Digital Signal and Image Processing
CSC 701

Subject In-charge

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Module II

Discrete Fourier Transform

Introduction

- Fourier Series is the mathematical way of representing periodic signal.
- Fourier Transform is the mathematical way of representing the aperiodic signal.
- Both Fourier series and Fourier Transform convert a signal from time domain to frequency domain.
- Fourier Transform of discrete signal is of two types:
  - Discrete Time Fourier Transform (DTFT)
  - Discrete Fourier Transform (DFT)

- For more information refer https://www.youtube.com/watch?v=spUNpyF58BY

Diagrammatic representation of Fourier Transform, DTFT, DFT

Fig. shows development of DFT formula

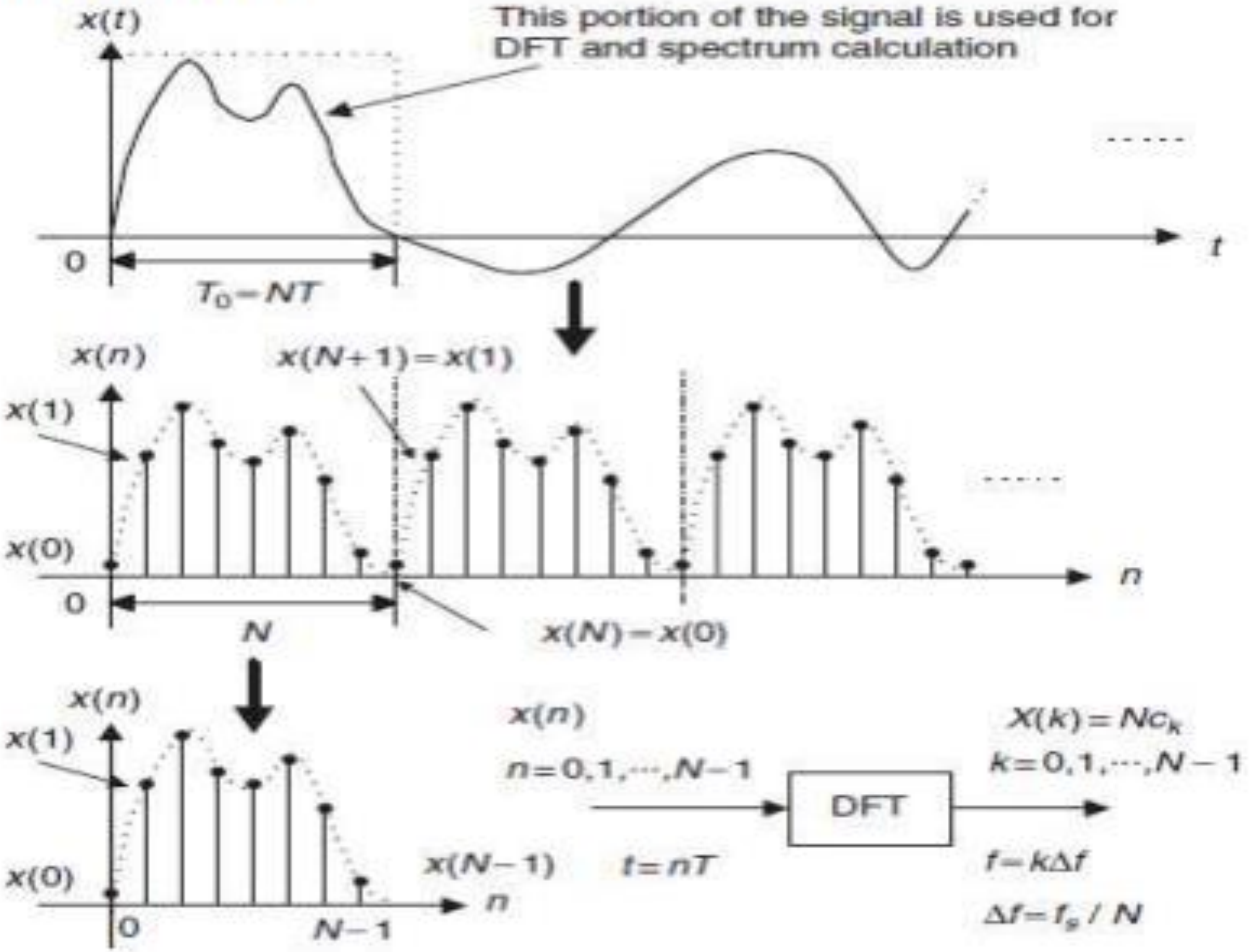


Fig. development of DFT formula

DTFT- Discrete Time Fourier Transform

- DTFT- Fourier transform of discrete-time signals is called the Discrete-Time Fourier Transform

$$x(n) \underset{\text{IDTFT}}{\overset{\text{DTFT}}{\rightleftarrows}} \underbrace{X(\omega) \text{ or } X(e^{j\omega})}_{\text{Signal in Frequency domain}}$$

