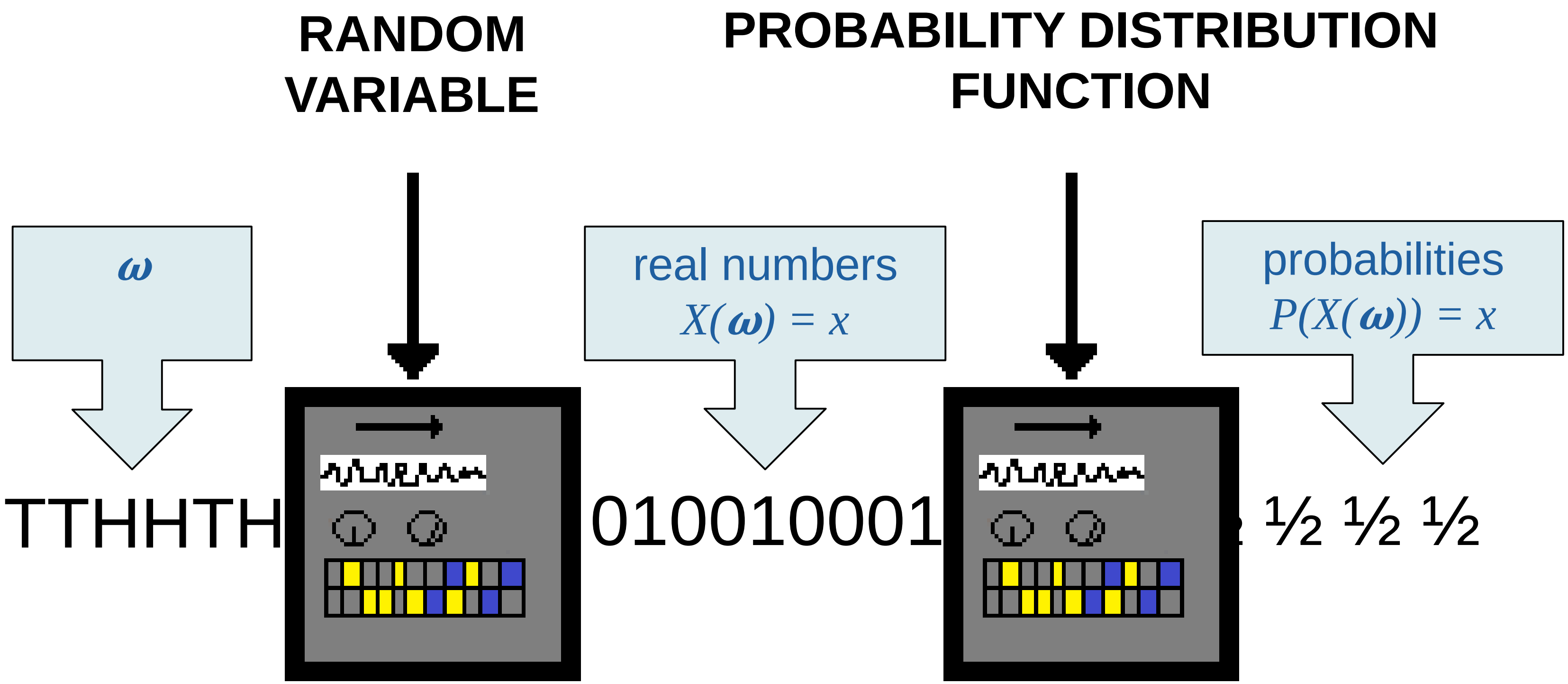
CM 3.2: Probability Functions

The probability triple

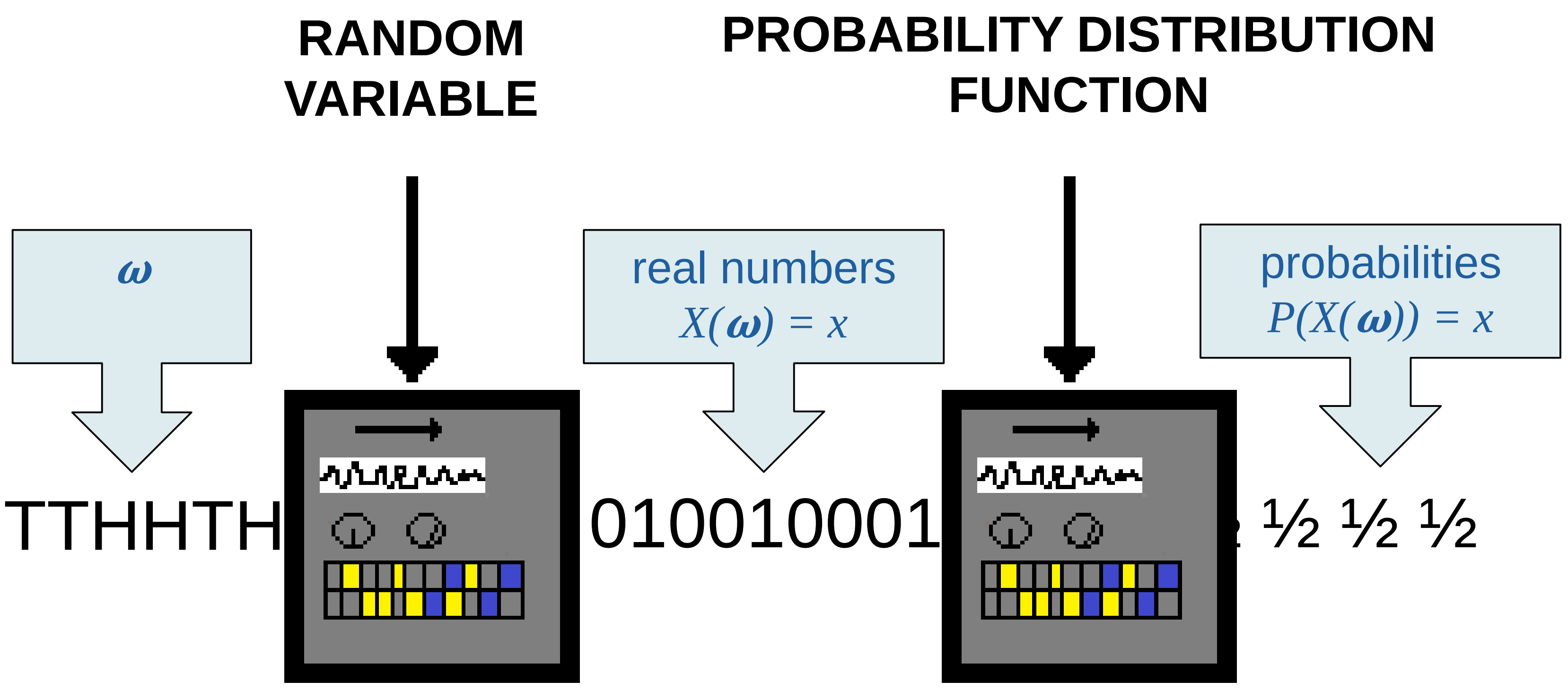
- Given a random experiment…

$\boldsymbol{\Omega}$	Sample space	What can possibly happen?
$\mathscr{F}$	Set of events	What are the sane questions I can ask about this probability distribution?
$\mathbf{P}$	Probability measure	Function that maps elements from $\mathscr{F}$ to the interval [0,1]


RANDOM VARIABLE
PROBABILITY DISTRIBUTION FUNCTION
ω
real numbers
X(ω) = x
probabilities
P(X(ω)) = x
TTHHTH
010010001
½ ½ ½

A function that associates a real number with an event.

Random Variable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Function that maps the output of a random variable to the interval [0, 1] in probability space.

