

# Analysis Notes

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# 1 Sequence and Series of Functions

## 1.1 Uniform Convergence

**Definition 1.1 (Pointwise Convergence).** A sequence of functions  $(f_n)_n$  **converges pointwise** to a function  $f$  on  $E$  if for every  $\varepsilon > 0$  and for all  $x \in E$  there is an integer  $N$  (which depends on  $x$ ) such that for all  $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

**Definition 1.2 (Uniform Convergence).** A sequence of functions  $(f_n)_n$  **converges uniformly** to a function  $f$  on  $E$  if for every  $\varepsilon > 0$  there is an integer  $N$  such that for all  $n \geq N$  and for all  $x \in E$  we have

$$|f_n(x) - f(x)| < \varepsilon$$

**Theorem 1.1 (Limit of uniformly convergent, bounded sequence of functions is bounded.).** If  $(f_n)_n$  converges uniformly to  $f$  and each  $f_n$  is bounded, then  $f$  is bounded.

*Proof.* Fix  $\varepsilon > 0$ . Uniform convergence implies that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have  $|f_n(x) - f(x)| < \varepsilon$ . Further, boundedness of the sequence of functions implies that  $\forall n \exists M_n \in [0, \infty)$  such that  $|f_n(x)| \leq M_n \forall x \in X$ . Fix an  $n \geq N$ . Then  $\forall x \in X$

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| && \text{(triangle inequality)} \\ &< \varepsilon + M_n && \text{(uniform convergence and boundedness)} \end{aligned}$$

Thus  $|f(x)| < \varepsilon + M_n \forall x \in X$ , so  $f(x)$  is bounded.  $\square$

**Theorem 1.2 (Limit of uniformly convergent, continuous sequence of functions is continuous.).** If  $(f_n)_n$  converges uniformly to  $f$  and each  $f_n$  is continuous, then  $f$  is continuous.

*Proof.* Fix  $\varepsilon > 0$ . Uniform convergence implies that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have  $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ . Continuity of each function  $f_n(x)$  at  $x \in X$  in the sequence implies that  $\exists \delta > 0$  such that  $\forall y$  for which  $|x - y| < \delta$  we have that  $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ . Then  $\forall y$  for which  $|x - y| < \delta$  we have that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| && (\Delta) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && \text{(uniform convergence and continuity of } f_n \text{ at } x) \\ &= \varepsilon \end{aligned}$$

Therefore  $f$  is continuous at  $x, \forall x \in X$ .  $\square$

**Definition 1.3 (Uniformly cauchy).** A sequence of functions  $(f_n)$  is **uniformly cauchy** if  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n, m \geq N$  and  $\forall x \in X$ , we have  $|f_n(x) - f_m(x)| < \varepsilon$ .

**Theorem 1.3 (Uniform convergence iff uniform cauchy).** A sequence  $(f_n)_n$  of functions on a metric space  $X$  converges uniformly if and only if it is uniformly Cauchy.

*Proof.* Fix  $\varepsilon > 0$ .

$\Rightarrow$  Uniform convergence implies  $\exists N$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have that  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ . Then  $\forall n, m \geq N$  and  $\forall x \in X$  we have that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore  $(f_n)_n$  is uniformly cauchy.

$\Leftarrow$  Notice that  $(f_n)_n$  is a cauchy sequence in a complete metric space  $(\mathbb{R})$ , therefore it converges pointwise. We need to show uniform convergence. **[[Incomplete]]**  $\square$

**Theorem 1.4 (Weierstrass M-test).** Let  $(f_n)_n$  be a sequence of functions on a metric space  $X$  such that there exists a sequence of non-negative real numbers  $(M_n)_n$  such that  $\forall n$  we have that

$$|f_n(x)| \leq M_n \tag{1.1}$$

If  $\sum_{n=1}^{\infty} M_n$  converges, then the series  $\sum_{n=1}^{\infty} f_n$  converges uniformly. That is, the sequence of partial sums  $(\sum_{n=1}^m f_n)_m$  converges uniformly.

*Proof.* We will show that the sequence of partial sums is uniformly cauchy, which then implies that sequence of partial sums converges uniformly. Fix  $\varepsilon > 0$ . The Cauchy criterion (adapted for series: sequences of partial sums) implies that since  $\sum_{n=1}^{\infty} M_n$  converges, there exists an  $N$  such that  $\forall m, n \geq n \geq N$  we have that

$$\left| \sum_{k=n}^m M_k \right| = \sum_{k=n}^m M_k < \varepsilon$$

Therefore  $\forall m, n \geq n \geq N$  and  $\forall x \in X$  we have that

$$\begin{aligned} \left| \sum_{k=n}^m f_k(x) \right| &\leq \sum_{k=n}^m |f_k(x)| \\ &\leq \sum_{k=n}^m M_k \\ &\leq \varepsilon \end{aligned} \tag{\Delta}$$

Therefore  $(\sum_{n=1}^m f_n)_m$  is uniformly cauchy, so the series converges uniformly.  $\square$

## 1.2 Power Series

**Definition 1.4 (Power Series).** A **power series** is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (1.2)$$

where  $c_n \in \mathbb{C}$  are complex coefficients.

**Definition 1.5 (Radius of convergence).** To a power series we can associate a number  $R \in [0, \infty]$  (thus  $R$  is an *extended* real number) called its **radius of convergence** such that

- (i)  $\sum_{n=0}^{\infty} c_n x^n$  converges for every  $|x| < R$ .
- (ii)  $\sum_{n=0}^{\infty} c_n x^n$  diverges for every  $|x| > R$ .

**Theorem 1.5 (Power series continuous on interval of convergence).** A power series with radius of convergence  $R$  converges uniformly on  $[-R + \varepsilon, R - \varepsilon]$  for every  $0 < \varepsilon < R$ . Therefore, power series are continuous on  $(-R, R)$ .

**Theorem 1.6 (Abel summation (summation by parts)).**

$$\sum_{n=0}^N (a_n - a_{n-1}) b_n = a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \quad (1.3)$$

*Proof.* Assume that  $a_{-1} = 0$ . We can derive this formula by reordering terms:

$$\begin{aligned} \sum_{n=0}^N (a_n - a_{n-1}) b_n &= a_0 b_0 + a_1 b_1 - a_0 b_1 + a_2 b_2 - a_1 b_2 + \cdots + a_N b_N - a_{N-1} b_N \\ &= a_0 (b_0 - b_1) + a_1 (b_1 - b_2) + a_{N-1} (b_{N-1} - b_N) + a_N b_N \\ &= a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \end{aligned}$$

□

**Theorem 1.7 (Abel).** Let  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  be a power series with radius of convergence  $R = 1$ . Assume  $\sum_{n=0}^{\infty} c_n$  converges. Then

$$\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} c_n \quad (1.4)$$

*Proof.* We use summation by parts. Set  $s_n = \sum_{k=0}^n c_k$  and by convention assume  $s_{-1} = 0$ . □

## 2 Compactness in Metric Spaces

### 2.1 Review of Basic Topology

**Definition 2.1** (Open, open relative to). Let  $E \subset U \subset X$ , where  $X$  is a metric space.

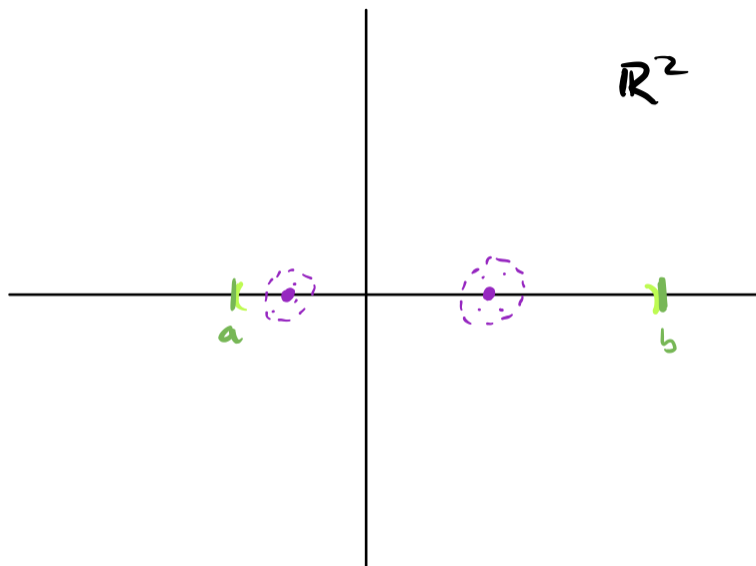
- (i)  $E$  is an **open** subset of  $X$  if for each point  $p \in E$  there exists an  $r > 0$  such that for all  $q \in X$  for which  $d(p, q) < r$  we have that  $q \in E$ .
- (ii)  $E$  is **open relative to**  $Y$  if for each point  $p \in E$  there exists an  $r > 0$  such that for all  $q \in Y$  for which  $d(p, q) < r$  we have that  $q \in E$ .

**Example 2.1** (Relative open sets). Let  $(a, b)$  be an interval on the real line. Notice that  $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$ .

- (i)  $(a, b)$  is an open subset of (or open relative to)  $\mathbb{R}$ .



- (ii)  $(a, b)$  is *not* an open subset of  $\mathbb{R}^2$ . Indeed, any ball around a point  $x \in (a, b) \subset \mathbb{R}$  will leave the  $x$ -axis and intersect with points in the second dimension. Thus no point of  $(a, b)$  is interior relative to  $\mathbb{R}^2$ .