Signal in Time domain

Signal in Frequency domain

Formula: $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

DTFT Example 1

$$x(n) = \{1, -2, 2, 3\}$$

$$X(\omega) = \mathrm{F}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= x(0) + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + x(3)\, e^{-j3\omega}$$

$$= 1 - 2e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega}$$

DTFT Example 2

Given $$x(n)=\begin{cases} n, & -4\le n\le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$X(\omega)=\mathrm{F}\{x(n)\}=\sum_{n=-4}^{4} ne^{-j\omega n}$$

$$=-4e^{j4\omega}-3e^{j3\omega}-2e^{j2\omega}-e^{j\omega}+e^{-j\omega}+2e^{-j2\omega}+3e^{-j3\omega}+4e^{-4j\omega}$$

$$=-2j\,\{4\sin 4\omega+3\sin 3\omega+2\sin 2\omega+\sin\omega\}$$

Limitations of DTFT

- DTFT transforms are defined for infinite-length signals
- DTFT’s are the function of continuous variable
- Hence they cannot be processed by computers and processors
- In other words, DTFT are not numerically computable

…………..these problems are overcome by DFT

- DFT is obtained by sampling DTFT

DFT- Discrete Fourier Transform

- DFT- The DFT is one of the most powerful tools in digital signal processing which enables us to find the spectrum of a finite-duration signal.

$$x(n) \underset{\textbf{IDFT}}{\overset{\textbf{DFT}}{\rightleftarrows}} X(k)$$

Signal in Time domain **Signal in Frequency domain**

Formula:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} = \sum_{n=0}^{N-1} x(n)\, e^{-j(2\pi/N)nk}$$

DFT Analysis and Synthesis

N= Period

The DFT transform:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}} \quad \text{analysis}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi\frac{kn}{N}} \quad \text{synthesis (IDFT)}$$

Alternative formulation:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W^{kn} \quad \longleftarrow \quad W = e^{-j\frac{2\pi}{N}}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W^{-kn}$$

Twiddle Factor: Used to speed up DFT and IDFT calculations

Types of DFT in our scope of syllabus

1. 2- point DFT
2. 4- point DFT
3. 8- point DFT

Few Basics..

$\cos \pi = -1$	$\sin \pi = 0$
$\cos \pi/2 = 0$	$\sin \pi/2 = 1$
$\cos 3\pi/2 = 0$	$\sin 3\pi/2 = -1$

Construction of Twiddle factor matrix (W) for different types of DFT

Twiddle factor Matrix for N-point DFT

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} n=0 & n=1 & n=2 & \cdots\cdots & n=N-1 \end{array} \\
\begin{array}{c} k=0 \\ k=1 \\ k=2 \\ \cdot \\ \cdot \\ \cdot \\ k=N-1 \end{array} & \underline{W_N^{kn}} = \begin{bmatrix}
W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn} \\
W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn} \\
W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
W_N^{kn} & W_N^{kn} & W_N^{kn} & \cdots\cdots & W_N^{kn}
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 0 & 1 & 2 & \cdot & \cdot & N \end{array} \\
\begin{array}{c} k=0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ N \end{array} & \begin{bmatrix} W_N^{kn} & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix}
\end{array}
$$

Twiddle factor Matrix for N-point DFT

$$
W_N^{kn} = \begin{array}{cc}
 & \begin{matrix} n=0 & n=1 & n=2 & \cdots\cdots & n=N-1 \end{matrix} \\
\begin{matrix} k=0 \\ k=1 \\ k=2 \\ \\ \vdots \\ \\ k=N-1 \end{matrix} &
\begin{bmatrix}
W_N^0 & W_N^0 & W_N^0 & \cdots\cdots & W_N^0 \\
W_N^0 & W_N^1 & W_N^2 & \cdots\cdots & W_N^{N-1} \\
W_N^0 & W_N^2 & W_N^4 & \cdots\cdots & W_N^{2(N-1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots\cdots & W_N^{(N-1)^2}
\end{bmatrix}
\end{array}
$$

