CM 3.3: Continuous Random Variables

Past: discrete rvs
Present: continuous rvs

- for discrete rvs, $P(X = x)$ is technically called the *probability mass function*

- for continuous rvs, the analogous concept is the *density*

Density

- The density of X can be seen as a value proportional to the chance of drawing from the population a number that is lying in the close proximity of X.

- Sadly, density does not give you probabilities directly

- Probabilities can only be obtained from densities by taking an integral

- Integrals are simply continuous sums

$$f(x)$$

For continuous X, $f(x)$ is a

Probability Density Function*

*Not a probability! (> 1)

Continuous rv pdf

$$1.\ \int_{-\infty}^{\infty} f(x)dx = 1$$

$$2. f(x) \geq 0 \quad always$$

What does $P(X = a) = ?$

$$P(a < X < b) = \int_a^b f(x)dx$$

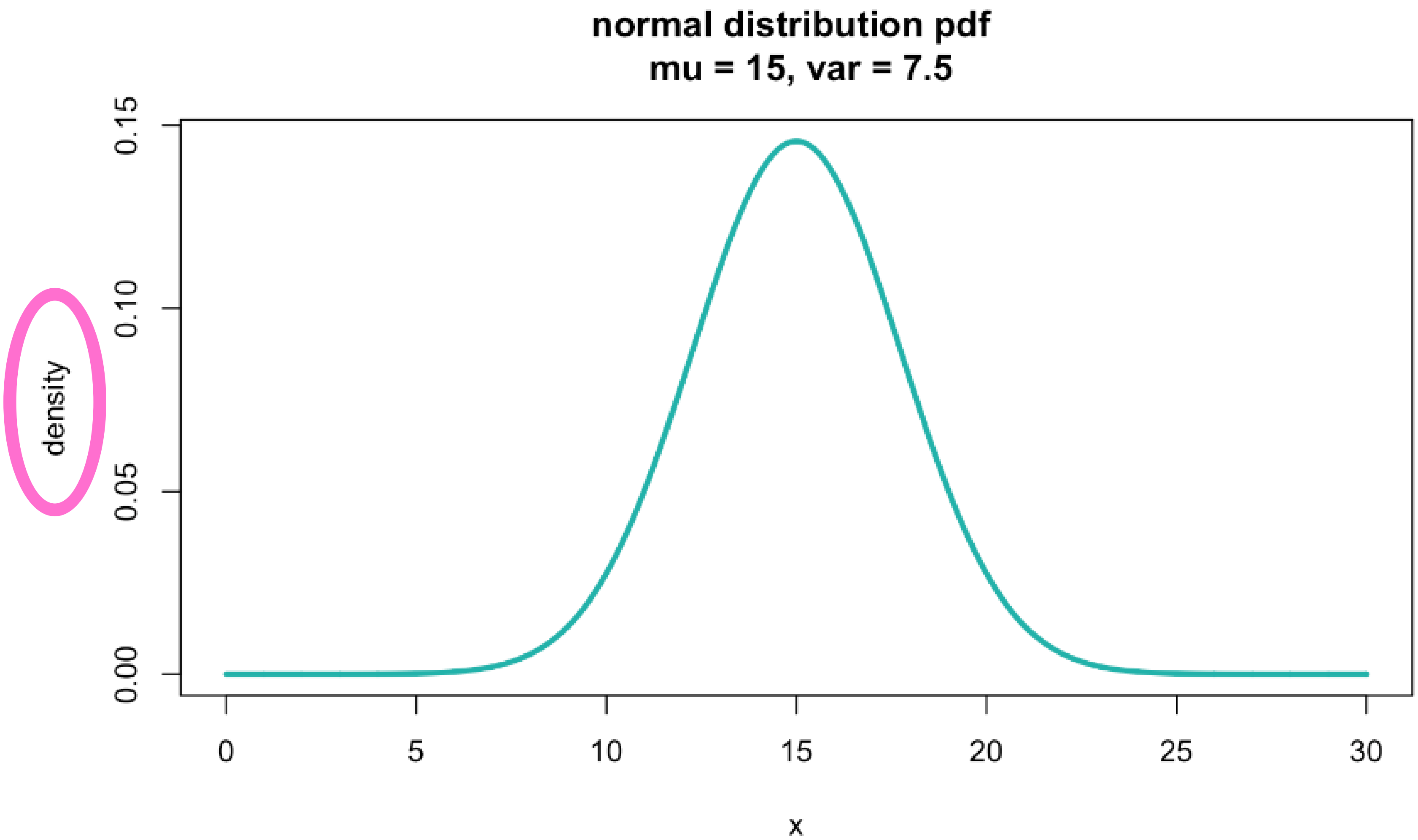
$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

One continuous rv distribution: Gaussian/normal

- Let's use the mean and sd from a coin flip example:

  - *Recall: X ~ Bin* (30, 0.5)
    - E(*X*) = np = 30 * 0.5 =15
    - Var(*X*) = np(1-p) = 30 * 0.5 (1-0.5) = 7.5
    - Sd(*X*) = $\sqrt{7.5} \cong 2.74$

$X \sim N(15, \sqrt{7.5})$



Questions we can ask the pdf

- "What is the height of the density curve at x = 10?" 0.028

`dnorm(10, 15, sqrt(7.5))`

$$F(x)$$

For all X, $F(x)$ is a

cumulative distribution function

Cumulative distribution function

$$1.\ F(x) \rightarrow 1\ as\ x \rightarrow \infty$$

$$2.\ F(x) \rightarrow 0\ as\ x \rightarrow -\infty$$

$$3.\ F(x)\ is\ monotonic;\ never\ decreasing$$

$$4.\ F(x)\ does\ not\ need\ to\ be\ smooth$$

$$but\ \ is\ continuous$$

$$F(x) = P(X \leq x)$$

Discrete

$$F(x) = \sum_{t \leq x} f(t)$$

Continuous

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

(since x is used as a variable in the upper limit of integration, we use some other variable, say "t", in the integrand)

cdf in practice

- $F(x)$ is the probability of values less than x
- Thus, $F(x)$ is the probability of an interval
- If $F(x)$ is the cdf for the age in months of fish in a lake, then $F(10)$ is the probability a random fish is 10 months or younger.
- Can F(10) be less than F(9)?