**Theorem 2.1 (Relative open sets).** Let  $Y \subset X$ . A subset  $E$  of  $Y$  is open relative to  $Y$  if and only if  $E = Y \cap G$  for some open subset  $G$  of  $X$ .

*Proof.* [[Todo]] □

**Example 2.2 (Relative open sets).** Let  $X = \mathbb{R}$  and  $A = [0, 1]$ . Observe that  $B = [0, \frac{1}{2}) \subset A \subset X$  is open in  $A$ , but not open in  $X$ . However, there exists a  $C$  open in  $X$  such that  $B = C \cap A$ . One example is  $C = (-\frac{1}{2}, \frac{1}{2})$ .

**Definition 2.2 (Continuous).** Suppose  $X$  and  $Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Then  $f$  is **continuous** at  $p$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $d_X(x, p) < \delta$ , we have that  $d_Y(f(x), f(p)) < \varepsilon$ .

**Definition 2.3 (Uniformly Continuous).** Let  $f$  be a mapping of a metric space  $X$  into a metric space  $Y$ . We say that  $f$  is **uniformly continuous** on  $X$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $p, q \in X$  for which  $d_X(p, q) < \delta$ , we have  $d_Y(f(p), f(q)) < \varepsilon$ .

**Definition 2.4 (Dense).** TFAE:  $E$  is **dense** in  $X$  if

- (i) Every point of  $X$  is a limit point of  $E$  or a point of  $E$  (or both).
- (ii)  $\bar{E} = X$ .
- (iii)  $\forall \varepsilon > 0$  and  $\forall x \in X$  we have that  $B(x, \varepsilon) \cap E \neq \emptyset$ .

**Theorem 2.2 (Functions, inverses, and subsets).** Let  $f : X \rightarrow Y$

- (i)  $E \subset Y \Rightarrow f(f^{-1}(E)) \subset E$
- (ii)  $E \subset X \Rightarrow f(f^{-1}(E)) \supset E$

## 2.2 Basic Definitions

**Definition 2.5 (Open Cover).** Let  $I$  be an arbitrary index set. A collection  $(G_i)_{i \in I}$  of open sets  $G_i \subset X$  is called an **open cover** of  $X$  if  $X \subset \cup_{i \in I} G_i$ .

**Definition 2.6 (Compact).**  $X$  is **compact** if every open cover of  $X$  contains a finite subcover. More explicitly, for every open cover  $(G_i)_{i \in I}$  there exists  $m \in \mathbb{N}$  and  $i_1, i_2, \dots, i_m \in I$  such that  $X \subset \cup_{j=1}^m G_{i_j}$ .

**Remark.** This is also called the Heine-Borel property.

**Definition 2.7 (Compact subset).** A subset  $A \subset X$  is called **compact subset** if  $(A, d|_{A \times A})$  is a compact metric space.  $d|_{A \times A}$  is the restriction of  $d$  to  $A \times A$ .

**Theorem 2.3 (Heine-Borel).** A subset  $A \subset \mathbb{R}^n$  is compact if and only if  $A$  is closed and bounded.

**Definition 2.8 (Relatively compact or precompact).** A subset  $A \subset X$  is called **relatively compact** or **precompact** if the closure  $\bar{A} \subset X$  is compact.

**Example 2.3 (Any finite metric space is compact).** Let  $X = \{x_i\}_{i=1}^n$  be a finite metric space. Let  $J$  be an arbitrary index set, and let  $(G_j)_{j \in J}$  be an open cover of  $X$ . Therefore, each  $x_i$  must be in at least one  $G_j$ . Suppose that  $x_i \in G_{j_i}, j_i \in J$ . Then  $\cup_{i=1}^n G_{j_i}$  is a finite subcover.

**Example 2.4 ( $K = \{0\} \cup \{1/n\}_{n=1}^\infty \subset \mathbb{R}$  is compact).** Let  $J$  be an arbitrary index set, and let  $(G_j)_{j \in J}$  be an open cover of  $K$ . First note that  $0$  must be contained in some element of the open cover; call it  $G_i$ . Since  $G_i$  is open, each element of  $G_i$  is an interior element, so there exists a ball around  $0$  of radius  $\varepsilon > 0$  contained in  $G_i$ . The ball also contains all the elements of  $K$  for which  $n > \frac{1}{\varepsilon}$ . Then, each for each of the finitely many  $n \leq \frac{1}{\varepsilon}$  there exists a  $G_{j_n}$  that contains  $\frac{1}{n}$ . Therefore every open cover has a finite subcover.

**Example 2.5 (Compactness and relative compactness in  $\mathbb{R}$ ).** Any closed and bounded interval  $[a, b]$  in  $\mathbb{R}$  is compact. Half open intervals  $[a, b), (a, b]$  and open intervals  $(a, b)$  in  $\mathbb{R}$  are relatively compact, since their closures are closed and bounded intervals (assuming  $b$  finite).

**Example 2.6 (Unit circle in  $\mathbb{R}^n$  compact).** The set  $C = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^2 = 1\right\} \subset \mathbb{R}^n$  is compact. To show closedness, consider the map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f(\mathbf{x}) = \sum_{i=1}^n x_i^2, \mathbf{x} \in \mathbb{R}^n$ . This map is continuous (since it is the sum of continuous functions). Then  $C = f^{-1}(\{1\})$ , and the singleton  $\{1\}$  is a closed set in  $\mathbb{R}$ . To show boundedness, note that no single element of a vector can have an absolute value greater than one. Therefore  $C \subset [-1, 1]^n$ .  $C$  is closed and bounded, and by Heine-Borel, compact.

**Example 2.7 (Orthogonal matrices).** The set of orthogonal  $n \times n$  matrices with real entries (call this  $O(n, \mathbb{R})$ ) is compact as a subset in  $\mathbb{R}^2$ . To see this, let  $M_n$  be the set of all  $n \times n$  matrices with real entries. Define a function  $f : M_n \rightarrow M_n$  where  $f(A) = A^T A$ . This mapping is continuous. To see this, first note that  $f : A \rightarrow A$  is the identity map and hence continuous. Also  $f : A \rightarrow A^T$  is continuous: this follows since  $\|A\| = \|A^t\|$ , so

$$\begin{aligned}\|f(A) - f(B)\| &= \|B^t - A^t\| \\ &= \|(B - A)^t\| \\ &= \|B - A\|\end{aligned}$$

Thus whenever  $\|B - A\| < \varepsilon$ , we have  $\|f(A) - f(B)\| < \varepsilon$  (hence we set  $\delta = \varepsilon$ ). The product of two continuous functions is continuous.

Since an orthogonal matrix  $O$  has inverse  $O^T$ , we have that  $f^{-1}(I) = O(n, \mathbb{R})$ . The continuity of  $f$  implies that  $O(n, \mathbb{R})$  is closed since it is the preimage of a singleton (which is a closed set).

To show boundedness, recall that the columns of orthogonal matrices are orthonormal. Therefore, the absolute value of an element cannot be larger than 1 in absolute value. Thus  $O(n, \mathbb{R}) \subset [-1, 1]^{n^2}$ . Then by Heine-Borel,  $O(n, \mathbb{R})$  is compact.

**Theorem 2.4** (Compact subsets of metric spaces are closed.).

*Proof.* Let  $X$  be a metric space and  $K$  a compact subset of  $X$ . To show  $K$  is closed, we will show that its complement  $K^c$  is open. To do this, we must show that for all  $p \in K^c$ , there exists a neighborhood of  $p$  completely contained in  $K^c$  (and hence does *not* intersect  $K$ ).

Let  $q \in K$ . We will construct two types of neighborhoods:

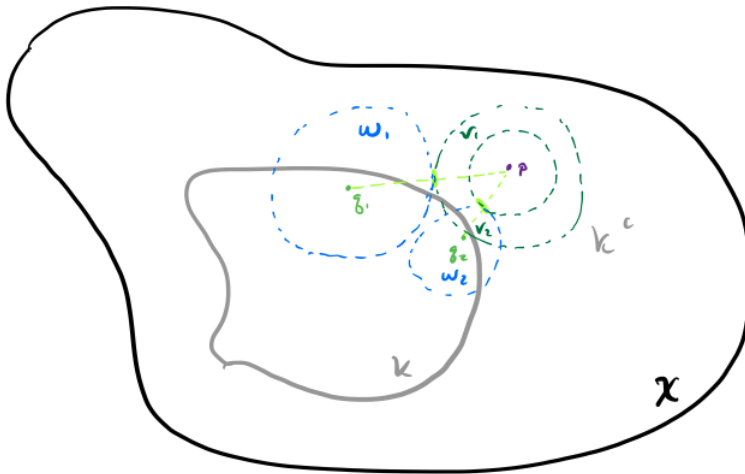
$$W_q = \left\{ x \in X \mid d(x, q) < \frac{1}{2}d(p, q) \right\} \quad (\text{neighborhood of } q)$$

$$V_q = \left\{ x \in X \mid d(x, p) < \frac{1}{2}d(p, q) \right\} \quad (\text{neighborhood of } p)$$

Notice that the union of all  $W_q$  forms an open cover of  $K$ . Since  $K$  is compact, this open cover must have a finite subcover,  $W = \bigcup_{i=1}^n W_{q_i}$ . Define  $V = \bigcap_{i=1}^n V_{q_i}$ , which is still a neighborhood of  $p$ . By construction (since we have used open balls),  $V_{q_i} \cap W_{q_i} = \emptyset$ . Since  $V \subseteq V_{q_i}$ ,  $V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \emptyset$ . Therefore

$$V \cap W = (V \cap W_1) \cup \dots \cup (V \cap W_n) = \emptyset$$

Since  $K \subseteq W$ , we have that  $V \subseteq K^c = \emptyset$ . Thus  $V$  is a neighborhood of  $p$  completely contained in  $K^c$ . Since  $p$  was arbitrary,  $K^c$  is open, and  $K$  is closed.



