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## Power series

**Th'm:** The p.s $\sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R = 1/A$,

where $A = \limsup_{n\to\infty} |a_n|^{1/n}$

If $A=0$, $R=\infty$

" $A=\infty$, $R=0$

**proof:**

For any pt. $z_0$, $\limsup_{n\to\infty} |a_n z_0^n|^{1/n} = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right) |z_0|$

$$= |z_0| A$$

According to root test, the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges absolutely when $|z_0|A < 1$

i.e $|z_0| < \frac{1}{A}$ and diverges when $|z_0| > \frac{1}{A}$

**Note:** $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is continuous at $z=z_0$ with $|z_0| < R$.

What about differentiation?

**Th'm:** If a function $f(z)$ is the pointwise limit of a p.s $\sum_{n=0}^{\infty} a_n z^n$ in $|z| < R$, then $f(z)$ is analytic for $|z| < R$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

The proof relies on the fact that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have the same radius of convergence, which follows from the fact that

$$\limsup_{n\to\infty} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} \limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$$

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\log n} \quad \text{Apply. (L'Hospital's rule)}$$

$$= \lim_{n\to\infty} e^{\frac{1/n}{1}} = \lim_{n\to\infty} e^{1/n} = 1$$

**Note:** The theorem states that every function defined by a power series is analytic inside of R.

**Note:** In fact, we can use this theorem repeatedly to get

$$f''(z) = 2a_2 + 6a_3 z + \cdots$$

$$\vdots$$

$$f^{(k)}(z) = k!\, a_k + (k+1)!\, a_{k+1} z + \cdots$$

which are valid inside R.

Now, letting $z=0$, we get

$$f^{(k)}(0) = k!\, a_k$$

$$\Rightarrow \boxed{a_k = \frac{f^{(k)}(0)}{k!}}$$

**Thm:** If a function $f(z)$ is pointwise limit of P.S in some nbhd of origin. Then $f(z)$ has derivates of all orders at each point interior of the circle $|z| = R$ and furthermore

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad |z| < R}$$

McLauren series; (Taylor series → when you expand about the point 0)

**Note:** Change of variables $z \to z-b$ leads to Taylor Series

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (z-b)^n, \quad |z-b| < R}$$

**Th'm:** Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence $R_1$ and $R_2$.

Then we can show that $f(z) + g(z)$ and $f(z) \cdot g(z)$ have P.S representations whose $R$ is given by $R = \min\{R_1, R_2\}$.

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum (a_n + b_n) z^n.$$

$$\left(\sum a_k z^k\right)\left(\sum b_k z^k\right) = \sum c_k z^k$$

where $c_k = \sum_{m=0}^{k} a_m b_{k-m}$

**Ex:** $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

we can show that this function is analytic everywhere.

$\hookrightarrow$ Radius of convergence, $R = \infty$

$\Rightarrow A = 0$.

$$A = \limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} \left(\frac{1}{n!}\right)^{1/n} = 0 \Rightarrow R = \infty$$

$\therefore$ Analytic everywhere.

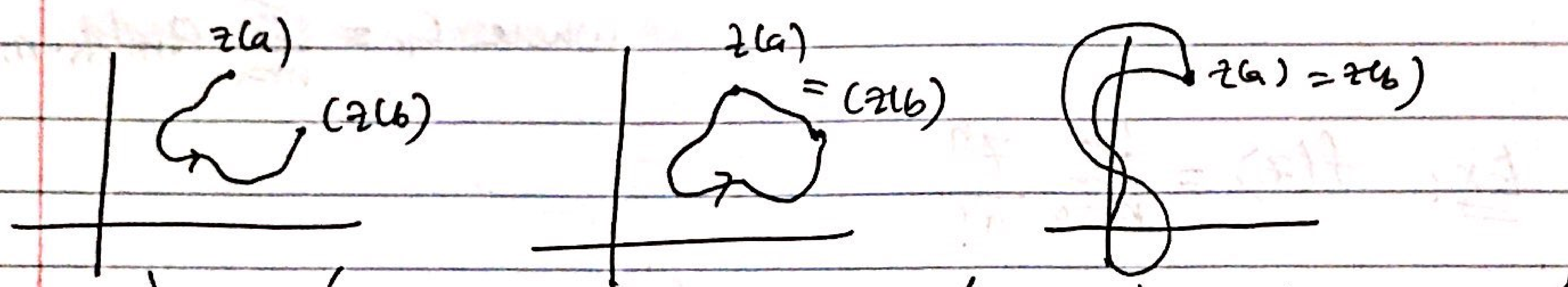
$$f'(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = f(z)$$