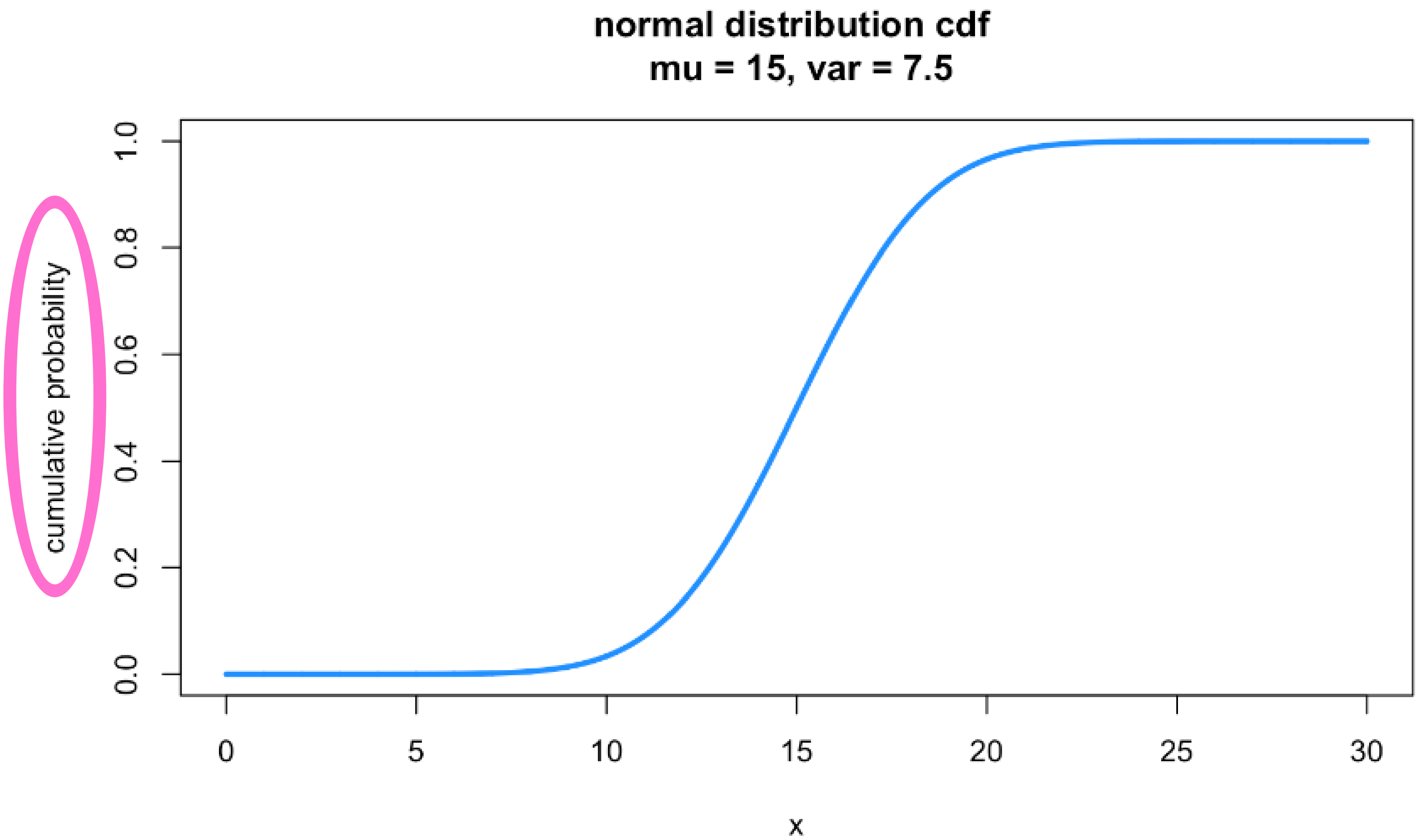
cdf for continuous rvs

- By the Fundamental Theorem of Calculus:

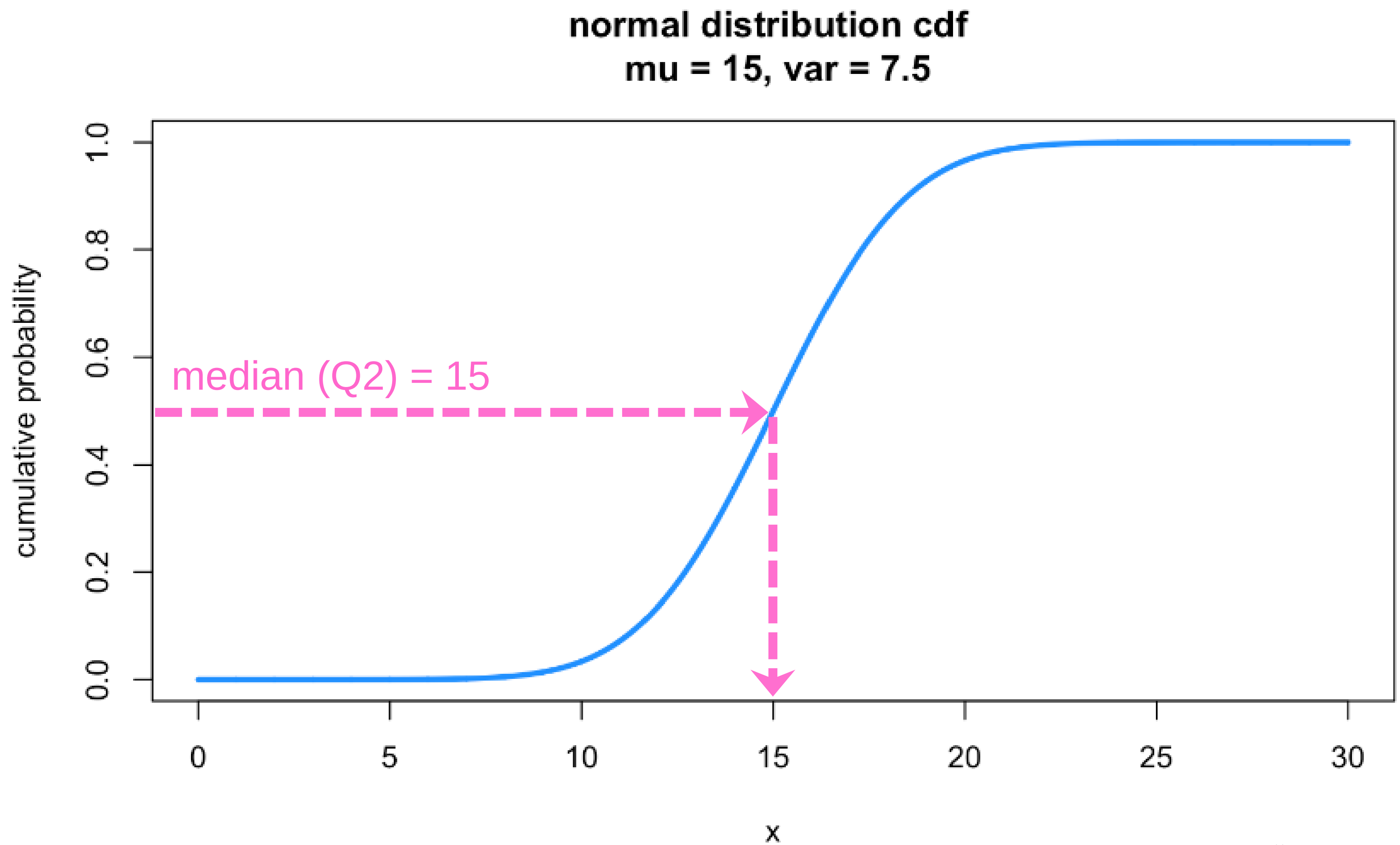
$$F(b) - F(a) = \int_a^b f(x)dx$$

The area under the curve from a to b of a function f is just the difference between the values of that function's antiderivative, F, at b and a.

$$X \sim N(15, \sqrt{7.5})$$

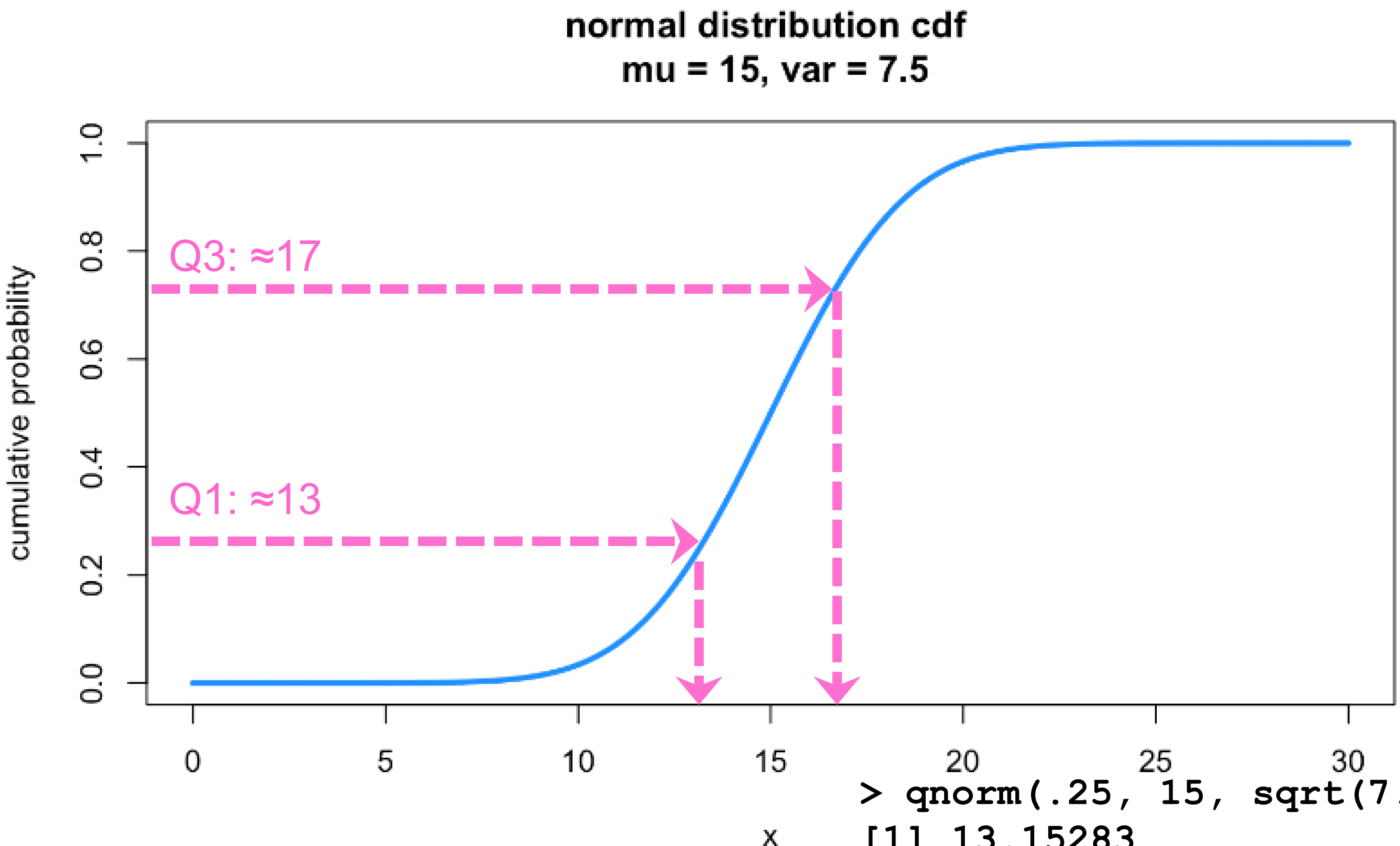


$X \sim N(15, \sqrt{7.5})$




```
> qnorm(.5, 15, sqrt(7.5)) #Q2 (median)
[1] 15
```

$$X \sim N(15, \sqrt{7.5})$$




```
> qnorm(.25, 15, sqrt(7.5)) #Q1
[1] 13.15283
> qnorm(.75, 15, sqrt(7.5)) #Q3
[1] 16.84717
```

Questions we can ask the cdf in R

- "What is the probability that x is exactly 7?" 0
- "What is the probability that x is between 18 to 24?" 0.136
- "What is the probability that x ends in 7?" 0
- "What is the probability that x is greater than 24?" 0.0005