Probability Distribution Function

Discrete or continuous?

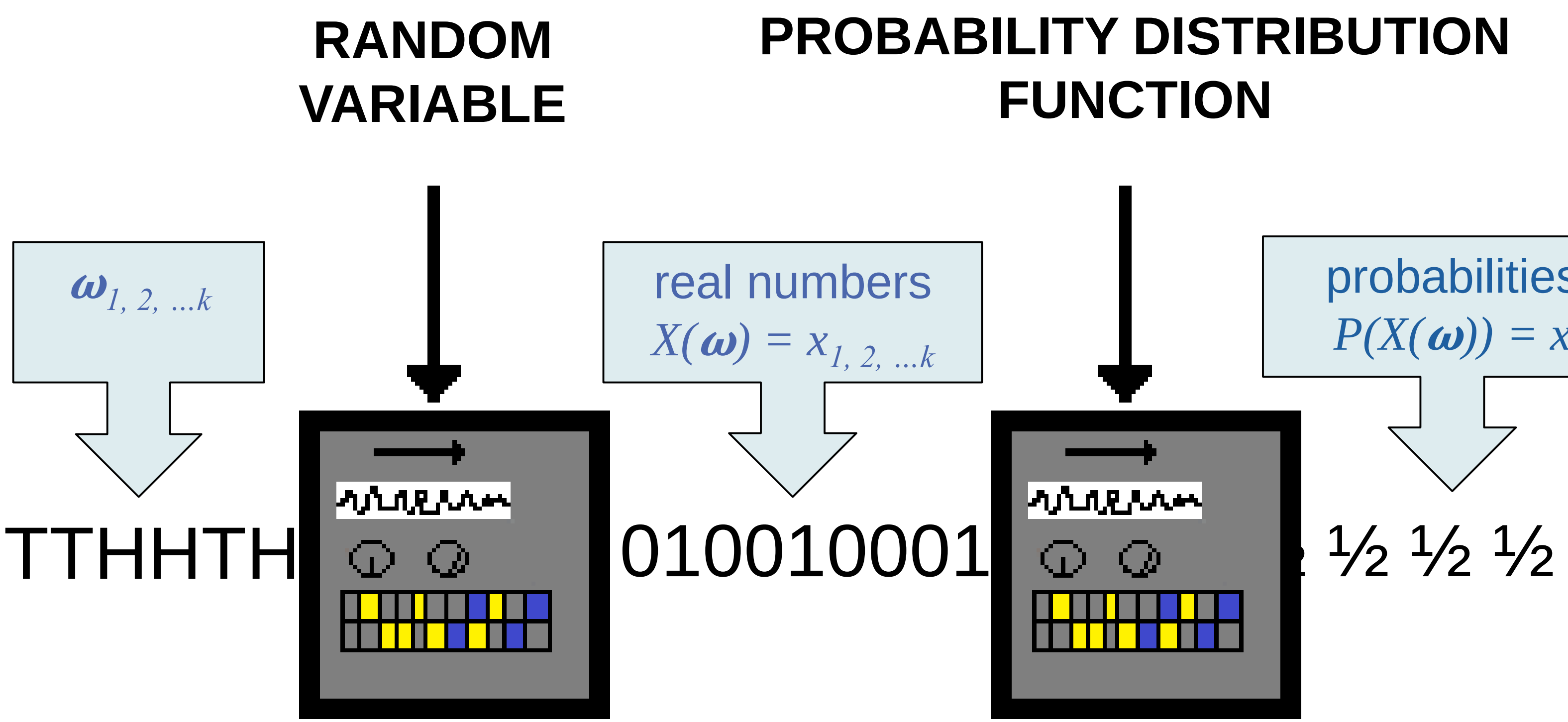
- A continuous rv can take any value in an interval or collection of intervals
- A discrete rv can take one of a countable list of distinct values
- Different things → different math required

pmf or pdf?

- Every discrete rv has a probability mass function (pmf)
- Every continuous rv has a probability density function (pdf)
- Different ways of defining the function that says how likely an event is.

Discrete rv

- Remember X is our random variable
- We feed it all the $\boldsymbol{\omega}_{1,\ 2,\ \ldots k}$ and it transforms each simple event into a value, call that $x_{1,\ 2,\ \ldots k}$
- Now each of these values is fed to the probability distribution function, which takes these $x_{1,\ 2,\ \ldots k}$ and tallies the number of values it sees, then divides by the total, such that we now have P($X = X_k$)
- So far we have only seen this as a column in a table, but it is a function! P($X = X_k$) = $f(X_k)$
- Called the *probability mass function* (pmf)
- What must be true about this function?



RANDOM VARIABLE
PROBABILITY DISTRIBUTION FUNCTION
$\omega_{1,\ 2,\ \ldots k}$
real numbers
$X(\omega) = x_{1,\ 2,\ \ldots k}$
probabilities
$P(X(\omega)) = x$
TTHHTH
010010001
½ ½ ½

$$\sum_{x_i \in \Omega} f(x_i) = 1$$

$$f(x_i) \geq 0, \forall x_i \in \Omega$$

$$f(x)$$

For discrete X, $f(x)$ is a

Probability Mass Function*

*It gives you a probability! (always between 0 to 1)

Which of these is a pmf?

x	f(x)
-1	0.3
0	0.3
2	0.3

x	f(x)
10	-0.1
20	0.9
30	0.2

x	f(x)
1	0.35
2	0.25
3	0.2
4	0.1
5	0.1

x	f(x)
10	0.1
20	0.9
30	0.2

x	f(x)
5	1

What does a pmf look like?

- Table/list of numbers
  - Rows = $x_{1,\ 2,\ \ldots k}$
- Function (large n)
- A plot
  - x-axis?
  - y-axis?

Bernoulli rv

- Special type of simple discrete rv

- Two possible outcomes: success ($X = 1$) or failure ($X = 0$)
  - P(1) = *p*
  - P(0) = *1- p*

Sequences of independent Bernoulli trials → other special types of discrete rvs with discrete distributions

Distribution	Definition
Binomial	Number of successes in n trials
Negative binomial	Number of failures before the x-th success
Geometric	Number of failures before the 1^{st} success

Notation

- The tilde or twiddle: “~”
  - English: “is distributed as”

- X ~ DistributionName(parameters)

- Coin tossing example: $X \sim Bin(n, p)$
  - English: rv X is distributed as a binomial rv with…
  - n number of trials and*…
  - p as the **single trial** probability of success

*n here can be number of coins tossed, or you could have 1 coin tossed n times

$$X(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is 2 heads} \\ 1, & \text{if } \omega \text{ is 1 head} \\ 0, & \text{if } \omega \text{ is 0 heads} \end{cases}$$

Ω

Ω	$X(\omega)$	probability
H T	1	.25
T H	1	.25
H H	2	.25
T T	0	.25

n = ?
p = ?

$$X(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is 2 heads} \\ 1, & \text{if } \omega \text{ is 1 head} \\ 0, & \text{if } \omega \text{ is 0 heads} \end{cases}$$

Ω

$X(\omega)$	probability
1	.25
1	.25
2	.25
0	.25

n = 2 (not an rv!)
p = .5
(parameter!)

