Informative hypotheses evaluation

Bayesian model selection

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bain

1. Hoijtink, H., Mulder, J., van Lissa, C., and Gu, X. (2018). A tutorial on testing hypotheses using the Bayes factor. *Psychological Methods, 24*, 539-556.

Bayes Factor

Balancing Fit and Complexity

The Bayes factor quantifies the relative support in the data for two hypotheses, for example,

$$H_i : \mu_1 > \mu_2 > \mu_3$$

$$H_u : \mu_1, \mu_2, \mu_3$$

with

$$BF_{iu} = \frac{f_i}{c_i} = \frac{\text{fit } H_i}{\text{complexity } H_i}$$

that is, after observing the data H_i is BF_{iu} times as likely as H_u, for example, .2, 5, 10.

Bayes Factor

Balancing Fit and Complexity

A (very) loose interpretation of the meaning of fit

$$H_i : \mu_1 > \mu_2 > \mu_3$$

if $\bar{x}_1 = 7$ & $\bar{x}_2 = 4$ & $\bar{x}_3 = 2$ the fit is good

if $\bar{x}_1 = 2$ & $\bar{x}_2 = 4$ & $\bar{x}_3 = 7$ the fit is bad

Bayes Factor

Balancing Fit and Complexity

A (very) loose interpretation of the meaning of complexity

$$H_1 : \mu_1 = \mu_2 = \mu_3$$

very parsimonious, the means have to be exactly equal.

$$H_1 : \mu_1 > \mu_2 > \mu_3$$

one ordering of three means: 1-2-3, thus is parsimonious.

$$H_2 : \mu_1 > (\mu_2, \mu_3)$$

2 orderings of three means: 1-2-3 and 1-3-2, less parsimonious.

$$H_u : \mu_1, \mu_2, \mu_3$$

contains all six possible orderings of three means, not parsimonious.

Bayes Factor

Balancing Fit and Complexity

Three forms of Hypotheses and Bayes factors involving

$$H_i : \mu_1 > \mu_2 > \mu_3$$

BF_{iu} evaluating H_i versus $H_u : \mu_1, \mu_2, \mu_3$

$BF_{ii'}$ evaluating H_i versus $H_{i'} : \mu_1 = \mu_2 = \mu_3$

BF_{ic} evaluating H_i versus $H_c :$ not H_i

Bayes Factor

Interpreting (the Size of) the Bayes Factor

1. Select the best of a set of hypotheses using BF_{iu}
2. Compare two competing hypotheses using $BF_{ii'}$
3. Compare "my theory" with "not my theory" using BF_{ic}

	f_i	c_i	BF_{iu}	BF_{ic}
H_1 : Sex Match	.0039	.012	.32	.32
H_2 : Gender Role Match	.0725	.012	5.85	6.44
H_3 : Sex Mismatch	.0007	.012	.06	.06
H_4 : Gender Role Mismatch	.0001	.012	.01	.01

Bayes Factor

Descriptives

Gender Role Match Effect

$$H_2 : (\mu_1, \mu_5) > (\mu_2, \mu_3, \mu_4, \mu_6) \text{ and } (\mu_7, \mu_{11}) > (\mu_8, \mu_9, \mu_{10}, \mu_{12})$$

$$H_2 : (166, 163) > (158, 154, 155, 164) \text{ and }$$
$$(157, 152) > (157, 150, 143, 149)$$

Gender Role Mismatch Effect

$$H_4 : (\mu_2, \mu_4) > (\mu_1, \mu_3, \mu_5, \mu_6) \text{ and } (\mu_8, \mu_{10}) > (\mu_7, \mu_9, \mu_{11}, \mu_{12})$$

$$H_4 : (158, 155) > (166, 154, 163, 164) \text{ and }$$
$$(157, 143) > (157, 150, 152, 149)$$

Bayes Factor

Interpreting (the Size of) the Bayes Factor

1. The Bayes factor **is** a measure of support (also for the null-hypothesis)
2. The Bayes factor **can be indecisive.** A value around 1 denotes "the data don't tell us which hypothesis to prefer"
3. One **can compare** more than two hypotheses
4. "Something is going on and **we do know what!**"
5. The Bayes factor **selects the best of the hypotheses under consideration.** Note that the "true" hypothesis may not be among them, and that all hypotheses may be "wrong"

Note on hypotheses

H_i contains 1 ordering of means:

1. $\mu_1 > \mu_2 > \mu_3$

H_c contains 5 orderings of means:

2. $\mu_1 > \mu_3 > \mu_2$
3. $\mu_2 > \mu_1 > \mu_3$
4. $\mu_2 > \mu_3 > \mu_1$
5. $\mu_3 > \mu_1 > \mu_2$
6. $\mu_3 > \mu_2 > \mu_1$

H_u combines H_i and H_c.

