

Analysis Notes

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March 22, 2019

Contents

1	Sequence and Series of Functions	2
1.1	Uniform Convergence	2
1.2	Power Series	3
2	Compactness in Metric Spaces	4
2.1	Review of Basic Topology	4
2.2	Basic Definitions	6
2.3	Compactness and Continuity	9
2.4	Sequential Compactness and Total Boundedness	11
2.5	Equicontinuity and the Arzela-Ascoli Theorem	14
3	Approximation Theory and Fourier Series	15
3.1	Orthonormal Systems	15
3.2	Trigonometric Polynomials	15
3.3	Weierstrass Theorem	18
3.4	Stone-Weierstrass Theorem	18
4	Linear Operators and Derivatives	20
5	Differential Calculus in \mathbb{R}^n	20
6	The Baire category theorem	20
A	Review From Elementary Analysis	20
A.1	The Real and Complex Number System	20
A.2	Basic Topology	21
A.3	Numerical Sequences and Series	21
A.4	Continuity	23
A.5	Differentiation	23

1 Sequence and Series of Functions

1.1 Uniform Convergence

Definition 1.1 (Pointwise Convergence). A sequence of functions $(f_n)_n$ **converges pointwise** to a function f on E if for every $\varepsilon > 0$ and for all $x \in E$ there is an integer N (which depends on x) such that for all $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

Definition 1.2 (Uniform Convergence). A sequence of functions $(f_n)_n$ **converges uniformly** to a function f on E if for every $\varepsilon > 0$ there is an integer N such that for all $n \geq N$ and for all $x \in E$ we have

$$|f_n(x) - f(x)| < \varepsilon$$

Theorem 1.1 (Limit of uniformly convergent, bounded sequence of functions is bounded.). If $(f_n)_n$ converges uniformly to f and each f_n is bounded, then f is bounded.

Proof. Fix $\varepsilon > 0$. Uniform convergence implies that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in X$ we have $|f_n(x) - f(x)| < \varepsilon$. Further, boundedness of the sequence of functions implies that $\forall n \exists M_n \in [0, \infty)$ such that $|f_n(x)| \leq M_n \forall x \in X$. Fix an $n \geq N$. Then $\forall x \in X$

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| && \text{(triangle inequality)} \\ &< \varepsilon + M_n && \text{(uniform convergence and boundedness)} \end{aligned}$$

Thus $|f(x)| < \varepsilon + M_n \forall x \in X$, so $f(x)$ is bounded. \square

Theorem 1.2 (Limit of uniformly convergent, continuous sequence of functions is continuous.). If $(f_n)_n$ converges uniformly to f and each f_n is continuous, then f is continuous.

Proof. Fix $\varepsilon > 0$. Uniform convergence implies that $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall x \in X$ we have $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. Continuity of each function $f_n(x)$ at $x \in X$ in the sequence implies that $\exists \delta > 0$ such that $\forall y$ for which $|x - y| < \delta$ we have that $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$. Then $\forall y$ for which $|x - y| < \delta$ we have that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| && (\Delta) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && \text{(uniform convergence and continuity of } f_n \text{ at } x) \\ &= \varepsilon \end{aligned}$$

Therefore f is continuous at x , $\forall x \in X$. \square

Definition 1.3 (Uniformly Cauchy). A sequence of functions (f_n) is **uniformly Cauchy** if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$ and $\forall x \in X$, we have $|f_n(x) - f_m(x)| < \varepsilon$.

Theorem 1.3 (Uniform convergence iff uniform Cauchy). A sequence $(f_n)_n$ of functions on a metric space X converges uniformly if and only if it is uniformly Cauchy.

Proof. Fix $\varepsilon > 0$.

\Rightarrow Uniform convergence implies $\exists N$ such that $\forall n \geq N$ and $\forall x \in X$ we have that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then $\forall n, m \geq N$ and $\forall x \in X$ we have that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore $(f_n)_n$ is uniformly Cauchy.

