# CLASS 36 - Shitty PDE Stuff

Monday, December 9, 2019 3:01 PM

## MULTIPLE TRANSFORM

let $f(x,y)$ DEFINED FOR

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$

THE F.T. OF $f$ W.R.T. $x$:

$$P(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} f(x,y)\, dx$$

F.T. OF $f$ W.R.T. $y$:

$$P(\xi,\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta y} P(\xi,n)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\xi x+\eta y)} f(x,y)\, dx\, dy$$

$F(\xi,\eta)$ IS DOUBLE FOURIER TRANSFORM OF $f(x,y)$

AND INVERSION THM

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int e^{-i(\xi x+\eta y)} F(\xi,\eta)\, d\xi\, d\eta$$

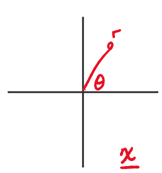
NOTE: RESULT SIMILAR TO HIGHER DIMENSIONS.

NOTE: USEFUL TO INTRODUCE VECTORS $\underset{\sim}{x}$ AND $\underset{\sim}{\xi}$ S.T. $F(\underline{\xi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int e^{i(\underline{\xi}\cdot\underline{x})} f(\underline{x})\, dA_{\underline{x}}$

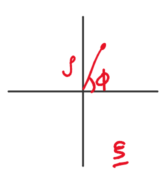
NOTE: POLAR COORDINATES:

let $f(\underline{x}) \to g(r,\theta)$

$$\boxed{F(\underline{\xi}) = \frac{1}{2\pi}\int_0^\infty r\,dr \int_0^{2\pi} e^{\overbrace{ir\rho\cos(\theta-\phi)}^{\text{DOT PRODUCT}}} g(r,\theta)\, d\theta}$$




$$\boxed{f(\underline{x}) = g(r,\theta) = \frac{1}{2\pi}\int_0^\infty \rho\,d\rho \int_0^{2\pi} e^{-ir\rho\cos(\theta-\phi)} G(\rho,\phi)\, d\phi}$$




EX) LAPLACE EQ IN 3D.

VECTOR

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \delta(\underline{x}-\underline{\xi})$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \delta(\underline{x} - \underline{\xi})$$

VECTOR

VARIABLES

3D DELTA FUNCTION

$$g(\underline{\xi}) = \iint f(\underline{x})\, \delta(\underline{x} - \underline{\xi})\, dx\, dy$$

B.C. $\phi(\underline{x}) \to 0$ AS $|\underline{x}| \to \infty$

TAKE $\mathcal{F}$ IN 3D!

$$= \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{i(\underline{\lambda} \cdot \underline{x})} f(\underline{x})\, dV_{\underline{x}}$$

$$= \frac{1}{(2\pi)^{3/2}} \iiint e^{i(\lambda_x x + \lambda_y y + \lambda_z z)} f(x,y,z)\, dx\, dy\, dz$$

$$(i\lambda_x)^2 \Phi(\underline{\lambda}) + (i\lambda_y)^2 \Phi(\underline{\lambda}) + (i\lambda_z)^2 \Phi(\underline{\lambda}) = \underbrace{\frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{i(\underline{\lambda}\cdot\underline{\xi})} \delta(\underline{x} - \underline{\xi})\, dV_{\underline{x}}}$$

$$-(\lambda_x^2 + \lambda_y^2 + \lambda_z^2)\, \Phi(\underline{\lambda}) = \frac{1}{(2\pi)^{3/2}} e^{i(\underline{\lambda}\cdot\underline{\xi})}$$

$$-|\underline{\lambda}|^2 \Phi(\underline{\lambda}) = \frac{e^{i(\underline{\lambda}\cdot\underline{\xi})}}{(2\pi)^{3/2}}$$

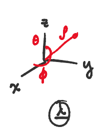
$$\Phi(\underline{\lambda}; \underline{\xi}) = \frac{e^{i(\underline{\lambda}\cdot\underline{\xi})}}{(2\pi)^{3/2} |\underline{\lambda}|^2}$$

NOW INVERT TO GET $\phi(\underline{x})$

$$\phi(\underline{x}, \underline{\xi}) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} e^{-i(\underline{\lambda}\cdot\underline{x})}\, \Phi(\underline{\lambda}, \underline{\xi})\, dV_{\underline{\lambda}}$$

$$= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\underline{\lambda}\cdot(\underline{x} - \underline{\xi})} \cdot \frac{1}{|\underline{\lambda}|^2}\, dV_{\underline{\lambda}}$$

REWRITE IN SPHERICAL COOR:




$$\therefore dV_{\underline{\lambda}} = \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$$

$(\underline{x} - \underline{\xi}) = \underline{r}$

$$\underline{r} = |\underline{x} - \underline{\xi}|$$

SO DOT PRODUCT BETWEEN $\underline{r}$ AND ABOVE IS JUST A MAGNITUDE * (ANGLE BETWEEN THEM)