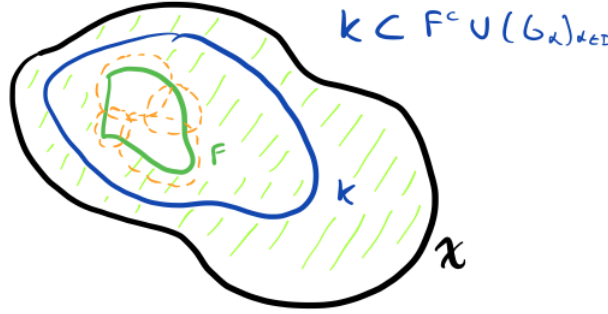
$$\begin{aligned} V &:= \bigcap_{i=1}^n V_i \\ W &:= \bigcup_{i=1}^n W_i \\ V_i \cap W_i &= \emptyset \\ \Rightarrow V \cap W &= \emptyset. \end{aligned}$$

□



**Theorem 2.5** (Closed subsets of compact sets are compact.).

*Proof with compactness.* Let  $F \subset K \subset X$ , where  $F$  is closed (relative to  $X$ ) and  $K$  is compact. Let  $(G_\alpha)_{\alpha \in I}$  be an open cover of  $F$ . Since  $F$  is closed,  $F^c$  is open and  $F^c \cup (G_\alpha)_{\alpha \in I}$  is an open cover of  $K$ .  $K$  is compact, so every open cover has a finite subcover:  $K \subset F^c \cup (G_{\alpha_i})_{i=1}^n$ . Since  $F \subset K$ , this finite open cover also covers  $F$ , but clearly we don't need  $F^c$  in the cover. Therefore  $F$  has a finite subcover:  $F \subset (G_{\alpha_i})_{i=1}^n$ .



□

*Proof with sequential compactness.* Let  $F \subset K \subset X$ , where  $F$  is closed (relative to  $X$ ) and  $K$  is compact. Let  $(x_n)$  be a sequence of points in  $F$  (so is also in  $X$ ). Since  $X$  is sequentially compact, there is a subsequence  $(x_{n_k})$  converging to some point  $x \in X$ . Since  $F$  is closed,  $x \in F$ . Therefore  $F$  is sequentially compact since every sequence in  $F$  has a convergent subsequence. □

**Theorem 2.6** (The intersection of a nested sequence of compact nonempty sets is compact and nonempty.).

*Proof.* Let  $(A_n)$  be a sequence of compact nonempty sets. **Compact:** We know that each  $A_n$  is closed. The intersection of closed sets is closed, so  $\cap A_n$  is closed. Notice that  $\cap A_n \subset A_1$  is a closed subset of a compact set, so is also compact.

**Nonempty:** Since each  $A_n$  is nonempty, fix an  $a_n \in A_n$ . The sequence  $(a_n) \subset A_1$ . Sequential compactness of  $A_1$  implies that there is a subsequence  $(a_{n_k})$  converging to some point  $a \in A_1$ . Notice that this limit must also be in  $A_2$ , since the sequence  $(a_{n_k})$  is also in  $A_2$  (except for potentially the first term, which does not affect convergence). This holds for all  $A_n$ , so  $p \in \cap A_n$ , which shows the intersection is nonempty. □

**Theorem 2.7.** Let  $X$  be a compact metric space. Then there exists a countable, dense set  $E \subset X$ .

*Proof.* We will show that  $\forall \varepsilon > 0$  and  $\forall x \in X$  we can find a set  $E$  such that for each open ball in  $X$  we have that  $B(x, \varepsilon) \cap E \neq \emptyset$ . Fix  $\varepsilon > 0$  and define an open cover of  $X$  as follows:

$$B_n = \bigcup_{x \in X} B(x, \frac{1}{n}) \quad (2.1)$$

$X$  is compact so the open cover  $B_n$  must have a finite subcover. Let  $E$  be the union of the centers of the balls of each finite subcover.  $E$  is the countable union of finite sets, so it is countable. Now fix  $x \in X$ . Choose  $n$  such that  $\varepsilon < \frac{1}{n}$ . Since  $B_n$  covers  $X$ , there must be some ball centered at a point of  $E$ , call it  $y$ , that contains  $x$ . Thus  $d(x, y) < \frac{1}{n} < \varepsilon$ . Thus  $y \in B(x, \varepsilon) \cap E$ .  $\square$

## 2.3 Compactness and Continuity

**Theorem 2.8** (Continuous mappings on compact sets are uniformly continuous.). Let  $X, Y$  be metric spaces and assume  $X$  is compact. If  $f : X \rightarrow Y$  is continuous, then it is uniformly continuous.

*Proof.* Fix  $\varepsilon > 0$ . **Goal:** We need to find a  $\delta > 0$  such that for all  $x, y \in X$  for which  $d_X(x, y) < \delta$ , we have  $d_Y(f(x), f(y)) < \varepsilon$ .

**Use continuity to create balls of points (which cover)  $X$  and are mapped by the function:** Since  $f$  is continuous, we know that for each  $x \in X$ , there exists a number  $\delta_x > 0$  such that for all  $y \in X$  for which  $d_X(x, y) < \delta_x$ , we have that  $d_Y(f(x), f(y)) < \varepsilon/2$ . Now, let  $B_x$  be a ball of radius  $\delta_x/2$  centered at  $x$ . Formally, we can write

$$B_x = B(x, \delta_x/2) = \{y \in X \mid d_X(x, y) < \delta_x/2\}$$

Then  $(B_x)_{x \in X}$  is an open cover of  $X$ .

**Use compactness to select a finite subcover of these balls and use the smallest radius to create uniformity:** By compactness, there exists a finite subcover of  $X$ . Let  $x_1, \dots, x_m$  be the  $x$  which generate this finite subcover. Define  $\delta > 0$  as follows

$$\delta = \frac{1}{2} \min(\delta_{x_1}, \dots, \delta_{x_m}) \quad (2.2)$$

(notice that  $\delta$  is indeed positive, since we are taking the minimum of finitely many positive numbers).

**Show that our choice of  $\delta$  works:** Fix  $x, y \in X$  such that  $d_X(x, y) < \delta$ . There exists an index  $i \in \{1, \dots, m\}$  such that  $x \in B_{x_i}$  (i.e., an element of  $X$  is in some ball of the finite subcover). Then

$$d_X(x_i, y) \leq d_X(x_i, x) + d_X(x, y) < \frac{1}{2}\delta_{x_i} + \delta < \delta_{x_i} \quad (2.3)$$

Now, using the definition of  $\delta_{x_i}$ , we have that

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (2.4)$$

since  $d_X(x_i, x) < \delta_x/2 < \delta_x$  so continuity implies  $d_Y(f(x), f(x_i)) < \varepsilon/2$ , and  $d_X(x_i, y) < \delta_{x_i}$  so continuity implies  $d_Y(f(x_i), f(y)) < \varepsilon/2$ .  $\square$

**Theorem 2.9** (The image of a continuous function which maps from a compact set is compact). Let  $X, Y$  be metric spaces and assume  $X$  is compact. If  $f : X \rightarrow Y$  is continuous, then  $f(X) \subset Y$  is compact.

*Proof with compactness.* Let  $(V_i)_{i \in I}$  be an open cover of  $f(X)$ . Since  $f$  is continuous, we have that  $U_i = f^{-1}(V_i) \subset X$  is open for each  $i$ . Note that  $U \subset f^{-1}(f(U))$  for every  $U \subset X$ . Therefore

$$X \subset f^{-1}(f(X)) \subset \cup_{i \in I} f^{-1}(V_i) = \cup_{i \in I} U_i \quad (2.5)$$

which shows that  $\cup_{i \in I} U_i$  is an open cover of  $X$  (the second subset relation uses that applying  $f^{-1}$  preserves unions). Since  $X$  is compact, there are finitely many indices, say up to  $m$ , such

$$X \subset \bigcup_{j=1}^m U_{i_j} \quad (2.6)$$

Applying  $f$  preserves inclusions and  $V \supset f(f^{-1}(V))$ , so we have that

$$f(X) \subset \bigcup_{j=1}^m f(U_{i_j}) \subset \bigcup_{j=1}^m V_{i_j} \quad (2.7)$$

Thus every open cover of  $f(X)$  has a finite subcover, so  $f(X)$  is compact.  $\square$

*Proof with sequential compactness.* Let  $(y_n)$  be a sequence in  $f(X) \subset Y$ . For each  $n \in \mathbb{N}$  choose a point  $x_n$  such that  $f(x_n) = y_n$ . By sequential compactness of  $X$ , there is some subsequence  $(x_{n_k})$  converging to a point  $x \in X$ . The continuity of  $f$  implies that  $f(x_{n_k})$  converges to  $f(x) \in f(X)$ . Therefore every sequence in  $f(X)$  has a convergent subsequence, so  $f(X)$  is sequentially compact.  $\square$

**Theorem 2.10 (Functions from compact sets to the reals achieve their minimum and maximum values).** Let  $X$  be a compact metric space and  $f : X \rightarrow \mathbb{R}$  a continuous function. Then there exists  $x_0 \in X$  such that  $f(x_0) = \sup_{x \in X} f(x)$ .

