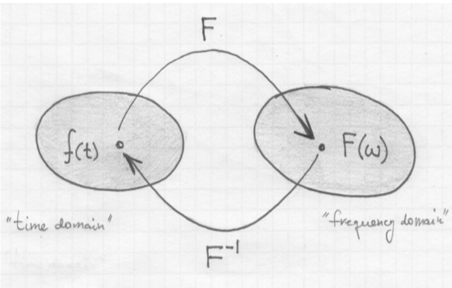
# IX.2 THE FOURIER TRANSFORM

## IX.2.1 DEFINITION — The Fourier Integral Transform Pair




The complex **Fourier transform** of a function $f(t)$ is defined as

$$F\{f(t)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

and the **inverse Fourier Transform** is defined as

$$f(t) = F^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t} d\omega$$

The function $f(t)$ from the *"time domain"* is translated to its *spectrum* – the function $F(\omega)$ in the *"frequency domain"*.

The inverse Fourier transform reconstructs the function $f(t)$ from its spectrum:

***Fourier integral formula***

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi\right] e^{i\omega t} d\omega$$

Jean-Baptiste Joseph Fourier
(1768 - 1830)

The question arises if the function reconstructed from its spectrum coincides with the original function. The answer to this question is given for functions satisfying certain conditions.

***Theorem 9.2 The Fourier Integral Theorem (Fourier, 1822)***

Let the function $f(t)$ satisfy the conditions (***Dirichlet's conditions***):

i) $f(t)$ has only a finite number of finite discontinuities (jumps) in the interval $(-\infty,\infty)$ and has no infinite discontinuities.

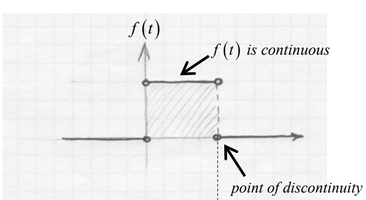
ii) $f(t)$ has only the finite number of extrema (maxima and minima) in the interval $(-\infty,\infty)$.

And let the function $f(t)$ be *absolutely integrable* on $(-\infty,\infty)$:

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty .$$

Then the Fourier integral converges to the function $f(t)$ at the points $t \in (-\infty,\infty)$ where $f(t)$ is continuous, and it converges to $\frac{1}{2}[f(t-0)+f(t+0)]$ (average value of the jump) at the points where the function $f(t)$ is discontinuous:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi\right] e^{i\omega t} d\omega \to \begin{cases} f(t) & f \textit{ is cont. at } t \\ \dfrac{f(t^-)+f(t^+)}{2} & f \textit{ is discont. at } t \end{cases}$$




$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi} d\xi\right] e^{i\omega t} d\omega = f(t)$$


The *Theorem 9.2* provides only the sufficient condition of the existence of the Fourier and the inverse Fourier transforms – there are many other functions for which the transform also exists.

## IX.2.2 PROPERTIES

Let $f(t)$ satisfy the conditions of the Fourier integral theorem on $(-\infty,\infty)$. Denote the Fourier transform and the inverse Fourier transform by

$$\hat{f}(\omega) \equiv F\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \qquad F^{-1}\{\hat{f}(\omega)\} \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

Consider some important properties of the Fourier transform:

**1)** *Linearity:*

$$F\{af(t)+bg(t)\} = aF\{f(t)\} + bF\{g(t)\}$$
$$F^{-1}\{\alpha\hat{f}(\omega)+\beta\hat{g}(\omega)\} = \alpha F^{-1}\{\hat{f}(\omega)\} + \beta F^{-1}\{\hat{g}(\omega)\}$$

**2)** *Shifting in* $\omega$:

$$F\{e^{i\omega_0 t} f(t)\} = \hat{f}(\omega-\omega_0) \qquad \omega_0 \in \mathbb{R}$$
$$F^{-1}\{\hat{f}(\omega-\omega_0)\} = e^{i\omega_0 t} F^{-1}\{\hat{f}(\omega)\}$$

**3)** *Shifting in t:*

$$F\{f(t-t_0)\} = e^{-i\omega t_0} F\{f(t)\} = e^{-i\omega t_0} \hat{f}(\omega) \qquad t_0 \in \mathbb{R}$$

**4)** *Scaling:*

$$F\{f(at)\} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right) \qquad a \in \mathbb{R}$$

**5)** *Duality:*

$$F\{\hat{f}(t)\} = f(-\omega)$$ *(not a misprint)*

**6)** ***Convolution***:

$$f * g = \int_{-\infty}^{\infty} f(t-x) g(x) dx$$
$$F\{f*g\} = F\{f\}F\{g\} = \hat{f}(\omega)\hat{g}(\omega)$$
$$F^{-1}\{\hat{f}(\omega)\hat{g}(\omega)\} = f*g$$

**Operational properties**

**7)** *Derivatives:* Assume $\lim_{t\to\infty} f(t) = 0$ and $\lim_{t\to\infty} f^{(k)}(t) = 0$, $k = 1,2,\ldots$

$$F\{f'(t)\} = i\omega\hat{f}(\omega)$$
$$F\{f''(t)\} = -\omega^2\hat{f}(\omega)$$
$$\vdots$$
$$F\{f^{(n)}(t)\} = (i\omega)^n \hat{f}(\omega)$$

*Fourier Transform in x variable of the function of two variables* $u(x,t)$

Denote the Fourier transform in the $x$ variable of the function $u(x,t)$ as

$$F\{u(x,t)\} \equiv \hat{u}(\omega,t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx$$

Assume that $\lim_{x\to\pm\infty} u(x,t) = 0$ and $\lim_{x\to\pm\infty} \frac{\partial^k u(x,t)}{\partial x^k} = 0$, $k = 1,2,\ldots$, then

**8)** *Derivative in t:*

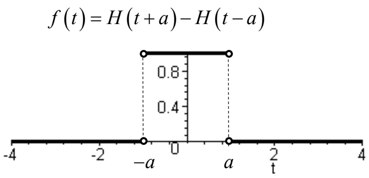
$$F\left\{\frac{\partial}{\partial t}u(x,t)\right\} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x} dx = \frac{\partial}{\partial t}\hat{u}(\omega,t)$$

**9)** *Derivatives in x:*

$$F\left\{\frac{\partial}{\partial x}u(x,t)\right\} = i\omega\hat{u}(\omega,t)$$
$$F\left\{\frac{\partial^2}{\partial x^2}u(x,t)\right\} = -\omega^2\hat{u}(\omega,t)$$
$$\vdots$$
$$F\left\{\frac{\partial^n}{\partial x^n}u(x,t)\right\} = (i\omega)^n \hat{u}(\omega,t)$$

## IX.2.3 EXAMPLES

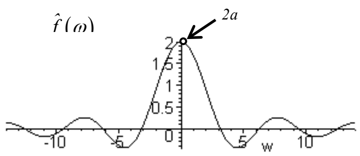
### 1. *Unit Pulse Function – gate function – filter function*

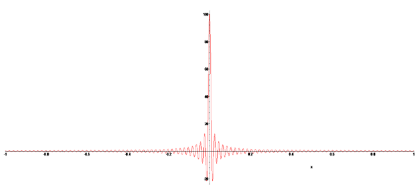



The unit pulse function can be defined with the help of the Heaviside unit step function

$$f(t)=H(t+a)-H(t-a)=\begin{cases} 0 & x<-a \\ 1 & |x|<a \\ 0 & x>a \end{cases} \qquad a>0$$

The Fourier transform of this function can be determined as



$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \qquad =\int_{-a}^{a} e^{-i\omega t}dt$$

$$=\frac{-1}{i\omega}\left[e^{-i\omega t}\right]_{-a}^{a}$$

$$=\frac{-1}{i\omega}\left(e^{-i\omega a}-e^{i\omega a}\right) \qquad \textit{use Euler's formula}$$

$$=\frac{2}{\omega}\left(\frac{\cancel{\cos\omega a}+i\sin\omega a-\cancel{\cos\omega a}+i\sin\omega a}{2i}\right)$$

$$=\frac{2}{\omega}\sin(a\omega)$$

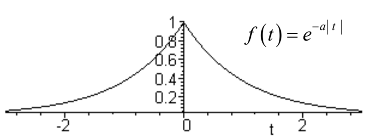
$$=2a\frac{\sin(a\omega)}{a\omega} \qquad \textit{see p.484 for sinc}$$

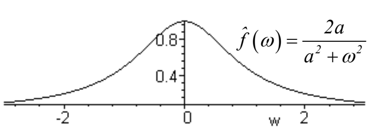
### 2. *Two-sided exponential function*

Consider the even two-sided exponential function:

$$f(t)=e^{-a|t|} \qquad a>0$$

Then the Fourier transform of this function can be evaluated as




$$\hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \qquad =\int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}dt$$

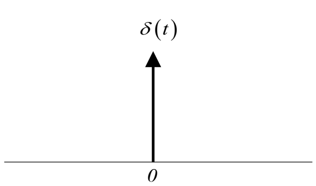
$$=\int_{-\infty}^{0} e^{(a-i\omega)t}dt+\int_{0}^{\infty} e^{-(a+i\omega)t}dt$$

$$=\frac{1}{a-i\omega}\left[e^{-(a+i\omega)t}\right]_{-\infty}^{0}-\frac{1}{a+i\omega}\left[e^{-(a+i\omega)t}\right]_{0}^{\infty}$$

$$=\frac{1}{a-i\omega}+\frac{1}{a+i\omega}$$

$$=\frac{2a}{a^2+\omega^2}$$

### 3. *Dirac delta* $f(t)=\delta(t)$



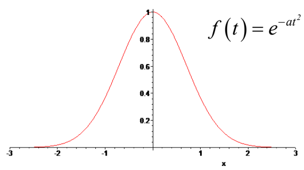

$$F\{\delta(t)\}=\int_{-\infty}^{\infty}\delta(t)\ e^{-i\omega t}dt=e^{0}=1$$

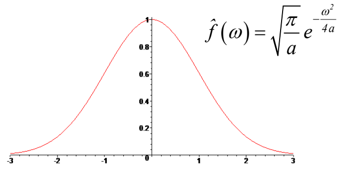
$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty}\delta(\xi)e^{-i\omega\xi}d\xi\right]}_{1}e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}d\omega=\underbrace{\left(\lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a}e^{i\omega t}d\omega=\lim_{a\to\infty}\frac{\sin a\omega}{\omega}\right)}_{\text{not confitmed}}$$

## 4. *Gaussian distribution function*

$$f(t)=e^{-at^2}, \qquad a>0$$

The Fourier transform of $f(t)=e^{-at^2}$ can be evaluated as




$$\begin{aligned}
F\left\{e^{-at^2}\right\} &= \int_{-\infty}^{\infty} e^{-at^2} e^{-i\omega t}\,dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega t-at^2}\,dt \\
&= \int_{-\infty}^{\infty} e^{-a\left(t+\frac{i\omega}{2a}\right)^2-\frac{\omega^2}{4a}}\,dt \\
&= e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(t+\frac{i\omega}{2a}\right)^2}\,dt \\
&= \frac{1}{\sqrt{a}} e^{-\frac{\omega^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-a\left(t+\frac{i\omega}{2a}\right)^2} d\left[\sqrt{a}\left(t+\frac{i\omega}{2a}\right)\right]}_{\sqrt{\pi}} \\
&= \sqrt{\frac{\pi}{a}}\, e^{\frac{-\omega^2}{4a}}
\end{aligned}$$

