

# Math 530/630: CM 4.1

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Sampling distributions

# So far... what do we know?

- Random Variables
  - Discrete and Continuous
  - Have pmf or pdf – with specific parameters
  - And a really useful cdf
  - Answer questions about the probability of an observation
    - $P(X = x) = ? \%$
  - $E(X)$  and  $\text{Var}(X)$
- Samples
  - Descriptive statistics
    - E.g. mean, median, variance, standard deviation, skew kurtosis

# What do we want to know?

- How to use our sample to understand a population
- Currently a disconnect
  - I can tell you all about my sample
  - And all about a known population
- But I can't tell you about how my sample is related to its' *unknown* population

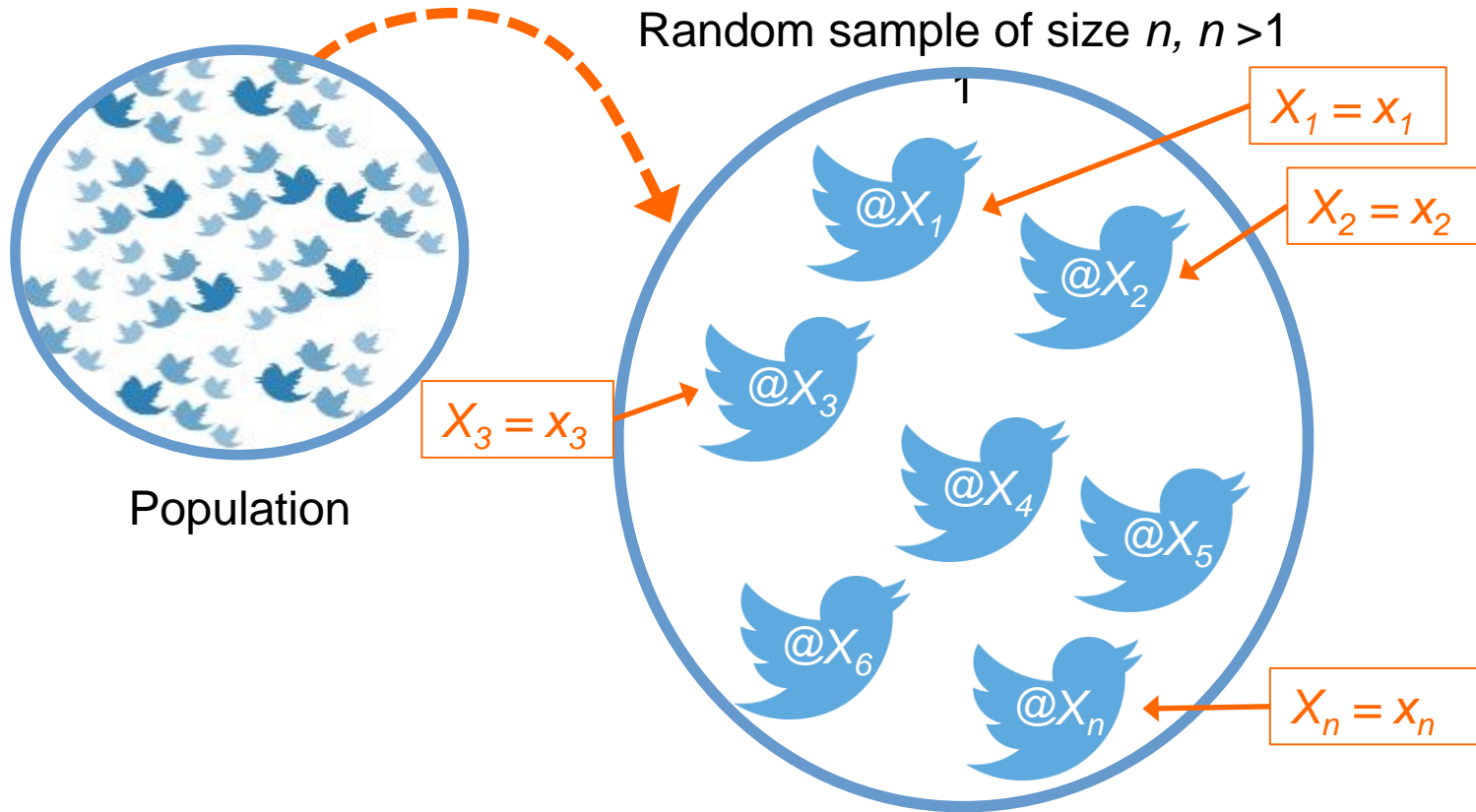
# Today

- Sampling Distributions
  - “Toy” examples to build concepts
    - Starts with a known population
- LLN (Law of Large Numbers)
- CLT (Central Limit Theorem)

# Samples

- A **sample** includes the results of  $n$  random experiments. A sequence of random variables.
- A **random sample** is the result of a random experiment **repeated**  $n$  times. Because experiment can be repeated, and observations are “exchangeable”, iid.

# $X = \# \text{ tweets per day}$



After the sample  $X_1, X_2, \dots, X_n$  is drawn, we usually want to create some summary of the  $x_i$  values



# Statistics

- The definition of a statistic is fairly broad
  - The main restriction is that it cannot be a function of a population parameter
  - Think of a sample as a collection/sequence of rvs: any function of an rv is an rv itself
- 2 main reasons we love them:
  - Sometimes they are **estimators** for population parameters we care about
  - Sometimes they are **test statistics**, i.e. the basis for a hypothesis test



- Formally, let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from a population, and let  $T(x_1, x_2, \dots, x_n)$  be a real-valued or vector-valued function whose domain includes the sample space of  $(X_1, X_2, \dots, X_n)$ .
- Then the random variable or random vector  $Y = T(x_1, x_2, \dots, x_n)$  is called a **statistic**.
- The probability distribution of a statistic  $Y$  is called the **sampling distribution** of  $Y$ .



# Example statistic: the sample mean

- Formally,  $Y = T(x_1, x_2, \dots, x_n)$  (this just says let's make a new variable, call it  $Y$ , and make it equal some function, call it  $T()$ , of the observed values of  $X_1, X_2, \dots, X_n$ )
- Let's make  $Y$  the arithmetic average of the values in our random sample. This is a common statistic, so we all tend to call it the same thing (instead of  $Y$ )...

$$Y = \bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$
$$= \frac{1}{n} \sum_{i=1}^n X_i$$

It's a  
**random  
variable**  
too



## Sample mean of $X$ = average # of tweets per day

- $X$  versus  $x$ : It is pretty standard at this point to use uppercase letters to denote that we are talking about a statistic, and to use lowercase letters when we are talking about observed value of a statistic.
- Let's say we observed 6 twitter users whose tweets totaled:  $\{X_1 = 5; X_2 = 3; X_3 = 0; X_4 = 10; X_5 = 3; X_6 = 2\}$
- Each numerical value above is an  $x_i$  that we have now observed

$$\begin{aligned}\bar{x} &= \frac{x_1 + x_2 + \dots + x_n}{n} \\ &= \frac{1}{6} \times 23 = 3.833\end{aligned}$$

Still a  
random  
variable



# Example statistic: the sample variance

- Formally,  $Y = T(x_1, x_2, \dots, x_n)$  (this just says let's make a new variable, call it  $Y$ , and make it equal some function, call it  $T()$ , of the observed values of  $X_1, X_2, \dots, X_n$ )
- Let's make  $Y$  the average squared deviations of each value from the mean in our random sample. This is another common statistic, so we all tend to call it the same thing (instead of  $Y$ )...

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

It's a  
random  
variable  
too



# Example statistic: the sample variance

- Again, we observed 6 twitter users whose tweets totaled:  
 $\{X_1 = 5; X_2 = 3; X_3 = 0; X_4 = 10; X_5 = 3; X_6 = 2\}$
- Each numerical value above is an  $x_i$  that we have now observed

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$
$$= \frac{1}{5} \times 58.8333 = 11.766$$

Still a  
random  
variable



# Expectation of a statistic

- Now we have an interesting problem- we have defined the formula for expectation (and the variance) for any rv (either discrete or continuous)
- How do we determine the expectation of a **statistic**, which is both an rv and a function of *other* rvs?

# A helpful proof

- Let  $X_1, X_2, \dots, X_n$  be a random sample from a population. Let  $g(x)$  be a function such that  $E[g(X_i)]$  and  $\text{Var}[g(X_i)]$  exist. Then:

$$E \left( \sum_{i=1}^n g(X_i) \right) = nE[g(X_i)]$$

- Since the  $X_i$ s are identically distributed, the expectation of each individual  $g(X_i)$  is the same for all  $i$ . We can arbitrarily pick  $i = 1$ .  $nE[g(X_i)] = nE[g(X_1)]$

For any given  $X_i$ ,  $E(X_i) = \mu$

# Expectation of a statistic: the mean

- Let  $X_1, X_2, \dots, X_n$  be a random sample from a population with mean  $\mu$  and variance  $\sigma^2 < \infty$ .
- We know that the formula for the sample mean is:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

- To get the expectation of the sample mean statistic, we just take the expectation on each side of this equation.

# Expectation of a statistic: the sample mean

- Let  $X_1, X_2, \dots, X_n$  be a random sample from a population with mean  $\mu$  and variance  $\sigma^2 < \infty$ . Then, taking the expectation of both sides of the equation:

$$\begin{aligned} E(\bar{X}) &= E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \\ &= \frac{1}{n} E\left(\sum_{i=1}^n X_i\right) \\ &= \frac{1}{n} n E(X_i) = \mu \end{aligned}$$

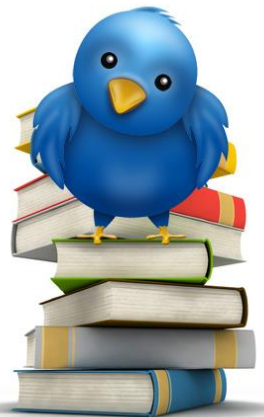
1. For any given  $X_i$ ,  $E(X_i) = \mu$
2.  $n$  can take on any value  $> 0$ : it will never change this result!



$$E(\overline{X}) = \mu$$



*“The expected value of the sample mean is the population mean”*



# Variance of a statistic: the sample mean

- So, we know that the expectation of our sample mean is the population mean
- What about the variance of the sample mean?
- We need another helpful proof...

$$\text{Var} \left( \sum_{i=1}^n g(X_i) \right) = n(\text{Var}[g(X_1)])$$

- Again, since the  $X_i$ s are identically distributed, the variance of each individual  $g(X_i)$  is the same for all  $i$ . We can arbitrarily pick  $i = 1$ .

# Variance of a statistic: the sample mean

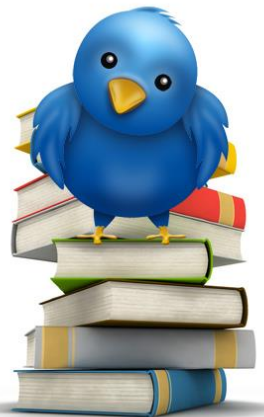
- So, we know that the expectation of our sample mean is the population mean
- What about the variance of the sample mean?

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{Var} \left( \frac{1}{n} \sum_{i=1}^n X_i \right) \\ &= \frac{1}{n^2} \text{Var} \left( \sum_{i=1}^n X_i \right) \\ &= \frac{1}{n^2} n \text{Var}(X_1) = \frac{\sigma^2}{n} \end{aligned}$$

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$



*“The variance of the sample mean is the population variance divided by the number of observations”*



# Expectation and variance of the sample mean

- Why are these statistics about other statistics helpful to know?
- How likely is it that the mean of our sample is close to our population mean?
- This probability is determined by the sampling distribution

# Three distributions to keep in mind simultaneously

1. The distribution of  $X$  in the population
2. The distribution of  $x$  in the particular sample
3. The sampling distribution of sample means across all possible samples



# The sampling distribution of sample means

- Let's perform the following random experiment:
  - Toss  $n$  fair dice;
  - Observe the number of dots (“pips”) showing for each die as  $x_i$ ;
  - Calculate the mean number of dots across observed  $n$  values (note: here,  $n = 1$ )

$$\bar{x} = \sum_{i=1}^n \frac{x_i}{n}$$

- Let's start with  $n = 1$  die



# The population distribution

- Here the population is infinite ( $n \rightarrow \infty$ ) and  $X$  has this probability distribution

$x_i$	$p_i$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
sum	1





# Mean of the population distribution

What is the mean of this distribution?

$x_i$	$p_i$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
sum	1



# Mean of the population distribution

What is the mean of this distribution?

$x_i$	$p_i$	$E(x_i)$
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6
sum	1	3.5

$$E(X) = \mu_X = \sum x_i \times p_i = 3.5$$



# Variance of population distribution

What is the variance/standard deviation of this distribution?

$x_i$	$p_i$	$E(x_i)$	$[x_i - E(X)]^2$	$\times p_i$
1	1/6	1/6	6.25	1.042
2	1/6	2/6	2.25	0.375
3	1/6	3/6	0.25	0.042
4	1/6	4/6	0.25	0.042
5	1/6	5/6	2.25	0.375
6	1/6	6/6	6.25	1.042
sum	1	3.5	not yet!	2.917

$$\begin{aligned} Var(X) &= \sum_{i=1}^n [x_i - E(X)]^2 p(x_i) \\ &= 2.917 \end{aligned}$$

