Informative hypotheses evaluation

Bayesian model selection

Rebecca M. Kuiper
(credits slides: Herbert Hoijtink and others)

Department of Methodology & Statistics
Utrecht University

Table of Contents

bain

1. Hoijtink, H., Mulder, J., van Lissa, C., and Gu, X. (2018). A tutorial on testing hypotheses using the Bayes factor. *Psychological Methods, 24*, 539-556.

Bayes Factor

Balancing Fit and Complexity

The Bayes factor quantifies the relative support in the data for two hypotheses, for example,

$$H_i : \mu_1 > \mu_2 > \mu_3$$

$$H_u : \mu_1, \mu_2, \mu_3$$

with

$$BF_{iu} = \frac{f_i}{c_i} = \frac{\text{fit } H_i}{\text{complexity } H_i}$$

that is, after observing the data H_i is BF_{iu} times as likely as H_u, for example, .2, 5, 10.

Bayes Factor

Balancing Fit and Complexity

A (very) loose interpretation of the meaning of fit

$$H_i : \mu_1 > \mu_2 > \mu_3$$

if $\bar{x}_1 = 7$ & $\bar{x}_2 = 4$ & $\bar{x}_3 = 2$ the fit is good

if $\bar{x}_1 = 2$ & $\bar{x}_2 = 4$ & $\bar{x}_3 = 7$ the fit is bad

Bayes Factor

Balancing Fit and Complexity

A (very) loose interpretation of the meaning of complexity

$$H_1 : \mu_1 = \mu_2 = \mu_3$$

very parsimonious, the means have to be exactly equal.

$$H_1 : \mu_1 > \mu_2 > \mu_3$$

one ordering of three means: 1-2-3, thus is parsimonious.

$$H_2 : \mu_1 > (\mu_2, \mu_3)$$

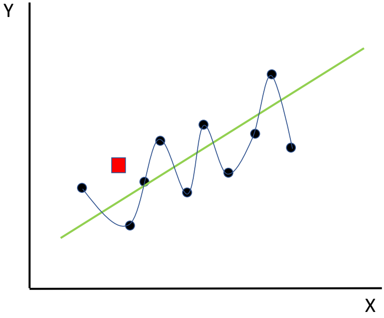
2 orderings of three means: 1-2-3 and 1-3-2, less parsimonious.

$$H_u : \mu_1, \mu_2, \mu_3$$

contains all six possible orderings of three means, not parsimonious.

Bayes Factor

Balancing Fit and Complexity



The straight line results from a linear regression model with 3 parameters (intercept, slope, residual variance).

The other line results from a polynomial regression models with 11 parameters (intercept, nine slopes, residual variance).

The red square is a new observation that is added to the original 10 observations.

What is the predictive value of both models?

Bayes Factor

Balancing Fit and Complexity

Three forms of Hypotheses and Bayes factors involving

$H_i : \mu_1 > \mu_2 > \mu_3$

BF_{iu} evaluating H_i versus $H_u : \mu_1, \mu_2, \mu_3$

$BF_{ii'}$ evaluating H_i versus $H_{i'} : \mu_1 = \mu_2 = \mu_3$

BF_{ic} evaluating H_i versus $H_c :$ not H_i

Bayes Factor

Interpreting (the Size of) the Bayes Factor

1. Select the best of a set of hypotheses using BF_{iu}
2. Compare two competing hypotheses using $BF_{ii'}$
3. Compare "my theory" with "not my theory" using BF_{ic}

	f_i	c_i	BF_{iu}	BF_{ic}
H_1 : Sex Match	.0039	.012	.32	.32
H_2 : Gender Role Match	.0725	.012	5.85	6.44
H_3 : Sex Mismatch	.0007	.012	.06	.06
H_4 : Gender Role Mismatch	.0001	.012	.01	.01

