ENGINEERING MATHEMATICS 521

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Problem Set 9

Due date: December 9

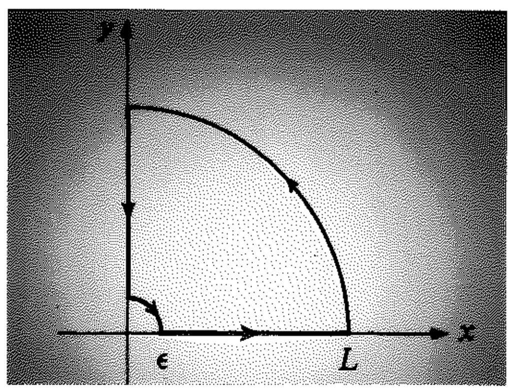
1. Prove the Poisson sum formula

$$\sqrt{a} \sum_{n=-\infty}^{+\infty} f(na) = \sqrt{b} \sum_{n=-\infty}^{+\infty} F(nb), \tag{1}$$

where F is the Fourier transform of f and $ab = 2\pi$.

2. For $0 < \alpha < 1$, compute the Fourier transform of the function

$$f(t) = |t|^{-\alpha}. \tag{2}$$



(a) First compute the Fourier transform of the function

$$f(t) = H(t)|t|^{-\alpha}, \tag{3}$$

where $H(t)$ is the Heavyside step function ($H(t) = 1$ for $t \geq 0$, and $H(t) = 0$ for $t < 0$). Make use of the contour shown in the figure below and express your answer in terms of the Gamma function $\Gamma(1-\alpha)$ and an appropriate function of the Fourier variable λ. (Check your answer against the result given by expression (7-15) in CKP for the special case when $\alpha = 1/2$.)

(b) Next, compute the Fourier transform of the function

$$f(t) = H(-t)|t|^{-\alpha}. \tag{4}$$

(c) Finally, put these two results together to derive the expression for the Fourier transform of (2).

3. p. 303 of Carrier, Krook and Pearson. Exercise 1 (a), (b) and (c).

1. Use Eq. (7-7) to derive the following transform pairs, and verify Eq. (7-8) in each case. Here t, λ, a are real, with $a > 0$.

(a) $f(t) = \dfrac{\sin at}{t}$, $F(\lambda) = \sqrt{\dfrac{\pi}{2}}$ for $|\lambda| < a$

$= 0$ for $|\lambda| > a$

(b) $f(t) = (a^2 + t^2)^{-1}$, $F(\lambda) = \sqrt{\dfrac{\pi}{2}}\dfrac{e^{-a|\lambda|}}{a}$

(c) $f(t) = \exp(-at^2)$, $F(\lambda) = \dfrac{1}{\sqrt{2a}}\exp(-\lambda^2/4a)$

(Observe the special case $a = \frac{1}{2}$.)

4. Consider the Dirichlet boundary value problem for the function $u(x, y)$ on the upper half plane:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < +\infty, y \geq 0, \tag{5}$$

$$u(x, 0) = f(x). \tag{6}$$

Make use of the Fourier transform in the variable x to show that the solution may be expressed as

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y f(\zeta)}{(\zeta - x)^2 + y^2}\, d\zeta. \tag{7}$$

Note: This is a version of the Poisson integral formula.

5. Make use of the Fourier transform to find the solution to the problem defined by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } -\infty < x < +\infty,\ t > 0, \tag{8}$$

where $c > 0$ is a constant, together with the initial conditions

$$u(x, 0) = h(x), \quad \text{and} \quad \frac{\partial u}{\partial t} = 0, \tag{9}$$

where the initial shape of the wave, as defined by h, is smooth and bounded in the sense that

$$\int_{-\infty}^{+\infty} |h(x)|\, dx < \infty.$$

For some appropriate choice of h, sketch the solution at $t = 0$, and at a later time $t = \tau$.