$$sd(X) = \sqrt{2.917} = 1.708$$



$$n = 1$$

$x_i$	$p_i$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6
sum	1

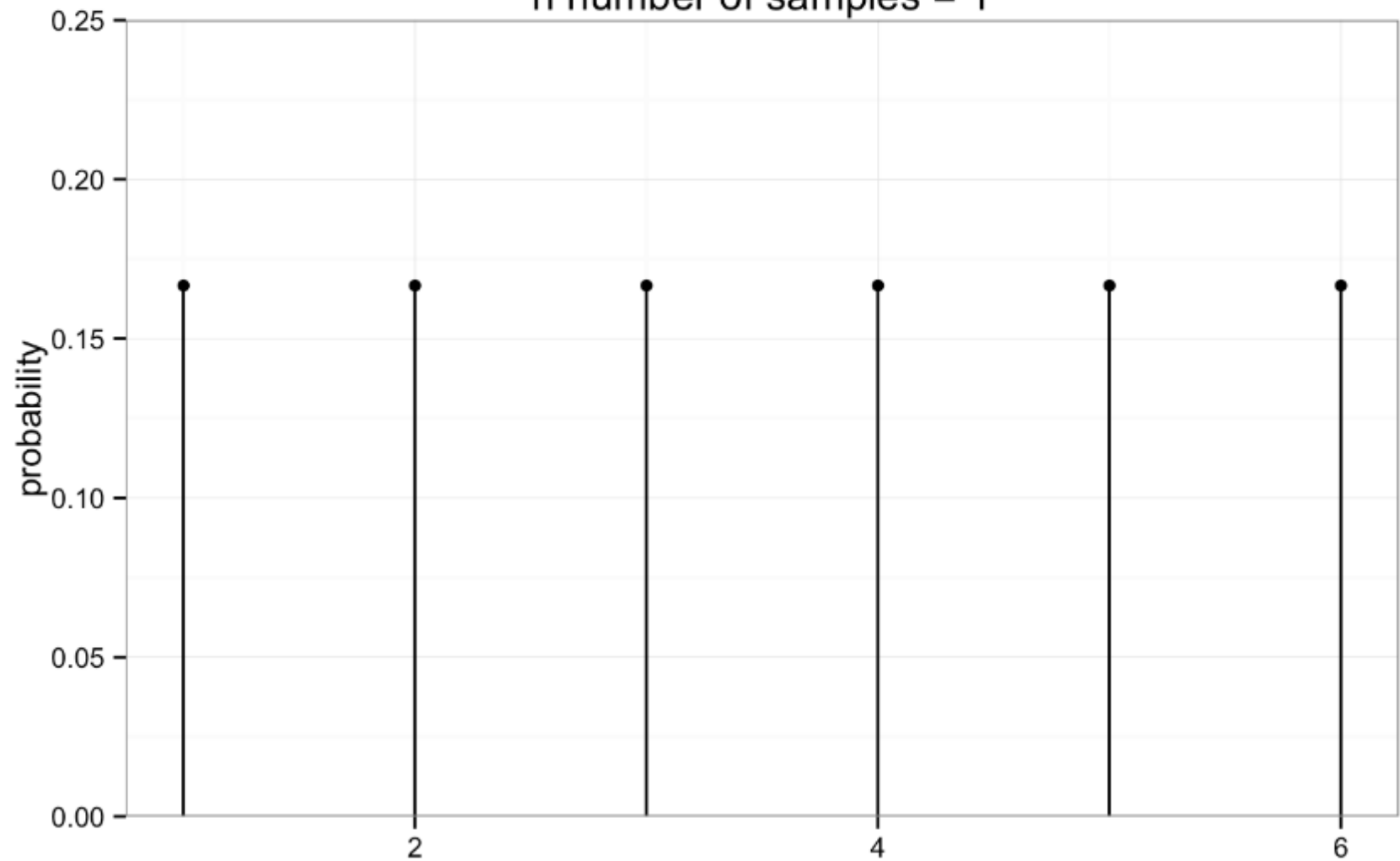
$$E(X) = \mu_X = 3.5$$

$$Var(X) = \sigma_X = 2.917$$

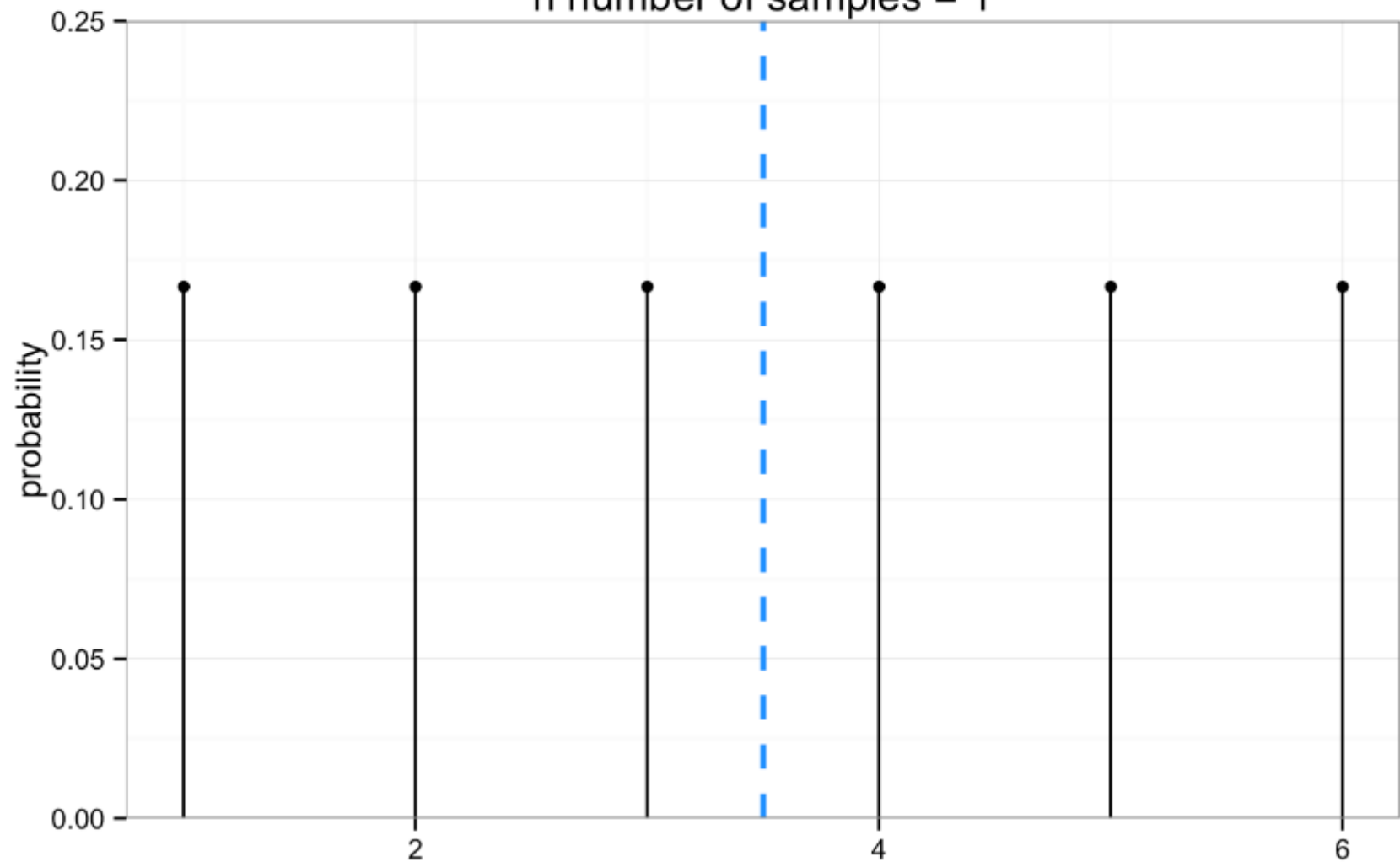
We can also consider this as the **sampling distribution of means** from samples of size  $n = 1$  (where the samples are independent draws from an infinite population)



Population distribution  
n number of samples = 1



Population distribution  
n number of samples = 1



- Now let's add a second die to our random experiment. Our random variable,  $X$ , is the mean number of pips across both die. What is the population mean for this sample?

(a) 2

(b) 3.5

(c) 2.5

(d) I have no idea!

# Step 1: what is the sample space? How can we count the number of elements in it?

- $n = 2$
- $6 \times 6 = 36$  possible combinations





```
omega <- expand.grid(rep(list(1:6), 2))
omega <- omega %>%
  data.frame()
names(omega) <- c("die1", "die2") #just renaming my 2 variables
```

```
> omega
```

	die1	die2
1	1	1
2	2	1
3	3	1
4	4	1
5	5	1
6	6	1
7	1	2
8	2	2
9	3	2
10	4	2
11	5	2
12	6	2
13	1	3
14	2	3
15	3	3
16	4	3
17	5	3
18	6	3
19	1	4
20	2	4
21	3	4
22	4	4
23	5	4
24	6	4
25	1	5
26	2	5
27	3	5
28	4	5
29	5	5
30	6	5
31	1	6
32	2	6
33	3	6
34	4	6
35	5	6
36	6	6



Step 2: what are the sample means of all possible samples in sample space?



```
omega <- omega %>%  
  mutate(xbar_i = (die1 + die2)/2)
```

```
> omega
```

	die1	die2	xbar_i
1	1	1	1.0
2	2	1	1.5
3	3	1	2.0
4	4	1	2.5
5	5	1	3.0
6	6	1	3.5
7	1	2	1.5
8	2	2	2.0
9	3	2	2.5
10	4	2	3.0
11	5	2	3.5
12	6	2	4.0
13	1	3	2.0
14	2	3	2.5
15	3	3	3.0
16	4	3	3.5
17	5	3	4.0
18	6	3	4.5
19	1	4	2.5
20	2	4	3.0
21	3	4	3.5
22	4	4	4.0
23	5	4	4.5
24	6	4	5.0
25	1	5	3.0
26	2	5	3.5
27	3	5	4.0
28	4	5	4.5
29	5	5	5.0
30	6	5	5.5
31	1	6	3.5
32	2	6	4.0
33	3	6	4.5
34	4	6	5.0
35	5	6	5.5
36	6	6	6.0



# Step 3a: how many unique values of $\bar{x}$ ?

```
> omega %>%  
  select(xbar_i) %>%  
  distinct()
```

	xbar_i
1	1.0
2	1.5
3	2.0
4	2.5
5	3.0
6	3.5
7	4.0
8	4.5
9	5.0
10	5.5
11	6.0



## Step 3b: what is the probability of each?

```
> xbar_prob <- omega %>%  
  group_by(xbar_i) %>%  
  summarise(count = n(), p_i = count/36) %>%  
  arrange(xbar_i)
```

```
> xbar_prob  
Source: local data frame [11 x 3]
```

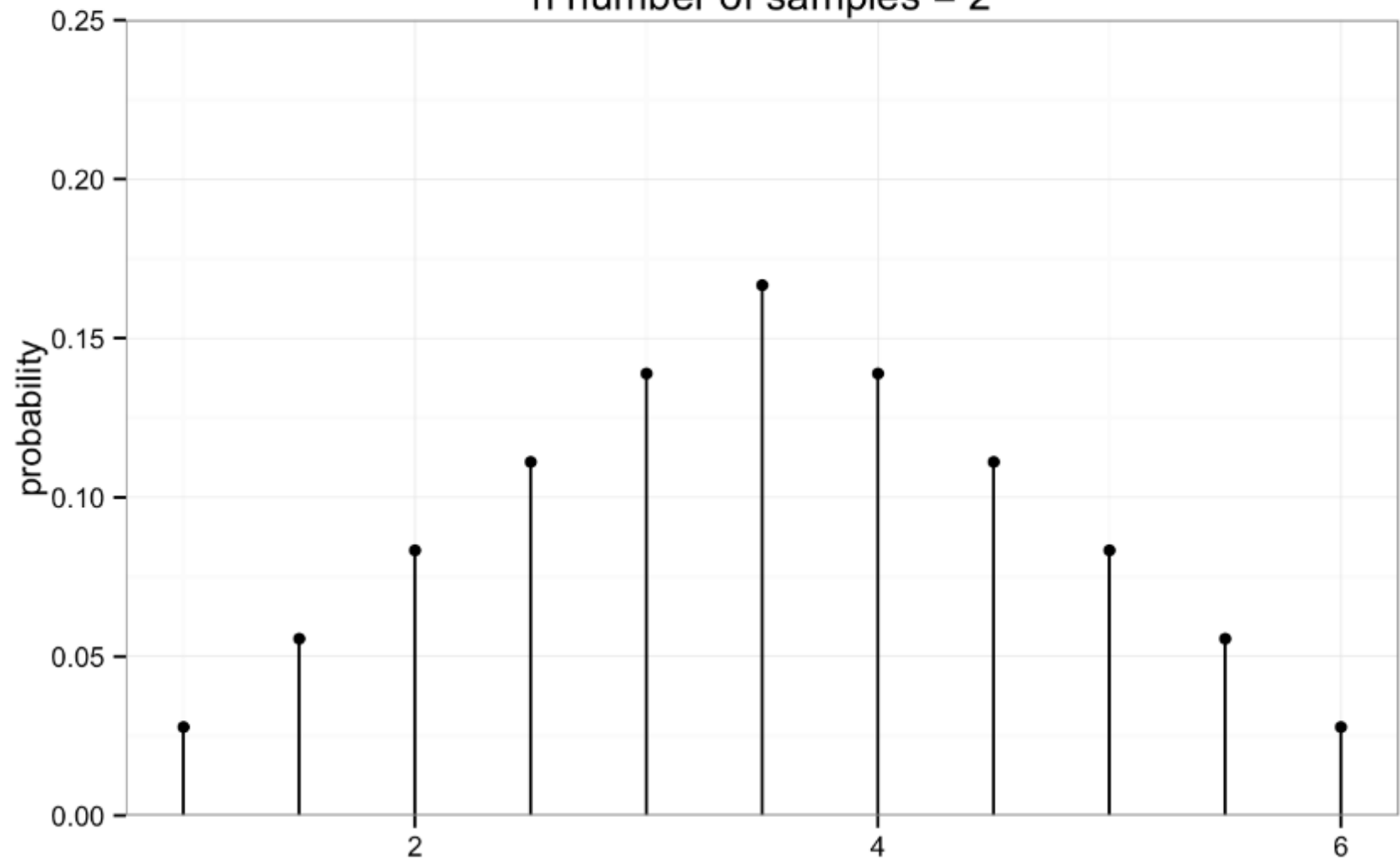
	xbar_i	count	p_i
1	1.0	1	0.02777778
2	1.5	2	0.05555556
3	2.0	3	0.08333333
4	2.5	4	0.11111111
5	3.0	5	0.13888889
6	3.5	6	0.16666667
7	4.0	5	0.13888889
8	4.5	4	0.11111111
9	5.0	3	0.08333333
10	5.5	2	0.05555556
11	6.0	1	0.02777778

```
> xbar_prob %>%  
  summarise(sum_of_ps =  
    sum(p_i))  
Source: local data frame [1 x 1]  
  
  sum_of_ps  
1      1
```

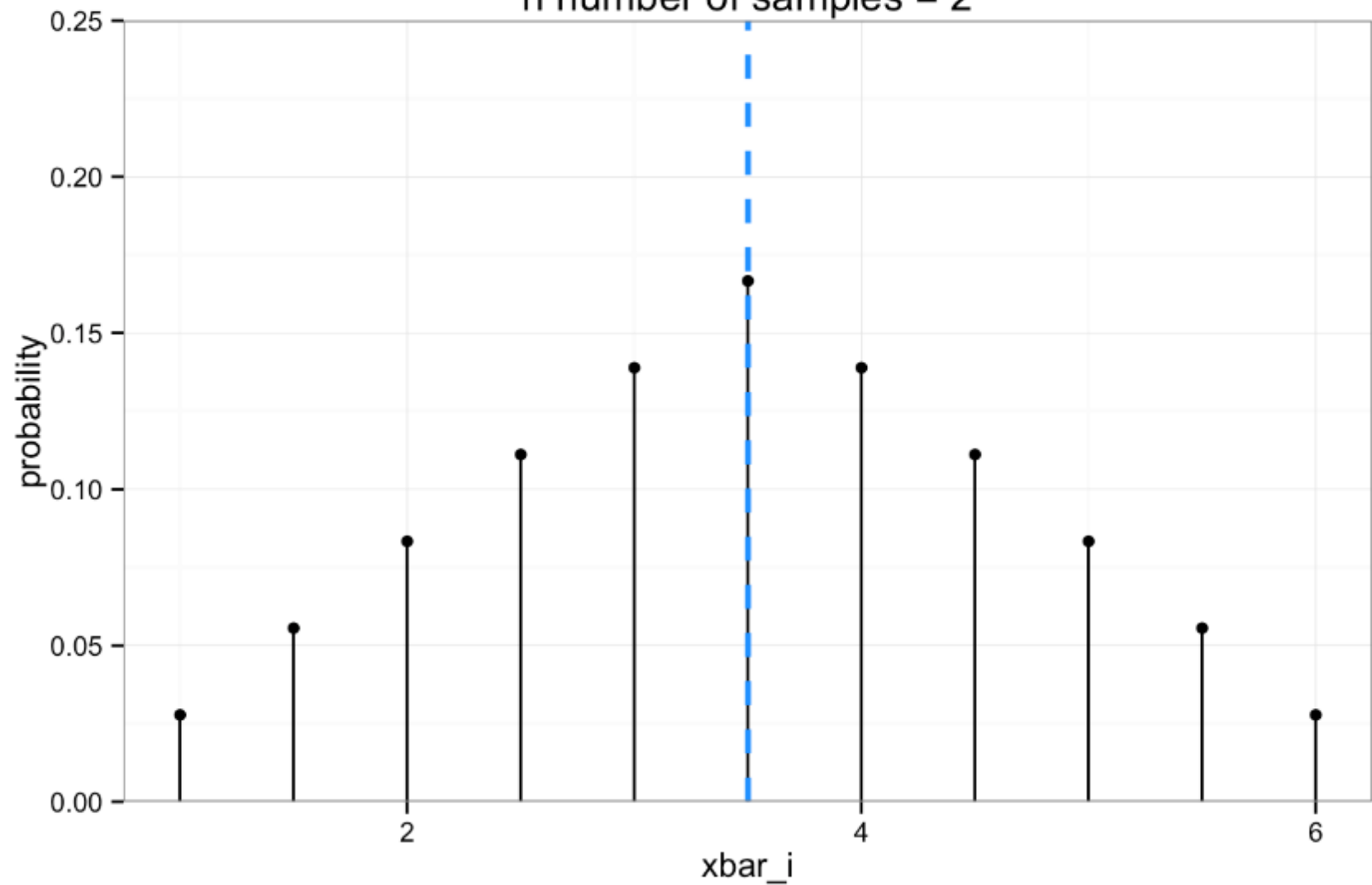
What is the sum here?



