ENGINEERING MATHEMATICS 521

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Practice Problems for Final Exam

1. The Laplace transform of a function $f(t)$ with $f(t) = 0$ for $t < 0$ is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \tag{1}$$

while the inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds, \tag{2}$$

where γ is taken to be to the right of all singularities of $F(s)$.

   (a) Show that the Laplace transform of the nth derivative of f is given by

$$\begin{aligned} F_n(s) &= \int_0^\infty f^{(n)}(t)e^{-st}dt \\ &= s^n F(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - ... - s^{n-2}f'(0) - s^{n-1}f(0). \end{aligned} \tag{3}$$

   (b) Define the convolution product of f and g by

$$h(t) = \int_0^t f(t')g(t-t')dt', \tag{4}$$

   and show that

$$H(s) = F(s)G(s). \tag{5}$$

2. Show that the inverse Laplace transform of the function

$$F(s) = \frac{e^{-s^{1/2}|x|}}{2s^{1/2}} \tag{6}$$

is given by

$$f(t) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}. \tag{7}$$

Hint. Select the negative real axis as the branch cut for $s^{1/2}$, and close the contour in expression (2) on the left-hand side, but "avoid" the branch cut by going around the branch point at $s = 0$. Show that the semicircular and circular contributions of radius R and ϵ vanish as $R \to \infty$ and $\epsilon \to 0$, respectively, leaving two integrals from $-\infty$ to zero that—by a change of variables—can be combined into a Gaussian-type integral. Recall that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}. \tag{8}$$

3. In class we considered the diffusion equation in an infinite rod ($-\infty < x < \infty$)

$$\frac{\partial \phi(x,t)}{\partial t} = \alpha \frac{\partial^2 \phi(x,t)}{\partial x^2}, \tag{9}$$

subject to the initial condition

$$\phi(x,0) = f(x), \tag{10}$$

and "asymptotic" boundary conditions

$$\phi(x,t) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{11}$$

To obtain the solution for *general* f, we first found the solution for a "unit" forcing function (i.e., a delta function), which we called the Green's function, and then generated the more general solution for arbitrary f by superposition. In class, we made use of the *Fourier transform* in the variable x to obtain the Green's function $g(x,t;\xi) = g(x-\xi,t)$. The objective of this problem is to obtain the same result by *first* making use of a *Laplace transform* in the time variable t.

Note. By the change of variable $t \to \alpha t$, we can obtain the solution of equation (9) by first solving the problem with $\alpha = 1$ and then changing back to the original variable.

(a) Making use of the result (3) of Problem 1, take the Laplace transform of equation (9) with $\alpha = 1$ and initial condition (10) replaced by $\phi(x,0) = \delta(x-\xi)$ to obtain an ODE in the variable x for $G(x,s;\xi)$ (the Laplace transform of $g(x,t;\xi)$ w.r.t. t).

(b) To solve the resulting ODE, take the Fourier transform of $G(x,s;\xi)$ with respect to the variable x, which we could call $\hat{G}(\lambda,s;\xi)$, and obtain the result

$$\hat{G}(\lambda,s;\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda^2 + s} e^{i\lambda\xi}. \tag{12}$$

(c) Now evaluate the inverse Fourier transform of this expression to obtain

$$G(x,s;\xi) = \frac{e^{-s^{1/2}|x-\xi|}}{2s^{1/2}} = G(x-\xi,s). \tag{13}$$

and make use of the result of Problem 2 to show that the Green's function for the problem is given by

$$g(x,t;\xi) = \frac{e^{-(x-\xi)^2/4t}}{2\sqrt{\pi t}} = g(x-\xi,t). \tag{14}$$

(d) Show that you can obtain the same result by first evaluating the inverse Laplace transform of (12) and then the inverse Fourier transform of the result.

(e) By changing back to the original time variable show that the Green function to the original problem with $\alpha \neq 1$ is given by

$$g(x,t;\xi) = \frac{e^{-(x-\xi)^2/4\alpha t}}{2\sqrt{\pi\alpha t}} = g(x-\xi,t), \tag{15}$$

and finally obtain the solution for the original problem with initial condition (10) for general f.

4. Consider the 3-D wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = f(x,y,z,t) \tag{16}$$

for the *unknown* function $\phi(x,y,z,t)$, which is defined in the infinite domain given by $-\infty < x < +\infty$, $-\infty < y < +\infty$, and $-\infty < z < +\infty$, for $t \geq 0$, and where f is a *known* function that decays sufficiently fast as $r = \sqrt{x^2+y^2+z^2}$ tends to ∞.

(a) Assume that the driving force is harmonic, i.e.,

$$f(x,y,z,t) = h(x,y,z)e^{i\omega t}, \tag{17}$$

and look for solutions of the form $\phi(x,y,z,t) = u(x,y,z)e^{i\omega t}$. What is the equation for u?

(b) Find the Green's function $g(x,y,z;\xi,\eta,\zeta)$ for the resulting problem, defined by

$$h = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta), \tag{18}$$

and by the asymptotic condition that $g \to 0$ as $r \to \infty$, is given by

$$g(x,y,z;\xi,\eta,\zeta) = -\frac{1}{4\pi R}e^{\frac{-iR\omega}{c}}, \tag{19}$$

where $R = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$.

Hint: Anticipating that $g(x,y,z;\xi,\eta,\zeta) = g(x-\xi, y-\eta, z-\zeta)$, set $\xi=\eta=\zeta=0$ in the PDE for g. Then, make use of a triple Fourier transform in x, y and z, and invert the resulting expression for the Fourier transform of g. When performing the inverse Fourier transform, make a change of variables to spherical coordinates (recall that $d\lambda_1\,d\lambda_2\,d\lambda_3 = \rho^2 \sin\theta\,d\rho\,d\theta\,d\phi$), such that $x\lambda_1 + y\lambda_2 + z\lambda_3 = r\rho\cos\theta$, and carry out the integrations with respect to θ and ϕ. Then, make use of the residue theorem and Jordan's lemma to perform the final integration with respect to ρ keeping in mind the appropriate "radiation" conditions.

(c) Write an integral expression for the solution $u(x,y,z)$ of the original equation (16) in terms of the solution (19) for $g(x,y,z;\xi,\eta,\zeta)$ and $h(\xi,\eta,\zeta)$.

5. Consider the 3-D diffusion equation for the unknown function $\phi(x, y, z, t)$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha} \frac{\partial \phi}{\partial t}, \tag{20}$$

together with initial condition

$$\phi(x, y, z, 0) = f(x, y, z), \tag{21}$$

where f is a *known* function that decays sufficiently fast as $r = \sqrt{x^2 + y^2 + z^2}$ tends to ∞, and "asymptotic" boundary conditions

$$\phi(x, y, z, t) \to 0 \quad \text{as} \quad r = \sqrt{x^2 + y^2 + z^2} \to \infty. \tag{22}$$

(a) First, by means of the Fourier transform, find the infinite-body Green's function, corresponding to

$$f(x, y, z) = \delta(x - \xi)\delta(y - \eta)\delta(x - \zeta). \tag{23}$$

(b) Then, make use of the convolution theorem to find the solution for general f.