TEST PREP

1) LAURENT SERIES EXPANSION

$f(z) = \frac{1}{z}$, FIND LAURENT SERIES EXPANSION IN

(i) $|z| > 3$

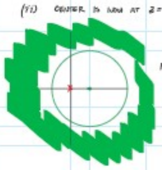
(ii) $|z-1| > 2$

(iii) $|z-2| < 1$

$f(z)$ HAS SIMPLE POLE AT $z = 0$

(i) $\frac{1}{z}$ B.C. CENTER IS AT 0.

(ii) CENTER IS NOW AT $z = 1$



NOW THE SINGULARITY IS OUTSIDE THE REGION.

USE A TRANSFORMATION TO CONVERT IT TO SOMETHING WE KNOW.

$$f(z) = \frac{1}{z} = \frac{1}{(z-1)+1} = \frac{1}{z-1} \cdot \frac{1}{1+\frac{1}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}\left(-\frac{1}{z-1}\right)^n$$

FACTOR OUT.

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{z-1}\right)^{n+1}$$

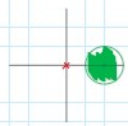
OR

$$\sum_{n=0}^{\infty}(-1)^n (z-1)^{-(n+1)}$$

OR

$$\sum_{n=-1}^{-\infty}(-1)^n (z-1)^{n}$$

(iii) HAS CENTER AT $z = 2$



$$f(z) = \frac{1}{z} = \frac{1}{(z-2)+2} = \frac{1}{2}\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n, \quad \left|\frac{z-2}{2}\right| < 1 \Rightarrow |z-2| < 2$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n$$

LOOKS LIKE $\frac{1}{1+w} = 1 - t + t^2 - t^3 \ldots = \sum_{n=0}^{\infty}(-t)^n$, $|t| < 1$

$$\boxed{f(z) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n}$$

PRESERVE MULTIPLE VALUED-NESS

$$\boxed{(1+t)^{\alpha} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}t^n (1)^{\alpha}, \quad |t| < 1}$$

$$\alpha(\alpha-1)\ldots(\alpha+1-n) = \prod_{k=0}^{n-1}(\alpha - k)$$

$$\boxed{\frac{1}{1+t} = \sum_{n=0}^{\infty}(-t)^n, \quad |t| < 1}$$

$$\boxed{\log(1+t) = \log 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}t^n, \quad |t| < 1}$$

PRESERVE MULTIPLE VALUED-NESS.

* FOR POWER INTEGRAT

* FOR LOG

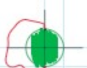
$$\boxed{\frac{1}{1-t} = 1 + t + t^2 + \ldots = \sum_{n=0}^{\infty}t^n, \quad |t| < 1}$$

SO FOR EXAMPLE: $f(z) = \frac{1}{1-z} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = -z^{-2} - z^{-2} \times z^{-3} \ldots = \sum_{n=0}^{\infty} z^{-n}$, $|z| > 1$

* HOMEWORK #5, PROBLEM #46

$f(z) = (z^2+1)^{1/2}$, B.P. @ $z = \pm i$

(i) $|z| < 1$



$$f(z) = (z^2+1)^{-1/2}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n)}(z^2)^n (1)^{-\frac{1}{2}}$$

(ii) $|z| > 1$ CREATES A PROBLEM BECAUSE WE ENCOUNTER B.P.

$$f(z) = (z^2+1)^{-1/2} = \left(z^2\left(1+\frac{1}{z^2}\right)\right)^{-1/2} = z^{-1}\left(1+\frac{1}{z^2}\right)^{-1/2} = z^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n)}\left(\frac{1}{z^2}\right)^n (1)^{-1/2}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-n)}z^{-2n-1}(1)^{-1/2}$$

* $f(z) = \log\left[\frac{z-3}{z+3}\right]$, B.P. @ $z = \pm 3$

$= \log(z-3) - \log(z+3)$

(i) $|z| < 3$

$= \log\left(-3\left(1-\frac{z}{3}\right)\right) - \log\left(3\left(1+\frac{z}{3}\right)\right)$

$= \log(-3) + \log\left(1-\frac{z}{3}\right) - \log(3) - \log\left(1+\frac{z}{3}\right)$

$= \log(-1) + \log 1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{z}{3}\right)^n$

(ii) $|z| > 3$

$= \log\left(z\left(1-\frac{3}{z}\right)\right) - \log\left(z\left(1+\frac{3}{z}\right)\right)$

$= \cancel{\log z} + \log\left(1-\frac{3}{z}\right) - \cancel{\log z} - \log\left(1+\frac{3}{z}\right)$