Sampling distribution  
n number of samples = 2



Sampling distribution  
n number of samples = 2



# Step 4: what is the expectation?

```
> xbar_samp <- xbar_prob %>%  
  mutate(e_x_i = xbar_i * p_i)  
> xbar_samp  
Source: local data frame [11 x 4]
```

	xbar_i	count	p_i	e_x_i
1	1.0	1	0.02777778	0.02777778
2	1.5	2	0.05555556	0.08333333
3	2.0	3	0.08333333	0.16666667
4	2.5	4	0.11111111	0.27777778
5	3.0	5	0.13888889	0.41666667
6	<u>3.5</u>	6	0.16666667	0.58333333
7	4.0	5	0.13888889	0.55555556
8	4.5	4	0.11111111	0.50000000
9	5.0	3	0.08333333	0.41666667
10	5.5	2	0.05555556	0.30555556
11	6.0	1	0.02777778	0.16666667

```
> xbar_samp %>%  
+   summarise(e_xbar =  
sum(e_x_i))  
Source: local data frame [1 x 1]  
  
e_xbar  
1      3.5
```

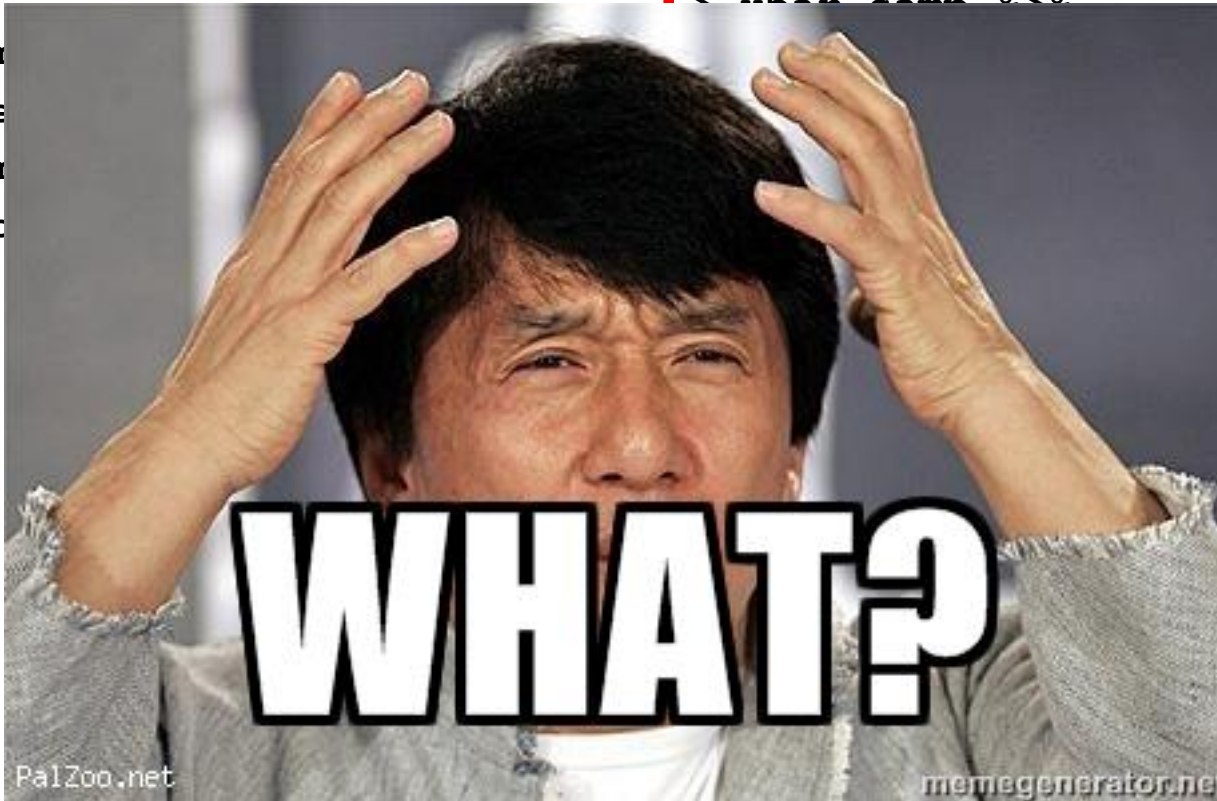




# Step 4: what is the expectation?

```
> xbar_sam  
mutate  
> xbar_sam  
Source: local
```

	xbar_i
1	1.0
2	1.5
3	2.0
4	2.5
5	3.0
6	3.5
7	4.0
8	4.5
9	5.0
10	5.5
11	6.0



# What did we just show?

Distribution: population  
Statistic: mean

$$E(X) = \mu_X = \sum x_i \times p_i = 3.5$$

$$E(X) = \mu_X = 3.5$$

$$E(\bar{x}) = \sum \bar{x}_i p_i = 3.5$$

Distribution: sampling  
distribution of the sample  
mean  
Statistic: mean

$$E(\bar{x}) = \sum \bar{x}_i p_i = 3.5$$

What if we now roll 3 dice ( $n = 3$ )?



To the in class exercise!

# Did you get...

Source: local data frame [16 x 4]

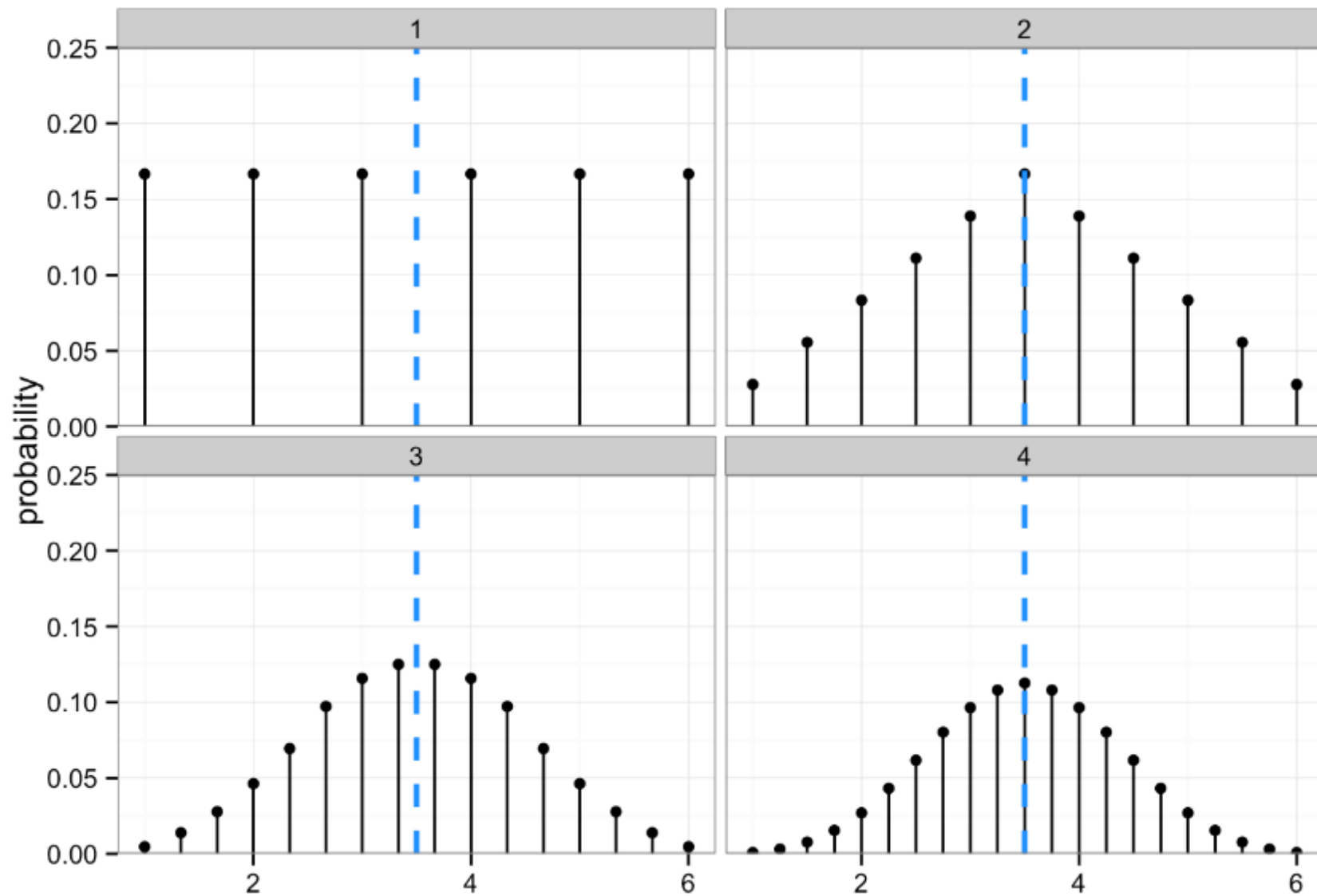
	xbar_i	count	p_i	e_x_i
1	1.000000	1	0.00462963	0.00462963
2	1.333333	3	0.01388889	0.01851852
3	1.666667	6	0.02777778	0.04629630
4	2.000000	10	0.04629630	0.09259259
5	2.333333	15	0.06944444	0.16203704
6	2.666667	21	0.09722222	0.25925926
7	3.000000	25	0.11574074	0.34722222
8	3.333333	27	0.12500000	0.41666667
9	3.666667	27	0.12500000	0.45833333
10	4.000000	25	0.11574074	0.46296296
11	4.333333	21	0.09722222	0.42129630
12	4.666667	15	0.06944444	0.32407407
13	5.000000	10	0.04629630	0.23148148
14	5.333333	6	0.02777778	0.14814815
15	5.666667	3	0.01388889	0.07870370
16	6.000000	1	0.00462963	0.02777778

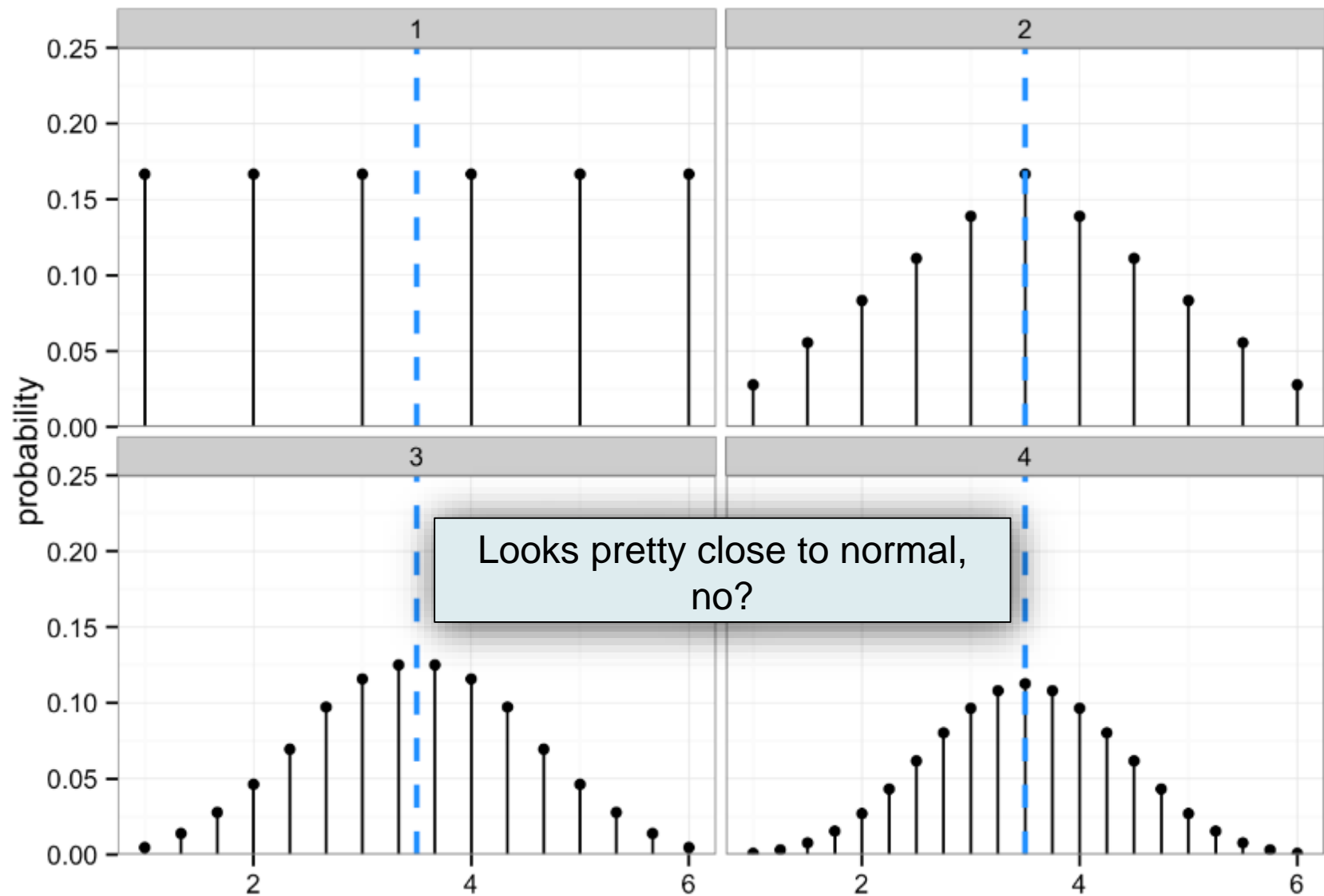
e_xbar	
1	3.5

With 3 dice, this count column should sum to the total number of elementary events in this sample space:  $6 \times 6 \times 6 = 216$

What if we now increase the sample size to  $n = 4$ ?

