Collision-Free Vehicle Formation Control under Arbitrarily Switching Network Topology

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Abstract— This paper is concerned with collision-free vehicle formation control when the inter-vehicle network topology is time-varying. Like typical collision-free formation control approaches, the proposed control approach involves two control phases: one for converging to a desired formation (no collision avoidance considered); the other for achieving collision avoidance. However, the proposed control approach has two distinctive features. First, the proposed control approach uses a fixed set of time-invariant control gains for achieving the desired formation but allows time-varying control gains only for avoiding collision. This feature lends itself well to the increase (and the decrease) of the controller's reliability (and complexity) during the most of manoeuvring periods. Second, the proposed control approach provably works for time-varying network topologies. Theoretical and numerical evidences are provided to justify these two features.

Key words: formation control; collision avoidance; time-varying network.

I. PROBLEM STATEMENT AND INTRODUCTION

The starting point of this paper comes from the problem considered in [1]. Therein, a static feedback control law is proposed to let m second-order vehicles change their positions and velocities to achieve a desired formation. Mathematically, each vehicle has the following dynamics: $\dot{x}_i = A_{veh}x_i + B_{veh}u_i$, $i = 1, 2, \ldots, m$, $x_i \in \mathbf{R}^{2n}$ where the entries of x_i represent n configuration variables for vehicle i and their derivatives, u_i represents the control inputs, and

$$A_{veh} = I_n \otimes \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix}, \quad B_{veh} = I_n \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Here, I_n is an identity matrix with dimension n and $\otimes$ the Kronecker product. Then, using a static feeback control law

$$u = -FL(x - h), \tag{1.1}$$

where u and x are the vectors of u_i's and x_i's, the m-vehicle dynamics can be represented as follows: $\dot{x} = Ax - BFL(x - h)$ or

$$\dot{x} = (A - BFL)x + BFLh. \tag{1.2}$$

Here, $A = I_m \otimes A_{veh}$, $B = I_m \otimes B_{veh}$, $L = L_G \otimes I_{2n}$ (L_G is the Laplacian matrix of an inter-vehicle communication graph G - see [2] for details; G is assumed to be connected and has an edge if two vehicles are apart by less than

a communication range of d), and h is a constant vector associated with a desired formation. In [1], it is shown that $a_{21} = 0$ and $F = I_m \otimes F_{veh}$ with $F_{veh} = I_n \otimes [f_1, f_2]$ with sufficiently large positive constants f_1 and f_2, can guarantee achieving any formation h for any connected static graph of G. For this reason, a_{21} is assumed to be zero in this paper, and the 2-norm of u_i ($i = 1, 2, \ldots, m$), $\|u_i\|$, is bounded by a constant u_{max} after large f_1 and f_2 are chosen. Also, the velocity of each vehicle is bounded by v_{max}. Note that the proposed control law in (1.1) is decentralized in the sense that each vehicle uses the information only from its neighbours (including itself) for the formation control.[1]

The purpose of this paper is twofold. First, the proposed control law in (1.1) is verified for time-varying network topologies, i.e. time-varying L_G. Unlike typical approaches in the literature, the main merit of this control law is that it requires no significant computational power to design the control law. Since F is a fixed vector throughout the formation manoeuvre, one can simply use the correct Laplacian representing the current communication topology. Second, the proposed control law is further modified to account for collision avoidance between vehicles. When a collision between two vehicles is imminent, a collision-avoidance rule replaces the control law in (1.1) to avoid the collision. This replacement is taken place in such a way that the convergence to be verified for the control law in (1.1) is not affected by the collision-avoidance manoeuvre.