\Leftarrow Notice that $(f_n)_n$ is a Cauchy sequence in a complete metric space (\mathbb{R}) , therefore it converges pointwise. We need to show uniform convergence. **[[Incomplete]]** \square

Theorem 1.4 (Weierstrass M-test). Let $(f_n)_n$ be a sequence of functions on a metric space X such that there exists a sequence of non-negative real numbers $(M_n)_n$ such that $\forall n$ we have that

$$|f_n(x)| \leq M_n \tag{1.1}$$

If $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly. That is, the sequence of partial sums $(\sum_{n=1}^m f_n)_m$ converges uniformly.

Proof. We will show that the sequence of partial sums is uniformly Cauchy, which then implies that sequence of partial sums converges uniformly. Fix $\varepsilon > 0$. The Cauchy criterion (adapted for series: sequences of partial sums) implies that since $\sum_{n=1}^{\infty} M_n$ converges, there exists an N such that $\forall m, n, m \geq n \geq N$ we have that

$$\left| \sum_{k=n}^m M_k \right| = \sum_{k=n}^m M_k < \varepsilon$$

Therefore $\forall m, n, m \geq n \geq N$ and $\forall x \in X$ we have that

$$\begin{aligned} \left| \sum_{k=n}^m f_k(x) \right| &\leq \sum_{k=n}^m |f_k(x)| \\ &\leq \sum_{k=n}^m M_k \\ &\leq \varepsilon \end{aligned} \tag{\Delta}$$

Therefore $(\sum_{n=1}^m f_n)_m$ is uniformly Cauchy, so the series converges uniformly. \square

1.2 Power Series

Definition 1.4 (Power Series). A **power series** is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1.2}$$

where $c_n \in \mathbb{C}$ are complex coefficients.

Definition 1.5 (Radius of convergence). To a power series we can associate a number $R \in [0, \infty]$ (thus R is an *extended* real number) called its **radius of convergence** such that

- (i) $\sum_{n=0}^{\infty} c_n x^n$ converges for every $|x| < R$.
- (ii) $\sum_{n=0}^{\infty} c_n x^n$ diverges for every $|x| > R$.

Theorem 1.5 (Power series continuous on interval of convergence). A power series with radius of convergence R converges uniformly on $[-R + \varepsilon, R - \varepsilon]$ for every $0 < \varepsilon < R$. Therefore, power series are continuous on $(-R, R)$.

Theorem 1.6 (Abel summation (summation by parts)).

$$\sum_{n=0}^N (a_n - a_{n-1})b_n = a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \quad (1.3)$$

Proof. Assume that $a_{-1} = 0$. We can derive this formula by reordering terms:

$$\begin{aligned} \sum_{n=0}^N (a_n - a_{n-1})b_n &= a_0 b_0 + a_1 b_1 - a_0 b_1 + a_2 b_2 - a_1 b_2 + \cdots + a_N b_N - a_{N-1} b_N \\ &= a_0(b_0 - b_1) + a_1(b_1 - b_2) + \cdots + a_{N-1}(b_{N-1} - b_N) + a_N b_N \\ &= a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \end{aligned}$$

□

Theorem 1.7 (Abel). Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series with radius of convergence $R = 1$. Assume $\sum_{n=0}^{\infty} c_n$ converges. Then

$$\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} c_n \quad (1.4)$$

Proof. We use summation by parts. Set $s_n = \sum_{k=0}^n c_k$ and by convention assume $s_{-1} = 0$. □

2 Compactness in Metric Spaces

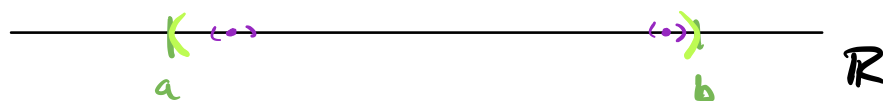
2.1 Review of Basic Topology

Definition 2.1 (Open, open relative to). Let $E \subset U \subset X$, where X is a metric space.