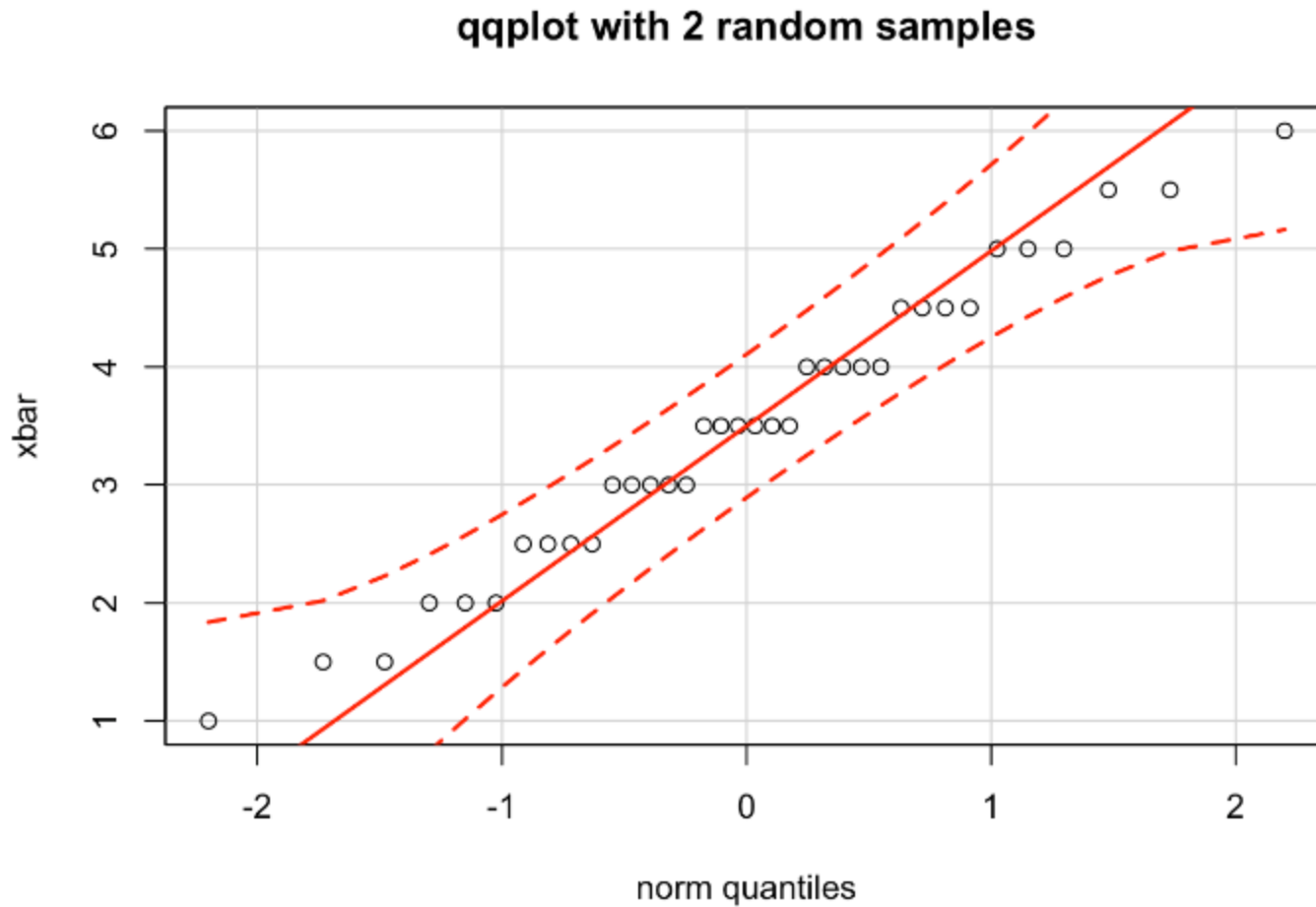




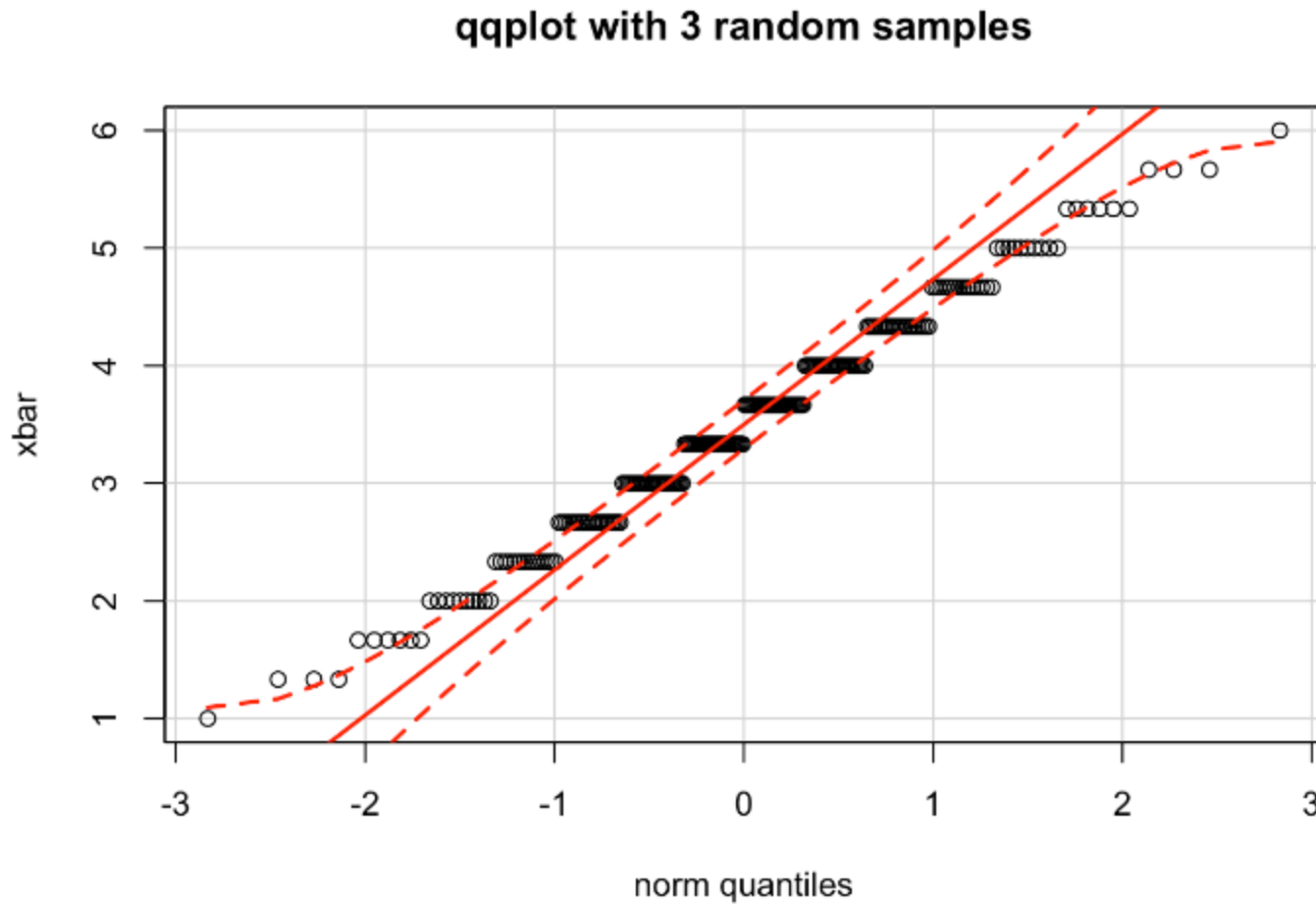




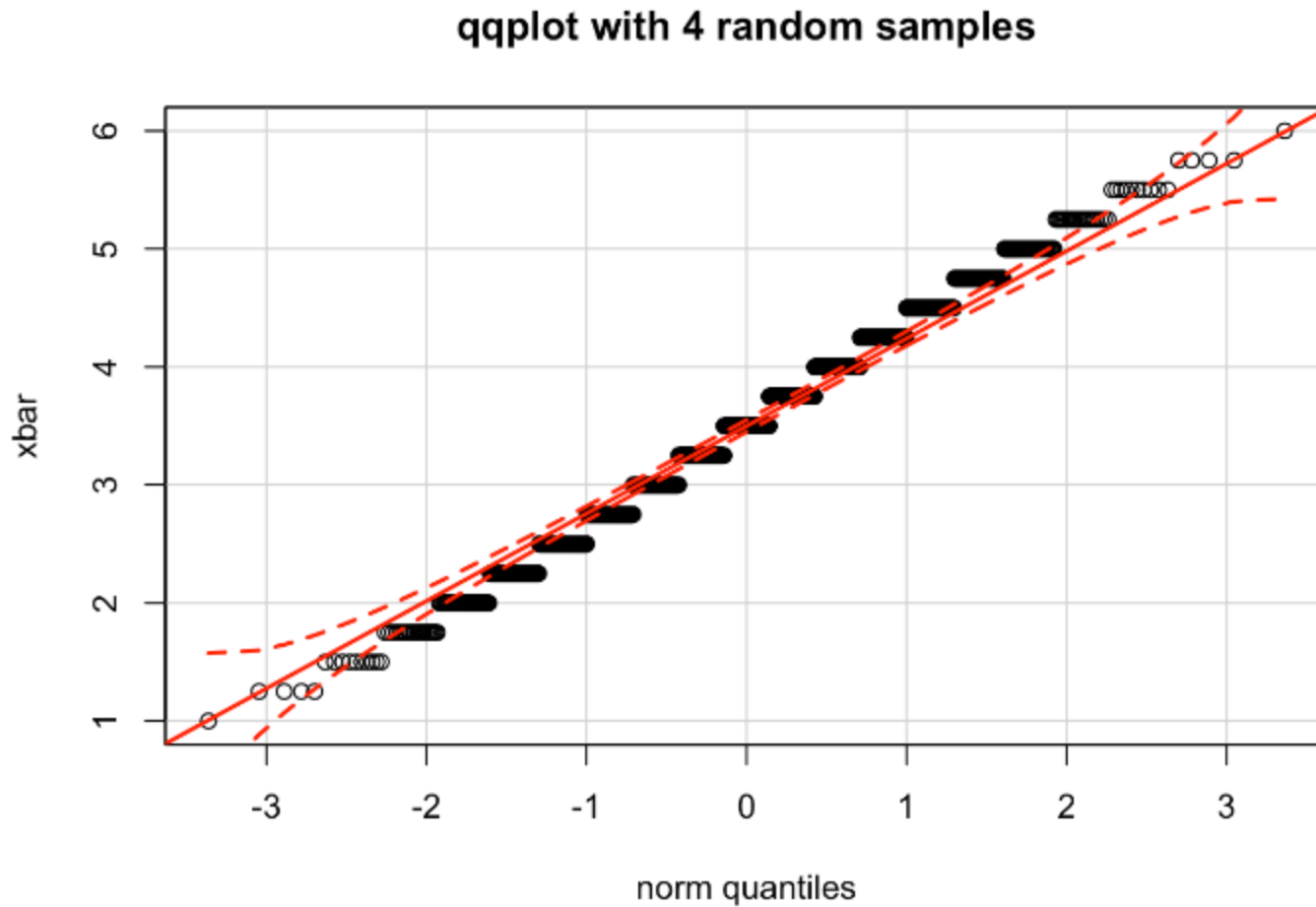
# “Close to normal?”



# “Close to normal?”

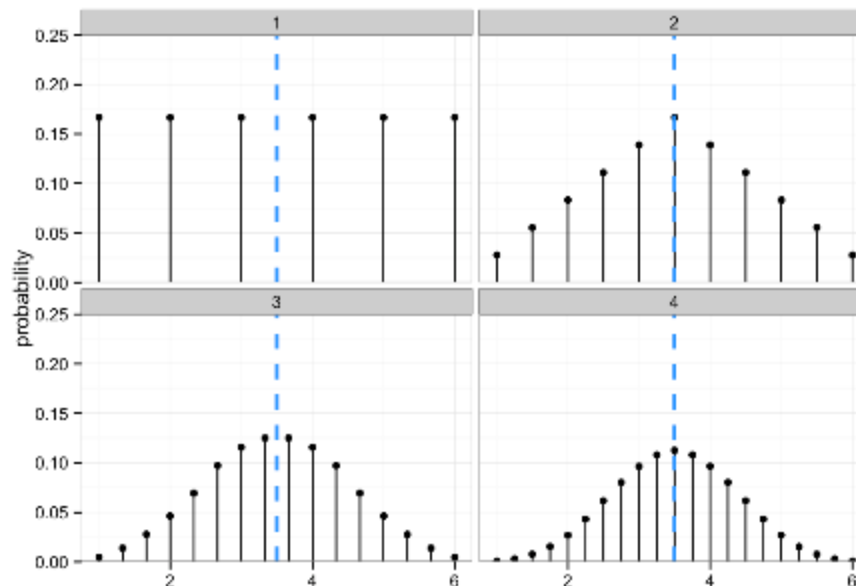


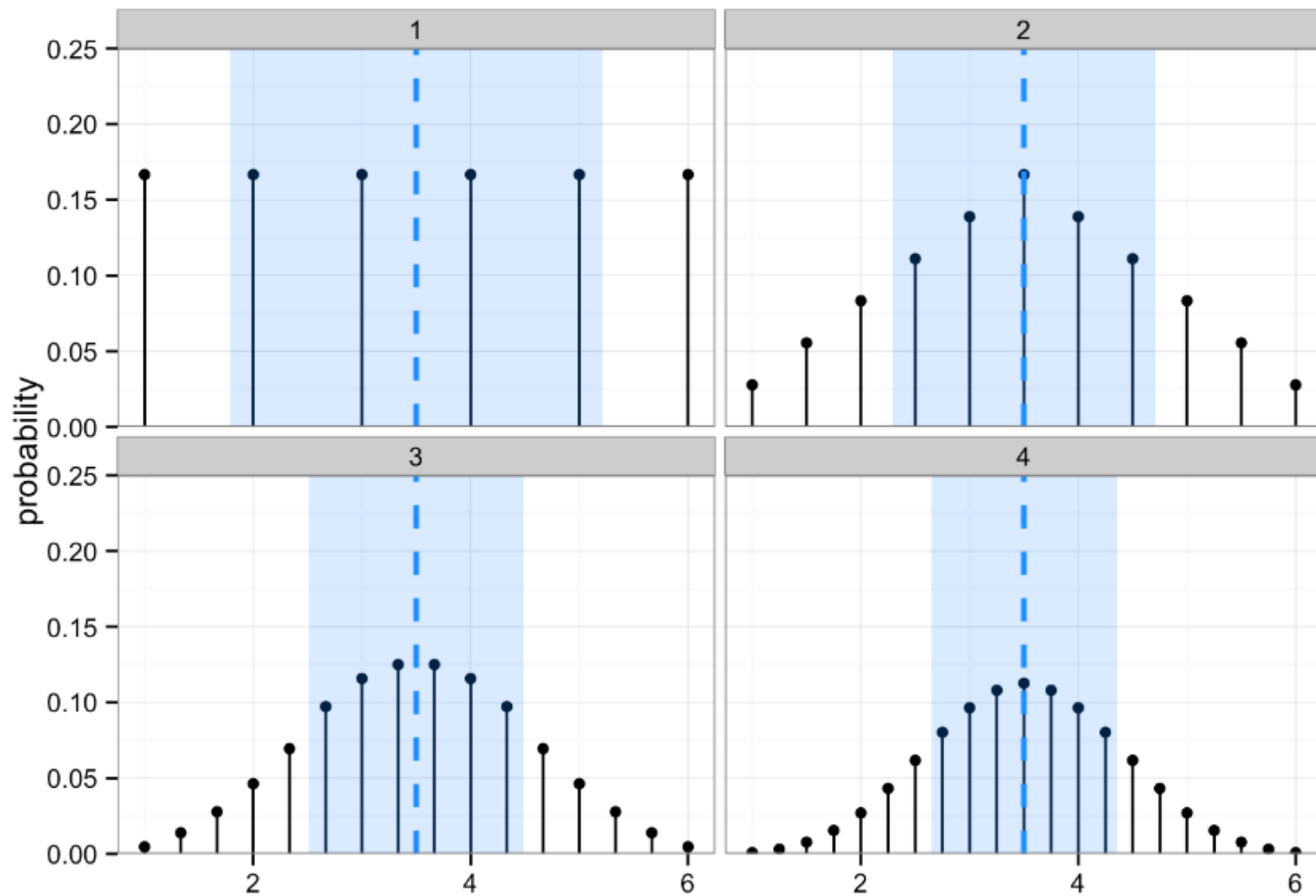
# “Close to normal?”



