CLASS 24 - ODEs

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ODEs (CH 5 CKP)

⇒ CONSIDER N^{th} ORDER ODE

$$w^{(n)} = P_{n-1}(z) w^{n-1} + \dots P_1(z) w + f(z)$$

DEF: HOMOGENEOUS EQ: $f(z) = 0$

NON-HOMOGENEOUS EQ: $f(z) \neq 0$

NOTE: GENERAL SOLN: HOM + NON-HOM

* HOMOGENEOUS EQ

- ASSUME THAT $P_j(z)$ ARE ANALYTIC FUNCTIONS WITHIN CIRCLE $|z - z_0| < R$

THEN A SOLUTION EXISTS WHICH IS ANALYTIC INSIDE CIRCLE
MOREOVER, IF $w(0)$, $w'(0) \dots w^{n-1}(0)$ EXIST THEN THE SOLUTION IS UNIQUE.

PROOF: GIVEN ANALYTIC, TAKE TAYLOR SERIES EXPANSION FOR W.

$$w(z) = a_0 + a_1 z + a_2 z^2 \dots$$
$$P_j(z) = b_0^j + b_1^j z + b_2^j z^2 \dots$$

SO $P_j(z) w(z) = (a_2 \dots \quad) z^0 + (\quad\quad) z^1 + (\quad\quad) z^2 \dots = 0$

∴ ALL THESE TERMS MUST BE $= 0$

∴ SOLVE FOR ALL $a_1 \dots a_n$, $b_1^j \dots b_n^j$

THEN SHOW THIS SERIES CONVERGES INSIDE THE CIRCLE.

NOTE: IF $w(0) \dots w^{n-1}(0) = 0$, THEN CLEARLY $w(z) = 0$

NOTE: IF WE DEFINE $w_0, w_1, \dots w_{n-1}$ BY MEANS OF INITIAL CONDITIONS

$$w_0(0) = 1 \dots \quad w_0'(0) = 0, \dots \quad w_0^{n-1}(0) = 0$$
$$w_1(0) = 0, \quad w_1'(0) = 1, \quad \dots \quad w_1^{n-1}(0) = 0$$
$$\vdots$$
$$w_{n-1}(0) = 0, \quad w_{n-1}'(0) = 0 \quad \dots \quad w_{n-1}^{n-1}(0) = 1$$

(IDENTITY)

THEN ANY SOLN MAY BE WRITTEN IN THE FORM:

$$w(z) = a_0 w_0(z) + a_1 w_1(z) + \dots a_{n-1} w_{n-1}(z)$$

AND ANY SET OF SOLUTIONS $w_0, w_1, \dots w_{n-1}$ IS CALLED A FUNDAMENTAL SET

* WRONSKIAN (FOR $n=2$)

→ LET $w_1(z)$ AND $w_2(z)$ BE TWO SOLUTIONS OF ODE (LINEARLY INDEPENDENT OR NOT)

→ WRONSKIAN $\boxed{\mathbb{W}(z) = \det\begin{pmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{pmatrix}} = w_1(z)\,w_2'(z) - w_2(z)\,w_1'(z)$

THEN

$$\mathbb{W}'(z) = w_1 w_2'' - w_1'' w_2 = P_1(w_1 w_2' - w_1' w_2) = P_1(z)\,\mathbb{W}(z)$$

(MIDDLE TERMS CANCEL)

S.T.

$$\mathbb{W}(z) = \overset{\text{CONSTANT}}{C'} \exp\left[\int P_1(z)\,dz\right]$$ IS A SOLUTION.

NOTE: SINCE EXP CANNOT VANISH FOR $z < R$, $\mathbb{W}(z)$ CANNOT VANISH FOR ANY VALUE OF z IN $|z| < R$ UNLESS IT VANISHES FOR ALL VALUES OF z.

NOTE: $\mathbb{W} = 0$ IFF w_1 AND w_2 ARE LINEARLY DEPENDENT SOLUTIONS!

NOTE: IF A SOLUTION w_1 OF ODE IS KNOWN, THEN A LINEARLY INDEPENDENT SOLN w_2 MAY BE FOUND FROM EQ FOR WRONSKIAN

$$w_2' w_1 - w_1' w_2 = C' \exp\left[\int P_1(z)\,dz\right]$$ {FIRST ORDER EQ IN w_2}

* NON - HOMOGENEOUS:

$$w'' - P_1(z)\,w' - P_0(z)\,w = f(z) \quad (0)$$

→ ASSUME THAT TWO LINEARLY INDEPENDENT SOLN w_1 / w_2 OF HOMOGENEOUS EQ ARE KNOWN.

→ NOW WRITE $w = a_1(z) w_1 + a_2(z) w_2$

FUNCTIONS OF z, NOT NECESSARILY HOMO SOLN TO OUR ODE!