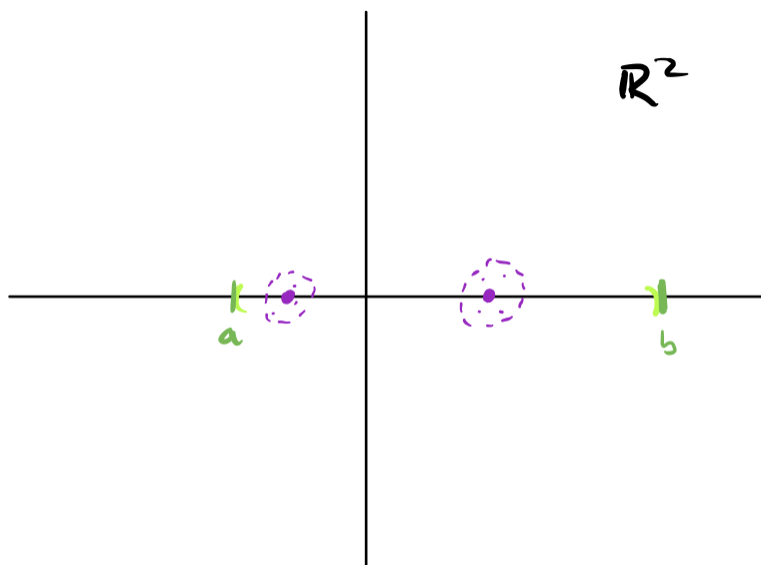
- (i) E is an **open** subset of X if for each point $p \in E$ there exists an $r > 0$ such that for all $q \in X$ for which $d(p, q) < r$ we have that $q \in E$.
- (ii) E is **open relative to** Y if for each point $p \in E$ there exists an $r > 0$ such that for all $q \in Y$ for which $d(p, q) < r$ we have that $q \in E$.

Example 2.1 (Relative open sets). Let (a, b) be an interval on the real line. Notice that $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$.

- (i) (a, b) is an open subset of (or open relative to) \mathbb{R} .



- (ii) (a, b) is *not* an open subset of \mathbb{R}^2 . Indeed, any ball around a point $x \in (a, b) \subset \mathbb{R}$ will leave the x -axis and intersect with points in the second dimension. Thus no point of (a, b) is interior relative to \mathbb{R}^2 .



Theorem 2.1 (Relative open sets). Let $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X .

Proof. [[Todo]]

□

Example 2.2 (Relative open sets). Let $X = \mathbb{R}$ and $A = [0, 1]$. Observe that $B = [0, \frac{1}{2}) \subset A \subset X$ is open in A , but not open in X . However, there exists a C open in X such that $B = C \cap A$. One example is $C = (-\frac{1}{2}, \frac{1}{2})$.

Definition 2.2 (Continuous). Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y . Then f is **continuous** at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ for which $d_X(x, p) < \delta$, we have that $d_Y(f(x), f(p)) < \varepsilon$.

Definition 2.3 (Uniformly Continuous). Let f be a mapping of a metric space X into a metric space Y . We say that f is **uniformly continuous** on X if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$ for which $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$.

Definition 2.4 (Dense). TFAE: E is **dense** in X if

- (i) Every point of X is a limit point of E or a point of E (or both).
- (ii) $\bar{E} = X$.
- (iii) $\forall \varepsilon > 0$ and $\forall x \in X$ we have that $B(x, \varepsilon) \cap E \neq \emptyset$.

Theorem 2.2 (Functions, inverses, and subsets). Let $f : X \rightarrow Y$

- (i) $E \subset Y \Rightarrow f(f^{-1}(E)) \subset E$
- (ii) $E \subset X \Rightarrow f(f^{-1}(E)) \supset E$

2.2 Basic Definitions

Definition 2.5 (Open Cover). Let I be an arbitrary index set. A collection $(G_i)_{i \in I}$ of open sets $G_i \subset X$ is called an **open cover** of X if $X \subset \cup_{i \in I} G_i$.

Definition 2.6 (Compact). X is **compact** if every open cover of X contains a finite subcover. More explicitly, for every open cover $(G_i)_{i \in I}$ there exists $m \in \mathbb{N}$ and $i_1, i_2, \dots, i_m \in I$ such that $X \subset \cup_{j=1}^m G_{i_j}$.

Remark. This is also called the Heine-Borel property.

Definition 2.7 (Compact subset). A subset $A \subset X$ is called **compact subset** if $(A, d|_{A \times A})$ is a compact metric space. $d|_{A \times A}$ is the restriction of d to $A \times A$.

