ENGINEERING MATHEMATICS 521

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Problem Set 8

Due date: November 25

1. Show that
$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)t^n \tag{1}$$
is a *generating function* for Bessel functions J_n.

2. p. 230 of Carrier, Krook and Pearson. Exercises 8, 9, 10, 11.

Obtain a similar inequality for $J_\nu(z)$, from Poisson's integral.

7. Show that

(a) $K_\nu(z)I_{\nu+1}(z) + K_{\nu+1}(z)I_\nu(z) = \dfrac{1}{z}$

(b) $J_{\frac{1}{2}+\nu}(z)J_{\frac{1}{2}-\nu}(z) + J_{-\frac{1}{2}+\nu}(z)J_{-\frac{1}{2}-\nu}(z) = \dfrac{2\cos\nu\pi}{\pi z}$

8. Show that, if n is an odd positive integer,

$$J_n(z) = (-1)^{\frac{1}{2}(n-1)}\frac{2}{\pi}\int_0^{\pi/2}\cos n\theta \sin(z\cos\theta)\,d\theta$$

9. Prove that

$$\int_0^\pi e^{\alpha\cos\theta}\cos(\beta\sin\theta)\,d\theta = \pi J_0[(\beta^2-\alpha^2)^{\frac{1}{2}}]$$

10. Show that

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z \qquad J_{\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\frac{\sin z}{z} - \cos z\right)$$
$$J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z \qquad J_{-\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(-\frac{\cos z}{z} - \sin z\right)$$

11. Prove the *addition formula*

$$J_n(\alpha+\beta) = \sum_{m=-\infty}^{\infty} J_m(\alpha)J_{n-m}(\beta)$$

12. Prove that any positive integral power of z may be expanded in a series of Bessel functions, via the formula

$$z^n = 2^n\sum_{m=0}^{\infty}\frac{(n+2m)(m+n-1)!}{m!}J_{n+2m}(z)$$

13. Find the generating function for

$$\sum_{n=-\infty}^{\infty} I_n(z)t^n$$

and show that

$$\sum_{m=-\infty}^{\infty} I_{n-m}(z)J_m(z) = \begin{cases}\dfrac{z^n}{n!} & \text{for } n \geq 0\\ 0 & \text{for } n < 0\end{cases}$$

14. J. C. Miller[1] has shown that the recurrence formula for the I_n functions may be used "backward" so as to generate $I_p, I_{p-1}, I_{p-2}, \ldots$, where p is some chosen number. Let p and x be real and > 0. Then the procedure is to define a sequence of functions

$$\varphi_{r-1}(x) = \frac{2r}{x}\varphi_r(x) + \varphi_{r+1}(x)$$

starting with $\varphi_{p+n+1}(x) = 0$, $\varphi_{p+n}(x) = 1$.

Show that this process yields

$$\varphi_p(x) = \alpha\left[I_p(x) + (-1)^n\frac{I_{p+n+1}(x)}{K_{p+n+1}(x)}K_p(x)\right]$$
$$\varphi_{p-1}(x) = \alpha\left[I_{p-1}(x) + (-1)^{n+1}\frac{I_{p+n+1}(x)}{K_{p+n+1}(x)}K_{p-1}(x)\right]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and find a compact expression for the common multiplier α. From Chap. 6, the asymptotic behavior of $I_n(x)$ and $K_n(x)$ is such that the ratio $I_{p+n+1}(x)/K_{p+n+1}(x)$ can be made as small as desired by choosing n sufficiently large. The functions $\varphi_p(x), \varphi_{p-1}(x), \ldots$ are then effectively multiples of $I_p(x), I_{p-1}(x), \ldots$, and the multiplier α can be found either by comparing $\varphi_0(x)$ with tabulated values of $I_0(x)$ or by the use of some such formula as $1 = I_0(x) - 2I_2(x) + 2I_4(x) - \cdots$. The method is particularly effective for evaluating a sequence of $I_p(x)$ functions for large values of p and x; it is not necessary to use different formulas depending on the relative sizes of p and x. The method has also been used for complex values of argument and for other kinds of Bessel functions, as well as for associated Legendre functions and repeated error integrals.

[1] See "Mathematical Tables," vol. X, pt. II, Bessel Functions, British Association for the Advancement of Science, Cambridge University Press, New York, 1960. These tables contain a general discussion of computational methods for Bessel functions.

Contour Integral Representation

One often needs an integral representation for a function $w(z)$ defined by a differential equation. A technique that is frequently useful is to write

$$w(z) = \int_C K(z,t)f(t)\,dt$$

where $K(z,t)$, $f(t)$, and the contour C in the complex t plane are so chosen that the differential equation for $w(z)$ is satisfied.

In the case of Bessel functions, several representations of this form have been found. We begin with the choice

$$w(z) = z^\nu\int_C e^{izt}f(t)\,dt \tag{5-137}$$

where the factor z^ν can be anticipated because of its occurrence in the series