we would like to say that $f(z) = e^z$.

# COMPLEX INTEGRATION

**DEF:** A continuous curve (an arc) in a complex plane is defined parametrically by
$$z(t) = x(t) + i\,y(t) \quad , a \le t \le b$$
where $x(t)$ and $y(t)$ are real valued continuous functions of real variable $t$.

**Note:** A curve is a continuous function as well as a compact, connected set.




'Simple, not closed' (doesn't intersect itself) — 'Simple, Closed' — 'Not Simple, closed'

## Jordan Curve theorem:-

If $C$ is a simple, closed curve, then the compliment of $C$ consists of 2 disjoint domains, a BDD domain called interior and an unbdd domain called exterior.

**DEF:** A domain $D$ is simply connected if each simple, closed curve contained in $D$ contains only points of $D$ inside.

**Note:** Topologically, a simply connected domain can be continuously shrunk to a point

Geometrically, Simply Connected domains (S.C.Ds) has no holes inside.

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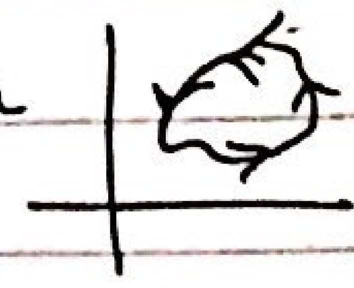
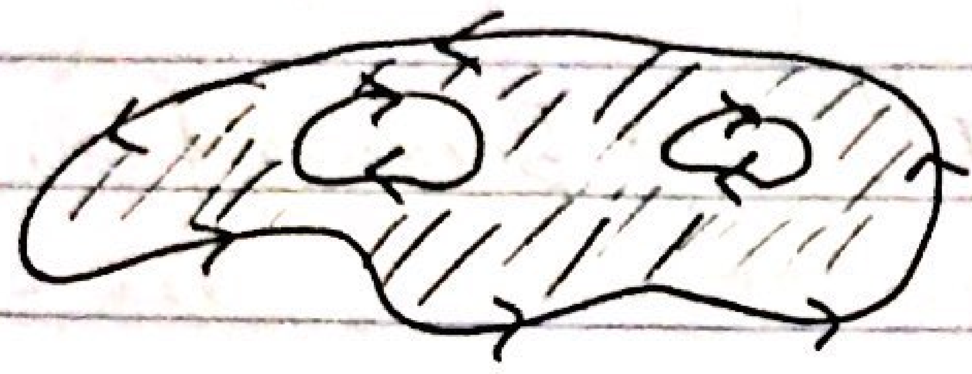
Note: For this definition, $z=\infty$ is not included.

Note: The Boundary $C$ of domain $D$ has +ve orientation if a person walking on it always has domain to left.





- Let $C$ be a smooth curve.

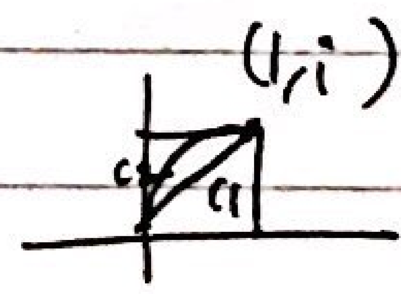
DEF: If complex function $f(z)$ is continuous on $C$, it follows that $f(z(t))\,z'(t)$ is continuous. For $a \le t \le b$, the integral of $f(z)$ on $C$ is defined by

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Note: The value $\int_C f(z)\,dz$ may depend on $f(z)$ as well as all points on $C$. (not just the end points).