Theorem 2.3 (Heine-Borel). A subset $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Definition 2.8 (Relatively compact or precompact). A subset $A \subset X$ is called **relatively compact** or **precompact** if the closure $\bar{A} \subset X$ is compact.

Example 2.3 (Any finite metric space is compact). Let $X = \{x_i\}_{i=1}^n$ be a finite metric space. Let J be an arbitrary index set, and let $(G_j)_{j \in J}$ be an open cover of X . Therefore, each x_i must be in at least one G_j . Suppose that $x_i \in G_{j_i}$, $j_i \in J$. Then $\cup_{i=1}^n G_{j_i}$ is a finite subcover.

Example 2.4 ($K = \{0\} \cup \{1/n\}_{n=1}^\infty \subset \mathbb{R}$ is compact). Let J be an arbitrary index set, and let $(G_j)_{j \in J}$ be an open cover of K . First note that 0 must be contained in some element of the open cover; call it G_i . Since G_i is open, each element of G_i is an interior element, so there exists a ball around 0 of radius $\varepsilon > 0$ contained in G_i . The ball also contains all the elements of K for which $n > \frac{1}{\varepsilon}$. Then, each for each of the finitely many $n \leq \frac{1}{\varepsilon}$ there exists a G_{j_n} that contains $\frac{1}{n}$. Therefore every open cover has a finite subcover.

Example 2.5 (Compactness and relative compactness in \mathbb{R}). Any closed and bounded interval $[a, b]$ in \mathbb{R} is compact. Half open intervals $[a, b)$, $(a, b]$ and open intervals (a, b) in \mathbb{R} are relatively compact, since their closures are closed and bounded intervals (assuming b finite).

Example 2.6 (Unit circle in \mathbb{R}^n compact). The set $C = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^2 = 1\right\} \subset \mathbb{R}^n$ is compact. To show closedness, consider the map $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(\mathbf{x}) = \sum_{i=1}^n x_i^2$, $\mathbf{x} \in \mathbb{R}^n$. This map is continuous (since it is the sum of continuous functions). Then $C = f^{-1}(\{1\})$, and the singleton $\{1\}$ is a closed set in \mathbb{R} . To show boundedness, note that no single element of a vector can have an absolute value greater than one. Therefore $C \subset [-1, 1]^n$. C is closed and bounded, and by Heine-Borel, compact.

Example 2.7 (Orthogonal matrices). The set of orthogonal $n \times n$ matrices with real entries (call this $O(n, \mathbb{R})$) is compact as a subset in \mathbb{R}^2 . To see this, let M_n be the set of all $n \times n$ matrices with real entries. Define a function $f : M_n \rightarrow M_n$ where $f(A) = A^T A$. This mapping is continuous. To see this, first note that $f : A \rightarrow A$ is the identity map and hence continuous. Also $f : A \rightarrow A^T$ is continuous: this follows since $\|A\| = \|A^t\|$, so

$$\begin{aligned}\|f(A) - f(B)\| &= \|B^t - A^t\| \\ &= \|(B - A)^t\| \\ &= \|B - A\|\end{aligned}$$

Thus whenever $\|B - A\| < \varepsilon$, we have $\|f(A) - f(B)\| < \varepsilon$ (hence we set $\delta = \varepsilon$). The product of two continuous functions is continuous.

Since an orthogonal matrix O has inverse O^T , we have that $f^{-1}(I) = O(n, \mathbb{R})$. The continuity of f implies that $O(n, \mathbb{R})$ is closed since it is the preimage of a singleton (which is a closed set).

To show boundedness, recall that the columns of orthogonal matrices are orthonormal. Therefore, the absolute value of an element cannot be larger than 1 in absolute value. Thus $O(n, \mathbb{R}) \subset [-1, 1]^{n^2}$. Then by Heine-Borel, $O(n, \mathbb{R})$ is compact.

Theorem 2.4 (Compact subsets of metric spaces are closed.).

