ENM 521

tricks

Take the derivative Bound to zero Do contours around the branch cut (keyhole) Exponentials are faster than polynomials

Summation/Integration techniques

$\sum_{j=1}^{n}\sum_{k=1}^{j} f(j,k) = \sum_{k=1}^{n}\sum_{j=k}^{n} f(j,k)$

$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f(j,k) = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} f(n+k,k)$

Change of variables: $\int_{\phi(a)}^{\phi(b)} f(u)du = \int_a^b f(\phi(x))phi'(x)dx$

$\iint_R f(x,y)dA = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\bar{A}$

Integration by Parts: $\int_a^b udv = uv|_a^b - \int_a^b vdu$

Integral in spherical over whole domain:
$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ \rho\cos\phi$
$\rho \in [0,\infty], \theta \in [0,\pi], \phi \in [0,2\pi]\ dV = \rho^2\sin\phi d\rho d\theta d\phi$

Bounding: $|e^{e^{i\theta/2}+tRe^{i\theta}}| \le e^{\cos(\theta/2)+Rt\cos\theta}$

Common functions

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- $\sin x = \frac{e^{ix}-e^{-ix}}{2i},\ \frac{\mathrm{d}}{\mathrm{d}x} asinx = \frac{1}{\sqrt{1-x^2}}$
- $\cos \frac{e^{ix}+e^{-ix}}{2},\ \frac{\mathrm{d}}{\mathrm{d}x} acosx = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{\mathrm{d}}{\mathrm{d}x} atanx = \frac{1}{1+x^2}$
- $\sinh x = \frac{e^x-e^{-x}}{2}$
- $\cosh x = \frac{e^x+e^{-x}}{2}$
- $\sin z = \sin x\cosh y + i\cos x\sinh y$,
  $\cos z = \cos x\cosh y - i\sin x\sinh y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $\log z$: B.P. at 0,∞. $D[\log x] = 1/x$.
- $\lim_{\epsilon\to 0}\epsilon\log\epsilon = 0$
- L'hopital's rule for indefinite limits
  $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$
- $(1-z)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n} z^n 1]\alpha$ where $\binom{\alpha}{n} = \frac{\alpha!}{n!(\alpha-n)!}$
- $\arctan z = \frac{1}{2i}\ln\frac{1+iz}{1-iz}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{x^{2n-1}}{(2n-1)!} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{x^n}{n} \stackrel{\text{or}}{=} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots = \sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{x^{2n-1}}{2n-1} \stackrel{\text{or}}{=} \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Sequences

Convergent sequence: A sequence $\{z_n\}$ is said to have a limit z_0 or converge to z_0 which we write as

$$\lim_{n\to\infty} z_n = z_0$$

if for every $\epsilon > 0, \exists\ M \in \mathbb{Z}$, such that:

$$|z_n - z_0| < \epsilon,\ \forall n > M$$

Cauchy Sequence: a sequence $\{z_m\}$ of complex numbers is a Cauchy sequence if for every $\epsilon > 0$, $\exists N \in \mathbb{Z}$ such that $|z_m - z_M| < \epsilon,\ \forall m, M > N$.
Cauchy Criterion for Series Convergence Let $s_m = \sum_{k=1}^{m} a_k$. The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}$ for $m, M > N$:

$$|s_M - s_m| = \left|\sum_{k=m+1}^{M} a_k\right| < \epsilon$$

Alternatively, letting $M = m + p$:

$$|s_{m+p} - s_m| = \left|\sum_{k=m+1}^{m+p} a_k\right| < \epsilon, \forall p = 1, 2, 3, ..$$

This is the most general test for convergence of a series.
Radius of Convergence: The power series $\sum_{m=0}^{\infty} a_m z^m$ has a radius of convergence $R = \frac{1}{A}$, where $A = \limsup_{m\to\infty} |a_m|^{1/m}$. If $A = \infty$, $R = 0$. Likewise, if $A = 0$, $R = \infty$.

Theorem 0.1 (Weirstrass M-test). *Let M_m be a sequence of real numbers. Suppose that $|f_m(z)| < M_m, \forall z \in E, m \in \mathbb{Z}^+$. If $\sum_{m=1}^{\infty} M_m$ converges, then $\sum_{m=1}^{\infty} f_m(z)$ converges uniformly and absolutely on E.*

Harmonic function: Satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **Analyticity**: Differentiable everywhere in neighborhood of point. **Roots of Unity**: The kth root of unity is ω s.t. $\sum_{n=1}^{k} w^n = 0$ **Entire Function** holomorphic/analytic all over

Branch Points and Branch Cuts

NOTE: only have to say what it is.

