Digital Signal and Image Processing
CSC 701

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Module III

Fast Fourier Transform

Introduction

- The N-point DFT of a sequence x(n) converts the time domain N-point sequence x(n) to a frequency domain N-point sequence X(k).

- The direct computation of an N-point DFT requires
  - N * N complex multiplications
  - N(N – 1) complex additions

- Many methods were developed for reducing the number of calculations involved.

- The most popular of these is the Fast Fourier Transform (FFT), a method developed by Cooley and Turkey.

- FFT is an algorithm that computes DFT in Nlog(N) time

- The computational efficiency is achieved by adopting a divide and conquer approach.

- This approach is based on the decomposition of an N-point DFT into successively smaller DFTs and then combining them to give the total transform.

- Basically there are two FFT algorithms;
    1. Decimation-in-time (DIT) FFT algorithm
    2. Decimation-in-frequency (DIF) FFT algorithm

Decimation-in-Time (DIT-FFT) FFT Algorithm

- This algorithm is also known as radix-2 DIT-FFT algorithm.
- As the name implies, the number of output points N can be expressed as a power of 2.
- $N = 2^m$ where m is an integer.

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

1. Select the number of input samples N such that $N= 2^m$ where m is an integer
2. The input sequence is shuffled through bit reversal
3. The number of stages in the flowgraph is given as $M=\log_2 N$
4. Each stage has N/2 Butterflies
5. Inputs/Outputs for each butterfly are separated by 2^{m-1} samples where m represents the stage index .
6. No. of butterflies in each stage = 2^{M-m}
7. Twiddle factor exponents is given by $k=Nt/2^m$, $t=0,1,2.. 2^{m-1}-1$

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

1. Select the number of input samples N such that $N= 2^m$ where m is an integer

   N=4,m=2

2. The input sequence is shuffled through bit reversal

Input Sample	Binary Representation	Bit Reversed Binary	Bit Reversed Index
0	00	00	0
1	01	10	2
2	10	01	1
3	11	11	3

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

3. The number of stages in the flowgraph is given as $M=\log_2 N$

$$M= \log_2 4 = 2$$

4. Each stage has N/2 Butterflies

Thus each stage has 2 butterflies

5. Inputs/Outputs for each butterfly are separated by 2^{m-1} samples where m represents the stage index .

Stage	Separation (2^{m-1})
1	1
2	2

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

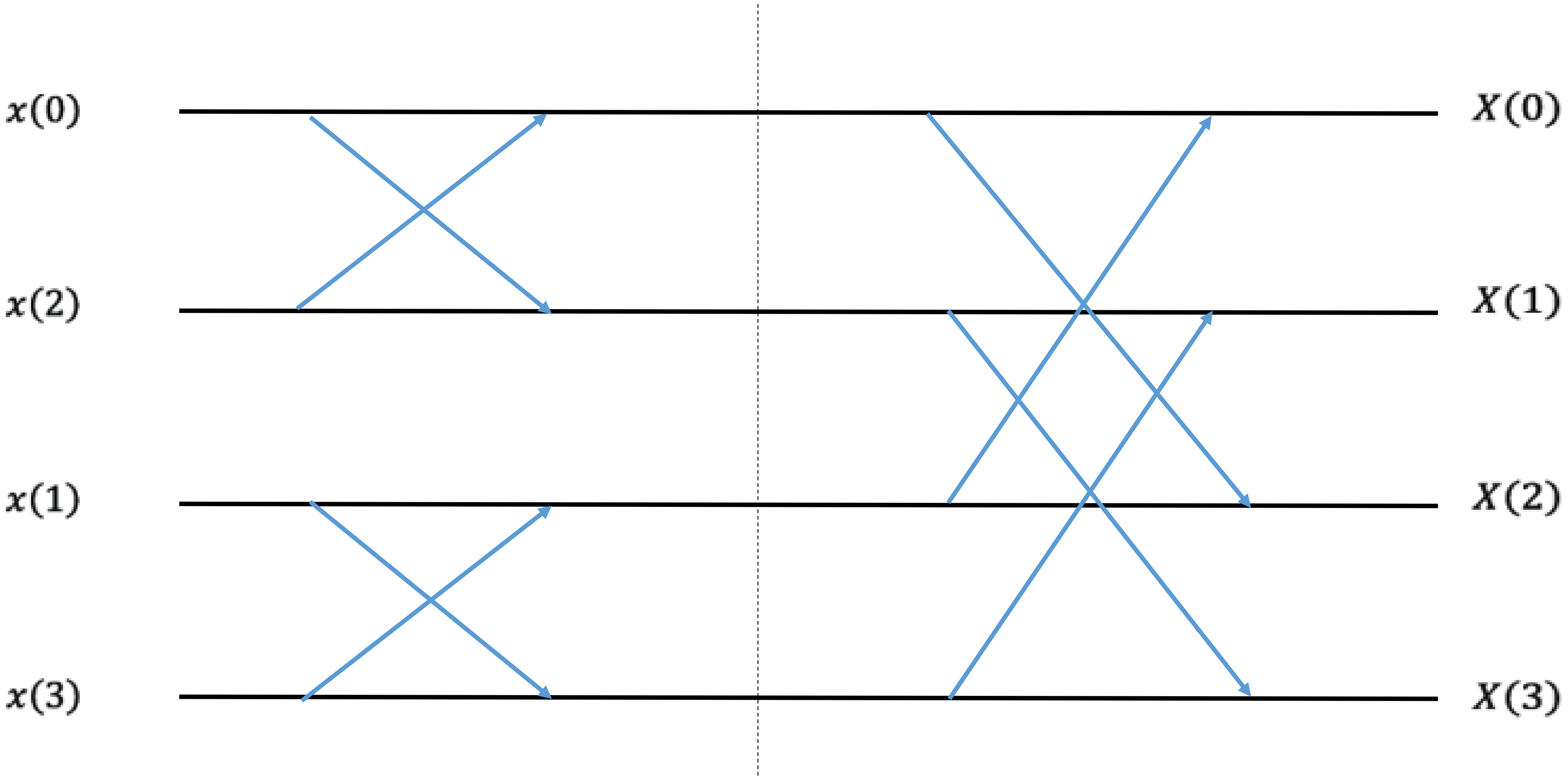
6. No. of butterflies in each stage = 2^{M-m}

