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Brief paper

A novel approach to multiparametric quadratic programming[☆]

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ABSTRACT

Multiparametric (mp) programming pre-computes optimal solutions offline which are functions of parameters whose values become apparent online. This makes it particularly well suited for applications that need a rapid solution of online optimization problems. In this work, we propose a novel approach to multiparametric programming problems based on an enumeration of active sets and use it to obtain a parametric solution for a convex quadratic program (QP). To avoid the combinatorial explosion of the enumeration procedure, an active set pruning criterion is presented that makes the enumeration implicit. The method guarantees that all regions of the partition are critical regions without any artificial cuts, and further that no region of the parameter space is left unexplored.

1. Introduction

Multiparametric (mp) programming is an approach for solving constrained optimization problems by computing a parameter-dependent solution a priori. It has emerged as a promising tool that is particularly suited for applications that need to solve optimization problems rapidly such as in model predictive control (MPC), where the value of the parameter becomes apparent online and the optimal control problem needs to be solved in a small fraction of the sampling period (Alessio & Bemporad, 2009; Bemporad, Morari, Dua, & Pistikopoulos, 2002; Pistikopoulos, 2009; Spjotvold, Kerrigan, Jones, Tondel, & Johansen, 2006; Tondel, Johansen, & Bemporad, 2003a). Applications of mp programming have also been reported for solving scheduling problems (Li & Ierapetritou, 2008), process design (Dua & Pistikopoulos, 1999, 2000), and energy management in presence of uncertainties (Moser, Thiele, Brunelli, & Benini, 2010), among others. The basic idea in the multiparametric approach is to decompose the parameter space into regions, each of which is characterized by a set of optimal active constraints in the decision space. The parameter-dependent solution can then be easily deduced using the necessary condition for optimality or its corresponding parametric sensitivity. Depending on the type of optimization problem, mp-programming problems are classified as mp-linear programming (LP), mp-quadratic programming (QP), mp-nonlinear programming (NLP), and mp-mixed integer nonlinear programming (MINLP).

All approaches reported in literature for solving multiparametric programming problems involve two basic steps: (1) determination of the optimal solution as a parameter-dependent function, valid over a certain region in the parameter space; and (2) exploration of the remaining parameter space. In the case of convex mp-QP, two types of algorithm have been reported in the literature: the first type is that of Bemporad et al. (2002) and Dua, Bozinis, and Pistikopoulos (2002), and the second type is that of Tondel et al. (2003a). Both use an active set approach to determine the optimal solution, and they differ only in the second basic step, namely, the parameter space exploration strategy. The first type of algorithm proposes exploration of the parameter space using artificial cuts while the second type of algorithm proposes a facet-to-facet property for parameter space exploration. Both these algorithms have certain lacunae. In the first type, the artificial cuts cause redundant partitioning of the parameter space, that is, multiple partitions in the parameter space correspond to an identical set of active constraints. On the other hand, the second type of algorithm, while offering significant computational advantages over the first type, relies upon a facet-to-facet property which fails to hold under certain conditions, resulting in unexplored regions in the parameter space. Spjotvold et al. (2006) combine the advantages of the two types of approach: their method begins by using the second type of algorithm and if the facet-to-facet property fails at a particular facet, it switches to the approach of Dua et al. (2002) in a user-defined region around that facet. The switch to the first type of approach reintroduces the potential for redundant partitioning and

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the accompanying computational complexity, although to a limited extent.

In this work, a novel approach for solving a strictly convex mp-QP problem is presented, which eliminates the need for a parameter exploration step. The main advantage of the proposed technique is that it guarantees arriving at the minimum number of partitions of the parameter space while ensuring that the parameter space is fully explored, thereby remaining free of the issues in the previously reported algorithms. The approach is based on an implicit enumeration of the set of active constraints. Further, it uses a pruning criterion that facilitates a significant reduction in the number of active sets that need be enumerated, thus avoiding the combinatorial complexity in the enumeration-based approach. The proposed approach also inherently identifies degeneracies arising out of violations of Strict Complementary Slackness (SCS) and Linear Independence Constraint Qualification (LICQ) conditions, which are commonly encountered in real problems.

The paper is organized as follows. Section 2 briefly reviews mp-QP along with the two categories of algorithm discussed above; Section 3 presents the proposed approach of implicit enumeration with special emphasis on degeneracy; Section 4 presents the final algorithm and examples. Finally, conclusions are presented in Section 5.

