ENM 521 - MechE Math part 2

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Contents

Organization

These notes are live-TeXed and not guaranteed to be correct. Likewise, sometimes symbols, particularly m and n may be mixed up. If you find an error, please contact rebeccali at alum dot mit dot edu.

Organization

- Instructors: Pedro Ponte Castaneda (Towne 235)

- Office Hours: M 4:30-5:30 or by appointment

- TA: Chuanpeng Sun

- References:
  Functions of a complex variable: Theory and Technique, Carrier.
  PDE: theory and technique, Carrier.
  Introduction to complex variables and Applications, Churchill.
  Boundary Value Problems of Mathematical Physics, Stakgold.

1   Complex Numbers

Real numbers obey the usual operations. In particular, $\mathbb{R}$ is closed under addition and multiplication. However, there are some operations that are not possible with real numbers:, e.g. Find $x \in \mathbb{R}$ such that $x^2 = -1$.

    In addition, there are many mysterious properties of power series:

Remark. *$f(x) = \frac{1}{1+x^2}$ is a nice function. The Taylor series expansion is:*

$$f(x) \ 1 - x^2 + x^4 + ...$$

*   This diverges for $|x| \geq 1$. Why? Even nice functions of a real variables have divergence properties which are related to the fact that for when you generalize to the complex numbers, the function has a singularity at $x = i$.*

Definition 1.1. *Complex Numbers*
A complex number z is a pair of reals (x, y) :

$$z = (x, y) \in \mathbb{C}$$

where

$$Re(z) = x \in \mathbb{R}, \ Im(z) = y \in \mathbb{R}$$

    This obeys the following:

- Equality: $z_1 = (x_1, y_1) = (x_2, y_2) \iff x_1 = x_2, \ y_1 = y_2$

- Addition: $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

- Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$

Real numbers are a special case of a complex number: $(x, 0)$ "=" x. Furthermore, $(1, 0)$ is the identity for the reals.

Similarly, we can define $(0, 1)$ to be the complex number i such that:

$$(x, y) = (x, 0) + (0, y) = x + y(0, 1) = x + iy$$

In terms of i, we can write addition and multiplication:

- Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

- Multiplication: $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Definition 1.2. *Argand Representation*
We can think of our complex number as a vector in a plane with magnitude $r = |z|$ and angle $\theta = \arg z$. This is such that:

$$\tan \theta = \frac{x}{y}$$

We note that *theta* is defined up to an arbitrary multiple of 2π radians. Princple value of θ such that $-\pi < \theta \leq \pi$.
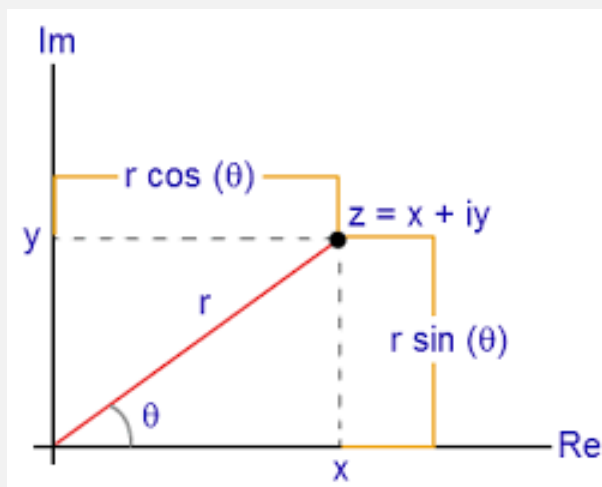

Figure 1: The Argand Plane.

Lemma 1. *The Triangle Inequality is:*

$$\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|$$

We can prove this using the argand representation.

Properties of complex numbers:

- Commutative

- Associative

- Distributive

- $z_1 - z_2 = z_1 + (-z_2)$.

- $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

- Complex Cojugate: $z^* = \bar{z} = x - iy = r(\cos\theta - i\sin\theta)$

- $zz^* = \|z\|^2$

- Division

Definition 1.3. *Roots of Unity*
Given a complex number z_0 characterized by r_0, θ_0, can we find a z such that $z^n = z_0$? Well, clearly:

$$r^n = r_0$$

$$\cos(n\theta) + i\sin(n\theta) = \cos(\theta_0) + i\sin(n\theta_0)$$

This, $\|z\| = r_0^{1/n}$, and $n\theta = \theta_0 + 2k\pi, \ k = 0, \pm 1, \pm 2...$) There are only n different solutions, meaning there are only n roots. If we let $z_0 = 0$, we can find the roots of unity:

$$\text{roots of unity} = z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

If we let $\omega = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, then $\sum_{k=0}^{n-1}\omega^k = 0$
  TODO: rebecca, this seems a bit wrong. See https://en.wikipedia.org/wiki/Root_of_unity for a better explanation.
  Essentially, geometrically, this means that the powers of ω sum to zero, as in Figure 2.
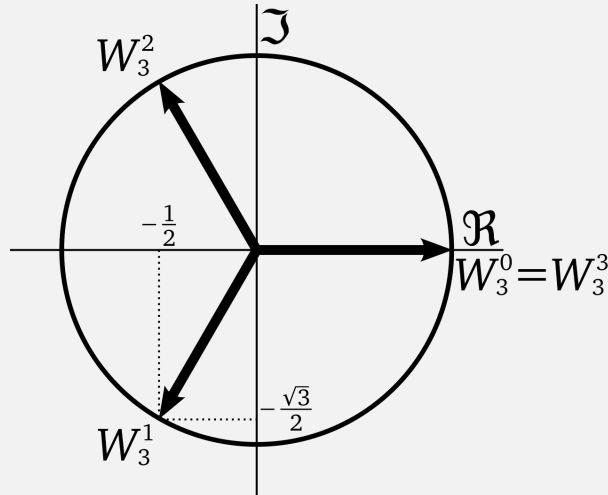
Figure 2: The third roots of unity sum to one - as do nth roots of unity. *Wikipedia.*

The complex numbers lack the notion of order, in the way the real numbers have order. In particular, the notion of ∞ is very different. For real numbers, we have $\pm\infty$, which serves as an upper and lower bound on $\mathbb{R}$. For complex numbers, we define a *point at infinity*. The point at infinity is the limit of what happens as r goes to infinity for $z = re^{i\theta}$, While it may seem like this goes to a different point for different θ, the point at infinity is in fact the same.

One way to imagine this is to consider the *stereographic projection* (Figure 3):

Figure 3: The stereographic projection. We can see that as the point picked to move away from the point at infinity (the north pole), becomes horizontal. The intersection point on the sphere will gradually move closer to the point at the north pole, hence the north pole is the point at infinity. *MathPages.*
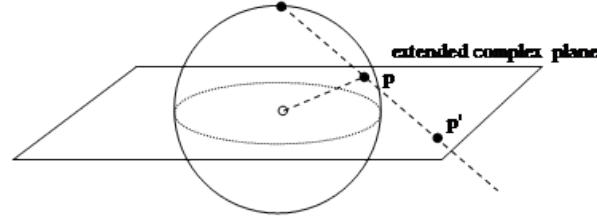
1.1 Real and imaginary pairs of Z

$$f(z) = u(x,y) + iv(x,y)$$

The continuity of $f(z)$ is continuous at $z = x_0 + iy_0 \iff u(x,y)$ and $v(x,y)$ are continuous at x_0, y_0

1.2 Differentiation of complex functions

Definition 1.4. *Differentiable*
A function f is differentiable at point z if

- $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = f'(z)$ exists

Note: $h \in \mathbb{C}$ and can approach zero along any path.

Example 1. $f(z) = z^2$ *is everywhere differentiable.*

Example 2. $f(z) = |z|$ *is continuous, but not differentiable at $z = 0$.*
If we look at the definition:
$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = \lim \frac{|h|}{h}$
but $|h|$ approaches 0 so the limit does not exist.

Theorem 1.1 (Differential of a Complex function). *1. If $f(z) = z^m$, m is integer, then $f'(z) = mz^{m-1}$*
If $f(z)$ and $g(z)$ are differentiable, then:

1. $(f+g)' = f' + g'$

2. $(fg)' = f'g + fg'$

3. $(f/g)' = \frac{f'g-fg'}{g^2}$, $g \neq 0$

Suppose g is differentiable at z and f is differentiable at g(z). If F(z)=f(g(z)), then:

$$F'(z) = f'(g(z))g'(z)$$

Suppose that $h \to 0$ along the real axis. Then:

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h}$$
$$= \frac{u(x+h,y) - u(x,y)}{h} + i\frac{v(x+h,y)v(x,y)}{h}$$

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If f is differentiable at $z = x+iy$, both limits (as $h \to 0$) must exist and:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Next, suppose that $h = ik$, $k \in \mathbb{R}$.

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x, y+k) + iv(x, y+k) - u(x,y) - iv(x,y)}{ik}$$
$$= \frac{u(x, y+k) - u(x,y)}{ik} + i\frac{v(x, y+k)v(x,y)}{ik}$$

Since f is differentiable, both limits as $k \to 0$ must exist. We can conclude that:

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

We conclude that:

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

*We can additionally conclude the **Cauchy-Riemann Conditions**:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Definition 1.5. *Cauchy-Riemann Equations*
For a function $f(z) = u(x,y) + iv(x,y)$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These furnish a necessary condition for differentiability at a point.

Remark. *The Cauchy-Riemann equations are necessary, but **not** sufficient for differentiability at a point.*

Example 3. *Consider the complex conjugate function:*

$$f(z) = z^* = \bar{z} = x - iy$$

We would like to determine if f is differentiable. If it is, it must satisfy Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = 1, \ \frac{\partial v}{\partial y} = -1$$

Already, we see that $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$. So $f(z)$ is not differentiable for any value of z.

Remark. *The Cauchy-Riemann equations can also be written as:*

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Theorem 1.2. *Let $f = u + iv$ be differentiable with complex partials at $z = re^{i\theta}$. Then:*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Definition 1.6. *Analyticity*
A function is **analytic (holomorphic)** at a point if it is differentiable everywhere in some neighborhood of the point. If the function is analytic for all values of $z \in \mathbb{C}$, then it is **entire**.

Note: Being differentiable in a neighborhood (analytic) is a stronger requirement than just differentiable.

Remark: If a function is analytic at a point, it is differentiable for all orders at that point.

Example 4. *Let $f(z) = |z|^2 = z\bar{z}$. This function is differentiable only at $z = 0$, but it is nowhere analytic.*

Theorem 1.3. *Let $f(z) = u(x,y) + iv(x,y)$ be defined in a domain D. Let $u(x,y)$ and $v(x,y)$ have continuous partials that satisfy Cauchy-Riemann equations for all points in D. Then, we can show that $f(z)$ is analytic in D. Thus, Cauchy-Riemann applied to the neighborhood is sufficient to show analyticity.*

Example 5. $f(z) = e^z = e^z = e^x(\cos y + i \sin y)$
Therefore, e^z is an entire function.

$$f'(z) = e^z$$

Note: if f is analytic of non-zero at a point z, then a branch may be chosen for which $\log(f(z))$ is also analytic in the neighborhood of z and $\frac{\mathrm{d}}{\mathrm{d}z}\log(f(z)) = \frac{f'(z)}{f(z)}$

1.3   Branch points and Branch Cuts

Explaining the epsilon method: `https://math.stackexchange.com/questions/2150639/branch-point-of-logz`
The easy way to do branch points is to remember that $log(z)$ and z^p where p is non integer both have branch points at $0, \infty$. So simply find when $z = 0, \infty$ for those functions to get the branch points. Then a branch cut must connect all branch points, but does not have to go in any particular way. For $log(z)$, the negative real line is often the branch cut that is picked.

Branch points are points where the function is non-analytic.

1.4   Harmonic functions

Later, we will see that analytic functions have derivatives of all orders. Recall:

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

We can take the derivative:

$$f''(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

And so:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Since $f(z) = u(x,y) + iv(x,y)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Definition 1.7. *Harmonic Function*
A continuous, real valued function u(x,y) defined on domain D is called **harmonic** if it has continuous first and second derivatives, satisfying Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Note: The real and imaginary parts of an analytic functions are harmonic functions.

Definition 1.8. *Harmonic Conjugate*
If $f(z) = u + iv$ is analytic, v is called the harmonic conjugate of u, and vice versa.

Remark. *Laplace's equation furnishes a **necessary** condition for a function to be real and imaginary parts of an analytic function.*

Example 6. $u(x, y) = x^2 + y$
   This cannot be the real part of any analytic function, since the Laplace equations are not satisfied.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0$$

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2 Sequences

Definition 2.1. *Sequence*
A sequence $\{z_n\}$ of complex numbers is an assignment of a complex number z_n to each positive integer n.

Definition 2.2. *Convergent sequence*
A sequence $\{z_n\}$ is said to have a limit z_0 or converge to z_0 which we write as

$$\lim_{n \to \infty} z_n = z_0$$

if for every $\epsilon > 0, \exists\, M \in \mathbb{Z}$, such that:

$$|z_n - z_0| < \epsilon, \ \forall n > M$$

This means for sufficiently large values of n, z_n is arbitrarily close to z_0.
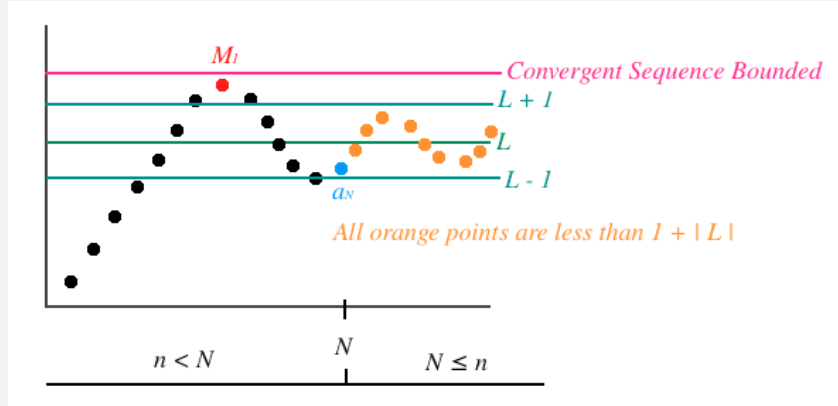
Figure 4: A convergent sequence example.

Alternative definition: Every neighborhood of z_0 contains all but a finite number of z_n's.

Example 7. $\{\frac{1}{m}\}$ *converges to 0. However,* $\{(-1)^m\}$ *does not converge.*

Theorem 2.1. *Let* $z_m = x_m + iy_m$ *be a sequence of complex numbers* $\{z_m\}$. *Then, this converges to* $z_0 = x_0 + iy_0$ *if and only if the real sequences* $\{x_m\}, \{y_m\}$ *converge to* x_0, y_0 *respectively.*

    Remark: *The properties of complex sequences can be derived from corresponding properties of real sequences. For example, the uniqueness of the limit (can only converge to one value).*

Theorem 2.2. *A convergent sequence is bounded.*
    Proof:
    If $\lim_{m \to \infty} z_m = z_0$, *then there must be a* $z_m \in N(z_0, 1)$, *where N means neighborhood, for* $m > N$, *let* $M = \max\{|z_1|, .., |z_N|\}$, *then*

$$|z_m| < M + |z_0| + 1, \forall m$$

    Note that the converse is not true. Counterexample:
    $\{1, 2, 1, 2, 1, 2\}$ *is bounded but does not converge.*

Definition 2.3. *Subsequence*
A subsequence of $\{z_m\}$ is a sequence $\{z_{m_k}\}$ whose terms are selected from the terms in an original sequence and arranged in the same order. Subsequences may converge even if sequences do not.

Example 8. *Given a sequence* $z_m = (-1)^m$, *a subsequence could be* $z_{m_k} = z_{2k}$, *or the even terms, which results in* $\{1, 1, ..1\}$ *which converges to 1. The subsequence* $z_{m_k} = z_{2k+1}$ *converges to -1.*

Theorem 2.3. *If a sequence* $\{z_m\}$ *converges to* z_0, *then every subsequence also converges to* z_0.