The main challenge in achieving the said purpose is to verify the stability of the proposed control system (a switched linear system) and the convergence to a desired formation. Although coordinated control (including vehicle formation control) under time-varying network topology has been progressed by many researchers, this type of stability analysis is very difficult in general, and so is often limited to special classes of matrices such as commuting matrices [5], a pair of matrices [6] or 2-by-2 matrices [7]. Related works can also be found for discrete-time linear systems. Note that continuous-time systems may be converted to discrete-time ones by introducing a proper sampling period. Among the recent works, [8], [9] provides an elegant theory for analyzing convergence behaviour of multi-agent systems under time-varying network topologies. The key fact used in [8] is that the product of an infinite number of ergodic matrices (belonging to a finite set) becomes a constant

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[1]In [3], a generalized static control law $u = K(x - h)$ with K not necessarily the same as FL is proposed. Also, there are some interesting works such as [4] in which neighbouring vehicles' velocity information is not used for formation control.

rank-one matrix. Unfortunately, the theory developed in [8] may not be directly used for the present purpose, as the matrices of current interest are not ergodic (not even non-negative). Convergence analysis about the product of general matrices is quite difficult (even boundedness of the product is undecidable [10]). However, there are special classes of matrices (including ergodic ones) which admit a nice convergence analysis such as [11]. In this paper, the matrices of our interest shall form such a special class which does admit a nice convergence analysis.

Collision-free vehicle formation control has also attracted many researchers, especially in aerospace engineering. See [12], [13], [14], [15], [16], [17] for some recent works in this direction. One noticeable work is [18] (and the references therein) in which the framework of ISS (input-to-state stability) and a small-gain theorem are given to control vehicle formations effectively. Comparing with those existing works using limit cycle, MPC and so on, it is stressed again that the present work has a practically or computationally favourable feature of using the simple control gain in (1.1) - a fixed set of time-invariant gains only depending on the inter-vehicle network topology L_G - for a desired formation, and allows time-varying control gains only when necessary for collision avoidance. This feature of no frequent switching between control gains during the most of manoeuvring periods, lends itself well to the significant increase (and decrease) of the controller's reliability (and complexity).

This paper is organized as follows. In §II, the convergence analysis for the m-vehicle dynamics (1.2) under time-varying network topology is presented. In §III, a collision-avoidance control law is combined with the static control law (1.1). Note that the collision-avoidance control law is active only when necessary. A numerical example is given in §IV to demonstrate the developments in the preceding sections, and some concluding remarks follow in §V.

II. FORMATION CONTROL UNDER TIME-VARYING NETWORK TOPOLOGY

As addressed in §I, the control law (1.1) with sufficiently large f_1 and f_2 lets the vehicle dynamics (1.2) converge to any formation prescribed by h, i.e. $x \rightarrow h + \mathbf{1}_m \otimes \gamma$, where $\mathbf{1}_m$ denotes the all one vector of size m and $\gamma \in \mathbf{R}^{2n}$ is a possibly time-varying vector. See [1] for more details on the definition of formation in the present context. It is assumed that such proper values of f_1 and f_2 can be found and used for the formation control, regardless of the inter-vehicle network topology.[2] To validate this assumption, the value of a_{21} is set to zero (according to [1]).

To facilitate the ensuing analysis, consider the dynamics of relative states $x_{i/j} = x_i - x_j$ with $i, j = 1, \ldots, m$ $(i \neq j)$, and define $x_{i/\bullet}$ as $[x_{i/1}^T, \ldots, x_{i/j}^T, \ldots, x_{i/n}^T]^T$ $(j \neq i)$. This consideration changes the m-vehicle dynamics (1.2) to the following:

$$\dot{x}_{i/\bullet} = A x_{i/\bullet} - BFT_i L_i (x_{i/\bullet} - h_{i/\bullet}) \qquad (2.3)$$

[2]For instance, $f_1 = 1$ and $f_2 = 2$ are used in the numerical example in §IV.