2. Multiparametric quadratic programming

Consider the following multiparametric quadratic programming (mp-QP) problem:

$$\begin{aligned} \min_x \quad & 0.5x^T Qx + c^T x + \theta^T Px \\ \text{s.t.} \quad & Ax = b + S\theta \\ & x \in R^n; \quad \theta \in \Theta \subset R^m, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ represents decision variables and $\theta \in \mathbb{R}^m$ is a vector of parameters; Θ is a closed polyhedral set which defines the bounds on θ. $\mathbb{R}^{n\times n}$ $Q > 0$, $P \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{p\times n}$, $S \in \mathbb{R}^{p\times m}$ and $b \in \mathbb{R}^p$ are real matrices. The term $\theta^T Px$ in the objective function can be eliminated by the transformation $x = u - Q^{-1}P^T$ (Dua et al., 2002), which converts the strictly convex mp-QP to the following standard form:

$$\begin{aligned} \min_u \quad & 0.5u^T Qu + c^T u \\ \text{s.t.} \quad & Au = b + F\theta \\ & u \in R^n; \quad \theta \in \Theta \subset R^m, \end{aligned} \tag{2}$$

with $F = S + AQ^{-1}P^T$. Let $\mathcal{M}$ refer to the set of indices of all constraints of the QP,

$$\mathcal{M} \triangleq \{1, 2, \ldots, p\}, \tag{3}$$

and let the active set $\mathcal{A}(u, \theta)$ denote the set of indices of the constraints that are active at (u, θ),

$$\mathcal{A}(u, \theta) \triangleq \{i \in \mathcal{M} | A_i u - b_i - F_i\theta = 0\}, \tag{4}$$

where A_i denotes the ith row of matrix A. The total number of active sets which can be constructed from $\mathcal{M}$ is given by its power set,

$$\begin{aligned} \mathcal{P}(\mathcal{M}) \triangleq \{&\mathcal{A}_1 = \{\}, \mathcal{A}_2 = \{1\}, \ldots, \mathcal{A}_{p+1} = \{p\}, \\ &\mathcal{A}_{p+2} = \{1, 2\}, \ldots, \mathcal{A}_{2^p} = \{1, 2, \ldots, p\}\}. \end{aligned} \tag{5}$$

The indices which are not in the active set $\mathcal{A}$ are members of the inactive set $\mathcal{I}$,

$$\mathcal{I}(u, \theta) = \mathcal{M} \setminus \mathcal{A}(u, \theta). \tag{6}$$

If $u^*(\theta), \lambda^*(\theta)$ correspond to an optimal solution of Eq. (2), then the optimal active set can be determined as

$$\mathcal{A}^*(\theta) = \{i | i \in \mathcal{A}(u^*(\theta), \theta)\}. \tag{7}$$

We can now form matrices $A_{\mathcal{A}}$, $b_{\mathcal{A}}$, and $F_{\mathcal{A}}$ from the rows of A, b, and F corresponding to the optimal active set $\mathcal{A}^*(\theta)$.

2.1. Preliminaries

Definition 1. Linear Independence Constraint Qualification (LICQ) (Nocedal & Wright, 1999): Given the point $u^*(\theta)$ and the corresponding active set $\mathcal{A}^*(\theta)$, we say that the LICQ holds if the set of active constraint gradients $\{A_i | i \in \mathcal{A}(u^*(\theta), \theta)\}$ is linearly independent, that is, $A_{\mathcal{A}}$ has full row rank.

The Karush–Kuhn–Tucker (KKT) conditions represent the first-order necessary conditions for constrained optimization problems, which form the foundation for the various algorithms used to solve the QP in Eq. (2) and can be stated as follows (Nocedal & Wright, 1999). Suppose that the LICQ holds at the solution $u^*(\theta')$ of the strictly convex QP in Eq. (2) corresponding to a given parameter value θ', then there is a Lagrange multiplier $\lambda^*(\theta')$ with components $\lambda_i{}^*(\theta')$, $i \in \mathcal{M}$, such that the following conditions are satisfied when evaluated at $u^*(\theta')$ and $\lambda^*(\theta')$:

$$\begin{aligned} & Qu + c + \sum_{i\in\mathcal{A}_k} \lambda_i A_i = 0 \\ & A_i u - b_i - F_i\theta = 0; \qquad \lambda_i \geq 0; \quad \forall i \in \mathcal{A}_k \\ & A_i u - b_i - F_i\theta \leq 0; \qquad \lambda_i = 0; \quad \forall i \in \mathcal{I}_k. \end{aligned} \tag{8}$$

The second and third statements in Eq. (8) are known as the complementarity condition. A stronger condition called strict complementarity is often invoked.