*u-substitution*

$$u=\sqrt{a}\left(t+\frac{i\omega}{2a}\right)$$

$$\frac{1}{\sqrt{a}}du=dt$$

$$\int_{-\infty}^{\infty} e^{-u^2}du=\sqrt{\pi}$$

The inverse Fourier transform of $e^{-a\omega^2}$ can be evaluated as

$$\begin{aligned}
F^{-1}\left\{e^{-a\omega^2}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a\omega^2} e^{i\omega t}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a\left[\left(\omega-\frac{it}{2a}\right)^2+\frac{t^2}{4a^2}\right]}\,d\omega \\
&= \frac{1}{2\pi} e^{\frac{-t^2}{4a}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\left(\omega-\frac{it}{2a}\right)\right]^2}\,d\omega \\
&= \frac{1}{2\pi\sqrt{a}} e^{-\frac{\omega^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\,du}_{\sqrt{\pi}} \\
&= \frac{1}{\sqrt{4\pi a}} e^{\frac{-t^2}{4a}}
\end{aligned}$$

*u-substitution*

$$u=\sqrt{a}\left(\omega-\frac{it}{2a}\right)$$

$$\frac{1}{\sqrt{a}}du=d\omega$$

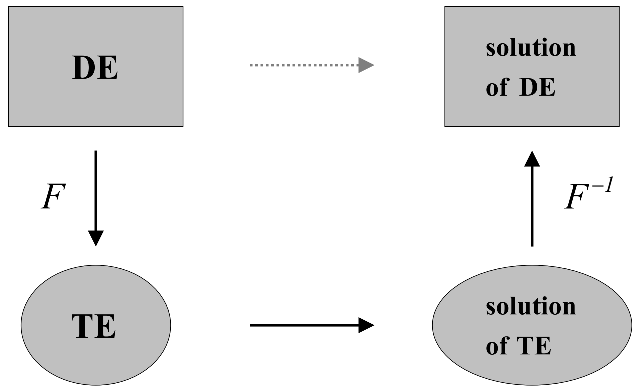
$$\int_{-\infty}^{\infty} e^{-u^2}du=\sqrt{\pi}$$

## 5. Derive the Fourier transform of the derivative (*property 7*):

$$\begin{aligned}
F\left\{\frac{d}{dt}f(t)\right\} &= \int_{-\infty}^{\infty}\left[\frac{d}{dt}f(t)\right]e^{-i\omega t}\,dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega t}\,d\,f(t) \\
&= \left[e^{-i\omega t}f(t)\right]_{-\infty}^{\infty}-\int_{-\infty}^{\infty} f(t)\,de^{-i\omega t} \qquad \textit{integration by parts} \\
&= 0+i\omega\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \\
&= i\omega\hat{f}(\omega)
\end{aligned}$$

## Application of integral transform for the solution of PDE

According to properties 7) and 9), application of the Fourier Transform eliminates the derivatives with respect to time or to space variables. This fact will be used for the solution of the differential equations. First, we transform a differential equation to eliminate the derivative of unknown function, then we solve (algebraic) transformed equation in the frequency space, and finally, the solution of the original problem will be obtained by the inverse transform:




## IX.2.4 SOLUTION OF THE ORDINARY DIFFERENTIAL EQUATIONS

***Example 4*** *(Steady-State Conduction)*

Solve the 2nd order ordinary differential equation

$$\frac{d^2y}{dx^2} - a^2 y + f(x) = 0 \qquad x \in (-\infty, \infty)$$

with the help of the Fourier transform.

***Solution:***

Apply the Fourier transform $\hat{y}(\omega) = \int_{-\infty}^{\infty} y(x) e^{-i\omega x} dx$ to the given equation, using *Property (7)* for the transform of the 2nd derivative, assuming $\lim\limits_{x\to\pm\infty} u(x) = 0$, $\lim\limits_{x\to\pm\infty} \frac{\partial^k u(x)}{\partial x^k} = 0$, $k = 1,2$:

$$-\omega^2 \hat{y}(\omega) - a^2 \hat{y}(\omega) + \hat{f}(\omega) = 0 \qquad \textit{transformed equation}$$

Solve the transformed equation for $\hat{y}(\omega)$

$$\boxed{\hat{y}(\omega) \quad = \frac{\hat{f}(\omega)}{a^2 + \omega^2}} \qquad \textit{transformed solution}$$

Note that the function $\hat{g}(\omega) = \frac{1}{a^2 + \omega^2}$ is the Fourier transform of the two-sided exponential function (*Example 2*):

$$\hat{g}(\omega) = \frac{1}{a^2 + \omega^2} = F\left\{\frac{1}{2a} e^{-a|x|}\right\} = F\{g(x)\}$$

Then the transformed solution can be written as the product of two transformed functions:

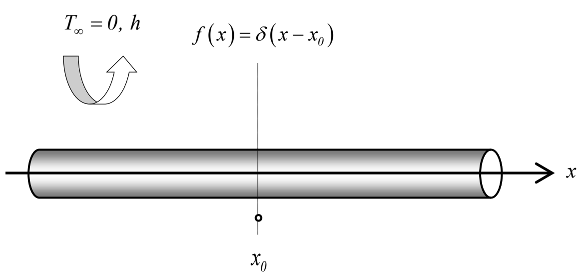
$$\hat{y}(\omega) \quad = \hat{g}(\omega) \hat{f}(\omega)$$

The solution of the differential equation can be found by inverse transform with application of the convolution theorem:

$$
\begin{aligned}
y(x) &= F^{-1}\{\hat{y}(\omega)\} = F^{-1}\{\hat{g}(\omega)\hat{f}(\omega)\} \\
&= g * f \qquad \textit{convolution} \\
&= \int_{-\infty}^{\infty} g(x-s) f(s) ds \\
&= \frac{1}{2a}\int_{-\infty}^{\infty} e^{-a|x-s|} f(s) ds
\end{aligned}
$$

*The problem can be interpreted as a steady state conduction in the thin rod infinite in both directions (lumped capacitance model) exposed to convective enviroment at 0 temperature,* $a^2 = \frac{hP}{kA_c}$ *with a point heat source at* $x = x_0$.

***Case of a point source*** $f(x) = \delta(x - x_0)$ *:*




Consider the case when the source function $f(x)$ is the Dirac delta function

$$f(x) = \delta(x - x_0)$$

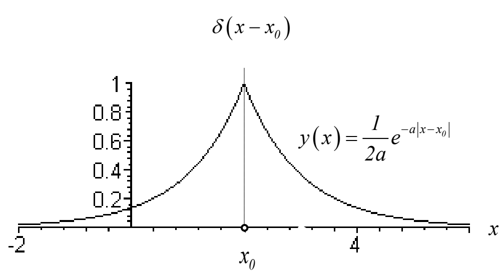
defining the constant point source at $x = x_0$.

Then integration using the property of the delta function

$$\int_{-\infty}^{\infty} u(s)\,\delta(s - x_0)\,ds = u(x_0)$$

yields the solution of the differential equation:

$$y(x) = \frac{1}{2a}\int_{-\infty}^{\infty} e^{-a|x-s|}\delta(s - x_0)\,ds$$



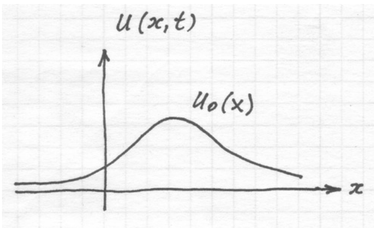

$$\boxed{y(x) = \frac{1}{2a}e^{-a|x-x_0|}} \qquad \textit{solution}$$

The graph of the solution is the graph of the double-sided exponential function (*Example 2*) centered at $x = x_0$.

## IX.2.5 SOLUTION OF PDEs IN THE INFINITE REGION

### IX.2.5.1 Heat Equation – Gauss's Kernel – Green's Function

***Example 5*** (*1-D heat conduction with heat generation in the infinite region - Gauss's Kernel*)




Consider the initial-boundary value problem for non-homogeneous heat equation in the infinite slab (see Chapter 4):

$$\frac{\partial^2 u(x,t)}{\partial x^2} + S(x,t) = a^2\,\frac{\partial u(x,t)}{\partial t} \qquad t > 0\,, \quad x \in (-\infty,\infty)$$

with the initial condition:

$$u(x,0) = u_0(x) \qquad x \in (-\infty,\infty)$$

The source function defines the heat generation in the slab:

$$S(x,t) = \frac{g(x,t)}{k}\,, \text{ where } g(x,t), \left[\frac{W}{m^3}\right] \text{ volumetric heat source}$$

There are no boundary conditions for the infinite region; but from physical considerations, we assume that both the unknown function and its derivative vanish when approaching $\pm\infty$ :

$$u(x,t)\,\Big|_{x\to\pm\infty} = 0 \qquad \frac{\partial u(x,t)}{\partial x}\Bigg|_{x\to\pm\infty} = 0$$

This assumption will allow application of property (7) of the Fourier transform which will be used for solution of the given IBVP.

*transformation w.r.t variable x*

$$F\left[u(x,t)\right] = \int_{-\infty}^{\infty} u(x,t)\; e^{-i\omega x} dx = \hat{u}(\omega,t)$$

**Transformed equation** Take the Fourier integral transform of the equation (applying properties (1) and (7) for the transform of the derivatives)

$$-\,\omega^2 \hat{u}(\omega,t) + \hat{S}(\omega,t) = a^2\,\frac{\partial \hat{u}(\omega,t)}{\partial t}$$

with the transformed initial condition:

$$\hat{u}(\omega,0) = \hat{u}_0(\omega)$$

where the following notations for the transformed functions are used

$$F\left[u(x,t)\right] = \int_{-\infty}^{\infty} u(x,t)\; e^{-i\omega x} dx = \hat{u}(\omega,t)$$

$$F\left[u_0(x)\right] = \int_{-\infty}^{\infty} u_0(x)\; e^{-i\omega x} dx = \hat{u}_0(\omega)$$

$$F\left[S(x,t)\right] = \int_{-\infty}^{\infty} S(x,t)\; e^{-i\omega x} dx = \hat{S}(\omega,t)$$

**Solution of transformed equation.** The obtained transformed equation is an ordinary linear differential equation in variable $t$ with constant coefficients:

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} + \frac{\omega^2}{a^2}\hat{u}(\omega,t) = \frac{\hat{S}(\omega,t)}{a^2} \qquad \hat{u}(\omega,0) = \hat{u}_0(\omega)$$

the general solution of which can be obtained by the variation of parameter formula (see Table ODE):

***transformed solution***

$$\hat{u}(\omega,t) = \hat{u}_0(\omega) e^{\frac{-\omega^2}{a^2}t} + \frac{1}{a^2} e^{\frac{-\omega^2}{a^2}t} \int_0^t e^{\frac{\omega^2}{a^2}t'}\; \hat{S}(\omega,t')dt' \qquad (\Diamond)$$

**Inverse transform** Solution of the given IBVP now can be obtained by the inverse Fourier integral transform of the expression (◊):

$$u(x,t) = F^{-1}\{\hat{u}(\omega,t)\} \qquad = F^{-1}\left\{\hat{u}_0(\omega)e^{\frac{-\omega^2}{a^2}t}\right\} + F^{-1}\left\{\frac{1}{a^2}\int_0^t e^{-\frac{\omega^2}{a^2}(t-t')}\hat{S}(\omega,t')dt'\right\}$$

Functions $e^{\frac{-\omega^2}{a^2}t}$ and $e^{\frac{-\omega^2}{a^2}(t-t')}$ which appear in the integrand are the Fourier transforms of the Gaussian distribution functions (p.725).