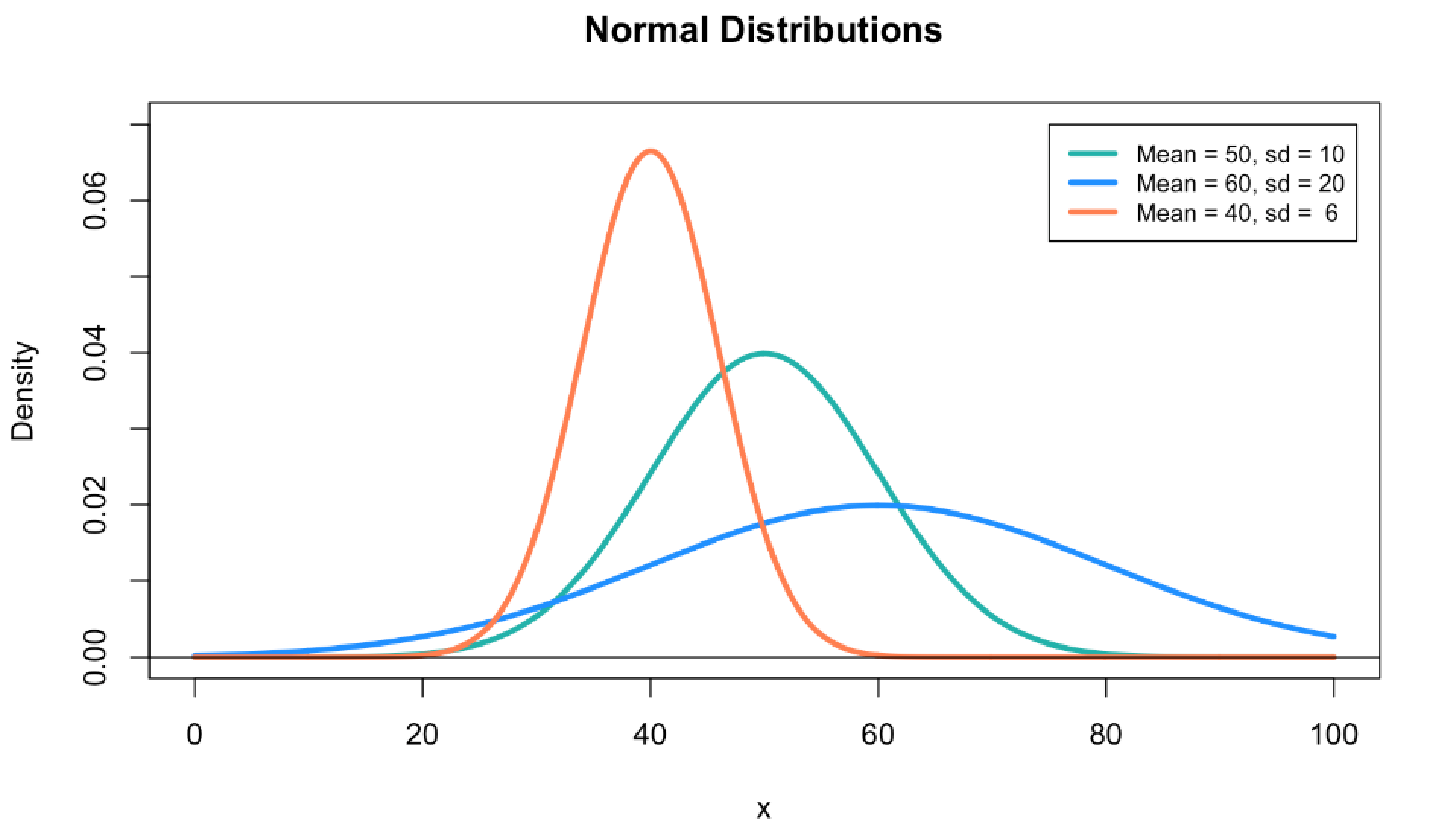
Questions we can ask the cdf in R

- "What is the probability that x is exactly 7?" 0
- "What is the probability that x is between 18 to 24?" 0.136

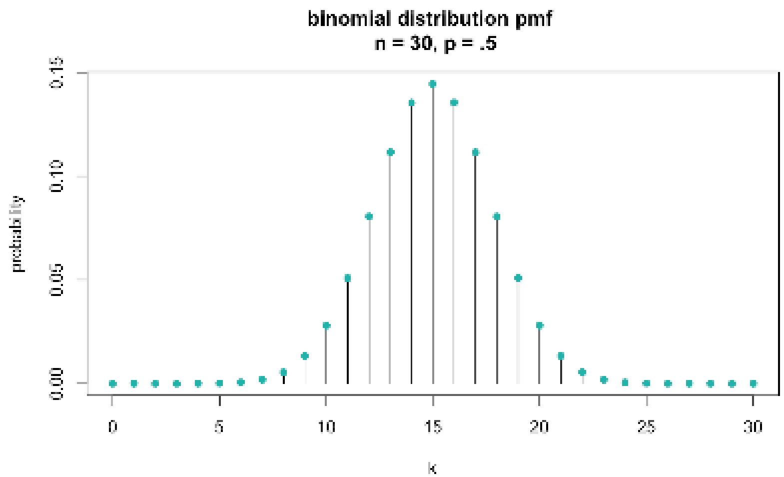
```
pnorm(24, 15, sqrt(7.5)) - pnorm(18, 15, sqrt(7.5))
```

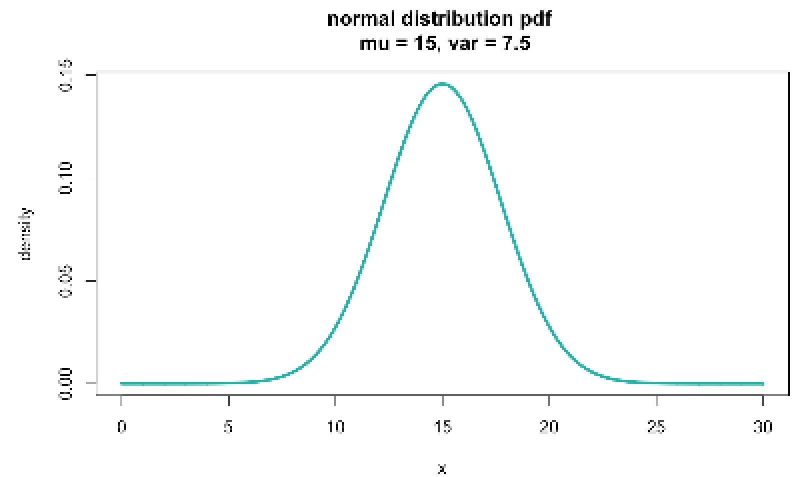
- "What is the probability that x ends in 7?" 0
- "What is the probability that x is greater than 24?" 0.0005