Bayes Factor

Descriptives

Gender Role Match Effect

$$H_2 : (\mu_1, \mu_5) > (\mu_2, \mu_3, \mu_4, \mu_6) \text{ and } (\mu_7, \mu_{11}) > (\mu_8, \mu_9, \mu_{10}, \mu_{12})$$

$$H_2 : (166, 163) > (158, 154, 155, 164) \text{ and }$$
$$(157, 152) > (157, 150, 143, 149)$$

Gender Role Mismatch Effect

$$H_4 : (\mu_2, \mu_4) > (\mu_1, \mu_3, \mu_5, \mu_6) \text{ and } (\mu_8, \mu_{10}) > (\mu_7, \mu_9, \mu_{11}, \mu_{12})$$

$$H_4 : (158, 155) > (166, 154, 163, 164) \text{ and }$$
$$(157, 143) > (157, 150, 152, 149)$$

Bayes Factor

Interpreting (the Size of) the Bayes Factor

1. The Bayes factor **is** a measure of support (also for the null-hypothesis)
2. The Bayes factor **can be indecisive.** A value around 1 denotes "the data don't tell us which hypothesis to prefer"
3. One **can update**, that is, collect more data and recompute the Bayes factor
4. One **can compare** more than two hypotheses (see extra comments later on)
5. "Something is going on and **we do know what!**
6. The Bayes factor **selects the best of the hypotheses under consideration.** Note that the "true" hypothesis may not be among them, and that all hypotheses may be "wrong"

Bayes Factor

Interpreting (the Size of) the Bayes Factor

When is the Bayes factor large enough?

1. Guidelines by Jeffreys (1969) and Kass and Raftery (1995), e.g., < 3 is ignorable, > 3 is positive evidence, > 10 is strong evidence ...
2. Will lead to a return of sloppy science and publication bias (when used without pre-registration or a pre-registered report)
3. Were does the 3 come from?

Bayes Factor

Interpreting (the Size of) the Bayes Factor

When is the Bayes factor large enough?

1. Before collecting or accessing the data, formulate informative hypotheses (and decide how large you would like the Bayes factor to be).
2. Insert this information in a pre-registration or pre-registered report.
3. Collect data and evaluate hypotheses.
   - Is one good and the best with a "large" Bayes factor: nice!
   - Are the Bayes factors "not large enough": follow up research or updating is needed.
   - Is none good: BIG news, well-constructed hypotheses have been rejected!

Extra: PMPs

Posterior Model Probabilities, e.g., $PMP(H_i \mid \text{data})$ and $PMP(H_c \mid \text{data})$ quantify the support in the data for each hypothesis.

$$\frac{PMP(H_i|\text{data})}{PMP(H_c|\text{data})} = \text{BF}_{ic} \times \frac{PRI(H_i)}{PRI(H_c)}, \tag{1}$$

where $PRI(H_i)$ and $PRI(H_c)$ denote the *prior* probabilities, that is, an evaluation of the support for the hypotheses *before* observing the data.

Usually equal prior model probabilities are used (which means that the PMP's convey the same information as the Bayes factors), but this is not a requirement.

PMPs

Bayesian Error Probabilities

PMPs can be interpreted as Bayesian error probabilities, that is, the Bayesian counterparts of the Type I and Type II errors.

	f_i	c_i	BF_{iu}	PMP_i	PRI_i
H_1 : Sex Match	.0039	.012	.32	.04	1/5
H_2 : Gender Role Match	.0725	.012	5.85	.81	1/5
H_3 : Sex Mismatch	.0007	.012	.06	.01	1/5
H_4 : Gender Role Mismatch	.0001	.012	.01	.00	1/5
H_u :				.14	1/5

Note on hypotheses

H_i contains 1 ordering of means:

1. $\mu_1 > \mu_2 > \mu_3$

H_c contains 5 orderings of means:

2. $\mu_1 > \mu_3 > \mu_2$
3. $\mu_2 > \mu_1 > \mu_3$
4. $\mu_2 > \mu_3 > \mu_1$
5. $\mu_3 > \mu_1 > \mu_2$
6. $\mu_3 > \mu_2 > \mu_1$

H_u combines H_i and H_c.

