ENM 521

tricks

Take the derivative Bound to zero Do contours around the branch cut (keyhole)

Common functions

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, $\frac{\mathrm{d}}{\mathrm{d}x} asinx = \frac{1}{\sqrt{1-x^2}}$
- $\cos \frac{e^{ix} + e^{-ix}}{2}$, $\frac{\mathrm{d}}{\mathrm{d}x} acosx = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x} atanx = \frac{1}{1+x^2}$
- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\sin z = \sin x \cosh y + i \cos x \sinh y$, $\cos z = \cos x \cosh y - i \sin x \sinh y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $\log z$: B.P. at 0,∞. $D[\log x] = 1/x$.
- $\lim_{\epsilon \to 0} \epsilon \log \epsilon = 0$
- L'hopital's rule
- Funny trig rule
- Geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Sequences

Convergent sequence: A sequence $\{z_n\}$ is said to have a limit z_0 or converge to z_0 which we write as

$$\lim_{n \to \infty} z_n = z_0$$

if for every $\epsilon > 0, \exists\, M \in \mathbb{Z}$, such that:

$$|z_n - z_0| < \epsilon, \ \forall n > M$$

Cauchy Sequence: a sequence $\{z_m\}$ of complex numbers is a Cauchy sequence if for every $\epsilon > 0$, $\exists N \in \mathbb{Z}$ such that $|z_m - z_M| < \epsilon$, $\forall m, M > N$.

Cauchy Criterion for Series Convergence Let $s_m = \sum_{k=1}^{m} a_k$. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}$ for $m, M > N$:

$$|s_M - s_m| = \left| \sum_{k=m+1}^{M} a_k \right| < \epsilon$$

Alternatively, letting $M = m + p$:

$$|s_{m+p} - s_m| = \left| \sum_{k=m+1}^{m+p} a_k \right| < \epsilon, \forall p = 1, 2, 3, ..$$

This is the most general test for convergence of a series.

Radius of Convergence: The power series $\sum_{m=0}^{\infty} a_m z^m$ has a radius of convergence $R = \frac{1}{A}$, where $A = \limsup_{m \to \infty} |a_m|^{1/m}$. If $A = \infty$, $R = 0$. Likewise, if $A = 0$, $R = \infty$.

Theorem 0.1 (Weirstrass M-test). *Let M_m be a sequence of real numbers. Suppose that $|f_m(z)| < M_m, \forall z \in E, m \in \mathbb{Z}^+$. If $\sum_{m=1}^{\infty} M_m$ converges, then $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly and absolutely on E.*

Harmonic function: Satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **Analyticity**: Differentiable everywhere in neighborhood of point. **Roots of Unity**: The kth root of unity is ω s.t. $\sum_{n=1}^{k} w^n = 0$ **Entire Function** (TODO)

Branch Points and Branch Cuts

NOTE: only have to say what it is.

- z^p, p non integer has BP at 0, ∞.
- $\log z$ BP at 0, ∞

Singularities

Isolated singularities

- **Pole**: f can be writen as $g/(z - z_0)$
- **Removable**: z_0 is analytic in neighborhood and limit exists. Chuanpeng says this is not a singularity.
- **Essential** Neither, yet isolated. Ex. $f(z) = \frac{\sin z}{z}$.

Non-isolated singularities

Taylor/Maclaurin series expansion

Maclaurin is just taylor about $z = 0$.

Laurent Series Expansions

Two cases:

1. f analytic in circle around expansion point. Use Taylor Series Expansion.
2. f analytic in annulus. Then find Laurent series through something like change of variables to get series.

Gauss Mean Value: Suppose $f(z)$ is analytic in the closed disk $|z - z_0| \leq r$. Then:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta} d\theta)$$

Cauchy Integral Formula If f analytic, C simply connected and closed, $z_0 \in C$, then $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$. Also $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

Cauchy Riemmann Equations Necessary but NOT sufficient for differentiability at a point. Applied to neighborhood is sufficient for analyticity. Or if u, v have continuous partials in domain D. Given $f(z) = u(x, y) + iv(x, y)$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

OR

$$f'(z) = \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

OR Let $f = u + iv$ be differentiable with complex partials at $z = re^{i\theta}$. Then:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \ \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

Indented Path lemma Let f have a simple pole at a with a residue Res$[f; a]$. Then given an upper half clockwise semi-circular contour around the pole C_ϵ, the resulting contour is:

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z) dz = -\operatorname{Res}[f; a] \pi i \tag{1}$$

Jordan's Theorem: If the only singularities in $g(z)$ are poles, $a > 0$, and $g(z) \to 0$ as $R \to \infty$, then

$$\lim_{R \to \infty} \int_{C_R} e^{iaz} g(z) dz = 0$$

Residue theorem: Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, .., z_n$ inside C. Let the residues at these points be $\alpha_1, \alpha_2, .., \alpha_n$ respectively. Then: $\int_C f(z) dz = 2\pi i \sum_{i=1}^{n} \alpha_i$

Example of sector integration

Example log expansion:Find the Maclaurin expansion of $f(z) = \log(z+1)$ valid for $|z| < 1$. We know that $f'(z) = \frac{1}{1+z}$. This is just the same as a geometric series: $f'(z) = \sum_{n=0}^{\infty}(-1)^n z^n$

Hence:

$$\begin{aligned} f(z) &= \log(1+z) && (2)\\ &= \int_0^z f'(\zeta)d\zeta + f(0) && (3)\\ &= \sum_{n=0}^{\infty}\int_0^z (-1)^n \zeta^n d\zeta + 0 && (4)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n \zeta^{n+1}}{n+1} && (5)\end{aligned}$$