# What did change?

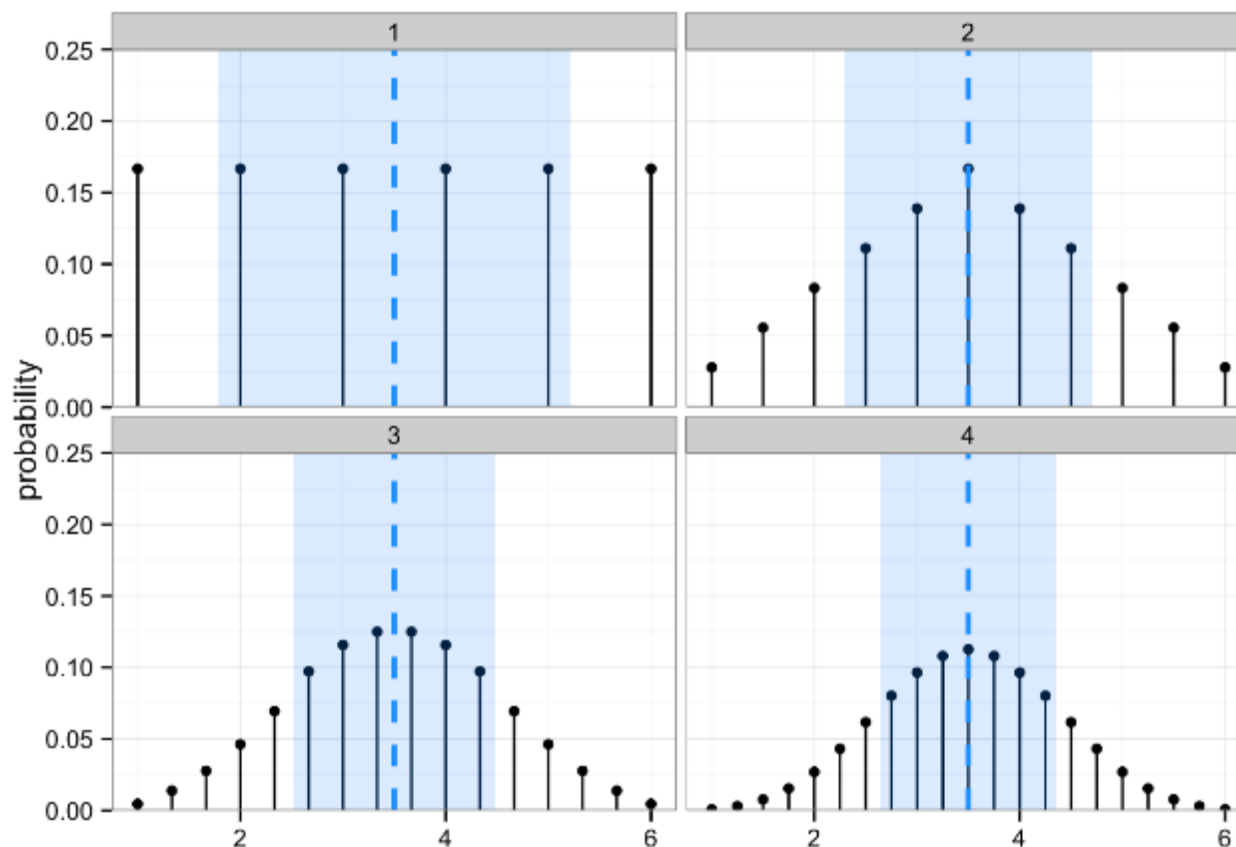
- Hopefully I've convinced you that no matter how many random samples we take, the expectation of our sample mean is the population mean.
- But clearly something else changed every time we added another random sample/observation. What was it?





# In other words...

- The sample mean is obviously a good guess at  $E(X) = \mu$ , but how good is it?



# Spread of the sampling distribution of means

# of dice	$\mu_{\overline{X}}$	$\sigma_{\overline{X}}^2$	$\sigma_{\overline{X}}$
1	3.5	2.917	1.708
2	3.5	1.458	1.207
3	3.5	0.972	0.986
4	3.5	0.729	0.854

$$\sigma_{\overline{X},n=2} = \frac{1.708}{\sqrt{2}} = 1.207$$

$$\sigma_{\overline{X},n=3} = \frac{1.708}{\sqrt{3}} = 0.986$$

$$\sigma_{\overline{X},n=4} = \frac{1.708}{\sqrt{4}} = 0.854$$

If we know the spread for the population distribution, can we calculate the spread of the sampling distributions of the mean?

# Standard error

- The standard error (SE) is the standard deviation of the sampling distribution of a statistic
- Here, the statistic is the mean, so the standard deviation of the sampling distribution of the mean is the standard error of the mean (often called SEM)

$$\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}}$$

- But realize that this is just a specific standard error -- it's a more general concept and they won't always have this exact form



# LAW OF LARGE NUMBERS

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# Weak LLN

- The average of a large, iid sample will be close to the true mean

$$\overline{X}_n \xrightarrow{P} \mu$$

# Strong LLN

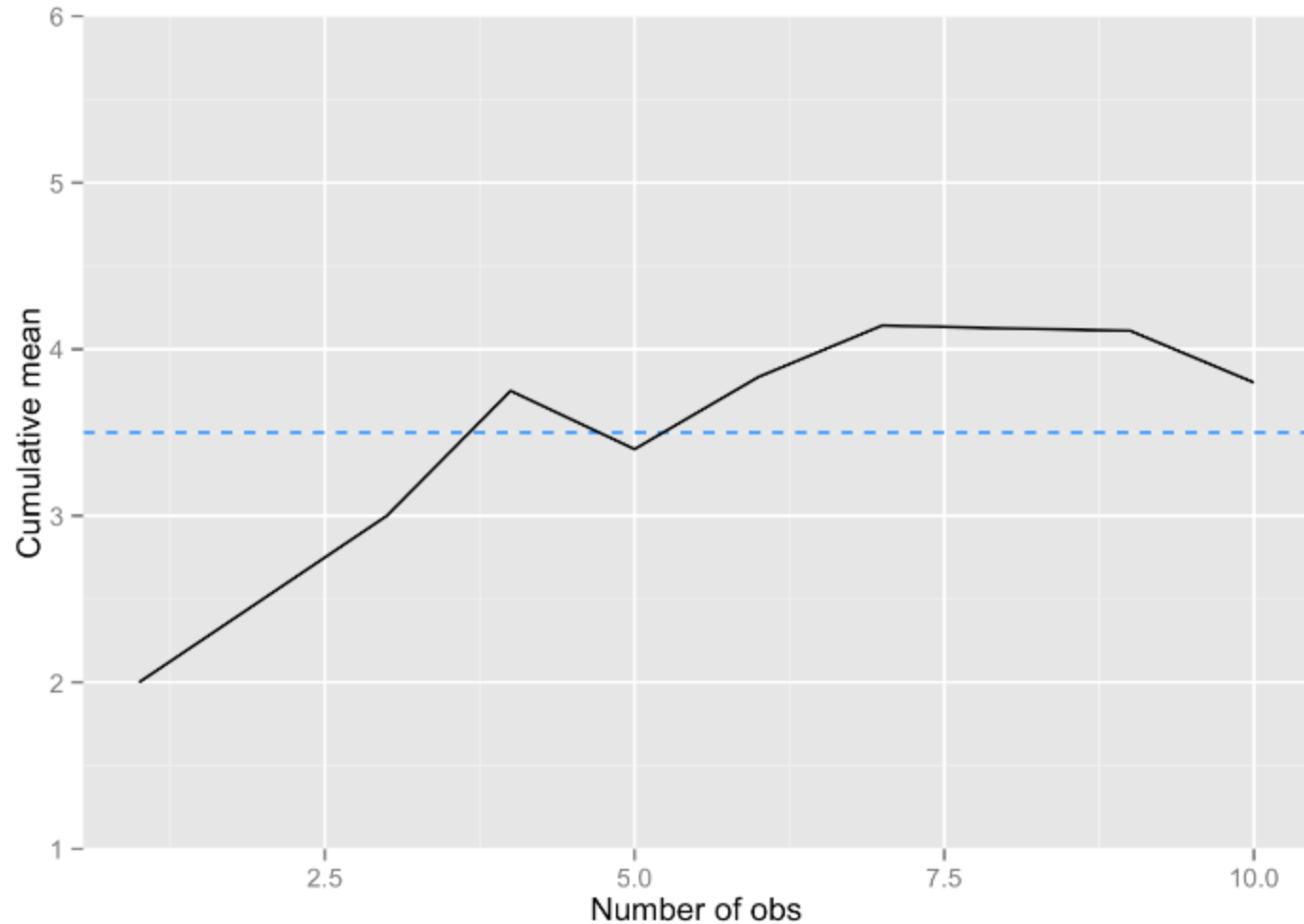
- Let  $X_1, X_2, \dots$  be iid rvs with  $E[X_i] = \mu < \infty$ . Then:

$$P \left( \lim_{N \rightarrow \infty} \frac{X_1 + X_2 + \dots + X_N}{N} = \mu \right) = 1$$

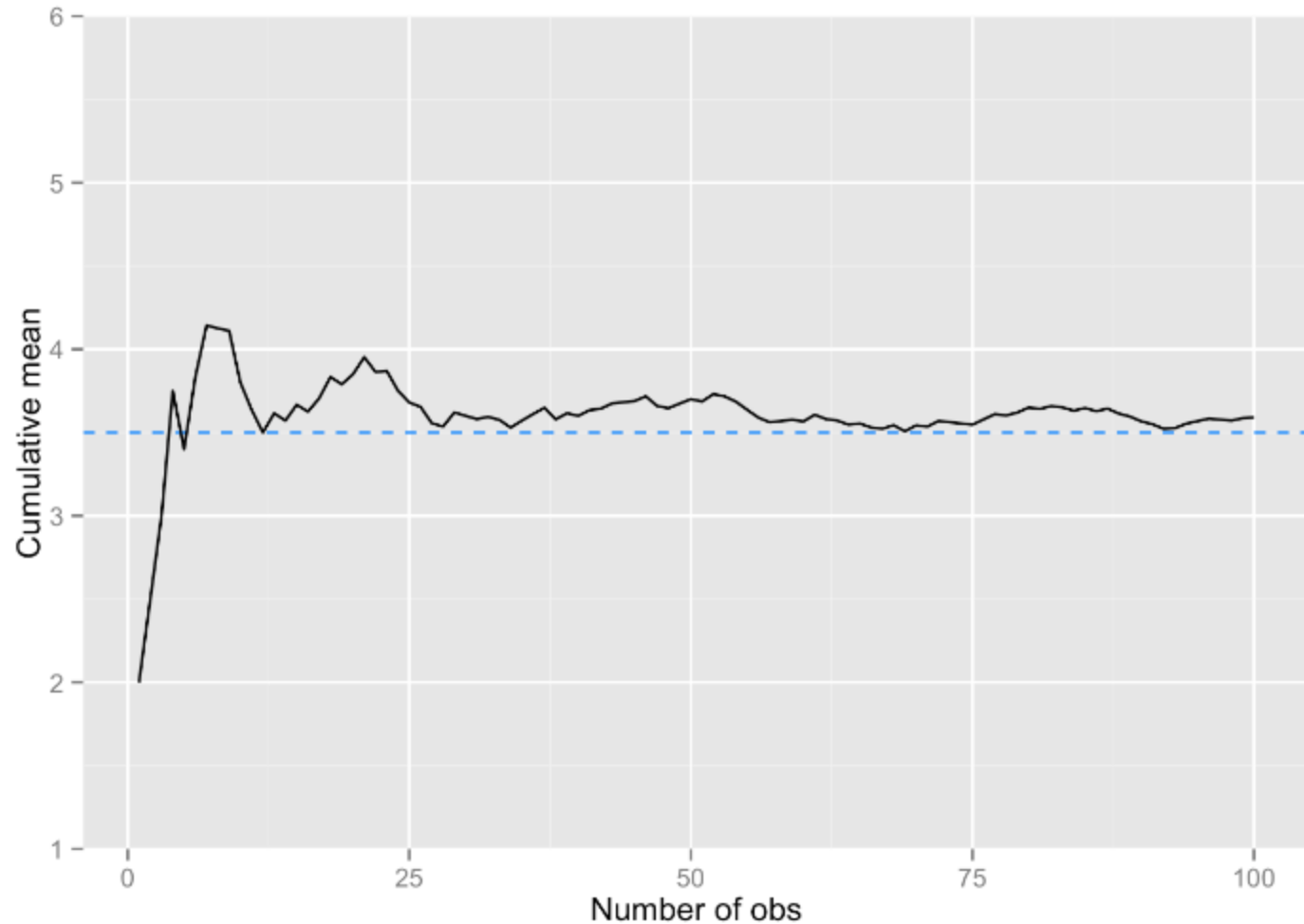
- Works by **swamping**, not **compensation**.
  - If you flip a fair coin and get 80 heads in the first 100 flips, then over the next 100 flips, the expected number of heads is 50—not 20\*