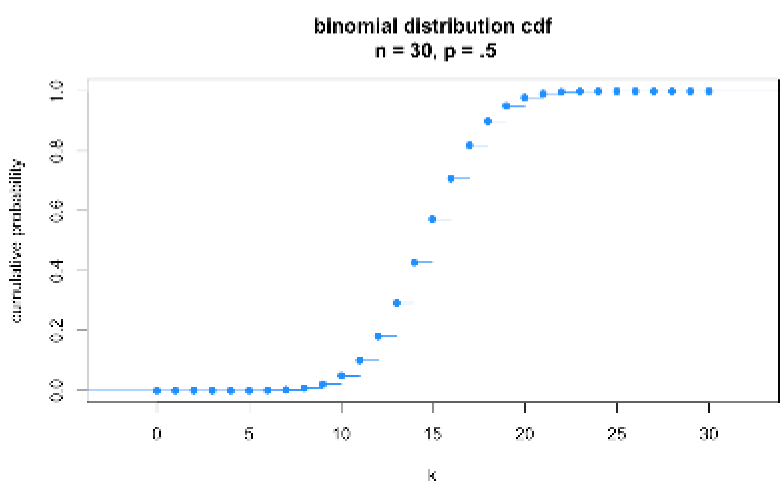
```
pnorm(24, 15, sqrt(7.5), lower.tail = FALSE)
```

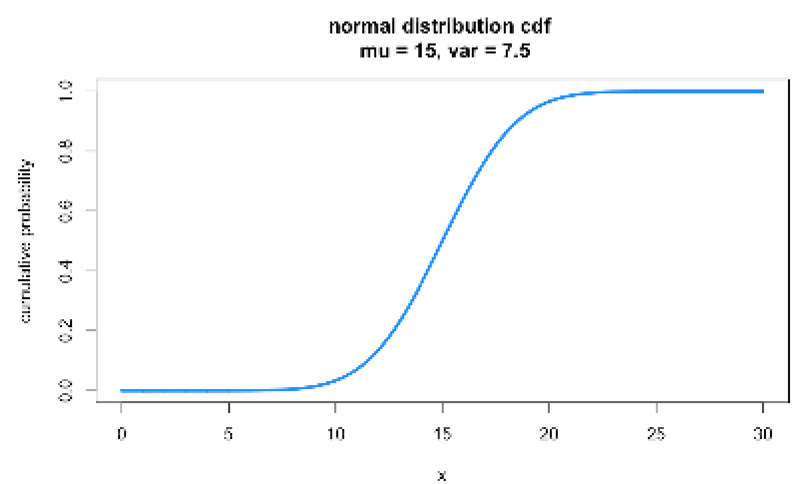


Normal Distributions
Mean = 50, sd = 10
Mean = 60, sd = 20
Mean = 40, sd = 6
Density
0.00
0.02
0.04
0.06
0
20
40
60
80
100
x


binomial distribution pmf
n = 30, p = .5
probability
0.00
0.05
0.10
0.15
0
5
10
15
20
25
30
k


normal distribution pdf
mu = 15, var = 7.5
density
0.00
0.05
0.10
0
5
10
15
20
25
30
x


binomial distribution cdf
n = 30, p = .5
cumulative probability
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
k


normal distribution cdf
mu = 15, var = 7.5
cumulative probability
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30
x

Transformations

- If $X \sim N(\mu, \sigma^2)$ and $Y = X + b$
  - Then $Y \sim N(\mu + b, \sigma^2)$

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX$
  - Then $Y \sim N(a\mu, a^2\sigma^2)$

- If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$
  - Then $Y \sim N(a\mu + b, a^2\sigma^2)$

Random variables

Discrete

- Finite or countably infinite sample space.
- Subset of integers-ish.
- Use sums.
- Has **pmf**.

Continuous

- Uncountably infinite sample space.
- Subset of the real number line.
- Use integrals.
- Has **pdf**.

continuous rvs

Outside support,
f(x)=0

$$f(x)$$

Integrate
from -∞ to x

Differentiate

$$F(x)$$

Outside support,
F(x)=0 or F(x)=1

X is a continuous rv

$$f(x) = \begin{cases} 4x & : 0 \le x \le 1/2 \\ -4x + 4 & : 1/2 \le x \le 1 \end{cases}$$

Integrate from -∞ to x

$$F(x) = \begin{cases} 2x^2 + A & : 0 \le x \le 1/2 \\ -2x^2 + 4x + C & : 1/2 \le x \le 1 \end{cases}$$

Wolfram Alpha:
integrate 4x = $2x^2$ + constant
integrate -4x + 4 = $-2x^2$ + 4x + constant

How to solve for A and C

x_{min}

$$F(x) = \begin{cases} 2x^2 + A & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x + C & : 1/2 \leq x \leq 1 \end{cases}$$

x_{max}

Where A and C are constants of integration. Now we use the properties of $F(x)$ to solve for those constants.