Theorem 2.4. *Every bounded sequence of complex numbers contains at least one convergent subsequence.*
    Proof: *Exercise.*

Definition 2.4. *Cauchy Sequence*
A sequence is Cauchy sequence if in the limit, the difference between the terms becomes arbitrarily small (each value becomes arbitrarily close to each other), which lets us not define the actual limit.

Formally, a sequence $\{z_m\}$ of complex numbers is a Cauchy sequence if for every $\epsilon > 0$, $\exists N \in \mathbb{Z}$ such that $|z_m - z_M| < \epsilon$, $\forall m, M > N$.

Theorem 2.5. *A sequence $\{z_m\}$ converges if and only if $\{z_m\}$ is a Cauchy sequence.*
Note: The two notions of convergence and Cauchy convergence are equivalent. Cauchy convergence is the most general test of convergence. It works even if the limit z_0 is not known.

3 Series

Definition 3.1. *Series*
Given a complex sequence $\{a_m\}$, we associate a new sequence defined by $s_m = \sum_{k=1}^{n} a_k$. Then, we say that $\sum_{k=1}^{\infty} a_k$ is a series. The series is said to converge or diverge according to whether the sequence $\{s_m\}$ converges or diverges. We call $\{s_m\}$ the **partial sum** of the series, and a_k the kth term.
Note: Every theorem about sequences can be rephrased to be a theorem about series, since series are sequences, and vice versa

Theorem 3.1. *Let $\{a_m\}$ be a sequence of complex numbers, with $a_m = \alpha_m + i\beta_m$ where $\{\alpha_m\}, \{\beta_m\}$ are sequences of real numbers. Then, by the earlier result, the complex series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} \alpha_k$ and $\sum_{k=1}^{\infty} \beta_k$ converge.*

Definition 3.2. *Cauchy Criterion.*
Let $s_m = \sum_{k=1}^{m} a_k$. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}$ for $m, M > N$:

$$|s_M - s_m| = \left| \sum_{k=m+1}^{M} a_k \right| < \epsilon$$

Alternatively, letting $M = m + p$:

$$|s_{m+p} - s_m| = \left| \sum_{k=m+1}^{m+p} a_k \right| < \epsilon, \forall p = 1, 2, 3, ..$$

This is the most general test for convergence of a series.
Note: Familiar properties of a series are immediate consequences of the Cauchy criterion.
Note: $a_k \to 0$ to converge. (set $p = 1$).

Definition 3.3. *Absolutely Convergence*
A series $\sum_{m=1}^{\infty} a_m$ is said to be absolutely convergent if $\sum_{m=1}^{\infty} |a_m|$ converges.
Note: Applying the triangle inequality:

$$\left| \sum_{k=m+1}^{m+p} a_k \right| \le \sum_{k=m+1}^{m+p} |a_k|$$

Applying this with the Cauchy criterion, we can deduce that absolute convergence of a series ensures its convergence.

Note: If $|a_m| \le |b_m|$, for every m, the convergence of $\sum_{m=1}^{\infty} |b_m|$ implies convergence of $\sum_{m=1}^{\infty} |a_m|$.

Theorem 3.2. *Suppose $a_m > 0 \; \forall m$ and that $\sum_{m=1}^{\infty} a_m$ diverges. If $s_m = \sum_{k=1}^{m} a_k$, then:*

1. *$\sum_{m=1}^{\infty} \frac{a_m}{s_m}$ also diverges.*

2. *$\sum_{m=1}^{\infty} \frac{a_m}{s_m^2}$ also converges. This is shown by Cauchy.*

Corollary 1. *The series $\sum_{m=1}^{\infty} \frac{1}{m}$ also diverges. But $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges.*

Theorem 3.3 (Geometric series). *Consider $s_m = \sum_{k=1}^{m} r^{k-1} = \frac{1-r^m}{1-r}$. This is a geometric series.*
  Proof:

$$(1-r)\sum_{k=1}^{m} r^{k-1} = \sum_{k=1}^{m} r^{k-1} - r^k$$
$$= 1 - r^m$$

Hence a series $\sum_{m=1}^{\infty} a_m$ converges absolutely if ther eexists a constant $r \in [0,1)$ and a real number M such that $|a_m| < Mr^m, \; m > M$

3.1 Limits of series and sequences

Definition 3.4. *Limit Superior*
Let $\{a_m\}$ be the real bounded sequence and let A be the set of subsequence limits of $\{a_m\}$. We define the limit sup of $\{a_m\}$ as the least upper bound of A:

$$\lim_{m \to \infty} \sup a_m = L.U.B.\{A\}$$

Note: If $\{a_m\}$ is unbounded above, then:

$$\lim_{m \to \infty} \sup a_m = +\infty$$

Note: If all but a finite number of a_m are less than any pre-assigned real number, we say that :

$$\lim_{m \to \infty} \sup a_m = -\infty$$

Definition 3.5. *Limit Inferior*
Let $\{a_m\}$ be the real bounded sequence and let A be the set of subsequence limits of $\{a_m\}$. We define the limit inf of $\{a_m\}$ as the greatest lower bound of A:

$$\lim_{m \to \infty} \inf a_m = G.LB.\{A\}$$

Note: in the extended reals ($\mathbb{R} \cup \pm\infty$), the limit superior and limit inferior of a real sequence always exists. This allows us to state and prove results without worrying about the existence of limits.

Theorem 3.4. *Let $\{a_m\}$ and $\{b_m\}$ be real valued sequences. Then,*

1. $\limsup_{m\to\infty}(a_n + b + m) \leq \limsup a_m + \limsup b_m$

2. $\liminf_{m\to\infty}(a_n + b + m) \leq \liminf a_m + \liminf b_m$

Remark. ***On the difference between limsup and sup*** $https://math.stackexchange.com/questions/1734087/difference-between-limsup-and-sup$

Theorem 3.5 (Root Test). *Let $\{a_m\}$ be a complex sequence and suppose that:*

$$\limsup_{m\to\infty} |a_m|^{1/m} = L$$

Then the series $\sum_{m=1}^{\infty} a_m$ converges absolutely if $L < 1$ and diverges if $L > 1$.
Proof:
Case 1:
If $L < 1$, choose r such that $L < r < 1$. For all but a finite number of m, we have that:

$$|a_m|^{1/m} < r$$
$$|a_m| < r^m$$

This is true since by the definition of L, this must be true.
The convergence of $\sum_{m=1}^{\infty} a_m$ now follows from convergence of $\sum_{m=1}^{\infty} r^m$, $r < 1$.
Case 2:
If $L > 1$, then $|a_m|^{1/m} > 1$ for infinitely many values of m. But then $|a_m| > 1$ infinitely often. Then $a_m \not\to 0$ and the series diverges.
Case 3:
If $L = 1$, the root test does not give any information.
Ex: *$\sum \frac{1}{m}$ diverges, but $\sum \frac{1}{m^2}$ converges.*
However, $\limsup_{m\to\infty} |\frac{1}{m}|^{1/m} = 1$. Similarly, $\limsup_{m\to\infty} |\frac{1}{m^2}| = 1$

3.2 Convergence

Definition 3.6. *Pointwise Convergence*
A sequence of functions $\{f_m(z)\}$ converges pointwise to a function f on a set E $(f_m \to f)$, if to each $z_0 \in E$, and $\epsilon > 0$, there corresponds and integer $N = N(\epsilon, z_0)$ for which $|f_m(z_0) - f(z_0)| < \epsilon$ whenever $m > N$.

Pointwise convergence implies $\lim_{m\to\infty} f_m(z_0) = f(z_0)$

Note: the integer N may, in general, vary with $z_0 \in E$. If one integer can be found that works for all $z_0 \in E$, then we have a special (stronger) type of convergence: **uniform convergence.**

Definition 3.7. *Uniform Convergence*
A sequence of functions $\{f_m\}$ converges uniformly to f on set E $(f_m \Rightarrow f)$, If for each $\epsilon > 0$ there corresponds an integer $N = N(\epsilon)$ such that for all $z \in E$, $|f_m(z_0) - f(z_0)| < \epsilon$ whenever $m > N$.

Example 9. *Consider $f_m(z) = \frac{1}{mz}$. This converges pointwise but not uniformly to $f(z) = 0$ in the set $0 < |z| < 1$. This is because uniform convergence would require existence of N such that $|\frac{1}{mz}| < \epsilon < 1$ valid for all z in the set. You can see this is not true in the case $z = \frac{1}{n}$. $f(\frac{1}{n})$ would be greater than 1, so yeah.*

The importance of uniform convergence is that it allows for the interchange of many limiting operations. which, in turn, compels the limit to retain many properties of sequences.

Theorem 3.6. *Suppose $\{f_m\}$ converges uniformly to f on E. If each f_m is continuous at point $z_0 \in E$, the limit of the function f is also continuous at z_0:*

$$\lim_{z \to z_0} \lim_{m \to \infty} f_m(z) = \lim_{m \to \infty} \lim_{z \to z_0} f_m(z)$$

The provides a necessary condition for uniform convergence.

Example 10. $f_m = \frac{1}{1_m z}$. *This converges pointwise to*

$$f = \begin{cases} 0 & if \ z \neq 0 \\ 1 & if \ z = 0 \end{cases}$$

Since f is not continuous, the convergence cannot be uniform in any region containing the point $z = 0$.

Note, this generalizes cauchy convergence.

3.3 Series of Complex Functions

Given a sequence of functions $\{f_m\}$ defined on the set E, we associate a new sequence $\{s_m(z)\}$ defined by

$$s_m(z) = \sum_{k=1}^{m} f_m(z)$$

For values for which $\lim_{m \to \infty} s_m(z)$ exists, we say the series converges.

$$f(z) = \lim_{m \to \infty} \sum_{k=1}^{m} f_k(z)$$

$$= \sum_{k=1}^{\infty} f_k(z)$$

Note, this is different than before. Now $f(z)$ is the limit of the sum, not just the limit.

If $\{s_m(z)\}$ converges uniformly on E, then the series $\sum_{k=1}^{\infty} f_k(z)$ is said to be **uniform** on E. Similarly, if $\sum_{k=1}^{\infty} f_k(z)$ is uniformly convergent on E, then $\sum |f_m(z)|$ converges.

Theorem 3.7. *The series $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly on a set E if and only if to each ϵ, there corresponds an integer $N = N(\epsilon)$, such that for all $z \in E$,*

$$\left| \sum_{k=m+1}^{m+p} f_k(z) \right| < \epsilon, \ m > N, \ p = 1, 2, 3..$$

By the triangle inequality, this also implies absolute convergence.

Theorem 3.8 (Weirstrass M-test). *Let M_m be a sequence of real numbers. Suppose that $|f_m(z)| < M_m, \forall z \in E, m \in \mathbb{Z}^+$. If $\sum_{m=1}^{\infty} M_m$ converges, then $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly and absolutely on E.*

Example 11. *The series $\sum_{m=1}^{\infty} z^m$ converges absolutely for $|z| < 1$. It converges uniformly for $|z| \leq r < 1$.*

Note: $\sum_{m=1}^{\infty} |z^m| = \sum_{m=1}^{\infty} |z|^m = \frac{|z|}{1-|z|}$, $|z| < 1$. This proves absolute convergence for $|z| < 1$.

Let $M_m = r^m$. Then $|z^m| \leq r^m$ for $|z| \leq r < 1$. But $\sum_{m=1}^{\infty} r^m$ converges for $r < 1$. So: $\sum_{m=1}^{\infty} z_m$ converges uniformly for $|z| \leq r < 1$. This follows from the Weirstrass M-test.

4 Power Series

Definition 4.1. *Power Series*
Let $f_m(z) = a_m(z-b)^m$ where a_m, b are complex. $\sum_{m=1}^{\infty} f_m(z) = \sum_{m=0}^{\infty} a_m(z-b)^m$ is a called a power series in $z - b$. WLOG, set $b = 0$.

Theorem 4.1. *Suppose a power series $\sum_{m=0}^{\infty} a_m z^m$ converges at point $z = z_0$. Then $\sum_{m=0}^{\infty} |a_m||z|^m$ converges for $|z| < |z_0|$.*

Basically, if it converges at z_0, then it converges in a disc centered at b with radius $z_0 - b$.

Proof: *Since $\sum_{m=0}^{\infty} a_m z_0^m$ converges, we have that $\lim_{m \to \infty} a_m z_0^m = 0$. Hence there is a constant M such that $|a_m z_0^m| < M$, $\forall m$. Also:*

$$|a_m||z|^m = \left| a_m z_0^m \left(\frac{z}{z_0} \right)^m \right| < M \left| \frac{z}{z_0} \right|^m$$

For $|z| < |z_0|$, the geometric series $\sum_{m=0}^{\infty} \left| \frac{z}{z_0} \right|^m$ converges. Thus:

$$\sum_{m=0}^{\infty} |a_m||z|^m \leq \sum_{m=0}^{\infty} \left| \frac{z}{z_0} \right|^m = \frac{M}{1 - \left| \frac{z}{z_0} \right|}, \quad |z| < |z_0|$$

Corollary 2. *If $\sum_{m=0}^{\infty} a_m z^m$ diverges at $z = z_0$, then $\sum_{m=0}^{\infty} a_m z^m$ diverges for $|z| > |z_0|$.*

Corollary 3. *Of $\sum_{m=0}^{\infty} a_m z^m$ converges for all real values of z, then the series also converges for all complex values.*

Theorem 4.2. *For every power series $\sum_{m=0}^{\infty} a_m z^m$, there correspondes a number $R \in [0, \infty]$ for which it satisfies:*

1. *It converges absolutely in $|z| < R$*

2. *It converges uniformly in $|z| \leq R_0 < R$*

3. *It diverges for $|z| > R$.*

Proof: *Let $s = \{r \text{ s.t. } \sum_{m=0}^{\infty} a_m z^m \text{ converges for } |z| < r\}$. Then $R = \begin{cases} L.U.B.s \text{ if } s \text{ is bounded} \\ \infty \text{ otherwise} \end{cases}$*

Note: R is the radius of convergence of a power series. It converges inside, diverges outside the circle $|z| = R$. On the other hand, it may converge at all some, or none of the points on the circle $|z| = R$.

Theorem 4.3. *The power series $\sum_{m=0}^{\infty} a_m z^m$ has a radius of convergence $R = \frac{1}{A}$, where $A = \limsup_{m \to \infty} |a_m|^{1/m}$. If $A = \infty$, $R = 0$. Likewise, if $A = 0$, $R = \infty$.*

Proof: *For any point z_0, $\limsup_{n \to \infty} |a_n z_0^n|^{1/n} = \left(\limsup_{n \to \infty} |a_n|^{1/n} \right)$. According to the root test, the series $\sum_{m=0}^{\infty} a_m z^m$ converges absolutely when $|z_0| A < 1$*

Note: *$f(z) = \sum_{m=0}^{\infty} a_m z^m$ is continuous at $z = z_0$ with $|z_0| < R$. what about differentiation?*

Theorem 4.4. *If a function $f(z)$ is the pointwise limit of a power series $\sum_{m=0}^{\infty} a_m z^m$ in $|z| < R$,*

See other notes of 9/23-9/25 for a description of what happened here.