*Proof.* Since  $X$  is compact, we know that the image  $f(X)$  is also compact. Since  $f(X) \subset \mathbb{R}$ , the Heine-Borel theorem tells us that  $f(X)$  is closed and bounded. Completeness of the real numbers (every nonempty subset of the real numbers that is bounded above has a supremum in  $\mathbb{R}$ ). Since  $f(X)$  is closed, this supremum must be contained in  $f(X)$ . Therefore there exists  $x_0 \in X$  such that  $f(x_0) = \sup_{x \in X} f(x)$ .  $\square$

## 2.4 Sequential Compactness and Total Boundedness

**Definition 2.9 (Sequentially compact).** A metric space  $X$  is **sequentially compact** if every sequence in  $X$  has a convergent subsequence.

**Remark.** This is also called the Bolzano-Weierstrass property. Recall that the Bolzano-Weierstrass theorem states that every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

**Example 2.8 (Finite sets are sequentially compact).** Suppose  $(x_n)$  is a sequence contained in a finite set. Then this sequence must repeat a term infinitely often. Define the subsequence which is constant at this term: this subsequence clearly converges.

**Theorem 2.11** ((Sequentially) compact sets are closed and bounded).

*Proof.* Let  $A$  be a sequentially compact subset of a metric space  $X$ .

**Closed:** Let  $p$  be a limit point of  $A$ . Then there is a sequence  $(a_n)$  in  $A$  that converges to  $p$ . Sequential compactness of  $A$  implies that there is a subsequence of  $(a_n)$  which converges to some  $q \in A$ . However every subsequence of a convergent sequence must converge to the same limit as the full sequence, which implies  $p = q \in A$ . Therefore  $A$  is closed.

**Bounded:** Fix a point  $x \in X$ . Suppose  $A$  is not bounded. Then for each  $n \in \mathbb{N}$  there must be some point  $a_n$  such that  $d(x, a_n) \geq n$ . However, sequential compactness implies that some subsequence  $(a_{n_k})$  must converge. Convergent sequences are bounded, so we cannot have that  $d(x, a_{n_k}) \rightarrow \infty$  as  $k \rightarrow \infty$ . Thus the sequence  $(a_n)$  cannot behave as assumed, and it must be that  $A$  is bounded for some  $r \in \mathbb{R}$ . □

**Definition 2.10** (Bounded metric space). A metric space  $X$  is **bounded** if it fits in a single fixed ball. More precisely, there exists some  $x_0 \in X$  and  $r > 0$  such that  $X \subseteq B(x_0, r)$ .

**Definition 2.11** (Totally bounded). A metric space  $X$  is **totally bounded** if for every  $\varepsilon > 0$  there exist finitely many balls of radius  $\varepsilon$  that cover  $X$ .

**Claim 2.1** (Totally bounded implies bounded). Let  $X$  be a totally bounded metric space. Then it is bounded.

*Proof.* Fix  $\varepsilon > 0$  and  $x, y \in X$ . Total boundedness implies there exists points  $(x_i)_{i=1}^n$  such that  $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$ . Suppose  $x \in B(x_i, \varepsilon)$  and  $y \in B(x_j, \varepsilon)$ . Then

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < d(x_i, x_j) + 2\varepsilon \quad (2.8)$$

There are only finitely many values of  $d(x_i, x_j)$  we can set  $M = \max_{i,j} d(x_i, x_j)$ . Therefore  $d(x, y) < M + 2\varepsilon$ . Therefore  $X$  is bounded. □

**Example 2.9** (Closed and bounded interval in  $\mathbb{R}$  is totally bounded). A closed and bounded interval  $I = [a, b] \subset \mathbb{R}$  is totally bounded.

**Example 2.10** (Bounded but not totally bounded). The following are examples of metric spaces which are bounded but not totally bounded.

- (i)  $\ell^1$ -space is defined to be the collection of all sequences  $(a_n)_n$  with  $\sum_{n=1}^{\infty} |a_n| < \infty$ . We define a distance  $d$  on  $\ell^1$  as follows:  $d((a_n)_n, (b_n)_n) = \sum_{n=1}^{\infty} |a_n - b_n|$ . The unit ball of  $\ell^1$  centered at the zero-element (call this call  $B_1$ ) (i.e., the zero sequence) is bounded but not totally bounded. More precisely,

$$B_1 = \left\{ (a_n)_n \mid \sum_{n=1}^{\infty} |a_n| \leq 1 \right\} \quad (2.9)$$

Boundedness is clear. Consider the set of sequences  $A = \{a_n\}$  where each  $a_n$  is zero except for a 1 in the  $n$ -th entry. Each  $a_n \in B_1$ . However,

$$d(a_n, a_m) = \begin{cases} 2 & n \neq m \\ 0 & n = m \end{cases} \quad (2.10)$$

If we take balls of radius  $\varepsilon = \frac{1}{2}$ , then a ball can cover at most one sequence in  $A$ . Therefore we can't cover  $A$  with finitely  $\varepsilon$ -balls and hence can't cover  $B_1$  with finitely many  $\varepsilon$ -balls, so that  $B_1$  is not totally bounded.

- (ii)  $\ell^\infty$ -space is defined to be the collection of all sequences  $(a_n)_n$  with  $\sup_n |a_n| < \infty$ . We define a distance  $d$  on  $\ell^\infty$  as follows:  $d((a_n)_n, (b_n)_n) = \sup_n |a_n - b_n|$ . Using the same set of sequences as the previous example, note that for all  $n$  we have that

$$d(a_n, \{0\}_{n=1}^\infty) = 1 \quad (2.11)$$

so that  $A \subset B_1$  (with the sup norm). Further,

$$d(a_n, a_m) = \begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases} \quad (2.12)$$

But again, if we take balls of radius  $\varepsilon = \frac{1}{2}$ , then a ball can cover at most one sequence in  $A$ . Therefore we can't cover  $A$  with finitely  $\varepsilon$ -balls and hence can't cover  $B_1$  with finitely many  $\varepsilon$ -balls, so that  $B_1$  is not totally bounded.

**Theorem 2.12 (Characterizations of compactness).** Let  $X$  be a metric space. The following are equivalent:

- (i)  $X$  is compact.
- (ii)  $X$  is sequentially compact.
- (iii)  $X$  is totally bounded and complete.

*Proof of  $X$  is compact  $\Rightarrow X$  is sequentially compact.* We argue by contradiction. Suppose  $X$  is compact but not sequentially compact. Thus there must exist some sequence  $(x_n)_n \subset X$  without a convergence subsequence. Let  $A$  be the range of the sequence (more explicitly,  $A = \{x_n \mid n \in \mathbb{N}\}$ ). Note that  $A$  has to be an infinite (if  $A$  were finite, then there would be a constant subsequence, which is convergent).

Further,  $A$  cannot have any limit points (by definition, a point is a limit point of a sequence if there exists a subsequence converging to it). Therefore for each  $x_n$ , we can find an open ball centered at  $x_n$  that only intersects  $A$  at  $x_n$ :  $B_n \cap A = \{x_n\}$ .  $A$  is also closed (since it has no limit points), so that  $X - A$  is open. We can construct an open cover of  $X$  as:

$$X \subseteq \bigcup_n B_n \cup \{X - A\} \quad (2.13)$$

Compactness of  $A$  implies this open cover has a finite subcover. Therefore the finite subcover can

only contain finitely many of the sets in  $\bigcup_n B_n$ , so can only contain finitely many of the points in  $A$ , which is a contradiction to the covering.  $\square$

*Proof of  $X$  is sequentially compact  $\Rightarrow X$  is totally bounded and complete.* Suppose  $X$  is sequentially compact.

**Complete:** Fix  $\varepsilon > 0$ . Let  $(x_n)_n \subset X$  be a Cauchy sequence. Sequential compactness implies  $(x_n)_n$  has a convergent subsequence. Call this subsequence  $(x_{n_k})$  and its limit  $x$ . Then there exists an  $N$  such that  $d(x_{n_k}, x) < \varepsilon$  for all  $n_k \geq N$ . Since  $(x_n)_n$  is Cauchy, there exists an  $N$  such that  $n, m \geq N$  implies that  $d(x_n, x_m) < \varepsilon$ . But then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < 2\varepsilon \quad (2.14)$$

for all  $n, n_k \geq \max\{N, M\}$ . Therefore  $(x_n)_n$  converges, so  $X$  is complete. **In words, Cauchy sequences with convergence subsequences also converge.**

**Totally Bounded:** We argue by contradiction. Suppose  $X$  is not totally bounded. Then there exists an  $\varepsilon > 0$  such that  $X$  cannot be covered by finitely many balls of radius  $\varepsilon$ . We know that 1 ball of radius  $\varepsilon$  cannot cover  $X$ . Therefore there must be a point of  $X$  outside of this ball: call it  $x_1$ . Similarly, 2 balls of radius  $\varepsilon$  cannot cover  $X$ , so pick a point outside of these two balls and call it  $x_2$ . Proceeding in this manner generates an infinite sequence. Now consider the  $n$ -th term of this sequence. When we choose  $n + 1$ , we must be able to choose a point such that  $d(x_i, x_{n+1}) \geq \varepsilon$  for all  $i \in \{1, \dots, n\}$ . Otherwise, we would have that  $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$ , which would be a contradiction to  $X$  not being totally bounded. Thus we can construct such a sequence.