Proof. Let X be a metric space and K a compact subset of X . To show K is closed, we will show that its complement K^c is open. To do this, we must show that for all $p \in K^c$, there exists a neighborhood of p completely contained in K^c (and hence does *not* intersect K).

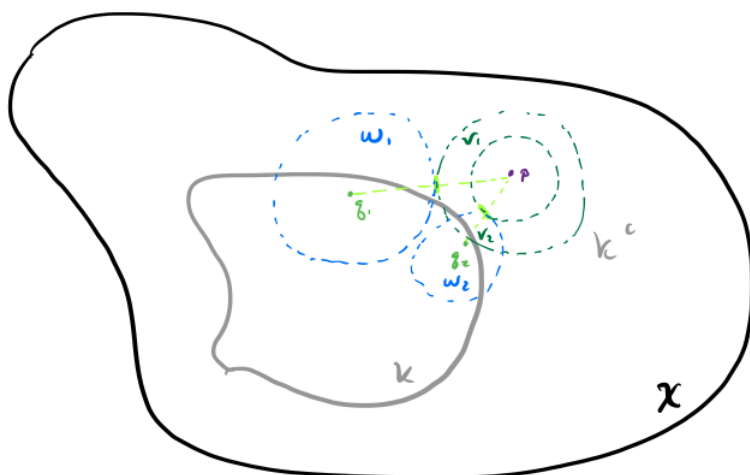
Let $q \in K$. We will construct two types of neighborhoods:

$$\begin{aligned}W_q &= \left\{ x \in X \mid d(x, q) < \frac{1}{2}d(p, q) \right\} && \text{(neighborhood of } q) \\ V_q &= \left\{ x \in X \mid d(x, p) < \frac{1}{2}d(p, q) \right\} && \text{(neighborhood of } p)\end{aligned}$$

Notice that the union of all W_q forms an open cover of K . Since K is compact, this open cover must have a finite subcover, $W = \bigcup_{i=1}^n W_{q_i}$. Define $V = \bigcap_{i=1}^n V_{q_i}$, which is still a neighborhood of p . By construction (since we have used open balls), $V_{q_i} \cap W_{q_i} = \emptyset$. Since $V \subseteq V_{q_i}$, $V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \emptyset$. Therefore

$$V \cap W = (V \cap W_1) \cup \cdots \cup (V \cap W_n) = \emptyset$$

Since $K \subseteq W$, we have that $V \subseteq K = \emptyset$. Thus V is a neighborhood of p completely contained in K^c . Since p was arbitrary, K^c is open, and K is closed.

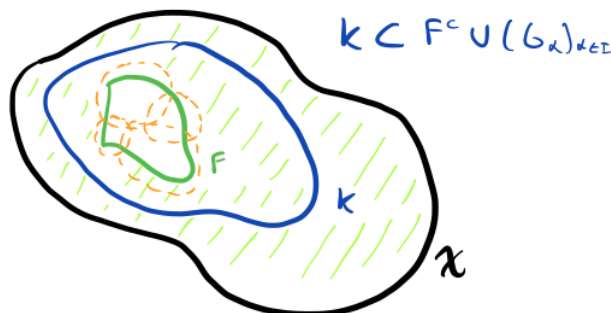


$$\begin{aligned} V &:= \bigcap_{i=1}^{\infty} V_i \\ W &:= \bigcap_{i=1}^{\infty} W_i \\ V_i \cap W_i &= \emptyset \\ \Rightarrow V \cap W &= \emptyset. \end{aligned}$$

□

Theorem 2.5 (Closed subsets of compact sets are compact.).

Proof with compactness. Let $F \subset K \subset X$, where F is closed (relative to X) and K is compact. Let $(G_\alpha)_{\alpha \in I}$ be an open cover of F . Since F is closed, F^c is open and $F^c \cup (G_\alpha)_{\alpha \in I}$ is an open cover of K . K is compact, so every open cover has a finite subcover: $K \subset F^c \cup (G_{\alpha_i})_{i=1}^n$. Since $F \subset K$, this finite open cover also covers F , but clearly we don't need F^c in the cover. Therefore F has a finite subcover: $F \subset (G_{\alpha_i})_{i=1}^n$.