2 –point DFT

Twiddle factor Matrix Calculation for 2- Point DFT

$N = 2$

$$W_N^{kn} = \begin{array}{c|cc} & n=0 & n=1 \\ \hline k=0 & W_2^0 & W_2^0 \\ k=1 & W_2^0 & W_2^1 \end{array}$$

$$W_2^0 = e^{-j2\pi \frac{kn}{N}} = e^{-j2\pi \cdot \frac{0}{2}}$$

$$= e^0 = 1.$$

$$W_2^1 = e^{-j2\pi \frac{kn}{N}} = e^{-j2\pi \cdot \frac{1}{2}}$$

$$= e^{-j\pi}$$

$$= \cos \pi - j \sin \pi$$

$$= -1$$

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Twiddle factor Matrix Calculation for 2- Point IDFT

$N = 2$

$$W_N^{kn} = \begin{array}{c|cc} & n=0 & n=1 \\ \hline k=0 & W_2^0 & W_2^0 \\ k=1 & W_2^0 & W_2^1 \end{array}$$

$$W_2^0 = e^{+j2\pi\frac{kn}{N}} = e^{+j2\pi\cdot\frac{0}{2}}$$

$$= e^0 = 1.$$

$$W_2^1 = e^{+j2\pi\frac{kn}{N}} = e^{+j2\pi\cdot\frac{1}{2}}$$

$$= e^{+j\pi}$$

$$= \cos\pi + j\sin\pi$$

$$= -1$$

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Problem on 2-point DFT

$x(n) = \{1, 1\}$ Find DFT & IDFT

$$X(k) = [W_N^{nk}]\, x(n)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

∴ 2 point DFT is

$X(k) = \{2, 0\}$

IDFT: $x(k) = \frac{1}{N} [W_N^{*kn}] \cdot X'(k)$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2+0 \\ 2-0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

4–point DFT

Twiddle factor Matrix Calculation for 4- Point DFT

$N = 4$

i.e. $n = 0$ to 3

$k = 0$ to 3

$$W_4 = \begin{array}{c|cccc} & n=0 & n=1 & n=2 & n=3 \\ \hline k=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ k=1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ k=2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ k=3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array}$$

$$W_4^0 = e^{-j 2\pi nk/N}$$
$$= e^{-j 2\pi (0)/N}$$
$$= e^0$$
$$= 1$$

$$W_4^1 = e^{-j 2\pi \cdot \frac{nk}{N}}$$
$$= e^{-j 2\pi (1)/4}$$
$$= e^{-j \pi/2}$$
$$= \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$
$$= 0 - j = -j$$

$$W_4^2 = e^{-j 2\pi \frac{nk}{N}}$$
$$= e^{-j 2\pi \frac{2}{4}}$$
$$= e^{-j\pi}$$
$$= \cos\pi - j\sin\pi$$
$$= -1$$