Definition 2. Strict Complementarity Slackness (SCS) (Nocedal & Wright, 1999): Given the pair $(u^*(\theta'), \lambda^*(\theta'))$ satisfying the KKT conditions, SCS holds if exactly one of $\lambda_i^*(\theta')$ and $(A_i u^*(\theta') - b_i - F_i\theta')$ is zero for each $i \in \mathcal{M}$, that is $\lambda_i{}^*(\theta') > 0$, for each $i \in \mathcal{A}^*(\theta')$

Basic Sensitivity Theorem (Fiacco, 1983). *Let θ' be a parameter vector in the convex mp-QP in Eq. (8) and $(u^*(\theta'), \lambda^*(\theta'))$ be the corresponding KKT pair. Also assume* (i) *that SCS is satisfied, and* (ii) *that the LICQ holds; then, in the neighborhood of θ', there exists a unique, once continuously differentiable function $[u(\theta'), \lambda(\theta')]$ satisfying the KKT conditions in Eq. (8), where $u(\theta')$ is a unique isolated minimizer for the mp-QP in Eq. (2) such that*

$$\begin{bmatrix} u(\theta) \\ \lambda(\theta) \end{bmatrix} = -M^{-1}N(\theta - \theta') + \begin{bmatrix} u^*(\theta') \\ \lambda^*(\theta') \end{bmatrix} \tag{9}$$

$$M = \begin{bmatrix} Q & A_1^T & \cdots & A_p^T \\ -\lambda_1 A_1 & -V_1 & & \\ \vdots & & \ddots & \\ -\lambda_p A_p & & & -V_p \end{bmatrix}$$

$$N = [Y, -\lambda_1 F_1, \ldots, -\lambda_p F_p]^T$$

$$V_i = A_i u(\theta') - b_i - F_i\theta' \quad \text{and} \quad Y = [0]_{n\times m}.$$

Note that satisfying the LICQ and SCS guarantees that the Jacobian M is invertible for $Q > 0$. It is evident from Eq. (9) that the mp-QP solution is affine in its parameters and holds only in a subset of Θ, where the set of optimal active constraints $\mathcal{A}^*(\theta')$ identified during the solution of the QP at θ' remains unchanged. The optimal active set and the corresponding optimal inactive set can be characterized as follows:

$$A_i u(\theta) \leq b_i - F_i\theta, \quad i \in \mathcal{I}^*(\theta') \tag{10}$$

$$\lambda_i(\theta) \geq 0, \quad i \in \mathcal{A}^*(\theta'). \tag{11}$$

These inequalities along with the original parameter bounds Θ, after removal of redundant constraints, represent a polyhedron in the parameter space, termed the critical region ($CR_{\mathcal{A}}$), which corresponds to the active set $\mathcal{A}$,

$$\begin{aligned} CR_{\mathcal{A}} \triangleq \triangle \{&\theta \in \Theta \subseteq \mathbb{R}^m : A_{\mathcal{I}} u(\theta) - b_{\mathcal{I}} - F_{\mathcal{I}}\theta \leq 0, \\ &\lambda_{\mathcal{A}}(\theta) \geq 0\}, \end{aligned} \tag{12}$$

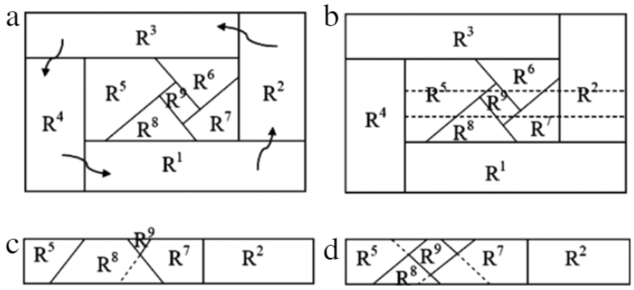

Fig. 1. Parameter space exploration using different approaches.

where Δ is a notional operator that eliminates redundant inequalities (see for example Cheng, 1987). $\lambda_{\mathcal{A}}$ represents the Lagrange multipliers that correspond to the constraints in the active set $\mathcal{A}$.

