Math 460: Complex Analysis
MWF 11am, Fulton Hall 425
Homework 6

Please write neatly, and in complete sentences when possible.

Do the following problems from the book: 3.2.1, 3.2.2, 3.2.8, 3.2.9

Problem D. Consider the curve γ and the points a, b, and c in Figure 1. Compute ind_γ for each of the points a, b, and c.

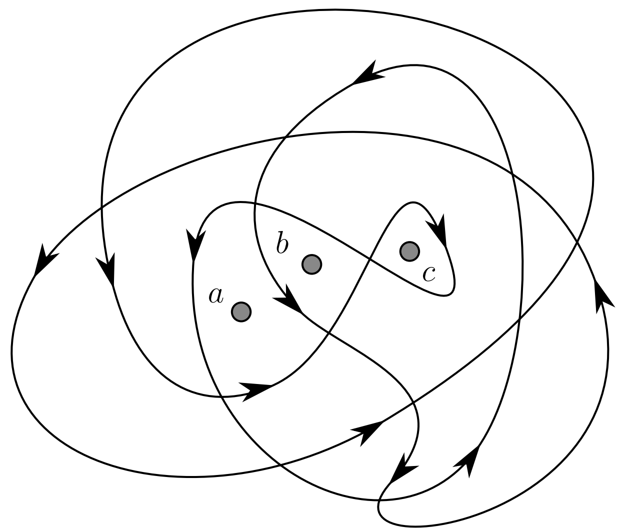

FIGURE 1. A curve γ in $\mathbb{C}$, and points $a, b, c \in \mathbb{C}$.

Problem E. Suppose f is a function which is analytic on a disk in $\mathbb{C}$ that contains the curve η, pictured in Figure 2 along with a pair of points z_1 and z_2. Use the Cauchy Integral Formula to write an expression for

$$\int_\eta \frac{f(w)}{(w-z_1)(w-z_2)} dw$$

that doesn't involve integrals.

Problem F.

(1) Find a power series expansion for $1/z$, centered at $z = 2$. What is its radius of convergence? (Hint: $z = 2 - (2 - z)$).
(2) Use the previous to find a power series expansion for $\log z$ centered at $z = 2$.

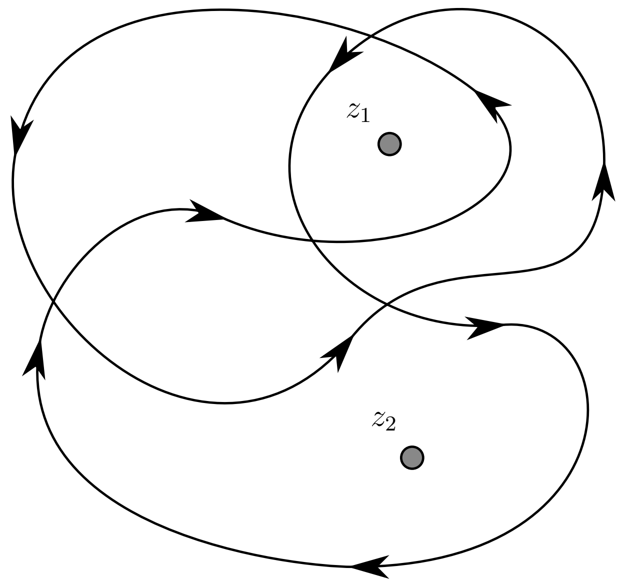

FIGURE 2. A curve η in $\mathbb{C}$, and a pair of points $z_1, z_2 \in \mathbb{C}$.

Problem G.

(1) Find a power series expansion for $1/(1+z^2)$, centered at $z = 0$. What is its radius of convergence?
(2) Use the previous to find a power series expansion for $\tan^{-1} z$ centered at $z = 0$, and note its radius of convergence. (You may take for granted that $\tan^{-1} z$ is an analytic function defined in some neighborhood of 0, whose derivative is $1/(1+z^2)$.

To think about: What happens in your power series when you plug in $z = 1$?

Problem H.

