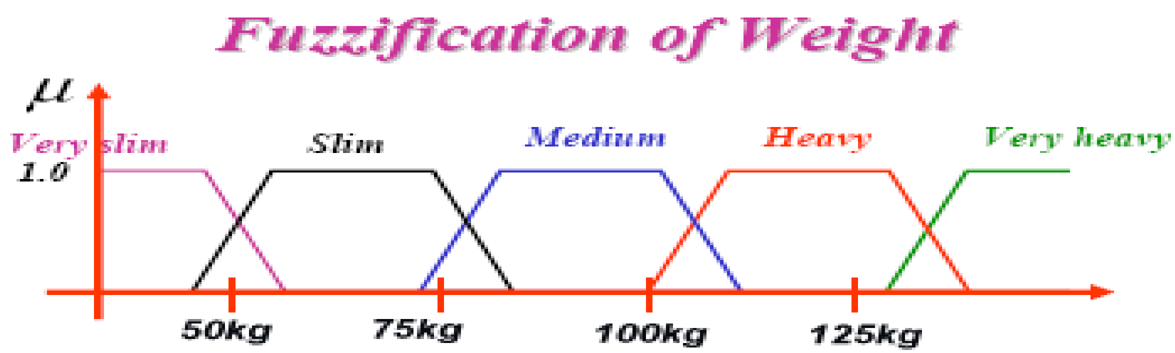
- Fuzzy controllers

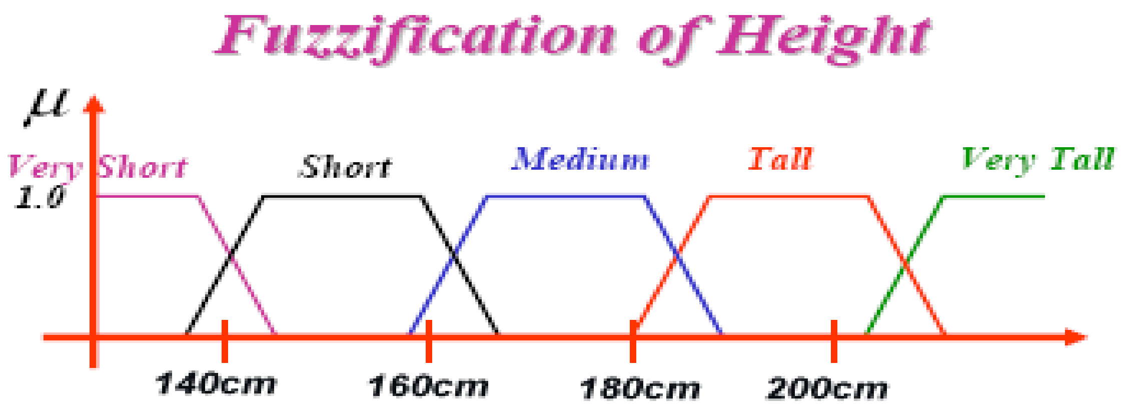
Fuzzy Inference system- Example 1

- Assume we want to evaluate the health of a person based on his height and weight.
- The input linguistic variables are the crisp numbers of the person’s height and weight.
- Output is percentage of healthiness.

Fuzzy Inference- Step 1: Fuzzification

- Fuzzification is a process by which the numbers are changes into linguistic words.

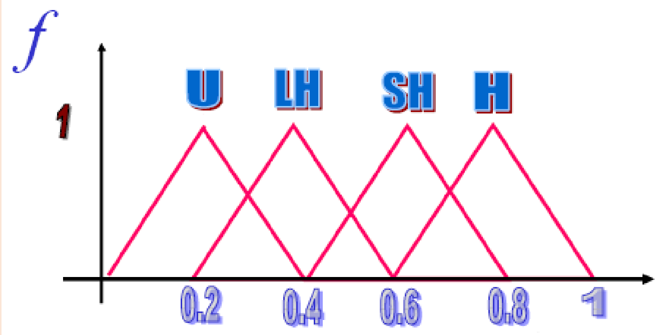




Fuzzy Inference- Step 2: Rules

- Rules reflect experts decisions.
- Rules are tabulated as fuzzy words
- Rules can be grouped in subsets
- Rules can be redundant
- Rules can be adjusted to match desired

- Rules are tabulated as fuzzy words
  - – Healthy (H)
  - – Somewhat healthy (SH)
  - – Less Healthy (LH)
  - – Unhealthy (U)
  - Rule function f
- $f = \{U, LH, SH, H\}$