5 Contours

Note: If f is analytic in domain including C, we can find F such that $F' = f$

Theorem 5.1 (Cauchy's Theorem). *If $f(z)$ is analytic in a Domain D and C is a closed contour lying on D, then*

$$\int_C f(z)dz = 0$$

Note: *the argument use to generalize the weaker version of Cauchy for multiply connected regions can be repeated for the stronger version*
Note: *this takes the work of parameterizing ugly contours, as will be seen in the next example.*

Example 12. *Evaluate $\int_C \frac{1}{z-z_0}dz$ for some arbitrary ugly contour Define a nice contour $C_1 : |z - z_0| = \epsilon$ which is a circle completely inside C, such that it does not divide the interior region.*
Now, $f(z) = \frac{1}{z-z_0}$ is analytic in a multiply connected region between C and C_1. Hence

$$\int_{C+C_1} f(z)dz = 0$$

Therefore, $\int_C \frac{1}{z-z_0}dz = -\int_{C_1} \frac{1}{z-z_0}dz$.
Since $C_1 : z(t) = z_0 + \epsilon e^{it}, \ t \in [0, 2\pi]$. Thus,

$$\int_C \frac{1}{z-z_0}dz = \int_0^{2\pi} \frac{z'(t)}{z(t)-z_0}dt = \int_0^{2\pi} \frac{i\epsilon e^{it}}{\epsilon e^{it}}dt = 2\pi i$$

Definition 5.1. *Laplace Transform*

$$F(s) = \int_0^\infty e^{-st}f(t)dt$$

The inversion of the Laplace transform involves an integral on the complex plane of a vertical contour to the right of all singularities.

$$f(t) = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds$$

There is a trick where you can close the contour to the right if there are no singularities to the right of the contour. Then, this causes $f(t)$ to have a zero contribution from the point at infinity. This only happens if $t < 0$. This is usually true as $f(t)$ is usually zero before $t = 0$.

Theorem 5.2 (Cauchy Integral Formula). *Let $f(z)$ be analytic in a simply connected domain containing the simple closed contour C. If $z_0 \in C$, then*
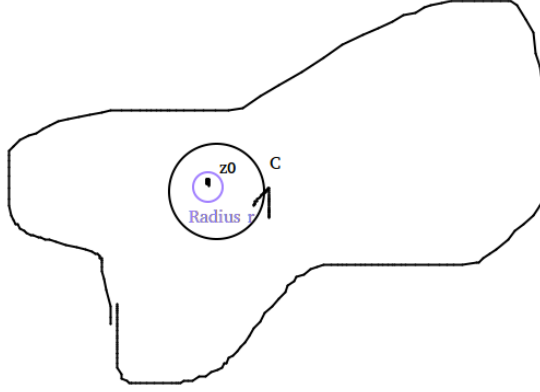
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

Figure 5: Contour for Cauchy integral Form on some domain.

Note: $f(z) = 1$ is the previous example.

Roughly speaking, if z_0 in the integral, then $\frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$. If it is outside C, the value of the integral is zero.

Proof:

Given $\epsilon > 0$, construct a circle $C_1 : |z - z_0| = r$. And small enough r such that $|f(z) - f(z_0)| < \epsilon$ for all z on C_1. By Cauchy's theorem:

$$\int_C \frac{f(z)}{z - z_0}dz = \int_{C_1} \frac{f(z)}{z - z_0}dz$$
$$= \int_{C_1} \frac{f(z_0)}{z - z_0}dz + \int_{C_1} \frac{f(z) - f(z_0)}{z - z_0}dz$$
$$= 2\pi i f(z_0) + \int_{C_1} \frac{f(z) - f(z_0)}{z - z_0}dz$$

However:

$$\left| \int_{C_1} \frac{f(z) - f(z_0)}{z - z_0}dz \right| \leq \int_{C_1} \frac{|f(z) - f(z_0)|}{|z - z_0|}dz$$
$$< \int_0^{2\pi} \frac{\epsilon}{r}|dz| = \epsilon 2\pi$$

This goes to zero as $\epsilon \to 0$. Thus, $\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

Note: *The theorem expresses the value of $f(z)$ at any point in C in terms of values of f on C (no real valued analog to this result).*

Theorem 5.3 (Cauchy Integral Formula for Derivative). *Let $f(z)$ be analytic on a simply connected domain containing a closed simple contour C. Then, $f(z)$ has derivatives of all orders at each point $z_0 \in C$. Moreover, the nth derivative at that point is:*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}dz$$

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Proof: *Use the previous result with the definition of the derivative. Assume that h is small enough such that $z_0 + h$ is inside C. Then:*

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i h} \left(\int_C \frac{f(z+h)}{z - z_0 - h} dz - \int_C \frac{f(z)}{z - z_0} dz \right)$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)(z - z_0 - h)} dz$$

As $h \to 0$, we see $f'(z_0) = \frac{1}{2\pi i h} \int_C \frac{f(z)}{(z - z_0)^2} dz$. However, it is hard to take the limit of an integral so this is hard, so we have to argue this.

Note: We know by hypothesis that $f(z)$ was differentiable at all z_0 inside C. The theorem demonstrates the existence of all derivatives of $f(z)$.

Note: If f is analytic at z_0, it is differentiable to all orders there.

Note: The Cauchy-integral formula is also valid for multiply connected domains.
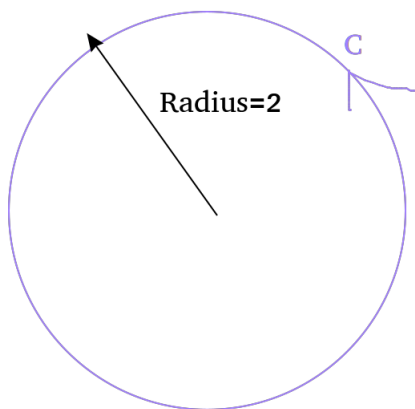

C

Radius=2

Figure 6:

Example 13.

$$\int_{|z|=2} \frac{z - 3\cos z}{(z - \pi/2)^2} dz$$

Let $f(z) = z - 3\cos z$. Then $f'(z) = 1 + 3\sin z$. By the Cauchy integral Formula,

$$\int_C \frac{f(z)}{(z - z_0)^2} dz = 2\pi i f'(\pi/2) = 8\pi i$$

"Almost like magic, isn't it?"

Definition 5.2. *Principal Value Integrals*

If z_0 is outside contour C, then $\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = 0$. If z_0 is on C, the integral does not exist. Instead, the integrand blows up "too strongly," much like the drink I'll be having tonight. However, we can re-interpret the integral so as to avoid the singularity, keeping in mind of course that such a reinterpretation

is a matter of definition. Given C and z_0 through C, and not having a corner there , we consider the modified contour C' excluding z_0, putting z_0 on the inside. By semi-circle of radius $\epsilon > 0$. Then:

$$\frac{1}{2\pi i} \int_{C'} \frac{f(z)}{z - z_0} dz = 0$$

As $\epsilon \to 0$, the part over the semicircle becomes

$$\frac{1}{2\pi i} \int_{\pi}^{0} \frac{f(z_0 + \epsilon e^{i\theta})}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = -\frac{f(z_0)}{2}$$

Using the previous theorems:

$$\frac{1}{2\pi i} \int_{C'} \frac{f(z)}{z - z_0} dz = PV \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz - \frac{f(z_0)}{2} = 0$$

And so:

$$PV \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz = \frac{f(z_0)}{2}$$

Note: This relies on putting the contour on the inside, but we could put it on the outside. Key feature, it doesn't matter! Wow. Now if z_0 is a corner point enclosing an **interior angle** of α radians, then the principal value is:

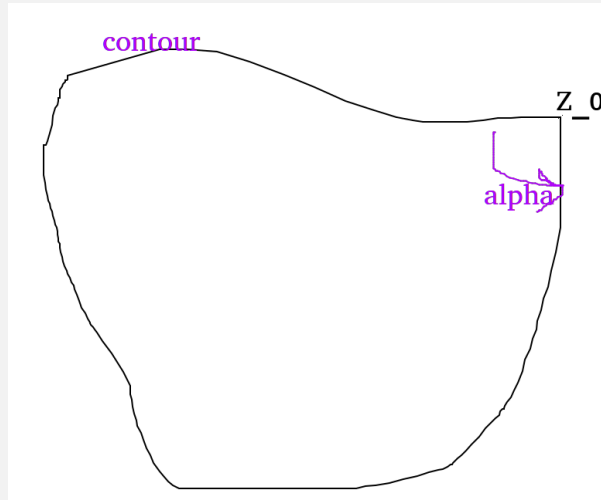$$PV \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz = \frac{f(z_0)}{2}$$

Figure 7:

Theorem 5.4 (Taylor's Theorem). *Let $f(z)$ be analytic in domain D whose boundary is C. If z_0 is a point in D, then $f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2!} f''(z_0)(z - z_0)^2 + ...$*

The series converges for $|z - z_0| < \delta$ where δ is the distance from z_0 to C.

Proof:

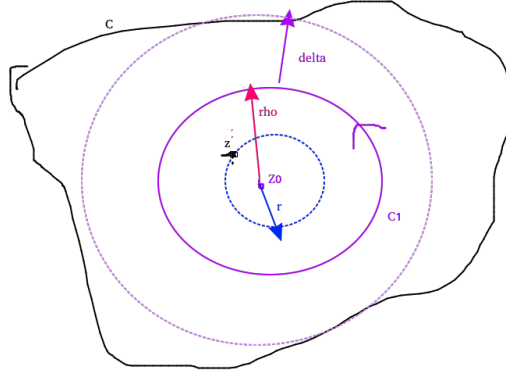Figure 8: δ is the largest circle that fits inside the contour C

Construct circle $C_1 : |\zeta - z_0| = \rho$ with $\rho < \delta$. Let z be any point inside C_1, then by C.T.M,

$$\rho(z) = \frac{1}{2\pi} \int_{C_1} \frac{\rho(\zeta)}{\zeta - z} d\zeta$$

.

We set $|z - z_0| = r$, then $r = |z - z_0| < |\zeta - z_0| = \rho$. Additionally,

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0 - (z - z_0)} \\
&= \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0} \left[1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0} \right)^2 + .. + \left(\frac{z - z_0}{\zeta - z_0} \right)^{n-1} + \sum_{k=m}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^k \right] \\
&= \frac{1}{\zeta - z_0} \left[1 + \frac{z - z_0}{\zeta - z_0} + .. + \left(\frac{z - z_0}{\zeta - z_0} \right)^{n-1} + \frac{\left(\frac{z - z_0}{\zeta - z_0} \right)^n}{1 - \frac{z - z_0}{\zeta - z_0}} \right]
\end{aligned}$$

Then

$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z_0} d\zeta + ... + \frac{(z - z_0)^{n-1}}{2\pi i} \int \frac{f(\zeta)}{\zeta - z_0} d\zeta + R_n$$

Where

$$R_n = \frac{1}{2\pi i} \int_{C_1} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \frac{f(\zeta)}{\zeta - z} d\zeta$$

By Cauchy Integral Formula, we get that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + ... + \frac{f^{(n-1)}(z_0)}{n - 1} (z - z_0)^{n-1} + R_n$$

However, we still need to show that $R_n \to 0$ as $n \to \infty$.
For ??? assume $|f(z)| \leq M$ on C_1. Then, We note that from triangle inequality

$$\frac{1}{|\zeta - z|} = \frac{1}{|\zeta - z - z_0 + z_0|} \leq \frac{1}{|\zeta - z_0| - |z - z_0|} = \frac{1}{\rho - r}$$

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Then we can conclude that

$$|R_n| \leq \frac{1}{2\pi i} \int_{C_1} \left| \frac{z - z_0}{\zeta - z_0} \right|^n \frac{|f(\zeta)|}{|\zeta - z|} |d\zeta|$$

$$\leq \frac{M}{2\pi i} \left(\frac{r}{\rho} \right)^n \int_{C_1} \frac{1}{|\zeta - z|} |d\zeta|$$

$$\leq \frac{M}{2\pi i} \left(\frac{r}{\rho} \right)^n \frac{1}{\rho - r} 2\pi \rho$$

Since $r/\rho < 1$ as $n \to \infty$, $|R_n| \to 0$

Note: *by the M-test, the Taylor series converges uniformly on compact subsets of disk $|z - z_0| < \delta$.*

What is this saying? We can chose any circle originating from z_0 with radius R, as long as it is less than δ which is the maximum radius to stay within the contour C, we will converge uniformly. (is this really true?)

We have seen before that a power series represents an analytic function inside its radius of convergence. The result in Taylor's theorem says essentially the converse. Thus, a function $f(z)$ is analytic at a point z_0 if and only if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in some disk $|z - z_0| \leq R$. a_n has a dual representation in both derivative and integral form:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{|z - z_0| = \delta} \frac{f(z)}{z - z_0} dz$$

Note: Taylor's theorem now fully justifies Maclaurin expansions such as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin z = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n-1)!}$$

Example 14. *Consider $\sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$. It converges within the disk of radius 1 about the origin. Can we define the function outside of this disk? This is called analytic continuation. We might expand this function around a different point! We see that on the circle $r = 1$, the function blows up. How might we do this when we do not know the non-power series form? Contour integrals. More on that later.*

Theorem 5.5. *Let $\{f_n(z)\}$ be a sequence of functions continuous on contour C. Suppose that the series $\{f_n(z)\}$ converges uniformly to some function $f(z)$ on the contour. Then, we can show that*

$$\lim_{n \to \infty} \int_C f_n(z) dz = \int_C \lim_{n \to \infty} f_n(z) dz$$

$$= \int_C f(z) dz$$

Essentially, this means we can interchange limits, but only for uniformly converging functions.
We can also do this for series:

$$\sum \int_C f_n(z) dz = \int_C \sum f_n(z) dz$$

$$= \int_C f(z) dz$$

We might expect a function $f(z)$ to be analytic at a point z_0 if $\int_C f(z)dz = 0$ along every simple closed contour in which z_0 is an interior point. However, this is **NOT TRUE!**

Counterexample: $f(z) = \frac{1}{z^2}$.

However, if we add the hypothesis that f is continuous, this is true:

Theorem 5.6 (Morera). *Let $f(z)$ be continuous on domain D. If $\int_C f(z)dz = 0$ along every simple closed contour C contained in D, then $f(z)$ must be analytic in D.*

Proof:

Fix z_0 in D, and $F(z) = \int_{z_0}^z f(\zeta)d\zeta$. We can consider the derivative of F. We can use the continuity of f, so it follows that F is analytic.

Corollary 4. *Let $f(z)$ be a function on a simply connected domain D. if z_0 is a point in D, then its integral, $F(z) = \int_{z_0}^z f(\zeta)d\zeta$ is also analytic in D.*

Theorem 5.7. *Let $\{f_n(z)\}$ be a sequence of analytic functions converging uniformly to a function $f(z)$ on all compact (closed and bounded) subsets of a domain D. Then, $f(z)$ is analytic in D.*

Corollary 5. *Given the same hypothesis as above, $\sum_{k=0}^{\infty} f_k(z)$ where $f_k(z)$ converges uniformly to $f(z)$ is analytic in D.*

Corollary 6. $f'(z) = \sum_{n=0}^{\infty} f_n'(z)$

In the complex case, the term by term differentiation of a sequence of analytic functions uniformly converging on compact subsets of the domain may be used to get the derivative of the sum. This is not true in the real case.

