CLASS 25 - ODES II

Wednesday, November 6, 2019 3:00 PM

FROBENIUS

$$w'' = P_1 w' + P_0 w$$

$$\left.\begin{aligned} S(z) &= zP_1(z) \\ T(z) &= z^2 P_0(z) \end{aligned}\right\} \text{ ANALYTIC } \begin{aligned} &= S_0 + S_1 z + S_2 z^2 + \dots \\ &= T_0 + T_1 z + \dots \end{aligned}$$

$$zw'' = zP_1 w' + zP_0 w = zS(z)w' + T(z)w$$

$\Downarrow$ SOLN

$$w(z) = z^{\rho}(a_0 + a_1 z + a_2 z^2 + \dots) \otimes$$

WHERE $a_0 \neq 0$

AND $\rho(\rho-1) - \rho S_0 - T_0 = 0$

TWO ROOTS $\rho_1 \neq \rho_2$ IN GENERAL, BUT THINGS CAN GO WRONG:

1) $S_1 - S_2 \neq$ INTEGER

THEN THERE ARE TWO SOLUTIONS OF THE FORM $\otimes$ CORRESPONDING TO ρ_1 & ρ_2 CHOICES.

POWER SERIES CONVERGE UNIF. IN $|z| < R$ AND ARE ANALYTIC

TWO LINEARLY INDEPENDENT SOLN.

AT $z=0$ THEY HAVE BRANCH POINTS

2) $\rho_1 - \rho_2 = n$, n IS NON NEGATIVE INTEGER

LABEL THE ROOTS S.T. $\rho_2 = \rho_1 - n$ $(n > 0)$ $\quad \rho_1 = \rho_2 + n$

SINCE NONE OF COEFF. IN EQ $\oplus$ VANISH IF WE SET $\rho = \rho_1$

$\oplus \quad a_0[\quad] = 0$

$\vdots$

$\oplus \quad a_n[(\rho+n)(\rho+n-1) - (\rho+n)S_0 - T_0] = \{\ \}$

THEN WE STILL HAVE A SOLUTION OF THE FORM $\otimes$

NOW IF WE CONSIDER $\rho_2 = \rho_1 - n \dots$

HOWEVER IN GENERAL $\rho_2 = \rho_1 - n$ DOES NOT PROVIDE A SOLUTION SINCE EACH TERM IN $\oplus$ VANISHES.

→ USE THE WRONSKIAN TO GET OUR OTHER SOLUTION FROM OUR SOLUTION TO ρ_1 (w_1)

$$\boxed{w_2(z) = C\, w_1(z) \int \frac{dz}{w_1^2(z)} \exp\left[\int^z \frac{S(z')}{z'}\, dz'\right]}$$

$P_1 = \frac{S}{z}$

NOW, EXCEPT FOR A CONSTANT OF INTEGRATION THAT CAN BE ABSORBED INTO C, WE HAVE THAT:

$$\int \frac{S(z')}{z'}\, dz' = S_0 \log(z) + S_1 z + \frac{S_2}{2} z^2 + \dots$$

AND $\left(\frac{1}{w(z)}\right)^2 = \frac{1}{z^{2\rho_1}} \cdot \frac{1}{a_0^2}\left[1 - \frac{2a_1}{a_0} z + \dots\right]$

$$w_2(z) = C\, w_1(z) \int z^{-2\rho_1 + S_0}[g(z)]\, dz$$

WHERE a_0 ABSORBED INTO C AND $g(z)$ IS ANALYTIC @ $z=0$. [IN FACT $g(0) = 1$]

RECALL FROM INITIAL EQ THAT $\rho_1 + \rho_2 = S_0 + 1$

FROM $\rho(\rho-1) + \rho S_0 - T_0 = 0$

AND WE KNOW $\rho_1 - \rho_2 = n$, SO THAT $-2\rho_1 + s_0 = -(n+2)$

NEXT, WRITING $g(z) = 1 + g_1 z + g_2 z^2 + \dots$

$$\boxed{\begin{aligned} &(a)\ n=0 \Rightarrow w_2(z) = C w_1(z)\left[\log z + h(z)\right] \\ &(b)\ n \neq 0 \Rightarrow w_2(z) = C\left[w_1(z)\ \ g_n \log z + z^{2\rho} h(z)\right] \end{aligned}}$$

WHERE $h(z)$ IS ANALYTIC AT $z=0$.