$$f = \{U, LH, SH, H\}$$

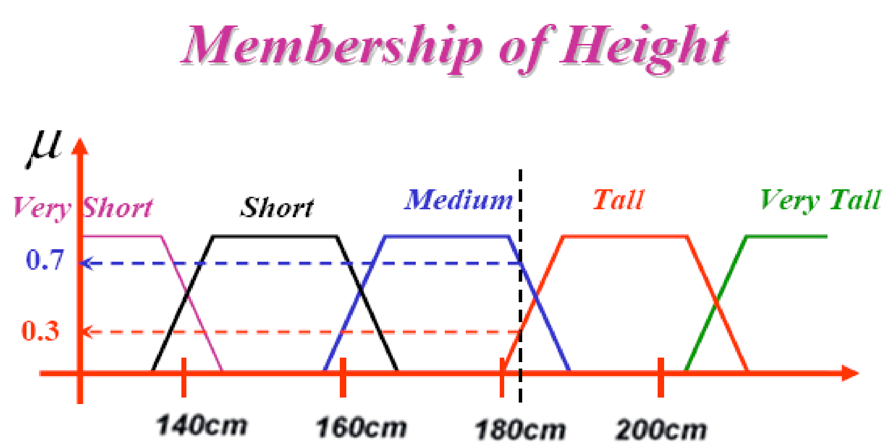
Fuzzy Inference- Step 2: Fuzzy Rule Table

		Weight				
Height		Very Slim	Slim	Medium	Heavy	Very Heavy
	Very Short	H	SH	LH	U	U
	Short	SH	H	SH	LH	U
	Medium	LH	H	H	LH	U
	Tall	U	SH	H	SH	U
	Very Tall	U	LH	H	SH	LH

Fuzzy Inference- Step 3: Calculation

- For a given person, compute the membership of his/her weight and height
- Assume that a person height is 185cm
- Assume that the person's weight is 49

$$\mu_{Weight} = \{\mu_{VS} \quad \mu_S \quad \mu_M \quad \mu_H \quad \mu_{VH}\}$$
$$\mu_{Weight} = \{0.8 \quad 0.2 \quad 0 \quad 0 \quad 0\}$$



$$\mu_{height} = \{\mu_{VS} \quad \mu_S \quad \mu_M \quad \mu_T \quad \mu_{VT}\}$$
$$\mu_{height} = \{0 \quad 0 \quad 0.7 \quad 0.3 \quad 0\}$$

Fuzzy Inference- Step 3: Calculation

- Rule Activation

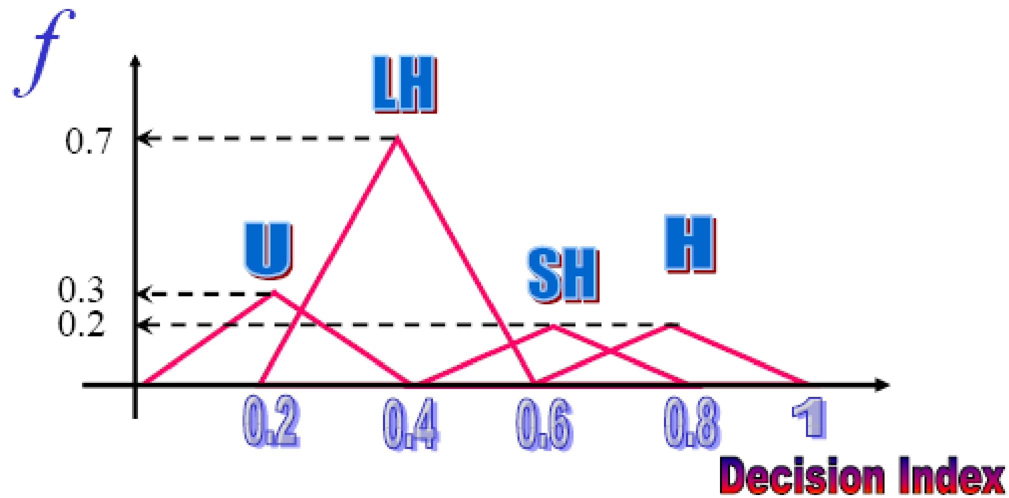
Height	Weight:	0.8	0.2	Medium	Heavy	Very Heavy
	Very Short	H	SH	LH	U	U
	Short	SH	H	SH	LH	U
	0.7	LH	H	H	LH	U
	0.3	U	SH	H	SH	U
	Very Tall	U	LH	H	SH	LH

- Min Operation

Height	Weight:	0.8	0.2	Medium (0)	Heavy (0)	V.Heavy (0)
	V. Short (0)	0	0	0	0	0
	Short (0)	0	0	0	0	0
	0.7	*0.7*	*0.2*	0	0	0
	0.3	*0.3*	*0.2*	0	0	0
	V. Tall (0)	0	0	0	0	0