Example 15. *We cannot do term by term differentiation of this real function: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$. Its derivative cannot be computed term by term.*

If f is entire, it has a power series representation of

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
$$= \frac{f^{(n)}(0)z^n}{n!}$$

which is valid for all z. Then,

1. $f'(z) = \sum_{n=0}^{\infty} na_n z^n$
2. $F(z) = \int_0^z f(\zeta)d\zeta = \int_0^z \sum_{n=0}^{\infty} a_n \zeta^n d\zeta = \sum_{n=0}^{\infty} a_n \int_0^z \zeta^n d\zeta = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n \zeta^{n+1}$

Example 16. *Find the Maclaurin expansion of $f(z) = \log(z+1)$ valid for $|z| < 1$. We know that $f'(z) = \frac{1}{1+z}$. This is just the same as a geometric series: $f'(z) = \sum_{n=0}^{\infty} (-1)^n z^n$*

Hence:

$$f(z) = \log(1+z)$$
$$= \int_0^z f'(\zeta)d\zeta + f(0)$$
$$= \sum_{n=0}^{\infty} \int_0^z (-1)^n \zeta^n d\zeta + 0$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n \zeta^{n+1}}{n+1}$$

6   Polynomial and Transcendental Functions

Theorem 6.1 (Cauchy's Inequality). *Suppose $f(z)$ is analytic inside and on circle C having z_0 and radius r. If $|f(z)| < M$ on C then:*

$$f^{(m)}(z) \leq \frac{MM!}{r^m}$$

Proof: *By Cauchy integral formula:*

$$\left| f^{(m)}(z_0) \right| \leq \frac{m!}{z\pi} \int_C \frac{|f(z)|}{|z - z_0|^{m+1}} |dz|$$
$$\leq \frac{m!}{z\pi} \frac{M(2\pi r)}{r^{m+1}}$$

Theorem 6.2 (Liouville's theorem). *A bounded entire function must be constant.*
   Proof: *Suppose $f(z)$ is entire with $|f(z)| \leq M$ for all z. It is sufficient to show that $f'(z) = 0$ given complex z_0. We have by Cauchy's inequality:*

$$f'(z_0) \leq \frac{M}{r}, \ r \in \mathbb{R}^+$$

   As $r \to \infty$, we deduce that $f'(z_0) = 0$. Since z_0 was arbitrary, the result follows.
   Note: *There is no real variable analog. For example $f(x) = \sin(x)$ is everywhere bounded, but not constant.*
   Note:
   Liouviolle's implies for a non-constant entire function $f(z)$, there is a set $\{z_m\}$ such that $f(z_m) \to \infty$.

Theorem 6.3. *Suppose that $f(z)$ is an entire function and that $|f(z)| \leq Mr^\lambda$, and that $|z| = r \leq r_0$ for some non-negative λ. Then $f(z)$ is a polynomial of degree at least λ.*
   Proof:
   Let $f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^\infty a_n z^n$. By Cauchy's inequality, on circle $|z| = r$, we have that:

$$|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right|$$
$$\leq \frac{Mr^\lambda}{r^m}$$

   Letting $r \to \infty$, we see that $a_n = 0$ whenever $m > \lambda$. Hence f is a polynomial of a degree no more than λ.
   Note:
   What if $\lambda = 0$? The function is constant. This is a special case.

Definition 6.1. *Transcendental Function*
A **transcendental** entire function is one that is not a polynomial.

   Note:
   Letting $M(r, f) = \max_{|z|=r} |f(z)|$. The theorem says that for a transcendental function, $M(r, f)$ frows faster than any ? of r. However, this does not necessarily imply that $f(z) \to \infty$ along any path going to ∞.

Example 17. $f(z) = e^z$. Then $M(r, f) = e^r$. But $e^z \to 0$ as $z \to \infty$ along negative real axis.

Theorem 6.4 (Picard). *A non-constant entire function assumes each complex value with one possible exception.*

Theorem 6.5. *Suppose that $P(z) = a_0 + a_1 z + .. + z_m z^m$, for $a_m \neq 0$. Then for $|z| = r$ sufficiently large,*

$$\frac{|a_m|}{z} r^m \leq |P(z)| \leq \frac{3|a_m|}{2} r^m$$

Note: *Along its "rest" path, transcendental functions approach ∞ more rapidly than any polynomial. However, polynomials may go to ∞ along any path. Transcendental functions grow more rapidly than polynomials, but polynomials grow more uniformly.*

Proof: *Exercise.*

Complex analysis leads to important results in algebra.

Theorem 6.6 (Fundamental theorem of Algebra). *Every non-constant polynomial has at least one zero.*

Proof: *Assume $P(z) \neq 0$, such that $\frac{1}{P(z)}$ is entire. Use previous to results to obtain a contradiction.*

Corollary 7. *Every polynomial has exactly n roots.*

Corollary 8. *Every polynomial of degree n assumes eat complex number explicitly n times.*

Theorem 6.7. *Suppose $f(z)$ is analytic in domain F. $\{z_n\}$ is a sequence of distinct points converging to point $z_0 \in D$. If $f(z_n) = 0 \forall n$, then $f(z) = 0$, $z \in D$.*

Proof: *Consider a disk $z - z_0 < R$. Then:*

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

Since $f(z)$ is continuous t z_0, it follows that $f(z_n) \to f(z_0)$, $z_n \to z_0$. Therefore:

$$\lim_{n \to \infty} f(z_n) = 0 = f(z_0) = a_0$$

$$\implies a_0 = 0$$

Hence:

$$f(z) = (z - z_0) \left(a_1 + \sum_{k=2}^{\infty} a_k (z - z_0)^k \right)$$

Setting $z = z_n$ and dividing by $z_n - z_0$:

$$\lim \frac{f(z_n)}{z_n - z_0} = 0 = a_1$$

$$\implies a_1 = 0$$

Repeating we get the result that $f(z) = 0$.

Corollary 9. *Suppose $f(z)$ is analytic at $z = z_0$, with $f(z_0) = 0$. Then, either $f(z) = 0$ in some neighborhood of z_0, or there exists a number f such that $f(z) \neq 0$ in $\delta < |z - z_0| < r$.*

Note: *No real-valued analog/*

Theorem 6.8 (Identity Theorem). *Suppose $\{z_n\}$ is a sequence of points with a limit point in domain D. If $f(z)$ and $g(z)$ are analytic in D, where $f(z_n) = g(z_n) \forall n$, then $f = g for all z \in D$.*

Proof: *Let $h(z) = f(z) - g(z)$. Recall the mean value theorem for real valued functions:*

$$\frac{1}{b-a} \int_a^b f(x) dx = \frac{F(b) - F(a)}{b-a} = F'\zeta$$

$F' = f$, $a < \zeta < b$.

Theorem 6.9 (Gauss Mean value). *Suppose $f(z)$ is analytic in the closed disk $|z - z_0| \leq r$. Then:*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta} d\theta)$$

Proof:

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz$$

$$|z - z_0| = r$$

We set $z = z_0 + re^{i\theta}$ and obtain the result.
Note: *If $f(z) = c$ is constant, $f(z_0) = c$.*

In fact, for non-constant f, there must be some points on the circle for which $|f(z)| < |f(z_0)|$

Theorem 6.10 (Maximum Modulus). *If $f(z)$ is analytic on domain D, then $|f(z)|$ cannot obtain a max on D unless $f(z)$ is constant.*
Proof: *First, we make use of the Gauss Mean value theorem.*

Theorem 6.11 (Maximum modulus theorem (Strong)). *If $f(z)$ is analytic in a bounded domain D and continuous in its closure $\bar{D}$ (domain plus boundary), then $|f(z)|$ obtain a maximum on its boundary. Additionally, $|f(z)|$ does not attain a maximum at any interior point, unless $f(z)$ is a constant.*
Note: *The domain need not be simply connected. The domain can be taken to be any small neighborhood surrounding a given point. It follows that a non-constant analytic function cannot attain even a local maximum at any point inside the region of analyticity.*

6.1   Classification of Singularities

Definition 6.2. *Singularity*
A single-valued function is said to have a singularity at a point if the function is not analytic at that point. Furthermore, if the function is analytic in some deleted neighborhood (neighborhood not including the point) of the point, then the singularity is said to be isolated. The behavior of the function near the isolated singularity z_0 can be :

1. $f(z)$ may be bounded in a deleted neighborhood of z_0. As an example, take $f(z) = \frac{num}{den}$, $z \neq 0$. $f(z)$ is bounded in the neighborhood of $z_0 = 0$. In fact, $f(z)$ can be made analytic at $z = 0$ by defining $f(0) = 1$.

2. $f(z)$ may approach ∞ as $z \to z_0$. Example being $f(z) = \frac{1}{z}$, $z \neq 0$. There is a singularity at $z_0 = 0$ and $\lim_{z \to 0} f(z) = \infty$.

3. $f(z)$ satisfies neither (1) nor (2). As an example, consider $f(z) = e^{1/z}, z \neq 0$.

Theorem 6.12 (Riemann's Theorem). *If $f(z)$ has an isolated singularity at z_0 and is bounded in some deleted neighborhood of z_0, then $f(z)$ can be defined at z_0 in such a way that it is analytic at z_0.*
Proof: *For some $R > 0$, $f(z)$ is analytic in a punctured disk $0 < |z - z_0| < R$. Given a point z in the punctured disk, choose $r > 0$ such that $r < |z_1 - z_0| < R$ The function is analytic in such annulus bounded by the circle C and C_1 (Figure 9).*
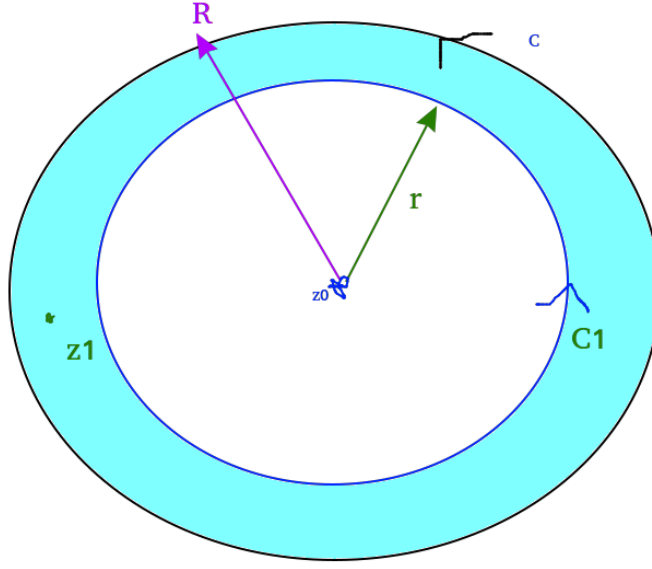
Figure 9: Riemann's theorem proof graphics. In the blue annulus, $|f(z)| \leq M$.

Hence, by Cauchy,

$$f(z_1) = \frac{1}{2\pi} \int_C \frac{f(\zeta)}{\zeta - z_1} d\zeta - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta - z_1} d\zeta$$

Note that the value of $f(z_1)$ is independent of r. We can say that:

$$\left| \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z_1} d\zeta \right| \leq \frac{M 2\pi r}{2\pi(|z_1 - z_0| - r)}$$

The RHS goes to zero as $r \to \infty$. And so:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_1} d\zeta$$

But the right hand side is analytic inside C. So define:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta$$

Thus, the function $f(z)$ becomes analytic in the whole disk $|z - z_0| < R$.

Corollary 10. *If $f(z)$ has an isolated singularity at $z = 0$ and is bounded in some neighborhood of z_0, then $\lim_{z \to z_0} f(z)$ exists. This singularity is a **removable singularity.***

Example 18. *An example of a removable singularity: let $f(z) = \frac{\sin z}{z}$ Since $\lim_{z \to z_0} f(z) = 1$, we can define $f(z)$ to be 1 at $z = 0$ and the function becomes entire.*
> ***Note:***

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Definition 6.3. *Pole*
If $f(z)$ is defined in the deleted neighborhood of z_0, and a positive integer can be found such that:

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq \begin{cases} 0 \\ \infty \end{cases}$$

Then we say $f(z)$ has a pole of order n at z_0. When $n = 1$, the pole is simple.

Example 19. $f(z) = \frac{1}{z}$ *has a simple pole at* $z = 0$.

Theorem 6.13. *If $f(z)$ has an isolated singularity at $z = z_0$ and $f(z) \to \infty$ as $z \to z_0$, then $f(z)$ has a pole at $z = z_0$.*
Proof: *If $f(z) \to \infty$ as $z \to z_0$, then $g(z) = 1/f(z)$ is bounded in some deleted neighborhood of z_0. By Riemann's theorem, $g(z)$ has a removable singularity at z_0 and we can write:*

$$g(z) = \frac{1}{f(z)} = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ..$$

It follows that $\lim_{z \to z_0} g(z) = a_0$
Now, the coefficients of $g(z)$ cannot all be zero. Suppose that z_k is the first coefficient to be nonzero. Then,

$$\frac{1}{f(z)(z - z_0)^k} = a_k + a_{k+1}(z - z_0) + a_{k+2}(z - z_0)^2$$

Hence, $\lim_{z \to z_0} (z - z_0)^n f(z) = 1/a_k$ and $f(z)$ has a pole of the order k at z_0.

Corollary 11. *If $f(z)$ has a pole at $z = z_0$, then $f(z)$ may be expressed as $f(z) = \sum_{n=-k}^{\infty} b_n(z - z_0)^n$ where k is the order of the pole.*

Definition 6.4. *Essential Singularity*
An isolated singularity that is neither removable nor a pole is called an isolated essential singularity.

Theorem 6.14. *If $f(z)$ has an isolated essential singularity $z = z_0$, then $f(z)$ comes arbitrarily close to every complex value in each deleted neighborhood of z_0.*
Proof: *Assume, that for some complex a, there is some $\epsilon > 0$ such that $|f(z) - a| \leq \epsilon$ for all z in $0|z - z_0| < \delta$.*
Set $g(z) = \frac{1}{f(z)-a}$. Steps show how singularity must either be removable or pole.

Theorem 6.15 (Generalization of previous theorem, Picard). *In the neighborhood of isolated essential singularities, a function assumes each complex value, with one possible exception, infinitely often!*

Example 20. $f(z) = e^{1/z}$ *has an isolated essential singularity at $z = 0$ and never assumes a value of zero. Picard says that it must assume every other value in the neighborhood of the origin infinitely often.*
Look for values of z close to zero such that $e^{1/z} = i$. First, we note that $|e^{1/z}| = |i| = 1$. This is only the case if the real part of $z = 0$. Now, set $z = iy$, $y \in \mathbb{R}$, then $e^{1/(iy)} = e^{i/y} = \cos(1/y) - i\sin(1/y) = i$. We see this is true when $1/y = -\pi/2 + 2n\pi$, $n \in \mathbb{Z}$.
Setting $z_n = \frac{1}{i(-\pi/2+2n\pi)}$, we have that $e^{1/z_n} = i$ for each n. Ovserve that there are infinitely many z_n in the neighborhood of $z_0 = 0$.

Example 21. $f(z) = z^2 + 1$. *This function has a pole of order 2 at $z = \infty$. This is true since $f(\frac{1}{z}) = \frac{1}{z^2} + 1$ has a double pole at $z = 0$.*

$g(z) = e^z$ *has an isolated essential singularity at $z = \infty$*

6.2 Entire Functions

Note: If $f(z) = \sum_{n=0}^{k} a_n z^n$ which is a polynomial of degree k, and as $z \to \infty$, the function blows up. If we look at

$$f(\frac{1}{z}) = \frac{1}{z^k} \left(a_k \sum_{n=0}^{k-1} a_n z^{k+n} \right)$$

We see that since these powers are positive on the RHS, there is a pole of order k at $z = 0$.

Hence, $f(z)$ has a pole of order k at $z = \infty$. This is consistent with an earlier result. $f(z) \to \infty$ as $z \to \infty$.

Note: A function $f(z)$ that has a pole of order k at $z = \infty$ if and only if

$$\lim_{z \to \infty} \frac{f(z)}{z^k} = A \neq \begin{cases} 0 \\ \infty \end{cases}$$

Also by an earlier result, for entire functions, if the above condition is satisfied, then $f(z)$ is a polynomial of degree k. That is, transcendental entire functions must have essential singularities at the point at infinity.

7 Laurent Series and Residues

7.1 Laurent Series

Laurent series are a "generalization" of power series. We have seen that if $f(z)$ has a pole of order k at $z = z_0$, then we find that $f(z)$ has a representation of the form $f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$, $(a_{-k} \neq 0)$.

This is valid in some deleted neighborhood about z_0.