PMPs

Replacing H_u by H_c

	f_i	c_i	BF_{iu}	PMP_i	PRI_i
H_1 : Sex Match	.0039	.012	.32	.04	1/5
H_2 : Gender Role Match	.0725	.012	5.85	.84	1/5
H_3 : Sex Mismatch	.0007	.012	.06	.00	1/5
H_4 : Gender Role Mismatch	.0001	.012	.01	.00	1/5
H_c :	.9200	.9500	.97	.12	1/5

Where H_c denotes the complement H_1 through H_4, that is, "not one of these four hypotheses".

PMPs

The Number of Hypotheses and PMPs

Look what happens if we compare many hypotheses, the PMPs become smaller and smaller, and thus the Bayesian error probabilities become larger and larger:

	f_i	c_i	BF_{iu}	PMP_i	PRI_i
H_1 : Sex Match	.0039	.012	.32	.013	1/13
H_2 : Gender Role Match	.0725	.012	5.85	.270	1/13
H_3 : Sex Mismatch	.0007	.012	.06	.003	1/13
H_4 : Gender Role Mismatch	.0001	.012	.01	.000	1/13
H_5 : Lets try this one too	.0521	.012	2.61	.180	1/13
...					
H_{12} : Don't miss something	.0164	.012	1.36	.040	1/13
H_u :				.047	1/13

PMPs

The Number of Hypotheses and PMPs

The same results as two slides up are in fact obtained by assigning PMPs of 0 to each hypothesis that is NOT considered:

	f_i	c_i	BF_{iu}	PMP_i	PRI_i
H_1 : Sex Match	.0039	.012	.32	.04	1/5
H_2 : Gender Role Match	.0725	.012	5.85	.81	1/5
H_3 : Sex Mismatch	.0007	.012	.06	.01	1/5
H_4 : Gender Role Mismatch	.0001	.012	.01	.00	1/5
H_5 : Lets try this one too	.0521	.012	2.61	.18	0
...					
H_{12} : Don't miss something	.0164	.012	1.36	.04	0
H_u :				.14	1/5

Subjectivity of Bayesian Hypotheses Evaluation

1. Which hypotheses to evaluate?
2. How to formalize hypotheses?
   E.g. $(\mu_1, \mu_2) > (\mu_3, \mu_4)$ or $\mu_1 = \mu_2 > \mu_3 = \mu_4$
3. The (implicit) choice for equal prior model probabilities
4. The specification of the prior distribution

Table of Contents

Informative Hypotheses

Example 1: ANOVA

What is the relation between "knowledge of numbers after watching Sesame Street for a year"

and

site from which the child originates (1 = disadvantaged inner city, 2 = advantaged suburban , 3 = advantaged rural, 4 = disadvantaged rural, 5 = disadvantaged Spanish speaking).

Informative Hypotheses

Example 1: ANOVA

```
library(bain)
sesamesim$site <- as.factor(sesamesim$site)
anov <- lm(postnumb~site-1,sesamesim)
coef(anov)
set.seed(100)
results <- bain(anov,
               "site1=site2=site3=site4=site5;
                site2>site5>site1>site3>site4")
print(results)
summary(results, ci = 0.95)
```

Informative Hypotheses

Example 1: ANOVA

coef(anov) renders

```
   site1    site2    site3    site4    site5
29.66667 38.98182 23.18750 25.32558 31.72222
```

summary(results) renders

```
  Parameter  n Estimate       lb       ub
1     site1 60 29.66667 26.82991 32.50343
2     site2 55 38.98182 36.01892 41.94472
3     site3 64 23.18750 20.44082 25.93418
4     site4 43 25.32558 21.97466 28.67650
5     site5 18 31.72222 26.54303 36.90141
```