Next, since  $X$  is sequentially compact, we must be able to find a convergence subsequence. However, the sequence we've constructed has  $d(x_i, x_j) \geq \varepsilon$  for all  $i, j \in \mathbb{N}$  (hence no subsequence can be Cauchy, and we know that convergence sequences are Cauchy). Therefore we have reached a contradiction and it holds that  $X$  is totally bounded.  $\square$

*Proof of  $X$  is totally bounded and complete  $\Rightarrow X$  is sequentially compact.* Suppose  $X$  is totally bounded and complete. Let  $(x_n)_n \subset X$  be a sequence. **We will construct an convergent subsequence.** By the definition of total boundedness, for all  $\varepsilon > 0$ ,  $X$  can be covered by finitely many balls of radius  $\varepsilon$ . **Observation:** One of these balls must contain infinitely many points of  $(x_n)_n$ . This inspires the following process:

- (i) **Step 1:** Cover  $X$  with balls of radius 1. One of these balls must contain infinitely many points of  $(x_n)_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(0)})_n$ .
- (ii) **Step 2:** Cover  $X$  with balls of radius  $\frac{1}{2}$ . One of these balls must contain infinitely many points of  $(x_n^{(0)})_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(1)})_n$ .
- (iii) **Step  $n$ :** Cover  $X$  with balls of radius  $\frac{1}{2^n}$ . One of these balls must contain infinitely many points of  $(x_n^{(n-1)})_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(n)})_n$ .

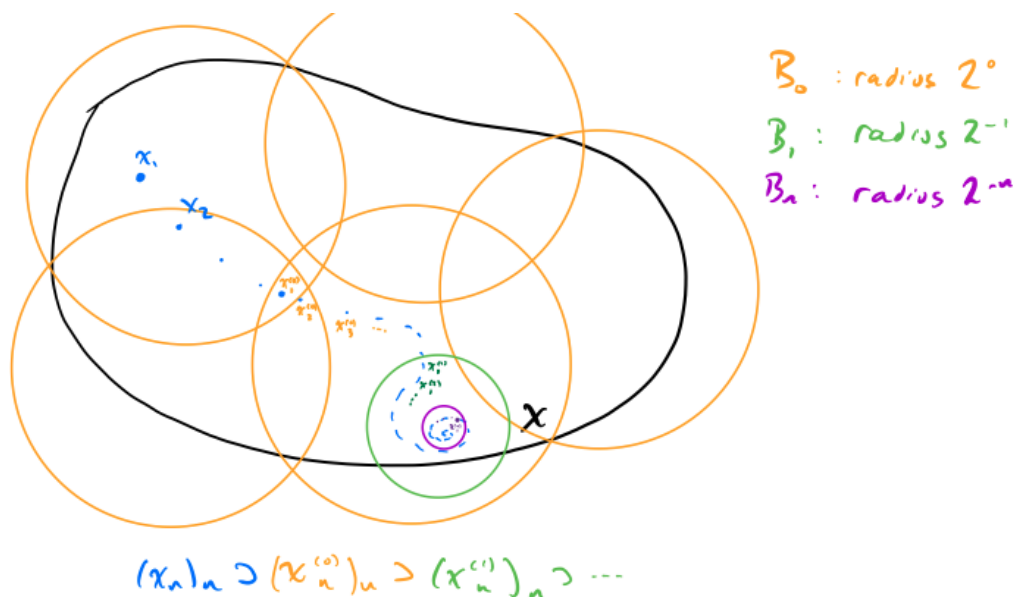
Therefore we have founded nested sequences:  $(x_n^{(0)})_n \subset (x_n^{(1)})_n \subset \dots \subset (x_n^{(n-1)})_n \dots$ . Set  $a_n = x_n^{(n)}$  (which is a subsequence of  $(x_n)_n$ ).

**Now we show that  $a_n$  is a Cauchy sequence.** Fix  $\varepsilon > 0$  and choose  $N$  such that  $2^{-N+1} < \varepsilon$ . Now for  $m > n \geq N$  we have that

$$d(a_m, a_n) < 2 \cdot 2^{-n} < 2^{-N+1} < \varepsilon \quad (2.15)$$

since  $a_m$  and  $a_n$  are contained in the same ball of radius  $2^{-n}$ . Therefore  $(a_n)_n$  is a Cauchy sequence, and completeness implies it converges.

**We have found a convergence subsequence, so  $X$  is sequentially compact.**



□

## 2.5 Equicontinuity and the Arzela-Ascoli Theorem

[[To prove relatively compact, we construct a sequence of functions that converges uniformly.]]

### 3 Approximation Theory and Fourier Series

#### 3.1 Orthonormal Systems

**Theorem 3.1** (Best  $L^2$ -approximation). Let

- $(\phi_n)_n$  be an orthonormal system
- $f \in pc([a, b])$
- $c_n = \langle f, \phi_n \rangle$

Define

- $s_N(x) = \sum_{n=1}^N c_n \phi_n(x)$
- $t_N(x) = \sum_{n=1}^N b_n \phi_n(x)$  for arbitrary coefficients  $b_1, \dots, b_N \in \mathbb{C}$

Then

$$\|f - s_N\|_2 \leq \|f - t_N\|_2 \quad (3.1)$$

and equality holds iff  $b_n = c_n$  for all  $n = 1, \dots, N$ .

*Proof.* We compute the following elements:

$$\langle f, t_N \rangle = \sum_{n=1}^N \bar{b}_n \langle f, \phi_n \rangle = \sum_{n=1}^N \bar{b}_n c_n$$

$$\langle f, s_N \rangle = \sum_{n=1}^N \bar{c}_n \langle f, \phi_n \rangle = \sum_{n=1}^N \bar{c}_n c_n = \sum_{n=1}^N |c_n|^2$$

$$\begin{aligned} \langle t_N, t_N \rangle &= \left\langle \sum_{n=1}^N b_n \phi_n, \sum_{m=1}^N b_m \phi_m \right\rangle \\ &= \sum_{n=1}^N \sum_{m=1}^N b_n \bar{b}_m \langle \phi_n, \phi_m \rangle \\ &= \sum_{n=1}^N |b_n|^2 \end{aligned} \quad ((\phi_n)_n \text{ orthonormal})$$

Then

$$\begin{aligned} \|f - t_N\|_2^2 &= \langle f - t_N, f - t_N \rangle \\ &= \langle f, f \rangle - \langle f, t_N \rangle - \langle t_N, f \rangle + \langle t_N, t_N \rangle \\ &= \langle f, f \rangle - \sum_{n=1}^N \bar{b}_n c_n - \sum_{n=1}^N b_n \bar{c}_n + \sum_{n=1}^N |b_n|^2 \end{aligned}$$



$$\begin{aligned}
|b_n - c_n|^2 &= (b_n - c_n)(\overline{b_n - c_n}) \\
&= (b_n - c_n)(\overline{b_n} - \overline{c_n}) \\
&= b_n \overline{b_n} - b_n \overline{c_n} - c_n \overline{b_n} + c_n \overline{c_n} \\
&= |b_n|^2 - b_n \overline{c_n} - c_n \overline{b_n} + |c_n|^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f - t_N\|_2^2 &= \langle f, f \rangle - \sum_{n=1}^N \overline{b_n} c_n - \sum_{n=1}^N b_n \overline{c_n} + \sum_{n=1}^N |b_n|^2 \\
&= \langle f, f \rangle + \sum_{n=1}^N |b_n - c_n|^2 - \sum_{n=1}^N |c_n|^2
\end{aligned}$$

And

$$\begin{aligned}
\|f - s_N\|_2^2 &= \langle f - s_N, f - s_N \rangle \\
&= \langle f, f \rangle - \langle f, s_N \rangle - \langle s_N, f \rangle + \langle s_N, s_N \rangle \\
&= \langle f, f \rangle - 2 \sum_{n=1}^N |c_n|^2 + \sum_{n=1}^N |c_n|^2 \\
&= \langle f, f \rangle - \sum_{n=1}^N |c_n|^2
\end{aligned}$$

Finally

$$\|f - t_N\|_2^2 = \|f - s_N\|_2^2 + \sum_{n=1}^N |b_n - c_n|^2$$

Thus the claim follows. Indeed, for equality to be achieved, we must have that  $b_n = c_n$  for  $n = 1, \dots, N$ .

□

**Theorem 3.2 (Bessel's Inequality).** If  $(\phi_n)_n$  is an orthonormal system on  $[a, b]$  and  $f \in pc([a, b])$ , then

$$\sum_n |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2 \quad (3.2)$$

*Proof.* In the proof of the previous theorem, we calculated that

$$0 \leq \|f - s_N\|_2^2 = \langle f, f \rangle - \sum_{n=1}^N |c_n|^2 \quad (3.3)$$

Therefore the following holds for all  $N$

$$\sum_{n=1}^N |c_n|^2 \leq \langle f, f \rangle \quad (3.4)$$

Take the limit  $N \rightarrow \infty$  to get the claim

$$\sum_n |c_n|^2 = \sum_n |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2 \quad (3.5)$$

(Notice that the sum  $\sum_{n=1}^N |c_n|^2$  converges.)  $\square$

**Corollary 3.3 (Riemann-Lebesgue Lemma).** Let  $(\phi_n)_n$  be an orthonormal system and  $f \in pc([a, b])$ , then

$$\lim_{n \rightarrow \infty} \langle f, \phi_n \rangle = 0 \quad (3.6)$$

*Proof.* In the proof of Bessel's inequality, we showed that the series  $\sum_n |\langle f, \phi_n \rangle|^2$  converges. Therefore, a necessary condition of convergence, is that the terms go to zero.  $\square$

**Definition 3.1 (Complete Orthonormal System).** An orthonormal system  $(\phi_n)_n$  is called **complete** if

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 = \|f\|_2^2 \quad \forall f \in pc([a, b]) \quad (3.7)$$

**Theorem 3.4.** Let

- $(\phi_n)_n$  be an orthonormal system
- $s_N(x) = \sum_{n=1}^N c_n \phi_n(x)$

Then  $(\phi_n)_n$  is complete if and only if  $(s_N)_N$  converges to  $f$  in the  $L^2$ -norm.