***Gauss's Kernel***

$$G(x,t) = \frac{a}{2\sqrt{\pi t}}e^{\frac{-a^2}{4t}x^2} \qquad G(x,t-t') = \frac{a}{2\sqrt{\pi(t-t')}}e^{\frac{-a^2}{4(t-t')}x^2}$$

$$\hat{G}(\omega,t) = e^{\frac{-\omega^2}{a^2}t} \qquad \hat{G}(\omega,t-t') = e^{\frac{-\omega^2}{a^2}(t-t')}$$

$$u(x,t) = F^{-1}\{\hat{u}(\omega,t)\} \qquad = F^{-1}\left\{\hat{u}_0(\omega)\hat{G}(\omega,t)\right\} + \frac{1}{a^2}F^{-1}\left\{\int_0^t \hat{S}(\omega,t')\hat{G}(\omega,t-t')dt'\right\}$$

$$= F^{-1}\left\{\hat{u}_0(\omega)\hat{G}(\omega,t)\right\} + \frac{1}{a^2}\int_0^t F^{-1}\left\{\hat{S}(\omega,t')\hat{G}(\omega,t-t')\right\}dt'$$

Then the first integral in the solution is the inverse Fourier transform of the product of two transformed functions $\hat{u}_0\hat{G}$ which according to property (3) is a convolution of these two functions

$$F^{-1}\left\{\hat{u}_0\hat{G}\right\} = u_0 * G = \int_{-\infty}^{\infty} u_0(s)G(x-s,t)ds \qquad = \frac{a}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{\frac{-a^2}{4t}(x-s)^2}u_0(s)ds$$

Consider the second term in the solution. It leads to the convolution:

$$F^{-1}\left\{\hat{G}\hat{S}\right\} = G * S = \int_{-\infty}^{\infty} G(x-s,t-t')S(s)ds \qquad = \frac{a}{2\sqrt{\pi(t-t')}}\int_0^t\int_{-\infty}^{\infty} e^{\frac{-a^2(x-s)^2}{4(t-t')}}S(s,t')dsdt'$$

Then a formal solution of the given IBVP is:

**Solution:**

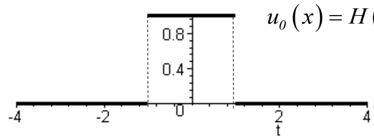
$$u(x,t) = \frac{a}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{\frac{-a^2}{4t}(x-s)^2}u_0(s)ds + \frac{1}{2ak\sqrt{\pi}}\int_0^t\int_{-\infty}^{\infty}\frac{e^{\frac{-a^2(x-s)^2}{4(t-t')}}}{\sqrt{(t-t')}}S(s,t')dsdt'$$

**Particular cases:**

Consider two particular cases:

**a) Homogeneous equation** (no heat generation, $S(x,t)=0$ )

Let the initial temperature distribution have a step-wise variation:




Then solution is given by the integral over the finite region:

$$u(x,t)=\frac{a}{2\sqrt{\pi t}}\int_{-1}^{1} e^{\frac{-a^2}{4t}(x-s)^2}ds$$

Use change of variable: $v=\frac{a}{2\sqrt{t}}(x-s)$ $dv=\frac{-a}{2\sqrt{t}}ds$

$$u(x,t)=\frac{a}{2\sqrt{\pi t}}\int_{-1}^{1} e^{-\frac{a^2}{4t}(x-s)^2}ds \qquad =-\frac{a}{2\sqrt{\pi t}}\frac{2\sqrt{t}}{a}\int_{\frac{a(x+1)}{2\sqrt{t}}}^{\frac{a(x-1)}{2\sqrt{t}}} e^{-v^2}dv$$

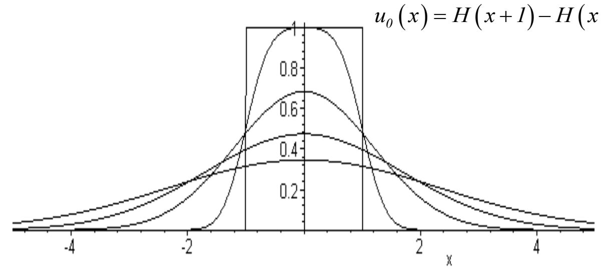
$$=\frac{1}{2}\left[\frac{2}{\sqrt{\pi}}\int_{0}^{\frac{a(x+1)}{2\sqrt{t}}} e^{-v^2}dv-\frac{2}{\sqrt{\pi}}\int_{0}^{\frac{a(x-1)}{2\sqrt{t}}} e^{-v^2}dv\right]$$

$$=\frac{1}{2}erf\left[\frac{a(x+1)}{2\sqrt{t}}\right]-\frac{1}{2}erf\left[\frac{a(x-1)}{2\sqrt{t}}\right]$$

Therefore, the solution of the Heat Equation in the infinite region with no heat source is given by

$$\boxed{u(x,t)=\frac{1}{2}erf\left[\frac{a(x+1)}{2\sqrt{t}}\right]-\frac{1}{2}erf\left[\frac{a(x-1)}{2\sqrt{t}}\right]}$$

The graph shows the temperature profiles for $a=2$



### b) Non-homogeneous equation - *Green's function*

Consider a heat conduction problem with a zero initial condition

$$u_0(x) = 0$$

and a point heat source located at $x = x_0$ which instantaneously released energy at time $t = t_0$ of strength $S_0$ (***impulse point source***):

***impulse point source***

$$S(x,t) = S_0 \delta(x - x_0)\delta(t - t_0)$$

Then the first term in the solution disappears because of the zero initial condition, and the solution becomes

$$u(x,t) = \frac{S_0}{2ak}\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi(t-t')}} e^{\frac{-a^2(x-s)^2}{4(t-t')}} \delta(s - x_0)\delta(t' - t_0)\,ds\,dt'$$

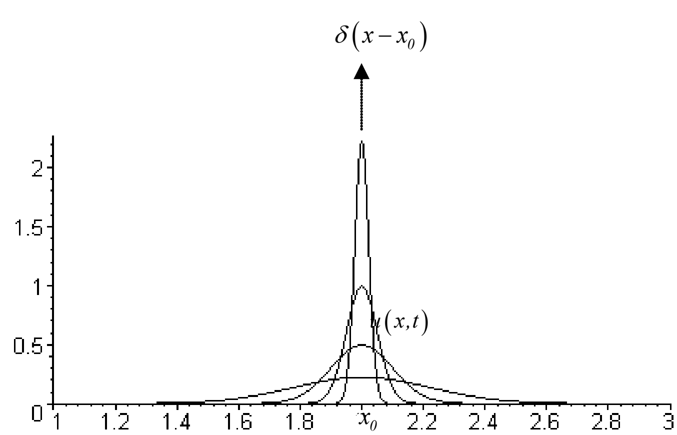
Then integration of the expression with the Dirac delta functions yields the solution

***Green's Function***

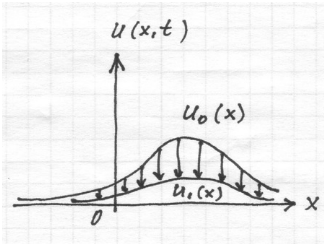
$$u(x,t) = \begin{cases} \dfrac{S_0}{2ak}\dfrac{e^{\frac{-a^2(x-x_0)^2}{4(t-t_0)}}}{\sqrt{\pi(t-t_0)}} & t > t_0 \\ 0 & t \le t_0 \end{cases}$$

This solution of the problem with the impulse point source is called the ***Green function*** for the infinite region in the Cartesian coordinate system. It is used for solution of non-homogeneous partial differential equations.

***Example*** The solution curves at different moments of time for $a = 2$, $S_0 = 1$, $k = 2$, $t_0 = 1$, $x_0 = 2$



## IX.2.5.2 Wave Equation *Example 6* Wave equation for an infinite string – **D'Alambert solution**




$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} \qquad t > 0, \quad -\infty < x < \infty$$

with initial conditions:

$$u(x,0) = u_0(x) \qquad -\infty < x < \infty \qquad \text{initial displacement}$$

$$\frac{\partial u(x,0)}{\partial t} = u_1(x) \qquad \text{initial velocity}$$

The coefficient in the Wave Equation $v$ is a physical property of the string and represents a speed of wave propagation along the string. It is determined through the equation

$$v^2 = \frac{Tg}{w} \quad \left[\frac{m^2}{s^2}\right]$$

where $T$ is a tension, $g$ is the acceleration of gravity, $w$ is a weight of the string per unit length.

**1) Transformed equation** Apply the Fourier transform to the wave equation and initial conditions:

$$-\omega^2 \hat{u}(\omega,t) = \frac{1}{v^2}\frac{\partial^2 \hat{u}(\omega,t)}{\partial t^2}$$

$$\hat{u}(\omega,0) = \hat{u}_0(\omega)$$

$$\frac{\partial \hat{u}(\omega,0)}{\partial t} = \hat{u}_1(\omega)$$

This is the initial value problem for a 2nd order linear ODE in $t$ with constant coefficients

$$\frac{\partial^2 \hat{u}(\omega,t)}{\partial t^2} + \omega^2 v^2 \hat{u}(\omega,t) = 0$$

The general solution is

$$\hat{u}(\omega,t) = c_1 \cos \omega vt + c_2 \sin \omega vt$$

where coefficients can be found from initial conditions.

$$c_1 = \hat{u}_0(\omega)$$

$$c_2 = \frac{\hat{u}_1(\omega)}{\omega v}$$

then the solution of the ODE becomes

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\cos \omega vt + \frac{\hat{u}_1(\omega)}{\omega v}\sin \omega vt$$

**2) Inverse transform** The solution of the IBVP can now be obtained using the inverse Fourier transform

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{u}_0(\omega)\cos \omega vt + \frac{\hat{u}_1(\omega)}{\omega v}\sin \omega vt\right]e^{i\omega x}d\omega}$$

This is the general form of the solution which includes in the integrand the Fourier transform of the initial conditions. For some functions, it can be integrated explicitly.

It can be shown (V.S.Vladimirov *Equation of Mathematical Physics, p.176*) that **D'Alambert solution** with arbitrary functions $u_0(x) \in C^2(\mathbb{R})$ and $u_1(x) \in C^1(\mathbb{R})$:

$$u(x,t) = \frac{1}{2}\left[u_0(x-vt) + u_0(x+vt)\right] + \frac{1}{2v}\int_{x-vt}^{x+vt} u_1(s)\,ds$$

is also a solution of the IBVP. Moreover, the solution of the IVP for the Wave Equation is unique.