2.2. Exploration of parameter space

The key difference between the two types of algorithm discussed in Section 1 lies in the manner in which they explore the remaining parameter set, defined as CR^{Rest}:

$$CR^{Rest} \triangleq \Theta \setminus CR_{\mathcal{A}}. \tag{13}$$

We use an illustrative example (adapted from Spjotvold et al. (2006)) to elicit the differences between the previously reported approaches and the proposed approach. Let the parameter-dependent solution of an mp-QP result in nine critical regions, each corresponding to a unique active constraint set, as shown in Fig. 1(a). After determining the facets of region R^1, the first type of algorithm (Bemporad et al., 2002; Dua et al., 2002) explores CR^{Rest} by extending the upper horizontal facet of R^1. However, this splits R^4 into two regions, resulting in redundant partitioning of the parameter space. The second type of algorithm, that of Tondel et al. (2003a), assumes that a facet-to-facet property holds, which requires that the facet lying at the intersection of two adjacent critical regions is exclusively a facet of the two intersecting regions. In Fig. 1(a), the upper horizontal facet of R^1 is shared by R^1 not only with R^5 but also with R^8, R^7, and R^2. Thus, the facet-to-facet property fails, and the algorithm can potentially result in unexplored regions. The arrows in the figure show a possible parameter space exploration strategy using the facet-based method, in which case regions R^5–R^9 remain unexplored. To overcome this drawback, Spjotvold et al. (2006) proposed combining the first type and the second type by exploring a small user-defined region along the facet where the facet-to-facet property fails by use of artificial cuts, as in Dua et al. (2002). The dashed lines in Fig. 1(b) show two possible choices of the user-defined regions across the upper horizontal facet of R^1, where the facet-to-facet property fails. Fig. 1(c) and (d) show these two user-defined regions across the facet of R^1, which are explored using artificial cuts. The dotted line in these two solutions shows the redundant partitioning. Thus, this hybrid approach introduces the potential for redundant partitioning, and the number of redundant partitions depends on the width of the user-defined region. Our proposed approach results in the partition as shown in Fig. 1(a), with all nine regions explored, and does not suffer from the various issues discussed above. This is a direct consequence of the fact that the exploration of the parameter space is implicit without requiring any ad hoc measures.

In the next section, we present a novel approach for solving the mp-QP that overcomes all limitations outlined above.

3. An implicit active set parametric region exploration strategy

The proposed approach differs from the previous algorithms in that we do not explicitly devise any strategy to explore the parameter space, Θ. Instead, we use the premise that, corresponding to each parameter $\theta \in \Theta$, at which the QP in Eq. (2) is feasible and SCS and the LICQ hold, there exists a unique set of constraints $\mathcal{A}^*(\theta)$ which are active at the optimal solution $u^*(\theta)$. Thus, if all candidate active sets are enumerated, then the parameter space is implicitly explored. In the case of LICQ or SCS failure, the active constraints may not be unique, but this does not limit our approach. Clearly, the total number of active sets that could be constructed from $\mathcal{M}$ is 2^p, which represent elements of the power set $\mathcal{P}(\mathcal{M})$. It should be noted that for a QP with n decision variables and p inequality constraints, with $p > n$, only a maximum of n linearly independent constraints can be strongly active at the optimal solution (Nocedal & Wright, 1999). Thus, the maximum number of optimal active sets is $\sum_{i=0}^{n}\binom{p}{i}$, which represent elements of a set, $\mathcal{P}'(\mathcal{M}) \subset \mathcal{P}(\mathcal{M})$,

$$\begin{aligned}\mathcal{P}'(\mathcal{M}) \triangleq \{&\mathcal{A}_1 = \{\}, \mathcal{A}_2 = \{1\}, \ldots, \mathcal{A}_{p+1} = \{p\},\\ &\mathcal{A}_{p+2} = \{1, 2\}, \ldots, \mathcal{A}_{\binom{p}{n}} = \{1, 2, \ldots, n\}\}.\end{aligned} \tag{14}$$

However, such an approach of exploring Θ by an enumeration of $\mathcal{P}'(\mathcal{M})$ as candidates for the optimal active set $\mathcal{A}^*(\theta)$ is impractical. We present a technique in Section 3.2 that makes the enumeration implicit, resulting in a reduction in the number of candidate optimal active sets that need be enumerated.