Specifically:

1. We know: $F(x_{min}) = 0$; so $F(0) = 0$
2. We also know: $F(x_{max}) = 1$; so $F(1) = 1$

Solve for A first

$$F(x) = \begin{cases} 2x^2 + A & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x + C & : 1/2 \leq x \leq 1 \end{cases}$$

$$F(x = 0) = 0 = 2(0^2) + A, \text{so } A = 0$$

Solve for C next

$$F(x) = \begin{cases} 2x^2 + A & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x + C & : 1/2 \leq x \leq 1 \end{cases}$$

$$F(x=1) = 1 = -2(1^2) + (4 \times 1) + C, \text{so } C = -1$$

Plug in: A = 0; C = -1

$$F(x) = \begin{cases} 2x^2 + A & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x + C & : 1/2 \leq x \leq 1 \end{cases}$$

$$F(x) = \begin{cases} 2x^2 & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x - 1 & : 1/2 \leq x \leq 1 \end{cases}$$

Alternative way to solve for C

- F(x) must be continuous, meaning that the graph of F(x) must touch when x = 1/2 (the two pieces of the piecewise graph must touch). So, to solve for C, we set x = 1/2 for each equation, plug in A = 0, and make them equal each other.

$$F(.5) = 2(.5^2) = -2(.5^2) + 4(.5) + C$$

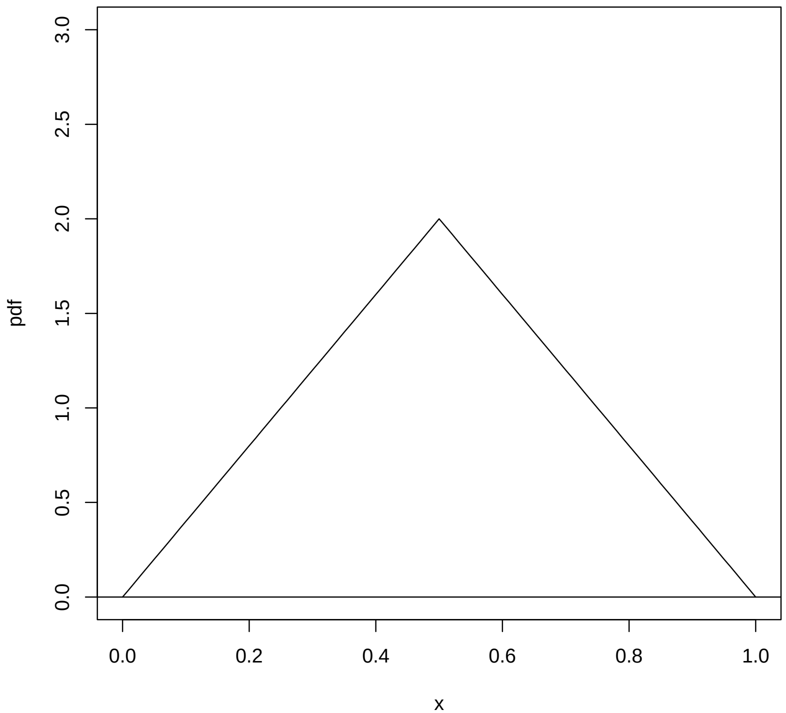
$$F(.5) = .5 = 1.5 + C$$

$$C = .5 - 1.5 = -1$$

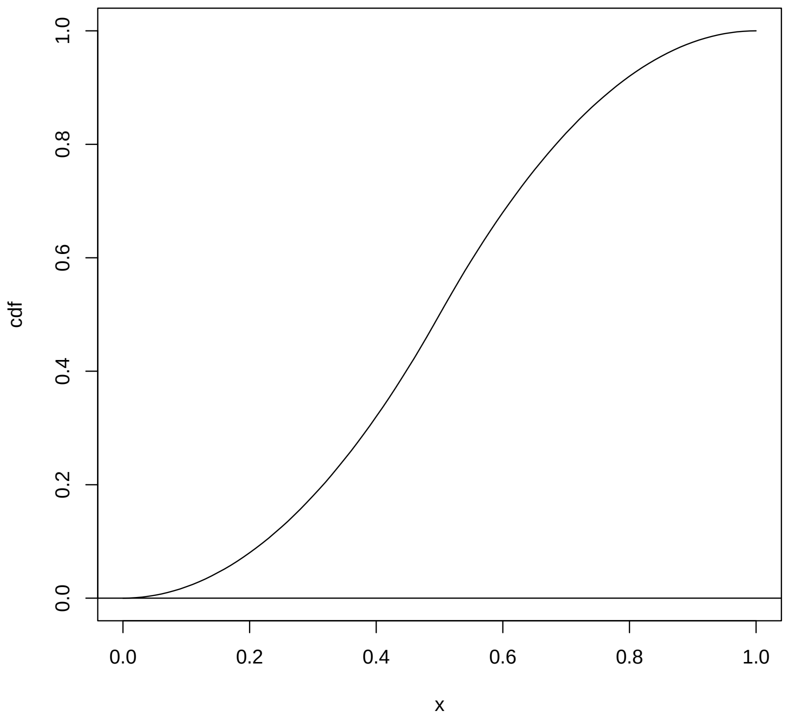
$$f(x) = \begin{cases} 4x & : 0 \leq x \leq 1/2 \\ -4x + 4 & : 1/2 \leq x \leq 1 \end{cases}$$

$$F(x) = \begin{cases} 2x^2 & : 0 \leq x \leq 1/2 \\ -2x^2 + 4x - 1 & : 1/2 \leq x \leq 1 \end{cases}$$

pdf



cdf



continuous rvs

Outside support, f(x)=0

$$\begin{tikzcd}
f(x) \arrow[d, bend right=60, "\text{Integrate from } -\infty \text{ to } x"'] \\
F(x) \arrow[u, bend right=60, "\text{Differentiate}"']
\end{tikzcd}$$

Outside support, F(x)=0 or F(x)=1

X is a continuous rv

$$f_X(x) = \begin{cases} 1/2 & -1 \leq x < 1, \\ 0 & \text{otherwise} \end{cases}$$

Differentiate

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

Wolfram Alpha:
differentiate 0 = 0
differentiate (x+1)/2 = ½
differentiate 1 = 0

Population median from pdf

- Halfway point: half the population has a lower value, and half has a higher value

- If $f(x)$ is a pdf, then the median of the distribution is the point M such that:

$$\int_{-\infty}^{M} f(x)dx = 0.5$$

Example: ladybug median life span

- Suppose a ladybug's life span (in months) has a pdf:

$$f(x) = \begin{cases} \frac{1}{72}x, & 0 \leq x \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

- What is the median life span for the ladybug population?