\*see gambler's fallacy; but caveat- Bayesian inference

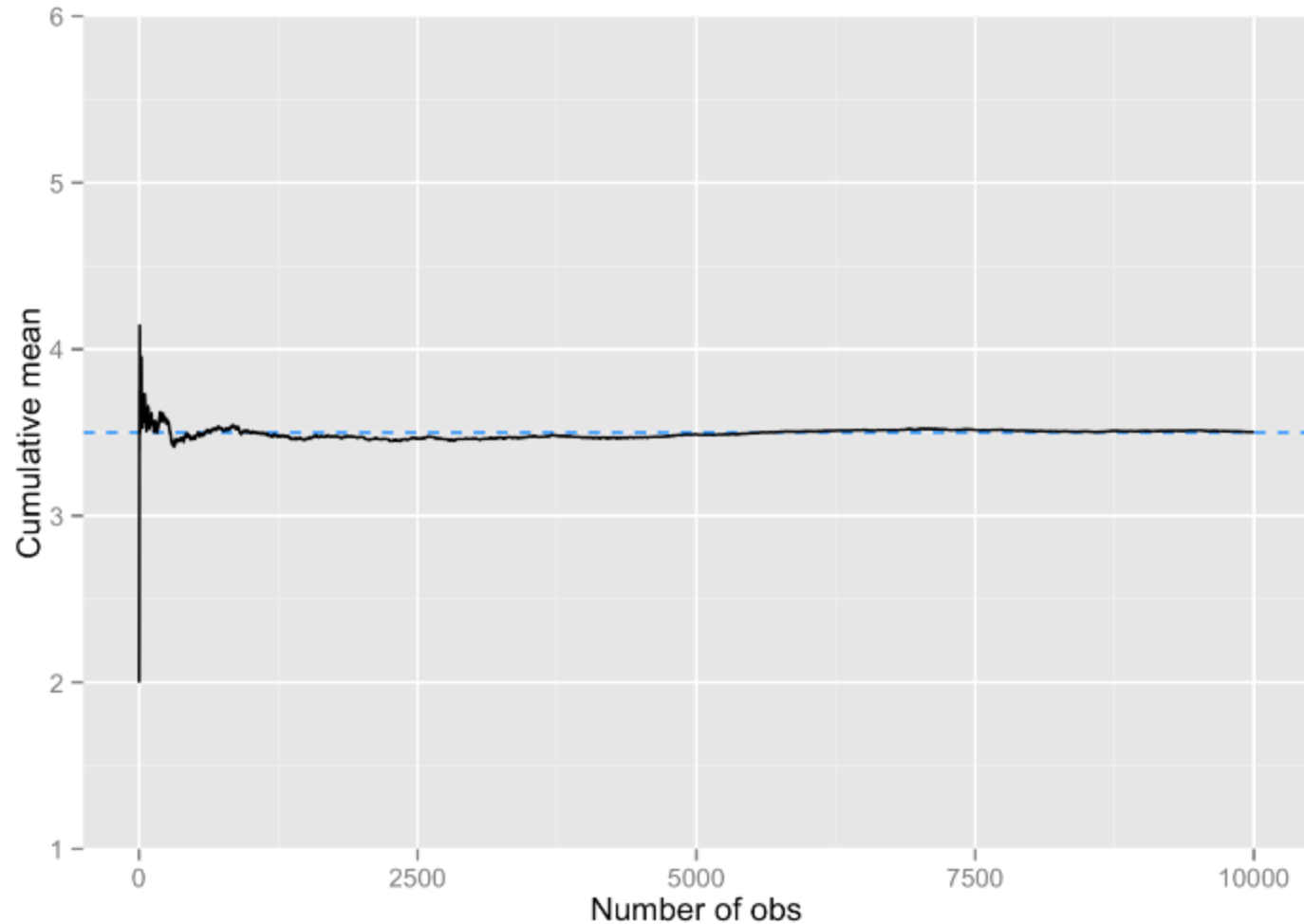
# LLN in action: 10 dice to estimate the known population mean



# LLN in action: 100 dice to estimate the known population mean



# LLN in action: 10,000 dice to estimate the known population mean



# What LLN does not say

- *“If I can just make my sample big enough, I won’t have to worry about error.”*
- There is no sample that is “big enough” in an unqualified sense
- Data still must be iid
- In stats, there are precious few fundamental constants, like there are in math (think:  $e$ ) or physics (think: speed of light)
- Context and goals always matter

# CENTRAL LIMIT THEOREM

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# CLT

- The sampling distribution for the mean (or sum) of a large, iid sample will be approximately a normal distribution, regardless of the shape of the population distribution of  $X$ :

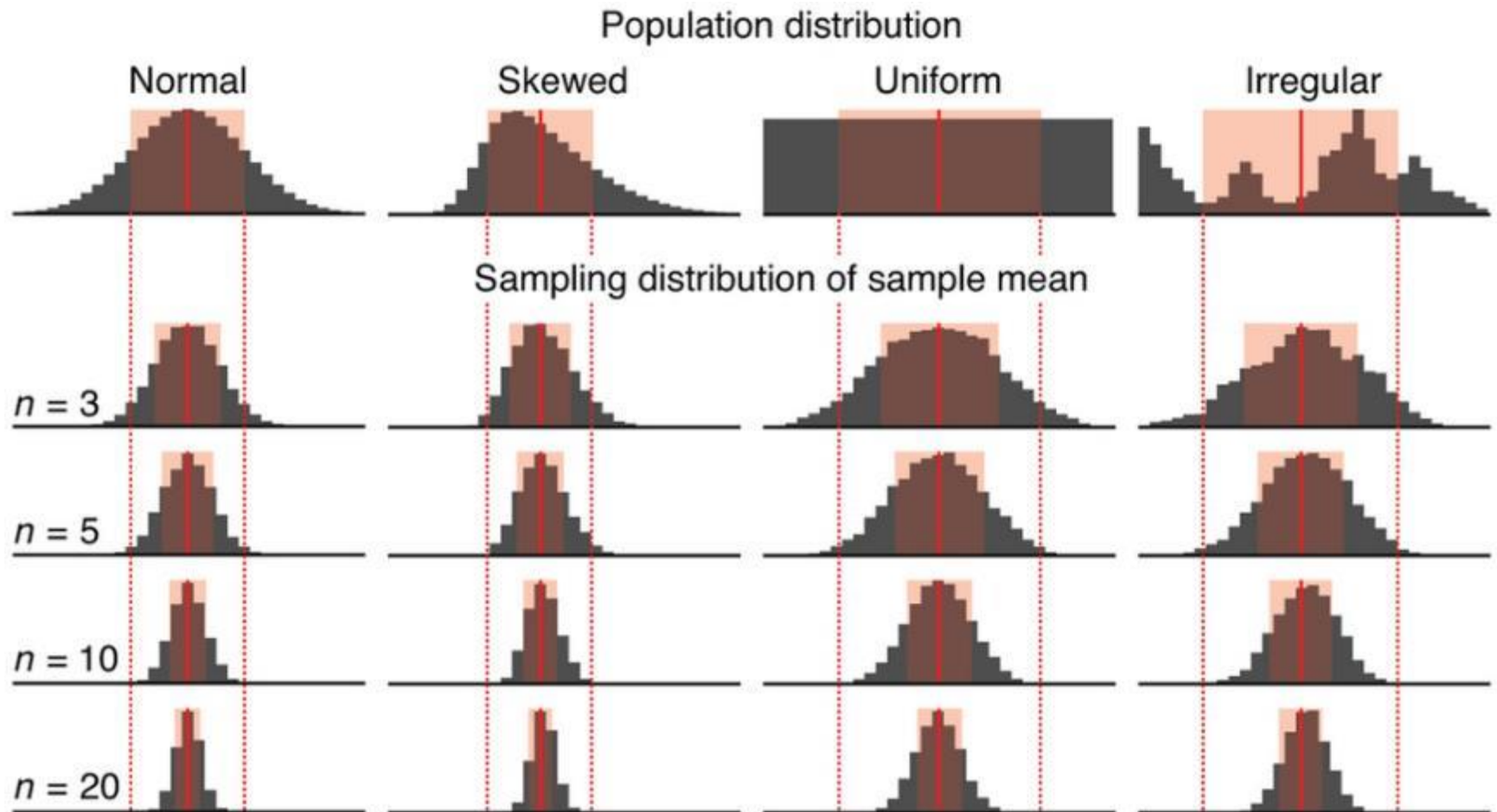
$$\bar{x} \rightarrow N(\mu, \sigma / \sqrt{n})$$

- How large  $n$  needs to be for the approximation to be good enough depends upon how far from normal the population distribution is, but  $n \geq 100$  almost always suffices.
- If the population distribution of  $X$  is itself normal, then, regardless of  $n$ , the sampling distribution of  $x$  is exactly normal.

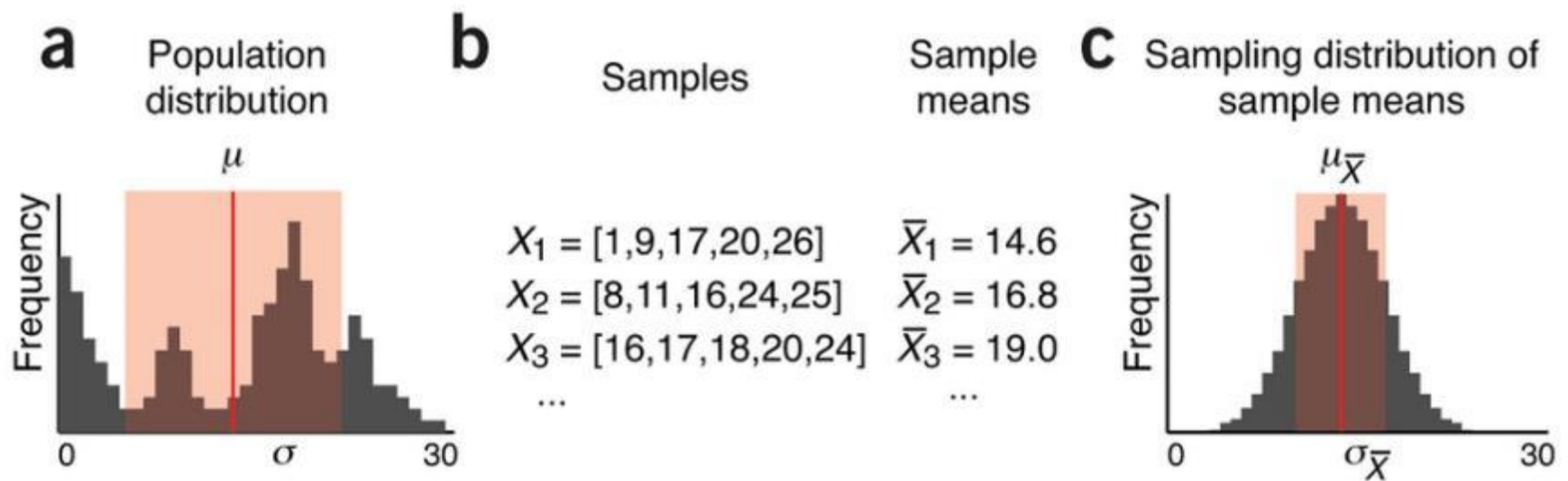
# CLT

- *“If we average more and more independent random quantities with a common distribution, and that common distribution isn’t too pathological, then the average becomes closer and closer to a Gaussian” – Cosma Shalizi*

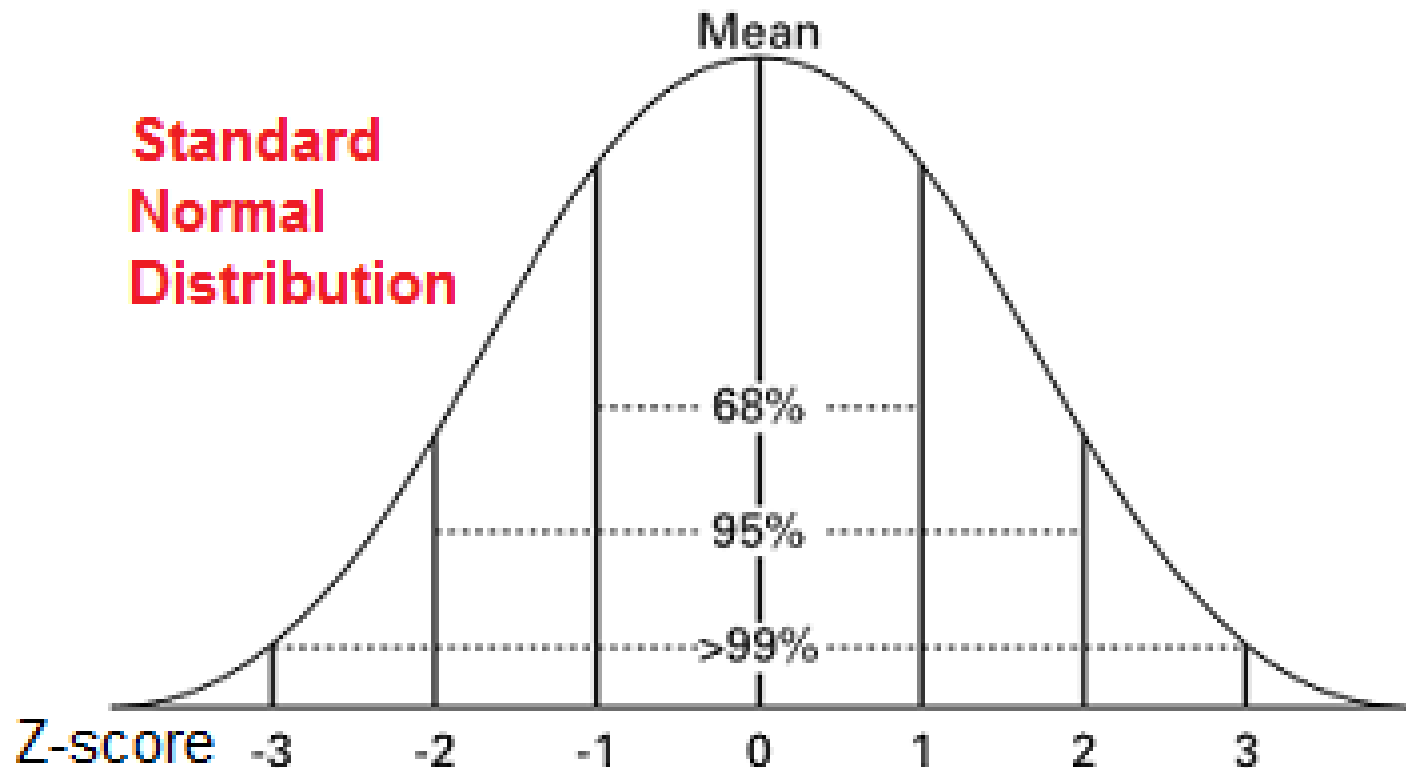
# Sample means for 10,000 samples



# Population parameters are estimated by sampling



# Why does it matter?



# Applying the CLT

- the CLT assumes you're averaging observations that are **iid**
- CLT applies in the case of:
  - Averaging gene expression for 1 gene across exchangeable subjects
  - Averaging disfluency use for 1 conversation task across exchangeable subjects
- CLT does not apply if:
  - Averaging gene expression for 1 subject across genes
  - Averaging disfluency use for 1 subject across conversation tasks

# QQplots in R

```
qqnorm(dataframe$variable)
```

```
qqline(dataframe$variable)
```

```
library(car)
```

```
qqPlot(dataframe$variable)
```

Do this!:

<https://xiongge.shinyapps.io/QQplots/>

Read this!:

<http://stats.stackexchange.com/questions/101274/how-to-interpret-a-qq-plot>

# INDEPENDENT AND IDENTICALLY DISTRIBUTED

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# iid sequences of random variables