3.1. Selection of optimal active sets

To implement the above approach, a candidate active set $\mathcal{A}_k(u, \theta) \in \mathcal{P}'(\mathcal{M})$ is chosen in the order of increasing cardinality of $\mathcal{P}'(\mathcal{M})$. The QP corresponding to $\mathcal{A}_k(u, \theta)$ may be infeasible, feasible, or feasible degenerate. Feasibility of the QP corresponding to $\mathcal{A}_k(u, \theta)$ can be easily verified by determining if the KKT conditions in Eq. (8) are feasible for any $\theta \in \Theta$. The issue of identifying if the feasible QP is degenerate (SCS failure) can be resolved by attempting to find a $\theta \in \Theta$ such that all constraints in the selected active set $\mathcal{A}_k(u, \theta)$ are strongly active, that is, $\lambda_{\mathcal{A}_k} > 0$, while those in $\mathcal{I}_k$ do not exhibit weak activity, that is, the slack variables of the inactive constraints, $s_{\mathcal{I}_k} > 0$. To implement the above checks as well as identify the parameter vector θ' necessary for determining the parameter dependent solution based on Eq. (9) and the corresponding critical region (Eq. (12)), the following optimization problem is formulated:

$$\begin{aligned}&\max_{u,\theta,\lambda_{\mathcal{A}_k},\, s_{\mathcal{I}_k}} \Big[\min_{i\in\mathcal{A}_k, j\in\mathcal{I}_k} [(\lambda_{\mathcal{A}_{k\,i}}) : (s_{\mathcal{I}_{k\,j}})]\Big]\\ &\text{s.t.}\quad Qu + c + A_{\mathcal{A}_k}^T \lambda_{\mathcal{A}_k} = 0\\ &\qquad A_{\mathcal{A}_k} u - b_{\mathcal{A}_k} - F_{\mathcal{A}_k}\theta = 0\\ &\qquad A_{\mathcal{I}_k} u - b_{\mathcal{I}_k} - F_{\mathcal{I}_k}\theta + s_{\mathcal{I}_k} = 0\\ &\qquad \lambda_{\mathcal{A}_k} \ge 0; \qquad s_{\mathcal{I}_k} \ge 0; \qquad \theta \in \Theta.\end{aligned} \tag{15}$$

Note that the KKT conditions in Eq. (8) form the constraints above. This max–min problem can be reformulated as an LP with the introduction of scalar variable t, as follows:

$$\begin{aligned}&\max_{t,u,\theta,\lambda_{\mathcal{A}_k},s_{\mathcal{I}_k}} \; t\\ &\text{s.t.}\quad te_1 \le \lambda_{\mathcal{A}_k}; \qquad te_2 \le s_{\mathcal{I}_k}\\ &\qquad Qu + c + A_{\mathcal{A}_k}^T \lambda_{\mathcal{A}_k} = 0\\ &\qquad A_{\mathcal{A}_k} u - b_{\mathcal{A}_k} - F_{\mathcal{A}_k}\theta = 0\\ &\qquad A_{\mathcal{I}_k} u - b_{\mathcal{I}_k} - F_{\mathcal{I}_k}\theta + s_{\mathcal{I}_k} = 0\\ &\qquad \lambda_{\mathcal{A}_k} \ge 0; \qquad s_{\mathcal{I}_k} \ge 0; \qquad t \ge 0; \qquad \theta \in \Theta,\end{aligned} \tag{16}$$

where e_1 and e_2 are column vectors of ones of appropriate sizes. The LP has three possible outcomes with different implications, as discussed below.

Case 1: *No feasible solution*. There does not exist any parameter vector $\theta \in \Theta$ for which the candidate active set is optimal. Thus, $CR_{\mathcal{A}_k} = \{\}$. The knowledge of the infeasible active set can be used to devise a fathoming strategy which makes the proposed enumeration of $\mathcal{P}'(\mathcal{M})$ implicit.

Case 2: *Feasible solution with* $t > 0$. A unique critical region exists where $\mathcal{A}_k$ is optimal.

Case 3: *Feasible solution with* $t = 0$. The mp-QP is degenerate with some constraints in the candidate active set or corresponding inactive set being weakly active, that is, $\lambda_{\mathcal{A}_k,i} = 0$ or $s_{\mathcal{I}_k,i} = 0$. This situation arises due to failure of SCS or the LICQ which may lead to non-unique or non-full-dimensional critical regions.

3.2. *Case 1: no feasible solution. An implicit enumeration strategy*

It is obvious that solving the LP for all elements in $\mathcal{P}'(\mathcal{M})$ is impractical even for a moderate number of constraints. A fathoming criterion that results in a reduction in the number of candidate active sets that need be checked for optimality is proposed and is based on the following result.