$X \sim Bin(2, .5)$: 1 experiment

In R, rbinom(number of experiments (`n`), number of trials per experiment (`size`), probability of success (`prob`). We'll start with one experiment (flipping 2 coins once):

```
> num_exp <- 1
> num_trials <- 2
> p <- .5
> rbinom(num_exp, num_trials, p)
[1] 2
> rbinom(num_exp, num_trials, p)
[1] 0
> rbinom(num_exp, num_trials, p)
[1] 1
```

A little boring- but seems to work!
Let's try flipping 2 coins 10 times…

$X \sim Bin(2, .5)$: 10 experiments

```
> num_exp <- 10
> num_trials <- 2
> x <- rbinom(num_exp,
num_trials, .5)
> length(x)
[1] 10
> table(x)
x
0 1 2
1 6 3
```

x	Expected $P(x)$	Observed $P(x)$
0		10%
1		60%
2		30%

$X \sim Bin(2, .5)$: 1,000 experiments

```
> num_exp <- 1000
> num_trials <- 2
> x <- rbinom(num_exp,
num_trials, .5)
> length(x)
[1] 1000
> table(x)
x
  0   1   2
234 512 254
```

x	Expected $P(x)$	Observed $P(x)$
0	25%	23.4%
1	50%	51.2%
2	25%	25.4%

Let X now count the number of heads (k) in $n = 30$ independent coin flips…

$$X(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is 2 heads} \\ 1, & \text{if } \omega \text{ is 1 head} \\ 0, & \text{if } \omega \text{ is 0 heads} \end{cases}$$

In R:

```
x <- rbinom(num_exp, 2,
.5)
```

$$X(\omega) = k, \text{ where k} = \text{total number of heads}$$

In R:

```
k <- rbinom(num_exp, 30,
.5)
```

What is the range of k?

Binomial pmf: $f_X(k; n, p)$ (equivalent notation)

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$$

1. How many ways are there to get k heads? (combination)

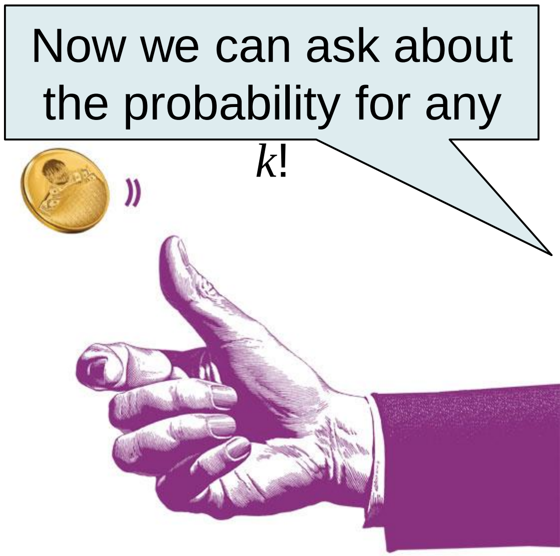
2. How likely is any particular combination of k heads and k - n tails? (multiplicative rule)

When we use the Binomial distribution we know *p* and *n* and the distribution describes the probabilities for all the *k*s we could have.

$X \sim Bin(30, .5)$: 1 experiment, 30 flips

- The distribution that describes the number of heads (k) after n tosses of a fair coin. Given that we have *n* = 30 trials, and we want *k* heads, and the probability of getting a head is *p* = .5, the probability of each *k* is:

$$P(X = k) = \binom{30}{k} .5^k (1 - .5)^{30-k}$$




```
> k <- rbinom(1, 30, .5)
> k # in my 1 experiment, I got 17 heads
[1] 17
```

In R…

R FUNCTION	DISCRETE RV	CONTINUOUS RV
`r`	Generates random deviates	
`d`	Probability (pmf)	Probability density (pdf)
`p`	Cumulative probability (cdf)	
`q`	Inverse cumulative probability (quantiles)	

Questions we can ask the pmf

- "What is the probability of getting exactly 17 heads?” 0.112
- "What is the probability of getting 18 to 24 heads?” 0.181
- "What is the probability that the number of heads ends in 7?” 0.113
- "What is the probability of getting 1.5 heads?” Huh?

Questions we can ask the pmf in R

- "What is the probability of getting exactly 17 heads?" 0.112

  `dbinom(17, 30, .5)`

- "What is the probability of getting 18 to 24 heads?" 0.181

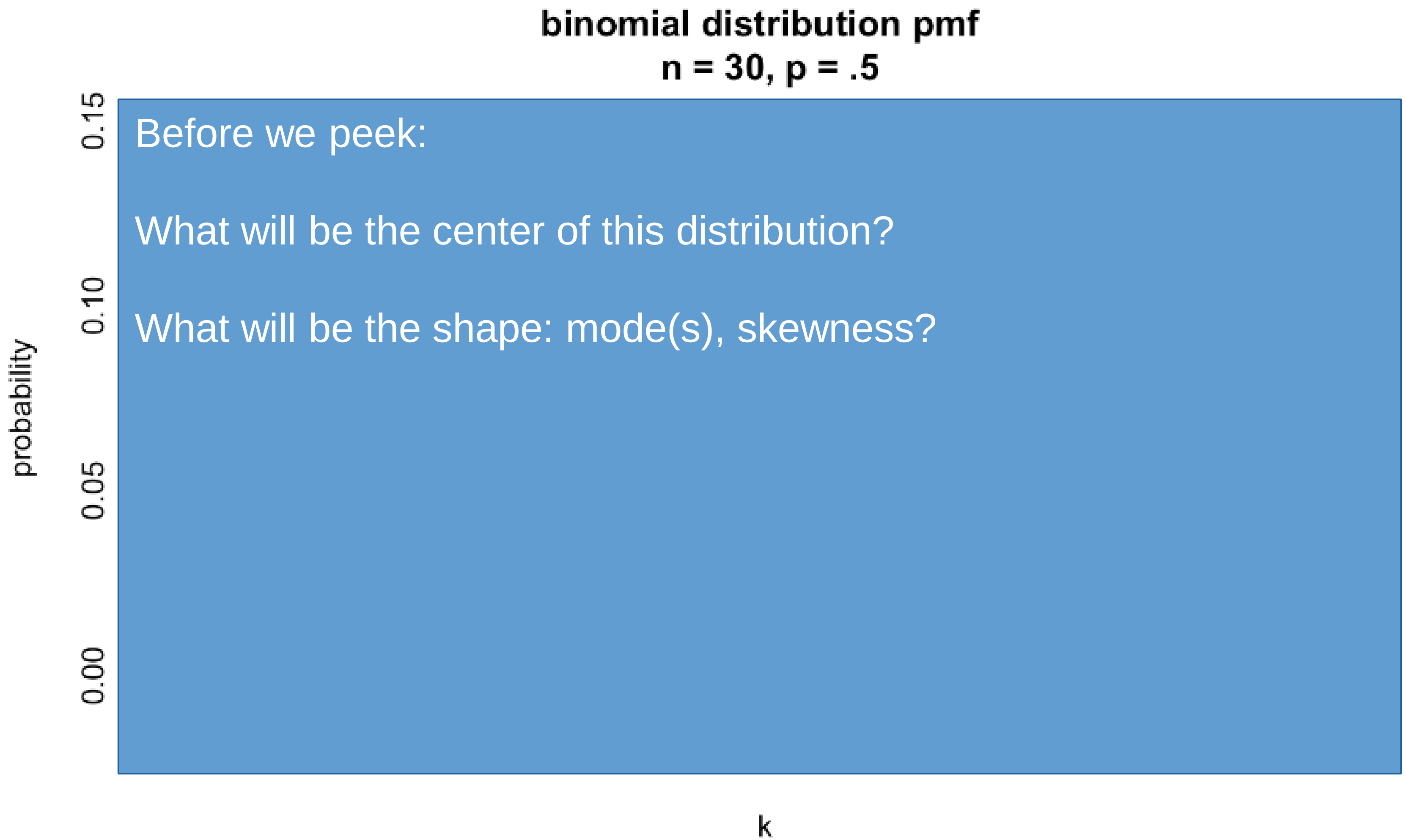
  `sum(dbinom(18:24, 30, .5))`

- "What is the probability that the number of heads ends in 7?" 0.113

  `sum(dbinom(c(7, 17, 27), 30, .5))`

- "What is the probability of getting 1.5 heads?" Huh?