- z^p, p non integer has BP at 0, ∞.
- $\log z$ BP at 0, ∞

Function	Branch Point
z^p, non-integer p	$z = 0, \infty$
$log(z)$	$z = 0, \infty$

function	branch cut(s)
$\csc^{-1} z$	$(-1, 1)$
$\cos^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
$\cot^{-1} z$	$(-i, i)$
csch^{-1}	$(-i, i)$
$\cosh^{-1}$	$(-\infty, 1)$
$\coth^{-1}$	$[-1, 1]$
sech^{-1}	$(-\infty, 0]$ and $(1, \infty)$
$\sinh^{-1}$	$(-i\infty, -i)$ and $(i, i\infty)$
$\tanh^{-1}$	$(-\infty, -1]$ and $[1, \infty)$
$\sec^{-1} z$	$(-1, 1)$
$\sin^{-1} z$	$(-\infty, -1)$ and $(1, \infty)$
$\tan^{-1} z$	$(-i\infty, -i]$ and $[i, i\infty)$
$\ln z$	$(-\infty, 0]$
$z^n, n \notin \mathbb{Z}$	$(-\infty, 0)$ for $\Re[n] \le 0$; $(-\infty, 0]$ for $\Re[n] > 0$
$\sqrt{z}$	$(-\infty, 0)$

Singularities

Isolated singularities

- **Pole**: f can be written as $g/(z - z_0)$
- **Removable**: z_0 is analytic in neighborhood and limit exists. Chuanpeng says this is not a singularity.
- **Essential** Neither, yet isolated. Ex. $f(z) = \frac{\sin z}{z}$.

Non-isolated singularities: each deleted neighborhood has a singularity. e.g. branch points, $2\frac{1}{\sin(1/z)}$

Taylor/Maclaurin series expansion

Maclaurin is just taylor about $z = 0$.

Laurent Series Expansions

Two cases:

1. f analytic in circle around expansion point. Use Taylor Series Expansion.
2. f analytic in annulus. Then find Laurent series through something like change of variables to get series.

Gauss Mean Value: Suppose $f(z)$ is analytic in the closed disk $|z - z_0| \leq r$. Then:

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta}d\theta)$$

Cauchy Integral Formula If f analytic, C simply connected and closed, $z_0 \in C$, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$. Also $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$

Cauchy Riemmann Equations Necessary but NOT sufficient for differentiability at a point. Applied to neighborhood is sufficient for analyticity. Or if u, v have continuous partials in domain D. Given $f(z) = u(x, y) + iv(x, y)$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

OR

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

OR Let $f = u + iv$ be differentiable with complex partials at $z = re^{i\theta}$. Then:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Indented Path lemma Let f have a simple pole at a with a residue $\text{Res}[f; a]$. Then given an upper half clockwise semi-circular contour around the pole C_ϵ, the resulting contour is:

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} f(z)dz = -\text{Res}[f; a]\pi i \tag{1}$$

Jordan's lemma: If C_R is the positive imaginary semicircular contour, the only singularities in $g(z)$ are poles, $a > 0$, and $g(z) \to 0$ as $R \to \infty$, then

$$\lim_{R\to\infty}\int_{C_R} e^{iaz}g(z)dz = 0$$

If you'd like to apply Jordan's lemma for $a < 0$, try taking the semicircular contour that goes through the negative semicircular contour. Likewise, we can also say very carefully that Jordan's lemma.

Residue theorem: Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, .., z_n$ inside C. Let the residues at these points be $\alpha_1, \alpha_2, .., \alpha_n$ respectively. Then: $\int_C f(z)dz = 2\pi i\sum_{i=1}^n \alpha_i$

Residue of simple pole of order k:
$a_{-1} = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}(z - z_0)^k f(z)$

Sector integration: take an arc-slice of a circle around a singularity (or two) and then go use residue theorem.