Stage	Number of Butterflies (2^{2-m})
1	2
2	1

7. Twiddle factor exponents is given by $k=Nt/2^m$, $t=0,1,2.. 2^{m-1}-1$

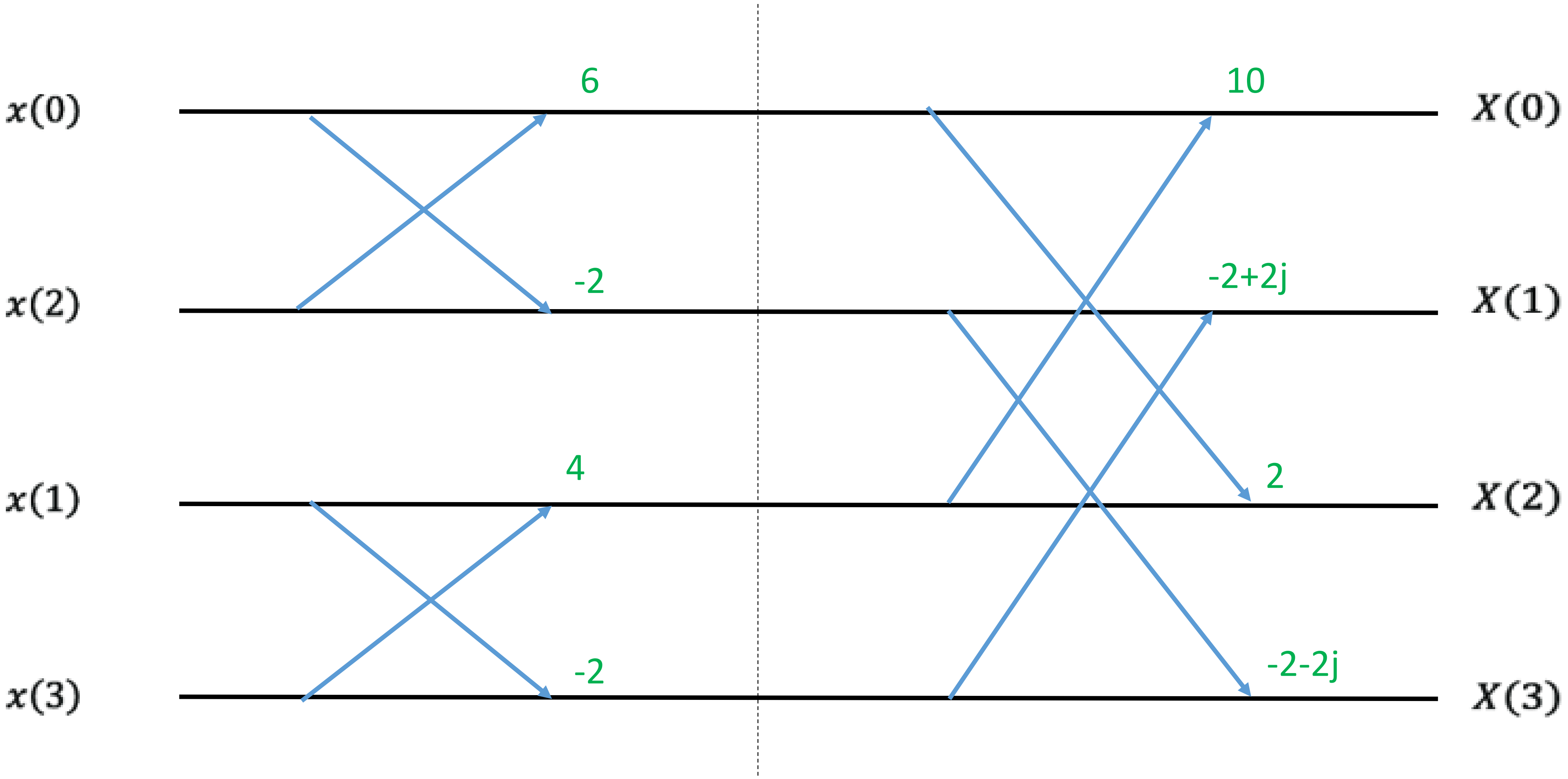
m	$2^{m-1} - 1$	t	$k=4t/2^m$
1	0	0	0
2	1	0,1	0,1

4-point DITFFT Butterfly



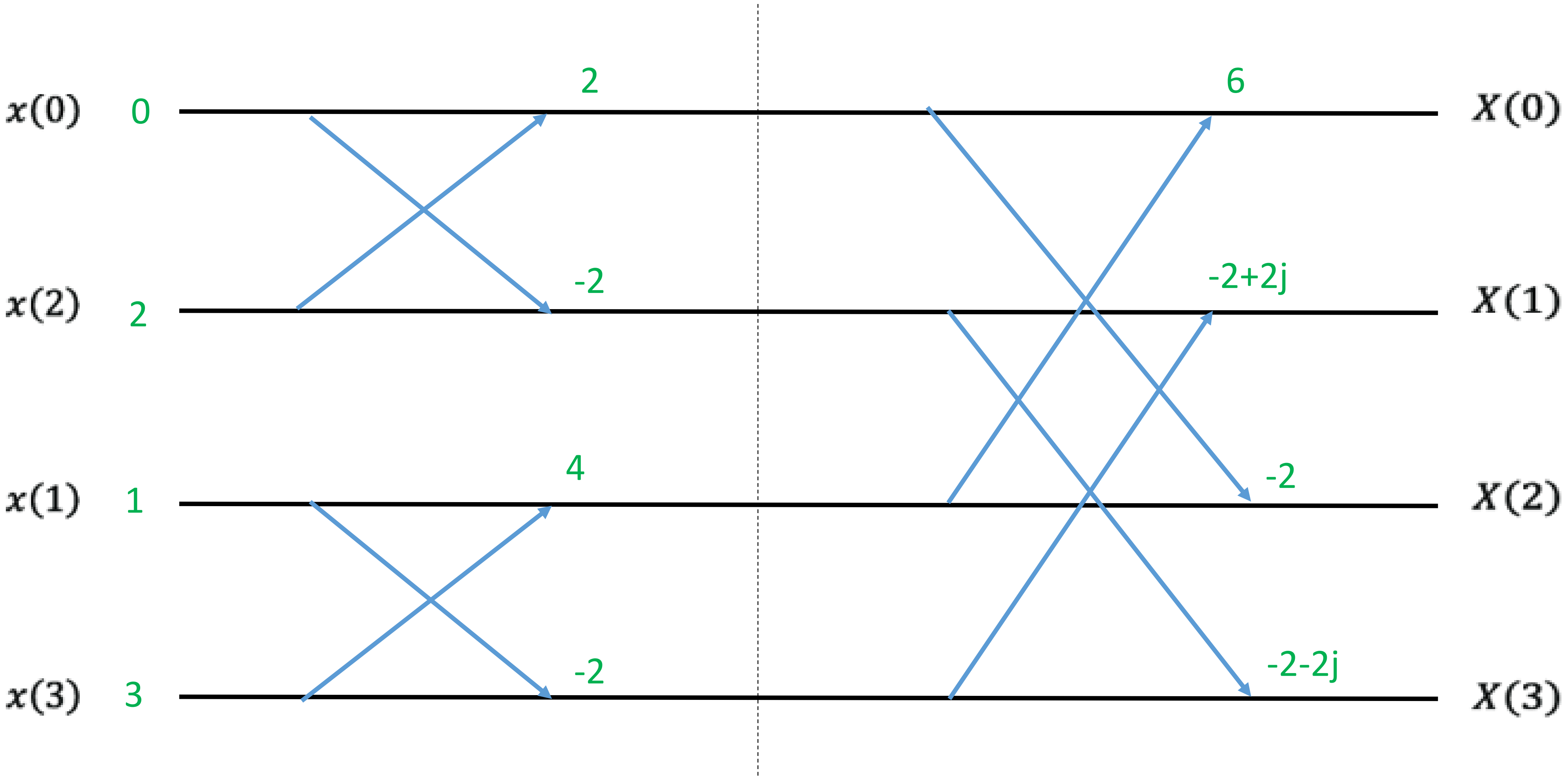
Problem 1:

Find the 4-point DFT of the sequence x(n) = {2, 1, 4, 3} by DIT FFT algorithm



Problem 2:

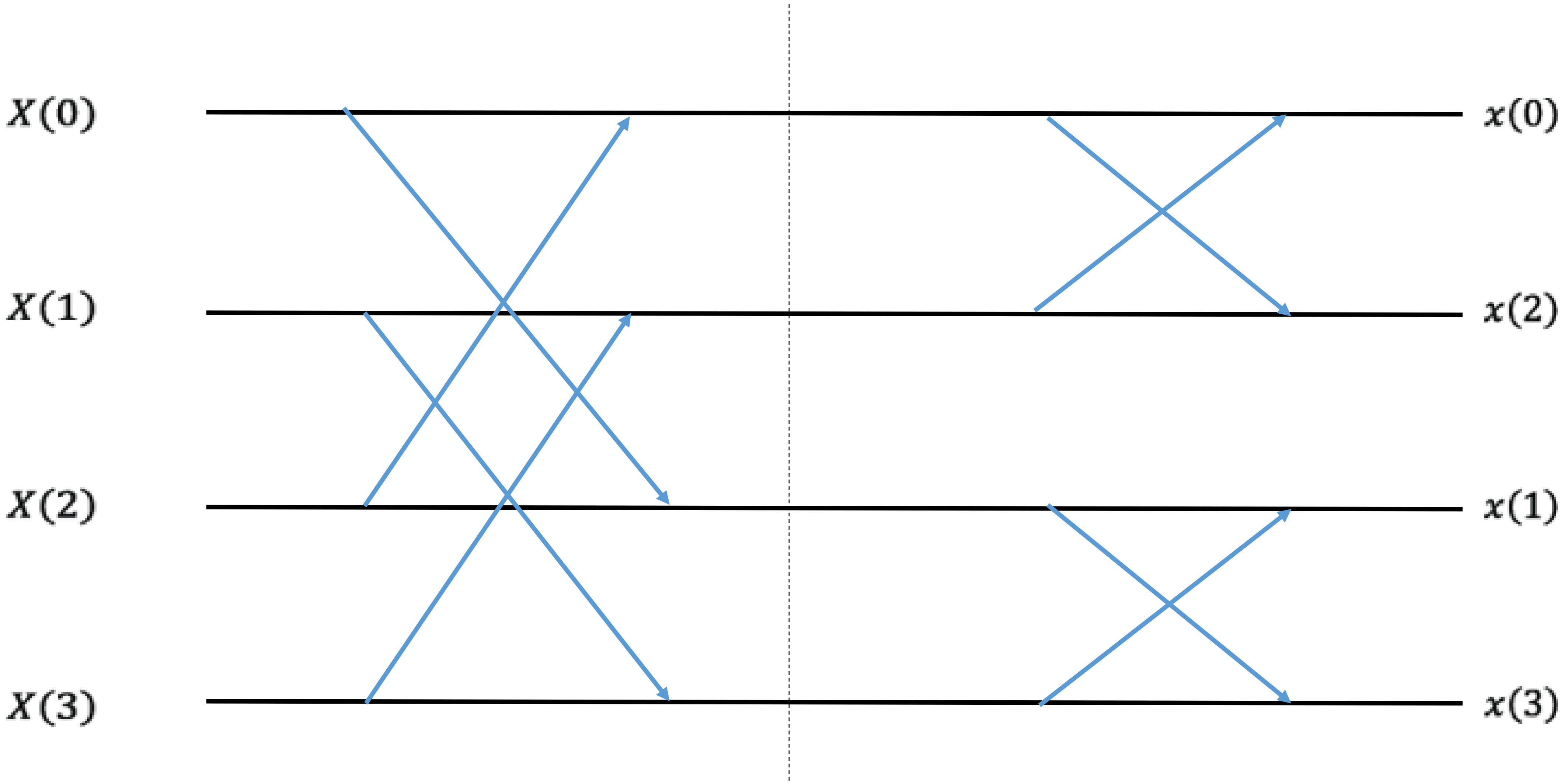
Find the 4-point DFT of the sequence x(n) = {0, 1, 2, 3} by DIT FFT algorithm