Precisely,  $\lim_{N \rightarrow \infty} \|f - s_N\|_2 = 0$  for every  $f \in pc([a, b])$ .

*Proof.* Again, from the best approximation theorem, we calculated that

$$\|f - s_N\|_2^2 = \langle f, f \rangle - \sum_n |\langle f, \phi_n \rangle|^2 \quad (3.8)$$

Then this converges to 0 if and only if  $(\phi_n)_n$  is complete.  $\square$

## 3.2 Trigonometric Polynomials

**Definition 3.2 (Trigonometric Polynomial, Degree).** A **trigonometric polynomial** is a function

of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad x \in \mathbb{R} \quad (3.9)$$

where  $N \in \mathbb{N}$  and  $c_n \in \mathbb{C}$ . The largest  $N$  for which either  $c_N$  or  $c_{-N}$  is non-zero is called the **degree** of  $f$ .

**Claim 3.1 (Alternative form of trigonometric polynomial).** Every trigonometric polynomial can also be written in the form

$$f(x) = a_0 + \sum_{n=1}^N a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \quad (3.10)$$

for coefficients  $a_n, b_n$ .

*Proof.* Using Euler's identity, recall that

$$e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x) \quad (3.11)$$

Then, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} f(x) &= \sum_{n=-N}^N c_n e^{2\pi i n x} \\ &= \sum_{n=-N}^N [\cos(2\pi n x) + i \sin(2\pi n x)] \\ &= c_0 + \sum_{n=1}^N [c_n (\cos(2\pi n x) + i \sin(2\pi n x)) + c_{-n} (\cos(2\pi i(-n)x) + i \sin(2\pi i(-n)x))] \\ &= c_0 + \sum_{n=1}^N [(c_n + c_{-n}) \cos(2\pi n x) + i(c_n - c_{-n}) \sin(2\pi n x)] \end{aligned}$$

Therefore the claim alternative holds for  $(1 \leq n \leq N)$

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned}$$

□

**Definition 3.3 (Partial sums).** For a 1-periodic function  $f \in \text{pc}$  we define the **partial sums**

$$S_N f(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x} \quad (3.12)$$

**Definition 3.4 (Convolution).** For two 1-periodic functions  $f, g \in \text{pc}$  we define their **convolution** by

$$f * g(x) = \int_0^1 f(t)g(x-t)dt \quad (3.13)$$

**Definition 3.5 (Approximation of Unity).** A sequence of 1-periodic continuous functions  $(k_n)_n$  is called **approximation of unity** if for all 1-periodic continuous functions  $f$  we have that  $f * k_n$  converges uniformly to  $f$  on  $\mathbb{R}$ . That is

$$\sup_{x \in \mathbb{R}} |f * k_n(x) - f(x)| \rightarrow 0 \quad (3.14)$$

as  $n \rightarrow \infty$

**Theorem 3.5 (Sufficient conditions for approximation of unity).** Let  $(k_n)_n$  be a sequence of 1-periodic continuous functions such that

- (i) Non-negative:  $k_n(x) \geq 0$ .
- (ii) Integrates to 1:  $\int_{-1/2}^{1/2} k_n(t)dt = 1$ .
- (iii) “Mass” of  $k_n$  concentrated near the origin: For all  $1/2 \geq \delta > 0$  we have

$$\int_{-\delta}^{\delta} k_n(t)dt \rightarrow 1 \text{ as } n \rightarrow \infty \quad (3.15)$$

*Proof.* Let  $f$  be a function that is 1-periodic and continuous. **What can we say about  $f$ ? Using continuity:** Consider the interval  $[-1/2, 1/2]$ .  $f$  is continuous on this compact set so that on  $[-1/2, 1/2]$  it is

- (i) Bounded
- (ii) Uniformly continuous

**Using periodicity:** by periodicity, on all of  $\mathbb{R}$ ,  $f$  is

- (i) Bounded
- (ii) Uniformly continuous

Thus there exists a  $\delta > 0$  such that

$$|f(x-t) - f(x)| \leq \varepsilon/2 \text{ for all } |t| < \delta, x \in \mathbb{R} \quad (3.16)$$

Then we can write

$$f * k_n(x) - f(x) = \int_{-1/2}^{1/2} (f(x-t) - f(x)) k_n(t)dt \quad (3.17)$$

Let’s break this integral up into two pieces:

$$A = \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t)dt$$

$$B = \int_{1/2 \geq \delta > 0} (f(x-t) - f(x)) k_n(t) dt$$

To bound  $A$ , we use uniform continuity and (ii):

$$|A| = \left| \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t) dt \right|$$

Incomplete.

□

**Theorem 3.6 (Fejér).** For every 1-periodic, continuous function  $f$  we have

$$\sigma_N f \rightarrow f \text{ as } N \rightarrow \infty \quad (3.18)$$

uniformly on  $\mathbb{R}$ .

In words, the sequence  $\sigma_n$  of Cesàro means of the sequence  $(s_n)$  of partial sums of the Fourier series of  $f$  converges uniformly to  $f$  on  $[0, 1]$ .

**Corollary 3.7.** Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials.

*Proof.*

□

**Corollary 3.8 (Fejér kernel an approximation of unity).**

*Proof.* We must verify the hypotheses of the above theorem.

Incomplete.

□

**Theorem 3.9 (Partial sum convoluted with 1-periodic, continuous function converges in 2-norm).** Let  $f$  be a 1-periodic and continuous function. Then

$$\lim_{N \rightarrow \infty} \|S_N f - f\|_2 = 0 \quad (3.19)$$

*Proof.* Let  $\varepsilon > 0$ . By Fejér's theorem, there exists a trigonometric polynomial  $p$  that uniformly approximates  $f$ : That is,  $|f(x) - p(x)| < \varepsilon/2$ . Then

$$\|f - p\|_2 = \left( \int_0^1 |f(x) - p(x)|^2 dx \right)^{1/2} \leq \varepsilon/2 \quad (3.20)$$

Suppose  $p$  has degree  $N$ . Then

$$S_N f - f = S_N f - S_N p + S_N p - f = S_N(f - p) + p - f \quad (3.21)$$

Then by Minkowski's inequality

$$\|S_N f - f\|_2 \leq \|S_N(f - p)\|_2 + \|p - f\|_2 \quad (3.22)$$

Bessel's inequality implies that  $\|S_N f\|_2 \leq \|f\|_2$ . Therefore  $\|S_N(f - p)\|_2 \leq \|f - p\|_2 = \|p - f\|_2$ . Then

$$\|S_N f - f\|_2 \leq 2\|f - p\|_2 \leq \varepsilon \quad (3.23)$$

which proves the claim.  $\square$

**Corollary 3.10 (Parseval's Theorem).** If  $f, g$  are 1-periodic, continuous functions, then

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \quad (3.24)$$

As a special case

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \quad (3.25)$$

*Proof.* We compute

$$\begin{aligned} \langle S_N f, g \rangle &= \left\langle \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x}, g \right\rangle \\ &= \sum_{n=-N}^N \left\langle \hat{f}(n) e^{2\pi i n x}, g \right\rangle \\ &= \sum_{n=-N}^N \hat{f}(n) \left\langle e^{2\pi i n x}, g \right\rangle \\ &= \sum_{n=-N}^N \hat{f}(n) \int_0^1 e^{2\pi i n x} \overline{g(x)} dx \\ &= \sum_{n=-N}^N \hat{f}(n) \overline{\hat{g}(n)} \end{aligned}$$

Now notice that

$$\langle S_N f, g \rangle \rightarrow \langle f, g \rangle \quad (3.26)$$

because

$$\begin{aligned} |\langle S_N f, g \rangle - \langle f, g \rangle| &= |\langle S_N f - f, g \rangle| \\ &\leq \|S_N f - f\|_2 \|g\|_2 && \text{(Cauchy-Schwarz)} \\ &\rightarrow 0 \text{ as } N \rightarrow \infty && \text{(previous theorem)} \end{aligned}$$

The special case follows from setting  $f = g$ . □

**Theorem 3.11.** Let  $f$  be a 1-periodic continuous function and let  $x \in \mathbb{R}$ . Suppose  $f$  is differentiable at  $x$ . Then

$$S_N f(x) \rightarrow f(x) \text{ as } N \rightarrow \infty \quad (3.27)$$

*Proof.* First by definition

Proof incomplete. □

### 3.3 Weierstrass Theorem

**Theorem 3.12 (Weierstrass Theorem).** For every  $f \in C([a, b])$  there exists a sequence of polynomials that converges uniformly to  $f$ .

*Proof.* Let  $\varepsilon > 0$ .

Incomplete. □

**Remark.** This shows that polynomials are dense in  $C([a, b])$ .

### 3.4 Stone-Weierstrass Theorem

**Theorem 3.13 (Stone-Weierstrass Theorem (Sufficient conditions for a subset of  $C(K)$ ,  $K$  compact, to be dense)).** Suppose

- $K$  is a compact metric space.
- $\mathcal{A} \subset C(K)$  such that
  - (i)  $\mathcal{A}$  is a self-adjoint algebra: for  $f, g \in \mathcal{A}, c \in \mathbb{C}$ , we have

$$f + g \in \mathcal{A}, f \cdot g \in \mathcal{A}, c \cdot f \in \mathcal{A}, \bar{f} \in \mathcal{A} \quad (3.28)$$

- (ii)  $\mathcal{A}$  separates points: for all  $x, y \in K$  with  $x \neq y$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .
- (iii)  $\mathcal{A}$  vanishes nowhere: for all  $x \in K$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq 0$ .