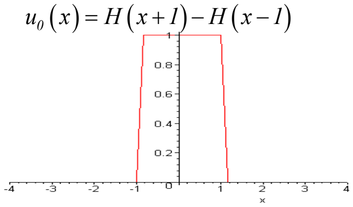
**<u>Particular cases</u>:**

**1.** Consider the IBVP for an infinite string with initial conditions

$$u_0(x) = H(x+1) - H(x-1)$$
$$u_1(x) = 0$$

*initial condition: (gate function)*

$u_0(x) = H(x+1) - H(x-1)$




Fourier transform of initial conditions yields

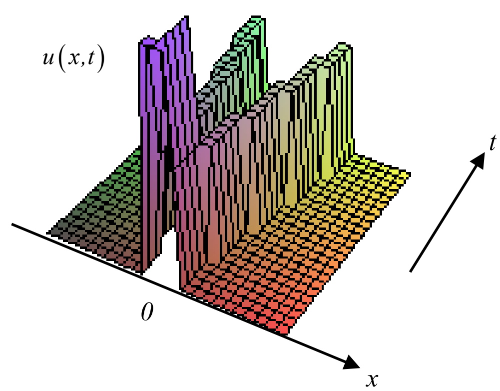
$$\hat{u}_0(\omega) = e^{i\omega}\left[\pi\delta(\omega) - \frac{1}{\omega}\right] - e^{-i\omega}\left[\pi\delta(\omega) - \frac{1}{\omega}\right] = 2\frac{\sin\omega}{\omega}$$
$$\hat{u}_1(\omega) = 0$$

Then the solution of the IBVP becomes (compare to D'Alambert soln):

$$u(x,t) = \frac{1}{2}\left[H(x+1+vt) + H(x+1-vt) - H(x-1+vt) - H(x-1-vt)\right]$$

$$u(x,t) = \frac{1}{2}\left[\underbrace{H(x+1-vt) - H(x-1-vt)}_{u_0(x-vt)} + \underbrace{H(x+1+vt) - H(x-1+vt)}_{u_0(x+vt)}\right]$$

Solution curves for $c = 2$ are shown in the figure (*F-1.mws*)




**2.** Consider the IBVP for an infinite string with initial conditions

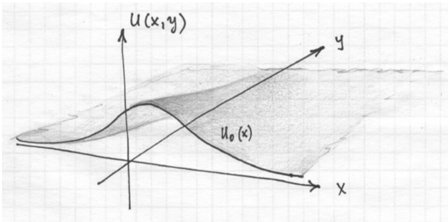
$$u_0(x) = exp(-x^2), \quad u_1(x) = 0$$

**3.** Consider the IBVP for an infinite string with initial conditions [Debnath, p.80]

$$u_0(x) = 0, \quad u_1(x) = \delta(x)$$

### IX.2.5.3 The Laplace Equation

***Example 7*** (*Laplace's Equation in a semi-infinite plane – Dirichlet problem – Poisson integral formula*)




Consider a 2-dimensional Laplace's equation in the semi-infinite plane $y>0$, but with $x\in(-\infty,\infty)$:

$$\boxed{\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0} \qquad x\in(-\infty,\infty),\ y\in(0,\infty)$$

with the boundary condition at $y=0$:

$$u(x,0)=u_0(x) \qquad \textit{Dirichlet}$$

**Transformed equation** We will apply the Fourier transform in the $x$ variable

$$F[u(x,y)]=\int_{-\infty}^{\infty}u(x,y)\ e^{-i\omega x}dx=\hat{u}(\omega,y)$$

$$F[f_0(x)]=\int_{-\infty}^{\infty}u_0(x)e^{-i\omega x}dx=\hat{u}_0(\omega)$$

Transformed equation

$$-\omega^2\hat{u}+\frac{\partial^2\hat{u}}{\partial y^2}=0$$

has the general solution:

$$\hat{u}=c_1e^{-\omega y}+c_2e^{\omega y} \qquad \text{The solution is bounded} \quad \Rightarrow \quad \begin{array}{l} c_2=0 \text{ for } \omega>0 \\ c_1=0 \text{ for } \omega<0 \end{array}$$

Then two terms of the solution can be combined

$$\hat{u}=ce^{-|\omega|y}$$

Application of the boundary condition yields

$$\hat{u}=\hat{u}_0e^{-|\omega|y}$$

**Inverse transform** Function $e^{-|\omega|y}$ is a Fourier transform of

***Poisson's kernel***

$$\frac{1}{\pi}\frac{y}{x^2+y^2}$$

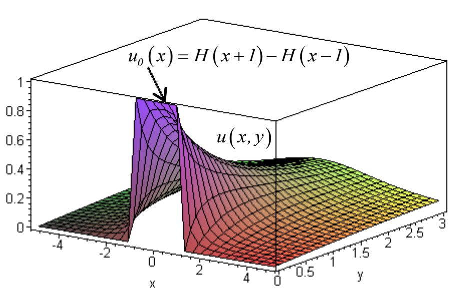
then the solution of the transformed equation can be written as

$$\hat{u}=\hat{u}_0F\left\{\frac{1}{\pi}\frac{y}{x^2+y^2}\right\}$$

which is a product of two Fourier transforms. Then the inverse transform of $\hat{u}$ can be written as a convolution of these two functions:

***Poisson's integral formula***

$$\boxed{u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u_0(s)}{(x-s)^2+y^2}ds}$$




It is a solution of the Dirichlet problem for Laplace's equation in the semi-plane which is called ***Poisson's integral formula*** for the upper half-plane.

For the initial condition $u_0(x)=H(x+1)-H(x-1)$

the solution is given by: $u(x,y)=\dfrac{1}{\pi}arctan\dfrac{x+1}{y}-\dfrac{1}{\pi}arctan\dfrac{x-1}{y}$

F-1 Poisson integral.mws **Solution of Laplace's Equation in the upper half-plane - Poisson's integral formula**

```
> restart;
> u0:=Heaviside(s+1)-Heaviside(s-1);
```

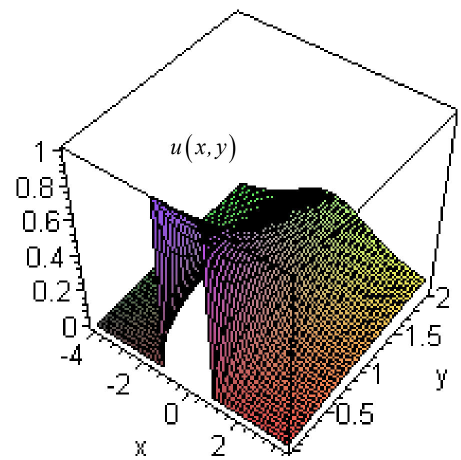
$$u0 := \text{Heaviside}(s+1) - \text{Heaviside}(s-1)$$

**Poisson's Integral Formula:**

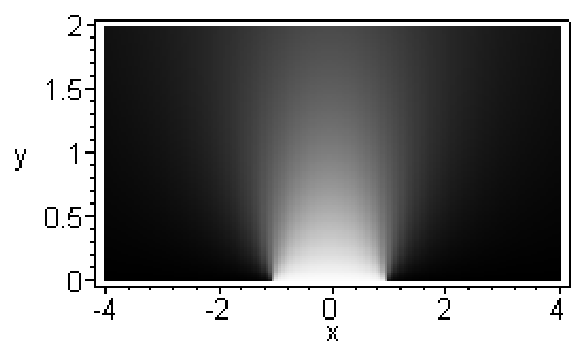
```
> u(x,y):=simplify(int(u0/((x-s)^2+y^2),s=-infinity..infinity)*y/Pi);
```

$$u(x,y) := -\frac{-\arctan\left(\frac{1+x}{y}\right) + \arctan\left(\frac{-1+x}{y}\right)}{\pi}$$

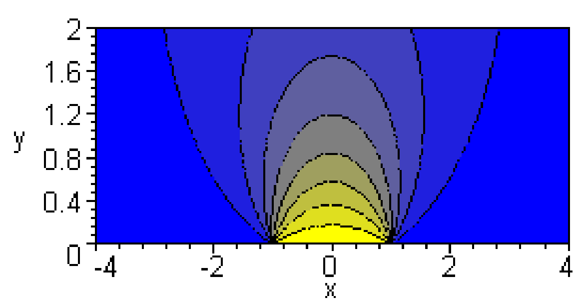
```
> plot3d(u(x,y),x=-4..4,y=0..2,grid=[50,50]);
```




```
> with(plots):
> densityplot(u(x,y),x=-4..4,y=0..2,grid=[150,150],style=patchnogrid,axes=boxed);
```



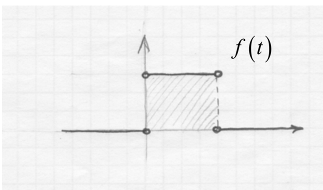

```
> contourplot(u(x,y),x=-4..4,y=0..2,axes=boxed,coloring=[blue,yellow],filled=true);
```



## IX.2.6 FOURIER INTEGRALS

**a) Complex Fourier integral representation** Suppose that the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the Dirichlet condition in every finite interval of $\mathbb{R}$ (see p.696) and suppose that there exists an improper integral

$$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty$$



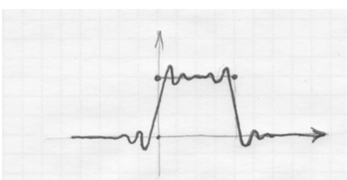

$$\boxed{f(t)=\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega}$$

where the coefficient function is given by

$$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$$

**b) Standard Fourier integral representation** Suppose that the function $f:(0,\infty)\to\mathbb{R}$ satisfies the Dirichlet condition in every finite interval of $\mathbb{R}$ and

$$\int_{0}^{\infty}|f(t)|\,dt<\infty$$



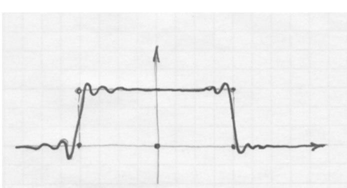
$$\boxed{f(t)=\int_{0}^{\infty}\left[A(\omega)\cos\omega t+B(\omega)\sin\omega t\right]d\omega}$$

where the real coefficient functions are given by

$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos\omega t dt$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin\omega t dt$$

**c) Fourier cosine integral representation**

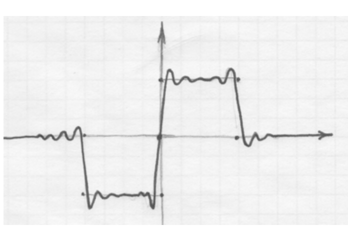


$$\boxed{f(t)=\int_{0}^{\infty}A(\omega)\cos\omega t d\omega}$$

where the real coefficient functions are given by

$$A(\omega)=\frac{2}{\pi}\int_{0}^{\infty}f(t)\cos\omega t dt$$

**d) Fourier sine integral representation**



$$\boxed{f(t)=\int_{0}^{\infty}B(\omega)\sin\omega t d\omega}$$

where the real coefficient functions are given by

$$B(\omega)=\frac{2}{\pi}\int_{0}^{\infty}f(t)\sin\omega t dt$$

***Convergence*** All given Fourier integral representations converge to:

$$f(t) \to \begin{cases} f(t) & \textit{if } f \textit{ is continuous at } t \\ \dfrac{f(t^-)+f(t^+)}{2} & \textit{if } f \textit{ is discontinuous at } t \end{cases}$$