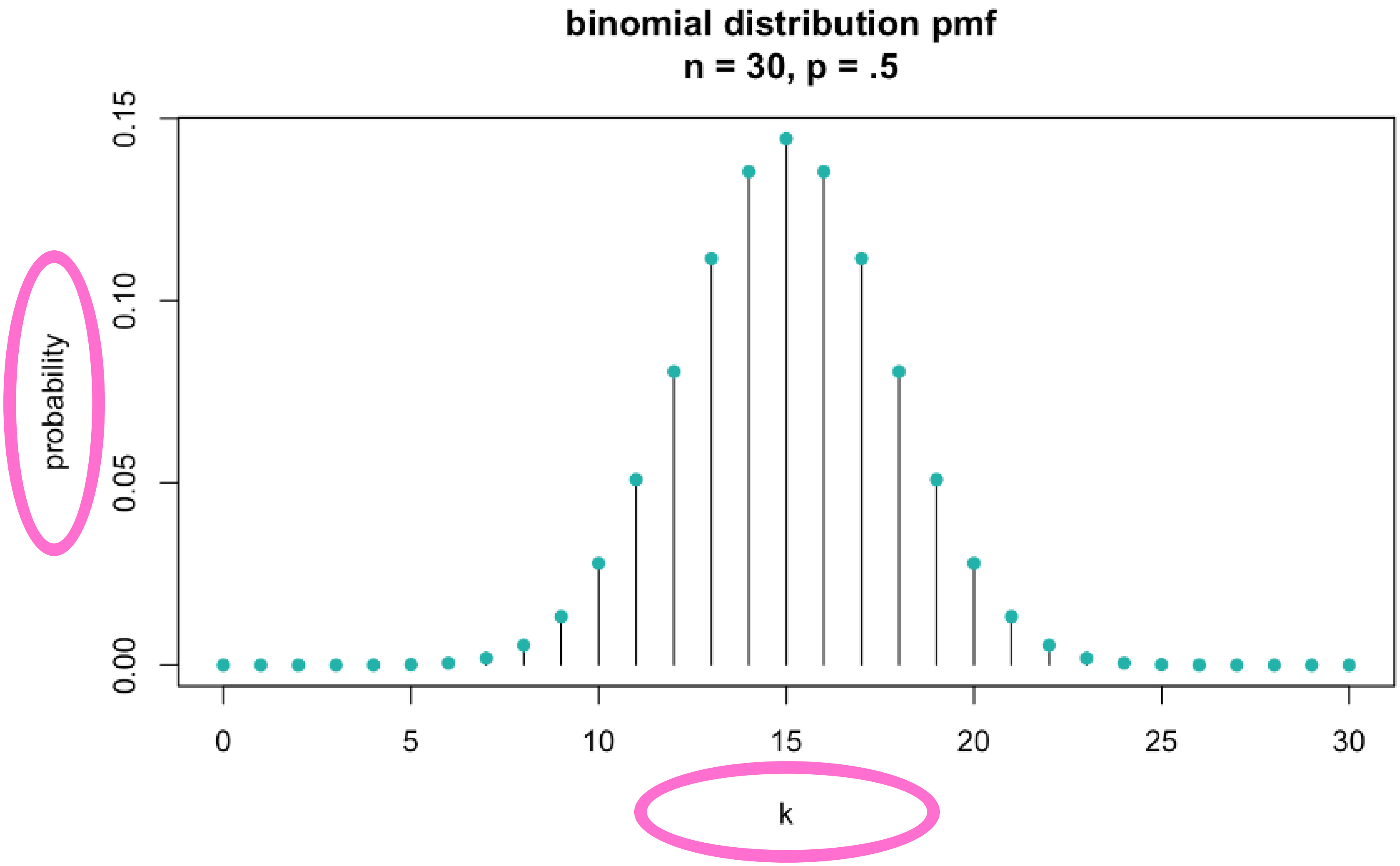
```
Warning message:
In dbino       non-integer x = 1.500000
```



Before we peek:

What will be the center of this distribution?

What will be the shape: mode(s), skewness?



binomial distribution pmf
n = 30, p = .5
probability
0.00
0.05
0.10
0.15
0
5
10
15
20
25
30
k

$$F(x)$$

For all X, $F(x)$ is a

Cumulative Distribution Function

Discrete rv: cumulative distribution function

$$F(x) = \sum_{t \leq x} f(t)$$

(since x is used as a variable in the summation, we use "t" just as some other variable)