□

Proof with sequential compactness. Let $F \subset K \subset X$, where F is closed (relative to X) and K is compact. Let (x_n) be a sequence of points in F (so is also in X). Since X is sequentially compact, there is a subsequence (x_{n_k}) converging to some point $x \in X$. Since F is closed, $x \in F$. Therefore F is sequentially compact since every sequence in F has a convergent subsequence. □

Theorem 2.6 (The intersection of a nested sequence of compact nonempty sets is compact and nonempty.).

Proof. Let (A_n) be a sequence of compact nonempty sets. **Compact:** We know that each A_n is closed. The intersection of closed sets is closed, so $\cap A_n$ is closed. Notice that $\cap A_n \subset A_1$ is a closed subset of a compact set, so is also compact.

Nonempty: Since each A_n is nonempty, fix an $a_n \in A_n$. The sequence $(a_n) \subset A_1$. Sequential compactness of A_1 implies that there is a subsequence (a_{n_k}) converging to some point $a \in A_1$. Notice that this limit must also be in A_2 , since the sequence (a_{n_k}) is also in A_2 (except for potentially the first term, which does not affect convergence). This holds for all A_n , so $a \in \cap A_n$, which shows the intersection is nonempty. \square

Theorem 2.7. Let X be a compact metric space. Then there exists a countable, dense set $E \subset X$.

Proof. We will show that $\forall \varepsilon > 0$ and $\forall x \in X$ we can find a set E such that for each open ball in X we have that $B(x, \varepsilon) \cap E \neq \emptyset$. Fix $\varepsilon > 0$ and define an open cover of X as follows:

$$B_n = \bigcup_{x \in X} B(x, \frac{1}{n}) \quad (2.1)$$

X is compact so the open cover B_n must have a finite subcover. Let E be the union of the centers of the balls of each finite subcover. E is the countable union of finite sets, so it is countable. Now fix $x \in X$. Choose n such that $\varepsilon < \frac{1}{n}$. Since B_n covers X , there must be some ball centered at a point of E , call it y , that contains x . Thus $d(x, y) < \frac{1}{n} < \varepsilon$. Thus $y \in B(x, \varepsilon) \cap E$. \square

2.3 Compactness and Continuity

Theorem 2.8 (Continuous mappings on compact sets are uniformly continuous.). Let X, Y be metric spaces and assume X is compact. If $f : X \rightarrow Y$ is continuous, then it is uniformly continuous.

Proof. Fix $\varepsilon > 0$. **Goal:** We need to find a $\delta > 0$ such that for all $x, y \in X$ for which $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \varepsilon$.

Use continuity to create balls of points (which cover) X and are mapped by the function: Since f is continuous, we know that for each $x \in X$, there exists a number $\delta_x > 0$ such that for all $y \in X$ for which $d_X(x, y) < \delta_x$, we have that $d_Y(f(x), f(y)) < \varepsilon/2$. Now, let B_x be a ball of radius $\delta_x/2$ centered at x . Formally, we can write

$$B_x = B(x, \delta_x/2) = \{y \in X \mid d_X(x, y) < \delta_x/2\}$$

Then $(B_x)_{x \in X}$ is an open cover of X .

Use compactness to select a finite subcover of these balls and use the smallest radius to create uniformity: By compactness, there exists a finite subcover of X . Let x_1, \dots, x_m be the x which generate this finite subcover. Define $\delta > 0$ as follows

$$\delta = \frac{1}{2} \min(\delta_{x_1}, \dots, \delta_{x_m}) \quad (2.2)$$

(notice that δ is indeed positive, since we are taking the minimum of finitely many positive numbers).