*FI-01.mws* **Fourier Integral Representation** W = the upper limit in the Fourier Integral (defines the accuracy of approximation)

```
> W:=20;
```

$$W := 20$$

```
> f(t):=Heaviside(t)-Heaviside(t-1);
```

$$f(t) := \text{Heaviside}(t) - \text{Heaviside}(t-1)$$

**Standard Fourier Integral:**

```
> A(w):=int(f(t)*cos(w*t),t=-infinity...infinity)/Pi;
```

$$A(w) := \frac{\sin(w)}{w\,\pi}$$

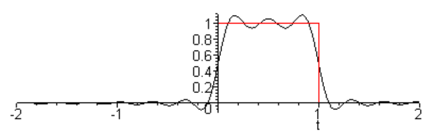
```
> B(w):=int(f(t)*sin(w*t),t=-infinity...infinity)/Pi;
```

$$B(w) := -\frac{-1+\cos(w)}{w\,\pi}$$

```
> u(t):=int(A(w)*cos(w*t)+B(w)*sin(w*t),w=0..W);
```

$$u(t) := -\frac{\text{Si}(20\,t-20)-\text{Si}(20\,t)}{\pi}$$

```
> plot({f(t),u(t)},t=-2..2,color=[red,black]);
```



**Fourier Cosine Integral:**

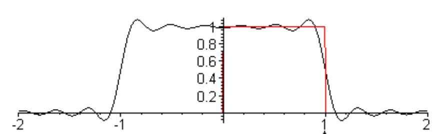
```
> A(w):=2*int(f(t)*cos(w*t),t=-infinity...infinity)/Pi;
```

$$A(w) := \frac{2\sin(w)}{w\,\pi}$$

```
> u(t):=int(A(w)*cos(w*t),w=0..W);
```

$$u(t) := \frac{\text{Si}(20+20\,t)-\text{Si}(20\,t-20)}{\pi}$$

```
> plot({f(t),u(t)},t=-2..2,color=[red,black]);
```



**Fourier Sine Integral:**

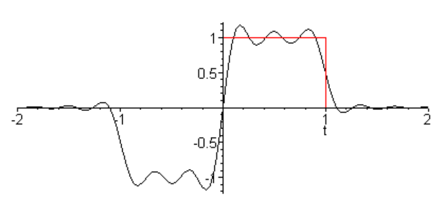
```
> B(w):=2*int(f(t)*sin(w*t),t=-infinity...infinity)/Pi;
```

$$B(w) := -\frac{2\,(-1+\cos(w))}{w\,\pi}$$

```
> u(t):=int(B(w)*sin(w*t),w=0..W);
```

$$u(t) := \frac{2\,\text{Si}(20\,t)-\text{Si}(20+20\,t)-\text{Si}(20\,t-20)}{\pi}$$

```
> plot({f(t),u(t)},t=-2..2,color=[red,black]);
```



## IX.2.7 INTEGRAL FOURIER TRANSFORM IN THE SEMI-INFINITE REGIONS


Consider a semi-infinite region $0 \le x < \infty$.

Define the integral transform pair of the function $u(x)$ as

$$F\{u\} = \qquad \hat{u}(\omega) = \int_0^\infty u(x) K(\omega, x) dx$$

$$F^{-1}\{\hat{u}\} = \qquad u(x) = \int_0^\infty \hat{u}(\omega) K(\omega, x) d\omega$$

**1. Fourier Integral Transform kernel**

We are looking for a specific form of the kernel $K(x,\omega)$ which can be applied for particular form of the boundary condition at $x = 0$. In the following Table, we specify the kernel $K(x,\omega)$ for three types of classical homogeneous boundary conditions:

| Boundary condition | Kernel $K(x,\omega)$ | Integral transform pair |
|---|---|---|
| **I** $[u]_{x=0} = 0$ | $\sqrt{\frac{2}{\pi}} \sin \omega x$ | $F_I\{u\} = \hat{u}_I(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \sin \omega x dx$ <br> $F_I^{-1}\{\hat{u}_I\} = u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_I(\omega) \sin \omega x d\omega$ |
| **II** $\left[\frac{du}{dx}\right]_{x=0} = 0$ | $\sqrt{\frac{2}{\pi}} \cos \omega x$ | $F_{II}\{u\} = \hat{u}_{II}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \cos \omega x dx$ <br> $F_{II}^{-1}\{\hat{u}_{II}\} = u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_{II}(\omega) \cos \omega x d\omega$ |
| **III** $\left[-\frac{du}{dx} + Hu\right]_{x=0} = 0$ | $\sqrt{\frac{2}{\pi}} \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}}$ | $F_{III}\{u\} = \hat{u}_{III}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}} dx$ <br> $F_{III}^{-1}\{\hat{u}\} = u(x_{III}) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_{III}(\omega) \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}} d\omega$ |

The kernel corresponding to the ***Dirichlet*** boundary condition is based on the Fourier sine integral transform pair (note, that the coefficient $2/\pi$ is split into two factors $\sqrt{2/\pi}$ ):

*Fourier sine transform $F_I$*

$$\hat{u}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \sin \omega x dx \qquad u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}(\omega) \sin \omega x d\omega$$

The kernel corresponding to the ***Neumann*** boundary condition is based on the Fourier cosine integral transform pair:

*Fourier cosine transform $F_{II}$*

$$\hat{u}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \cos \omega x dx \qquad u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}(\omega) \cos \omega x d\omega$$

The integral transform pair with the kernel corresponding to ***Robin*** boundary condition:

*Fourier transform $F_{III}$*

$$\hat{u}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}} dx \qquad u(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}(\omega) \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}} d\omega$$

The kernel of integral transform for the case of a Robin boundary condition is obtained from the solution of the auxiliary boundary value problem for the semi-infinite region with parameter $\omega \in [0,\infty)$ :

$$X''(x) + \omega^2 X(x) = 0 \qquad x \in [0,\infty)$$

$$-X'(0) + HX(0) = 0$$

It can be verified that the function

$$X = \omega \cos \omega x + H \sin \omega x$$

is a solution of the auxiliary BVP.

**Convolution Formulas:** [See also Debnath, p.94]

Let
$$F_I\{u\} = \hat{u}_I(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \sin \omega x dx$$

$$F_{II}\{v\} = \hat{v}_{II}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty v(x) \cos \omega x dx$$

$$F_I^{-1}\{\hat{u}_I(\omega) \cdot \hat{v}_{II}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(s) \left[ v(s-x) - v(s+x) \right] ds \qquad \text{(I-I-II)}$$

$$F_{II}^{-1}\{\hat{u}_{II}(\omega) \cdot \hat{v}_{II}(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(s) \left[ v(x+s) + v(|x-s|) \right] ds \qquad \text{(II-II-II)}$$

$$F_{II}^{-1}\{\hat{u}_I(\omega) \cdot \hat{v}_I(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(s) \left[ v(s+x) + v(s-x) \right] ds \qquad \text{(II-I-I)}$$

$$F_I^{-1}\{\hat{u}_I(\omega) \cdot \hat{v}_I(\omega)\} \qquad \text{(I-I-I)}$$

$$F_I^{-1}\{\hat{u}_{II}(\omega) \cdot \hat{v}_{II}(\omega)\} \qquad \text{(I-II-II)}$$

$$F_I^{-1}\{\hat{u}_I(\omega) \cdot \hat{v}_{II}(\omega)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_I(\omega) \hat{v}_{II}(\omega) \sin \omega x d\omega \qquad \text{(I-I-II)}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty u(s) \sin \omega s ds \right] \hat{v}_{II}(\omega) \sin \omega x d\omega$$

$$= \frac{2}{\pi} \int_0^\infty u(s) \left[ \int_0^\infty \hat{v}_{II}(\omega) \sin \omega s \sin \omega x d\omega \right] ds$$

$$= \frac{1}{\pi} \int_0^\infty u(s) \left\{ \int_0^\infty \hat{v}_{II}(\omega) \left[ \cos \omega(s-x) - \cos \omega(s+x) \right] d\omega \right\} ds$$

$$= \frac{1}{\pi} \int_0^\infty u(s) \left\{ \int_0^\infty \hat{v}_{II}(\omega) \cos \omega(s-x) d\omega - \int_0^\infty \hat{v}_{II}(\omega) \cos \omega(s+x) d\omega \right\} ds$$

$$= \frac{1}{\pi} \sqrt{\frac{\pi}{2}} \int_0^\infty u(s) \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{v}_{II}(\omega) \cos \omega(s-x) d\omega - \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{v}_{II}(\omega) \cos \omega(s+x) d\omega \right] ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty u(s) \left[ v(s-x) - v(s+x) \right] ds$$

$$F_I^{-1}\left\{\hat{u}_{II}(\omega)\cdot\hat{v}_{II}(\omega)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_{II}(\omega)\hat{v}_{II}(\omega)\sin\omega x d\omega \tag{I-II-II}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\int_0^\infty u(s)\cos\omega s ds\right]\hat{v}_{II}(\omega)\sin\omega x d\omega$$

$$= \frac{2}{\pi}\int_0^\infty u(s)\left[\int_0^\infty \hat{v}_{II}(\omega)\sin\omega x\cos\omega s d\omega\right]ds$$

$$= \frac{1}{\pi}\int_0^\infty u(s)\left\{\int_0^\infty \hat{v}_{II}(\omega)\left[\sin\omega(x-s)+\sin\omega(x+s)\right]d\omega\right\}ds$$

$$= \frac{1}{\pi}\int_0^\infty u(s)\left\{\int_0^\infty\left[\sqrt{\frac{2}{\pi}}\int_0^\infty v(p)\cos\omega p dp\right]\left[\sin\omega(x-s)+\sin\omega(x+s)\right]d\omega\right\}ds$$

$$= \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty u(s)\left\{\int_0^\infty\int_0^\infty v(p)\cos\omega p\left[\sin\omega(x-s)+\sin\omega(x+s)\right]d\omega dp\right\}ds$$

$$= \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty u(s)\left\{\int_0^\infty v(p)\underbrace{\int_0^\infty \cos\omega p\left[\sin\omega(x-s)+\sin\omega(x+s)\right]d\omega}_{undefined} dp\right\}ds$$

$$F_{II}^{-1}\left\{\hat{u}_{II}(\omega)\cdot\hat{v}_{II}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}\int_0^\infty u(s)\left[v(x+s)+v(|x-s|)\right]ds \tag{II-II-II}$$

[Debnath, 2.14.7, p.94]

$$F_{II}^{-1}\left\{\hat{u}_I(\omega)\cdot\hat{v}_I(\omega)\right\} \quad = \frac{1}{\sqrt{2\pi}}\int_0^\infty u(s)\left[v(s+x)+v(s-x)\right]ds \tag{II-I-I}$$

[Debnath, 2.14.10-11, p.95]

$$F_I^{-1}\left\{\hat{u}_I(\omega)\cdot\hat{v}_I(\omega)\right\} \tag{I-I-I}$$

## OPERATIONAL PROPERTIES OF THE INTEGRAL FOURIER TRANSFORMS

We consider the function $u(x)$ as a function describing some physical quantity in the semi-infinite region $x \in [0,\infty)$. At $x=0$, it is specified by one of the boundary conditions. When $x$ approaches infinity, we assume that both the function $u(x)$ and its derivative are equal to zero:

$$u(x)\big|_{x\to\infty} = 0 \qquad \left.\frac{\partial u(x)}{\partial x}\right|_{x\to\infty} = 0$$