Maclaurin expansion of $f(z) = \log(z + 1)$ valid for $|z| < 1$. We know that $f'(z) = \frac{1}{1+z}$. This is just the same as a geometric series: $f'(z) = \sum_{n=0}^{\infty}(-1)^n z^n$

Hence:

$$\begin{aligned} f(z) &= \log(1 + z) & (2)\\ &= \int_0^z f'(\zeta)d\zeta + f(0) & (3)\\ &= \sum_{n=0}^{\infty}\int_0^z (-1)^n\zeta^n d\zeta + 0 & (4)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\zeta^{n+1}}{n+1} & (5)\end{aligned}$$

Special Functions

Gamma function: $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$

The infinite form:

$\lim_{n\to\infty}\Gamma(z; n) := \int_0^n (1 - \frac{t}{n})^n t^{z-1}dt = \Gamma(z) =$ Integer: $n \in \mathbb{Z}^+$, $\Gamma(n) = (n + 1)!$ Properties of Gamma function

- $\Gamma(1) = 1$
- $\Gamma(z + 1) = z\Gamma(z)$
- $\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}$
- $\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad z \notin \mathbb{Z}$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
- $\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}$
- $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \frac{\pi}{\cos\pi z}$
- $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$
- $(1 + z)^\alpha = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)}z^n$ (use the factorial identity)
- $\log\Gamma(z) = -\log z - \gamma z - \sum_{k=1}^{\infty}\left[\log\left(1 + \frac{z}{k}\right) - \frac{z}{k}\right]$
- $\text{Res}[\Gamma, -n] = \frac{(-1)^n}{n!}$

Beta Function: $\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Psi Function:$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{\mathrm{d}}{\mathrm{d}z}(\log\Gamma(z))$, with $\Psi(1) = -\gamma$, where γ ie the Euler-Mascheroni Constant.

Riemann Zeta Function: $\zeta(s) = \sum_{n=0}^{\infty}\frac{1}{(1+n)^\zeta}$ Relation to prime numbers $\frac{1}{\zeta(s)} = \prod_p(1 - \frac{1}{p^s}),\ p \in$ Primes

Bessel Equation Definitions 1st kind:
$J_\nu(z) = \left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n+1)}\left(\frac{z}{2}\right)^{2n}$ $J_{-n}(x) = (-1)^n J_n(x)$
$J_n(x) = \frac{1}{\pi}\int_0^\infty\cos(n\tau - x\sin\tau)$ $J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(x\sin\tau n\tau)}d\tau$
2nd Kind: $Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}$

Legendre Polynomials: $P_n = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}[(z^2 - 1)^n]$
$P_0(z) = 1,\ P_1(z) = z,\ P_2(z) = \frac{1}{2}(3z^2 - 1)$

Generating Function:
$\phi(\zeta, z) = P_0(z) + \zeta P_1(z) + \zeta^2 P_2(z) + ...$

Fourier Transform

Fourier Transform:$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda\tau}f(\tau)d\tau$ Inverse Fourier Transform: $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\lambda)e^{-i\lambda\tau}d\lambda$

Properties of the Fourier Transform

For a function $f(t)$ with Fourier Transform $F(\lambda)$

- $I.F.T.[f(t)] = F(-\lambda)$
- For constants a, b:
  $F.T.[af(t) + bg(t)] = aF.T.[f(t)] + bF.T.[g(t)]$
- $F.T.[f(t)g(t)] = F.T.[f(t)] * F.T.[g(t)]$ where $*$ is the convolution operator.
- $F.T.[f(t) * g(t)] = F.T.[f(t)]F.T.[g(t)]$
- $F.T.[\frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t)] = (-i\lambda)^n F(\lambda)$
- $F.T.[\delta(t)] = \frac{1}{\sqrt{2\pi}}$
- $F.T.[1] = \frac{1}{\sqrt{2\pi}}\delta(-t) = \frac{1}{\sqrt{2\pi}}\delta(t)$
- $F.T.[f(t - t_0)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda t}f(t - t_0)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\lambda(t'+t_0)}f(t')dt' = e^{i\lambda t_0}F(\lambda)$
- $F.T.[\delta(x-\xi)\delta(y-\nu)\delta(z-\zeta)] = \left(\frac{1}{\sqrt{(2\pi)}}\right)^3 e^{-(X\xi+Y\nu+Z\zeta)}$