with $A = I_{m-1} \otimes A_{veh}$, $B = I_{m-1} \otimes B_{veh}$, $F = I_{m-1} \otimes F_{veh}$, $T_i = T_{veh_i} \otimes I_{2n}$ with T_{veh_i} having 1's in the ith column, a -1 at the (i, j) position for $j = 1, \ldots, m$ $(j \neq i)$, and zeros at the other positions, and $L_i = L_{Gi} \otimes I_{2n}$ where L_{Gi} is L_G with the ith column of L_G being removed. For instance, for $m = 5$, $i = 1$ and $L_G = [1, -1, 0, 0, 0; -1, 2, -1, 0, 0; 0, -1, 2, -1, 0; 0, 0, -1, 2, -1; 0, 0, 0, -1, 1]$, $T_1 = [1, -1, 0, 0, 0; 1, 0, -1, 0, 0; 1, 0, 0, -1, 0; 1, 0, 0, 0, -1] \otimes I_{2n}$ and $L_1 = [-1, 0, 0, 0; 2, -1, 0, 0; -1, 2, -1, 0; 0, -1, 2, -1; 0, 0, -1, 1] \otimes I_{2n}$. The fact that $L(\mathbf{1}_m \otimes I_{2n}) = 0_{(2nm) \times 2n}$ or $L(z_i \otimes I_{2n}) = -L(\bar{z}_i \otimes I_{2n})$, where $z_i \in \mathbf{R}^m$ is the zero vector except for the 1 at the ith entry and $z_i + \bar{z}_i = \mathbf{1}_m$, can be used to derive (2.3). Here, $0_{m \times n}$ denotes an m-by-n zero matrix.

In what follows, it will be shown that $x_{i/\bullet}$ in (2.3) converges to $h_{i/\bullet}$ for time-varying *connected* L_i, i.e. time-varying and connected graphs.[3] To this end, several preliminary results are needed. Without loss of generality, n is assumed to be 1 for the rest of this section.[4]

Lemma 2.1: Suppose an underlying graph G is connected. Then, for each $i \in [1, m]$, $A - BFT_i L_i$ with sufficiently large f_1 and f_2 is Hurwitz.

Lemma 2.2: Suppose an underlying graph G is connected. Then, for each $i \in [1, m]$, $T_i L_i$ is Hurwitz. Moreover, $(T_i L_i)^T Q + Q(T_i L_i) < 0$ for $Q = I_{m-1} + L_K > 0$, where L_K is the Laplacian matrix of a complete graph K of $m-1$ nodes and $L_K \leq m I_{m-1}$.[5]

Lemma 2.3: For all $i = 1, \cdots, m$, there exists a permutation matrix P such that

$$P^T (A - BFT_i L_i) P = \begin{bmatrix} 0_{m-1} & I_{m-1} \\ f_1 T_{veh_i} L_{Gi} & a_{22} I_{m-1} + f_2 T_{veh_i} L_{Gi} \end{bmatrix}.$$

Lemma 2.4: [19] If $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{m \times m}$ have eigenvalues λ_i $(i = 1, 2, \ldots, n)$ and μ_j $(j = 1, 2, \ldots, m)$, respectively, then the mn eigenvalues of $A \otimes B$ are $\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_m$.

The main result of this section is now stated.

Theorem 2.5: Suppose underlying time-varying graphs G are connected at all times.[6] Then, $x_{i/\bullet}$ in (2.3) converges to $h_{i/\bullet}$, i.e. $e_i = x_{i/\bullet} - h_{i/\bullet} \rightarrow 0$, for sufficiently large f_1 and f_2.

III. COLLISION-FREE FORMATION CONTROL

Although the state feedback control law (1.1) guarantees the convergence under time-varying network topology, it

[3]Mathematical proofs of the following results shall be found in a journal version of this paper.

[4]This assumption is due to the fact that the same control law can be used for each axis of the vehicle's configuration space.

[5]The expression $A > 0$ for a symmetric matrix A means that A is positive definite.

[6]The supposition can also be translated into that each vehicle has a certain communication range such that G remains connected all the time. The requirement of 'connected at all times' could certainly be relaxed using the concept of "jointly connected" found in [8]. But this direction is not further studied in this work.
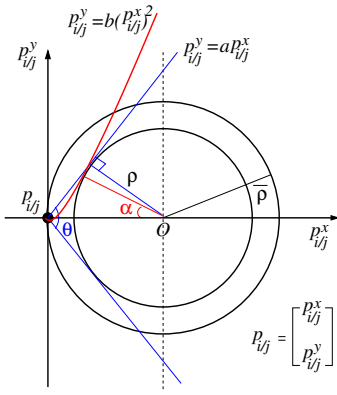
Fig. 1. Definition of θ, ρ and $\bar{\rho}$

may allow collisions between vehicles in the course of manoeuvres. In this section, a collision-avoidance (CA) rule is coupled with the state feedback law while making sure the convergence under time-varying network topology. For practical purposes (simplicity, no frequent switching between different controllers, etc), this coupling is made such that the state feedback control is replaced by the CA action only when necessary.