WHERE $a_1(z)$ & $a_2(z)$ ARE UNKNOWN (FOR NOW)

THEN $w' = a_1' w_1 + a_2' w_2 + a_1 w_1' + a_2 w_2' \quad (1)$

NOW RESTRICT a_1 & a_2 S.T.

$$a_1' w_1 + a_2' w_2 = 0 \quad (2) \qquad \text{(RESTRICTION)}$$

∴ WE HAVE: $a_1 w_1' + a_2 w_2' = w'$

$$w'' = a_1' w_1' + a_2' w_2' + a_1 w_1'' + a_2 w_2'' \quad (3)$$

SUBSTITUTE (1) & (3) INTO EQ (0)

USING FACT THAT w_1 & w_2 ARE SOLUTION TO HOMO EQ, WE GET RESULT

$$a_1' w_1' + a_2' w_2' = f(z) \quad (4)$$

THEN (2) & (4) PROVIDE TWO EQ FOR a_1' AND a_2'

$$\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(z) \end{bmatrix}$$

WRONSKIAN!! $W(z) \neq 0$ BECAUSE w_1/w_2 LINEARLY INDEPENDENT.

$\therefore$ WE CAN FIND UNIQUE SOLN FOR a_1' AND a_2'

THEN WE CAN GET a_1 AND a_2 BY ONE INTEGRATION

$$\boxed{a_1 = -\int \frac{f(z) w_2}{W} dz \qquad a_2 = \int \frac{f(z) w_1}{W} dz}$$

USING THE FACT THAT ANY TWO SOLUTIONS OF EQ. 0

THE MOST GENERAL SOLN FOR SYS OF EQ INCLUDING HOMO AND PARTICULAR IS OF FORM:

$$w'' - P_1(z) w' - P_0(z) w = f(z) \quad (0)$$

$$w = -w_1 \int \frac{f w_2}{W} dz + w_2 \int \frac{f w_1}{W} dz + A w_1 + B w_2$$

A + B ARE ARBITRARY CONSTANTS TBD BY INITIAL CONDITIONS ON $w(0)$, $w'(0)$

NOTE: EASILY GENERALIZE TO N^{th} ORDER SYSTEM.

* SINGULARITIES OF COEFFICIENTS. (FROBENIOUS THM)

→ CONSIDER 2^{nd} ORDER ODE WHERE $P_1(z)$ HAS AT MOST A SIMPLE POLE AT $z = z_0$ $(=0)$ AND $P_0(z)$ HAS AT MOST A DOUBLE POLE AT $z = z_0$

z_0 IS CALLED "REGULAR SINGULAR POINTS"

→ FOR SIMPLICITY let $z_0 = 0$

→ let $\underbrace{S(z) = z P_1(z) \quad \& \quad T(z) = z^2 P_0(z)}$

THESE ARE BOTH ANALYTIC IN A DISK $|z| < R$ WITH EXPANSIONS

$$S(z) = S_0 + S_1 z + S_2 z^2 \dots$$

$$T(z) = T_0 + T_1 z + T_2 z^2 \dots$$

THE ODE BECOMES: $z^2 w'' = z S(z) w' + T(z) w$

NOTE: WE CAN SHOW THAT THE ANSATZ "GUESS"

$$w(z) = w_0 + w_1 z + w_2 z^2 + \dots$$

FAILS IN GENERAL

→ INSTEAD TRY $w(z) = z^p (a_0 + a_1 z + a_2 z^2 + \dots)$

AN ANALYTIC FCT.

WHERE $a_0 \neq 0$

p = COMPLEX UNKNOWN CONSTANT.

→ SUB INTO ODE:

$$\left.\begin{aligned} &a_0\left[\rho(\rho-1)-\rho S_0-T_0\right]=0 \\ &a_1\left[(\rho+1)\rho-(\rho+1)S_0-T_0\right]=(\rho S_1+T_1)a_0 \\ &\quad\vdots \text{ TO THE } n^{th} \text{ EQ} \\ &a_n\left[(\rho+n)(\rho+n-1)-(\rho+n)S_0-T_0\right]=(\rho S_n+T_n)a_0+\dots+\left[(\rho+n-1)S_1+T_1\right]a_{n-1} \end{aligned}\right\}$$ SYSTEM OF EQ, SOLVE FOR $a_0 \dots a_n, \rho$

SINCE $a_0 \neq 0$, WE ASSUME $\boxed{\rho(\rho-1)-\rho S_0-T_0=0}$

INITIAL EQ

→ THIS EQ HAS TWO ROOTS, ρ_1 AND ρ_2. CHOOSE EITHER AND COMPUTE $a_1 = a_1(a_0)$

THEN COMPUTE $a_2 = a_2(a_0)$

$\vdots$

$a_n = a_n(a_0)$

– IN THIS WAY WE GENERATE TWO SOLN. (FOR EACH ρ)

NOTE: LOOKS GOOD HOWEVER PROCEDURE WILL FAIL IF $\boxed{(\rho+n)(\rho+n-1)-(\rho+n)S_0-T_0=0}$ FOR SOME n!

IF SO, IT SUGGESTS WE CANNOT FIND a_1 OR a_n ... ETC.