F.T. of Gaussian: $g(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}}, G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}}$

linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
time scaling	$f(at)$	$\frac{1}{\|a\|}F(\frac{\omega}{a})$
time shift	$f(t - T)$	$e^{-j\omega T}F(\omega)$
differentiation	$\frac{df(t)}{dt}$	$j\omega F(\omega)$
	$\frac{d^k f(t)}{dt^k}$	$(j\omega)^k F(\omega)$
integration	$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
multiplication with t	$t^k f(t)$	$j^k\frac{d^k F(\omega)}{d\omega^k}$
convolution	$\int_{-\infty}^{\infty}f(\tau)g(t - \tau)\,d\tau$	$F(\omega)G(\omega)$
multiplication	$f(t)g(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\widetilde{\omega})G(\omega - \widetilde{\omega})\,d\widetilde{\omega}$

Parseval's theorem for Fourier:
$\int_{-\infty}^{\infty}|F(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty}|f(x)|^2 dx$
This is analogous to **Parseval's identity for Fourier series:**
$\frac{1}{T}\int_{-T}^{T}f^2(t)dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

3D Fourier: $F(\lambda) = \frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty}e^{i\lambda\mathbf{x}}f(\mathbf{x})dv_{\mathbf{x}}$ **Fourier Coefficients**:

Green's function: Solution to $LG = \delta$ for linear differential operator L. To find this for some ODE, set the forcing function to $\delta(x-\xi)$, then the solution u is the green's function. $u_t - ku_xx = 0$ with $u(x,0) = \delta(x)$ has the kernel function: $G(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$, For an n dimensional equation: $U(x_1,, ...x_n, t) = \frac{1}{\sqrt{(4\pi kt)^n}} \exp\left(-\frac{\mathbf{x}\cdot\mathbf{x}}{4kt}\right)$, Then the forced solution is the convolution with the initial condition$u(x,0) = g(x)$ $u(x,t) = \int U(x-y,t)g(y)dy$
The Green's function $G(x,t;\xi)$ is the solution with $\delta(x-\xi)$:
$G(x,t;\xi) = G(x-\xi,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-\xi)^2}{4kt}\right)$,

Homogeneous ODE	Green's Function $g(x,u)$
$y' - ay$	$e^{a(x-u)}$
y''	$x-u$
$y'' - 2ay' + a^2$	$(x-u)e^{a(x-u)}$
$y'' - (a+b)y' + aby$	$\frac{e^{a(x-u)} - e^{b(x-u)}}{a-b}$
$y'' + b^2y$	$\frac{1}{b}\sin[b(x-u)]$
$y'' - b^2y$	$\frac{1}{b}\sinh[b(x-u)]$
$y'' - 2ay' + (a^2+b^2)y$	$\frac{1}{b}e^{a(x-u)}\sin[b(x-u)]$
$y'' - 2ay' + (a^2-b^2)y$	$\frac{1}{b}e^{a(x-u)}\sinh[b(x-u)]$
$x^2y'' + xy' - b^2y$	$\frac{u}{2b}\left(\frac{x^b}{u^b} - \frac{u^b}{x^b}\right)$
$x^2y'' - (b+a-1)xy' + aby$	$\frac{u}{b-a}\left(\frac{x^b}{u^b} - \frac{x^a}{u^a}\right)$

Laplace Transform

$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ Inverse L.T.:
$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$
Common Laplace Transforms:
Gaussian: $\mathcal{L}[e^{-\pi x^2}] = \frac{1}{2}e^{\frac{\pi s^2}{4}}[\frac{1}{2} - \frac{1}{2}\,\text{erf}\,\frac{s\sqrt{\pi}}{2}]$

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}\,dt$
$f+g$	$F+G$
αf $(\alpha \in \mathbf{R})$	αF
$\frac{df}{dt}$	$sF(s) - f(0)$
$\frac{d^kf}{dt^k}$	$s^kF(s) - s^{k-1}f(0) - s^{k-2}\frac{df}{dt}(0) - \cdots - \frac{d^{k-1}f}{dt^{k-1}}(0)$
$g(t) = \int_0^t f(\tau)\,d\tau$	$G(s) = \frac{F(s)}{s}$
$f(\alpha t)$, $\alpha > 0$	$\frac{1}{\alpha}F(s/\alpha)$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-\frac{dF}{ds}$
$t^kf(t)$	$(-1)^k\frac{d^kF(s)}{ds^k}$
$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
$g(t) = \begin{cases} 0 & 0 \le t < T \\ f(t-T) & t \ge T \end{cases}$, $T \ge 0$	$G(s) = e^{-sT}F(s)$