Informative Hypotheses

Example 1: ANOVA

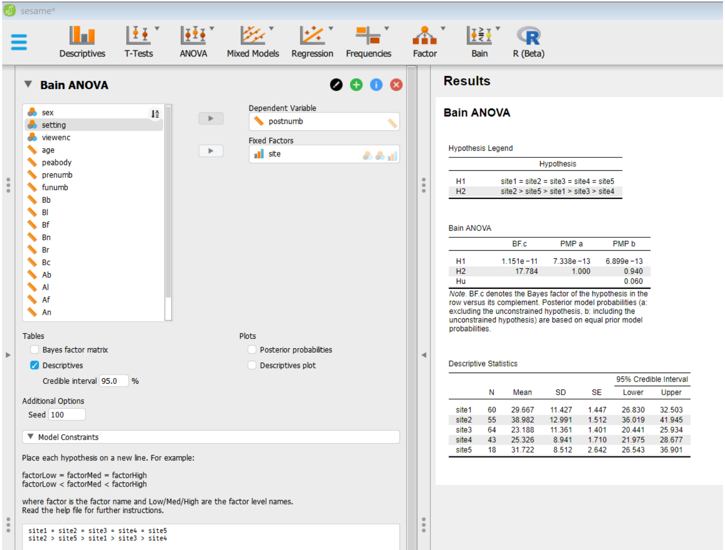
The main output is

```
   Fit   Com   BF.u   BF.c   PMPa  PMPb  PMPc
H1 0.000 0.000 0.000  0.000  0.000 0.000 0.000
H2 0.121 0.008 14.559 16.428 1.000 0.936 0.943
Hu                                 0.064
Hc 0.879 0.992 0.886                     0.057

Hypotheses:
  H1: site1=site2=site3=site4=site5
  H2: site2>site5>site1>site3>site4
```

Informative Hypotheses

Example 1: ANOVA



Informative Hypotheses

Example 2: ANOVA Interaction Effect

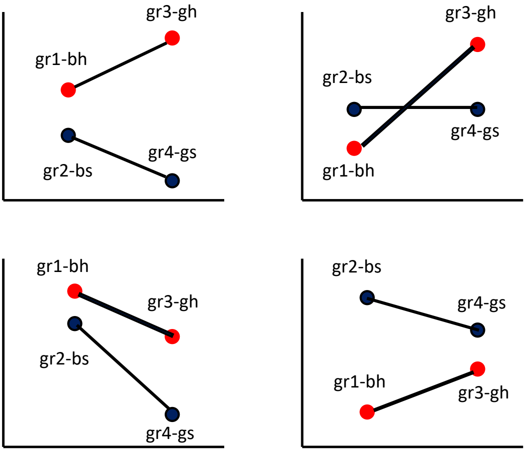
Dependent variable: Knowledge of numbers.

Factors: sex (boy, girl) and setting (watching at home, watching at school).

Gr: 1=boyhome, 2= boyschool, 3= girlhome, 4=girlschool.

Informative Hypotheses

Example 2: ANOVA Interaction Effect



Hi :

$$gr2 - gr1 > gr4 - gr3$$

and

$$gr2 > gr1$$

$$gr2 > gr4$$

Informative Hypotheses

Example 2: ANOVA Interaction Effect

```
sesamesim$gr <- as.factor(sesamesim$gr)
anov <- lm(postnumb~gr-1,sesamesim)
results <- bain(anov,
"gr2 - gr1 > gr4 - gr3 & gr2 > gr1 & gr2 > gr4")
```

Informative Hypotheses

Example 2: ANOVA Interaction Effect

The main output is

```
   Fit   Com   BF.u  BF.c   PMPa  PMPb  PMPc
H1 0.922 0.283 3.262 29.984 1.000 0.765 0.968
Hu                                0.235
Hc 0.078 0.717 0.109                    0.032

Hypotheses:
  H1: gr2-gr1>gr4-gr3&gr2>gr1&gr2>gr4
```

Informative Hypotheses

How to write down an hypothesis

bain can handle hypotheses build using constraints on (linear combinations) of parameters. Suppose the parameter names are "a", "b", "c".

