Solutions for Math 311 Assignment #3

(1) Compute the limits:

(a) $\lim_{z\to i} \frac{iz^3 - 1}{z^2+1}$;

(b) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2}$.

Solution. (a)

$$\begin{aligned}
\lim_{z\to i} \frac{iz^3-1}{z^2+1} &= \lim_{z\to i} \frac{i(z^3-i^3)}{z^2+1} \\
&= \lim_{z\to i} \frac{i(z-i)(z^2+iz+i^2)}{(z-i)(z+i)} \\
&= \lim_{z\to i} \frac{i(z^2+iz+i^2)}{z+i} = -\frac{3}{2}
\end{aligned}$$

or by complex L'Hospital,

$$\lim_{z\to i} \frac{iz^3-1}{z^2+1} = \lim_{z\to i} \frac{3iz^2}{2z} = -\frac{3}{2}$$

(b)

$$\begin{aligned}
\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} &= \lim_{z\to 0} \frac{4z^{-2}}{(z^{-1}-1)^2} \\
&= \lim_{z\to 0} \frac{4}{(1-z)^2} = 4
\end{aligned}$$

(2) Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

Solution. Let $z = x + yi$. When $x \to 0$ and $y = 0$,

$$\lim_{\substack{x\to 0\\ y=0}} \left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

When $x = y$ and $x \to 0$,

$$\lim_{\substack{x\to 0\\ x=y}} \left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0} \frac{(x+xi)^2}{(x-xi)^2} = -1$$

Therefore, $\lim_{z\to 0} f(z)$ does not exist.

(3) Let
$$T(z) = \frac{az+b}{cz+d}$$
where $ad - bc \neq 0$. Show that
(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$;
(b) $\lim_{z\to\infty} T(z) = \dfrac{a}{c}$ if $c \neq 0$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

Proof. Since
$$\begin{aligned}\lim_{z\to\infty} \frac{1}{T(z)} &= \lim_{z\to 0} \frac{1}{T(z^{-1})}\\ &= \lim_{z\to 0} \frac{cz^{-1}+d}{az^{-1}+b}\\ &= \lim_{z\to 0} \frac{c+dz}{a+bz} = \frac{c}{a}\end{aligned}$$
it follows that $\lim_{z\to\infty} T(z) = \infty$ when $c = 0$ and $\lim_{z\to\infty} T(z) = a/c$ if $c \neq 0$.

Since $ad - bc \neq 0$, $a(-d/c) + b \neq 0$. Therefore,
$$\lim_{z\to -d/c} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = 0$$
and hence $\lim_{z\to -d/c} T(z) = \infty$ when $c \neq 0$. □

(4) Find $f'(z)$ when
(a) $f(z) = 3z^2 - 2z + 4$;
(b) $f(z) = (1 - 4z^2)^3$;
(c) $f(z) = \dfrac{z-1}{2z+1}$ $(z \neq -\frac{1}{2})$;
(d) $f(z) = \dfrac{(1+z^2)^4}{z^2}$ $(z \neq 0)$.

Answer. (a) $6z - 2$ (b) $-24z(1-4z^2)^2$ (c) $3(2z+1)^{-2}$ (d) $2(3z^2-1)(1+z^2)^3 z^{-3}$

(5) Show that $f'(z)$ does not exist at any point when
(a) $f(z) = \operatorname{Im}(z)$;
(b)
$$f(z) = \begin{cases} \overline{z}^2/z & \text{if } z \neq 0\\ 0 & \text{if } z = 0\end{cases}$$

Proof. (a) Since

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(x-yi)=2\neq 0$$

for all $z=x+yi$, $f'(z)$ does not exist at any point.

(b) For $z=x+yi\neq 0$,

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f&=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\frac{(x-yi)^2}{x+yi}\\&=\frac{2(x-yi)(x+yi)-(x-yi)^2}{(x+yi)^2}\\&\quad+i\frac{-2i(x-yi)(x+yi)-i(x-yi)^2}{(x+yi)^2}\\&=\frac{4(x-yi)}{x+yi}\neq 0\end{aligned}$$

Therefore, $f'(z)$ does not exist for $z\neq 0$.