Subjectivity of Bayesian Hypotheses Evaluation

1. Which hypotheses to evaluate?
2. How to formalize hypotheses?
   E.g. $(\mu_1, \mu_2) > (\mu_3, \mu_4)$ or $\mu_1 = \mu_2 > \mu_3 = \mu_4$
3. The (implicit) choice for equal prior model probabilities
4. The specification of the prior distribution

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Informative Hypotheses

Example 1: ANOVA

What is the relation between "knowledge of numbers after watching Sesame Street for a year"

and

site from which the child originates (1 = disadvantaged inner city, 2 = advantaged suburban , 3 = advantaged rural, 4 = disadvantaged rural, 5 = disadvantaged Spanish speaking).

Informative Hypotheses

Example 1: ANOVA

```
library(bain)
sesamesim$site <- as.factor(sesamesim$site)
anov <- lm(postnumb~site-1,sesamesim)
coef(anov)
set.seed(100)
results <- bain(anov,
               "site1=site2=site3=site4=site5;
                site2>site5>site1>site3>site4")
print(results)
summary(results, ci = 0.95)
```

Informative Hypotheses

Example 1: ANOVA

coef(anov) renders

```
   site1    site2    site3    site4    site5 
29.66667 38.98182 23.18750 25.32558 31.72222 
```

summary(results) renders

```
  Parameter  n Estimate       lb       ub
1     site1 60 29.66667 26.82991 32.50343
2     site2 55 38.98182 36.01892 41.94472
3     site3 64 23.18750 20.44082 25.93418
4     site4 43 25.32558 21.97466 28.67650
5     site5 18 31.72222 26.54303 36.90141
```

Informative Hypotheses

Example 1: ANOVA

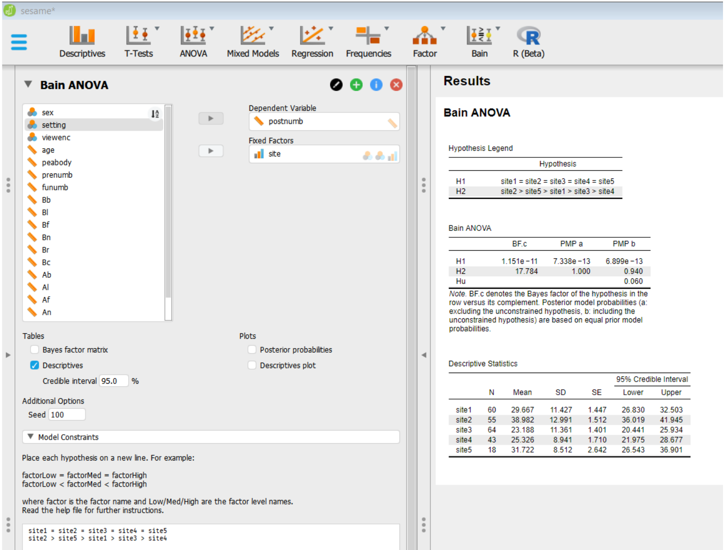
The main output is

	Fit	Com	BF.u	BF.c	PMPa	PMPb	PMPc
H1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
H2	0.121	0.008	14.559	16.428	1.000	0.936	0.943
Hu						0.064	
Hc	0.879	0.992	0.886				0.057

```
Hypotheses:
  H1: site1=site2=site3=site4=site5
  H2: site2>site5>site1>site3>site4
```

Informative Hypotheses

Example 1: ANOVA



Informative Hypotheses

Example 2: ANOVA Interaction Effect

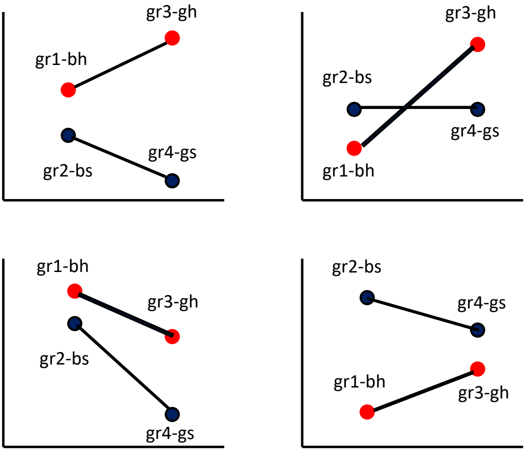
Dependent variable: Knowledge of numbers.