(1) Let $f(z) = e^{z^2}$. Show that f is analytic on the entire complex plane, and deduce that f has an anti-derivative defined on $\mathbb{C}$.
(2) Now let $F(z)$ be an anti-derivative for f. Find a power series expansion for F about $z = 0$. What is its radius of convergence?

Solution 3.2.1. We know from considerations of geometric series that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,$$

which converges on the disk $\mathbb{D}_1(0)$. By Weierstrass' Theorem, this convergence is uniform, and so we may compute the derivative by term-by-term differentiation:

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n.$$

Solution 3.2.2. The function $f(z) = \sqrt{1+z}$ is defined using the principal branch of the logarithm via

$$\sqrt{1+z} = e^{\frac{1}{2}\log(1+z)}.$$

We may compute derivatives of this function in a straightforward way:

$$\begin{aligned}
f'(z) &= \frac{1}{2(1+z)}e^{\frac{1}{2}\log(1+z)} = \frac{e^{\frac{1}{2}\log(1+z)}}{2e^{\log(1+z)}} \\
&= \frac{1}{2}e^{-\frac{1}{2}\log(1+z)} \\
f''(z) &= -\frac{1}{4(1+z)}e^{-\frac{1}{2}\log(1+z)} = -\frac{e^{-\frac{1}{2}\log(1+z)}}{4e^{\log(1+z)}} \\
&= -\frac{1}{4}e^{-\frac{3}{2}\log(1+z)}.
\end{aligned}$$

In fact, we claim that, for $k \geq 2$, we have

$$f^{(k)}(z) = (-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-3)}{2^k}e^{-\frac{2k-1}{2}\log(1+z)}.$$

This has been demonstrated for $k=2$, so it only remains to check the inductive step:

$$\begin{aligned}
f^{(k+1)}(z) &= \left((-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-3)}{2^k}e^{-\frac{2k-1}{2}\log(1+z)}\right)' \\
&= (-1)^{k+1}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-3)}{2^k}\cdot\left(-\frac{2k-1}{2}\right)\cdot\frac{e^{-\frac{2k-1}{2}\log(1+z)}}{1+z} \\
&= (-1)^{k+2}\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-3)\cdot(2k-1)}{2^{k+1}}e^{-\frac{2k+1}{2}\log(1+z)}.
\end{aligned}$$

Plugging in $z = 0$, and with a bit of rearranging, we have

$$
\begin{aligned}
f^{(k)}(0) &= (-1)^{k+1} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-3)}{2^k} \\
&= (-1)^{k+1} \frac{(2k-3)!}{2^k \cdot 2 \cdot 4 \cdot \ldots \cdot (2k-4)} \\
&= (-1)^{k+1} \frac{(2k-3)!}{2^k \cdot 2^{k-2} \cdot 1 \cdot 2 \cdot \ldots \cdot (k-2)} \\
&= (-1)^{k+1} \frac{(2k-3)!}{2^{2k-2} \cdot (k-2)!}
\end{aligned}
$$

Finally, if $\sum c_n z^n$ is a power series expansion of $\sqrt{1+z}$ about $z = 0$, then $c_n = f^{(n)}(0)/n!$, so that, for $n \geq 2$,

$$
c_n = (-1)^{n+1} \frac{(2n-3)!}{2^{2n-2} \cdot n!(n-2)!} = \frac{(-1)^{n+1}}{2^{2n-1}(n-1)} \binom{2n-2}{n}.
$$

Putting this into the power series, we find

$$
f(z) = 1 + \frac{1}{2} z + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{2^{2n-1}(n-1)} \binom{2n-2}{n} z^n.
$$

By Theorem 3.2.5, this power series converges on the largest disk around $z = 0$ on which $f(z)$ is analytic. Since $\sqrt{1+z}$ is not analytic at $z = 0$, the radius of convergence is 1.