Show that our choice of δ works: Fix $x, y \in X$ such that $d_X(x, y) < \delta$. There exists an index

$i \in \{1, \dots, m\}$ such that $x \in B_{x_i}$ (i.e., an element of X is in some ball of the finite subcover). Then

$$d_X(x_i, y) \leq d_X(x_i, x) + d_X(x, y) < \frac{1}{2}\delta_{x_i} + \delta < \delta_{x_i} \quad (2.3)$$

Now, using the definition of δ_{x_i} , we have that

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (2.4)$$

since $d_X(x_i, x) < \delta_x/2 < \delta_x$ so continuity implies $d_Y(f(x), f(x_i)) < \varepsilon/2$, and $d_X(x_i, y) < \delta_{x_i}$ so continuity implies $d_Y(f(x_i), f(y)) < \varepsilon/2$. \square

Theorem 2.9 (The image of a continuous function which maps from a compact set is compact). *Let X, Y be metric spaces and assume X is compact. If $f : X \rightarrow Y$ is continuous, then $f(X) \subset Y$ is compact.*

Proof with compactness. Let $(V_i)_{i \in I}$ be an open cover of $f(X)$. Since f is continuous, we have that $U_i = f^{-1}(V_i) \subset X$ is open for each i . Note that $U \subset f^{-1}(f(U))$ for every $U \subset X$. Therefore

$$X \subset f^{-1}(f(X)) \subset \cup_{i \in I} f^{-1}(V_i) = \cup_{i \in I} U_i \quad (2.5)$$

which shows that $\cup_{i \in I} U_i$ is an open cover of X (the second subset relation uses that applying f^{-1} preserves unions). Since X is compact, there are finitely many indices, say up to m , such

$$X \subset \bigcup_{j=1}^m U_{i_j} \quad (2.6)$$

Applying f preserves inclusions and $V \supset f(f^{-1}(V))$, so we have that

$$f(X) \subset \bigcup_{j=1}^m f(U_{i_j}) \subset \bigcup_{j=1}^m V_{i_j} \quad (2.7)$$

Thus every open cover of $f(X)$ has a finite subcover, so $f(X)$ is compact. \square

Proof with sequential compactness. Let (y_n) be a sequence in $f(X) \subset Y$. For each $n \in \mathbb{N}$ choose a point x_n such that $f(x_n) = y_n$. By sequential compactness of X , there is some subsequence (x_{n_k}) converging to a point $x \in X$. The continuity of f implies that $f(x_{n_k})$ converges to $f(x) \in f(X)$. Therefore every sequence in $f(X)$ has a convergent subsequence, so $f(X)$ is sequentially compact. \square

Theorem 2.10 (Functions from compact sets to the reals achieve their minimum and maximum values). *Let X be a compact metric space and $f : X \rightarrow \mathbb{R}$ a continuous function. Then there exists $x_0 \in X$ such that $f(x_0) = \sup_{x \in X} f(x)$.*

Proof. Since X is compact, we know that the image $f(X)$ is also compact. Since $f(X) \subset \mathbb{R}$, the Heine-Borel theorem tells us that $f(X)$ is closed and bounded. Completeness of the real numbers (every nonempty subset of the real numbers that is bounded above has a supremum in \mathbb{R}). Since $f(X)$ is closed, this supremum must be contained in $f(X)$. Therefore there exists $x_0 \in X$ such that $f(x_0) = \sup_{x \in X} f(x)$. \square

2.4 Sequential Compactness and Total Boundedness

Definition 2.9 (Sequentially compact). A metric space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Remark. This is also called the Bolzano-Weierstrass property. Recall that the Bolzano-Weierstrass theorem states that every bounded sequence in \mathbb{R}^n has a convergent subsequence.

Example 2.8 (Finite sets are sequentially compact). Suppose (x_n) is a sequence contained in a finite set. Then this sequence must repeat a term infinitely often. Define the subsequence which is constant at this term: this subsequence clearly converges.

Theorem 2.11 ((Sequentially) compact sets are closed and bounded).

Proof. Let A be a sequentially compact subset of a metric space X .

Closed: Let p be a limit point of A . Then there is a sequence (a_n) in A that converges to p . Sequential compactness of A implies that there is a subsequence of (a_n) which converges to some $q \in A$. However every subsequence of a convergent sequence must converge to the same limit as the full sequence, which implies $p = q \in A$. Therefore A is closed.