Factors: sex (boy, girl) and setting (watching at home, watching at school).

Gr: 1=boyhome, 2= boyschool, 3= girlhome, 4=girlschool.

Informative Hypotheses

Example 2: ANOVA Interaction Effect



Hi :

$$gr2 - gr1 > gr4 - gr3$$

and

$$gr2 > gr1$$

$$gr2 > gr4$$

Informative Hypotheses

Example 2: ANOVA Interaction Effect

```
sesamesim$gr <- as.factor(sesamesim$gr)
anov <- lm(postnumb~gr-1,sesamesim)
results <- bain(anov,
"gr2 - gr1 > gr4 - gr3 & gr2 > gr1 & gr2 > gr4")
```

Informative Hypotheses

Example 2: ANOVA Interaction Effect

The main output is

```
   Fit   Com   BF.u  BF.c   PMPa  PMPb  PMPc
H1 0.922 0.283 3.262 29.984 1.000 0.765 0.968
Hu                                0.235
Hc 0.078 0.717 0.109                    0.032

Hypotheses:
  H1: gr2-gr1>gr4-gr3&gr2>gr1&gr2>gr4
```

Informative Hypotheses

How to write down an hypothesis

bain can handle hypotheses build using constraints on (linear combinations) of parameters. Suppose the parameter names are "a", "b", "c".

Step 1: Construct the elements of the linear combination. E.g. "a" or "a + 2" or "3 * a" or "2 * a + 4"

Step 2: Constrain the resultsing elements. E.g. a > b > c

or a > b + 2 & b > c + 2

or 2 * a > b + c & b > 0 & c > 0

or a > (b, c) & b - c > 0

Informative Hypotheses

Example 3: Repeated Measures

Development of depression

	Measurement			
	8 years	12 years	16 years	20 years
Men	μ_1	μ_2	μ_3	μ_4
Women	μ_5	μ_6	μ_7	μ_8

$$H_1 : \mu_5 - \mu_1 > \mu_6 - \mu_2 > \mu_7 - \mu_3 < \mu_8 - \mu_4$$

$$H_2 : \mu_6 - \mu_5 < \mu_7 - \mu_6 > \mu_8 - \mu_7$$

Informative Hypotheses

Example 4: Multiple Regression

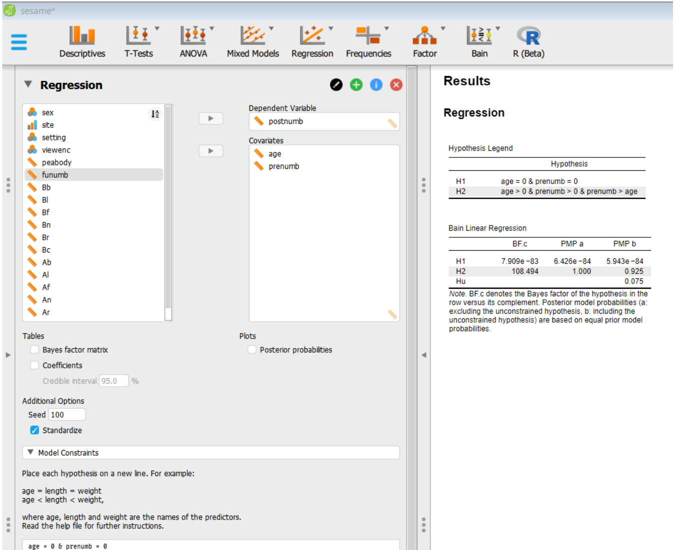
$$\text{postnumb}_i = \beta_0 + \beta_1 \times \text{age}_i + \beta_2 \times \text{prenumb}_i + \epsilon_i$$

$$H_1 : \beta_1 > 0, \beta_2 > 0, \beta_1 < \beta_2$$

Note: β_1 and β_2 are only comparable if age and prenumb are standardized

Informative Hypotheses

Example 4: Multiple Regression



Informative Hypotheses

Example 5: About Equality Constraints

Is the difference in number knowledge relevantly different between boys and girls?