EX: $f(z) = |z|^2$ ; $C_1 : z_1(t) = t + it$, $0 \le t \le 1$

$C_2 : z_2(t) = t^2 + it$, $0 \le t \le 1$




$$\int_{C_1} |z|^2 dz = \int_0^1 |t+it|^2 (1+i)\,dt$$

$$= \int_0^1 t^2 |1+i|^2 (1+i)\,dt = \int_0^1 2(1+i)\,t^2\,dt$$

$$= 2(1+i)\left[\frac{t^3}{3}\right]_0^1 = \frac{2}{3}(1+i)$$

$$\int_{C_2} |z|^2 dz = \int_0^1 |t^2+it|^2 (2t+i)\,dt$$

$$= \int_0^1 \left(t\sqrt{t^2+1}\right)^2 (2t+i)\,dt = \int_0^1 2t(t^4+t^2)\,dt + i\int_0^1 (t^4+t^2)\,dt$$

$$= 5/6 + \frac{8}{15} i \neq \int_{C_1} |z|^2 dz$$

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Ex: Let $f(z) = z^2$, we can check that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = -\frac{2}{3} + \frac{2}{3}i$$

NOT a COINCIDENCE!! → Because $f(z)$ is analytic.

Properties of integral :-

Suppose that $C : z(t)$ is a smooth curve on $[a,b]$.
Breaking up interval into $[a,c]$ & $[c,b]$, we obtain 2 curves $C_1$ & $C_2$.
By restricting $t$ to 2 different intervals. For any function $f(z)$ continuous on $C$,

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$$

$$= \int_a^c + \int_c^b f(z(t))\, z'(t)\,dt$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Valid for 'n' curves.

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Recall

For C smooth

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dc$$

where $z = z(t)$ is parametrization of curve. $(a \le t \le b)$

DEF: A function is sectionally continuous on an interval if it has atmost a finite number of discontinuities, with left & right-handed limits.

DEF: A contour is a curve having sectionally continuous derivative.

- Since every contour C may be expressed as a sum of a finite number of smooth curves $C_1 + C_2 + C_3 + \cdots + C_n$, the integral of a continuous function is defined by

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

- Recall that arclength L of a smooth curve in plane defined parametrically by $x = \phi(t)$ and $y = \psi(t)$ $(a \le t \le b)$ is given by

$$L = \int_a^b \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}\, dt$$

- Using complex parametrization,

$$C : z(t) = x(t) + i(y(t)) \quad (a \le t \le b)$$

for smooth curve 'C'.

The length L is

$$L = \int_a^b |z'(t)| \, dt$$

$$= \int_a^b \left| \frac{dx}{dt} + i \frac{dy}{dt} \right| dt$$

$$= \int_C |dz|$$

Note: If the curve is not a contour, its length may be $\infty$.

Theorem: Suppose $f(z)$ is continuous on contour $C$ having length $L$, with $|z| \le M$ on $C$. Then you can estimate the integral

$$\left| \int_C f(z)\, dz \right| \le \int_C |f(z)| \, |dz|$$

$$\le \int_C M \, dz = M \int_C dz$$

$$= ML$$

$$\Rightarrow \left| \int_C f(z) \, dz \right| \le ML$$

Theorem. Let $z(t) = x(t) + i\,y(t)$ $(a \le t \le b)$, be a contour $C$. Suppose $t = \phi(r)$ with $a = \phi(c)$ and $b = \phi(d)$ and $\phi'(r) > 0$, so that $t$ increases with $r$. If $\phi'(r)$ is sectionally continuous on $[d, c]$, the length of $C$ is

$$L = \int_c^d |\, z'(\phi(r)) \; \phi'(r) \; dr$$

proof: by chain rule

Note: Suppose $f(z)$ is continuous on contour C, and $t = \phi(r)$ satisfies hypothesis of above thm, then, also by chain rule,

$$\int_{C} f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$= \int_c^d f\big(z(\phi(r)) \, z'(\phi(r)) \, \phi'(r)\, dt$$