Next, consider the series:

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m$$

First note that

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n} = \sum_{m=-\infty}^{-1} a_m (z - z_0)^m$$

This represents a function $f_1(z)$ which is analytic for some $|z - z_0| > R_1$. This is because $\sum_{m=1}^{\infty} b_m (z - z_0)^{-m}$ is analytic for $|z - z_0|^{-1} < R = \frac{1}{R_1}$.

Next, suppose $f_2(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic for $|z - z_0| > R_2$. Now we have broken up our function into circles bounded by R_1, R_2. So now we can say that if it so happens that $R_2 > R_1$, then we get an annulus, and $f(z) = f_1(z) + f_2(z)$ in the annulus $R_1 < |z - z_0| < R_2$.

Definition 7.1. *Laurent Series*
Suppose $f(z)$ is analytic in the annulus $R_1 < |z - z_0| < R_2$. The Laurent series of a function $f(z)$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where a_n and z_0 are constants, with a_n defined by a line integral that generalizes Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

C is any simple closed contour inside of the annulus that makes a complete counterclockwise revolution about z_0.

Proof: Much like Taylor's theorem.

$$f(z) = \frac{1}{2\pi i} \oint_{R_1 = |z - z_0|} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta - \frac{1}{2\pi i} \oint_{R_2 = |z - z_0|} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$

Note: A series of positive powers of $z - z_0$ converges inside $R_2 = |z - z_0|$. It is called the *analytic part*. The series of negative powers converges outside $R_1 = |z - z_0|$ and is called the *principal part*. Both are analytic, just have funny names.

Remark. *If $R_1 = 0$, the point z_0 is an isolated singularity. There are three cases:*

1. *The principal part vanishes if and only if the singularity is removable.*

2. *The principal part has finitely many terms if and only if the singularity at z_0 is a pole.*

3. *If it is neither of the other cases, the principal part has an infinite number of terms if and only if the singularity at z_0 is an essential singularity.*

Note: The Laurent series representation is unique.

Example 22 (Laurent Series expansion). *Let $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$. Then, f has simple poles at $z = \pm i$. We note that $f(z)$ is analytic for $0 < |z - i| < 2$. This is an annulus with outer radius almost 2 expanded around $z = i$. And we can expand about $z = i$:*

$$\frac{1}{z + i} = \frac{1}{2i[1 + \frac{z-i}{2i}]}$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z - i}{2i} \right)^n$$

Hence:

$$f(z) = \frac{1}{2i(z - i)} \sum_{n=0}^{\infty} \left(\frac{-1}{2i} \right)^n (z - i)^n, \ \forall z \ s.t. \ 0 < |z - i| < 2$$

$$= \sum_{n=-1}^{\infty} \left(\frac{i}{2} \right)^{n+2} (z - i)^n, \ \forall z \ s.t. \ 0 < |z - i| < 2$$

This converges for $0 < |z - i| < 2$. We can also expand about $z = -i$ to get the Laurent series:

$$f(z) = -\sum_{n=-1}^{\infty} \left(\frac{-i}{2}\right)^{n+2} (z+i)^n, \ \forall z \ s.t. \ 0 < |z+i| < 2$$

7.2   Residues

Definition 7.2. *Residue*
The coefficient a_{-1} is a special case, or the integral of the function in the Laurent series. It is called the residue of $f(z)$ at $z = z_0$. It is such that

$$\frac{1}{2\pi i} \int_C f(\zeta)d\zeta = a_{-1}$$

Theorem 7.1 (Residue Theorem). *Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, .., z_n$ inside C. Let the residues at these points be $\alpha_1, \alpha_2, .., \alpha_n$ respectively. Then we can compute the following:*

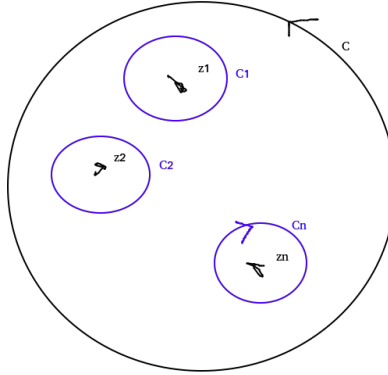$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} \alpha_i$$

Proof:

Figure 10: Contours for proof

By the Cauchy Integral Formula,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + .. + \int_{C_n} f(z)dz$$

We note the terms are residues, so

$$\int_C f(z)dz = \sum_{k=1}^{n} \alpha_k$$

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Note: The Cauchy Integral formula is the special case of the residue theorem.

1. Suppose $f(z)$ is analytic inside of a simple closed contour C containing z_0. Then :

$$g(z) = \frac{f(z)}{z - z_0}$$

$g(z)$ has a simple pole at $z = z_0$ provided $f(z)$ is analytic and $f(z_0) \neq 0$. The residue of $g(z)$ at z_0 is:

$$\lim_{z \to z_0} (z - z_0)g(z) = f(z_0)$$

Therefore, $\int_C g(z)dz = 2\pi i f(z_0)$. Thus concludes the Cauchy Integral Formula.

Note: Suppose f has a pole of order k at z_0. To find the residue a_{-1} in terms of f, consider:

$$(z - z_0)^k f(z) = a_{-k} + ... + a_{-1}(z - z_0)^{k-1} + g(z)(z - z_0)^k$$

where g is analytic. We differentiate this $k - 1$ times and evaluating $z = z_0$, we find that:

$$a_{-1} = \frac{1}{(k - 1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}(z - z_0)^k f(z)$$

Example 23. *Suppose:*

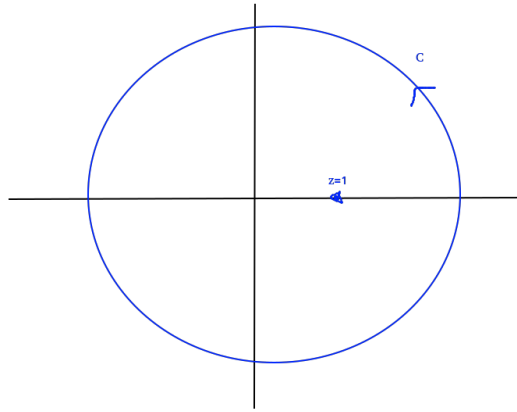$$\int_C \frac{z^4 - z^3 + 17z + 2}{(z - 1)^3} dz$$

.

Figure 11: Contour in example.

We would like to find the residue of this.
Method 1:
Set $f(z) = \frac{z^4 - z^3 + 17z + 2}{(z-1)^3}$. Then:

$$a_{-1} = \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2}(z - 1)^3 f(z)$$
$$= 3$$

By the residue theorem:

$$\int_C f(z)dz = 2\pi i a_{-1} = 6\pi i$$

Method 2:

Let $g(z) = z^4 - z^3 + 17z + 2$. *By the Cauchy Integral Formula:*

$$g''(z) = \frac{2!}{2\pi i} \int_C \frac{g(z)}{(z-1)^3} dz$$

Theorem 7.2. *Let $g(z)$ and $h(z)$ be analytic at $z = z_0$, such that $g(z_0) \neq 0$, $h(z_0) \neq 0$ and $h'(z_0) = 0$. This implies that $f(z) = g(z)/h(z)$ has a simple pole with residue $g(z_0)/h'(z_0)$ at $z = z_0$.*

Note that since g, h are analytic at $z = z_0$, they have Taylor series expansions of the form:

$$g(z) = \sum_{n=0}^{\infty} \gamma_n (z - z_0)^n$$

$$h(z) = \sum_{n=0}^{\infty} \delta_n (z - z_0)^n$$

By hypothesis, we have that $\gamma_0 \neq 0$. But $\delta_0 = 0$, and $g(z_0) = \gamma_0$ and $h'(z_0) = \delta_1$. Therefore,

$$f(z) = \frac{g(z)}{h(z)} = \frac{\sum \gamma_n (z - z_0)^n}{\sum \delta_n (z - z_0)^n}$$

$$= \frac{\gamma_0}{\delta_1 (z - z_0)} \frac{1 + \sum \frac{\gamma_n}{\gamma_0} (z - z_0)^n}{\sum \frac{\delta_n}{\delta_1} (z - z_0)^n}$$

Then $f(z)$ has a simple pole at $z = z_0$, which gives us the result that

$$a_{-1} = \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \frac{\gamma_0}{\delta_1}$$

7.3   Non-isolated Singularities

So far, everything we have done only applies to *isolated singularities*. What about non-isolated singularities? Consider the function:

$$f(z) = \frac{1}{\sin(1/z)}$$

This has a singularity at $z = 0$ since $\lim_{z \to 0} f(z)$ does not exist. Furthermore, the points $z_n = \frac{1}{n\pi}$ are also singularities of $f(z)$. Since $z_n \to 0$ as $n \to \infty$, each deleted neighborhood of the origin contains a singularity of $f(z)$.

Consider an additional example of a branch point. Let $f(z) = \log z$. This has a branch point at $z = 0$. From this, we can define a *branch* of f such as the *principal branch*, Log z, which is the single-valued in the cut plane (branch cut on the negative real axis.) If we exclude the origin, is Log z analytic? Is it analytic in a deleted neighborhood around the branch point? No, because it is discontinuous on the branch cut (negative real axis). The origin is *not* an isolated singularity. A branch point is *never* an isolated singularity.

8   Real Integrals

8.1   Evaluation of Real Integrals

Example 24. *Evaluate*

$$\int_0^\infty \frac{\sin x}{x} dx$$

This is kind of hard, so let's just exploit the $e^{iz} = \sin z + i \cos z$. Consider $\int_C \frac{e^{iz}}{z} dz$, where C is a semicircular annulus enclosing the positive imaginary plane excluding the origin. It has an inner radius of ϵ and outer radius of R, so we can characterize it with $z = re^{i\theta}$, where $r \in [\epsilon, R]$ and $\theta \in [0, \pi]$ This has no singularities, so it equals zero.

Also:

$$\int_C \frac{e^{iz}}{z} dz = 0 \text{ (no singularities, so zero)}$$

$$= \int_\epsilon^R \frac{e^{ix}}{x} dx + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta + \int_{+R}^{+\epsilon} \frac{e^{-ix}}{x} dx + \int_\pi^0 ie^{i\epsilon e^{i\theta}} d\theta$$

$$= \int_\epsilon^R \frac{e^{ix} - e^{-ix}}{x} dx + i \int_0^\pi e^{iRe^{i\theta}} d\theta - i \int_0^\pi e^{i\epsilon e^{i\theta}} d\theta$$

$$= 2i \int_\epsilon^R \frac{\sin x}{x} dx + i \int_0^\pi e^{iRe^{i\theta}} d\theta - i \int_0^\pi e^{i\epsilon e^{i\theta}} d\theta$$

Now consider that:

$$\left| \int_0^\pi e^{iRe^{i\theta}} d\theta \right| \leq \int_0^\pi \left| e^{iR\cos\theta - R\sin\theta} \right| d\theta$$

$$= \int_0^\pi e^{-R\sin\theta} d\theta$$

$$= 2 \int_0^{\pi/2} e^{-R\sin\theta} d\theta \text{ (integral of halves are equal, so only need to do 2*half integral)}$$

$$\leq 2 \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta} d\theta \text{ (linear function vs sine function)}$$

$$= \frac{\pi}{R}(1 - e^{-R})$$

This is true since $|e^{iR\cos\theta}| = 1$. Also $\frac{\pi}{R}(1 - e^{-R}) \to 0$ as $R \to \infty$.
Taking the limit as $R \to \infty$ for our original problem, we get that:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{1}{2} \int_0^\pi e^{i\epsilon e^{i\theta}} d\theta$$

As $\epsilon \to \infty$, $\int_0^\pi e^{i\epsilon e^{i\theta}} d\theta = \pi$. In conclusion, we get that:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{pi}{2}$$

Example 25. *Consider with $a, b \in \mathbb{R}$ and $a < |b|$,*

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta$$

We set $z = e^{i\theta}$ and let C be the unit circle. Thus:

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta = \int_C \frac{1}{iz[a + \frac{b}{2}(z + \frac{1}{z})]} d\theta$$

TODO

We can find the residue at z:

$$\lim_{z \to z_1} (z - z_1)f(z) = \frac{b}{2\sqrt{a^2 - b^2}}$$

Hence, by the residue theorem,

$$\int_0^{2\pi} \frac{1}{a + b\cos\theta} d\theta = \frac{2\pi i b}{2\sqrt{a^2 - b^2}} \left(\frac{2}{bi}\right)$$
$$= \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Example 26. *Find the value of:*

$$\int_0^\infty \sin u^2 \, du$$

Solution: *We note that $F(x) = \int_0^x \sin u^2 \, du$.*

Let $f(z) = e^{iz^2}$ and C be a contour that is a sector of a circle with an angle 0 to pi/4 and radius R. So by Cauchy's theorem, since there are no singularities in the contour:
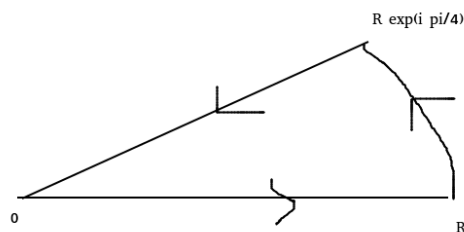

R exp(i pi/4)

0   R

Figure 12: Contour for Example

$$\int_C e^{iz^2} dz = 0$$

We want to relate this to our original integral. We can break up all parts of our contours

$$\int_0^R e^{ix^2} dx + \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} d\theta + \int_R^0 e^{i(te^{i\pi/4})^2} e^{i\pi/4} dt = 0$$

Since we've chosen the angle $\pi/4$, we can have the nice form of the gaussian integral on the RHS:

$$\int_0^R e^{ix^2} dx + \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} d\theta = e^{i\pi/4} \int_0^R e^{-t^2} dt$$

We note te RHS can also be computed using contour integration. Looking at the second term, utilizing $e^{i2\theta} = \cos 2 \ theta + i\sin 2\theta$:

$$\left| \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} d\theta \right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \theta} d\theta \ \textit{(change of variables)}$$

$$\le \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \theta / \pi^2} d\theta$$

$$= \frac{R}{2} \frac{\pi}{2R^2} (1 - e^{-R^2})$$

$$= \frac{\pi}{4R} (1 - e^{-R^2})$$

Finally, if we take the limit, we get zero: $\lim_{R \to \infty} \frac{\pi}{4R} (1 - e^{-R^2}) = 0$.
Now considering the integral again:

$$\int_0^R e^{ix^2} dx = e^{i\pi/4} \int_0^R e^{-t^2} dt$$

However, we note that $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$, *which is a Gaussian Integral. And so:*

$$\int_0^\infty (\cos x^2 + i \sin x^2) dx = \frac{1}{\sqrt{2}} (1 + i) \sqrt{\pi}/2$$

Taking just the real parts or imaginary parts to get the integrals of $\sin x^2$ *or* $\cos x^2$.
Note: *We can also compute a gaussian integral using contour integration. Let* $a = \sqrt{\pi i} = (1 + i)\sqrt{\pi/2}$ *and:*

$$g(z) = \frac{e^{-z^2}}{1 + e^{-2az}}$$

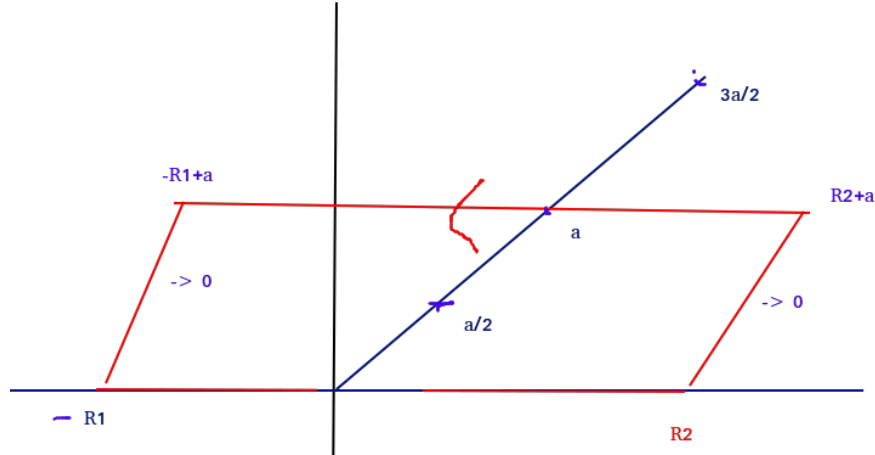*The contour we are integrating over is:*