Solution 3.2.8. The function $\sin z$ is defined via (see page 17)

$$
\sin z = \frac{e^{iz} - e^{-iz}}{2i},
$$

so that

$$
(\sin z)' = \frac{ie^{iz} + ie^{iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2},
$$

which is the definition of $\cos z$. Similarly, it is straightforward to check that $(\sin z)^{(2k)} = (-1)^k \sin z$ and $(\sin z)^{(2k+1)} = (-1)^k \cos z$, for any $k \in \mathbb{N}$. The coefficients of a power series expansion about $z = 0$ for $\sin z$ are given by

$$
c_n = (\sin z)^{(n)}\Big|_{z=0} / n!.
$$

If n is even, we see that $c_n = 0$. If $n = 2k+1$ is odd, then we have $c_n = (-1)^k / n!$. Summing over only odd terms, we find

$$
\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.
$$

The latter has infinite radius of convergence, since $\sin z$ is analytic on every disk around 0.

Now let $g(z)$ be equal to $\sin z/z$ for $z \neq 0$, and $g(0) = 1$. We can factor z out of the above power series (to think about: why is this possible?), and we obtain, for $z \neq 0$,

$$g(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

The latter power series has the same radius of convergence as that of $\sin z$ (namely, infinite), which we may check by the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n z^{2n}}{(2n+1)!} \cdot \frac{(2n-1)!}{(-1)^{n-1} z^{2n-2}} \right| = \lim_{n\to\infty} \frac{|z|^2}{2n(2n+1)} = 0.$$

Moreover, the value of the power series above at 0 agrees with that of g. Thus g defines an analytic function on the whole complex plane.

Solution 3.2.9. If f is analytic on $\mathbb{D}_r(z_0)$, then it has a convergent power series expansion on $\mathbb{D}_r(z_0)$ by Theorem 3.2.5., i.e. we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

converging on $\mathbb{D}_r(z_0)$. Since f is not identically zero on $\mathbb{D}_r(z_0)$, we must have $c_n \neq 0$, for some values of n. Let k indicate the smallest such value of n, so that $c_1 = c_2 = \ldots = c_{k-1} = 0$ and $c_k \neq 0$. Consider the power series

$$\sum_{n=0}^{\infty} c_{n+k} (z - z_0)^n.$$

In fact, this series has the same radius of convergence as the original series, since

$$\begin{aligned} \limsup |c_{n+k}|^{1/n} &= \limsup |c_n|^{1/(n-k)} \\ &= \limsup \left(|c_n|^{1/n} \right)^{\frac{n}{n-k}} = \limsup |c_n|^{1/n}. \end{aligned}$$

It thus defines an analytic function g on $\mathbb{D}_r(z_0)$, such that $g(z_0) = c_k \neq 0$. Moreover, by design we have $f(z) = (z - z_0)^k g(z)$.

Solution D. In order to compute the index of winding of γ around each of these points, we use the method described in class: The winding number of γ is constant on connected components of $\mathbb{C} \setminus \gamma$, and equal to zero on the unbounded component. Moreover, we know how the winding number changes when we cross the curve (see Figure 3).

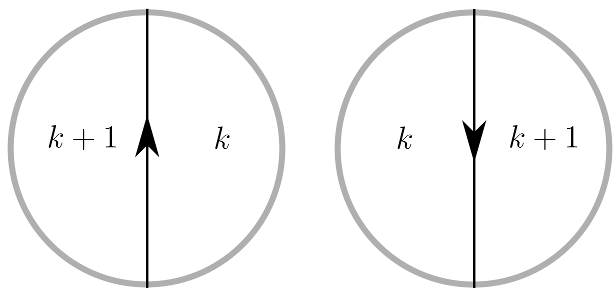

FIGURE 3. The 'local' rules for computing ind_γ.

We draw arcs in $\mathbb{C}$ that cross over the curve, connecting the components containing a, b, and c, respectively, to the unbounded component of $\mathbb{C} \setminus \gamma$. We know the value of ind_γ at one endpoint of this arc system (the one that lands in the unbounded component). For each crossing of the arcs across the curve γ, we apply the local moves from Figure 3. The result is pictured in Figure 4, and we conclude that $\mathrm{ind}_\gamma(a) = 3$, $\mathrm{ind}_\gamma(b) = 4$, and $\mathrm{ind}_\gamma(c) = 2$.