1	$\frac{1}{s}$
δ	1
$\delta^{(k)}$	s^k
t	$\frac{1}{s^2}$
$\frac{t^k}{k!}$, $k \ge 0$	$\frac{1}{s^{k+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2} = \frac{1/2}{s-j\omega} + \frac{1/2}{s+j\omega}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2} = \frac{1/2j}{s-j\omega} - \frac{1/2j}{s+j\omega}$
$\cos(\omega t+\phi)$	$\frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$

Multiplication and convolution of Laplace Transform:
$f(t)g(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(x)G(s-x)dx$
$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = F(s)G(s)$

Convolution

: $f \circledast g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi$
Properties of δ function:

- $\delta(-t) = \delta(t)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$
- $t\delta(t) = 0$
- $\delta(t^2-a^2) = \frac{1}{2|a|}[\delta(t+a) + \delta(t-a)]$
- $\delta(t) = -t\delta'(t)$ $(I.B.P.)$

Solving differential equations

To solve $y''(x) + xy'... = 0$, write out as series and solve:
$y(x) = \sum_{n=0}^\infty a_nx^n$
$y'(x) = \sum_{n=0}^\infty (n+1)a_{n+1}x^n = \sum_{n=1}^\infty (n)a_nx^{n-1}$
$y''(x) = \sum_{n=0}^\infty (n+1)(n+2)a_{n+2}x^n = \sum_{n=2}^\infty n(n-1)a_nx^n$
Substitute in and then we see that the sum $\sum Ca_n + Ba_{n+k} = 0$, then $Ca_n + Ba_{n+k} = 0$ so then solve for a_{n+k} in terms of a_n to get a recursive equation. There is also usually an a_0 case or whatever, and you can go solve for the odd and even series or every $n = 2k$ series or whatever.
How do we know when these infinite sums terminate? if it's $(a-1)...(a-2k-2)$ or something, then it terminates if a is a positive odd integer, then maybe expand to say that it is about all integers.
Laplacian Operator: $\Delta f = \nabla\nabla f$, where $\nabla = (\frac{\mathrm{d}}{\mathrm{d}x_1}, ..., \frac{\mathrm{d}}{\mathrm{d}x_n})$.
Cartesian: $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
Cylindrical: $\Delta f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$
Spherical:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$
$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$F.T.[\Delta f(x,y,z) + k^2\frac{\mathrm{d}f}{\mathrm{d}t}] = -(X^2+Y^2+Z^2)F + k^2F$

ODE

Solving IVP:

1. Transform D.E. using initial conditions and laplace
2. Solve for $Y(s)$
3. IFT for $y(t)$.

Also try $u(x,t) = X(x), T(t)$, then for u_t, u_{xx} you can integrate separately.

First order ODE

$\frac{\mathrm{d}U}{\mathrm{d}t} = -k\lambda^2U$ Then $U(\lambda,t) = C(\lambda)e^{-k\lambda^2t}$

Second Order ODE

Constant coefficient:$ay''(x) + by'(x) + cy(x) = f(x)$
$y(x) = e^{rx} \to ar^s + br + c = 0$ Cases:
Distinct, real roots: $r = r_{1,2}, y_h(x) = c_1x^{r_1} + c_2x^{r_2}$
One real root: $y_h(x) = (c_1 + c_2\ln|x|)x^r$
Complex roots: $r = \alpha + i\beta$,
$y_h(x) = (c_1\cos()\beta\ln|x|) + c_2\sin(\beta\ln|x|))x^\alpha$
Further: $-k^2U(\lambda,y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2}U(\lambda,y)$ has solution:
$U(\lambda,y) = A(\lambda)e^{-|\lambda|y}$ Furthermore: $y'' + \lambda y = 0$

- $\lambda > 0$: $y = a\cos\left(\sqrt{\lambda}x\right) + b\sin\left(\sqrt{\lambda}x\right)$
- $\lambda = 0$: $y = a + bx$
- $\lambda < 0$: $y = a\cosh(\sqrt{-\lambda}x) + b\sinh(\sqrt{-\lambda}x)$