Additional Problems

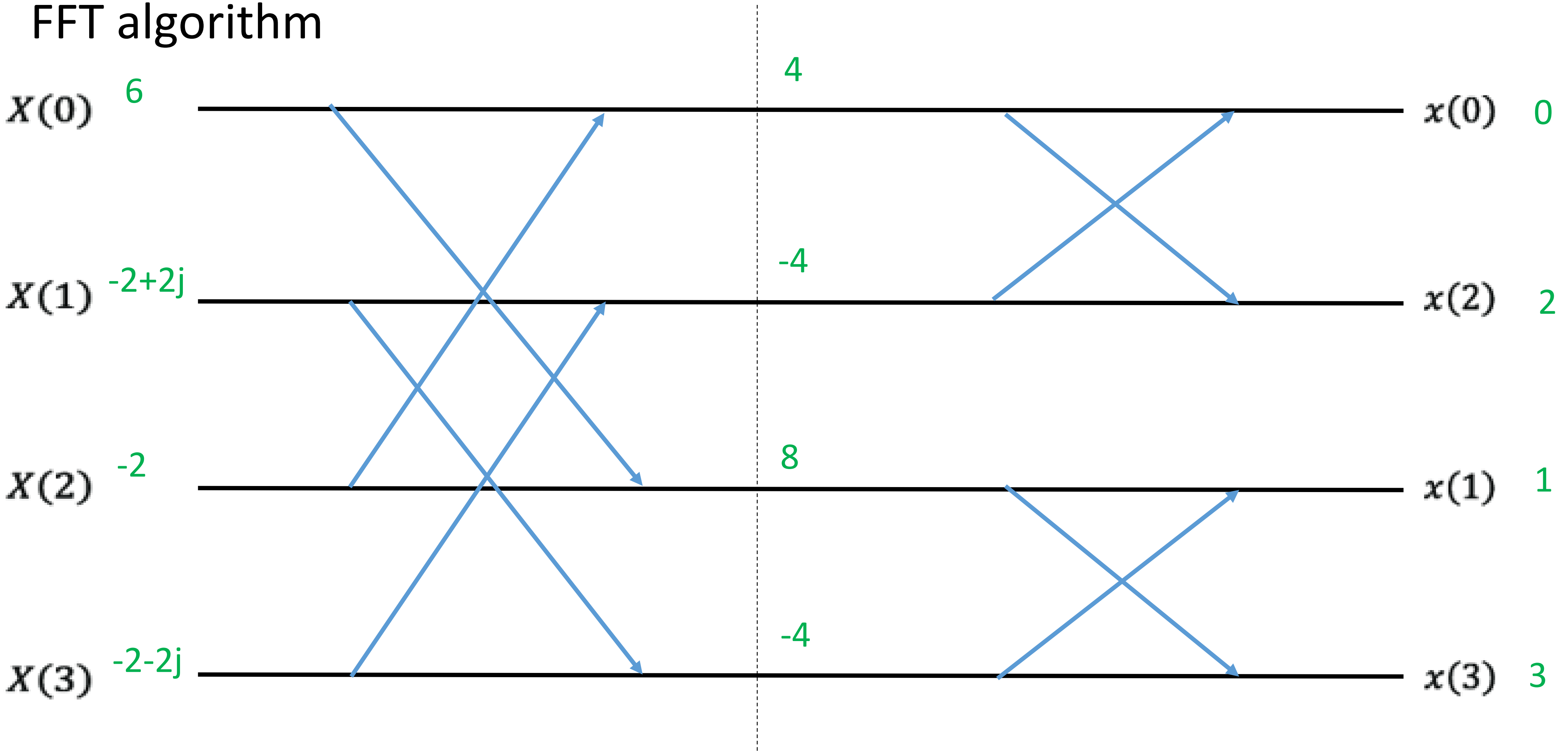
- Compute the 4-point DFT of a sequence x(n)={1,2,3,4} using DIT-FFT algorithm
- Find the 4-point DFT of the sequence x(n) = {2, 1, 4, 3} by DIT FFT algorithm
- For the causal signal x(n) = {2,2,4,4} compute 4-point DFT using DIT-FFT algorithm
- Draw the radix2 DIT flowgraph and find the DFT of the sequence x(n) = {10,11,8,5}
- Find the value of x(n)= Cos (0.25πn) for n=0,1,2,3 . Compute the DFT of x(n) using FFT flowgraph

4 point Inverse DIT FFT Butterfly



Problem 3:

Find the 4-point Inverse FFT of the sequence X(k) = {6, -2+2j, -2, -2-2j} by DIT FFT algorithm



Problem 4:

Find the 4-point Inverse FFT of the sequence $X(k) = \{10, -2+2j, -2, -2-2j\}$ by DIT FFT algorithm

Ans: $\{ 2, 1, 4, 3\}$

8 point DIT FFT Algorithm

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

1. Select the number of input samples N such that $N= 2^m$ where m is an integer

    N=8,m=3

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

2. The input sequence is shuffled through bit reversal

Input Sample	Binary Representation	Bit-reversed Binary	Bit-reversed Index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

3. The number of stages in the flowgraph is given as $M=\log_2 N$

$M= \log_2 8 = 3$

4. Each stage has N/2 Butterflies

Thus each stage has 4 butterflies

5. Inputs/Outputs for each butterfly are separated by 2^{m-1} samples where m represents the stage index .

Stage	**Separation (2^{m-1})**
1	1
2	2
3	4

Steps in drawing the flowgraph of radix-2 DIT FFT algorithm

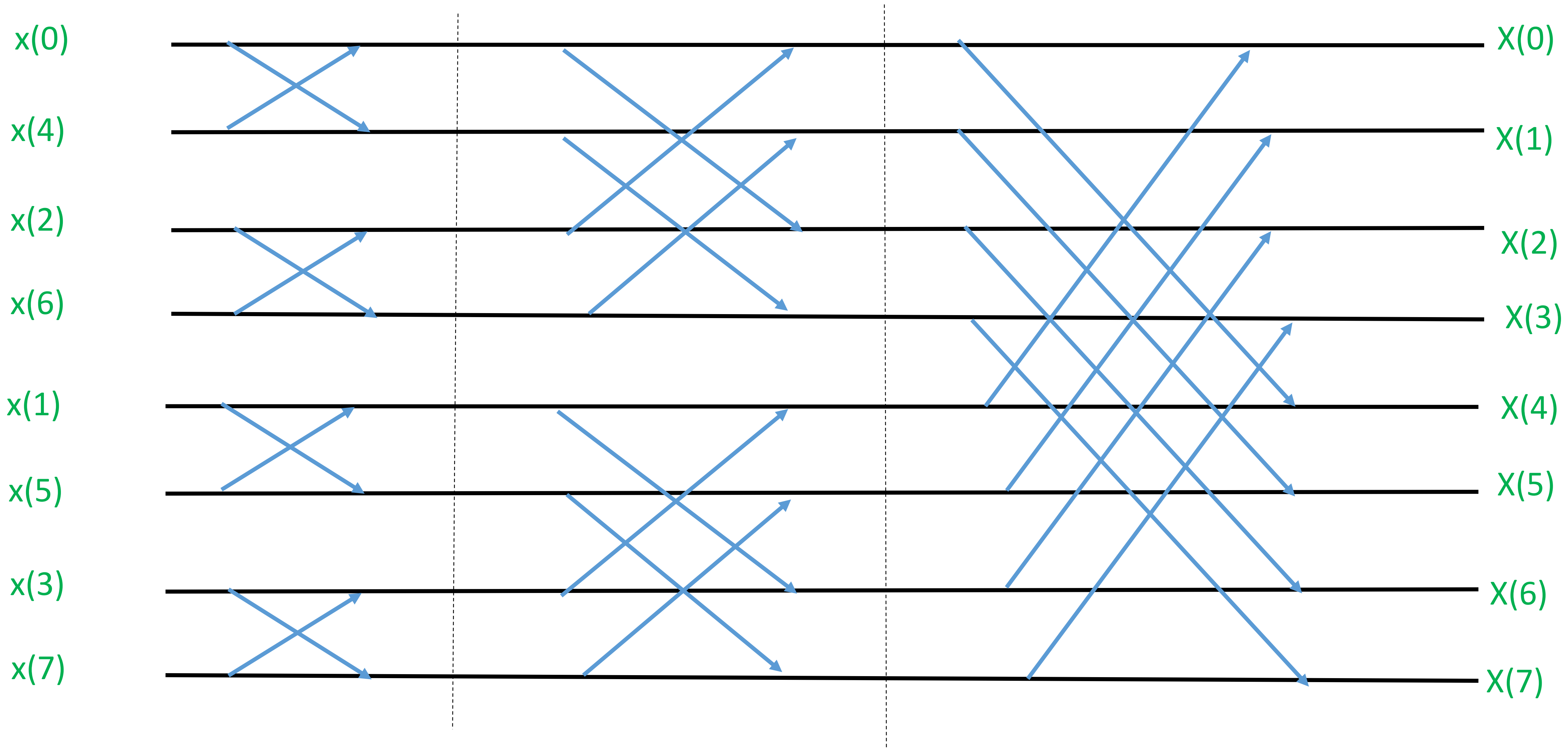
6. No. of butterflies in each stage = 2^{M-m}