CHOOSE COOR. SYS. SUCH THAT $\underline{r}$ LIES ON PLANE:

$$= \frac{-1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\infty} d\rho \int_0^{\pi} d\theta\; e^{-i\rho r\cos\theta}\, \frac{\cancel{\rho^2}\sin\theta}{\cancel{\rho^2}}$$

NO DEPENDENCE ON $\phi$, JUST MULTIPLE OF $2\pi$

$$= \frac{-1}{(2\pi)^2} \int_0^{\infty} d\rho \int_0^{2\pi} e^{-i\rho r\cos\theta} \sin\theta\, d\theta$$

let $u = -\cos\theta$
$du = +\sin\theta \, d\theta$

$$\therefore \quad \phi(\underline{x};\underline{\xi}) = \frac{-1}{(2\pi)^2} \int_0^\infty d\rho \int_{-1}^{1} e^{-i\rho r u} \, du$$

$$= -\frac{1}{(2\pi)^2} \int_0^\infty d\rho \left[ \frac{1}{-i\rho r} e^{-i\rho r u} \right]_{-1}^{1}$$

$$= -\frac{1}{(2\pi)^2} \frac{1}{r} \int_0^\infty \frac{1}{\rho} \sin(\rho r) \, d\rho$$

$$= -\frac{1}{(2\pi)^2} \frac{1}{r} \int_0^\infty \frac{1}{r\rho} \sin(\rho r) \, d(r\rho)$$

$$= -\frac{1}{(2\pi)^2} \frac{1}{r} \int_0^\infty \frac{\sin x}{x} \, dx \quad , \quad x = r\rho$$

$$= -\frac{1}{(2\pi)^2} \frac{\pi}{2} \cdot \frac{1}{r}$$

$$= \frac{-1}{4\pi r}$$

$$\boxed{\phi(\underline{x};\underline{\xi}) = -\frac{-1}{4\pi|\underline{x}-\underline{\xi}|}} = G(\underline{x},\underline{\xi})$$

NOTE: SINCE WE SOLVED THIS FOR $\delta$, $\phi(\underline{x};\underline{\xi})$ IS THE GREENE'S FUNCTION!

$\therefore$ IF WE WANNA SOLVE ARBITRARY $\phi(\underline{x};\underline{\xi})$ W/ PREVIOUS RESULT

$$\boxed{\phi(\underline{x}) = \iiint_{-\infty}^{\infty} f(\underline{\xi}) \frac{-1}{4\pi|\underline{x}-\underline{\xi}|} \, dV_{\xi}}$$

FINAL EXAM:

- COMPREHENSIVE
  - COVERS TOPICS AFTER MIDTERM.
  - PROBLEMS INVOLVE PRIOR RESULTS LIKE RESIDUES | CONTOUR INT | SINGULARITY ETC.

# WAVE EQUATION

(HYPERBOLIC)

$-\infty < x < \infty$
$t \geq 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} :$$

INFINITE STRING

$c = \sqrt{\frac{T}{\rho}}$

I.C: $u(x,0) = h(x)$

$\frac{\partial u}{\partial x}(x,0) = p(x)$

$u \to \infty$ AS $|x| \to \infty$

FOR C CONSTANT: GENERAL SOLN FOUND BY CHANGE OF VARIABLES.

$$\xi = x + ct \; ; \; \eta = x - ct$$

$$\phi(\xi, \eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$$

THEN PDE BECOMES:

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = 0$$

SOLN IS: $\phi(\xi, \eta) = \alpha(\xi) + \beta(\eta)$

WHERE $\alpha$ AND $\beta$ ARE ARB. FUNCTIONS.

IN TERMS OF $u$:

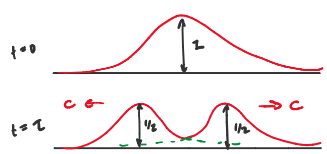
$$u(x,t) = \alpha(x+ct) + \beta(x-ct)$$

SIGNAL PROPAGATING W/ $\pm c$

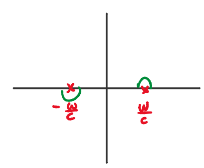
IN TERMS OF INITIAL CONDITIONS

$$u(x,t) = \frac{1}{2}\left[h(x+ct) + h(x-ct)\right] + \frac{1}{2c}\cdot\int_{x-ct}^{x+ct} P(\tau)\, d\tau$$

Ex: $\frac{\partial u}{\partial t}(x,0) = 0 \Rightarrow u(x,t) = \frac{1}{2}\left(h(x+ct) + h(x-ct)\right)$




CATCH: INVERSE F.T. WHEN DRIVING FUNCTION IS AT EXTREME.




INDENT ONE ABOVE THE OTHER BELOW.

WRONG INDENTATION PROVIDES BACKWARDS RESULTS.

OR

ADDING DAMPING MOVES SINGULARITIES OFF REAL AXIS SO NO INDENTATION NECESSARY.