NOTE: IF COEFF. g_n VANISHES, THEN WE JUST GET OUR SECOND SOLUTION: $z^{\rho_2} h(z)$

ANALYTIC

⋮ BRB BATHROOM

REIMANN P EQUATION

→ ODES OF TYPE $w'' = P_1 w' + P_0 w$

Let P_1 & P_0 BE ANALYTIC EVERYWHERE EXCEPT AT 3 DISTINCT POINTS, $\langle a, b, c \rangle$

ONE POINT (c) MAY CORRESPOND TO ∞, BUT OTHERWISE P_1, P_0 ARE ANALYTIC AT $z = \infty$

FOR NOW, ASSUME THAT a, b, c ARE NOT INFINITE

$$\Rightarrow \quad P_1(z) = \frac{S(z)}{(z-a)(z-b)(z-c)} \qquad P_0(z) = \frac{T(z)}{(z-a)^2(z-b)^2(z-c)^2}, \quad S \ \&\ T \text{ ARE ENTIRE}$$

CHANGE VARIABLES $z = \frac{1}{\xi}$

ODE BECOMES:

$$w_{\xi\xi} = \left[\frac{S(\frac{1}{\xi})}{\xi^2(\frac{1}{\xi}-a)(\frac{1}{\xi}-b)(\frac{1}{\xi}-c)} + \frac{2}{\xi}\right] w_\xi + \left[\frac{T(\frac{1}{\xi})}{\xi^4(\frac{1}{\xi}-a)^2(\frac{1}{\xi}-b)^2(\frac{1}{\xi}-c)^2}\right] w$$

SINCE $\xi = 0$ IS AN ORDINARY POINT, WE CAN SHOW THAT $T(z)$ IS AT MOST A SECOND-ORDER POLYNOMIAL, AND $S(z)$ IS PRECISELY A SECOND-ORDER POLYNOMIAL WITH COEFFICIENT -2 FOR TERM z^2.

THIS MEANS
$$\boxed{P_1(z) = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} \quad \& \quad P_0(z) = \left(\frac{A'}{z-a} + \frac{B'}{z-b} + \frac{C'}{z-c}\right)\frac{1}{(z-a)(z-b)(z-c)}}$$

PARTIAL FRACTIONS EXPANSION

WHERE A, A', B, B', C, C' ARE ARBITRARY EXCEPT THAT:

$$\boxed{A + B + C = -2}$$

AT THE POINT $z = a$, THE INDICIAL EQ IS: $\rho^2 - \rho(A+1) - \frac{A'}{(a-b)(a-c)} = 0$

TWO SOLUTIONS α, α'

CONVENIENT TO WRITE A, A' IN TERMS OF α, α', a, b, c

RESULT: $A = \alpha + \alpha' - 1$

$A' = (a-b)(c-a)\alpha\alpha'$

DENOTE SIMILAR EXP. W/ β, β' γ, γ'

∴ REIMANN P EQ:

$$\boxed{w'' = \left(\frac{\alpha+\alpha'-1}{z-a} + \frac{\beta+\beta'-1}{z-b} + \frac{\gamma+\gamma'-1}{z-c}\right) w' + \left(\frac{\alpha\alpha'(a-b)(c-a)}{z-a} + \frac{\beta\beta'(b-c)(a-b)}{z-b} + \frac{\gamma\gamma'(c-}{z-c}\right)\frac{w}{(z-a)(z-b)(z-c)}}$$

THE SOLUTION IS CONVENTIONALLY WRITTEN:

$$\boxed{w = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}}$$

NOTE: ONE POINT (c) MAY GO TO ∞

PROPERTIES

* TRANSFORMATION RULE $\Rightarrow$ C

CHANGE DEPENDENT VARIABLES: $w \to w_1$

$$w_1 = \frac{(z-a)^{\lambda}(z-b)^{\mu}}{(z-c)^{\lambda+\mu}}$$

$$\therefore\ w_1 = P\begin{Bmatrix} a & b & c & \\ \alpha+\lambda & \beta+\mu & \gamma-\lambda+\mu & z \\ \alpha'+\lambda & \beta'+\mu & \gamma'-\lambda+\mu & \end{Bmatrix}$$

CHANGE $z_1 \Rightarrow z = \dfrac{A_0 z_1 + A_1}{B_0 z_1 + B_1}$ ALLOWS YOU TO SHIFT THREE REG. POINTS $z = a, b, c$ TO $z_1 = a_1, b_1, c_1$

ALLOWS YOU TO CONSIDER C GOING TO ∞