		Number of random variables	
		1 rv	> 1 rv
iid?	no	Body weight of Angela $\{X = 150\}$	Weight and height of Angela  <i>Data is a sequence of rvs:</i> $\{X = 150; Y = 65\}$
	yes	Body weight of $n = 4$ randomly selected people  <i>Data is a sequence of iid rvs:</i> $\{X_1 = 150, X_2 = 125, X_3 = 208, X_4 = 180\}$	Weight and height of $n = 4$ randomly selected people  <i>Data is a sequence of iid rvs:</i> $\{X_1 = 150, X_2 = 125, X_3 = 208, X_4 = 180\}$ $\{Y_1 = 65, Y_2 = 60, Y_3 = 72, Y_4 = 70\}$

Both of these are  
**sequences** of iid  
random variables



## Sequences of rvs

$$X_i \sim N(\mu_i, \sigma_i)$$

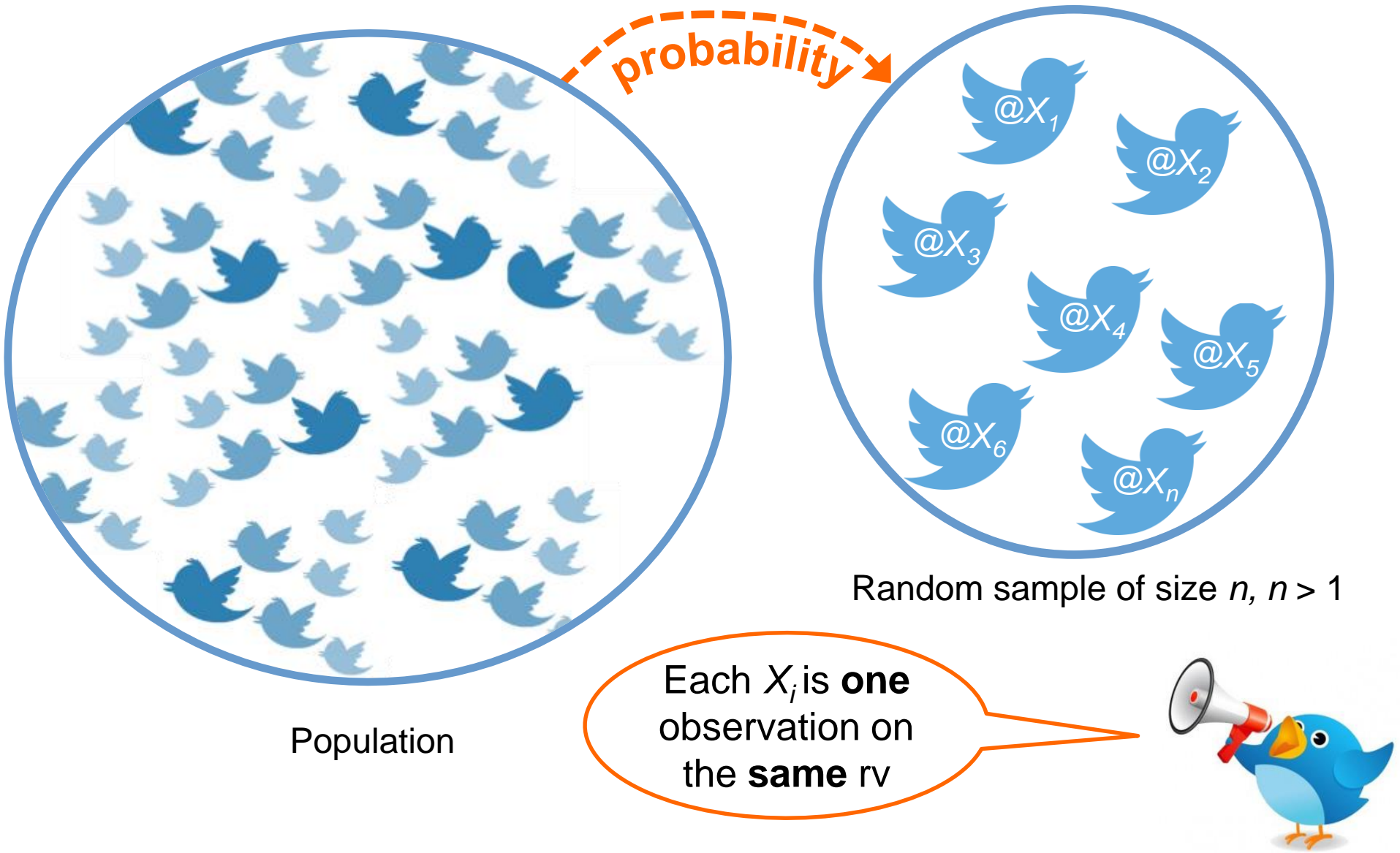
$$X_i \sim N(\mu, \sigma_i)$$

$$X_i \sim N(\mu_i, \sigma)$$

$$X_i \sim N(\mu, \sigma) \leftarrow \text{iid}$$



# Random variable, $X = \#$ of tweets per day

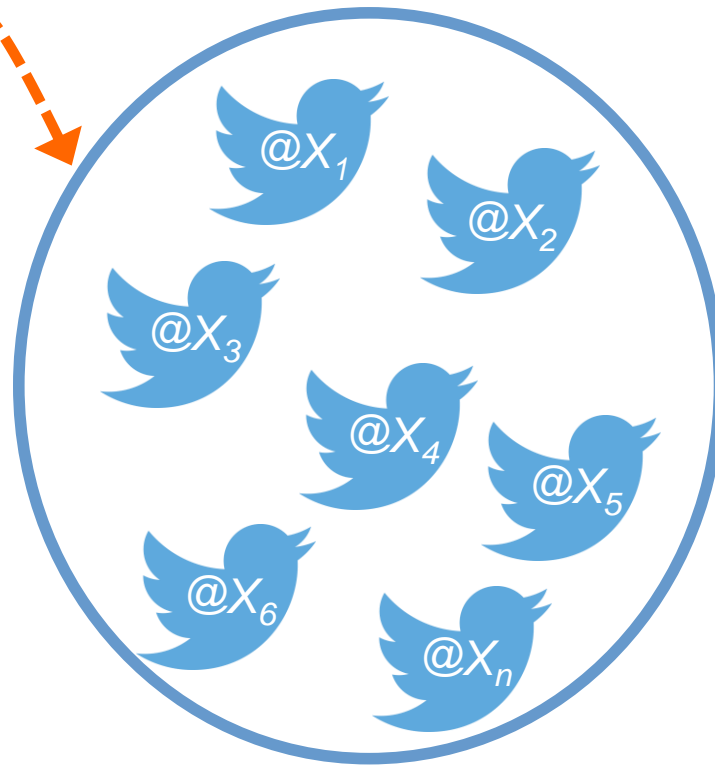


# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



Population

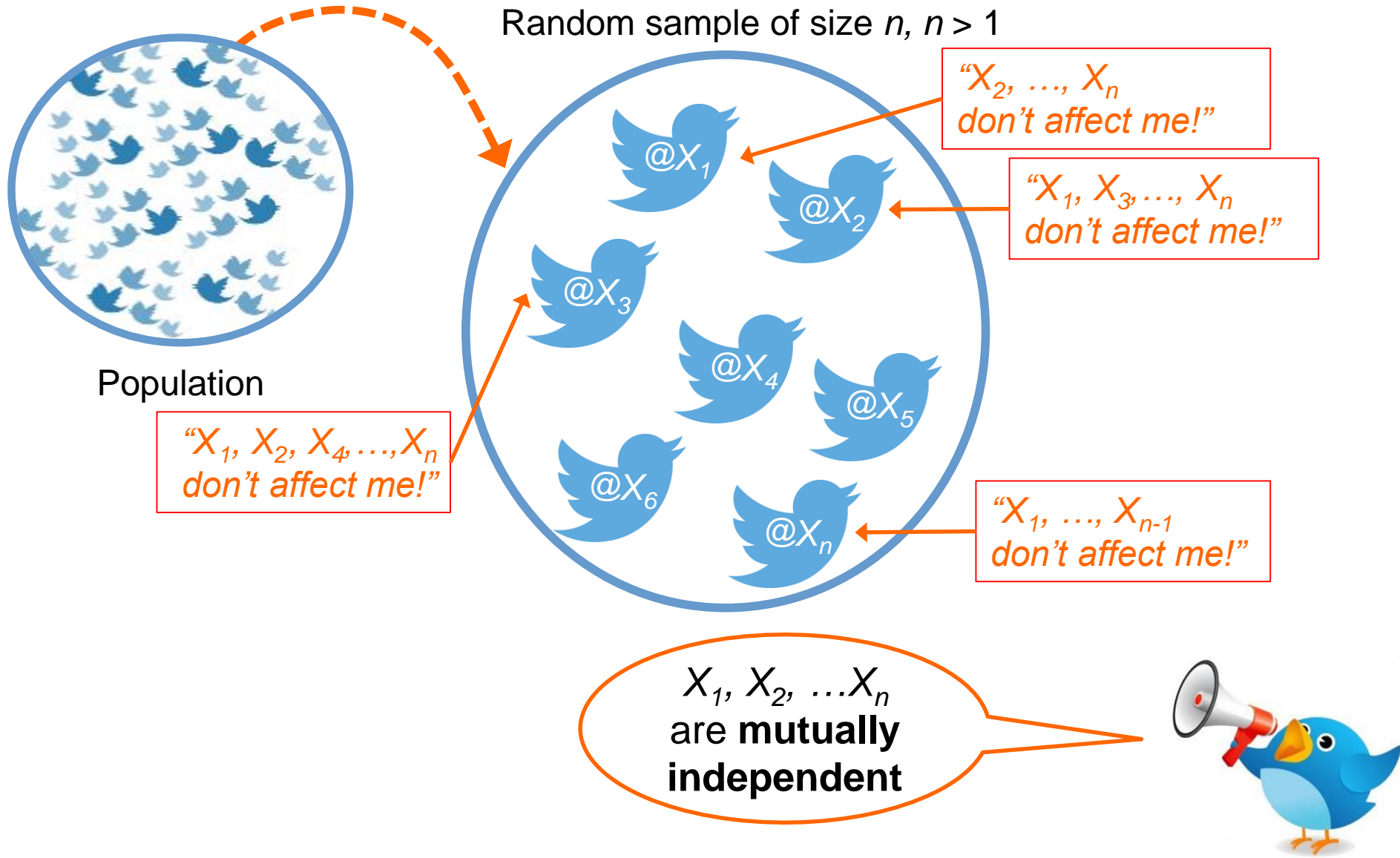


$X_1, X_2, \dots, X_n$   
is a sequence of  
observations of  
the **same** rv



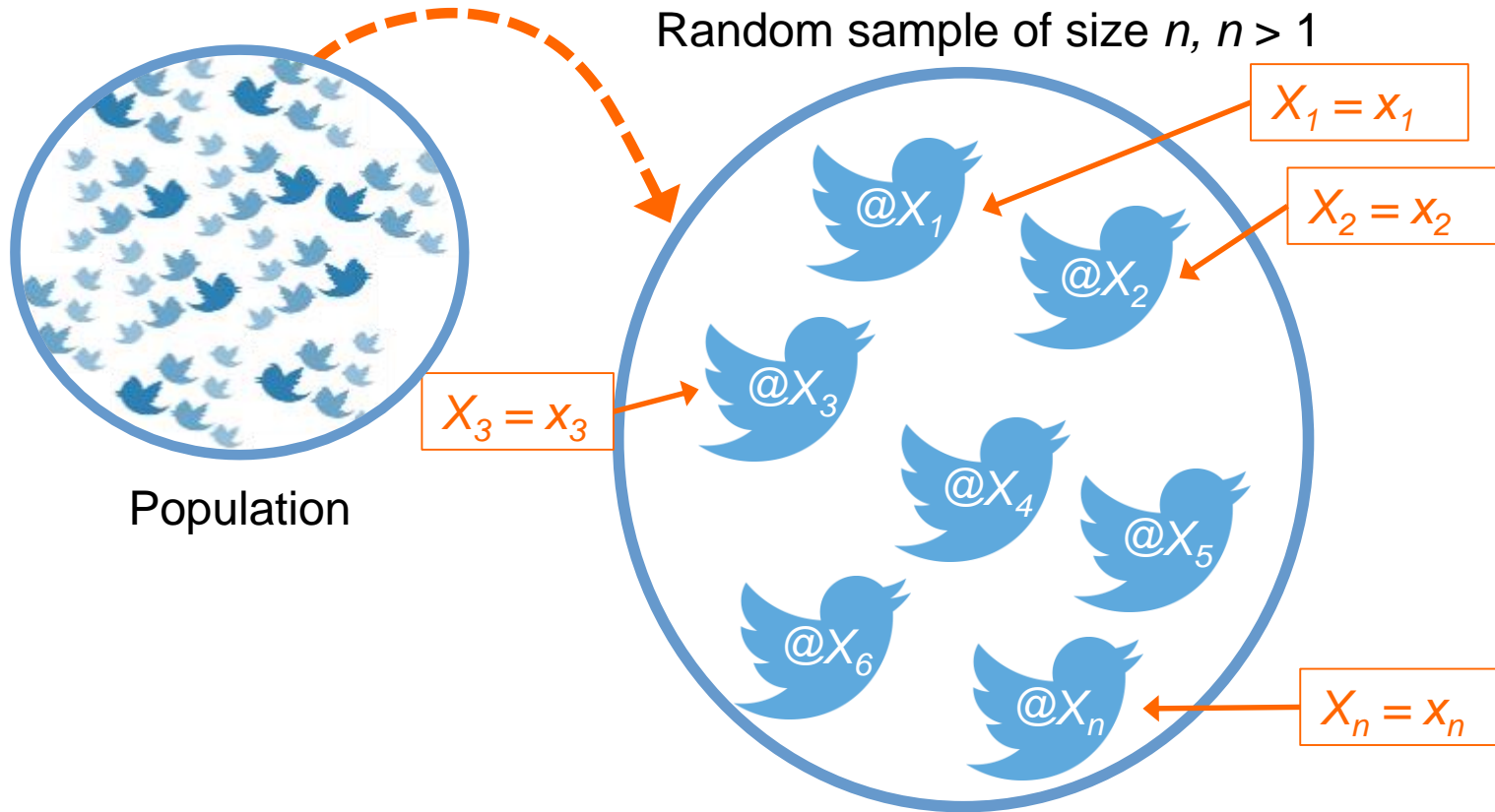
# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