Cumulative distribution function

$$1.\, F(x) \rightarrow 1\, as\, x \rightarrow \infty$$

$$2.\, F(x) \rightarrow 0\, as\, x \rightarrow -\infty$$

$$3.\, F(x)\, is\, monotonic;\, never\, decreasing$$

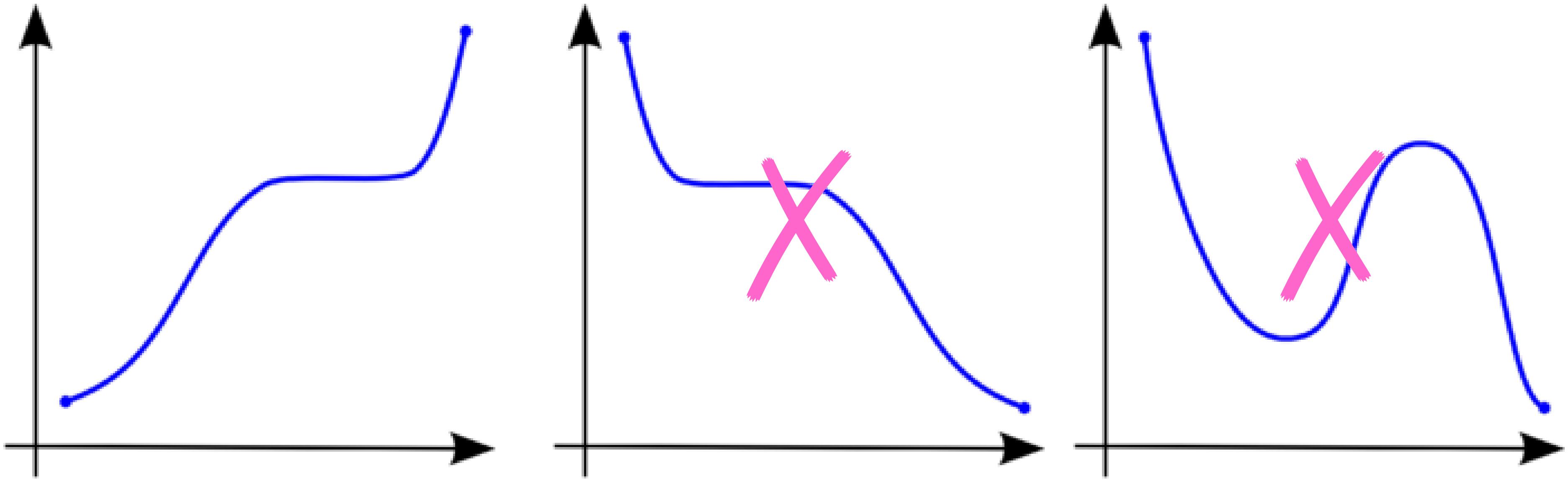
$1.\ F(x) \to 1\ as\ x \to \infty$

$$\lim_{x \to +\infty} F(x) = 1$$

$2.\ F(x) \to 0\ as\ x \to -\infty$

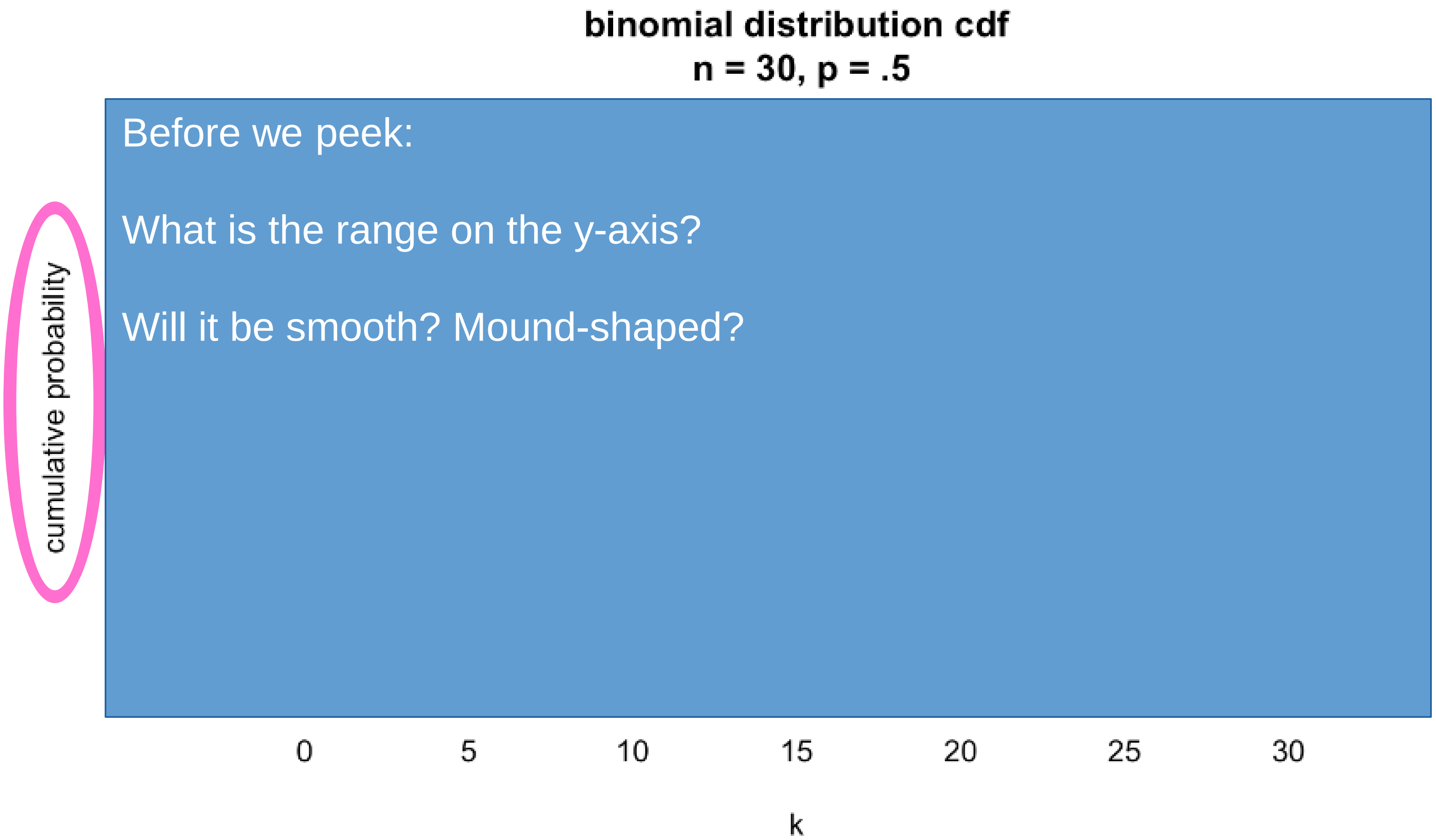
$$\lim_{x\to-\infty} F(x) = 0$$

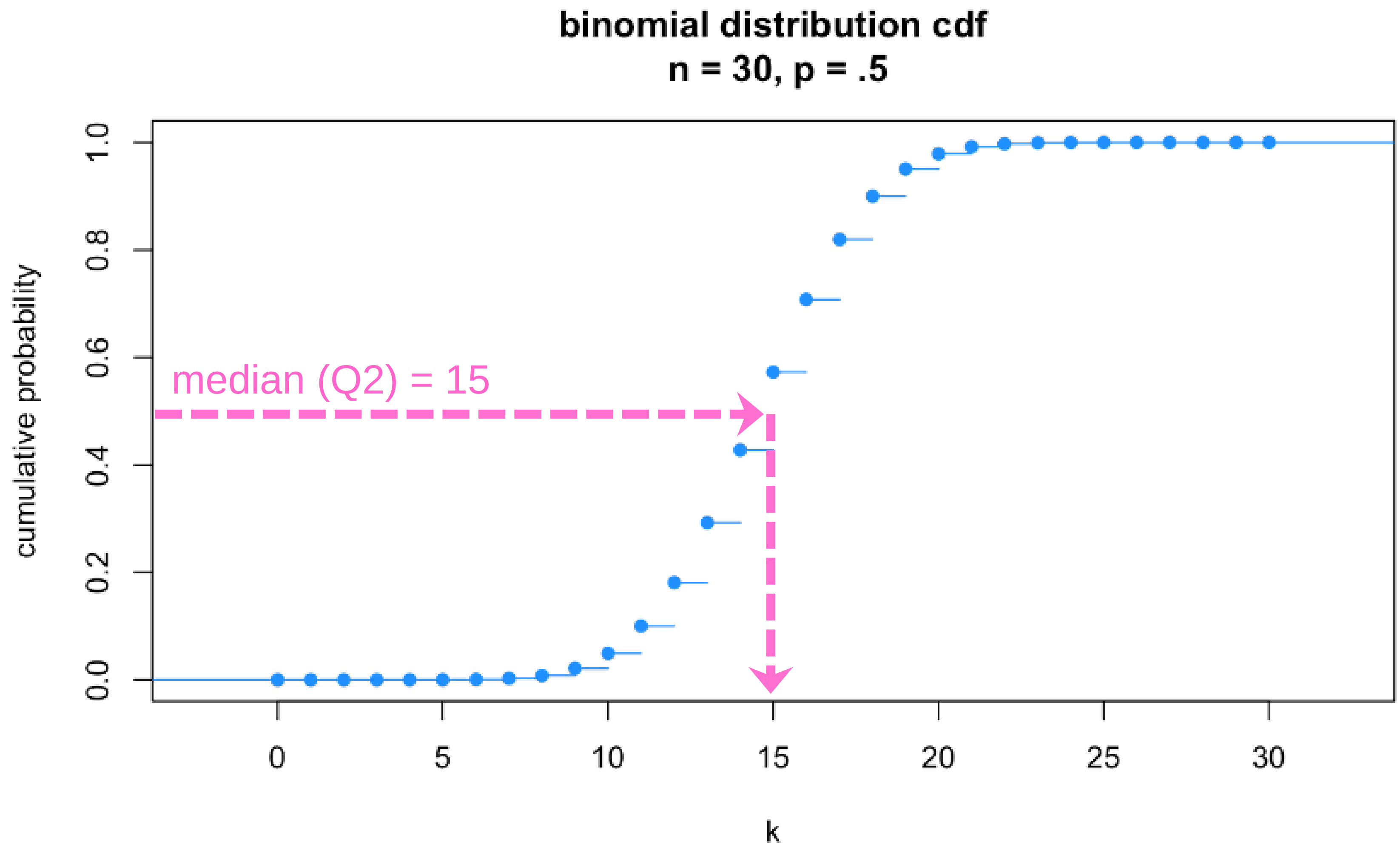
$$3.\, F(x)\, is\, monotonic;\; never\, decreasing$$