Stage	**Number of Butterflies (2^{3-m})**
1	4
2	2
3	1

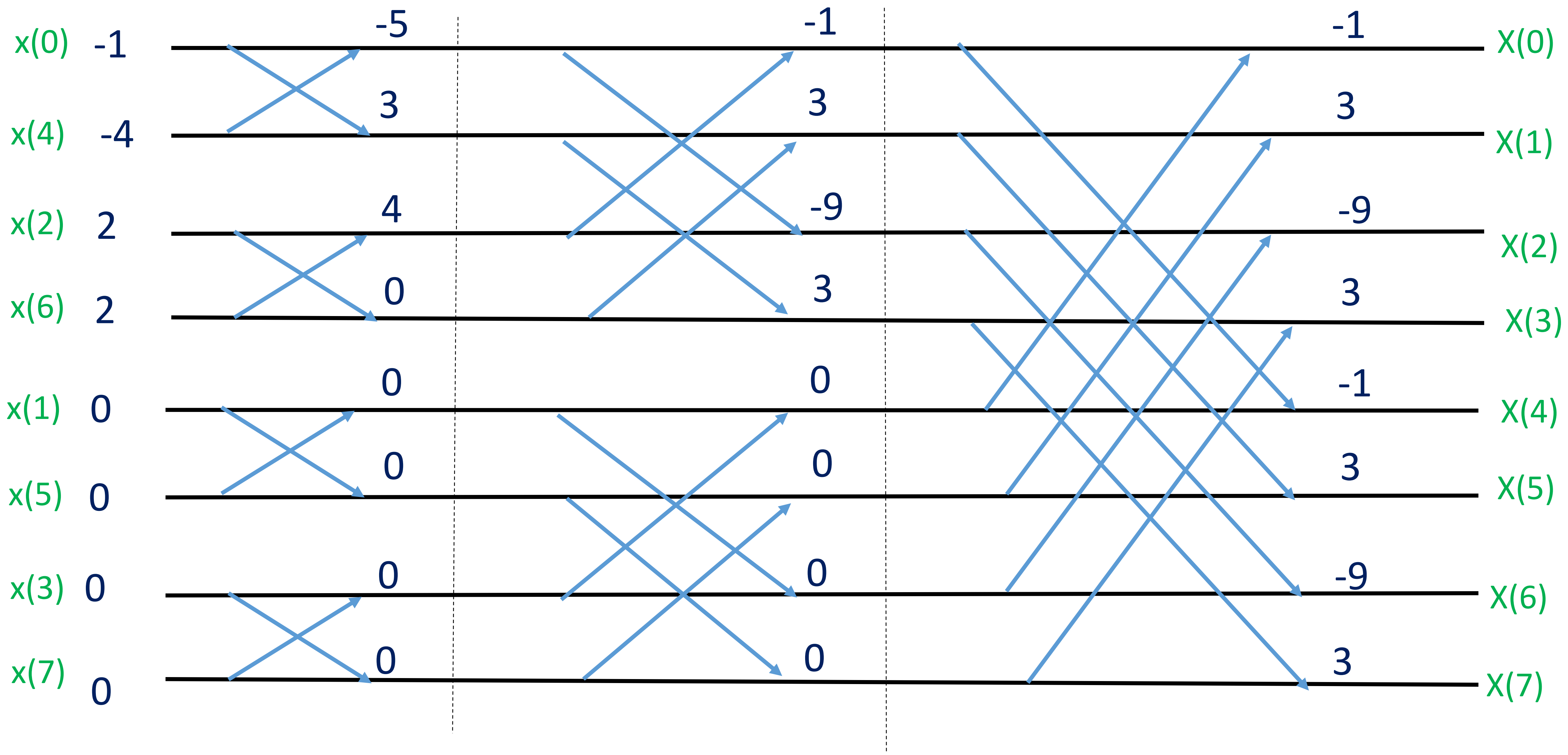
7. Twiddle factor exponents is given by $k=Nt/2^m$, $t=0,1,2..\ 2^{m-1}-1$

m	**$2^{m-1} - 1$**	**t**	**$k=8t/2^m$**
1	0	0	0
2	1	0,1	0,2
3	3	0,1,2,3	0,1,2,3

8 point DIT FFT Algorithm



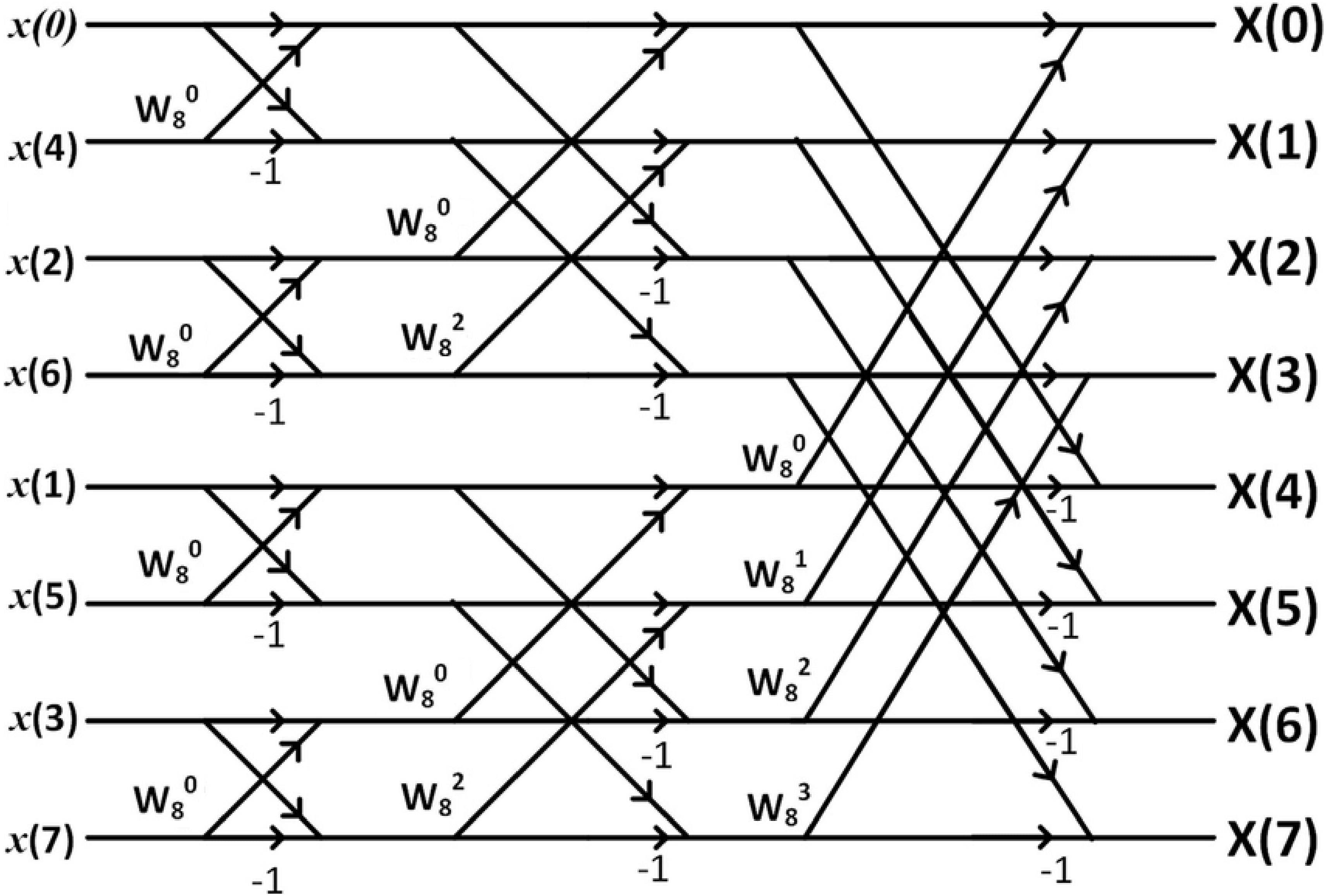
Problem 5: x(n)= {-1, 0, 2, 0, -4, 0, 2,0}. Find X(k) using DIT FFT Algorithm