**Then**  $\mathcal{A}$  is dense in  $C(K)$  (that is,  $\bar{\mathcal{A}} = C(K)$ ).

**Remark.** Polynomials and trigonometric polynomials satisfy the conditions of the Stone-Weierstrass Theorem (Theorem 3.13).

Show this.

We prove a sequence of lemmas before proving the theorem.

**Lemma 3.14** (Sequence of polynomial (zero intercept) uniformly converging to absolute value function). For every  $a > 0$  there exists a sequence of polynomials  $(p_n)_n$  with real coefficients such that  $p_n(0) = 0 \forall n$  and

$$\sup_{x \in [-a, a]} |p_n(x) - |x|| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.29)$$

*Proof.*  $f(x) = |x|$  is a continuous function on  $[-a, a]$ . By Weierstrass' theorem, there exists a sequence of polynomials  $(q_n)_n$  converging uniformly to  $f(x) = |x|$  on  $[-a, a]$ . Now set  $p_n(x) = q_n(x) - q_n(0)$ . It is clear that  $p_n(0) = 0$ . Further,  $p_n(x)$  converges uniformly to  $|x|$ , since  $q_n(x)$  uniformly converges to  $|x|$  and  $q_n(0)$  converges to 0.

More formally: fix  $\varepsilon > 0$ . Then  $\exists N$  such that  $|q_n(x) - |x|| < \varepsilon$  for all  $n > N$  and  $x \in [-a, a]$ . Thus, for this  $N$ ,  $|q_n(0)| < \frac{\varepsilon}{2}$  and  $|q_n(x)| < \frac{\varepsilon}{2}$  for all  $x \neq 0$ . Therefore

$$|p_n(x)| = |q_n(x) - q_n(0)| \leq |q_n(x)| + |q_n(0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (3.30)$$

□

**Lemma 3.15.** If  $f \in \bar{A}$ , then  $|f| \in \bar{A}$ .

*Proof.* By the previous lemma, there exists a sequence of polynomials (with zero intercept) converging uniformly to the absolute value function. Thus fix  $\varepsilon > 0$ . Let  $a = \max_{x \in K} |f(x)|$ . Then there exist coefficients  $c_1, c_2, \dots, c_N \in \mathbb{R}$  such that

$$\left| \sum_{i=1}^n c_i y^i - |y| \right| \leq \varepsilon \quad \forall |y| \leq a \quad (3.31)$$

**Note:** This sum does not have an intercept/constant term. Since  $f \in \bar{A}$  and  $A$  is a self-adjoint algebra, we have that

$$g = \sum_{i=1}^n c_i f^i \in \bar{A} \quad (3.32)$$

But since Equation 3.31 holds for all values  $y \in [-a, a]$ , and we have that  $|f(x)| \leq a$ , the same inequality holds for  $y^i = f^i(x)$ ,  $x \in K$ . **Note:** This holds for every  $x \in K$ .

$$|g(x) - |f(x)|| = \left| \sum_{i=1}^n c_i f^i(x) - |f(x)| \right| \leq \varepsilon \quad x \in K \quad (3.33)$$

This shows that  $|f|$  can be uniformly approximated by functions in  $\bar{A}$ . Since  $\bar{A}$  is closed, we have that  $|f| \in \bar{A}$ . □

**Lemma 3.16** ( $\bar{A}$  closed under max and min operations). If  $f_1, \dots, f_m \in \bar{A}$  then  $\min\{f_1, \dots, f_n\} \in \bar{A}$  and  $\max\{f_1, \dots, f_n\} \in \bar{A}$



*Proof.* We prove for  $m = 2$  (and the general case follows by induction). Let  $f, g \in \bar{A}$ . We can write  $\min\{f, g\}$  and  $\max\{f, g\}$  as linear combinations of functions in  $\bar{A}$ . Indeed, observe that

$$\begin{aligned}\min\{f, g\} &= \frac{f+g}{2} - \frac{|f-g|}{2} \\ \max\{f, g\} &= \frac{f+g}{2} + \frac{|f-g|}{2}\end{aligned}$$

Therefore, since  $\bar{A}$  is a self-adjoint algebra and is closed under taking the absolute value, we have that  $\bar{A}$  is also closed under taking the max and min of finitely many functions.  $\square$

**Lemma 3.17** (Any two points that could lie on the graph of a function in  $\bar{A}$  do lie on the graph of a function in  $\bar{A}$ ). For every  $x_0, x_1 \in K$ ,  $x_0 \neq x_1$  and  $c_0, c_1 \in \mathbb{R}$ , there exists  $f \in \bar{A}$  such that  $f(x_i) = c_i$  for  $i = 0, 1$ .

*Proof.* Using conditions (ii) and (iii) we have that there exist functions  $g, h_0, h_1 \in \bar{A}$  such that

- (i)  $g$  separates points:  $g(x_0) \neq g(x_1)$
- (ii)  $h_0$  and  $h_1$  don't vanish:  $h_0(x_0) \neq 0$  and  $h_1(x_1) \neq 0$ .

Define

$$\begin{aligned}u_0(x) &= (g(x) - g(x_1))h_0(x) \Rightarrow u_0(x_1) = 0, u_0(x_0) \neq 0 \\ u_1(x) &= (g(x) - g(x_0))h_1(x) \Rightarrow u_1(x_0) = 0, u_1(x_1) \neq 0\end{aligned}$$

Now let

$$f(x) = \frac{c_0 u_0(x)}{u_0(x_0)} + \frac{c_1 u_1(x)}{u_1(x_1)} \quad (3.34)$$

It is clear that

$$\begin{aligned}f(x_0) &= c_0 \\ f(x_1) &= c_1\end{aligned}$$

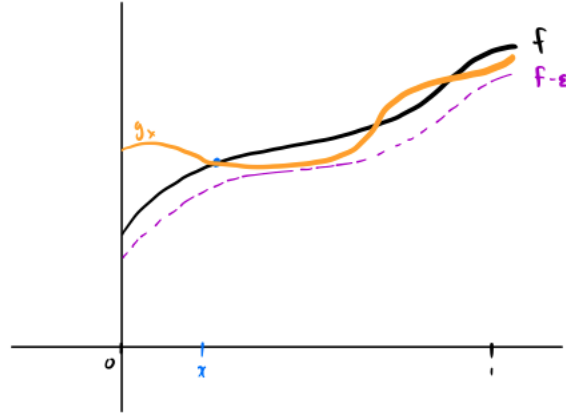
and these terms are well-defined (i.e., no issues with zero denominators).  $\square$

**Remark.** We can extend this lemma to finitely many points. Thus if  $K$  were finite, we would have proved the Stone-Weierstrass theorem.

**Claim 3.2.** Let  $f \in C(K)$  and  $\varepsilon > 0$ . For every  $x \in K$  there exists  $g_x \in \bar{A}$  such that

- (i)  $g_x(x) = f(x)$
- (ii)  $g_x(t) > f(t) - \varepsilon$

We are looking for  $g_x$  like shown in the following figure:



*Proof.* Take another point  $y \in K, y \neq x$ . By the previous lemma, we can find an  $h_y \in \bar{A}$  such that

$$(i) \ h_y(x) = f(x)$$

$$(ii) \ h_y(y) = f(y)$$

Since  $h_y$  is continuous, there exists a  $\delta_y > 0$  such that

$$t \in B_{\delta_y}(y) \Rightarrow |h_y(t) - f(t)| < \varepsilon \quad (3.35)$$

Notice that this implies that  $h_y(t) > f(t) - \varepsilon$ . Now notice that  $(B_{\delta_y}(y))_{y \in K}$  is an open cover of  $K$ . Compactness implies the existence of a finite subcover, identified by points  $y_1, \dots, y_m$ . Now let

$$g_x = \max\{h_{y_1}, \dots, h_{y_m}\} \quad (3.36)$$

We have, by a previous lemma, that  $g_x \in \bar{A}$ . Further, it is clear that

$$g_x(x) = f(x) \quad (3.37)$$

since each  $h_{y_i}(x) = f(x)$  and

$$g_x(t) > f(t) - \varepsilon \quad (3.38)$$

since by taking the max (pointwise) of elements for which this is true. This proves the claim.  $\square$

**Finally ready to complete the whole proof.**

*Proof of Stone-Weierstrass Theorem.*  $\square$

## 4 Linear Operators and Derivatives

## 5 Differential Calculus in $\mathbb{R}^n$

## 6 The Baire category theorem

### A Review From Elementary Analysis

#### A.1 The Real and Complex Number System

**Definition A.1 (Supremum, Infimum).** Let  $S$  be an ordered set,  $E \subset S$ , and  $E$  be bounded above. Suppose there exists an  $\alpha \in S$  with the following properties:

- (i)  $\alpha$  is an upper bound of  $E$ .
- (ii) If  $\gamma < \alpha$  then  $\gamma$  is not an upper bound of  $E$ .

Then  $\alpha$  is called the least upper bound of  $E$  or **supremum** of  $E$  and we write  $\alpha = \sup E$ . Similarly,  $\alpha$  is the greatest lower bound of  $E$  or **infimum** of  $E$  if

- (i)  $\alpha$  is a lower bound of  $E$ .
- (ii) If  $\beta > \alpha$  then  $\beta$  is not a lower bound of  $E$ .

and we write  $\alpha = \inf E$ .