Twiddle factor Matrix Calculation for 4- Point DFT

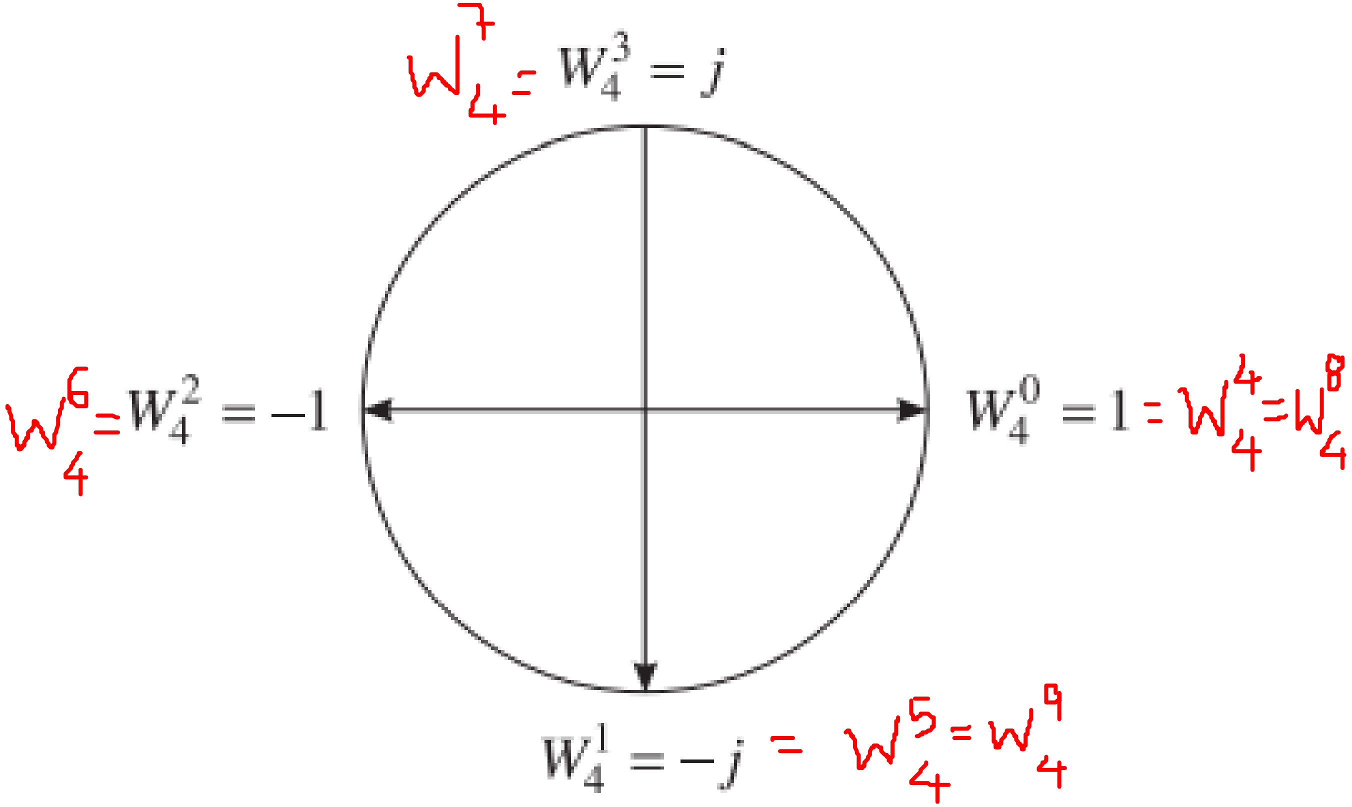
$$W_4^3 = e^{-j 2\pi \frac{nk}{N}}$$

$$= e^{-j \cdot 2\pi \frac{3}{4}}$$

$$= e^{-j3\pi/2}$$

$$= \cos\frac{3\pi}{2} - j\sin 3\pi/2$$

$$= j$$



(a) 4-point DFT

Twiddle factor Matrix for 4- Point DFT & IDFT

DFT

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

IDFT

- Find complex conjugate i.e. change the sign of ‘j’

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Problem-1: Calculate the four point DFT of four point sequence x(n)=(0,1,2,3).

Solution: The four point DFT in matrix form is given by

$$X_4 = [W_4]x_4$$

$$\therefore X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore X_4 = \begin{bmatrix} 0+1+2+3 \\ 0-j-2+3j \\ 0-1+2-3 \\ 0+j-2-3j \end{bmatrix} = \begin{bmatrix} 6 \\ 2j-2 \\ -2 \\ -2j-2 \end{bmatrix}$$

$$\therefore X_4 = \{6,\ 2j-2,\ -2,\ -2j-2\}$$

Find IDFT of above sequence.

$$\text{IDF} = x_n = \frac{1}{N}\left[W_N^*\right] X_N$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 6 - 2+2j - 2 - 2 - 2j \\ 6 - 2j + 2j^2 + 2 + 2j + 2j^2 \\ 6 + 2 - 2j - 2 + 2 + 2j \\ 6 + 2j - 2j^2 + 2 - 2j - 2j^2 \end{bmatrix}$$

$j^2 = -1$

$$= \frac{1}{4}\begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

EXAMPLE 6.8 Find the DFT of the sequence

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

Find IDFT of above sequence.

$$IDF(x(n)) = \frac{1}{N}\left[W_N^{*nk}\right] X(k)$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 - 2j + 0 + 2j \\ 4 - 2j^2 - 0 - 2j^2 \\ 4 + 2j + 0 - 2j \\ 4 + 2j^2 + 0 + 2j^2 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE 6.9 Find the DFT of $x(n) = \{1, -1, 2, -2\}$.

Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$$\therefore \quad \text{DFT}\,\{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$$

EXAMPLE 6.10 Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

Solution: Given $x(n) = \{1, -2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix as shown below.