Theorem. *If a system S, consisting of inequalities and equalities, is infeasible, then the new system S′ formed by treating some of the inequality constraints of S as equality constraints will also be infeasible.*

Proof. Let S consist of p inequalities and m equalities:

$$a_i^T u = b_i, \quad i \in \mathcal{I}$$

$$a_i^T u = b_i, \quad i \in \mathcal{A} \in \mathcal{P}'(\mathcal{M}). \tag{17}$$

All solutions of S, $u_s \in \mathbb{R}^n$, form a polyhedron in an $n - m$-dimensional linear variety. If l inequalities are removed from $\mathcal{I}$ and inserted as equalities in $\mathcal{A}$, the solution of the resulting system S', namely $u_{s'} \subseteq u_s$, corresponds to a polyhedron in an $n - m - l$-dimensional linear variety. Hence, if system S with $n - m$ degrees of freedom is infeasible, then system S' with $n - m - l$ degrees of freedom will also be infeasible. □

Note that the system S in the above theorem deals only with the feasibility of constraints in $\mathcal{I}$. However, the LP in Eq. (16) consists of feasibility ($s_{\mathcal{I}} \geq 0$) as well as optimality ($\lambda_{\mathcal{A}} \geq 0$) constraints. Thus, infeasibility of the LP could also arise from violation of the optimality constraint. Therefore, an additional check is performed for candidate sets for which the LP in Eq. (16) results in Case 1 as follows:

$$A_{\mathcal{A}_k} u - b_{\mathcal{A}_k} - F_{\mathcal{A}_k}\theta = 0$$

$$A_{\mathcal{I}_k} u - b_{\mathcal{I}_k} - F_{\mathcal{I}_k}\theta \leq 0, \quad \theta \in \Theta. \tag{18}$$

Thus, if an active set $\mathcal{A}_k$ is infeasible with respect to constraints in Eq. (18), then all $\mathcal{A}_l \supset \mathcal{A}_k$ will also be infeasible. This fathoming criterion is illustrated using a tree diagram, in Fig. 2, where increasing tree levels correspond to increasing cardinality of the candidate active set starting from the root node at $\mathcal{A}_k = \{\}$. If one of the nodes in the tree is found to be infeasible, then the entire branch can be deleted from $\mathcal{P}'(\mathcal{M})$, as shown in Fig. 2, making the enumeration of $\mathcal{P}'(\mathcal{M})$ implicit.

3.3. *Case 2: feasible solution with* $t > 0$

The feasibility of the LP assures us that the candidate active set $\mathcal{A}_k$ is indeed optimal at some $\theta' \in \Theta$. Further, since both $\lambda_{\mathcal{A}_k}$ and $s_{\mathcal{I}_k}$ are strictly positive, the LICQ as well as SCS holds, and substituting the corresponding optimal (u^*, λ^*) in Eq. (9) provides the parameter-dependent optimal solution. The corresponding critical region is characterized by the inequalities in Eq. (12).

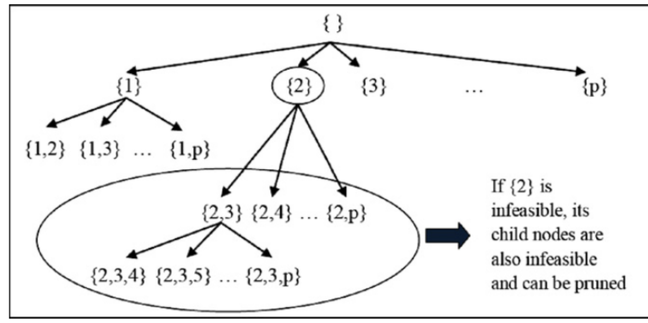

Fig. 2. Active set nodal tree.

3.4. *Case 3: feasible solution with* $t = 0$. *Degenerate QP*

Prior to solving the LP, we ensure that the candidate active set matrix $A_{\mathcal{A}}$ is full rank, thus excluding QPs for which the LICQ fails to hold. Then, Case 3 corresponds to failure of the SCS condition, that is, at least one of $\lambda_{\mathcal{A}_k}$ and $s_{\mathcal{I}_k}$ equals zero. Here, either one or more of the constraints is weakly active, and thus M in Eq. (9) is not invertible, as both V_k and λ_k become zero for some $k \in \mathcal{M}$. In this case, we can find the parameter-dependent solution by solving the equality constrained QP as follows:

$$\min_u \ \frac{1}{2} u^T Q u + c^T u$$

$$\text{s.t.} \quad A_{\mathcal{A}_k} u = b_{\mathcal{A}_k} + F_{\mathcal{A}_k}\theta, \tag{19}$$

for which the sensitivity theorem (Fiacco, 1983) yields the following:

$$M = \begin{bmatrix} Q & A_{\mathcal{A}_k}^T \\ A_{\mathcal{A}_k} & 0 \end{bmatrix}; \qquad N = \begin{bmatrix} 0 & -F_{\mathcal{A}_k} \end{bmatrix}. \tag{20}$$