Therefore, the curve could be equally will be parametrized by $z(\phi(r))$, $c \le r \le d$

- The contour

$$-C : z(-t) = x(-t) + i\,y(-t) \qquad -b \le t \le -a$$

represents the same curve but traversed in the opposite direction, as the original curve

$$C : z(t) = x(t) + iy(t) \qquad a \le t \le b.$$

we have

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f(z(-t))\, z'(-t)\,(-1)\, dt.$$

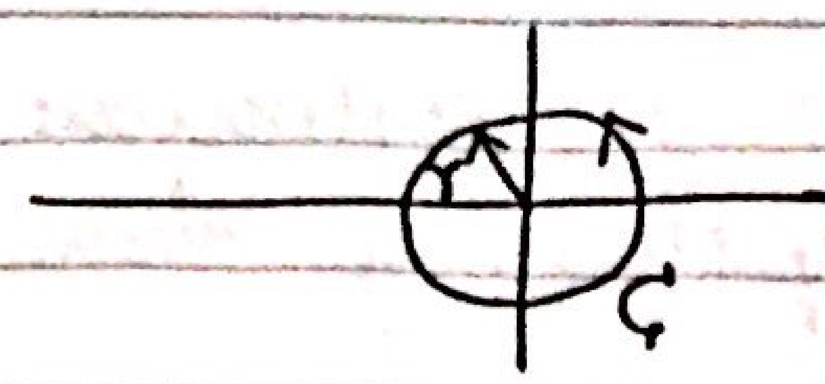
make the substitution $t = -r$

$$\int_{-C} f(z)\,dz = \int_b^a f(z(r))\, z'(r)\, dr$$

$$= -\int_a^b f(z(r))\, z'(r)\, dr$$

$$\Rightarrow \boxed{\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz}$$

Ex: $\displaystyle\oint_{|z|=r} z^n \, dz$



$$z = re^{i\theta} \Rightarrow dz = re^{i\theta}(i)\, d\theta$$
$$0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi} \left(re^{i\theta}\right)^n \left(i\, re^{i\theta}\, d\theta\right)$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} \, d\theta \qquad (\text{if } n \neq -1)$$

$$= i r^{n+1} \frac{1}{i(n+1)} \left[ e^{i(n+1)\theta} \right]_0^{2\pi} = 0$$

$$\oint_{|z|=r} z^n \, dz = \begin{cases} i r^{n+1}(2\pi) & n = -1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\oint_{|z|=r} z^n \, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{otherwise} \end{cases}$$

what's happening with $n = -1$?

$$\int \frac{1}{z}\, dz = \log z$$

multiple valued function.

## LINE INTEGRALS :-

Suppose $f(z) = u(x,y) + i\,v(x,y)$ is continuous on contour C.

$$C : z(t) = x(t) + i\,y(t) \quad (a \le t \le b)$$

Then,

$$\int_C f(z)\,dz = \int_a^b f(z(t))\; z'(t)\,dt .$$

$$= \int_a^b \left[u(z(t)) + i\,v(z(t))\right]\left[x'(t) + i\,y'(t)\right] dt$$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Recall (Fundamental thm of Calculus):

Suppose that $f(x)$ is continuous on $[a,b]$. Then $\exists$ a function $F(x)$ such that $F'(x) = f(x)$ on $[a,b]$ and

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

## GREEN'S THEOREM :-

(Nottingham : Bus tours to Green's Mill) ☺

Let $P(x,y)$ and $Q(x,y)$ be continuous with conti. partials in a simply connected region R whose boundary is the contour, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underbrace{dx\,dy}_{dA}}$$

where $C$ is 

Cauchy Theorem (weak version):-

If $f(z)$ is $\boxed{\text{analytic}}$ (with a continuous derivative) in a simply connected domain $D$, and $C$ is is a $\boxed{\text{closed contour}}$ lying in $D$, then

$$\int_C f(z)\,dz = 0$$

proof (use green's thm):-

Set $f = u(x,y) + i\,v(x,y)$

By C-R equations:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

For all pts. in domain since $f'(z)$ is presumed continuous, then 4 partials must be cont.