Example: ladybug median life span

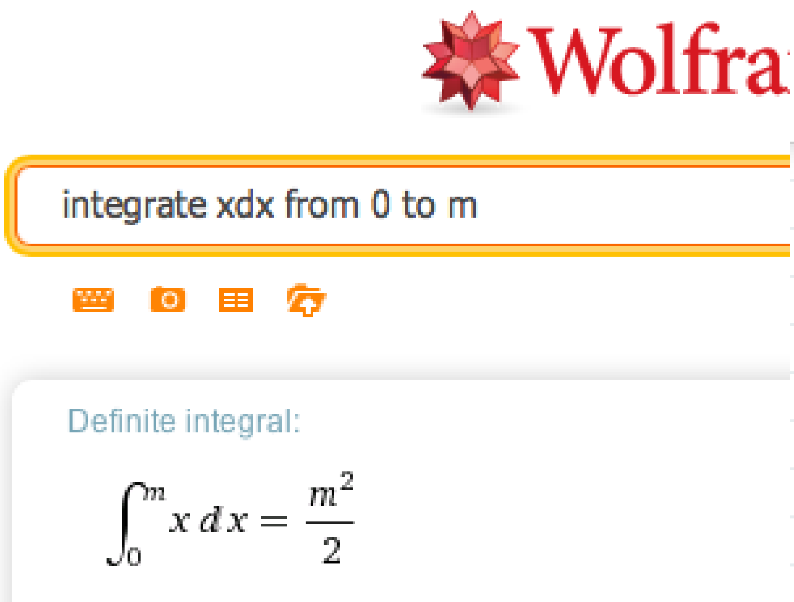
$$\frac{1}{2} = \int_0^M \frac{1}{72} x dx = \frac{1}{144} x^2 |_0^M = \frac{1}{144} M^2$$

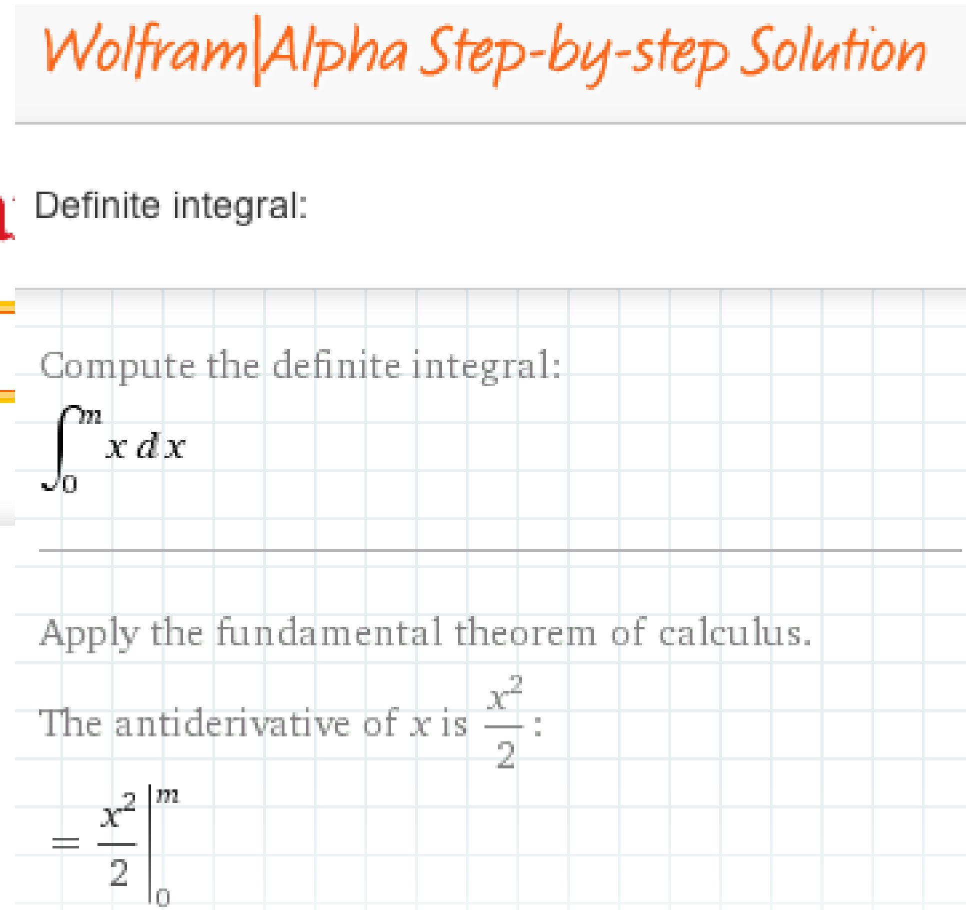
$$\frac{1}{2} = \frac{1}{144} M^2$$

$$72 = M^2$$

$$M = \sqrt{72} = 8.49$$

Aside: wolfram alpha





$$f(8.49) = \int_0^{8.49} \frac{1}{72} x dx = 0.5$$

- If we had the cdf, we could check this result:

$$F(M) = F(8.49) = 0.5$$

- How do we get there from the pdf? Integrate!

I know what you are thinking- I do so wish we could do more integration!

$$f(x) = \begin{cases} \frac{1}{72}x, & 0 \le x \le 12 \\ 0, & \text{otherwise} \end{cases}$$

pdf → cdf:
Integrate!

$$F(x) = \begin{cases} \frac{1}{144}x^2 + \text{constant}, & 0 \le x \le 12 \\ 0, & \text{otherwise} \end{cases}$$