Problem

- Find the DFT of the sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIT-FFT algorithm

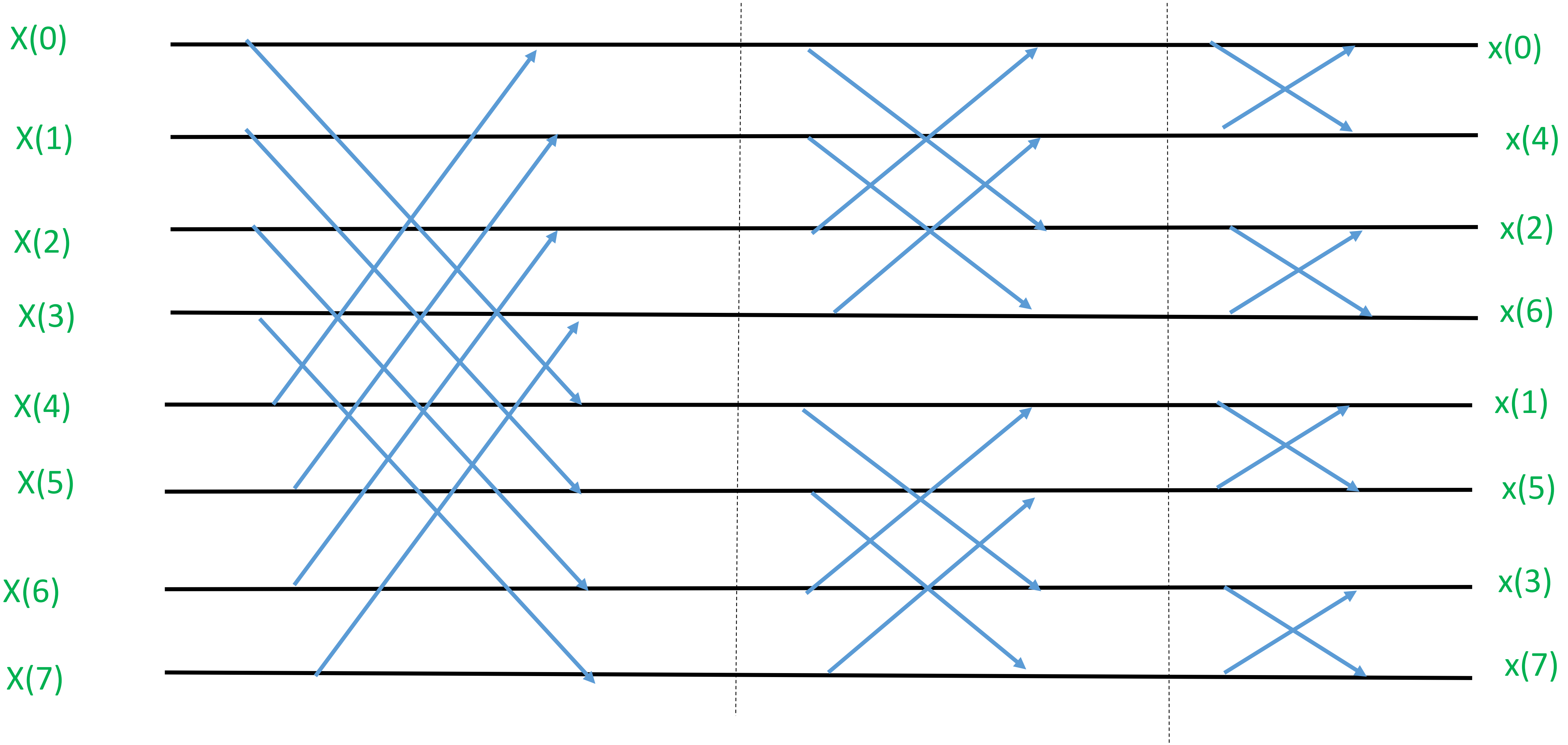
Solution



Additional Problems

- Find the DFT of the sequence x(n) = {1,2,2,1,1,2,2,1} using DIT-FFT algorithm
- Find the DFT of the sequence x(n) = {0,1,2,3,4,5,6,7} using DIT-FFT algorithm
- Find the DFT of the sequence x(n) = {2,3,4,1,0,0,0,0} using DIT-FFT algorithm
- Find the DFT of the sequence x(n) = {2,1,2,1,1,2,1,2} using DIT-FFT algorithm

8 point Inverse DIT FFT Algorithm



Problem:

Let X(k)={ 0.5, 2+j, 3+2j, j, 3, -j, 3-2j, 2-j }. Find x(n) using Inverse DIT FFT.