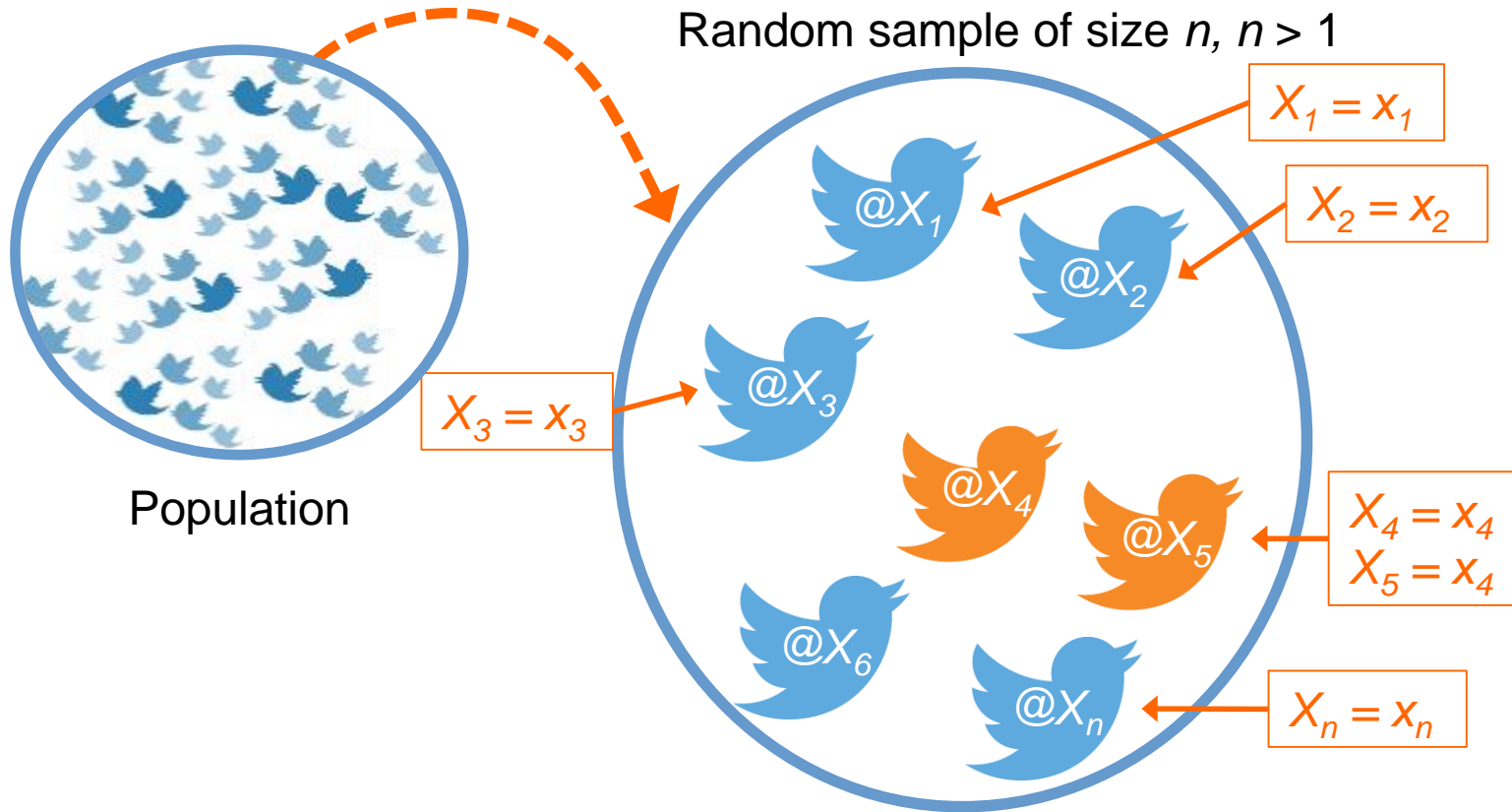
Population

The values ( $x_i$ )  
of  $X_1, X_2, \dots, X_n$   
are each  
observed



# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



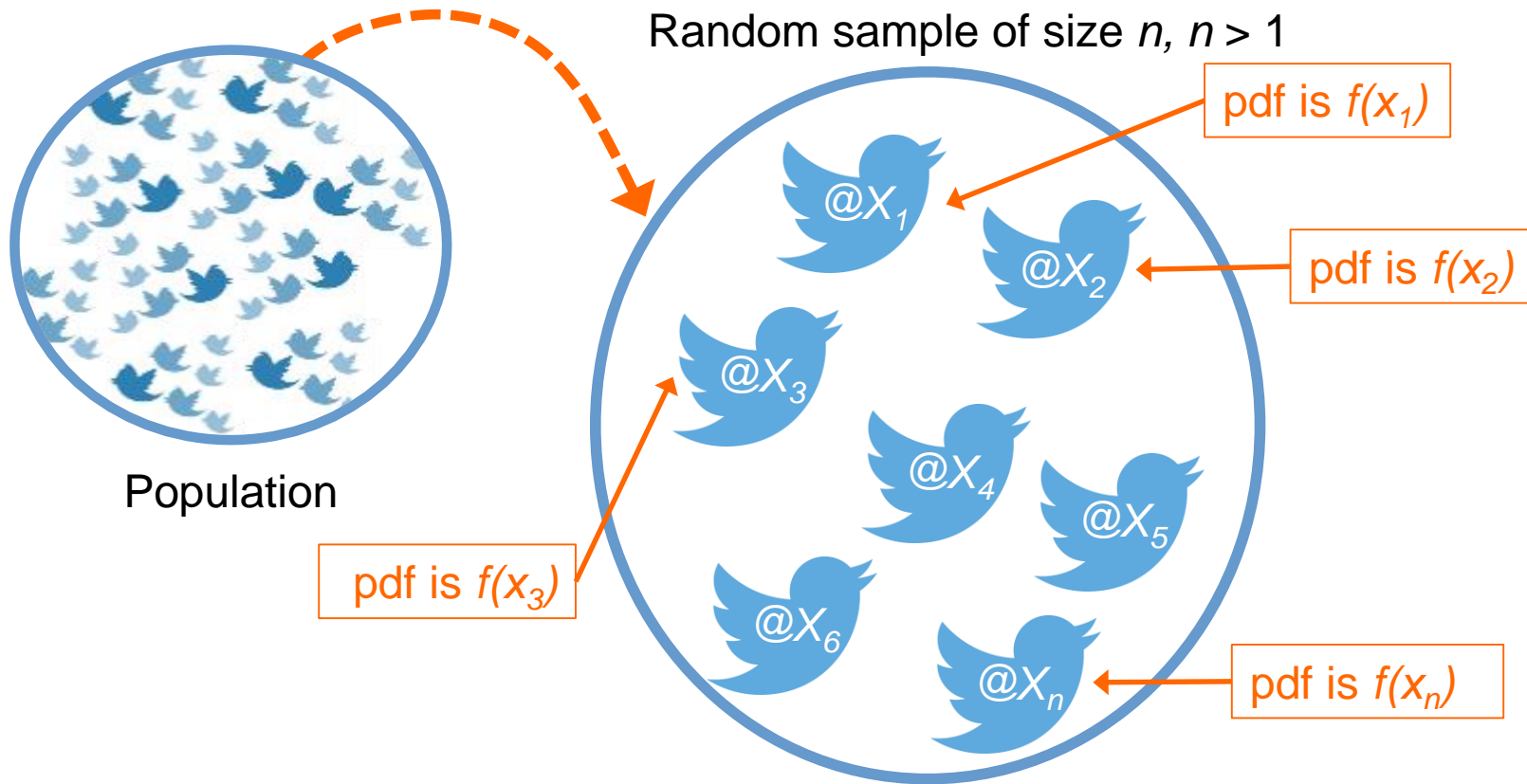
Population

If two values of  $x_i$  are the same...



# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



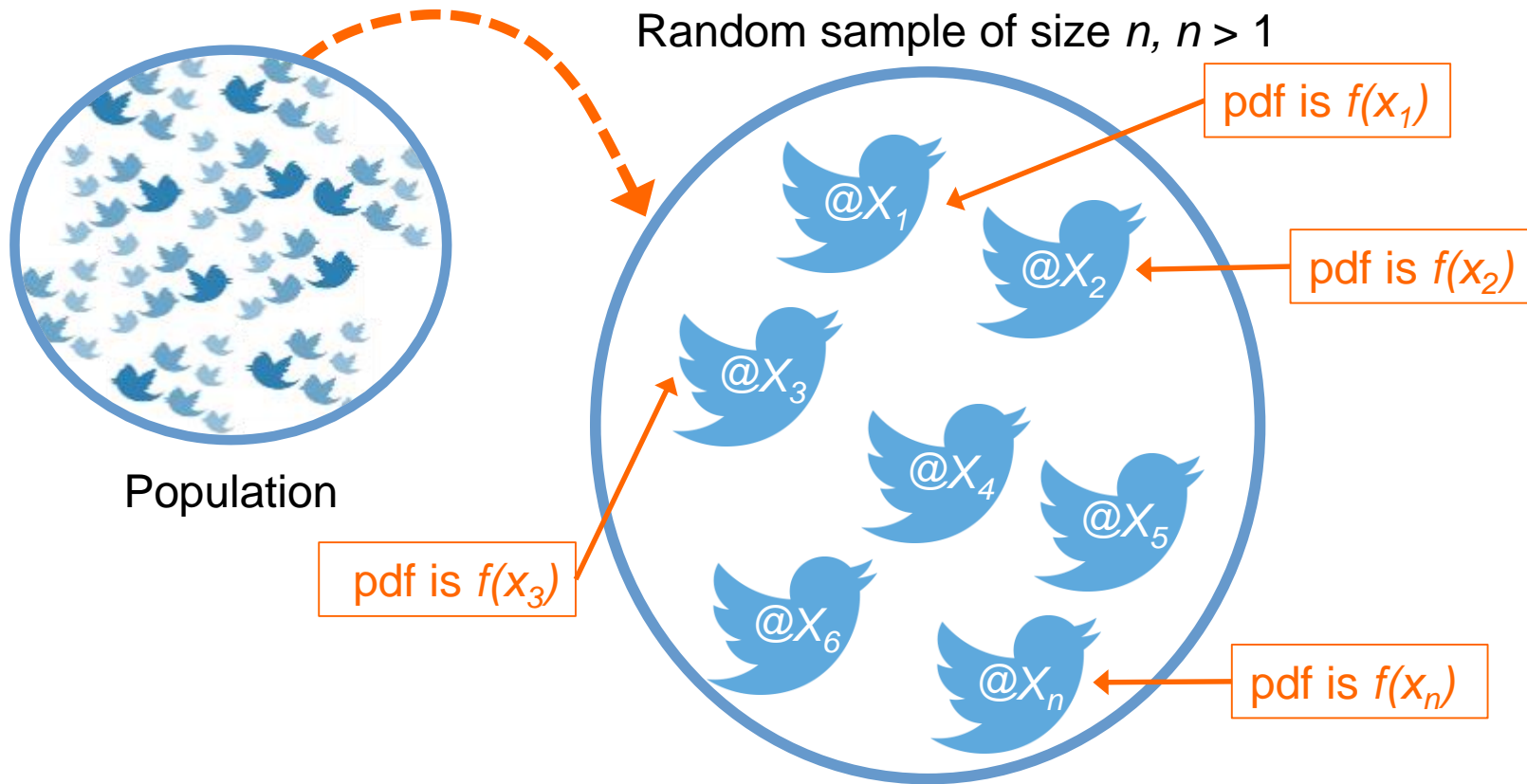
Each  $X_i$  has a  
marginal  
distribution,  $f(x_i)$





# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$

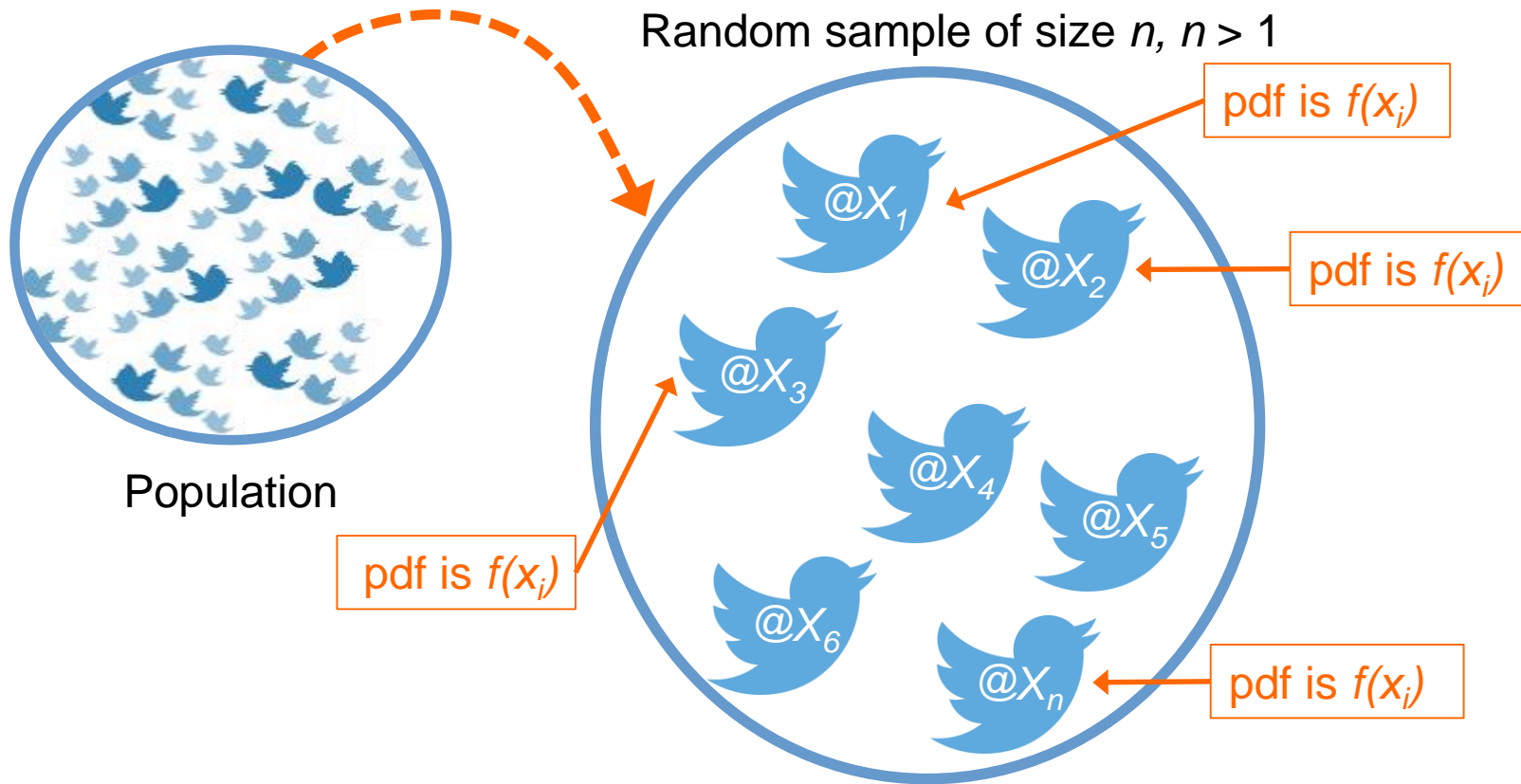


Hooray! Because each observation  $X_i$  is **independent**, we can just multiply each  $f(x_i)$  get joint pdf!



# Random variable, $X = \#$ of tweets per day

Random sample of size  $n$ ,  $n > 1$



And because each observation  $X_i$  is **identically distributed**,

$$f(x_1) = f(x_2) = \dots = f(x_n) = f(x_i)$$



# Random variable, $X = \#$ of tweets per day

$$f(x_1, \dots, x_n) = f(x_1) \times f(x_2) \times \dots \times f(x_n) = \prod_{i=1}^n f(x_i)$$

Since  $X_1, X_2, \dots, X_n$  are **mutually independent**, the probabilities  $P(X_1), P(X_2), \dots, P(X_n)$  can simply be multiplied to get the joint pdf of the sequence of observations of the rv of interest

Since  $X_1, X_2, \dots, X_n$  are **identically distributed**, all  $f(x_i)$  are the exact same function