Step 1: Construct the elements of the linear combination. E.g. "a" or "a + 2" or "3 * a" or "2 * a + 4"

Step 2: Constrain the resultsing elements. E.g. a > b > c

or a > b + 2 & b > c + 2

or 2 * a > b + c & b > 0 & c > 0

or a > (b, c) & b - c > 0

Informative Hypotheses

Example 3: Repeated Measures

Development of depression

	Measurement			
	8 years	12 years	16 years	20 years
Men	μ_1	μ_2	μ_3	μ_4
Women	μ_5	μ_6	μ_7	μ_8

$$H_1 : \mu_5 - \mu_1 > \mu_6 - \mu_2 > \mu_7 - \mu_3 < \mu_8 - \mu_4$$

$$H_2 : \mu_6 - \mu_5 < \mu_7 - \mu_6 > \mu_8 - \mu_7$$

Informative Hypotheses

Example 4: Multiple Regression

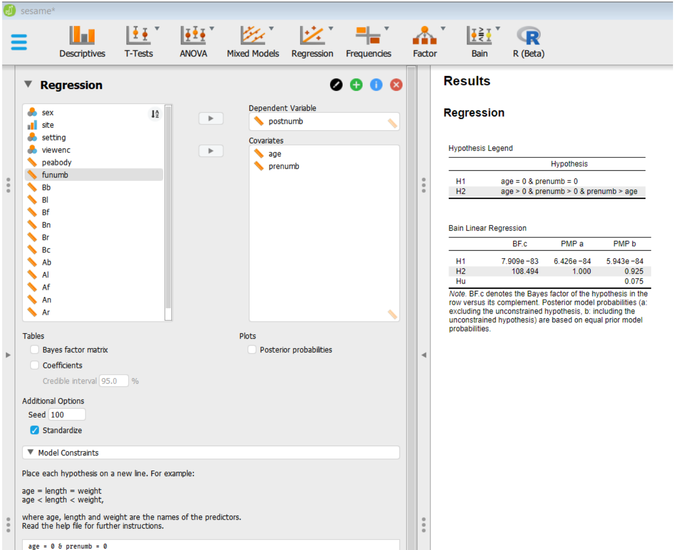
$$\text{postnumb}_i = \beta_0 + \beta_1 \times \text{age}_i + \beta_2 \times \text{prenumb}_i + \epsilon_i$$

$$H_1 : \beta_1 > 0, \beta_2 > 0, \beta_1 < \beta_2$$

Note: β_1 and β_2 are only comparable if age and prenumb are standardized

Informative Hypotheses

Example 4: Multiple Regression



Informative Hypotheses

Example 5: About Equality Constraints

Is the difference in number knowledge relevantly different between boys and girls?

Informative Hypotheses

Example 5: About Equality Constraints

```
sesamesim$sex <- as.factor(sesamesim$sex)
anov <- lm(postnumb~sex-1,sesamesim)
results <- bain(anov, "-2 < sex1 - sex2 < 2")
```

Informative Hypotheses

Example 5: About Equality Constraints

```
sex1     sex2
30.09565 28.85600

   Fit   Com   BF.u  BF.c   PMPa  PMPb  PMPc
H1 0.664 0.091 7.304 19.735 1.000 0.880 0.952
Hu                                0.120
Hc 0.336 0.909 0.370                    0.048

Hypotheses:
  H1: -2<sex1-sex2<2
```