Note that, regardless of violation of the SCS condition, M in Eq. (20) is non-singular as long as Q is positive definite and $A_{\mathcal{A}_k}$ has full row rank (the LICQ is satisfied). Here, some critical regions do not uniquely correspond to the optimal active set, and such critical regions may partially overlap with critical regions corresponding to a different set of active constraints. The fact that a given parameter may have more than one active sets associated with it is a direct consequence of SCS failure. Let $t = \lambda_i = 0$ be the solution of the LP in (16), that is, the ith active constraint in $\mathcal{A}_k$ is weakly active. Then, the LP corresponding to the active constraint set $\mathcal{A}_{k'} = \mathcal{A}_k \setminus \{i\}$ will also correspond to Case 3, with $t = s_i = 0$, and will yield an identical value for θ' as the LP for $\mathcal{A}_k$. This is because the KKT conditions for the QP with $\mathcal{A}_k$ and $\mathcal{A}_{k'}$ result in identical systems of equations, and thus both are optimal active sets at θ'. Thus, in the case of overlapping regions, although the parametric solution is unique, the critical regions themselves do not uniquely correspond to a set of active constraints (see Tondel, Johansen, and Bemporad (2003b) for an example).

3.4.1. *LICQ failure*

A related case of degeneracy in the mp-QP occurs when the LICQ condition is violated, which is also referred to as primal degeneracy. In such an event, Tondel et al. (2003a) show that, if $A_{\mathcal{A}_k}$ is rank deficient and $[A_{\mathcal{A}_k} \mid \ -b_{\mathcal{A}_k} \mid \ -F_{\mathcal{A}_k}]$ is full row rank, then the critical region is not full dimensional. Thus, if the above condition is satisfied, then the critical region corresponding to $\mathcal{A}_k$ will only form a facet of a full-dimensional region and can be ignored from further consideration. It is obvious that if the LICQ corresponding to an active set $\mathcal{A}_k$ is violated and the above rank condition is satisfied, then all $\mathcal{A}_l$ such that $A_{\mathcal{A}_k} \subset A_{\mathcal{A}_l}$ need not be considered further. We present an example that illustrates the impact of LICQ failure.

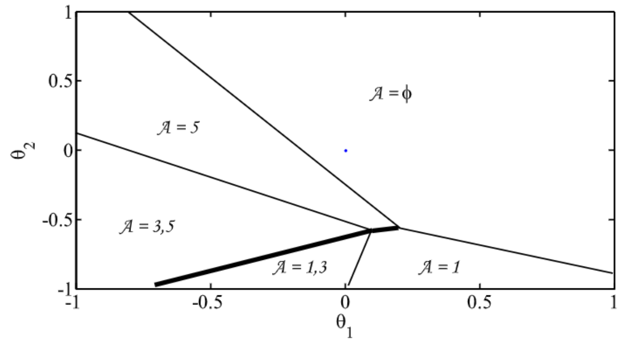

Fig. 3. LICQ failure results in lower-dimensional regions.

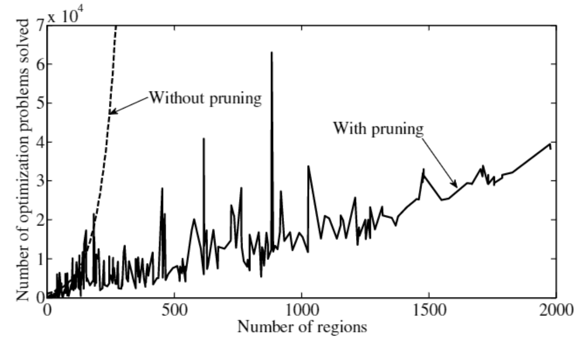

Fig. 4. Scalability of the algorithm with number of critical regions.