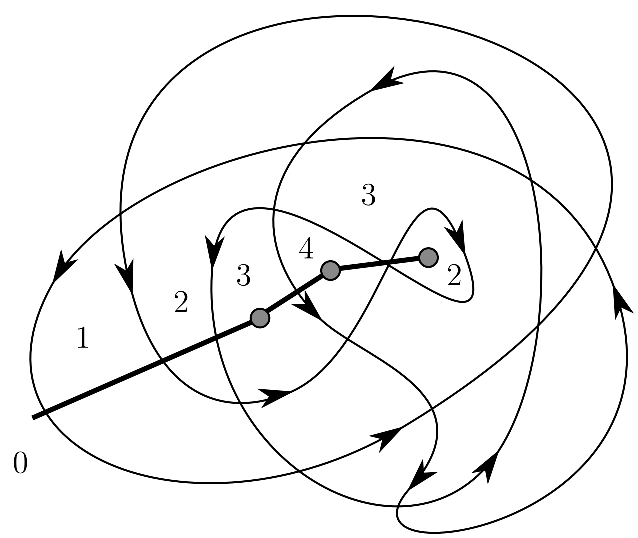

FIGURE 4. Computing the winding numbers in Problem D.

Solution E. We decompose η into its constituent arcs, each of which is embedded (i.e. the map from the interval to $\mathbb{C}$ that the arc represents is injective). The integral in question is a contour integral around η which itself is the sum of contour integrals over the arcs obtained above. We may reorganize the arcs into a different sequence, forming a different set of contours, and so long as we see each arc exactly once in this reorganization, the integral over η will equal the integral over the contours so obtained. In Figures 5 and 6 we make a specific choice of this reorganization, into two closed curves α and β. We have

$$\int_\gamma \frac{f(w)}{(w-z_1)(w-z_2)}dw = \int_\alpha \frac{f(w)}{(w-z_1)(w-z_2)}dw + \int_\beta \frac{f(w)}{(w-z_1)(w-z_2)}dw.$$

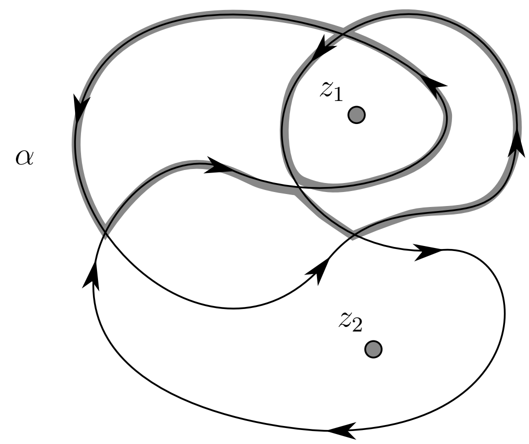

FIGURE 5. The curve α is formed by a union of some of the arcs that form η.

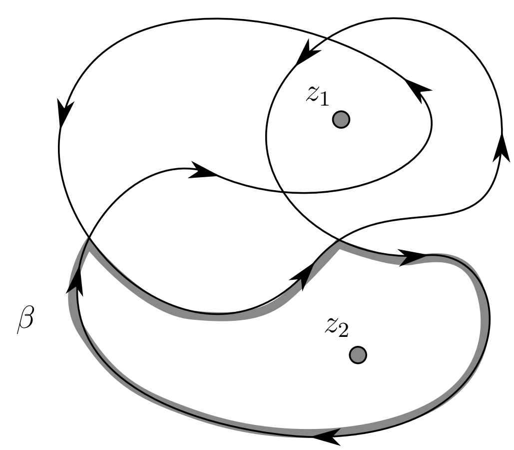

FIGURE 6. The curve β is formed by a union of the remaining arcs of η.