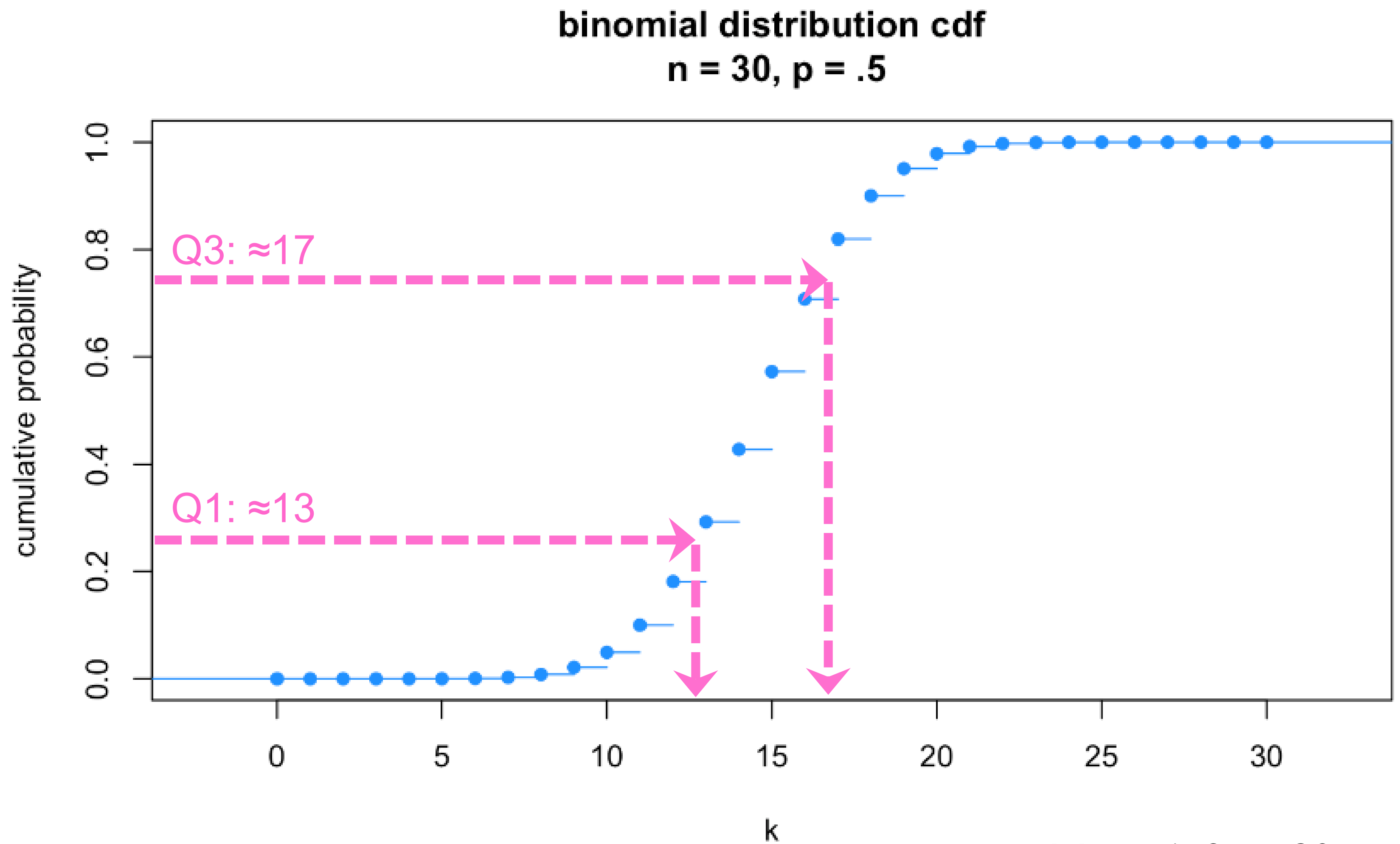
Let X count the number of heads (k) in n independent coin flips (again)…

$$X(\omega) = k, \text{ where k} = \text{total number of heads}$$

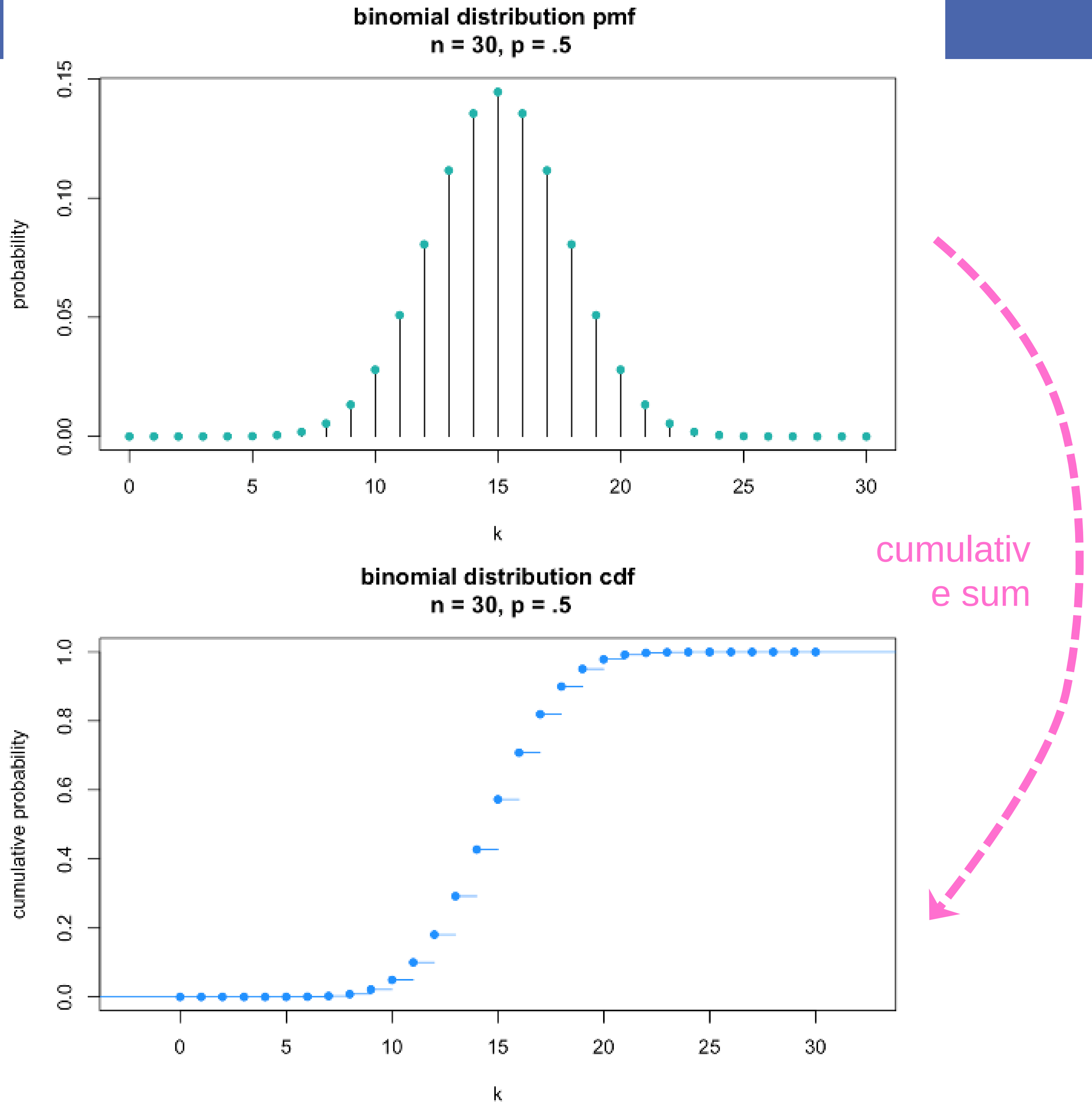

binomial distribution cdf
n = 30, p = .5
Before we peek:
What is the range on the y-axis?
Will it be smooth? Mound-shaped?
cumulative probability
0
5
10
15
20
25
30
k

