tricks

Take the derivative Bound to zero Do contours around the branch cut (keyhole) Exponentials are faster than polynomials

Common functions

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\frac{\mathrm{d}}{\mathrm{d}x} asinx = \frac{1}{\sqrt{1-x^2}}$
- $\cos \frac{e^{ix}+e^{-ix}}{2}$, $\frac{\mathrm{d}}{\mathrm{d}x} acosx = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x} atanx = \frac{1}{1+x^2}$
- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\sin z = \sin x \cosh y + i \cos x \sinh y$, $\cos z = \cos x \cosh y - i \sin x \sinh y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $\log z$: B.P. at 0,∞. $D[\log x] = 1/x$.
- $\lim_{\epsilon \to 0} \epsilon \log \epsilon = 0$
- L'hopital's rule for indefinite limits $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$
- Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \stackrel{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Sequences

Convergent sequence: A sequence $\{z_n\}$ is said to have a limit z_0 or converge to z_0 which we write as

$$\lim_{n \to \infty} z_n = z_0$$

if for every $\epsilon > 0, \exists\ M \in \mathbb{Z}$, such that:

$$|z_n - z_0| < \epsilon,\ \forall n > M$$

Cauchy Sequence: a sequence $\{z_m\}$ of complex numbers is a Cauchy sequence if for every $\epsilon > 0$, $\exists N \in \mathbb{Z}$ such that $|z_m - z_M| < \epsilon$, $\forall m, M > N$.
Cauchy Criterion for Series Convergence Let $s_m = \sum_{k=1}^{m} a_k$. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}$ for $m, M > N$:

$$|s_M - s_m| = \left| \sum_{k=m+1}^{M} a_k \right| < \epsilon$$

Alternatively, letting $M = m + p$:

$$|s_{m+p} - s_m| = \left| \sum_{k=m+1}^{m+p} a_k \right| < \epsilon, \forall p = 1, 2, 3, ..$$

This is the most general test for convergence of a series.
Radius of Convergence: The power series $\sum_{m=0}^{\infty} a_m z^m$ has a radius of convergence $R = \frac{1}{A}$, where $A = \limsup_{m \to \infty} |a_m|^{1/m}$. If $A = \infty$, $R = 0$. Likewise, if $A = 0$, $R = \infty$.

Theorem 0.1 (Weirstrass M-test). *Let M_m be a sequence of real numbers. Suppose that $|f_m(z)| < M_m, \forall z \in E, m \in \mathbb{Z}^+$. If $\sum_{m=1}^{\infty} M_m$ converges, then $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly and absolutely on E.*

Harmonic function: Satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **Analyticity**: Differentiable everywhere in neighborhood of point. **Roots of Unity**: The kth root of unity is ω s.t. $\sum_{n=1}^{k} w^n = 0$ **Entire Function** holomorphic/analytic all over

Branch Points and Branch Cuts

NOTE: only have to say what it is.

- z^p, p non integer has BP at 0, ∞.
- $\log z$ BP at 0, ∞

Singularities

Isolated singularities

- **Pole**: f can be written as $g/(z - z_0)$
- **Removable**: z_0 is analytic in neighborhood and limit exists. Chuanpeng says this is not a singularity.
- **Essential** Neither, yet isolated. Ex. $f(z) = \frac{\sin z}{z}$.

Non-isolated singularities: each deleted neighborhood has a singularity. e.g. branch points, $2\frac{1}{\sin(1/z)}$

Taylor/Maclaurin series expansion

Maclaurin is just taylor about $z = 0$.

Laurent Series Expansions

Two cases:

1. f analytic in circle around expansion point. Use Taylor Series Expansion.
2. f analytic in annulus. Then find Laurent series through something like change of variables to get series.

Gauss Mean Value: Suppose $f(z)$ is analytic in the closed disk $|z - z_0| \le r$. Then:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta} d\theta)$$

Cauchy Integral Formula If f analytic, C simply connected and closed, $z_0 \in C$, then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$. Also $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$
Cauchy Riemmann Equations Necessary but NOT sufficient for differentiability at a point. Applied to neighborhood is sufficient for analyticity. Or if u, v have continuous partials in domain D. Given $f(z) = u(x, y) + iv(x, y)$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

OR

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

OR Let $f = u + iv$ be differentiable with complex partials at $z = re^{i\theta}$. Then:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Indented Path lemma Let f have a simple pole at a with a residue Res$[f; a]$. Then given an upper half clockwise semi-circular contour around the pole C_ϵ, the resulting contour is:

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = -\operatorname{Res}[f; a]\pi i \tag{1}$$

Jordan's lemma: If C_R is the positive imaginary semicircular contour, the only singularities in $g(z)$ are poles, $a > 0$, and $g(z) \to 0$ as $R \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} e^{iaz} g(z) dz = 0$$

If you'd like to apply Jordan's lemma for $a < 0$, try taking the semicircular contour that goes through the negative semicircular contour. Likewise, we can also say very carefully that Jordan's lemma.