Fuzzy Inference- Step 3: Calculation

- Scaled Fuzzified Decision

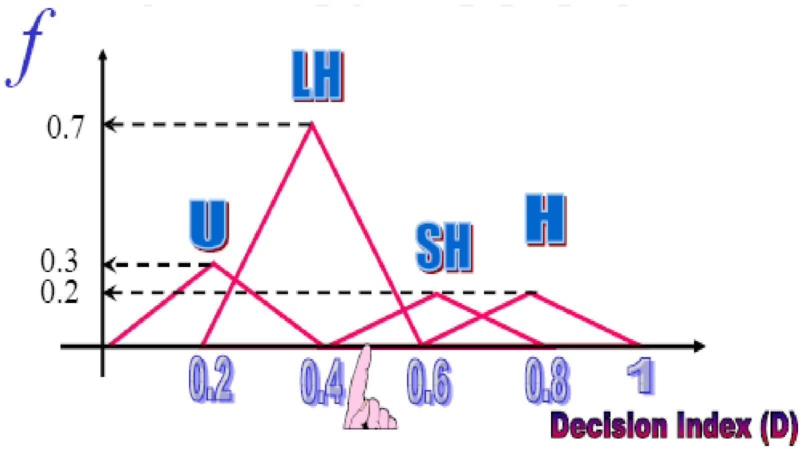


$$f = \{U, LH, SH, H\}$$

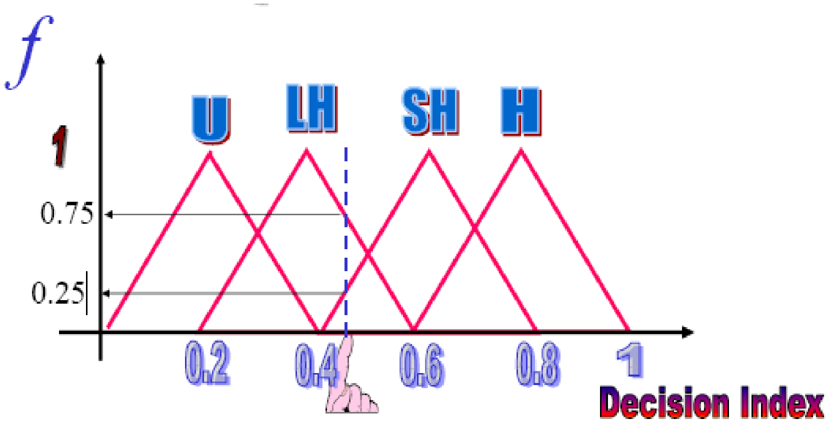
$$f = \{0.3, 0.7, 0.2, 0.2\}$$

Fuzzy Inference- Step 4: Final Decision

- De-fuzzification



Crisp Decision Index (D) is the centroid
D = 0.4429



Fuzzy Decision Index (D)
75% in Less Healthy group
25% in Somewhat Healthy group

Fuzzy Inference- Problem 2

Defining Inputs and Outputs For Fuzzy Logic Control

This step involves the declaration of the range of inputs and outputs. This process of
declaring is called **Universe of Discourse.**

Name	Input/Output	Minimum Value	Maximum Value
Water Level	Input	0	8
Temperature	Input	0	125
Relay Time	Output	0	100

Table5.1 Universe discourse of input and output

Fuzzy Inference- Step 1

Fuzzify The Inputs

In this step the range of Fuzzy variables as related to crisp inputs are determined. There is some general guide lines have to be followed for declaring Fuzzy variables. They are,

Symmetrically distribute the fuzzified values across universe of discourse.

Overlap adjacent sets.

Crisp Input Range	Fuzzy Variable
0-2	Extra Small (XL)
1-3	Small (S)
2.5-5	Medium (M)
4-7	Large (L)
6-8	Extra Large (XL)

Table 5.2 Fuzzy variable range for water level

Fuzzy Inference- Step 1

Crisp Input Range	Fuzzy Variable
0-20	Extra Small (XL)
10-35	Small (S)
30-75	Medium (M)
60-95	Large (L)
85-125	Extra Large (XL)

Table 5.3 Fuzzy variable range for temperature level

Fuzzy Inference- Step 2

Fuzzification Of Output :

This step determines the fuzzy variable set for the output.

Crisp Input Range	Fuzzy Variable
0-20	Very Little
15-40	Little Time
35-60	Optimum Time
55-80	Large Time
80-100	Extra Large

Table 5.4 Fuzzification of output

Fuzzy Inference- Step 3

Temperature Water level	XS	S	M	L
XS	OPTIMUM	LITTLE TIME	VERY LITTLE	VERY LITTLE
S	LARGE TIME	OPTIMUM	OPTIMUM	VERY LITTLE
M	EXTRA LARGE	LARGE TIME	LARGE TIME	OPTIMUM
L	EXTRA LARGE	LARGE TIME	LARGE TIME	OPTIMUM

Table 5.5 Fuzzy rule base

The table entries can be translated in to IF-THEN Rules.

IF TEMPERATURE is EXTRA SMALL(XS) and Water Level is EXTRA SMALL THEN Relay must be kept ON for OPTIMUM.

IF TEMPERATURE is EXTRA SMALL (XS) and Water Level is SMALL THEN Relay must be kept ON for LARGE.

IF TEMPERATURE is SMALL and Water Level is EXTRA SMALL THEN Relay must be kept ON for LITTLE TIME.

IF TEMPERATURE is MEDIUM and Water Level is EXTRA SMALL THEN Relay must be kept ON for VERYLITTLE TIME.

Fuzzy Inference- Step 4

Defuzzify the Output :

For any given crisp input value there may be Fuzzy membership in several input variables, and each will cause several Fuzzy output cell to fire or to be activated. This brings the process of Defuzzification of output to single crisp value.

Fuzzy Inference- MU Asked Question

Explain fuzzy controller system for a tipping example. Consider service and food quality rated between 0 and 10. use this to leave a tip of 25%.

Thank you