Suppose $C$ is a simple, closed contour and Application of Green's thm gives

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C u\,dy + v\,dx$$

(using Green's thm):

$$= \iint_R \cancelto{0}{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} dx\,dy + i \iint_R \cancelto{0}{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)} dx\,dy$$

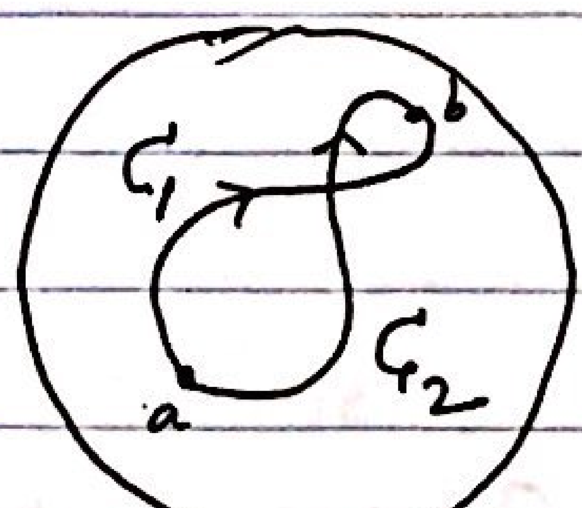
$= 0$ (using C-R conditions)

<u>Note:</u> The assumption that contour is simple can be relaxed.

- Result is true for closed contours.

$\Rightarrow$ <u>Corollary:</u> Under same hypothesis, Let $C_1$ and $C_2$ be any contours in the domain with same initial and terminal points, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$




<u>proof:</u> $C_1 - C_2$ is a closed contour and use Cauchy.

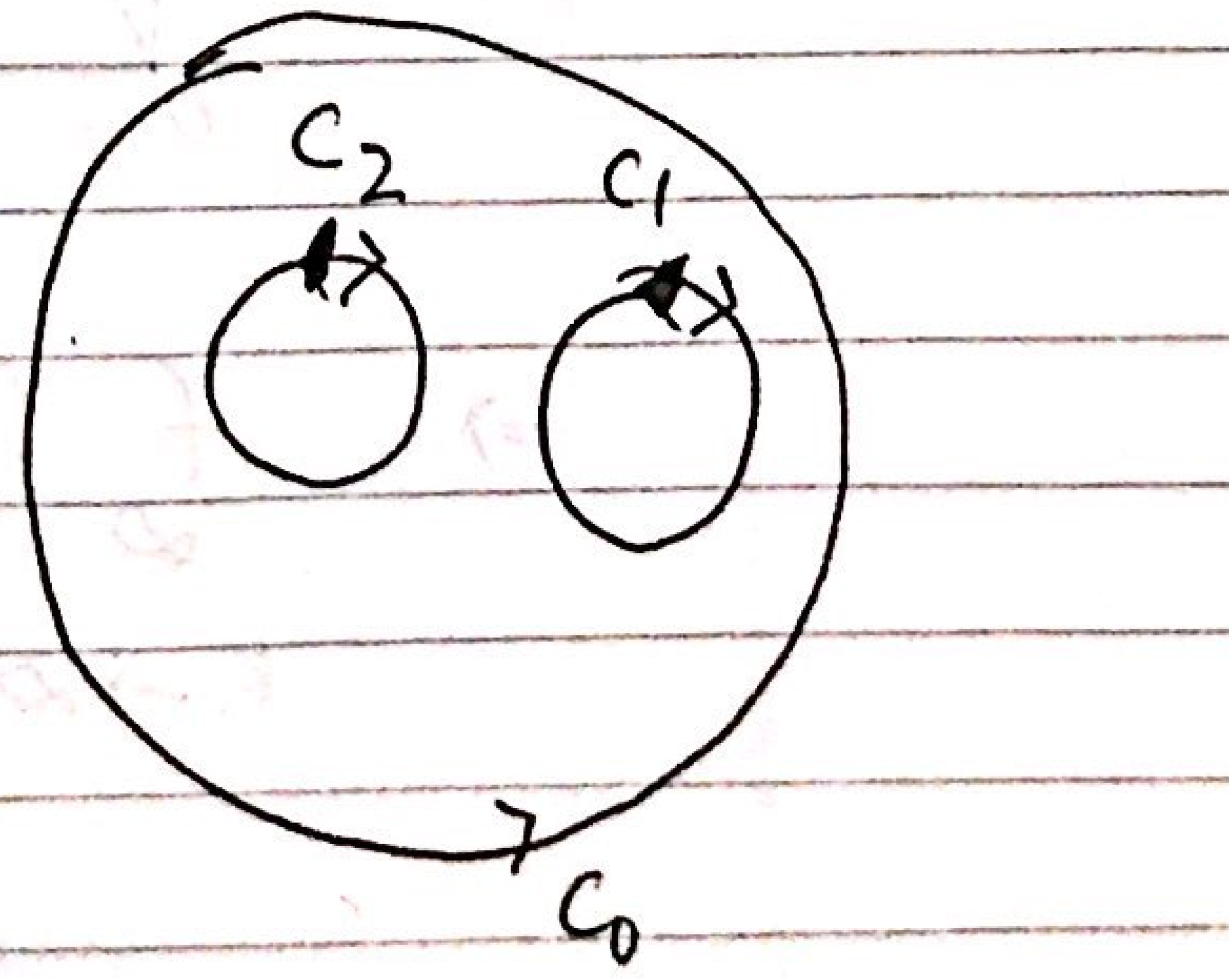
<u>Note:</u> The result can be generalized for multiply connected domains