Residue theorem: Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, .., z_n$ inside C. Let the residues at these points be $\alpha_1, \alpha_2, .., \alpha_n$ respectively. Then: $\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} \alpha_i$
Sector integration: take an arc-slice of a circle around a singularity (or two) and then go use residue theorem.
Maclaurin expansion of $f(z) = \log(z+1)$ valid for $|z| < 1$. We know that $f'(z) = \frac{1}{1+z}$. This is just the same as a geometric series: $f'(z) = \sum_{n=0}^{\infty} (-1)^n z^n$
Hence:

$$
\begin{aligned}
f(z) &= \log(1+z) && (2)\\
&= \int_0^z f'(\zeta)d\zeta + f(0) && (3)\\
&= \sum_{n=0}^{\infty} \int_0^z (-1)^n \zeta^n d\zeta + 0 && (4)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{n+1}}{n+1} && (5)
\end{aligned}
$$

Special Functions

Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$
The infinite form:
$\lim_{n\to\infty} \Gamma(z;n) := \int_0^n (1 - \frac{t}{n})^n t^{z-1} dt = \Gamma(z)$ = Integer: $n \in \mathbb{Z}^+$, $\Gamma(n) = (n+1)!$ Properties of Gamma function

- $\Gamma(1) = 1$
- $\Gamma(z+1) = z\Gamma(z)$
- $\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}$
- $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad z \notin \mathbb{Z}$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$
- $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos \pi z}$
- $\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
- $(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} z^n$ (use the factorial identity)
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Bessel Equation Definitions 1st kind:
$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n}$
Legendre Polynomials: $P_n = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n}[(z^2-1)^n]$
$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \frac{1}{2}(3z^2 - 1)$
Generating Function:
$\phi(\zeta, z) = P_0(z) + \zeta P_1(z) + \zeta^2 P_2(z) + ...$

Fourier Transform

Fourier Transform:$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\tau)d\tau$ Inverse Fourier Transform: $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda\tau} d\lambda$

Properties of the Fourier Transform

For a function $f(t)$ with Fourier Transform $F(\lambda)$

- $I.F.T.[f(t)] = F(-\lambda)$
- For constants a, b: $F.T.[af(t) + bg(t)] = aF.T.[f(t)] + bF.T.[g(t)]$
- $F.T.[f(t)g(t)] = F.T.[f(t)] * F.T.[g(t)]$ where $*$ is the convolution operator.
- $F.T.[f(t) * g(t)] = F.T.[f(t)]F.T.[g(t)]$
- $F.T.[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)] = (-i\lambda)^n F(\lambda)$
- $F.T.[\delta(t)] = \frac{1}{\sqrt{2\pi}}$
- $F.T.[1] = \frac{1}{\sqrt{2\pi}}\delta(-t) = \frac{1}{\sqrt{2\pi}}\delta(t)$
- $F.T.[f(t-t_0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f(t-t_0)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda(t'+t_0)} f(t')dt' = e^{i\lambda t_0} F(\lambda)$

linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$
time shift	$f(t-T)$	$e^{-j\omega T}F(\omega)$
differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
	$\frac{d^k f(t)}{dt^k}$	$(j\omega)^k F(\omega)$
integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
multiplication with t	$t^k f(t)$	$j^k \frac{d^k F(\omega)}{d\omega^k}$
convolution	$\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau$	$F(\omega)G(\omega)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\widetilde{\omega})G(\omega-\widetilde{\omega})\, d\widetilde{\omega}$

Fourier Coefficients:
Green's function: Solution to $LG = \delta$ for linear differential operator L.

Homogeneous ODE	Green's Function $g(x,u)$
$y' - ay$	$e^{a(x-u)}$
y''	$x - u$
$y'' - 2ay' + a^2$	$(x-u)e^{a(x-u)}$
$y'' - (a+b)y' + aby$	$\frac{e^{a(x-u)} - e^{b(x-u)}}{a-b}$
$y'' + b^2 y$	$\frac{1}{b}\sin[b(x-u)]$
$y'' - b^2 y$	$\frac{1}{b}\sinh[b(x-u)]$
$y'' - 2ay' + (a^2+b^2)y$	$\frac{1}{b}e^{a(x-u)}\sin[b(x-u)]$
$y'' - 2ay' + (a^2-b^2)y$	$\frac{1}{b}e^{a(x-u)}\sinh[b(x-u)]$
$x^2y'' + xy' - b^2y$	$\frac{u}{2b}\left(\frac{x^b}{u^b} - \frac{u^b}{x^b}\right)$
$x^2y'' - (b+a-1)xy' + aby$	$\frac{u}{b-a}\left(\frac{x^b}{u^b} - \frac{x^a}{u^a}\right)$

Laplace Transform

$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ Inverse L.T.:
$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$
TODO: ???