Informative Hypotheses

Example 5: About Equality Constraints

```
sesamesim$sex <- as.factor(sesamesim$sex)
anov <- lm(postnumb~sex-1,sesamesim)
results <- bain(anov, "-2 < sex1 - sex2 < 2")
```

Informative Hypotheses

Example 5: About Equality Constraints

```
sex1     sex2
30.09565 28.85600

   Fit   Com   BF.u  BF.c   PMPa  PMPb  PMPc
H1 0.664 0.091 7.304 19.735 1.000 0.880 0.952
Hu                                0.120
Hc 0.336 0.909 0.370                    0.048

Hypotheses:
  H1: -2<sex1-sex2<2
```

Informative Hypotheses

Example 6: Structural Equation Modelling

```
library(bain)
library(lavaan)

model <- '
    A  =~ Ab + Al + Af + An + Ar + Ac
    B  =~ Bb + Bl + Bf + Bn + Br + Bc
    A  ~ B + age + peabody'
fit <- sem(model, data = sesamesim, std.lv = TRUE)

hypotheses <- "A~B = A~peabody = A~age = 0;
               A~B > A~peabody > A~age = 0"
set.seed(100)
y1 <- bain(fit, hypotheses, standardize = TRUE)
```

Hands-on/Demo: BMS

Let's practice.

- If needed: Start Rstudio again (optional: make project).
- Open '**Hands-on_1_BMS**_Unc_ANOVA_bain.R' (in 'Hands-on files').
- Install packages and load them.
- Read and inspect data. Use Data_Lucas.txt.
- Run model (lm()).
- Specify hypotheses (make up your own).
  Note: Use names used in the model.
- Run bain().
- Inspect and interpret output.

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Your hypothesis of interest

If you have your own data

Before:

- What is your research question?
- What is your theory / expectation?
- What is your statistical hypothesis?
- Is there a competing statistical hypothesis?

Additionally:

- Are you able to specify your statistical hypothesis/-es?
- How will you evaluate it/them?
  (preference GORIC(A) or BMS or both?)

What's next

Depending on time and wishes:

- Some extra information
- Demo in JASP (GORIC(A) and/or BMS)
- Demo in R (GORIC(A) and/or BMS)

We end with:

- Lab: `https://github.com/rebeccakuiper/EMLaR---Informative-Hypothesis-Evaluation`

The End

BMS

Thanks for listening!

Are there any questions?

Websites

https://github.com/rebeccakuiper/Tutorials
www.uu.nl/staff/RMKuiper/Software
www.uu.nl/staff/RMKuiper/Extra2
informative-hypotheses.sites.uu.nl/software/goric/

E-mail

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Bayes Factor (BF)

comparing two informative hypotheses

The BF quantifies the relative support in the data for two hypotheses.

$$BF_{12} = \frac{BF_{1u}}{BF_{2u}} = \frac{f_1/f_2}{c_1/c_2}$$

using

$$BF_{iu} = \frac{f_i/f_u}{c_i/c_u} = \frac{f_i}{c_i}$$

A Closer Look at the Bayes Factor

A Closer Look at the Bayes Factor

Three Simple Hypotheses

Consider the hypotheses:

$$H_1 : \mu_1 \approx \mu_2, \text{ that is, } |\mu_1 - \mu_2| < .1$$

$$H_2 : \mu_1 > \mu_2$$

$$H_3 : \mu_1, \mu_2$$

A Closer Look at the Bayes Factor

Information in the Data about the Two Means

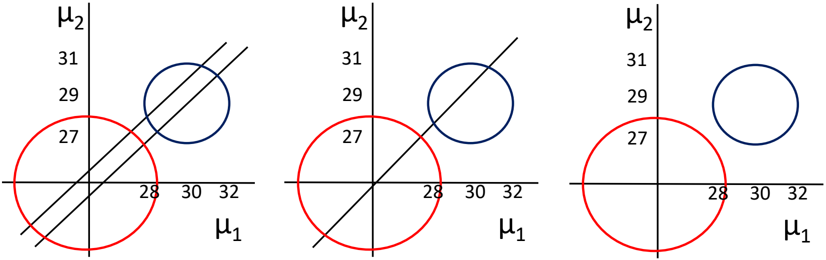
	N	Mean	SD	SE	95% Credible Interval	
					Lower	Upper
sex1	115	30.096	13.058	1.175	27.793	32.398
sex2	125	28.856	12.162	1.127	26.647	31.065

$$g(\mu_1, \mu_2 \mid \text{data}) \approx \mathcal{N}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} se_1^2 = \frac{SD_1^2}{N_1} & 0 \\ 0 & se_2^2 = \frac{SD_2^2}{N_2} \end{bmatrix} \right),$$