<u>Theorem:</u> Same hypotheses except this is multiply connected domain with bound $C$.

Let $C = C_1 + C_2 + C_0$

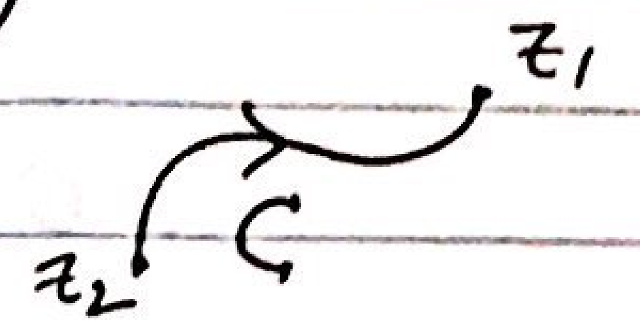
and apply Cauchy.

$$\int_C f(z)\,dz = 0$$



**Th'm:** Let $f(z)$ be continuous on Domain $D$, and suppose there's a different function $F(z)$ such that $F'(z) = f(z)$ in $D$. Then, for any contour $C$ in $D$, we have that

$$\int_C f(z)\,dz = F(z_1) - F(z_0)$$




**Counter example:**

When $f(z)$ is not analytic

$$f(z) = \bar{z}$$

$$\int_{|z|=1} \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta}\,d\theta = 2\pi i$$