**Integral transform of $\frac{\partial^2 u}{\partial x^2}$**

Apply the integral transform to the second partial derivative of $u$

$$F\left\{\frac{\partial^2 u(x)}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega,x)\, dx$$

| Boundary Condition at $x=0$ | Kernel $K(x,\omega)$ | $\int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(x,\omega) dx$ |
|---|---|---|
| I Dirichlet $[u]_{x=0} = f_0(t)$ | $\sqrt{\frac{2}{\pi}} \sin \omega x$ | $-\omega^2 \hat{u}(\omega) + \omega\sqrt{\frac{2}{\pi}} f_0(t)$ |
| II Neumann $\left[\frac{\partial u}{\partial x}\right]_{x=0} = f_0(t)$ | $\sqrt{\frac{2}{\pi}} \cos \omega x$ | $-\omega^2 \hat{u}(\omega) - \sqrt{\frac{2}{\pi}} f_0(t)$ |
| III Robin $\left[-\frac{\partial u}{\partial x} + Hu\right]_{x=0} = \frac{f_0(t)}{k}$ | $\sqrt{\frac{2}{\pi}} \frac{\omega \cos \omega x + H \sin \omega x}{\sqrt{\omega^2 + H^2}}$ | $-\omega^2 \hat{u}(\omega) + \sqrt{\frac{2}{\pi}} \frac{\omega}{\sqrt{\omega^2+H^2}} \frac{f_0(t)}{k}$ |

**1) <u>Dirichlet boundary condition</u>,** $[u]_{x=0} = 0$, $K(\omega,x) = \sqrt{\frac{2}{\pi}} \sin \omega x$ :

$$F_I\left\{\frac{\partial^2 u(x)}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega,x) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} \sin \omega x dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \omega x d\left[\frac{\partial u(x)}{\partial x}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin \omega x \frac{\partial u(x)}{\partial x}\right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u(x)}{\partial x} d[\sin \omega x]$$

$$= -\omega \sqrt{\frac{2}{\pi}} \int_0^\infty \cos \omega x \frac{\partial u(x)}{\partial x} dx$$

$$= -\omega \sqrt{\frac{2}{\pi}} \int_0^\infty \cos \omega x d[u(x)]$$

*in the case of non-homogeneous condition* $[u]_{x=0} = f_0(t)$

$$= -\omega \sqrt{\frac{2}{\pi}} [\cos \omega x u(x)]_0^\infty + \omega \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) d[\cos \omega x]$$

$$= -\omega \sqrt{\frac{2}{\pi}} \left[\cos(\omega x) u(x)\big|_{x\to\infty} - \cos(0) f_0(t)\right] + \omega \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) d[\cos \omega x]$$

$$= \omega\sqrt{\frac{2}{\pi}} f_0(t) - \omega^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \sin x dx$$

$$= \omega\sqrt{\frac{2}{\pi}} f_0(t) - \omega^2 \hat{u}(\omega)$$

$$\int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega, x) dx = \omega\sqrt{\frac{2}{\pi}} f_0(t) - \omega^2 \hat{u}(\omega)$$

$$\int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega, x) dx = -\omega^2 \hat{u}(\omega) \quad \text{in a case of } f_0(t) = 0$$

The integral transform eliminates the derivative. Similar results can be obtained for the remaining two boundary conditions:

**2) <u>Neumann b.c.</u>,** $\left[\frac{du}{dx}\right]_{x=0} = 0: \qquad K(\omega, x) = \sqrt{\frac{2}{\pi}} \cos \omega x$

$$F_{II}\left\{\frac{\partial^2 u(x)}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega, x) dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} \cos \omega x dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \cos \omega x d\left[\frac{\partial u(x)}{\partial x}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos \omega x \frac{\partial u(x)}{\partial x}\right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u(x)}{\partial x} d\left[\cos \omega x\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\cos \omega x \frac{\partial u(x)}{\partial x}\bigg|_{x\to\infty} - \cos 0 \frac{\partial u(0)}{\partial x}\right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u(x)}{\partial x} d\left[\cos \omega x\right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(0)}{\partial x} + \omega\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u(x)}{\partial x} \sin \omega x dx$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(0)}{\partial x} + \omega\sqrt{\frac{2}{\pi}} \int_0^\infty \sin \omega x d\left[u(x)\right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(0)}{\partial x} + \omega\sqrt{\frac{2}{\pi}} \left[\sin \omega x u(x)\big|_{x\to\infty} - \sin 0 u(0)\right] - \omega\sqrt{\frac{2}{\pi}} \int_0^\infty u(x) d\left[\sin \omega x\right]$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(0)}{\partial x} - \omega^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u(x) \cos \omega x dx$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\partial u(0)}{\partial x} - \omega^2 \hat{u}(\omega)$$

$$= -\omega^2 \hat{u}(\omega)$$ (for homogeneous boundary condition, $\frac{\partial u(0)}{\partial x} = 0$ )

If condition is non-homogeneous, $\left[\frac{du}{dx}\right]_{x=0} = f_0(t)$, then

$$\int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} K(\omega, x) dx = -\sqrt{\frac{2}{\pi}} f_0(t) - \omega^2 \hat{u}(\omega)$$

**3) Robin b.c.,**

$$\left[-\frac{\partial u}{\partial x}+Hu\right]_{x=0}=\frac{f_0(t)}{k},\qquad K(\omega,x)=\sqrt{\frac{2}{\pi}}\frac{\omega\cos\omega x+H\sin\omega x}{\sqrt{\omega^2+H^2}}$$

$$\left[-\frac{du}{dx}+Hu\right]_{x=0}=f_0(t),\ H=\frac{h}{k}$$

[see Ozicik (2-124), p.74]

$$K(\omega,x)=\sqrt{\frac{2}{\pi}}\frac{\omega\cos\omega x+H\sin\omega x}{\sqrt{\omega^2+H^2}}\qquad K(\omega,x)\Big|_{x=0}=\sqrt{\frac{2}{\pi}}\frac{\omega}{\sqrt{\omega^2+H^2}}$$

$$\frac{\partial}{\partial x}K(\omega,x)=\sqrt{\frac{2}{\pi}}\frac{-\omega\sin\omega x+H\cos\omega x}{\sqrt{\omega^2+H^2}}\omega\qquad \frac{\partial}{\partial x}K(\omega,x)\Big|_{x=0}=\sqrt{\frac{2}{\pi}}\frac{H}{\sqrt{\omega^2+H^2}}\omega$$

$$\frac{\partial^2}{\partial x^2}K(\omega,x)=\sqrt{\frac{2}{\pi}}\frac{-\omega\cos\omega x-H\sin\omega x}{\sqrt{\omega^2+H^2}}\omega^2=-\omega^2K(\omega,x)$$

$$F_{II}\left\{\frac{\partial^2 u}{\partial x^2}\right\}=\int_0^\infty\frac{\partial^2 u}{\partial x^2}K(\omega,x)\,dx$$

$$=\int_0^\infty K(\omega,x)\,d\left[\frac{\partial u}{\partial x}\right]$$

$$=\left[\frac{\partial u}{\partial x}K(\omega,x)\right]_0^\infty-\int_0^\infty\frac{\partial u}{\partial x}d\left[K(\omega,x)\right]$$

$$=-\left[\frac{\partial u}{\partial x}K(\omega,x)\right]_0-\int_0^\infty\frac{\partial u}{\partial x}\frac{\partial}{\partial x}K(\omega,x)\,dx$$

$$=-\frac{\partial u}{\partial x}\Big|_{x=0}K(\omega,0)-\int_0^\infty\frac{\partial}{\partial x}K(\omega,x)\,du$$

$$=-\frac{\partial u}{\partial x}\Big|_{x=0}K(\omega,0)-\left[u\frac{\partial}{\partial x}K(\omega,x)\right]_0^\infty+\int_0^\infty u\,d\left[\frac{\partial}{\partial x}K(\omega,x)\right]$$

$$=-\frac{\partial u}{\partial x}\Big|_{x=0}K(\omega,0)+\left[u\frac{\partial}{\partial x}K(\omega,x)\right]_0+\int_0^\infty u\frac{\partial^2}{\partial x^2}K(\omega,x)\,dx$$

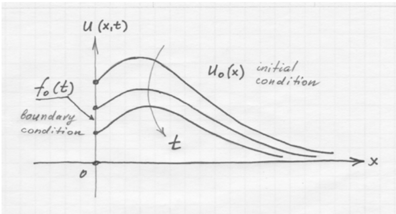
$$=-\frac{\partial u}{\partial x}\Big|_{x=0}K(\omega,0)+u\big|_{x=0}\frac{\partial}{\partial x}K(\omega,0)-\omega^2\int_0^\infty uK(\omega,x)\,dx$$

$$=-\frac{\partial u}{\partial x}\Big|_{x=0}\sqrt{\frac{2}{\pi}}\frac{\omega}{\sqrt{\omega^2+H^2}}+u\big|_{x=0}\sqrt{\frac{2}{\pi}}\frac{H}{\sqrt{\omega^2+H^2}}\omega-\omega^2\int_0^\infty uK(\omega,x)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{\omega}{\sqrt{\omega^2+H^2}}\left[-\frac{\partial u}{\partial x}+Hu\right]_{x=0}-\omega^2\int_0^\infty uK(\omega,x)\,dx$$

$$=\sqrt{\frac{2}{\pi}}\frac{\omega}{\sqrt{\omega^2+H^2}}\frac{f_0(t)}{k}-\omega^2\hat{u}(\omega)$$

**IX.2.7.1 Heat equation in the semi-infinite region** Consider the homogeneous one-dimensional heat equation




$$\frac{\partial^2 u(x,t)}{\partial x^2} = a^2 \frac{\partial u(x,t)}{\partial t}, \qquad x \in [0,\infty), \qquad t > 0$$

initial condition: $\quad u(x,0) = u_0(x), \quad t = 0$

boundary condition: $\quad u(0,t) = f_0(t), \quad t > 0 \quad$ (II)

**1.** Case $f_0(t) = 0$

1) Transformed equation. Apply the integral transform corresponding to the Dirichlet problem (**Fourier sine transform**) to the differential equation:

$$-\omega^2 \hat{u}(\omega,t) = a^2 \frac{\partial \hat{u}(\omega,t)}{\partial t}$$

This is an ODE for the transformed function $\hat{u}(\omega,t)$ as a function of $t$ with the variable $\omega$ treated as a parameter:

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} + \frac{\omega^2}{a^2}\hat{u}(\omega,t) = 0$$

with an initial condition which can be obtained by integral transform of the original initial condition:

$$\hat{u}(\omega,0) = \hat{u}_0(\omega) = \int_0^\infty u_0(x) K(\omega,x) dx$$

Solution of this linear 1st order homogeneus ODE with constant coefficients is given by

$$\hat{u}(\omega,t) = \hat{u}_0(\omega)\, e^{\frac{-\omega^2}{a^2}t} \qquad \textit{transformed solution}$$

2) Inverse transform – solution of IBVP.

$$\begin{aligned} u(x,t) &= \int_0^\infty \hat{u}_0(\omega)\, e^{\frac{-\omega^2}{a^2}t} K(\omega,x)\, d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_0(\omega)\, e^{\frac{-\omega^2}{a^2}t} \sin \omega x\, d\omega \end{aligned}$$

This solution can be reduced to the traditional form [Ozisik]:

$$\begin{aligned} u(x,t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{u}_0(\omega) e^{\frac{-\omega^2}{a^2}t} \sin \omega x\, d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty u_0(x') \sin \omega x' dx' \right] e^{\frac{-\omega^2}{a^2}t} \sin \omega x\, d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_0(x') \left[ \sqrt{\frac{2}{\pi}} \int_0^\infty e^{\frac{-\omega^2}{a^2}t} \sin \omega x' \sin \omega x\, d\omega \right] dx' \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_0(x') \left[ \frac{a}{\sqrt{8t}} \left( e^{\frac{-a^2}{4t}(x-x')^2} - e^{\frac{-a^2}{4t}(x+x')^2} \right) \right] dx' \\ &= \frac{a}{2\sqrt{\pi t}} \int_0^\infty u_0(x') \left[ e^{\frac{-a^2}{4t}(x-x')^2} - e^{\frac{-a^2}{4t}(x+x')^2} \right] dx' \end{aligned}$$