**Definition A.2 (Limit Superior, Inferior).** Let  $(x_n)$  be a sequence of real numbers.

- (i) The **limit superior** of the sequence is defined by

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} x_m \right) \quad (\text{A.1})$$

or

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \left( \sup_{m \geq n} x_m \right) = \inf \{ \sup \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.2})$$

**Alternatively**, the limit superior of the sequence is the smallest  $b \in \mathbb{R}$  such that  $\forall \varepsilon > 0$   $\exists N$  such that  $x_n < b + \varepsilon \forall n > N$ . Thus, any number larger than the limit superior is an upper bound for the sequence after a finite number of terms (hence, only a finite number of elements are greater than  $b + \varepsilon$ ).

**Alternatively**, the limit superior of the sequence is the supremum of the set of subsequential limits.

- (ii) The **limit inferior** of the sequence is defined by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} x_m \right) \quad (\text{A.3})$$

or

$$\liminf_{n \rightarrow \infty} x_n = \sup_{n \geq 0} \left( \inf_{m \geq n} x_m \right) = \sup \{ \inf \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.4})$$

**Theorem A.1 (Properties of limit superiors).** Let  $(x_n)$  and  $(y_n)$  be sequence of real numbers. Then

$$(i) \limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \text{ (as long as the RHS is not of the form } \infty - \infty \text{)}$$

*Proof.* We prove each item in turn.

(i) We have that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} \right) \quad (\text{A.5})$$

Then

$$M_n := \sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} = \sup_{m \geq n} \{x_m + y_m\} \leq \sup_{m \geq n} \{x_m\} + \sup_{m \geq n} \{y_m\} \quad (\text{A.6})$$

Take the limit of both sides to get

$$\lim_{n \rightarrow \infty} M_n \leq \lim_{n \rightarrow \infty} \sup_{m \geq n} \{x_m\} + \lim_{n \rightarrow \infty} \sup_{m \geq n} \{y_m\} \quad (\text{A.7})$$

Thus

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \quad (\text{A.8})$$

□

## A.2 Basic Topology

In what follows, assume  $X$  is a metric space.

**Definition A.3 (Limit point).** A point  $p$  is a **limit point** of the set  $E$  if every neighborhood of  $p$  contains a point  $q \neq p$  such that  $q \in E$ .

**Definition A.4 (Closed).**  $E$  is **closed** if every limit point of  $E$  is a point of  $E$ .

**Definition A.5 (Interior).** A point  $p$  is an **interior point** of  $E$  if there is a neighborhood  $N$  of  $p$  such that  $N \subset E$ .

**Definition A.6 (Open).**  $E$  is **open** if every point of  $E$  is an interior point of  $E$ .

**Definition A.7 (Bounded).**  $E$  is **bounded** if there is a real number  $M$  and a point  $q \in X$  such that  $d(p, q) < M$  for all  $p \in E$ .

**Definition A.8 (Separated, Connected).** Two subsets  $A$  and  $B$  of a metric space  $X$  are said to be **separated** if both  $A \cap \bar{B}$  and  $\bar{A} \cap B$  are empty (i.e., if no point of  $A$  lies in the closure of  $B$  and no point of  $B$  lies in the closure of  $A$ ). A set  $E \subset X$  is said to be **connected** if  $E$  is not a union of two nonempty separated sets.

### A.3 Numerical Sequences and Series

#### A.3.1 Sequences

**Definition A.9 (Convergent Sequence).** A sequence  $(p_n)$  in a metric space  $X$  is said to **converge** if there is a point  $p \in X$  such that for every  $\varepsilon > 0$  there is an integer  $N$  such that  $n \geq N$  implies  $d(p_n, p) < \varepsilon$ .

**Definition A.10 (Subsequence, Subsequential Limit).** Given a sequence  $(p_n)$ , consider a sequence  $(n_k)$  of positive integers such that  $n_1 < n_2 < n_3 < \dots$ . Then the sequence  $(p_{n_i})$  is called a **subsequence** of  $(p_n)$ . If  $(p_{n_i})$  converges, its limit is called a **subsequential limit** of  $(p_n)$ .

Observations:

- $(p_n)$  converges to  $p$  if and only if every subsequence of  $(p_n)$  converges to  $p$ .

**Theorem A.2 (Sequences in compact metric spaces have a convergent subsequence).** If  $(p_n)$  is a sequence in a compact metric space  $X$ , then some subsequence of  $(p_n)$  converges to a point of  $X$ .

**Theorem A.3 (Bolzano-Weierstrass).** Every bounded sequence in  $\mathbb{R}^k$  has a convergent subsequence.

**Definition A.11 (Cauchy Sequence).** A sequence  $(p_n)$  in a metric space  $X$  is said to be a **Cauchy Sequence** if for every  $\varepsilon > 0$  there is an integer  $N$  such that  $d(p_n, p_m) < \varepsilon$  if  $n \geq N$  and  $m \geq N$ .

**Definition A.12 (Diameter).** Let  $E$  be a nonempty subset of a metric space  $X$ , and let  $S$  be the set of all real numbers of the form  $d(p, q)$  with  $p \in E$  and  $q \in E$ . The sup of  $S$  is called the **diameter** of  $E$ .

**Theorem A.4 (Facts about Cauchy sequences).** We have that

- (i) In any metric space  $X$ , every convergent sequence is a Cauchy sequence.
- (ii) If  $X$  is a compact metric space and if  $(p_n)$  is a Cauchy sequence in  $X$ , then  $(p_n)$  converges to some point of  $X$ .
- (iii) In  $\mathbb{R}^k$ , every Cauchy sequence converges.

**Definition A.13 (Complete).** A metric space in which every Cauchy sequence converges is **complete**.

**Definition A.14 (Monotonically increasing, decreasing).** A sequence  $(s_n)$  of real numbers is said to be

- (i) **Monotonically increasing** if  $s_n \leq s_{n_1}$  for all  $n$ .
- (ii) **Monotonically decreasing** if  $s_n \geq s_{n_1}$  for all  $n$ .

**Theorem A.5 (Convergence of monotonic sequences).** Let  $(s_n)$  be a monotonic sequence. Then  $(s_n)$  converges if and only if it is bounded.

### A.3.2 Series

**Definition A.15 (Convergent Series).** Let  $\sum_{n=1}^{\infty} a_n$  be an infinite series. Define  $s_n = \sum_{k=1}^n a_k$  to be the  $n$ th partial sum of the series. If the sequence of partial sums  $\{s_n\}$  converges to  $s$ , we say the series **converges**.

**Theorem A.6 (Cauchy Criterion for Series).**  $\sum a_n$  converges iff

$$\forall \varepsilon > 0 \exists N \text{ s.t. } m, n \geq N \Rightarrow \left| \sum_{k=n}^m a_k \right| \leq \varepsilon \quad (\text{A.9})$$

**Theorem A.7** (Necessary condition for convergence: individual terms of series go to 0.). If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Theorem A.8.** A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

**Theorem A.9 (Root Test).** Given  $\sum a_n$ , let  $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . Then

- (i) If  $\alpha < 1$ ,  $\sum a_n$  converges.
- (ii) If  $\alpha > 1$ ,  $\sum a_n$  diverges.
- (iii) If  $\alpha = 1$ , the test gives no information.

## A.4 Continuity

**Definition A.16 (Limit).** Let  $X$  and  $Y$  be metric spaces,  $E \subset X$ ,  $f$  map  $E$  into  $Y$ , and  $p$  be a limit point of  $E$ . We write  $f(x) \rightarrow q$  as  $x \rightarrow p$  or  $\lim_{x \rightarrow p} f(x) = q$  if there is a point  $q \in Y$  such that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $0 < d_X(x, p) < \delta$ , we have  $d_Y(f(x), q) < \varepsilon$ .

**Definition A.17 (Continuous).** Suppose  $X$  and  $Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Then  $f$  is **continuous** at  $p$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $d_X(x, p) < \delta$ , we have that  $d_Y(f(x), f(p)) < \varepsilon$ .

**Definition A.18 (Uniformly Continuous).** Let  $f$  be a mapping of a metric space  $X$  into a metric space  $Y$ . We say that  $f$  is **uniformly continuous** on  $X$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $p, q \in X$  for which  $d_X(p, q) < \delta$ , we have  $d_Y(f(p), f(q)) < \varepsilon$ .

Observations:

- Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.
- If  $f$  is continuous on  $X$ , then for each  $\varepsilon > 0$  and each  $p \in X$ , we can find a  $\delta > 0$  that satisfies the condition in the definition. For uniform continuity, we can find one  $\delta > 0$  that works for all points  $p \in X$ .
- Every uniformly continuous function is continuous. The two concepts are equivalent on compact sets.

## A.5 Differentiation

**Definition A.19 (Differentiable, Derivative).** Let  $f$  be defined (and real-valued) on  $[a, b]$ . For any  $x \in [a, b]$  define

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \quad (\text{A.10})$$

provided this limit exists. If  $f'$  is defined at a point  $x$ , we say that  $f$  is **differentiable** at  $x$ .  $f'$  is called the **derivative** of  $f$ .

**Theorem A.10 (Mean Value Theorem).** If  $f$  is a real continuous function on  $[a, b]$  which is differentiable on  $(a, b)$ , then there is a point  $x \in (a, b)$  at which

$$f(b) - f(a) = (b - a)f'(x) \quad (\text{A.11})$$



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