A Closer Look at the Bayes Factor

Posterior Distribution, Prior Distribution, and Hypotheses

$H_1: \mu_1 \approx \mu_2$ $\quad H_2: \mu_1 > \mu_2$ $\quad H_u: \mu_1, \mu_2$



$BF_{1u} = f_1/c_1 = .25/.05 = 5$ $\quad BF_{2u} = f_2/c_2 = .75/.5 = 1.5$

$BF_{12} = 5/1.5 = 3.33$

A Closer Look at the Bayes Factor

Fit and Complexity

1. The **fit** of a hypothesis is the proportion of the **posterior** distribution in agreement with the hypothesis.
   Note: posterior = likelihood $\times$ prior.

2. The **complexity** of a hypothesis is the proportion of the **prior** distribution in agreement with the hypothesis.

A Closer Look at the Bayes Factor

The Prior Distribution

$$h(\mu_1, \mu_2 \mid \text{data}) \approx \mathcal{N}\left(\begin{bmatrix} m \\ m \end{bmatrix}, \begin{bmatrix} \frac{SD_1^2}{J} & 0 \\ 0 & \frac{SD_2^2}{J} \end{bmatrix}\right),$$

where μ_1 and μ_2 have the same prior mean m, and where J denotes the size of the training sample.

Possible choices for J for the example at hand:

- use the default in `bain`: J = the number of independent constraints, here, 1. This is a conservative choice.
  Sensitivity check: e.g., J, 2*J, and 3*J (fraction = 1, 2, and 3, resp.).
- or use J = the minimal training sample size, here, 4 because four observations are needed to estimate two means and two variances.
- or use J = *Jref*, which renders $BF_{0u} = 19$ if the effect size in the sample equals 0.

A Closer Look at the Bayes Factor

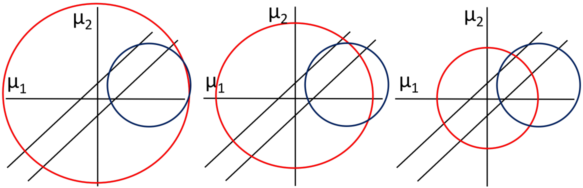
Posterior Distribution (f), Prior Distribution (c), and Hypotheses

Prior Sensitivity for = Constrained Hypotheses

Here, used bain: J = 1.
Sensitivity check: J, 2*J, and 3*J (i.e., fraction = 1, 2, and 3).

$H_1: \mu_1 \approx \mu_2$

J = 1 | 2*J = 2 | 3*J = 3

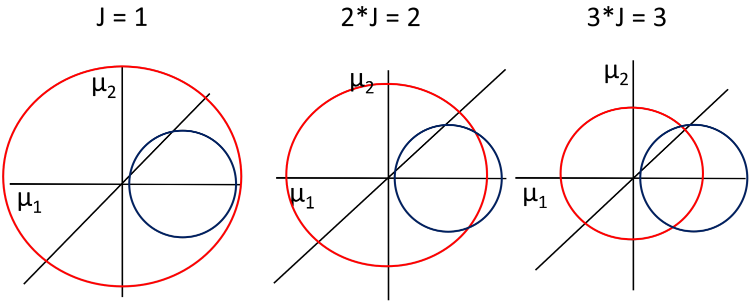


BF_{1u} = .2/.01 = 20 | BF_{1u} = .2/.05 = 4 | BF_{1u} = .2/.2 = 1

A Closer Look at the Bayes Factor

Posterior Distribution (f), Prior Distribution (c), and Hypotheses
Prior In-Sensitivity for > < Constrained Hypotheses

$H_2: \mu_1 > \mu_2$



$BF_{2u} = .9/.5 = 1.8$ $BF_{2u} = .9/.5 = 1.8$ $BF_{2u} = .9/.5 = 1.8$