To this end, consider the following CA rule which is active only when vehicle(s) j is (are) coming towards and close enough to vehicle i, i.e. $\dot{\hat{p}}_{i/j}^T \hat{p}_{i/j} \leq -\cos(\theta/2)$ and $\|p_{i/j}\| \leq \bar{\rho}$, where $p_{i/j}$ denotes the relative position of vehicle i with respect to vehicle j, $\hat{p}_{i/j} = p_{i/j}/\|p_{i/j}\|$, and $\bar{\rho}$, θ are some pre-determined positive constants: compute and apply u_i $(i = 1, 2, \ldots, m)$ such that

$$\dot{x}_i = A_{veh} x_i + B_{veh} u_i; \tag{3.4}$$

$$\|u_i\| \leq u_{max}; \tag{3.5}$$

$$r_{11} \bar{e}_{vi/j} \bar{e}_{pi/j} + r_{22} \dot{\bar{e}}_{vi/j} \bar{e}_{vi/j} < 0 \quad \forall j; \tag{3.6}$$

$$\dot{\hat{p}}_{i/j}^T \hat{p}_{i/j} \geq \beta_j \quad \forall j \tag{3.7}$$

with $\sum_j \beta_j$ being maximized, where u_{max} is the maximally possible control energy, and

$$\bar{e}_i = Pe_i = [\underbrace{\bar{e}_{pi/1}, \cdots, \bar{e}_{pi/j}, \cdots, \bar{e}_{pi/n},}_{\bar{e}_{pi}^T}$$

$$\underbrace{\bar{e}_{vi/1}, \cdots, \bar{e}_{vi/j}, \cdots, \bar{e}_{vi/n}}_{\bar{e}_{vi}^T}]^T,$$

where P is the permutation matrix used in Lemma 2.3, and r_{11}, r_{12}, r_{22} are defined so that (3.6) is a sufficient condition for $e_i \to 0$ as $t \to \infty$. The constraint in (3.7) with nonnegative β_j renders vehicle i move away from all the vehicles nearby, so that $\|p_{i/j}\| > \rho$ is always guaranteed. See Figure 1 for the definition of θ, ρ and $\bar{\rho}$. The size of $\bar{\rho}$ must be chosen to reflect v_{max} and u_{max}. A conservative choice of $\bar{\rho}$ is such that $\bar{\rho} > \rho(2\sin\alpha\tan\alpha + \cos\alpha)$, where $b = u_{max}/(2v_{max}^2)$ and $\cot^2\alpha/\sin\alpha = 4b\rho$. Then, one can show that the following theorem holds for the chosen $\bar{\rho}$.

Theorem 3.1: Suppose underlying time-varying graphs G are connected at all times, and a desired formation vector h

is given such that the desired distance between any pair of two vehicles is greater than $\bar{\rho}$. Then, $e_i = x_{i/\bullet} - h_{i/\bullet} \to 0$ without collision, i.e. $\|p_{i/j}\| > \rho$ for all j at all times, for sufficiently large f_1 and f_2, provided that u_i satisfying (3.4)–(3.7) exists during any CA manoeuvre.

It should be noted that (3.6) may not allow fully decentralized control, seeing that vehicle i may require state information from other vehicles with no direct communication link during the CA manoeuvre. However, if vehicles i and j have no direct link but there exists a vehicle k which indirectly links i and j, the following modified version of (3.6) for vehicle j, $r_{11}(\bar{e}_{vi/k} + \bar{e}_{vk/j})(\bar{e}_{pi/k} + \bar{e}_{pk/j}) + r_{22}(\dot{\bar{e}}_{vi/k} + \dot{\bar{e}}_{vk/j})(\bar{e}_{vi/k} + \bar{e}_{vk/j}) < 0$, can be included in the optimization problem for vehicle k to necessitate no direct communication between i and j. As discussed in [14], one can certainly imagine a situation in which formation re-configuration is not possible without collision. Some remedy, e.g. found in [22], could be considered to get around this situation, but it is assumed in this paper that such situation does not occur so as to guarantee the existence of u_i satisfying (3.4)–(3.7).