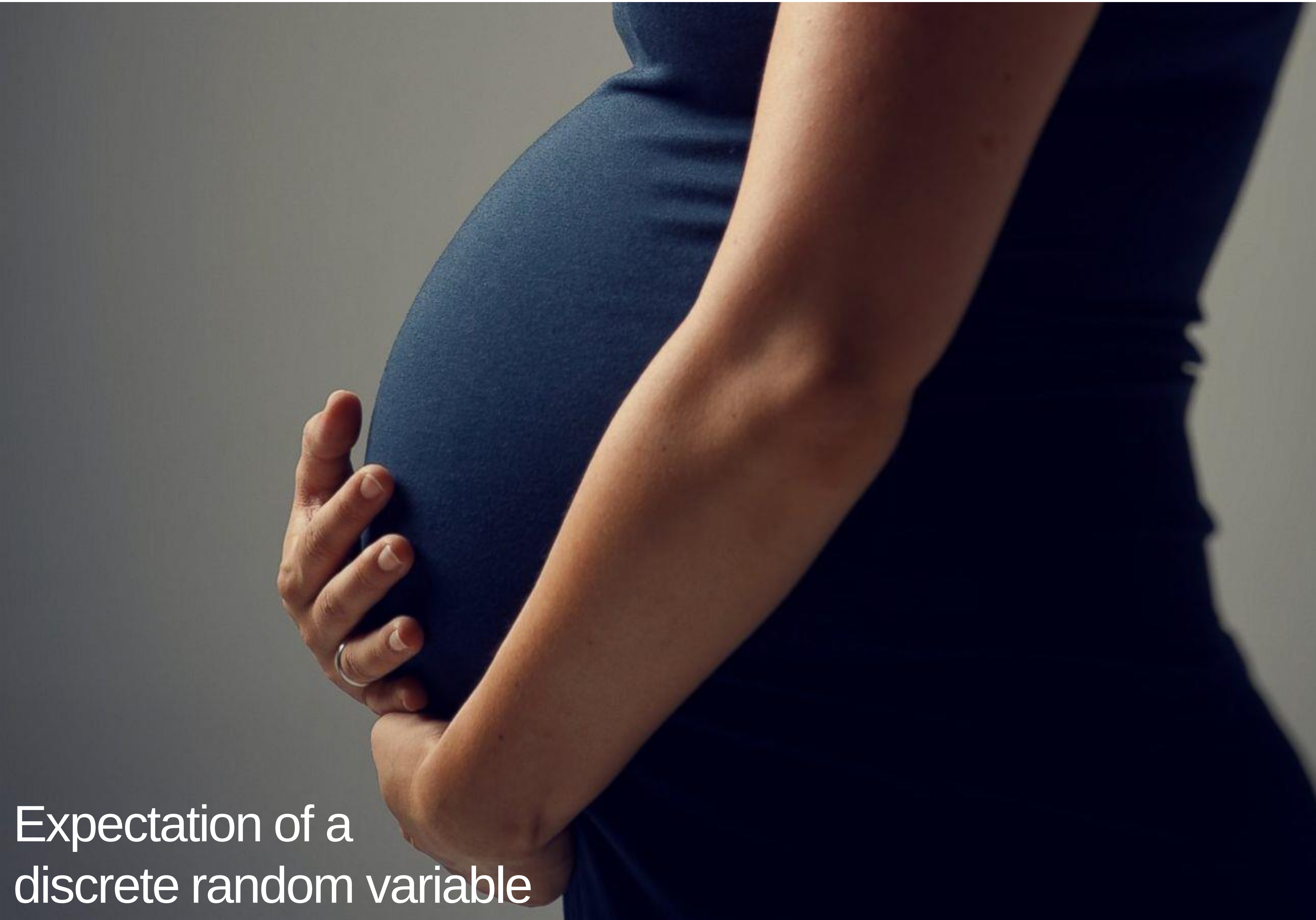



```
> qbinom(.5, 30, .5) #Q2 (median)
[1] 15
```




```
> qbinom(.25, 30, .5) #Q1
[1] 13
> qbinom(.75, 30, .5) #Q3
[1] 17
```

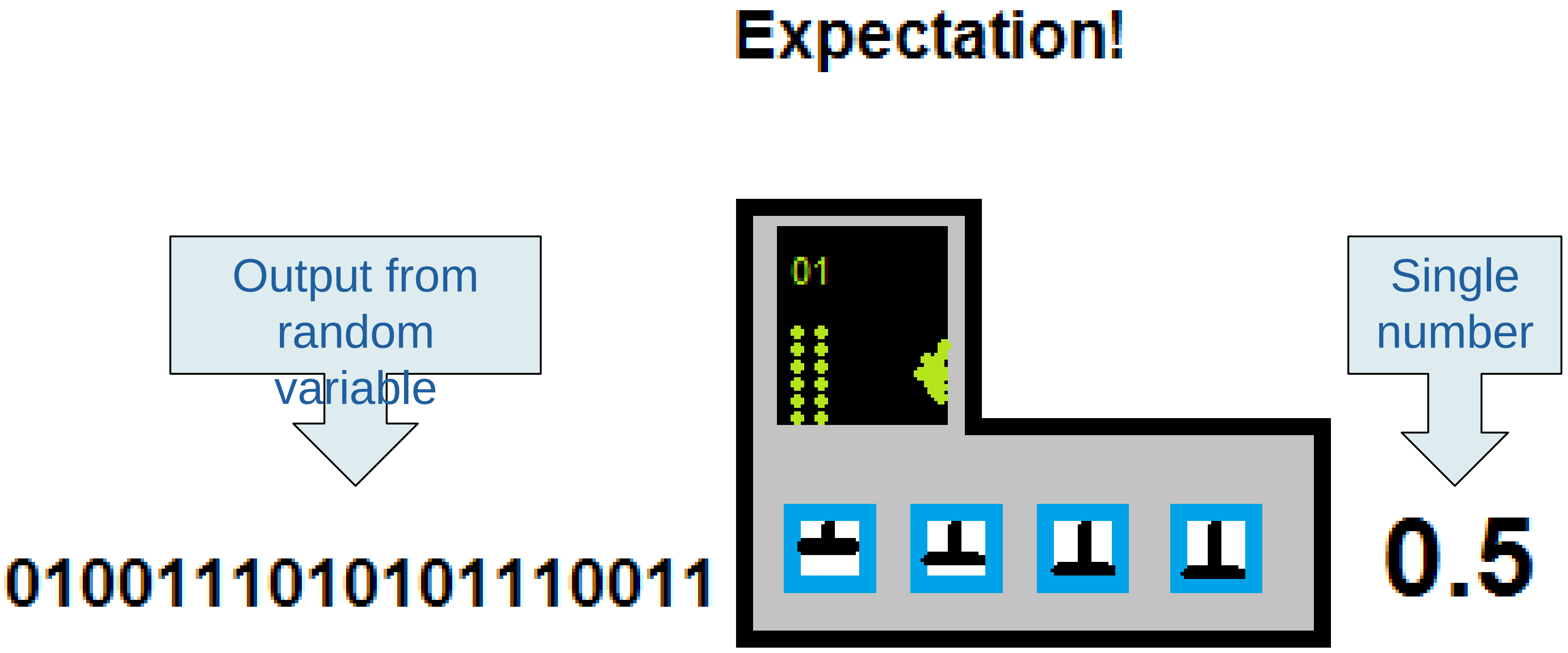

binomial distribution pmf
n = 30, p = .5
probability
k
cumulativ
e sum
binomial distribution cdf
n = 30, p = .5
cumulative probability
k