constant = 0 *(do this one on your own)*

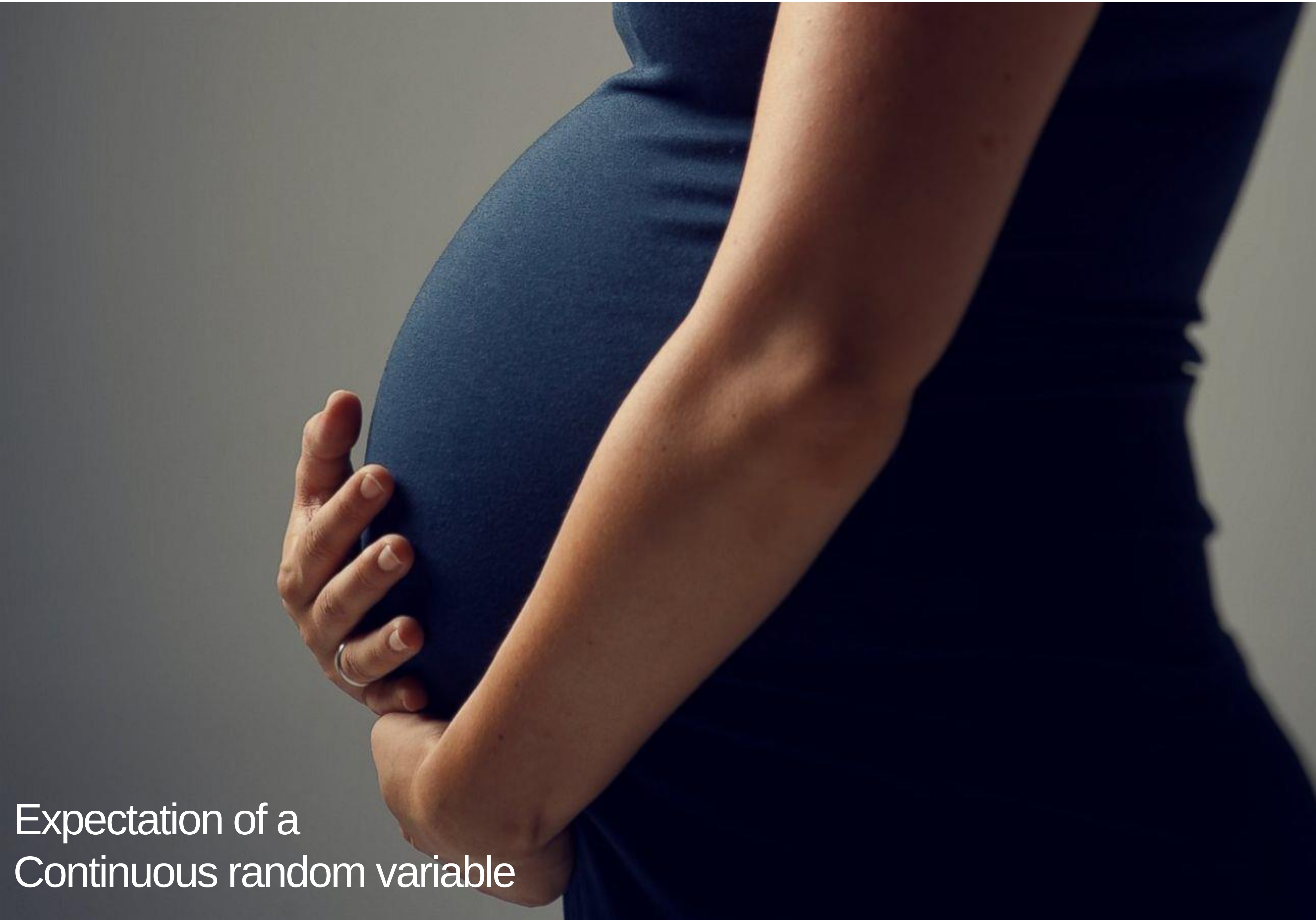
$$F(x) = \begin{cases} \frac{1}{144}x^2, & 0 \le x \le 12 \\ 0, & \text{otherwise} \end{cases}$$

```
> (1/144)*(8.49^2)
[1] 0.5005563
```

$$F(M) = F(8.49) = 0.5$$

Using the cdf

- Now we can answer so many more questions without having to integrate each time…
- Q1? 6
- Q3? 10.4
- IQR? 4.4



Expectation of a Continuous random variable

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx$$

Example: ladybug mean life span

- Suppose a ladybug's life span has a pdf:

$$f(x) = \begin{cases} \frac{1}{72}x, & 0 \leq x \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

- What is the mean life span for the ladybug population?

$$\int_{-\infty}^{+\infty} x f(x) dx$$

Example: ladybug mean life span

$$\int_{-\infty}^{+\infty} xf(x)dx = \int_0^{12} x(\frac{1}{72}x)dx$$

$$= \int_0^{12} \frac{1}{72}x^2 dx$$

$$= \frac{1}{216}x^3|_0^{12} = 8$$

Example: ladybug life spans

- What is the median life span? 8.49 months
- What is the mean life span? 8 months
- The mean is < median, so more than 50% of ladybugs live longer than 8 months
- What is the variance of ladybug life spans?

Variance of ladybug life spans

I'll leave this for you to solve…

12

8

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx$$

0

$$f(x) = \begin{cases} \frac{1}{72}x, & 0 \le x \le 12 \\ 0, & \text{otherwise} \end{cases}$$

Mini-review: probability functions

Discrete rvs:
probability mass function (pmf)

- $f(x)$ is defined by the distribution!
- Famous ones:
  - Bernoulli
  - Binomial
  - Negative binomial
  - Geometric
  - Hypogeometric
  - Poisson

Continuous rvs:
probability density function (pdf)

- $f(x)$ is defined by the distribution!
- Famous ones:
  - Normal/Gaussian
  - Chi-squared
  - F
  - Student's t
  - Gamma
  - Beta

Mini-review: conditions for probability functions

Probability	Discrete rv: *f(x)* is a pmf if…	Continuous rv: *f(x)* is a pdf if…
$P(A) \geq 0$ for all $\mathrm{A} \in \Omega$	$f(x) > 0$ for all $x \in \Omega$	$f(x) > 0$ for all $\mathrm{x} \in \mathbb{R}$
$P(\Omega) = 1$	$\sum_{x_i \in \Omega} f(x_i) = 1$	$\int_{\mathbb{R}} f(x) = 1$

Mini-review: cumulative density functions

Discrete rvs

$$F(x) = \sum_{t \leq x} f(t)$$

Continuous rvs

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

where *f(t)* is just the pmf/pdf

Mini-review: Expectation of an rv

Discrete rvs

$$E(X) = \sum_{\text{all } x} x f(x)$$

Continuous rvs

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Mini-review: Variance of an rv

Discrete rvs

$$Var(X) = \sum_{\text{all } x} (x - \mu_X)^2 f(x)$$

Continuous rvs

$$Var(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 f(x) dx$$

What you need to know about distributions

- That lots of distributions exist (not just the normal!), and what types of rvs can typically be represented by them
- What the pmf/pdf represents in terms of probability
- What the cdf represents in terms of probability
- How to calculate the expectation value and variance of the distribution, given the pmf/pdf
- How to determine the pdf from the cdf for any function
  - Requires differentiation
- How to determine the cdf from the pdf for any function
  - Requires integration