At $z=0$,

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{\overline{z}^2}{z^2}$$

When $x\to 0$ and $y=0$,

$$\lim_{\substack{x\to 0\\ y=0}}\frac{\overline{z}^2}{z^2}=\lim_{x\to 0}\frac{x^2}{x^2}=1$$

When $x=y$ and $x\to 0$,

$$\lim_{\substack{x\to 0\\ x=y}}\frac{\overline{z}^2}{z^2}=\lim_{x\to 0}\frac{(x-xi)^2}{(x+xi)^2}=-1$$

Therefore, $f'(0)$ does not exist. In conclusion, $f'(z)$ does not exist anywhere. □

(6) Use Cauchy-Riemann equations to verify that $f(z)$ is analytic when
 (a) $f(z)=z^3$ in $\mathbb{C}$;
 (b) $f(z)=z^{-1}$ for $z\neq 0$;
 (c) $f(z)=e^{-z^2}$ in $\mathbb{C}$.

Proof. (a) Since

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)z^3&=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(x+iy)^3\\&=3(x+yi)^2+i^2 3(x+yi)^2=0\end{aligned}$$

and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous everywhere, $f(z)$ is entire.

(b) Since

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)z^{-1} &= \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(x+iy)^{-1}\\ &= -(x+yi)^{-2}-i^2(x+yi)^{-2}=0\end{aligned}$$

and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous for $z\neq 0$, $f(z)$ is analytic for $z\neq 0$.

(c) Since

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{-z^2} &= \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{-(x+iy)^2}\\ &= -2(x+yi)e^{-(x+iy)^2}-2i^2(x+yi)e^{-(x+iy)^2}=0\end{aligned}$$

and $\partial f/\partial x$ and $\partial f/\partial y$ are continuous everywhere, $f(z)$ is entire. □

(7) Show that if both $f(z)$ and $g(z)$ satisfy the Cauchy-Riemann equations at z_0, so does $f(z)g(z)$.

Proof. Since C-R equations hold for $f(z)$ and $g(z)$ at z_0,

$$\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=\frac{\partial g}{\partial x}+i\frac{\partial g}{\partial y}=0$$

at z_0. Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(fg) &= \left(\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x}\right)+i\left(\frac{\partial f}{\partial y}g+f\frac{\partial g}{\partial y}\right)\\ &= \left(\frac{\partial f}{\partial x}g+i\frac{\partial f}{\partial y}g\right)+\left(f\frac{\partial g}{\partial x}+if\frac{\partial g}{\partial y}\right)\\ &= g\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)+f\left(\frac{\partial g}{\partial x}+i\frac{\partial g}{\partial y}\right)=0\end{aligned}$$

at z_0. Therefore, C-R equations hold for $f(z)g(z)$ at z_0. □

(8) Suppose that $f(z)=u+iv$ is analytic at z_0. Show that

$$f'(z_0)=-\frac{i}{z_0}\left(\frac{\partial u}{\partial\theta}+i\frac{\partial v}{\partial\theta}\right)$$

at $z=z_0$, where (r,θ) are the polar coordinates.

Proof. By chain rule,

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \text{ and } \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Since $f(z)$ is analytic at z_0,

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$$

and hence

$$\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) f = 0 \Rightarrow \frac{\partial f}{\partial r} = -\frac{i}{r}\frac{\partial f}{\partial\theta}$$

in $|z - z_0| < a$ for some $a > 0$. Therefore,

$$\begin{aligned} f'(z) = \frac{\partial f}{\partial x} &= \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta} \\ &= -\frac{i\cos\theta}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta} \\ &= -i\frac{e^{-i\theta}}{r}\frac{\partial f}{\partial\theta} = -\frac{i}{re^{i\theta}}\frac{\partial f}{\partial\theta} \\ &= -\frac{i}{z}\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) \end{aligned}$$

□