$= \log\left(1-\frac{3}{z}\right) - \log\left(1+\frac{3}{z}\right)$

CONSIDER $\int_0^{\infty}\frac{(\ln x)^2}{1+x^2}dx$

let $f(z) = \frac{(\ln z)^2}{1+z^2}$

$\oint f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), i)$

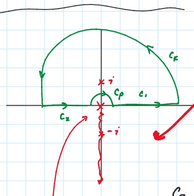
$$(\ln R)^n < R^\alpha \quad , \quad \begin{matrix}\alpha > 0 \\ n > 0\end{matrix}$$

$$\varepsilon^\alpha (\ln \varepsilon)^n = 0 \quad , \quad \begin{matrix}\alpha > 0 \\ n > 0\end{matrix}$$

FUNCTION WE TRY TO
E ALONG B.C.
WE TRY TO AVOID B.C.

$$\big||a|-|b|\big| \leq |a+b| \leq |a| + |b|$$

TRIANGLE INEQUALITY



NOT SIMPLE POLE, SO WE CAN'T USE INDENTED PATH THM.

$$C_R \Rightarrow \left| \int_0^\pi \frac{(\ln Re^{i\theta})^2}{1+R^2e^{i2\theta}} iRe^{i\theta} d\theta \right| \leq \int_0^\pi \frac{|\ln R + i\theta|^2}{R^2-1} R\, d\theta \leq \left(\frac{(\ln(R))^2 + \pi^2}{R^2-1} \right) R\pi \to 0 \text{ AS } R \to \infty$$

B.C. $\ln(R)$ IS WEAKER THAN ANY POLYNOMIAL.

TRIANGLE INEQ

$$C_\rho \Rightarrow \left| \int_0^\pi \frac{(\ln \varepsilon e^{i\theta})^2}{1+\varepsilon^2 e^{i2\theta}} i\varepsilon e^{i\theta} d\theta \right| \leq \int_0^\pi \frac{(\ln^2\varepsilon + \theta^2)\varepsilon}{1-\varepsilon^2} d\theta \leq \frac{\varepsilon}{1-\varepsilon^2} (\ln^2\varepsilon + \pi^2)\pi \to 0 \text{ AS } \varepsilon \to 0$$

B.C. $\varepsilon \ln^2 \varepsilon \to 0$

$\varepsilon^\alpha \ln^n \varepsilon = 0$ FOR $\alpha > 0$, $n > 0$

ε VERY SMALL

$C_2 \Rightarrow \quad z = x \ , \ x: -R \to -\varepsilon$
$\qquad\qquad\qquad\qquad -\infty \to 0$

FACTOR OUT THIS

$$\int_{-\infty}^0 \frac{\ln^2 x}{1+x^2} dx = \int_\infty^0 - \frac{\ln^2(-t)}{1+(-t)^2} dt = \int_0^\infty \frac{\ln^2(-t)}{1+t^2} dt = \int_0^\infty \frac{(\ln t + i\pi)^2}{1+t^2} dt = \int_0^\infty \frac{\ln^2 t}{1+t^2} dt + \int_0^\infty \frac{2\pi i \ln t}{1+t^2} dt + \underbrace{\int_0^\infty -\pi^2 \frac{1}{1+t^2} dt}_{-\frac{\pi^3}{2}}$$

$t = -x$

$\ln(-t) = \ln(te^{i\pi}) = \ln t + \ln e^{i\pi} \Rightarrow i\pi$

$\text{ARCTAN}(t)\Big|_0^\infty = \frac{\pi}{2}$

$$= 2\int_0^\infty \frac{(\ln x)^2}{1+x^2} dx + 2\pi i \int_0^\infty \frac{\ln x}{1+x^2} dx - \frac{\pi^3}{2}$$

$$2\pi i \cdot \text{Res}(f(z), i) = 2\pi i \left(\frac{(\ln i)^2}{2i} \right) = \cancel{2}\pi\cancel{i} \left(\frac{(i\frac{\pi}{2})^2}{\cancel{2i}} \right) = -\frac{\pi^3}{4}$$

$\frac{f}{g'}\Big|_{\text{POLE}}$

$$2\pi i \cdot \text{Res}(f(z), i) = 2\pi i \left(\frac{(\ln i)^2}{2i} \right) = \cancel{2}\pi \cancel{i} \left(\frac{\left(i \frac{\pi}{2}\right)^2}{\cancel{2i}} \right) = -\frac{\pi^3}{4}$$

$$\left.\frac{f}{g'}\right|_{\text{POLE}}$$

$$-\frac{\pi^3}{4} = 2\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx + \cancel{2\pi i \int_0^\infty \frac{\ln x}{1+x^2}\,dx} - \frac{\pi^3}{2}$$

0 BECAUSE THIS IS IMAGINARY

$$\therefore \quad \boxed{\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}}$$