Informative Hypotheses

Example 6: Structural Equation Modelling

```
library(bain)
library(lavaan)

model <- '
    A  =~ Ab + Al + Af + An + Ar + Ac
    B  =~ Bb + Bl + Bf + Bn + Br + Bc
    A  ~ B + age + peabody'
fit <- sem(model, data = sesamesim, std.lv = TRUE)

hypotheses <- "A~B = A~peabody = A~age = 0;
               A~B > A~peabody > A~age = 0"
set.seed(100)
y1 <- bain(fit, hypotheses, standardize = TRUE)
```

Hands-on/Demo: BMS

- Let's practice. Go to
  `https://github.com/rebeccakuiper/Tutorials`:
  1. Click on green button called Code.
  2. Download zip (last option in list).
  3. Unzip it on your machine (that folder is now your working directory).
- Start Rstudio. Optional: make project.
- Open 'Hands-on_1_BMS_Unc_ANOVA_bain.R' (in 'Hands-on files').
- Install packages and load them.
- Read and inspect data. Use Data_Lucas.txt.
- Run model (lm()).
- Specify hypotheses (make up your own).
  Note: Use names used in the model.
- Run bain().
- Inspect and interpret output.

Table of Contents

The End

BMS

Thanks for listening!

Are there any questions?

Websites

https://github.com/rebeccakuiper/Tutorials
www.uu.nl/staff/RMKuiper/Software
www.uu.nl/staff/RMKuiper/Websites%20%2F%20Shiny%20apps
informative-hypotheses.sites.uu.nl/software/goric/

What's next

- Model selection using information criteria
- Possibly: Evidence synthesis / Support aggregation

Depending on time and wishes:

- Some extra information
- Demo in R
- Demo in JASP

We end with:

- Lab

Table of Contents

A Closer Look at the Bayes Factor

A Closer Look at the Bayes Factor

Three Simple Hypotheses

Consider the hypotheses:

$$H_1 : \mu_1 \approx \mu_2, \text{ that is, } |\mu_1 - \mu_2| < .1$$

$$H_2 : \mu_1 > \mu_2$$

$$H_3 : \mu_1, \mu_2$$

A Closer Look at the Bayes Factor

Information in the Data about the Two Means

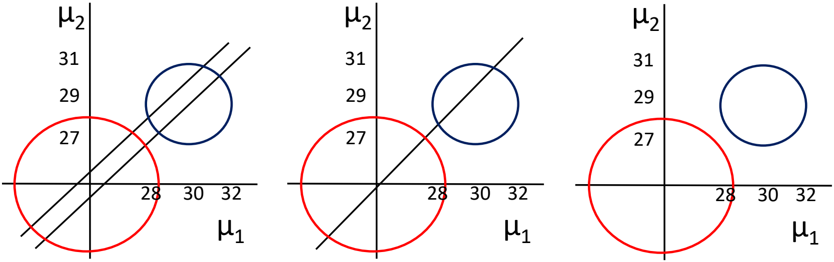
	N	Mean	SD	SE	95% Credible Interval	
					Lower	Upper
sex1	115	30.096	13.058	1.175	27.793	32.398
sex2	125	28.856	12.162	1.127	26.647	31.065

$$g(\mu_1, \mu_2 \mid \text{data}) \approx \mathcal{N}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} se_1^2 = \frac{SD_1^2}{N_1} & 0 \\ 0 & se_2^2 = \frac{SD_2^2}{N_2} \end{bmatrix} \right),$$

A Closer Look at the Bayes Factor

Posterior Distribution, Prior Distribution, and Hypotheses

$H_1: \mu_1 \approx \mu_2$ $H_2: \mu_1 > \mu_2$ $H_u: \mu_1 , \mu_2$



$BF_{1u} = f_1/c_1 = .25/.05 = 5$ $BF_{2u} = f_2/c_2 = .75/.5 = 1.5$

$BF_{12} = 5/1.5 = 3.33$

A Closer Look at the Bayes Factor

Fit and Complexity

1. The fit of a hypothesis is the proportion of the posterior distribution in agreement with the hypothesis.
2. The complexity of a hypothesis is the proportion of the prior distribution in agreement with the hypothesis.