Application of the convolution formula (I-I-II):

$$\begin{aligned} & u(x,t) \\ &= F_I^{-1}\{\hat{u}\} \\ &= F_I^{-1}\left\{ \hat{u}_0(\omega) e^{\frac{-\omega^2}{a^2}t} \right\} \\ &= F_I^{-1}\left\{ F_I\{\hat{u}_0(\omega)\} \cdot e^{\frac{-\omega^2}{a^2}t} \right\} \\ &= F_I^{-1}\left\{ F_I\{\hat{u}_0(\omega)\} \cdot F_{II}\overbrace{\left\{ \frac{a}{\sqrt{2t}} e^{\frac{-a^2}{4t}x^2} \right\}}^{G(x,t)} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty u_0(s) \left[ G(s-x,t) - G(s+x,t) \right] ds \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty u_0(s) \left[ \frac{a}{\sqrt{2t}} e^{\frac{-a^2}{4t}(x-s)^2} - \frac{a}{\sqrt{2t}} e^{\frac{-a^2}{4t}(x+s)^2} \right] ds \\ &= \frac{a}{2\sqrt{\pi t}} \int_0^\infty u_0(s) \left[ e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2} \right] ds \end{aligned}$$

**FS-1-1h.mws** **HE in the semi-infinite space - integral transform solution** (Fourier sine transform) B.C.: $f_0=0$

$$u_0(x)=1-H(x-1)$$

$$\hat{u}_0(\omega)=\sqrt{\frac{2}{\pi}}\frac{1-\cos\omega}{\omega}$$

```
> restart;
> a:=2;
```

$$a := 2$$

**Kernel of integral transform**

```
> K(omega,x):=sqrt(2/Pi)*sin(omega*x);
```

$$K(\omega, x) := \frac{\sqrt{2}\,\sin(\omega\, x)}{\sqrt{\pi}}$$

**Initial condition:**

```
> u0(x):=1-Heaviside(x-1);
```

$$u0(x) := 1 - \mathrm{Heaviside}(x-1)$$

```
> u(omega):=factor(int(u0(x)*K(omega,x),x=0..infinity));
```

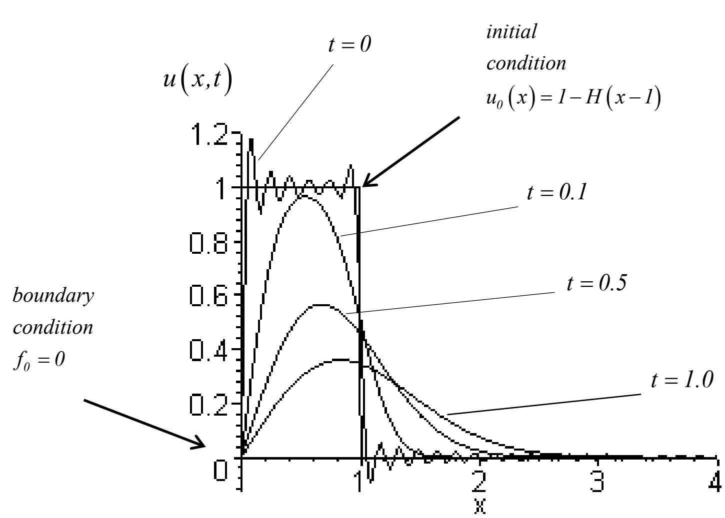
$$u(\omega) := -\frac{\sqrt{2}\,(\cos(\omega)-1)}{\sqrt{\pi}\,\omega}$$

**Inverse transform - solution of IBVP:** the upper limit defines the accuarcy of approximation (similar to the number of terms in the truncated Fourier series)

```
> u(x,t):=int(u(omega)*exp(-omega^2/a^2*t)*K(omega,x),
omega=0..40);
```

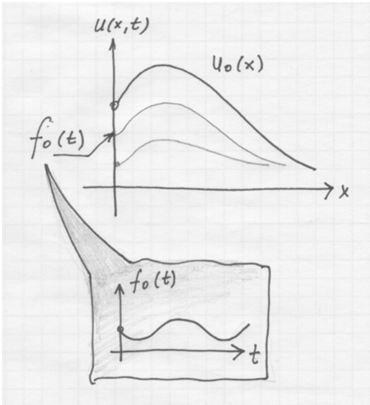
$$u(x, t) := \int_0^{40} -2\,\frac{(\cos(\omega)-1)\,\mathbf{e}^{(-1/4\,\omega^2\,t)}\,\sin(\omega\, x)}{\pi\,\omega}\,d\omega$$

```
> u00:=subs(t=0.0,u(x,t)):u1:=subs(t=0.1,u(x,t)):
u2:=subs(t=0.5,u(x,t)):u3:=subs(t=1.0,u(x,t)):
> plot({u0(x),u00,u1,u2,u3},x=0..4,color=black);
```



**2.** Case $f_0(t) \neq 0$

Non-homogeneous time-dependent boundary condition.




The transform of the derivative:

$$F_I\left\{\frac{\partial^2 u(x)}{\partial x^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u(x)}{\partial x^2} \sin \omega x dx = \sqrt{\frac{2}{\pi}}\omega f_0(t) - \omega^2 \hat{u}(\omega)$$

Then the transformed equation is

$$\sqrt{\frac{2}{\pi}}\omega f_0(t) - \omega^2 \hat{u}(\omega,t) = a^2 \frac{\partial \hat{u}(\omega,t)}{\partial t}$$

$$\frac{\partial \hat{u}(\omega,t)}{\partial t} + \frac{\omega^2}{a^2}\hat{u}(\omega,t) = \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2} f_0(t)$$

The initial condition is the same:

$$\hat{u}(\omega,0) = \hat{u}_0(\omega)$$

Then the transformed solution is obtained by variation of parameter

$$\hat{u}(\omega,t) = \hat{u}_0(\omega) e^{\frac{-\omega^2}{a^2}t} + \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2} e^{\frac{-\omega^2}{a^2}t}\int_0^t e^{\frac{\omega^2}{a^2}\tau} f_0(\tau) d\tau$$

The solution of the IBVP can be found by the inverse transform:

$$\begin{aligned}
u(x,t) &= F_I^{-1}\left\{\hat{u}(\omega,t)\right\} \\
&= F_I^{-1}\left\{\hat{u}_0(\omega) e^{\frac{-\omega^2}{a^2}t}\right\} + F_I^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{\omega}{a^2}\int_0^t e^{-\frac{\omega^2}{a^2}(t-\tau)} f_0(\tau) d\tau\right\} \\
&= \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + F_I^{-1}\left\{\sqrt{\frac{2}{\pi}}\frac{\omega}{a^2}\int_0^t e^{-\frac{\omega^2}{a^2}(t-\tau)} f_0(\tau) d\tau\right\} \\
&= \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + \sqrt{\frac{2}{\pi}}\frac{1}{a^2}\int_0^t F_I^{-1}\left\{\omega e^{-\frac{\omega^2}{a^2}(t-\tau)}\right\} f_0(\tau) d\tau \\
&= \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + \sqrt{\frac{2}{\pi}}\frac{1}{a^2}\frac{\sqrt{2}}{4}a^3\int_0^t \frac{x e^{-\frac{a^2x^2}{4(t-\tau)}}}{\sqrt{(t-\tau)^3}} f_0(\tau) d\tau \quad \textit{integration with Maple}
\end{aligned}$$

$$u(x,t) = \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + \frac{ax}{2\sqrt{\pi}}\int_0^t \frac{e^{-\frac{a^2x^2}{4(t-\tau)}}}{\sqrt{(t-\tau)^3}} f_0(\tau) d\tau$$

*see p.706*

Case $f_0 = const$

$$u(x,t) = \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + \frac{ax}{2\sqrt{\pi}} f_0 \frac{2\sqrt{\pi}}{ax}\left[1 - erf\left(\frac{ax}{2\sqrt{t}}\right)\right]$$

*integration with Maple*

$$u(x,t) = \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + f_0 erfc\left(\frac{ax}{2\sqrt{t}}\right)$$

*see p.707*

*Attempt of application of convolution formula (failed so far):*

$$= \frac{a}{2\sqrt{\pi t}}\int_0^\infty u_0(s)\left[e^{\frac{-a^2}{4t}(x-s)^2} - e^{\frac{-a^2}{4t}(x+s)^2}\right] ds + \sqrt{\frac{2}{\pi}}\frac{1}{a^2}\int_0^t F_I^{-1}\left\{F_{II}\left\{-\sqrt{\frac{2}{\pi}}\frac{1}{x^2}\right\} F_{II}\overbrace{\left\{\frac{a}{\sqrt{2t}} e^{\frac{-a^2}{4(t-\tau)}x^2}\right\}}^{G(x,t-\tau)}\right\} f_0(\tau) d\tau$$

**IX.2.7.2 Heat conduction with heat generation in a semi-infinite region** See Ozisik, p.72.


I. Conduction with heat generation

$$\frac{\partial^2 u}{\partial x^2}+\frac{g(x,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad t>0, \qquad x>0$$

initial condition:

$$u(x,0)=u_0(x), \qquad t=0$$

boundary condition:

$$[u]_{x=0}=f_0(t), \qquad t>0 \qquad \textbf{(I)}$$

II. Conduction with heat generation

$$\frac{\partial^2 u}{\partial x^2}+\frac{g(x,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad t>0, \qquad x>0$$

initial condition:

$$u(x,0)=u_0(x), \qquad t=0$$

boundary condition:

$$\left[-\frac{\partial u}{\partial x}\right]_{x=0}=f_0(t), \qquad t>0 \qquad \textbf{(II)}$$

III. Conduction with heat generation

$$\frac{\partial^2 u}{\partial x^2}+\frac{g(x,t)}{k}=\frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad t>0, \qquad x>0$$

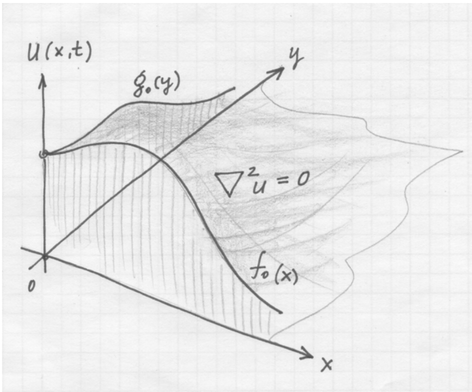
initial condition:

$$u(x,0)=u_0(x), \qquad t=0$$

boundary condition:

$$\left[-\frac{du}{dx}+Hu\right]_{x=0}=\frac{f_0(t)}{k}, \qquad t>0 \qquad \textbf{(III)}$$

**Laplace's Equation in the 1st quadrant**




Consider 2-D Laplace's Equation in semi-infinite strip:

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad x \in [0,\infty),\ y \in [0,\infty]$$

with boundary conditions:

$u\big|_{x=0} = 0 \qquad t > 0 \qquad$ Dirichlet

$u(x,0) = f_0(x) \qquad t > 0 \qquad$ Dirichlet

**Solution**: We will apply the *Fourier sine transform* in the $x$ variable corresponding to the Dirichlet problem in the semi-infinite region:

$$\hat{u}(\omega,y) = \int_0^\infty u(x,y)K(\omega,x)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\sin\omega x dx$$

The transformed equation:

$$-\omega^2\hat{u} + \frac{\partial^2\hat{u}}{\partial y^2} = 0$$

The integral transform of the second boundary condition:

$$\hat{f}_0(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_0(x)\sin\omega x dx$$

The general solution of the ODE is

$$\hat{u} = c_1 e^{-\omega y} + c_2 e^{\omega y}$$

The solution is bounded $\quad \Rightarrow \quad c_2 = 0$

Apply the first boundary condition

$$y = 0 \qquad \Rightarrow \quad c_1 = \hat{f}_0(\omega)$$

The solution of the transformed equation:

$$\hat{u} = \hat{f}_0(\omega)\ e^{-\omega y}$$

The solution of the BVP for Laplace's Equation

$$\boxed{u(x,y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_0(\omega)e^{-\omega y}\sin\omega x d\omega}$$

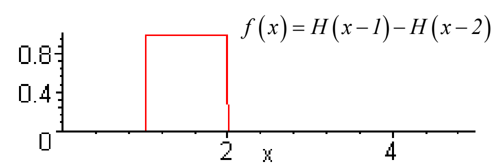
*FS-1-2h.mws*

## LE in the 1st quadrant - Dirichlet problem - Fourier sine transform

```
> restart;
> f(x):=Heaviside(x-1)-Heaviside(x-2);
```

$$f(x) := \text{Heaviside}(x-1) - \text{Heaviside}(x-2)$$

```
> plot(f(x),x=0..5);
```




```
> ft:=simplify(int(f(x)*sin(omega*x),x=0..infinity));
```

$$ft := -\frac{-\cos(\omega) + 2\cos(\omega)^2 - 1}{\omega}$$

Transformed solution:

```
> ut:=ft*exp(-omega*y);
```

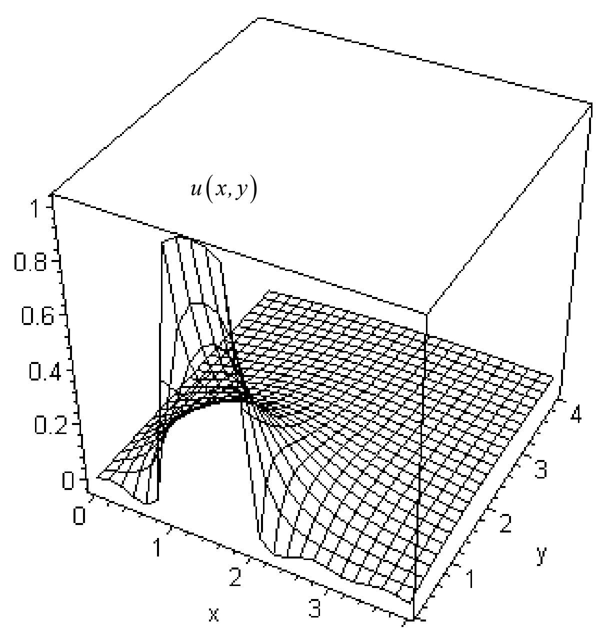
$$ut := -\frac{(-\cos(\omega) + 2\cos(\omega)^2 - 1)\, \mathbf{e}^{(-\omega\, y)}}{\omega}$$

**Inverse transform -solution of BVP**:

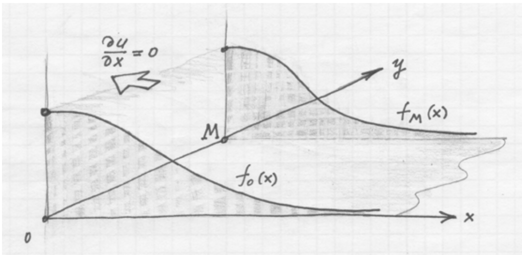
the upper limit of integral defines the accuracy of approximation

```
> u(x,y):=int(ut*sin(omega*x),omega=0..30)*2/Pi:
```

```
> plot3d(u(x,y),x=0..4,y=0..4,axes=boxed);
```



**Laplace's Equation in the semi-infinite strip**




Consider a 2-D Laplace's Equation in the semi-infinite strip:

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad x \in (0,\infty),\ y \in (0,M)$$

with boundary conditions:

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \qquad t>0 \qquad \textit{Neumann} \quad (\textit{insulation})$$

$$u(x,0) = f_0(x) \qquad t>0 \qquad \textit{Dirichlet}$$
$$u(x,M) = f_M(x) \qquad t>0 \qquad \textit{Dirichlet}$$

We will apply the *Fourier cosine transform* in the *x* variable which corresponds to the Neumann problem in the semi-infinite region:

$$\hat{u}(\omega,y) = \int_0^\infty u(x,y)K(\omega,x)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,y)\cos\omega x dx$$

Transformed equation:

$$-\omega^2\hat{u} + \frac{\partial^2 \hat{u}}{\partial y^2} = 0 \qquad \text{(2}^{\text{nd}}\text{ order ODE)}$$

The integral transform of the boundary conditions:

$$\hat{f}_0(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_0(x)\cos\omega x dx$$

$$\hat{f}_M(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f_M(x)\cos\omega x dx$$

Then the $2^{nd}$ order ODE has to be solved with boundary conditions. The general solution of the ODE is

$$\hat{u} = c_1\cosh\omega y + c_2\sinh\omega y$$

Find coefficients $c_1$ and $c_2$ from boundary conditions, then the solution of the IBVP will be given by the inverse integral transform:

$$u(x,y) = \int_0^\infty \hat{u}(\omega,y)K(\omega,x)d\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}(\omega,y)\cos\omega x d\omega$$

---

***Example*** Let $f_0(x) = 0$ $\qquad \hat{f}_0(\omega) = 0$

$f_M(x) = H(x-1) - H(x-2)$ $\qquad \hat{f}_M(\omega) =$

Apply first boundary condition

$$y = 0 \quad \Rightarrow \quad c_1 = 0$$

$$y = M \quad \Rightarrow \quad c_2\sinh\omega M = \hat{f}_M(\omega) \quad \Rightarrow \qquad c_2 = \frac{\hat{f}_M(\omega)}{\sinh\omega M}$$

Then the transformed solution is

$$\hat{u} = \frac{\hat{f}_M(\omega)}{\sinh\omega M}\sinh\omega y$$

Solution of IBVP (inverse transform):

$$u(x,y) \quad = \sqrt{\frac{2}{\pi}}\int_0^\infty [\hat{u}]\cos\omega x d\omega$$

## IX.2.8 REVIEW QUESTIONS

1) How is the Fourier transform defined?
2) What conditions guarantee the existence of the Fourier transform?
3) How is the inverse Fourier transform defined?
4) What are the main properties of the Fourier transform and the inverse Fourier transform?
5) How can the convolution theorem be applied for evaluation of the inverse Fourier transform?
6) What property allows application of the Fourier transform for solution of differential equations?
7) What are the main steps in the procedure of application of the Fourier transform for solution of the differential equations?

## EXERCISES

1. a) Derive the scaling property

$$F\{f(at)\} = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)$$

   b) Using integration by parts derive the Fourier transform of the second order derivative (*property 7b*):

$$F\left\{\frac{d^2}{dt^2}f(t)\right\} = -\omega^2\hat{f}(\omega)$$

2. Using the definition of the Fourier transform derive the transform of the one-sided exponential function:

$$f(t) = \begin{cases} e^{-at} & t \ge 0 \\ 0 & t < 0 \end{cases} = H(t)\cdot e^{-at}, \; a > 0$$

and sketch the graph of the real part $Re\,\hat{f}(\omega)$ and the imaginary part $Im\,\hat{f}(\omega)$ of the transformed function for $a = 1$.

3. Using the result of the previous problem and the convolution theorem, evaluate the inverse Fourier transform:

$$F^{-1}\left\{\frac{1}{(a+i\omega)^2}\right\}$$

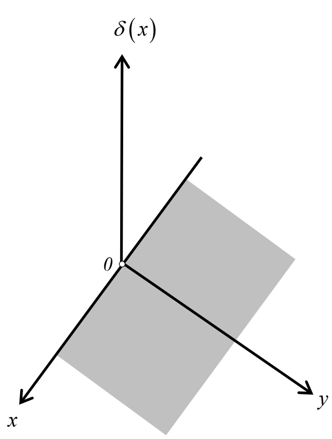
and sketch the graph of the solution for $a = 1$.

4. The current $y(t)$ in the electrical circuit is modeled with the help of the 1st order differential equation

$$\frac{dy}{dt} + ay = g(t) \qquad a > 0 \qquad t \in (-\infty, \infty)$$

where $f(t)$ represents the applied electromagnetic force.

Use the Fourier transform for solution. Write the solution in the form of a convolution. Then evaluate the solution for the case of instantaneous force defined by the Dirac delta function $g(t) = \delta(t)$. Sketch the graph of the solution curve.




5. Use the Poisson integral formula (p. 734) to derive the solution of the Laplace equation in the semi-infinite region ($y > 0$) with the boundary condition defined by the Dirac delta function:

$$u(x,0) = u_0(x) = \delta(x)$$

and sketch the graph of the solution.

6. Consider IX.2.7.1, p.744. Find the solution of the IBVP for :

   $a = 2$ , $u_0(x) = H(x-2) - H(x-4)$ , $f_0(t) = 2[H(t) - H(t-1)]$

   and visualize.

7. Verify operational property of Integral Fourier transform in a case of non-homogeneous Robin boundary condition (p.744).

8. Following Section IX.2.6, find the standard Fourier integral representation, the Fourier cosine integral representation, and the Fourier sine integral representation of the function

$$f(t) = \begin{cases} 0 & t < 0 \\ \cos t & 0 < t < \pi/2 \\ 0 & t > \pi/2 \end{cases} = \left[ H(t) - H(t - \pi/2) \right] \cdot \cos t$$

9. Neutron transfer in infinite medium. $u(x,t)$ is the number of neutrons per unit volume per unit time. $\delta(x)\delta(t)$ is the source function.

$$\frac{\partial^2 u}{\partial x^2} + \delta(x)\delta(t) = \frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = \delta(x), \qquad -\infty < x < \infty, \qquad t = 0$$

$$u(x,t) \to 0, \qquad x \to \pm\infty, \qquad t > 0$$

   Sketch the solution curves for $a = 1$ and time $t = 1, 4$.

10. Solve the initial-boundary problem for one-dimensional heat equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = H(x), \qquad -\infty < x < \infty, \qquad t = 0$$

   Sketch the solution curves for $a = 1$ and for time $t = 1$, $t = 2$ .

11. Solve the initial-boundary problem for one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = e^{-x^2}, \qquad -\infty < x < \infty, \qquad t = 0$$

$$\frac{\partial}{\partial t}u(x,0) = 0, \qquad -\infty < x < \infty, \qquad t = 0$$

   Sketch the solution curves for $v = 1$ and several moments of time.

12. Solve the initial-boundary problem for one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}, \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = 0, \qquad -\infty < x < \infty, \qquad t = 0$$

$$\frac{\partial}{\partial t}u(x,0) = \delta(x), \qquad -\infty < x < \infty, \qquad t = 0$$

   Sketch the solution curves for $v = 1$ and several moments of time.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