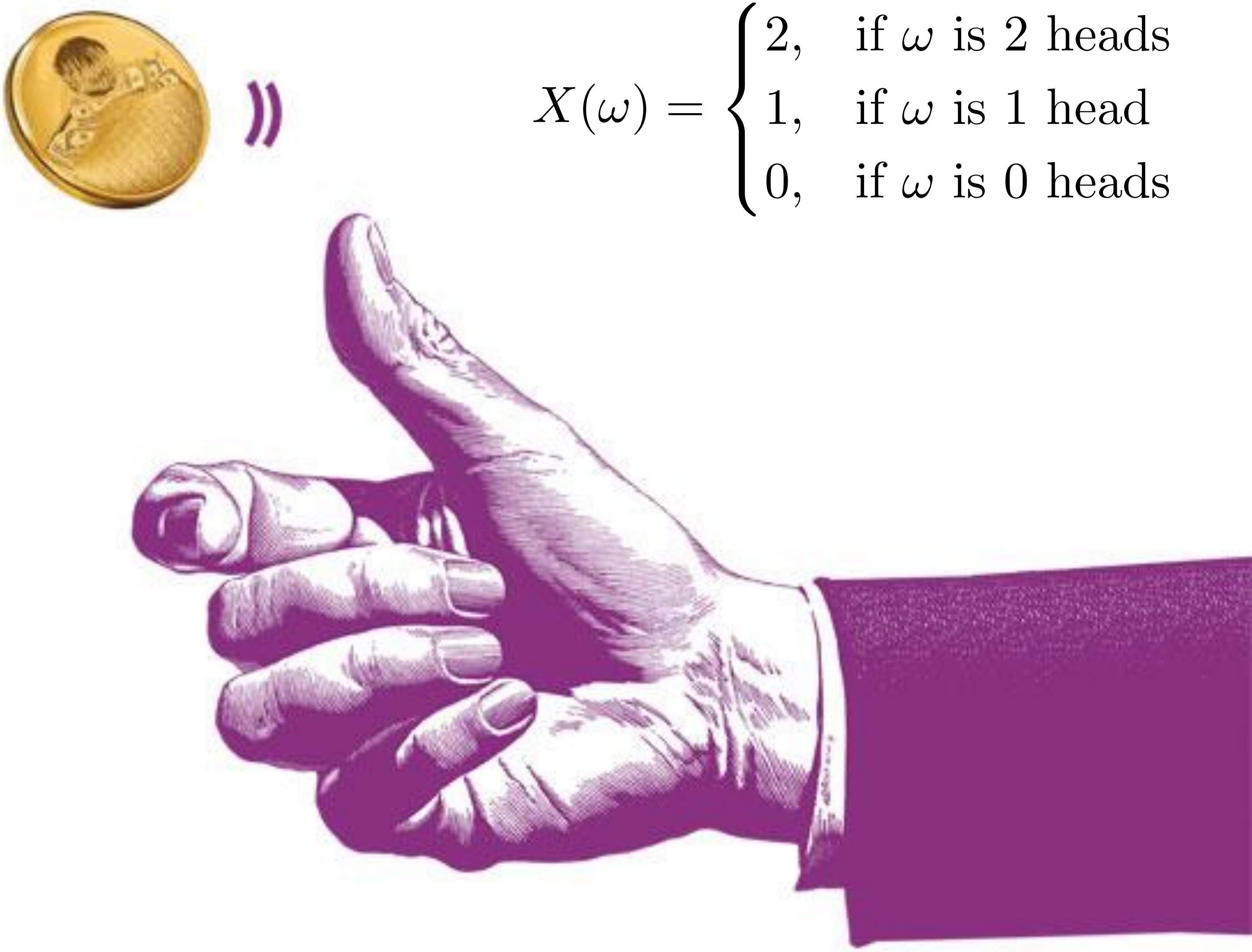
Expectation of a discrete random variable

Expectation

The expectation of a random variable (X) is the sum of its values (x_i) weighted by their probability (p_i).



Expectation: Toss a fair coin 2 times



$$X(\omega) = \begin{cases} 2, & \text{if } \omega \text{ is 2 heads} \\ 1, & \text{if } \omega \text{ is 1 head} \\ 0, & \text{if } \omega \text{ is 0 heads} \end{cases}$$

$$E(X) = (1 \times .5) + (2 \times .25) + (0 \times .25) = 1$$

x	$P(x)$	$E(x)$
	$1 \times .5$	$.5$
	$2 \times .25$	$.5$
	$0 \times .25$	0
	$E(X)$	1.0

Expected values

- The expected value estimates the population mean, μ_X, for an rv with a given distribution
- Our example of families with fraternal twins
- Random variable X: # of girls (discrete!)
- Population: all families with fraternal twins
- Expected value: the mean number of girls per families across all families with fraternal twins
- Given that boys and girls are equally likely, what is μ_X?

$$E(X) = (1 \times .5) + (2 \times .25) + (0 \times .25) = 1$$

	x	$P(x)$	$E(x)$
♀	1 ×	.5	.5
♀ ♀	2 ×	.25	.5
~~♀~~	0 ×	.25	0
		$E(X)$	1.0

Expected values: discrete distributions

Population mean

μ_X

pmf

$$\underbrace{E(X)}_{\mu_X} = \sum_{i=1}^{n} x_i \overbrace{f_X(x_i)}^{\text{pmf}}$$

$\boxed{\text{For binomial, plug in pmf: } \binom{n}{k} p^k (1-p)^{n-k}, \quad n = 2, \quad p = .5}$

For binomial, plug in pmf: $\binom{n}{k} p^k (1-p)^{n-k}$

n = 2

p = .5

Expected value of a binomial rv: the long way

$$E(X) = \sum_{i=0}^{2} x_i \binom{2}{x_i} \left(\frac{1}{2}\right)^2 = \underbrace{\left(0 \times \left(1 \times \frac{1}{4}\right)\right)}_{= 0} + \underbrace{\left(1 \times \left(2 \times \frac{1}{4}\right)\right)}_{= 1/2} + \underbrace{\left(2 \times \left(1 \times \frac{1}{4}\right)\right)}_{= 1/2}$$

R can help:

```
> choose(2,0)
[1] 1
> choose(2,1)
[1] 2
> choose(2,2)
[1] 1
```

$$E(X) = \sum_{i=0}^{2} x_i \binom{2}{x_i} \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Population variance for discrete rv

- The variance is the expected value of the square of the deviation of X from its own mean

$$Var(X) = \sigma_X^2 = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Population variance for discrete rv

- The variance is the expected value of the square of the deviation of X from its own mean

$$Var(X) = \sigma_X^2 = \sum_{i=1}^{n} [x_i - \mu_X]^2 f_X(x_i)$$

pmf → probabilities!

$$Var(X) = \sigma_X^2 = \sum_{i=1}^{n} [x_i - \mu_X]^2 \times p(x_i)$$

Expected value of a binomial rv: the short way

$$E(X) = np$$

$$Var(X) = np(1-p)$$

Expected values

- If you sample from a distribution, and your sample is sufficiently large, then you have a reasonable expectation that your sample mean will be close to the population mean
- We can calculate the expected value for the underlying distribution and use this to form hypotheses about our current sample

$$\mu_X \neq \overline{x}$$