Figure 13: Contour for Example. The Right and left sides go to zero, so subtract top from bottom to get the value of the integral over the red contour.

$$\int_C g(z) dz = 2\pi i \operatorname{Res}[g(z = a/2)]$$

Example 27. *Consider evaluating:*

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x} dx, \ \lambda \in (0,1)$$

Consider $f(z) = \frac{z^{\lambda-1}}{1+z}$, with branch cut $0 < \arg z < 2\pi$. There is a branch cut from 0 to ∞, so let us take the branch cut to the positive reals, such that we do not intersect our pole at $z = -1$:
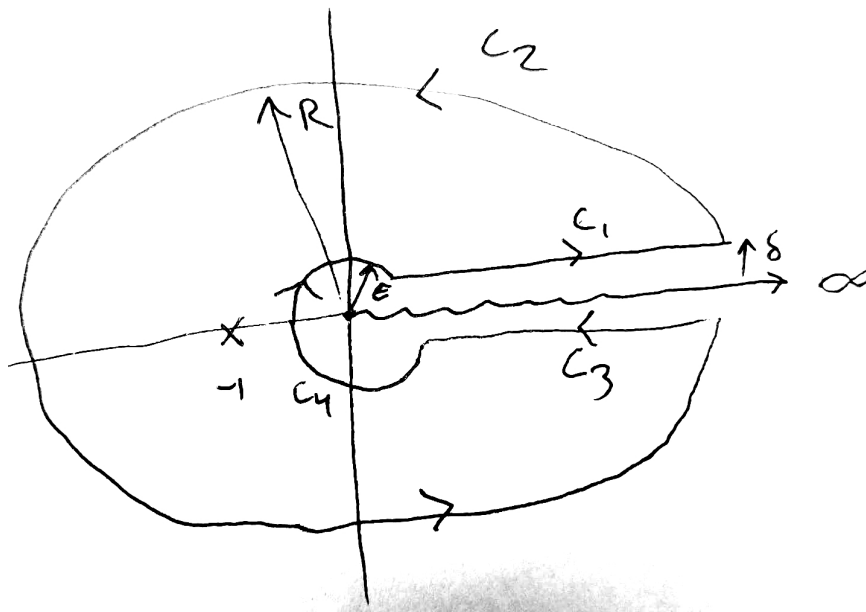

Figure 14:

$f(z)$ has a simple pole at $z = -1$ with a residue:

$$\text{Res}[f, -1] = \lim_{z \to -1} (1+z)f(z) = e^{\pi i(\lambda-1)}$$

We pick the contour which is the whole plane excluding the branch cut $C = C_1 + C_2 + C_3 + C_4$. By the residue theorem:

$$\int_C \frac{z^{\lambda-1}}{1+z} dx = 2\pi i e^{i\pi(\lambda-1)}$$
$$= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

Considering the C_1 integral, let $\delta, \epsilon \to 0$, and $R \to \infty$.

$$\int_{C_1} \frac{z^{\lambda-1}}{1+z} dx = \int_{C_1} \frac{(x+i\delta)^{\lambda-1}}{1+(x+i\delta)} dx$$
$$= \int_\epsilon^R \frac{x^{\lambda-1}}{1+x} dx \ (as \ \delta \ to \ 0)$$

Also we note that:

$$\int_{C_1} \frac{z^{\lambda-1}}{1+z} dx = \int_R^\epsilon \frac{(x-i\delta)^{\lambda-1}}{1+(x-i\delta)} dx$$

$$= \int_R^\epsilon \frac{e^{i2\pi(\lambda-1)}}{1+x} dx \ (as \ \delta \ to \ 0)$$

$$= -e^{i2\pi\lambda} \int_R^\epsilon \frac{x^{(\lambda-1)}}{1+x} dx \ (as \ \delta \ to \ 0)$$

We still have to deal with the other integrals. Let $z = Re^{i\theta}$

$$\left| \int_{C_2} \frac{z^{\lambda-1}}{1+z} dz \right| = \int_0^{2\pi} \frac{(Re^{i\theta})^{\lambda-1} iRe^{i\theta}}{1+Re^{i\theta}} d\theta$$

$$\leq R^\lambda \left| \frac{1}{R-1} \right|$$

$$= \frac{2\pi R^\lambda}{R-1}$$

As we take $R \to \infty$:

$$\lim_{R\to\infty} \left| \int_{C_2} \frac{z^{\lambda-1}}{1+z} dz \right| \leq 0$$

Finally, we look at C_4:

$$\left| \int_{C_4} \frac{z^{\lambda-1}}{1+z} dz \right| = \left| \int_0^{2\pi} \frac{(\epsilon e^{i\theta(\lambda-1)})\epsilon e^{i\theta}}{1+\epsilon e^{i\theta}} d\theta \right|$$

$$\leq \frac{\epsilon^\lambda}{1-\epsilon} \int_0^{2\pi} d\theta = 2\pi \frac{\epsilon^\lambda}{1-\epsilon}$$

This goes to 0 as $\epsilon \to 0$. *Now:*

$$\int_C \frac{z^{\lambda-1}}{1+z} dx = 2\pi i e^{i\pi(\lambda-1)}$$

$$= \int_{C_1} + \int_{C_3} + \int_{C_2} + \int_{C_4}$$

$$= (1-e^{i2\pi\lambda}) \int_0^\infty \frac{x^{(\lambda-1)}}{1+x} dx + 0 + 0$$

Finally, we see that:

$$\int_0^\infty \frac{x^{(\lambda-1)}}{1+x} dx = \frac{2\pi i e^{i\pi\lambda}}{e^{i2\pi\lambda}-1}$$

$$= \pi \left(\frac{2i}{e^{i\pi\lambda} - e^{-i\pi\lambda}} \right) = \frac{\pi}{\sin \pi\lambda}$$

These ideas, while useful for evaluating real integrals, can also be used to generate series representations for functions with poles. Or, we can do representations for functions with zeros. If a function has zeros which converges to a limit point, the function must be a zero function.

However, if the series of zeros do not converge, this identity theorem does not apply. We can use the Weirstrass Factorization theorem to generate an expression for the function in terms of the values of the zeros. The expansion will be dependent on the zeros of the function:

$$\sin z = \sum_{n=-\infty}^\infty (f(zeros))$$

8.2 Special Contour Integrals

Lemma 2 (Jordan's). *Let $f(z) = e^{iaz}g(z)$, $a > 0$ be a continuous function defined on a **semi-circular** counter-clockwise contour in the upper half plane centered around the origin:*

$$C_r = \{Re^{i\theta}|\theta \in [0, \pi]\}$$

Then we can say that:

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi}{a}M_R$$

where $M_R := \max_{\theta \in [0,\pi]} |g(Re^{i\theta})|$

Alternative Jordan's lemma:
If the only singularities in $g(z)$ are poles, $a > 0$, and $g(z) \to 0$ as $R \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} e^{iaz}g(z)dz = 0$$

Lemma 3 (Indentation Lemma). *Let f have a simple pole at a with a residue $\mathrm{Res}[f; a]$. Then given an upper half clockwise semi-circular contour around the pole C_ϵ, the resulting contour is:*

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz = -\mathrm{Res}[f; a]\pi i$$

9 Special Functions

9.1 Gamma Functions

Note:

$$e^{-t} = \lim_{n \to \infty} (1 - \frac{t}{n})^n$$

Let us define a Gamma function as:

$$\Gamma(z; n) := \int_0^n (1 - \frac{t}{n})^n t^{z-1}dt$$

Then:

$$\lim_{n \to \infty} \Gamma(z; n) = \Gamma(z)$$

Now, let $t = m\tau$, $t^{z-1}dt = n^z\tau^{z-1}d\tau$. Then:

$$\Gamma(z; n) = n^z \int_0^1 (1 - \tau)^n \tau^{z-1}d\tau$$

By I.B.P.:

$$\Gamma(z;n) = n^z \left[\frac{1}{z} \left((1-\tau)^n \tau^z \right) \right]_0^1 + \frac{n}{z} \int_0^1 (1-\tau)^{n-1} \tau^z d\tau$$

$$= n^z \left(\frac{n}{z} \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \right)$$

$$= n^z \left(\frac{n}{z} \frac{n-1}{z+1} \int_0^1 (1-\tau)^{n-2} \tau^{z+1} d\tau \right)$$

$$...$$

$$= \frac{n(n-1)...(2)(1)}{z(z+1)...(z+n-1)} n^z \int_0^1 \tau^{z+n-1} d\tau$$

We can then conclude that $\Gamma(z)$ converges on:

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{z(z+1)...(z+n-1)} \tag{1}$$

$$= \lim_{n \to \infty} \frac{(n-1)!(n-1)^z}{(z)_n} \tag{2}$$

Note:

$$\Gamma(z) = \lim_{n \to \infty} \left(\frac{n-1}{n} \right)^z \frac{(n-1)!(n)^z}{(z)_m}$$

$$= \lim_{n \to \infty} \frac{(n-1)!(n)^z}{(z)_n}$$

Recall that $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$, And thus with some algebraic manipulation:

$$\Gamma(z) = \lim_{n \to \infty} \frac{(n-1)! n^z}{\Gamma(z+n)} \Gamma(z)$$

$$1 = \lim_{n \to \infty} \frac{(n-1)! n^z}{\Gamma(z+n)}$$

Now we see:

$$\Gamma(z) = \lim_{n \to \infty} \frac{(n)!(n+1)^z}{(z)_{n+1}}$$

$$= \frac{1}{z} \lim_{n \to \infty} \frac{(n)!(n+1)^z}{(z+1)...(z+n)}$$

$$= \frac{1}{z} \lim_{n \to \infty} \frac{((n+1)^z}{(1+z)(1+z/2)...(1+z/n)}$$

$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{(1+\frac{1}{k})^z}{(1+\frac{z}{k})}$$

This is shown by **Gauss**. This infinite product (take the log) converges and therefore represents an analytic function for $z \neq 0, -1, -2, ...$ The function also provides analytic continuation into $\text{Re}\{z\} \leq 0$.

Definition 9.1. *Gamma Function*

A Gamma function is a special function of the form:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

If z is complex with positive real part, then the integral converges absolutely, and $\gamma(n) = (n-1)!$ for all positive integer n.

This formulation allows us to extend the idea of factorials to the complex numbers.

Definition 9.2. *Euler-Mascheroni Constant*

Let $s_n = \sum_{k=1}^n \frac{1}{k}$. This diverges as $n \to \infty$. However,

$$\lim_{n \to \infty} (s_n - \log n) = \gamma = 0.5772...$$

We define the Euler-Mascheroni Constant as γ. We can also define the constant γ as:

$$\gamma = \int_0^\infty e^{-t} \left[\frac{1}{1 - e^{-t}} - t\right] dt$$

Now, from Equation 1, we can see that:

$$\frac{1}{\Gamma(n)} = z \lim_{n \to \infty} m^{-z} \frac{(z+1)...(z+n-1)}{m!}$$

$$= z \lim_{n \to \infty} (1 + \frac{z}{1})(1 + \frac{z}{2})...(1 + \frac{z}{n}) e^{-z \log m}$$

$$= z \lim_{n \to \infty} \left[\prod_{k=1}^n (1 + \frac{z}{k}) e^{-z/k}\right] e^{z(1 + \frac{1}{2} + ... + \frac{1}{m} - \log m)}$$

We note that $1 + \frac{1}{2} + ... + \frac{1}{m} - \log m \to \gamma$, so:

$$\frac{1}{\Gamma(n)} = z e^{\gamma z} \lim_{n \to \infty} \left[\prod_{k=1}^n (1 + \frac{z}{k})\right]$$

This is provided by Weirstrass. This provides a factorization of $1/\Gamma(z)$ which is entire, with first order zeros only with $\Gamma(z)$ that has poles $z = 0, -1, -2....$ We note that since $1/\Gamma(z)$ has no poles, $\Gamma(z)$ can have no zeros!

Now, recall that:

$$\sin \pi z = \pi z \prod_{n=1}^\infty (1 - \frac{z^2}{n^2})$$

On the other hand, we have shown that:

$$\sin \pi z = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

$$= \frac{-\pi}{z\Gamma(z)\Gamma(-z)}$$

$$= \frac{\pi z^2}{z} \prod_{k=1}^{\infty} (1 + \frac{z}{n})(1 - \frac{z}{n})$$

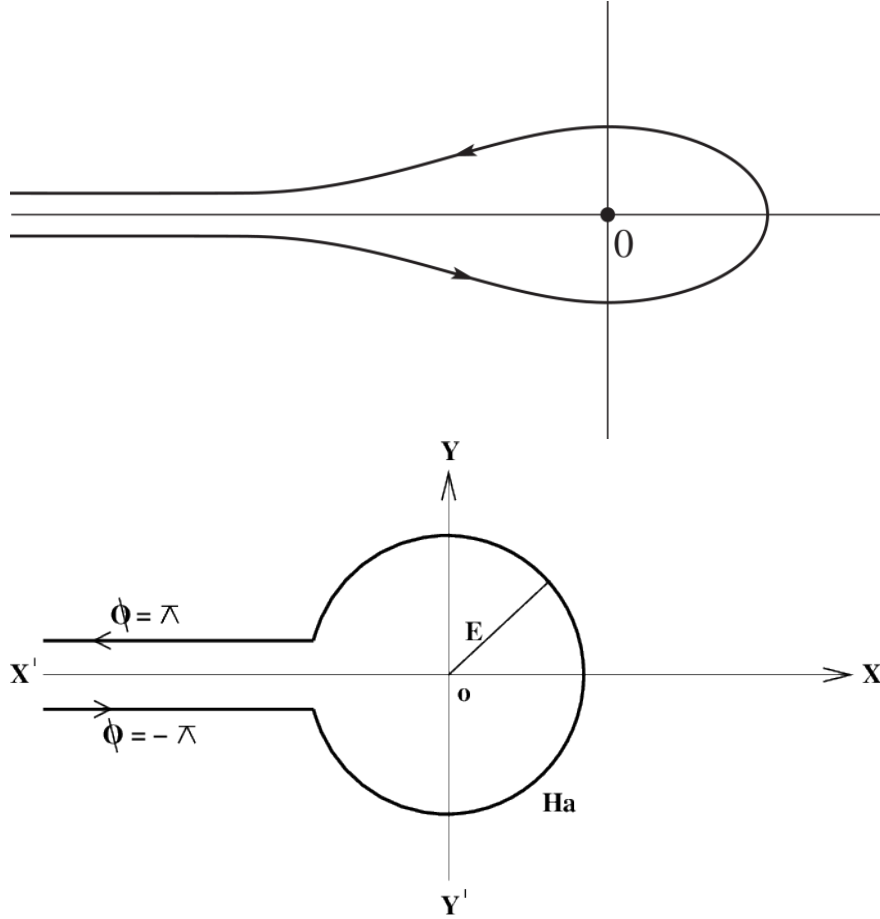**Theorem 9.1 (Contour Integral Representation (Hankel)).** *Consider a contour C:*

Figure 15: The Hankel contour in general, and then parameterized by a radius E or ϵ. Call the circle contour C_ϵ or γ_ϵ

The contour lets us take a branch cut along the negative real axis. We let $\zeta = e^{-\pi r} = r$ on the top side, and $\zeta = e^{-i\pi} r = -r$

$$I = \int_C e^\zeta \zeta^{-z} d\zeta$$

$$= \int_\infty^\epsilon e^{-r}(re^{-i\pi})^{-z}e^{-i\pi}dr + \int_{C_\epsilon} f(\zeta) + \int_\epsilon^\infty e^{-r}(re^{i\pi})^{-z}e^{i\pi}dr$$

As $\epsilon \to \infty$, $\int_{C_\epsilon} f(\zeta) \to 0$. Therefore:

$$I = \int_0^\infty e^{-r} r^{-z} e^{i\pi z} dr - \int_0^\infty e^{-r} r^{-z} e^{-\mathfrak{1}piz} dr$$

$$= 2i \sin \pi z \int_0^\infty e^{-r} r^{-=z} e^{i\pi z} dr$$

$$= 2i \sin \pi z \Gamma(1-z)$$

$$= 2i \sin \pi z \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(z)} \qquad\qquad = 2i \sin \pi z \Gamma(z) \sin \pi z = 2i\Gamma(z)$$

We then rearrange to get:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^\zeta \zeta^{-z} d\zeta$$

This form is valid for any value of z.