IV. NUMERICAL EXAMPLE

To test the proposed control strategy in a simulated environment, the following data is used: $a_{21} = 0$, $a_{22} = 0.1$, $d = 1.5$ (communication range), $\rho = 0.6$, $\bar{\rho} = 0.66$, $u_{max} = \sqrt{50}$, and five vehicles are initially lined up in a row at the start (marked with an 'x' in Fig. 2-(a)) and are required to form a prescribed formation defined by $h = [0, 0, 1.4, 0, 0, 1.4, 1.4, 1.4, 0, .7]^T \otimes [1, 0]^T$ (marked with circles in Fig. 2-(a)).[7] Note that as five vehicles move around, their neighbours may change. The control parameters f_1 and f_2 for the static output feedback control in (1.1) are set to 1 and 2 (small values although required to be sufficiently large), respectively, and the same values are used when the network topology changes.

Fig. 2 and 3 show simulation results without and with CA manoeuvre, respectively. Each figure has four subfigures showing the vehicles' trajectories, the relative distances between vehicles, the consumed control energy u_i for all i and the trajectories of $p_{2/5}$ and $p_{4/5}$, respectively. As shown in Fig. 2, the normal static output feedback control law under time-varying topologies allows the five vehicles to achieve the desired formation using relatively little control effort, but the distance between vehicles 2 and 5 and the distance between vehicles 4 and 5 become less than ρ. However, when the CA action is combined with the static output feedback control, the desired formation with no collision can be achieved. Note that the CA phase occurs only for a short period during which almost u_{max} is used (see Fig. 3).

V. CONCLUDING REMARKS

In this paper, collision-free vehicle formation control was considered when the inter-vehicle network topology is time-