The curve α is contained in a convex open set that excludes z_2, so that $f(w)/(w-z_2)$ is analytic on this set, and we may apply the Cauchy

Integral Formula. Similarly, β is contained in a convex open set that excludes z_1. We find:

$$\begin{aligned}\int_\gamma \frac{f(w)}{(w-z_1)(w-z_2)}dw &= \int_\alpha \frac{f(w)/(w-z_2)}{w-z_1}dw + \int_\beta \frac{f(w)/(w-z_1)}{w-z_2}dw \\ &= 2\pi i \cdot \mathrm{ind}_\alpha(z_1) \cdot \frac{f(z_1)}{z_1-z_2} + 2\pi i \cdot \mathrm{ind}_\beta(z_2) \cdot \frac{f(z_2)}{z_2-z_1} \\ &= \frac{2\pi i}{z_1-z_2}(2f(z_1)+f(z_2)),\end{aligned}$$

since $\mathrm{ind}_\alpha(z_1) = 2$ and $\mathrm{ind}_\beta(z_2) = -1$ (calculated via the method above in the solution to Problem D).

Solution F(1). Since $z = 2-(2-z)$, for any $z \neq 0$ we have

$$\frac{1}{z} = \frac{1}{2-(2-z)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{2-z}{2}\right)}.$$

When $|(2-z)/2| < 1$, we may use the geometric series to compute the latter:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2-z}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(2-z)^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n.\end{aligned}$$

The latter converges exactly when $|z-2| < 2$, so its radius of convergence is 2.

Solution F(2). Since $(\log z)' = 1/z$, (note that this is for any branch of the logarithm), we may compute the power series for $\log z$ by antiderivating the power series for $1/z$. Namely, we must have

$$\log z = c_0 + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^{n+1}}(z-2)^{n+1}.$$

Evidently, $c_0 = \log 2$, so after re-indexing, we find:

$$\log 2 = \log 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}(z-2)^n.$$

Solution G(1). We have

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)},$$

so that the geometric series applies when $|-z^2| < 1$:

$$\begin{aligned}\frac{1}{1+z^2} &= \frac{1}{1-(-z^2)} \\ &= \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n},\end{aligned}$$

whose radius of convergence is 1.

Solution G(2). Since $\tan^{-1} z$ is an anti-derivative of $1/(1+z^2)$, we may compute its power series by term-by-term anti-differentiation:

$$\tan^{-1} z = c_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1},$$

where $c_0 = \tan^{-1} 0 = 0$. This power series converges in the same disk that the power series of $1/(1+z^2)$ converges, so its radius of convergence is also 1.

Note that when $z = 1$, we have $\tan^{-1} 1 = \pi/4$ (there are some choices to define $\tan^{-1} z$ that are being suppressed here, but this subtlety is beyond the scope of this question). On the other hand, the value $z = 1$ is on the boundary of the disk $\mathbb{D}_1(0)$, so that we can't a priori know whether the power series converges. In fact, with a bit of real analysis, one can show that it does converge, and is equal to $\tan^{-1} 1$, so that we actually have:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Solution H(1). The function $f(z)$ is the composition of two analytic functions (namely $z \mapsto z^2$ and $z \mapsto e^z$), each of which are analytic on the entire complex plane. We may conclude by the chain rule that f is analytic on the entire complex plane. By Theorems 2.5.8 and 2.6.1, the f has an anti-derivative given by

$$F(z) = \int_p^z f(w)dw,$$

for any $p \in \mathbb{C}$.

Solution H(2). Since

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

converges for all $w \in \mathbb{C}$, it has an infinite radius of convergence. We have

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!},$$

which also converges for all $z \in \mathbb{C}$. The anti-derivative can be computed by anti-differentiating term-by-term, so we obtain:

$$F(z) = c_0 + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1) \cdot n!}.$$

Note that the decision of the value of c_0 is equal to the value of $F(0)$, which depends on the choice of p in the solution of Problem H(1). In general, we have

$$c_0 = \int_p^0 e^{w^2} dw.$$

In particular, if the choice $p = 0$ is made, then $c_0 = 0$.