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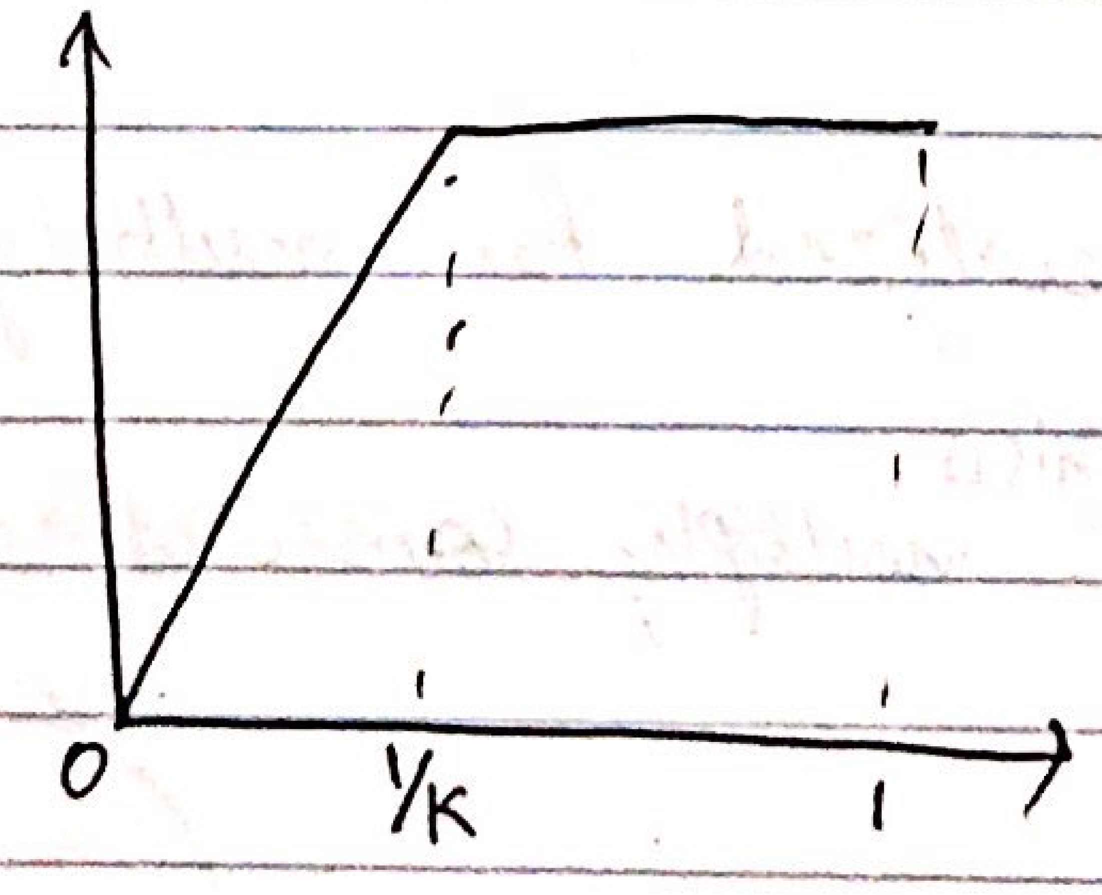
## RECITATION

Convergence and stuff:-

**Ex:**




$$f_k(x) = \begin{cases} kx & 0 \le x \le 1/k \\ 1 & 1/k < x \le 1 \end{cases}$$

$$\Rightarrow \lim_{k\to\infty} f_\infty(x) = \begin{cases} 1 & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

If a sequence of continuous functions are $\boxed{\text{uniformly}}$ convergent, then the limit of the sequence of these functions is also continuous.

Ex: Ex: $f_k(x) = \frac{1}{kx+1}$ $x \in [0,1]$

First find pointwise limit and check if convergence is uniform.

as $k \to \infty$ $f(x) = \begin{cases} 0 & x \in (0,1] \\ 1 & x = 0 \end{cases}$ $\Rightarrow$ limit function is not continuous.

$\hookrightarrow$ convergence is non-uniform.

Ex: $f_k(x) = \frac{1}{kx+1}$ $x \in [10^{-23}, 1]$

**Note:** Continuity of limit function of sequence of cont-fns is just a necessary but not sufficient condition.

- For a function that converges uniformly, the following must hold true.

$$\lim_{k \to \infty} \lim_{z \to z_0} f_k(z) = \lim_{z \to z_0} \lim_{k \to \infty} f_k(z)$$

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## Power Series :-

$$\sum_{n=0}^{\infty} a_n z^n$$

radius of convergence : $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$

Converges Uniformly in $|z| \leq r < R$

" Non-" in $|z| < R$

Ex: $1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z}$ (pointwise limit)

$R = 1$

check for uniform convergence in $|z|<1$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\frac{1}{(m+1)!} \leq \frac{1}{(m+1)\,m}$$

$$= \frac{1}{m} - \frac{1}{m+1}$$ use this trick!!