9.2   Beta, Digamma, Riemann Zeta functions

Definition 9.3. *Beta Function*
We define the beta function as:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Definition 9.4. *Digamma Function*
The Digamma function (or Psi Function) is

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\left(\log \Gamma(z)\right)$$

This is meneomorphic. It has simple poles at $z = 0, -1, -2..$. The residue at each pole is -1.
Now, recall

$$\log \Gamma(z) = -\log z - \gamma z - \sum_{k=1}^\infty \left[\log\left(1 + \frac{z}{k}\right) - \frac{z}{k}\right]$$

Then,

$$\Psi(z) = -\frac{1}{z} - \gamma - \sum_{k=1}^\infty \left[\frac{1}{k(1+\frac{z}{k})} - \frac{1}{k}\right] \qquad \Psi(z) = -\frac{1}{z} - \gamma + \sum_{k=1}^\infty \frac{z}{k(z+k)}$$

This has simple poles at $z = 0, -1, -2, ...$
We note that $\Psi(1) = -\gamma$. *("It's true by the way!")*
Additionally, recall that:

$$\Gamma(z+1) = z\Gamma(z)$$

By direct differentiation:

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma(z) + z\Gamma'(z)}{z\Gamma(z)} = \frac{1}{z} + \gamma(z)$$

Then we can compute:

$$\Psi(1) = -\gamma$$
$$\Psi(2) = 1 - \gamma$$
$$\Psi(3) = \frac{1}{2} + 1 - \gamma$$
$$...$$
$$\Psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma$$

We see that $\Psi(n+1) \sim \log n$.

Definition 9.5. *Riemann Zeta Function*

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{(1+n)^\zeta}$$

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}), \ p \in \text{Primes}$$

The poles of the Riemann Zeta Function are $-2, -4...$

10 ODEs

11 Frobenius Theory

12 Hypergeometric Equations

Definition 12.1. *Hypergeometric Equation*

$$z(1-z)w'' + [C - (A+B+1)z]w' - ABw = 0 \tag{3}$$

where A, B, C are complex constants.

Note: It is a P Equation with $a = 0, b = 1, c \to \infty$, and its solution can be written as:

$$w = P \begin{bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{bmatrix}$$

Note: By suitable changes of dependent and independent variables, the general. P equation can be put in the above form for the hyper geometric equation. Therefore, solutions of the P equation can be expressed in terms of the sums of the hypergeometric equation.

Note: It can be shown that the hypergeometric series:

$$F(A, B, C; z) = 1 + \frac{AB}{1C}z + \frac{A(A+1)B(B+1)}{(1)(2)C(C+1)}z^2 + \dots$$

Or as from Wikipedia:

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

This series is convergent for $|z| < 1$, and divergent $|z| > 1$. This is a solution of the hypergeometric equation, or Equation 3.

We note that C cannot be a negative integer.

We can use the transformation rules to get a second solution to the hypergeometric equation of the form:

$$z^{1-C} F(1 + A - C, 1 + B - C, 2 - C; z)$$

This solution, together with the previous solution of $F(A, B, C; z)$ provide 2 fundamental solutions of Equation 3 in the vicinity of $z = 0$.

We note that expressions for the function F, which for $|z| > 1$, may be obtained by contour integration.

12.1 Examples of ODE theory

Definition 12.2. *Legendre Polynomials*
The following is the definition of Legendre polynomials using Rodriguez[french]'s formula:

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dz^n}[(z^2 - 1)^n]$$

Clearly, we see that:

$$P_0(z) = 1$$
$$P_1(z) = z$$
$$P_2(z) = \frac{1}{2}(3z^2 - 1)$$

Since by Cauchy's formula's formula:

$$(z^2 - 1)^n = \frac{1}{2\pi i} \oint \frac{(\zeta^2 - 1)^n}{\zeta - z} d\zeta$$

We take the contour integral representation (Schlafli's Integral):

$$P_n(z) = \frac{1}{2\pi i n!} \frac{d^n}{dz^n} \oint \dots dz$$
$$= \frac{1}{2\pi i} \frac{d^n}{dz^n} \int_C \frac{(t^2 - 1)^n}{2^n (t - z)^{n+1}} dz$$

Where C is any contour that circles the point $t = z$ once in the counter clockwise direction.
Note:

$$\int_C \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t^2 - 1)^{n+1}}{(t - z)^{n+2}} dt = 0$$

This is because $\frac{(t^2 - 1)^{n+1}}{(t - z)^{n+2}}$ returns to its original value when t circles z once.

Carrying out differentiation and using Schlafli's integral for $P_n(z)$, it can be seen that P_n satisfies the **Legendre equation:**

$$(1 - z^2)P_n'' - 2zPn' + n(n + 1)P_n = 0$$

This proof shoes that Schlafli's representation for P_n still gives a solution to Legendre's equation for non-integer values of n and provided that C is still a close contour for which the expression $\frac{(t^2 - 1)^{n+1}}{(t - z)^{n+2}}$ returns to its original value after a complete circuit.

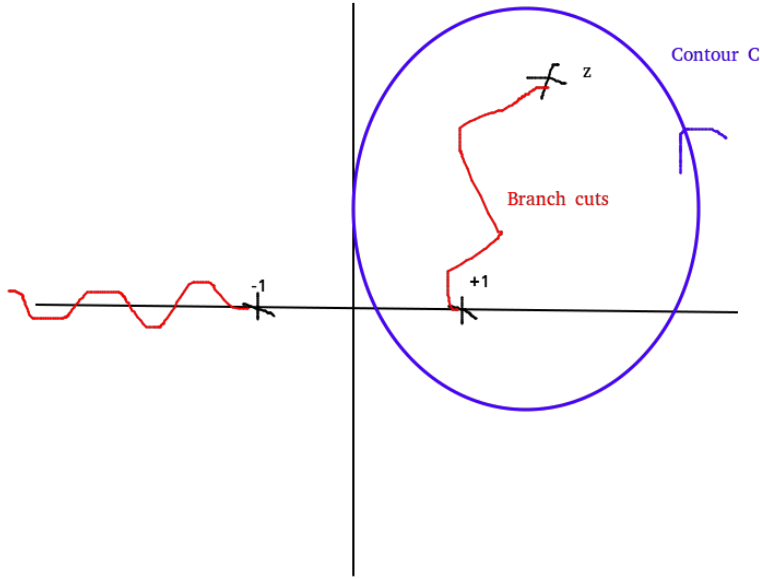One such contour is the following in Figure 16.

Figure 16: There are poles at $-1, 1, z$, and branch point at infinity. So we must do the contour around it. Branch points are in red. This allows us to make a contour that surrounds z

This choice of Branch cuts and contour C still leads to solutions of the Legendre Equations when n is not an integer. The solution is called a *Legendre function of the first kind.* We not that we can use the the Schlafli form to derive a recursive equation:

$$(2n + 1)P_n(z) = P_{n+1}'(z) + P_{n-1}'(z)$$
$$(z^2 - 1)P_n'(z) = nzP_n(z) - nP_{n+1}(z)$$
$$(n + 1)P_{n+1}(z) - (2n + 1)zP_n(z) + nP_{n+1}(z) = 0$$
$$nP_n(z) = zP_n'(z) - P_{n-1}'(z)$$

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Definition 12.3. *Generating Function*

A function $\pi(\zeta, z)$ such that

$$\phi(\zeta, z) = P_0(z) + \zeta P_1(z) + \zeta^2 P_2(z) + \ldots$$

for small values of $|\zeta|$ is a generating function.

Using the Schlafli representation:

$$\phi(\zeta, z) = \frac{1}{2\pi i} \int_C \left[\sum_{n=0}^{\infty} \frac{\zeta^n (t^2 - 1)^n}{2^n (t - z)^{n+1}} \right] dt$$

$$= \frac{1}{2\pi i} \int_C \frac{1}{t - z} \left[1 - \frac{\zeta (t^2 - 1)^n}{2(t - z)} \right]^{-1} dt$$

$$= \frac{1}{2\pi i} \int_C \frac{dt}{t - z - \frac{\zeta(t^2 - 1)}{2}}$$

The denominator vanishes for two values of t, but only 1 is inside c. The calculation of the residue gives the result:

$$\phi(\zeta, z) = (1 - 2z\zeta + \zeta^2)^{-1/2}$$

$$= 1 + \frac{1}{2} \left[\frac{-2z + 2\zeta}{(1 - 2z\zeta + \zeta^2)^{3/2}} \right]_{\zeta=0}$$

where the principal branch is taken. Then, for small $|\zeta|$, then

$$\phi(\zeta, z) = P_0(z) + \zeta P_1(z) + \zeta^2 P_2(z)$$

From this, we can check that $P_n(1) = 1$:

$$P_n(-1) = (-1)^n$$
$$P_n(-1) = (-1)^n$$
$$P_n(-z) = (-1)^n P_n(z)$$
$$P_{2n+1}(0) = 0$$

12.2 Laplace's Integrals

If we use Schlafli's and let contour C be a circle of radius $|z^2 - 1|^{1/2}$. Let us set $t = z + |z^2 - 1|^{1/2} e^{i\theta}$, where $\theta \in [-\pi, \pi]$. Then,

$$P_n(z) = \frac{1}{2^{n+1}\pi} \int_{-\pi}^{\pi} \left[\frac{z^2 - 1 + 2z(z^2 - 1)^{1/2} e^{i\theta} + (z^2 - 1)^{1/2} e^{i2\theta}}{(z^2 - 1)^{1/2} e^{i\theta}} \right]^n d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} [z + (z^2 - 1)^{1/2} \cos\theta]^n d\theta$$

This is called **Laplace's first integral.**

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Definition 12.4. *Laplace's Second Integral*

$$P_n(z) = \pm \frac{1}{\pi} \int_0^\pi \frac{d\theta}{[z + (z^2 - 1)^{1/2} \cos \theta]^n}$$

12.3 Legendre functions and their kinds

Consider again Legendre's equation:

$$(1 - z^2)w'' - 2zw' + \nu(\nu + 1)w = 0$$

This function is a Legendre function of the first kind. THis equation has 3 regular singularities at $z = \pm 1, \infty$ in the finite plane. its solutions are analytic, but not necessarily single valued except at $z = \pm 1$.

The indicial equation gives both roots $\rho = 0$. The branch points at ± 1 are logarithmic. In the vicinity og he origin, the solution is analytic and we can look for expansions of the form:

$$w = a_0 + a_1 z + a_2 z^2 + \dots$$

As we expand, we get two solutions, where F is the hypergeometric equation:

$$w_1(z) = 1 - \frac{\nu(\nu + 1)}{2!}z^2 + \frac{\nu(\nu - 2)(\nu + 1)(\nu + 3)}{4!}z^4 + \dots$$
$$= F(-\frac{\nu}{2}, \frac{\nu + 1}{2}, \frac{1}{2}; z^2)$$

$$w_1(z) = zF(-\frac{1 - \nu}{2}, \frac{\nu + 2}{2}, \frac{3}{2}; z^2)$$

These solutions are linearly independent. The general solution for $|z| < 1$ of the form

$$w = Aw_1 + Bw_2, \ A, B \in \mathbb{C}$$

As we have seen, Schlafli's representation provides the analytic continuation of one solution.

Note: at the point $z = 1$, the integrand does not have a branch point. By taking derivatives of P_ν and evaluating the taylor expansion about $z = 1$, we get

$$P_\nu(z) = F(\nu + 1, -\nu, 1; \frac{1 - z}{2})$$

This equation is analytic at $z = 1$, but not necessarily analytic at $z = -1$.

Now let us consider **Legendre functions of the second kind**, with contour in Figure 17:

$$Q_\nu(z) = \frac{1}{2i \sin \nu\pi} \int_C \frac{(t^2 - 1)^\nu}{2^\nu (z - t)^{\nu+1}} dt$$
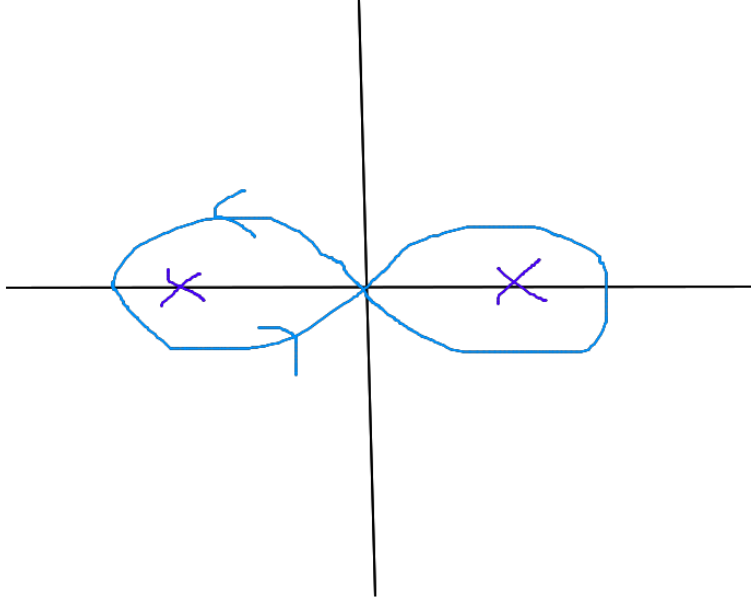
Figure 17: Contour of Legendre function of the second kind. It is ok to do this kind of contour since the multi-valuedness cancels out.

$Q_\nu(z)$ is a L.I. solution of Legendre equations such that P_ν, Q_ν provide a fundamental set.

12.4   Bessel Functions

Using separation of variables (e.g. for Helmholtz equation) leads to Bessel's equation:

$$z^2 w'' + zw' + (z^2 - \nu^2)z = 0, \ \nu \in \mathbb{C} \tag{4}$$

We note that this function has regular singular point at $z = 0$, however the point $z = \infty$ is not a regular singular point. Since the only singularity in the finite plane is at $z = 0$, solutions are expected to be entire except for the branch point at $z = 0$. Therefore, look for solutions of the form

$$w = z^\rho(a_0 - a_1 z + a_2 z^2 ...)$$

We plug this into the Bessel Equation (Equation 4). We find the first equation is:

$$a_0(\rho^2 - \nu^2) = 0 \tag{5}$$
$$a_1[(\rho + 1)^2 - \nu^2] = 0 \tag{6}$$
$$a_2[(\rho + 2)^2 - \nu^2] = -a_0 \tag{7}$$
$$a_n[(\rho + n)^2 - \nu^2] = -a_{n-2} \tag{8}$$

The indicial equation is $\rho = \pm\nu$. We denote $\rho_1 = +\nu$ to be the root with the largest real part. $\rho_2 = -\nu$ is the second root. Then we choose $\rho = +\nu$ in Equation 5. None of the a coefficients vanishes and we always have a solution of the form:

$$1 + \frac{1}{1 - (\nu + 1)} \left(\frac{z}{2}\right)^2 + \frac{1}{2(\nu + 1)(\nu + 2)} \left(\frac{z}{2}\right)^4 + ...$$

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Conventionally, we get:

$$a_0 = \frac{1}{2^\nu \Gamma(\nu + 1)}$$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n}$$

This is called **Bessel's equation of the first kind** (of order ν). **Note:** Apart from the factor $\left(\frac{z}{2}\right)^0$, the power series has an infinite radius of convergence, so that $\left(\frac{z}{2}\right)^{-\nu} J_\nu(z)$ is entire.