Stage 1 output: { 0.44, 0.25, 0.75, 0.25, -0.31, j0.35, -0.5, -j0.35}

Stage 2 output: { 1.19, 0.5, -0.31, 0, 0.81, 0, 0.91, -0.71}

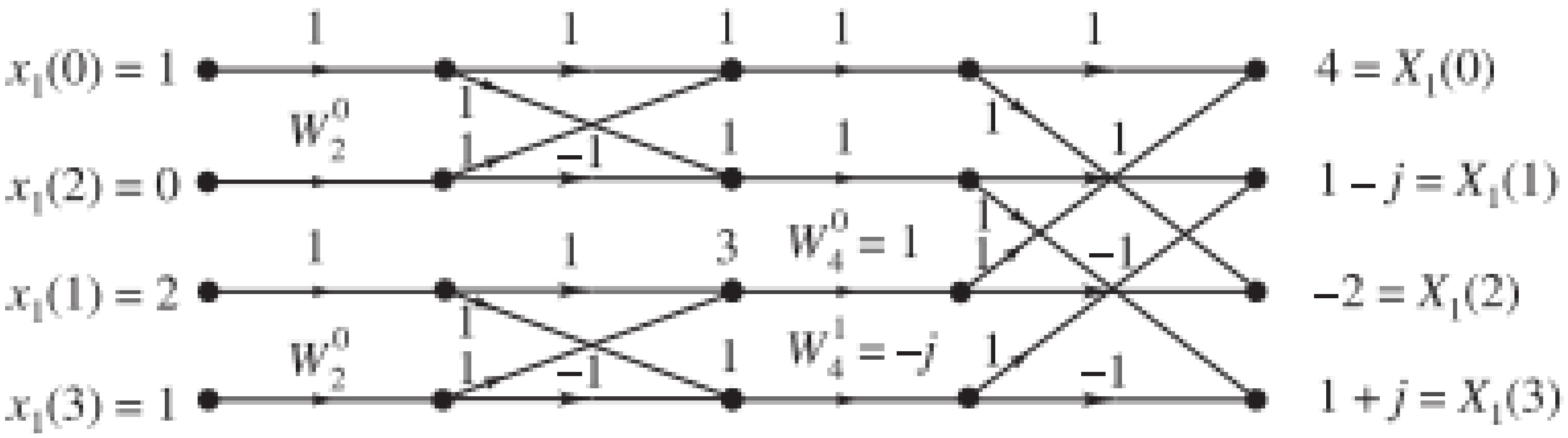
Stage 3 Output:

$x(n) = \{1.69, -0.81, -0.31, -0.51, 0.69, -0.81, -0.31, 0.89\}$

Problem: Compute the circular convolution of the two sequences x1(n) = {1, 2, 0, 1} and x2(n) = {2, 2, 1, 1} using FFT approach.

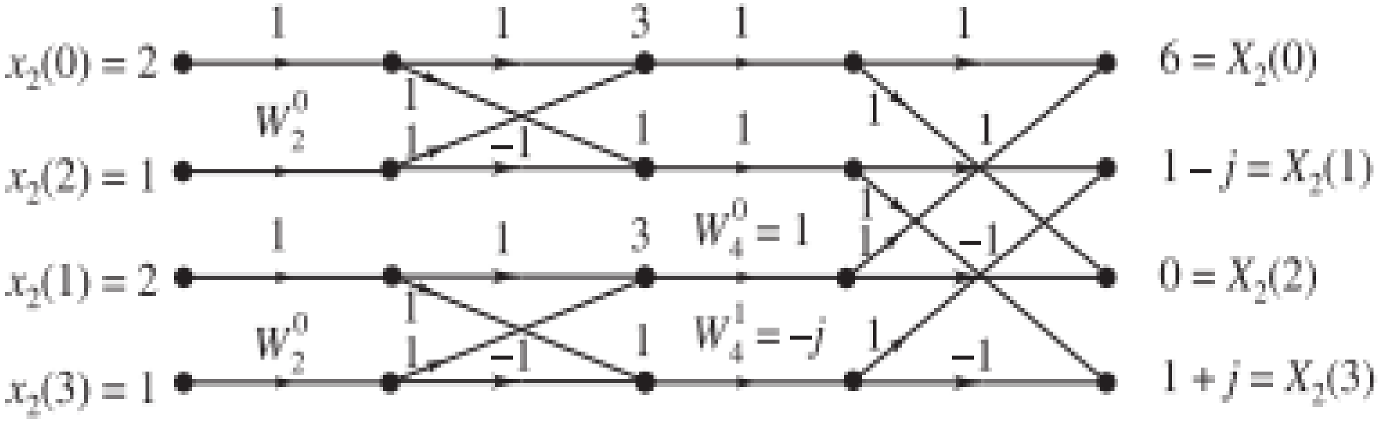
Step 1

$$x_1(n) \oplus x_2(n) \quad X_1(k)X_2(k)$$



(a)

Step 2



(b)

Step 3: $X(k) = \{4, 1-j, -2, 1+j\}\{6, 1-j, 0, 1+j\} = \{24, -j2, 0, j2\}$

Step 4: Find Inverse FFT

Step 5: Final Answer: x(n) = {6, 7, 6, 5}

Problem:

- In an LTI system, the input $x(n) = \{2, 2, 2\}$ and the impulse response $h(n) = \{-2, -2\}$. Determine the response of LTI system by radix-2, DIT FFT

Solution:

- Response of LTI system $y(n) = x(n) * h(n)$
- length of $y(n)$ is $3 + 2 - 1 = 4$

  Append zeros $x(n) = \{2, 2, 2, 0\}$

  $h(n) = \{-2, -2, 0, 0\}$

- The various steps in computing $y(n)$ are
  - Step 1: Determine $X(k)$ using radix-2 DIT FFT algorithm
  - Step 2: Determine $H(k)$ using radix-2 DIT FFT algorithm
  - Step 3: Determine the product $X(k)H(k)$
  - Step 4: Take IDFT of the product $X(k)H(k)$ using radix-2 DIT FFT algorithm

Step 1:

$$x(n) = \{2, 2, 2, 0\};\ x_r(n) = \{2, 2, 2, 0\}$$

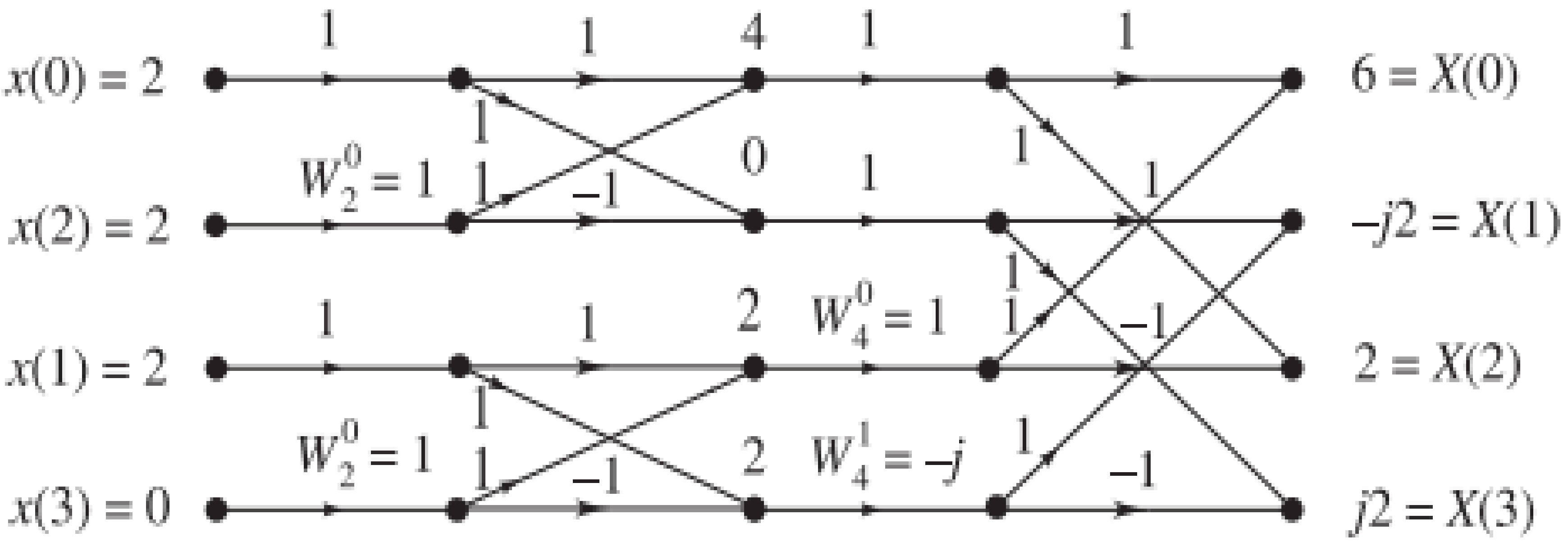


Figure 7.25 Computation of 4-point DFT of $x(n)$ by radix-2, DIT FFT.

From Figure 7.25, $X(k) = \{6, -j2, 2, j2\}$.

Step 2:

$$h(n) = \{-2, -2, 0, 0\};\ h_r(n) = \{-2, 0, -2, 0\}$$

From Figure 7.26, $H(k) = \{-4, -2 + j2, 0, -2 - j2\}$

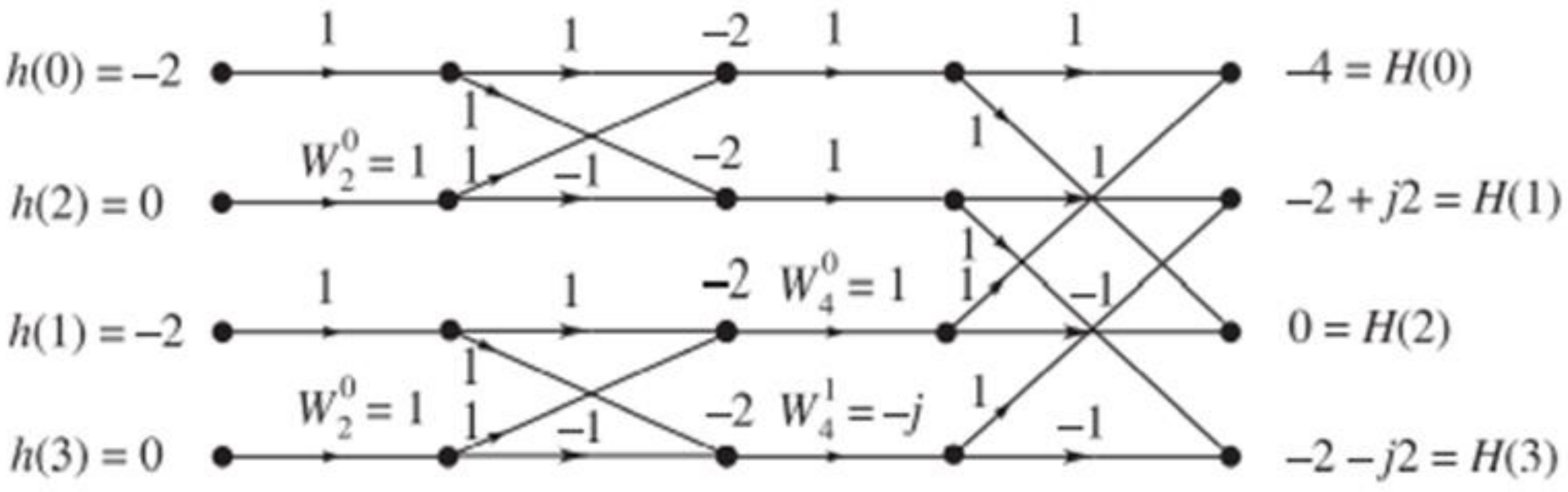


Figure 7.26 Computation of 4-point DFT of $h(n)$ by radix-2, DIT FFT.

Step 3:

$$Y(k) = X(k)H(k) = \{6, -j2, 2, j2\}\ \{-4,\ -2+j2, 0, -2-j2\}$$
$$= \{-24, 4+j4, 0, 4-j4\}$$

Step 4: Find Inverse FFT

Step 5: Final Answer: x(n) = {-4, -8, -8, -4}

Comparison of direct DFT computation & computation using FFT algorithms.

→ Table

Number of pts 'N'	Direct Computation		Using FFT	
	Complex multipl^n N^2	Complex Additions $(N^2 - N)$	Complex multipl^n $(N/2 \log_2 N)$	Complex Addition $(N \log_2 N)$
4	16	12	2	4
8	64	56	12	24
16	256	240	32	64
32	1024	992	40	80

Show & compare computational complexity is reduced if 32 point DFT is computed using Radix-2 DIT FFT alg.

32 point DFT using direct computation

Given ⇒ no. of pt. = 32

Number of complex multiplic^n = $N^2 = 32^2 = 1024$

Number of complex additions = $N^2 - N = 32^2 - 32$

$= 1024 - 32 = 992$

In case of Radix 2 DIT FFT alg.

Number of complex multiplic^n = $\frac{N}{2} \log_2 N$

$= \frac{32}{2} \log_2 32$

$= 16 \log_2 32 = 16 \times 5 = 80$

Number of complex add^n = $N \log_2 N = 32 \log_2 32$

$= 32 \times 5 = 160$

Comparing both answers, it is proved that computational complexity is reduced if 32 point DFT is computed using Radix-2 DIT FFT alg.

Find the number of complex additions & complex multiplicⁿ required to find DFT for 16 point signal. Compare them with no. of computation required if FFT alg. is used.

$N = 16$

Using DFT:

no. of complex multiplicⁿ $= N^2 = 16^2 = 256$

Using FFT:-

Number of Complex multiplications $= \frac{N}{2} \log_2 N = \frac{16}{2} \log_2 N$

$= 8 \log_2 16 = 32$

Number of Complex additions $= N \log_2 N = 16 \log_2 16 = 64$


THANKS