$$\text{DFT}\{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

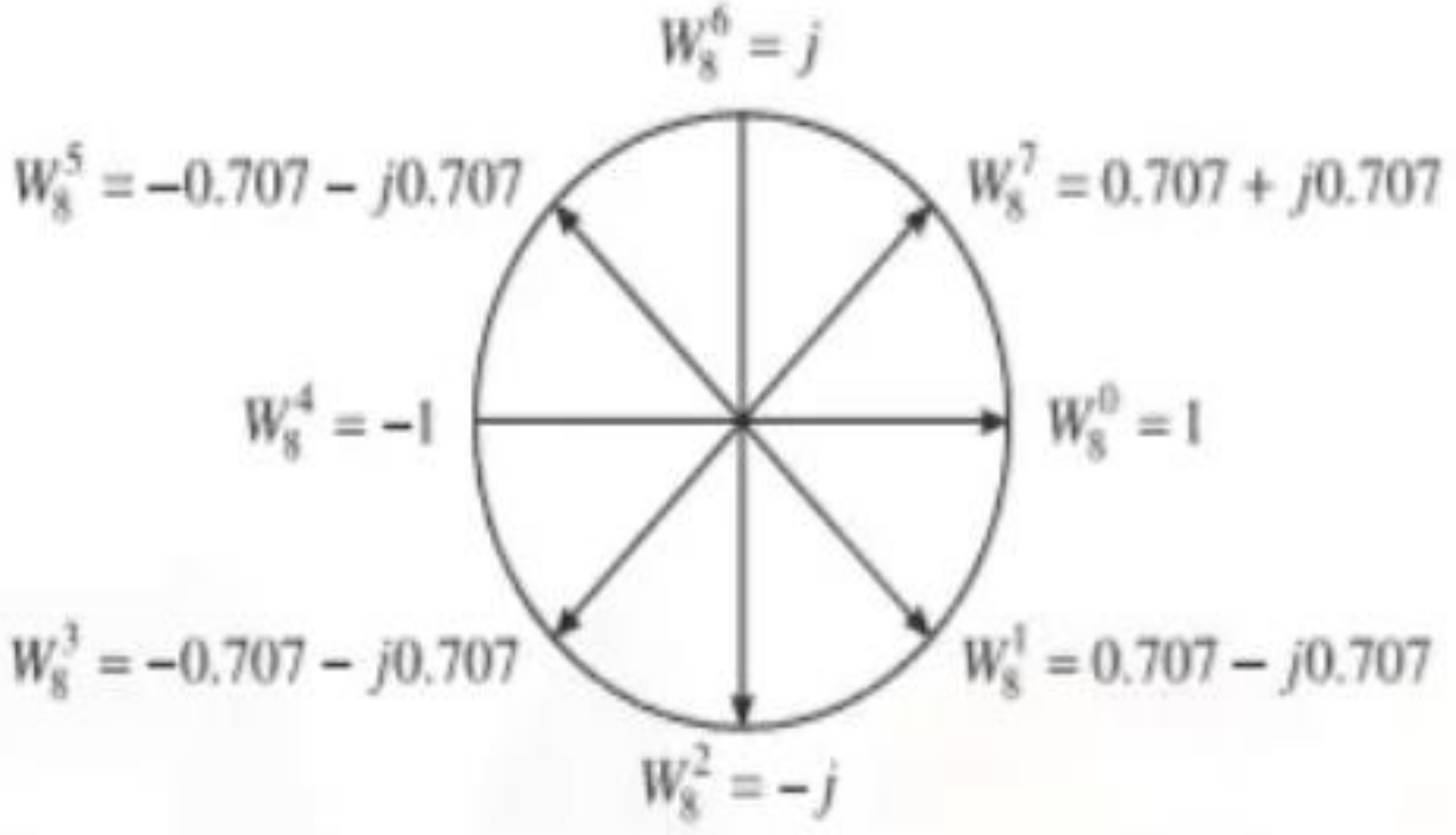
$$\therefore \quad \text{DFT}\{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$$

8–point DFT

Twiddle factor Matrix for 8- Point DFT

$$
W_8 = \begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \\ k=5 \\ k=6 \\ k=7 \end{array}
\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\
W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\
W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\
W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\
W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\
W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\
W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\
W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49}
\end{bmatrix}
$$

Twiddle factor Matrix for 8- Point DFT



(b) 8-point DFT

Twiddle factor Matrix for 8- Point DFT

$$
X_8 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\
1 & -j & -1 & j & 1 & -j & -1 & j \\
1 & -0.707+j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\
1 & j & -1 & -j & 1 & j & -1 & -j \\
1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707
\end{bmatrix}
$$

Comparison between DTFT and DFT

- DFT is a sampled version of DTFT, where the frequency term ω is sampled. But, we know that DTFT is obtained by using the sampled form of input signal $x(t)$. So, we find that DFT is obtained by the double sampling of $x(t)$.
- DFT gives only positive frequency values, whereas DTFT can give both positive and negative frequency values.
- DTFT and DFT coincide at intervals of $\omega = 2\pi k/N$, where $k = 0, 1, \ldots, N - 1$.
- To get more accurate values of DFT, number of samples N must be very high but when N is very high, the required computation time will also be very high.