For *integer* $\nu = n > 0$, then the Bessel function has the property:

$$J_n(z) = \left(\frac{z}{2}\right)^\nu \left[\frac{1}{0!n!} - \frac{1}{1!(n+1)!} \left(\frac{z}{2}\right)^2 + ...\right]$$

This is called the Bessel Coefficient of order n. It is an entire function.

Note: If the difference between indicial roots 2ν is not an integer, the choice of $\rho = -\nu$ leads to a 2nd linearly independent solution $J_{-\nu}$ Even when 2ν is an odd integer, J_ν and $J_{-\nu}$ are linearly independent.

We note since $2\nu = 2n$, then, the only case left is when $\nu = n \geq 0$. Then, one solution is still J_n as given above. The choice $\rho = -\nu$ leads to $J_{-n}(z) = (-1)^n J_n(z)$. This does not provide a linearly independent solution. This also suggests considering:

$$\lim_{\nu \to n} = \frac{J_\nu(z) - (-1)^n J_{-\nu}(z)}{\nu - n}$$

Which leads to the function of the second kind:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

When $\nu \neq n$, Y_ν is linearly independent from J_ν. In the limit as $\nu \to n$,

$$Y_n(z) = \frac{1}{\pi} \left[\frac{\partial J_\nu}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}}{\partial \nu}\right]_{\nu = n}$$

And this is linearly independent from J_n. For the case $n \geq 0$, we get:

$$Y_n(z) = -\frac{\left(\frac{z}{2}\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z^2}{4}\right)^k + \frac{2}{\pi} J_n(z) \ln\frac{z}{2} - \frac{\left(\frac{z}{2}\right)^n}{\pi} \sum_{k=0}^{\infty} (\psi(k+1) + \psi(n+k+1)) \frac{\left(-\frac{z^2}{4}\right)^k}{k!(n+k)!}$$

Where $\phi(z)$ is the digamma function.

13 Caucy Data on Arbitrary Curve Γ

Let Γ be defined by

$$\xi(x, y) = 0$$

and define adjacent curves $\xi(x, y) =$ constant. We also defined a family:

$$\eta(x, y) = \text{constant}$$

This is defined in such a way that ξ, η define a log coordinate system:

$$J(x, y) = \xi_x \eta_y - \xi_y \eta_x \neq 0$$

The cauchy data on Γ prescribe ϕ and normal derivative $\frac{\mathrm{d}\phi}{\mathrm{d}m}$ on Γ. Can we compute $\phi_{\eta\eta}, \phi_{\eta\xi}, \phi_{\eta\eta\xi}, \dots$ in order to find $\phi_{\xi\xi}$ from the PDE:

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0$$

We have to make a change of variables $(x, y) \to (\xi, \eta)$. We note that $\phi_x = \phi_\xi \xi_z + \phi_\eta \eta_x$ and that $\phi_{xx} = \phi_{\xi\xi}\xi_x^2 + 2\phi\phi_x\eta_x + \phi\eta_x^2 + \phi_\xi \xi_{xx} + \phi_\eta \eta_{xx}$

The PDE now becomes:

$$(A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2)\phi_{xx} + 2(stuff)\phi_{\xi\eta} + (stuff)\phi_{\eta\eta} + \dots + \sigma = 0$$

Now, in order to be able to solve for $\phi_{\xi\xi}$ in terms of $\phi_\eta, \phi_\xi, \phi_{\xi\eta}, \phi_{\eta\eta}\dots$ it is necessarily that the coefficient exist:

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 \neq 0$$

On the other hand, if the left hand side $= 0$, then the procedure fails. We note the coefficient can be written as

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + 2B\left(\frac{\xi_x}{\xi_y}\right)^2 + C = 0$$

This is quadratic in $\frac{\xi_x}{\xi_y}$. The solution depends on the discriminant $B^2 - AC$. There are three cases:

Case 1: $B^2 - AC > 0$

In this case,

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

Now, on $\xi(x, y) = const$, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y}$$

From this, we conclude that the Cauchy data is insufficient on any curve for which:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B + \sqrt{B^2 - AC}}{A} \tag{9}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{B - \sqrt{B^2 - AC}}{A} \tag{10}$$

These curved can be obtained by integration and are called **characteristics.** We choose Equation 9 to define $\xi(x, y) = const$ and Equation 10 to define $\eta(x, y) = const.$ We note that these equations are never tangent.

Note: In this special coordinate system, the PDE becomes:

$$\phi_{\xi\eta} + \alpha + \phi_\xi + \beta + \phi_\eta + \gamma\phi + \delta = 0$$

We note that if $B^2 - AC > 0$ in a region, the PDE is hyperbolic. In this case, there are 2 families of characteristics.

Example 28 (Wave Equation). *Given:*

$$\frac{1}{c^2}\phi_{tt} = \phi_{xx}$$

This has the characteristic:

$$\xi = x + ct$$
$$\eta = x - ct$$

The PDE becomes:

$$\phi_{\xi\eta} = 0$$
$$\phi(\xi, \eta) = F(\xi)G(\eta)$$

Case 2: $B^2 - AC = 0$
Without loss of generality, we take $A \neq 0$. (If $A = 0$, take $C \neq 0$). Then,

$$\frac{\xi_x}{\xi_y} = -\frac{B}{A} = -\frac{\mathrm{d}y}{\mathrm{d}x}$$

In this case, there is only one family of characteristics, or $\xi(x, y) = $ const. Then the equation is said to be *parabolic.* The canonical form is:

$$\phi_{\eta\eta} + \alpha\phi_\xi + \beta\phi_\eta + \gamma\phi + \delta = 0$$

Example 29 (Diffusion Equation).

$$\phi_x = a^2\phi_{xx}$$

Case 3: $B^2 - AC < 0$
In the region there is no characteristic and the PDE is elliptioc.

Example 30 (Laplace's Equation).

$$\phi_{xx} + \phi_{yy} = 0$$

14 Fourier Transform

Definition 14.1. *Fourier Transform*

Given a function $f(t)$, the fourier transform $F(\lambda)$ on $\hat{f}(\lambda)$ is defined by:

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda z} f(\tau) d\tau$$

We note that $F(\lambda)$ is an analytic function. This makes it very powerful! We can use it to convert PDE to ODE, or rather an ODE of the Fourier transform.

14.1   Heuristic for Fourier Transform Inversion Formula

Recall the Fourier series representation. A function $f(t)$, assumed to be square integrable (L2 integrable), defined on an interval $t \in [-T, T]$, has a Fourier series expansion representation given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right)$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(\tau) \cos\left(\frac{n\pi \tau}{T}\right) d\tau$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(\tau) \sin\left(\frac{n\pi \tau}{T}\right) d\tau$$

If we combine the results, we get that:

$$f(t) = \frac{1}{2T} \int_{-T}^{T} f(\tau) d\tau + \sum_{n=1}^{\infty} \frac{1}{T} \int_{-T}^{T} f(\tau) \cos\left(\frac{n\pi}{T}\right)(\tau - t) d\tau$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2T} \int_{-T}^{T} f(\tau) \cos\left(\frac{n\pi}{T}\right)(\tau - t) d\tau$$

Let $\frac{n\pi}{T} = \lambda_n$, and $\frac{\pi}{T} = \lambda_{n+1} - \lambda_n = \delta\lambda_n$. Then, we get:

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \delta\lambda_n \int_{-T}^{T} f(\tau) \cos \lambda_n (\tau - t) d\tau$$

As we let $n, T \to \infty$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \cos \lambda(\tau - t) + d\tau d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) \left[\cos \lambda(\tau - t) + i \sin \lambda(\tau - t) \right] d\tau d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) e^{i\lambda \tau} e^{-i\lambda t} d\tau d\lambda$$

Therefore,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda \tau} d\lambda$$

This is the **Fourier Inversion Formula.**

Theorem 14.1. *Assume that $f(t)$ is a real, piecewise-smooth function with $\int_{-\infty}^{\infty} |f(t)| dt < \infty$.*
  Then,

$$\lim_{L \to \infty} \frac{1}{2\pi} \int_{-L}^{L} \int_{-\infty}^{\infty} f(\tau) \cos \lambda(\tau - t) dt d\lambda = \frac{1}{2} [f(t_+) + f(t_-)]$$

Where:

$$f(t_+) = \lim_{\epsilon \to 0} f(t + \epsilon)$$
$$f(t_-) = \lim_{\epsilon \to 0} f(t - \epsilon)$$

Definition 14.2. *Piecewise smooth:*
$f and f'$ *are continuous in any finite interval are continuous in any finite interval, except at any finite number of discontinuities.*

Lemma 4. *Under certain hypothesis on real, piecewise-smooth f, then:*

$$\lim_{L \to \infty} \int_{-T}^{T} f(t + \tau) \frac{\sin L\tau}{\tau} = \frac{\pi}{2} [f(t_+) + f(t_-)]$$

This is related to the fact for delta functions:

$$\int_{-\infty}^{\infty} f(t - \tau) \delta(\tau) d\tau = f(t)$$

The reason we can think of this is thinking of a Gaussian that becomes infinitely thin and tall as the limit of the Gaussian. The $\frac{\sin L\tau}{\tau}$ is a sort of approximation of a delta function.

We note that since $\sin \lambda(\tau - t)$ is odd in λ, the Fourier inversion theorem can also be written as:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) e^{i\lambda(\tau - t)} d\tau d\lambda = \frac{1}{2} (f(t_+) + f(t_-))$$

We can rewrite this using the definition of the Fourier transform:

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\tau) d\tau$$

This provides the inversion formula:

$$\frac{1}{2} (f(t_+) + f(t_-)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\tau) d\tau$$

Note, there are other possible definitions, and that $f(t)$ need not be real valued.

14.2   Extensions of Fourier

The theorem does not say that the inversion result is not valid under more general hypotheses. As an example of a non-working inversion:

$$g(t) = \begin{cases} t^{-1/2} & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\int_{-\infty}^{\infty} |g(t)|\, dt = \int_{0}^{\infty} \frac{1}{\sqrt{t}} dt = 2\sqrt{t}\big|_{0}^{\infty} \to \infty$$

Hence, the theorem does not guarantee inversion.

However, contour integration can be used to show:

$$G(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda\tau} \tau^{-1/2} d\tau$$

$$= \frac{1}{\sqrt{\lambda}} \left(1 + i\frac{\lambda}{|\lambda|} \right), \ \lambda \in \mathbb{R}^{+}, \lambda \neq 0$$

Can we invert $G(\lambda)$ to get back $g(t)$. This does in fact work:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\lambda) e^{i\lambda t} d\lambda = g(t)$$

Thus, the inversion results works even though the hypothesis of the Fourier Inversion Theorem are not satisfied. What is the moral of this story? We must try to see if it works.

Example 31.

$$f(t) = \begin{cases} t^{1/2} & t > 0 \\ 0 & t < 0 \end{cases}$$

Then,

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{1/2} e^{i\lambda t} dt$$

But this integral does not make sense. It does not converge for $\lambda \in \mathbb{R}$. However, the integral converges when $\mathrm{Im}\{\lambda\} > 0$. For such a λ, in fact, it can be shown that $F(\lambda) = \frac{1}{2\sqrt{2}} e^{-i\frac{3\pi}{4}} \lambda^{-\frac{3}{2}}$ for $0 < \arg \lambda < \pi$. By analytic continuation, this definition can be extended to the complex plane, except for $\lambda = 0$.

Next, we write $\lambda = \lambda_1 + i\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$. If we hold $\lambda_2 > 0$ fixed, we can then define $G(\lambda_1) = F(\lambda_1 + i\lambda_2)$. This is the Fourier transform of $g(t) = e^{-\lambda_2 t} f(t)$ for $\lambda_1 \in \mathbb{R}$. Then, by the Fourier Inversion Theorem:

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda_1 t} G(\lambda_1) d\lambda_1$$

Additionally,

$$f(t) = e^{\lambda_2 t} g(t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\lambda_2}^{\infty+i\lambda_2} e^{\lambda t} F(\lambda) d\lambda \qquad\qquad = \frac{1}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda t} F(\lambda) d\lambda$$

We have now turned this into a contour integral along the real line with contour Γ. It can be checked that the inversion formula does in fact work.

14.3 Sine and Consine Transform

Often, $f(t)$ is defined for $t > 0$. Then, it is useful to extend f such that $f(-t) = f(t)$ (i.e., f is even). Then the following is also even:

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda\tau} f(\tau) d\tau$$

$$F(-\lambda) = F(\lambda)$$

And so we can write the cosine and sine transformation:

Definition 14.3. *Cosine Transformation*

We can consider the cosine Fourier transform of f:

$$F_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\cos \lambda\tau) f(\tau) d\tau \, f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\cos \lambda\tau) F_c(\lambda) d\lambda$$

Definition 14.4. *Sine Transformation*

We can consider the cosine Fourier transform of f:

$$F_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\sin \lambda\tau) f(\tau) d\tau \, f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\sin \lambda\tau) F_s(\lambda) d\lambda$$

14.4 Fourier transform of derivatives

Consider functions f that vanish as $|t| \to \infty$. Then,

$$F_1(\lambda) = \mathcal{F}\{f'\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f'(t) dt$$

By integration by parts:

$$F_1(\lambda) = \frac{1}{\sqrt{2\pi}} \left[e^{i\lambda t} f(t) \right]_{-\infty}^{\infty} - i\lambda \int_{-\infty}^{\infty} e^{i\lambda t} f(t) dt \qquad\qquad = (-i\lambda) F(\lambda)$$

Similarly, the Fourier transform of higher derivatives $f^{(n)}(t)$, this is given by:

$$F_n(\lambda) = \mathcal{F}\{f^{(n)}(t)\} = (-i\lambda)^n F(\lambda) \tag{11}$$

Note, this result is most useful for ODEs and PDEs.

Example 32 (1-D Conductivity).

$$\frac{1}{\alpha} \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$$

We need to find $\phi(x,t)$ satisfying initial conditions $\phi(x,0) = g(x)$ and boundary condition $\phi(x,t) \to 0$ as $|x| \to \infty$.

Thus, we define a Fourier transofrm of ϕ with respect ot x:

$$\Phi(\lambda,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \phi(x,t)dx$$

Then, let us multiply the PDE by $\frac{1}{\sqrt{2\pi}}e^{i\lambda x}$. Then, we can integrate. We get:

$$\frac{\partial \Phi(\lambda,t)}{\partial t} = \alpha(-i\lambda)^2 \Phi(\lambda,t)$$

We note, that we interchanged diff with respect to t and integrated in r.t. x. This is an assumption that can be verified after the fact.

Note, we transformed the PDE in two variables into an ODE in one variable. The Form of the ODE is:

$$\Phi(\lambda,t) = A(\lambda)e^{-\alpha\lambda^2 t}$$

where $A(\lambda)$ is an unknown function to be defined from the initial conditions. Thus, for $t = 0$, we have:

$$A(\lambda) = \Phi(\lambda,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \phi(x,t)dx$$

We let $g(x) = \phi(x,t)$. Then, let $g(x) = Be^{-\beta x^2}$.
We do the integration and get:

$$\phi(x,t) = \frac{B}{\sqrt{1+4\alpha\beta t}}e^{-\frac{\beta x^2}{1+4\alpha\beta t}}$$