Example. Consider the following problem:

$$\min_{u\in\mathbb{R}^3} 0.5u^T u$$
$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & 1 \\ -2 & 4 & 4 \\ 4 & 0 & 0 \\ 2 & 4 & 6 \end{bmatrix} u \le \begin{bmatrix} 0.2 \\ 1 \\ 0.2 \\ 0.5 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.2 & 0.5 \\ 0.4 & 0.8 \\ 0.5 & 0.5 \\ 0.2 & 0.3 \\ 0.7 & 0.5 \end{bmatrix} \theta$$
$$\Theta := \{\theta \in \mathbb{R}^2 | -1 \le \theta_i \le 1, i = 1, 2\}. \quad (21)$$

For optimal active sets $\mathcal{A}_k^* = \{\}, \{1\}, \{5\}, \{1, 3\}$ and $\{3, 5\}$, there exist unique regions as shown in Fig. 3. The LICQ fails for $\mathcal{A}_k^* = \{1, 5\}$, for which the critical region lies on the facet $[0.161 \quad -0.269]\theta = 0.157$, which is the common facet between critical regions corresponding to $\mathcal{A}_k^* = \{1\}$ & $\{5\}$, as shown by the bold line in Fig. 3. The active set $\mathcal{A}_k = \{1, 3, 5\} \supset \{1, 5\}$ also exhibits LICQ failure, and will also result in a low-dimensional critical region. All other candidate active sets resulted in the infeasible LP of Eq. (16).

4. Algorithm construction

The pseudo-code for the algorithm to solve the mp-QP, based on elements in Section 3, is as follows.

Define non_opt = { }.
For $k = 1$ to $\sum_{i=0}^{n} {}^{P}C_i$, choose $\mathcal{A}_k \in \mathcal{P}'$ from Eq. (14)
If $A_{\mathcal{A}_k} \not\ni$ non_opt AND If $A_{\mathcal{A}_k}$ is full row rank, solve Eq. (16)
  If Case 1, then
    Check the fathoming criteria based on feasibility of Eq. (18). If infeasible then append non_opt $\rightarrow$ non_opt + $\mathcal{A}_k$.
  ElseIf Case 2, then
    Use Eq. (9) to obtain $u^*(\theta)$ and $\lambda^*(\theta)$. Use Eq. (12) to define the critical region.
  ElseIf Case 3, then
    Use Eq. (20)to get $u^*(\theta)$ and $\lambda^*(\theta)$. Eq. (12) provides the critical region. If it is full dimensional then accept it as a critical region; otherwise, discard.
  EndIf
EndFor

Since all optimal active sets are considered by the proposed technique, the solution guarantees that no unexplored regions exist in the parameter space. Since the algorithm makes critical decisions based on the LP in Eq. (16), its solution should be robust to numerical errors, especially since the difference between Case 2 and Case 3 lies in determining if the constraints in the active set are strongly active ($t > 0$) or show weak activity ($t = 0$). It is therefore essential that the LP solution corresponds to the vertex of the polytope, describing the feasible region, exactly. If interior point methods are used, special care must be taken that the interior point solution converges to the vertex. The above algorithm was implemented in MATLAB (R2007a), wherein the Simplex method was used to solve all LPs, which implicitly ensures vertex solutions.

A key concern in the current algorithm is the scalability of the method with respect to the number of critical regions, which increase with the number of inequality constraints or the dimension of the parameter space. To test the scalability of the algorithm, we implemented the algorithm on a four-state, two-input, infinite-horizon MPC problem with a horizon ranging between 1 and 4, randomly relaxing the input and state constraints over the chosen horizon in each case. A total of 600 mp-QPs were solved. The computational complexity was measured by the number of LPs that needed to be solved in arriving at the final solution of the mp-QP. The dashed line in Fig. 4 shows the number of LPs that would need to be solved if the pruning criterion of Section 2.2 is not implemented. In this case, the number of LPs corresponds to the number of elements of $\mathcal{P}'$ (Eq. (14)). On the other hand, the solid line refers to the case when the pruning criterion is used, which makes the enumeration of $\mathcal{P}'$ implicit. Clearly, the implicit enumeration approach scales well with the number of critical regions, as the pruning criterion affords a dramatic reduction in the offline computational expense.

5. Conclusion

This paper presents a novel approach to mp-programming problems. It uses an implicit enumeration of active sets to avoid the combinatorial complexity of the enumeration-based approach. The advantage of the method is that it ensures an exhaustive and exclusive partition of parameter space and is not susceptible to over-partitioning and failure of the facet-to-facet property. Moreover, the enumeration approach is not limited to mp-QPs, and can be extended to other multiparametric programming problems.

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