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[7]Simulations were done using SeDuMi [23] and YALMIP [24], but could be done using a computationally inexpensive LP (linear program) solver if the ∞-norm was used in (3.5).
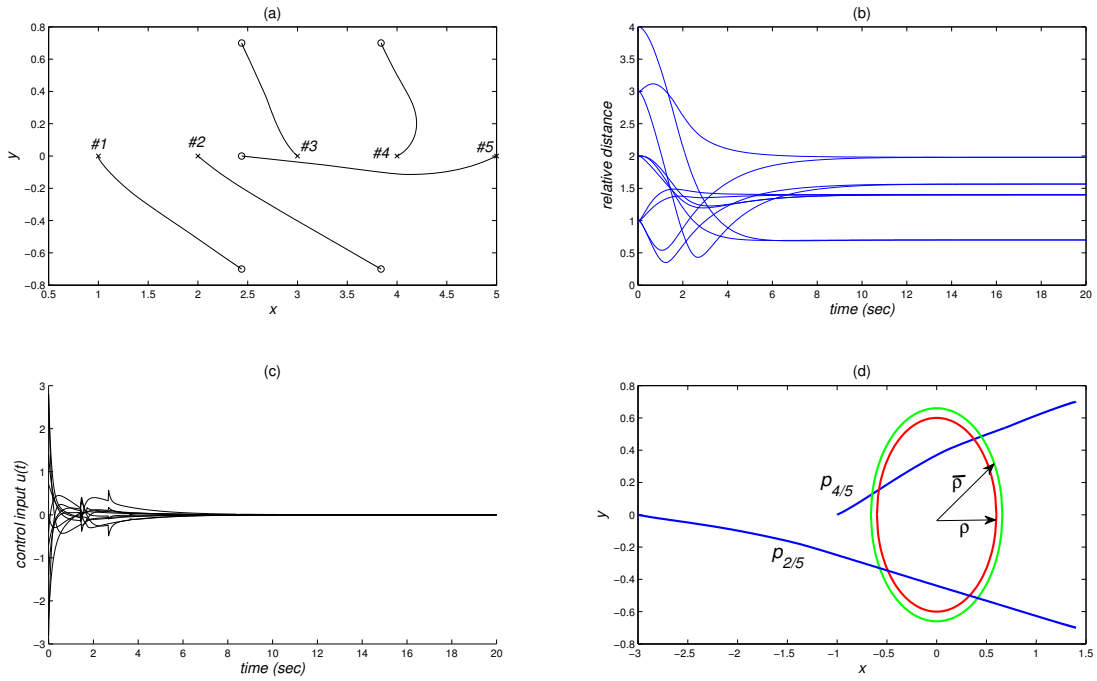
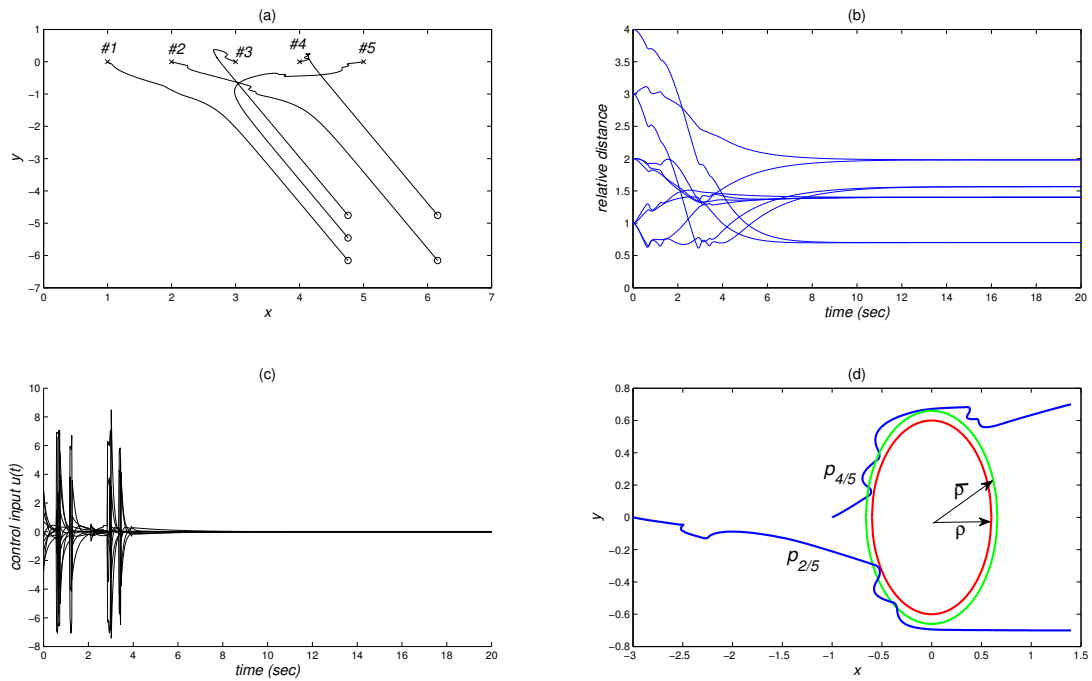Fig. 2. Simulation results with no CA manoeuvre

Fig. 3. Simulation results with CA manoeuvre

varying. Unlike typical collision-free formation control approaches, the proposed control approach uses a fixed set of time-invariant control gains for achieving a desired forma-

tion, and allows time-varying control gains only for avoiding collision. Since the collision-avoidance phase normally takes place for a short period, the proposed control approach can

allow the increase (and the decrease) of the controller's reliability (and complexity). Also, the proposed control approach provably works for time-varying network topologies. Unlike existing proof ideas which normally work for very special classes of matrices, such as ergodic matrices, this work finds a common Lyapunov-function to prove the convergence. Further research is required to handle real-world issues such as switching and time-delays in communications, fully decentralized control during collision-avoidance manoeuvres for large-scale formation control, and so on.

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