A Closer Look at the Bayes Factor

The Prior Distribution

$$h(\mu_1, \mu_2 \mid \text{data}) \approx \mathcal{N}\left(\begin{bmatrix} m \\ m \end{bmatrix}, \begin{bmatrix} \frac{SD_1^2}{J} & 0 \\ 0 & \frac{SD_2^2}{J} \end{bmatrix}\right),$$

Where μ_1 and μ_2 have the same prior mean m, and where J denotes the size of the training sample.

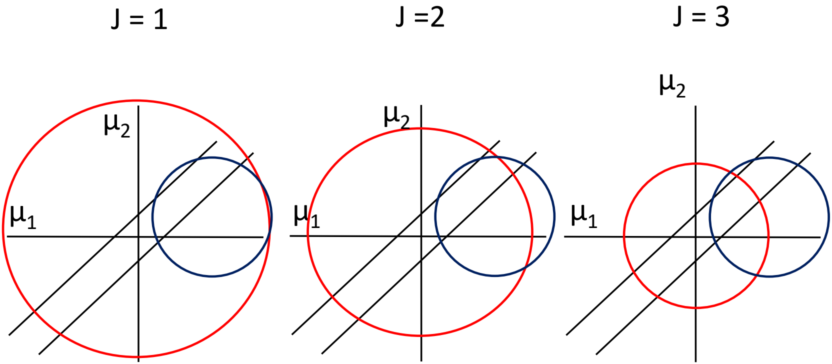
Choices for J for the example at hand:

- Default in `bain`: number of independent constraints, that is, 1, this is a conservative choice (NB. $.5 * J$)
- Minimal training sample size, that is, 4, because four observations are neede to estimate two means and variances
- *Jref*, which renders $BF_{0u} = 19$ if the effect size in the sample equals 0

A Closer Look at the Bayes Factor

Prior Sensitivity for = Constrained Hypotheses

$H_1: \mu_1 \approx \mu_2$

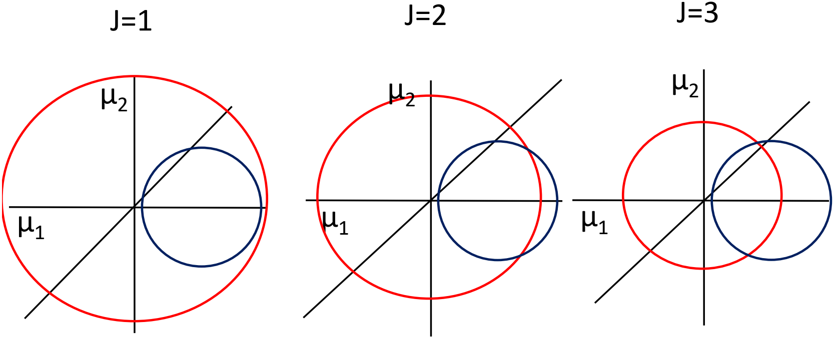


$BF_{1u} = .2/.01 = 20$ $BF_{1u} = .2/.05 = 4$ $BF_{1u} = .2/.2 = 1$

A Closer Look at the Bayes Factor

Prior In-Sensitivity for > < Constrained Hypotheses

$H_2: \mu_1 > \mu_2$



$BF_{2u} = .9/.5 = 1.8$ $BF_{2u} = .9/.5 = 1.8$ $BF_{2u} = .9/.5 = 1.8$

Bayes Factor (BF)

comparing two informative hypotheses

The BF quantifies the relative support in the data for two hypotheses.

$$BF_{12} = \frac{BF_{1u}}{BF_{2u}} = \frac{f_1/f_2}{c_1/c_2}$$

using

$$BF_{iu} = \frac{f_i/f_u}{c_i/c_u} = \frac{f_i}{c_i}$$