Bounded: Fix a point $x \in X$. Suppose A is not bounded. Then for each $n \in \mathbb{N}$ there must be some point a_n such that $d(x, a_n) \geq n$. However, sequential compactness implies that some subsequence (a_{n_k}) must converge. Convergent sequences are bounded, so we cannot have that $d(x, a_{n_k}) \rightarrow \infty$ as $k \rightarrow \infty$. Thus the sequence (a_n) cannot behave as assumed, and it must be that A is bounded for some $r \in \mathbb{R}$. □

Definition 2.10 (Bounded metric space). A metric space X is **bounded** if it fits in a single fixed ball. More precisely, there exists some $x_0 \in X$ and $r > 0$ such that $X \subseteq B(x_0, r)$.

Definition 2.11 (Totally bounded). A metric space X is **totally bounded** if for every $\varepsilon > 0$ there exist finitely many balls of radius ε that cover X .

Claim 2.1 (Totally bounded implies bounded). Let X be a totally bounded metric space. Then it is bounded.

Proof. Fix $\varepsilon > 0$ and $x, y \in X$. Total boundedness implies there exists points $(x_i)_{i=1}^n$ such that $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$. Suppose $x \in B(x_i, \varepsilon)$ and $y \in B(x_j, \varepsilon)$. Then

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < d(x_i, x_j) + 2\varepsilon \quad (2.8)$$

There are only finitely many values of $d(x_i, x_j)$ we can set $M = \max_{i,j} d(x_i, x_j)$. Therefore $d(x, y) < M + 2\varepsilon$. Therefore X is bounded. □

Example 2.9 (Closed and bounded interval in \mathbb{R} is totally bounded). A closed and bounded interval $I = [a, b] \subset \mathbb{R}$ is totally bounded.

Example 2.10 (Bounded but not totally bounded). The following are examples of metric spaces which are bounded but not totally bounded.

- (i) ℓ^1 -space is defined to be the collection of all sequences $(a_n)_n$ with $\sum_{n=1}^{\infty} |a_n| < \infty$. We define a distance d on ℓ^1 as follows: $d((a_n)_n, (b_n)_n) = \sum_{n=1}^{\infty} |a_n - b_n|$. The unit ball of ℓ^1 centered at the zero-element (call this call B_1) (i.e., the zero sequence) is bounded but not totally bounded. More precisely,

$$B_1 = \left\{ (a_n)_n \left| \sum_{n=1}^{\infty} |a_n| \leq 1 \right. \right\} \quad (2.9)$$

Boundedness is clear. Consider the set of sequences $A = \{a_n\}$ where each a_n is zero except for a 1 in the n -th entry. Each $a_n \in B_1$. However,

$$d(a_n, a_m) = \begin{cases} 2 & n \neq m \\ 0 & n = m \end{cases} \quad (2.10)$$

If we take balls of radius $\varepsilon = \frac{1}{2}$, then a ball can cover at most one sequence in A . Therefore we can't cover A with finitely ε -balls and hence can't cover B_1 with finitely many ε -balls, so that B_1 is not totally bounded.

- (ii) ℓ^∞ -space is defined to be the collection of all sequences $(a_n)_n$ with $\sup_n |a_n| < \infty$. We define a distance d on ℓ^∞ as follows: $d((a_n)_n, (b_n)_n) = \sup_n |a_n - b_n|$. Using the same set of sequences as the previous example, note that for all n we have that

$$d(a_n, \{0\}_{n=1}^{\infty}) = 1 \quad (2.11)$$

so that $A \subset B_1$ (with the sup norm). Further,

$$d(a_n, a_m) = \begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases} \quad (2.12)$$

But again, if we take balls of radius $\varepsilon = \frac{1}{2}$, then a ball can cover at most one sequence in A . Therefore we can't cover A with finitely ε -balls and hence can't cover B_1 with finitely many ε -balls, so that B_1 is not totally bounded.

Theorem 2.12 (Characterizations of compactness). *Let X be a metric space. The following are equivalent:*

- (i) X is compact.
- (ii) X is sequentially compact.
- (iii) X is totally bounded and complete.

Proof of X is compact $\Rightarrow X$ is sequentially compact. We argue by contradiction. Suppose X is compact but not sequentially compact. Thus there must exist some sequence $(x_n)_n \subset X$ without a convergence subsequence. Let A be the range of the sequence (more explicitly, $A = \{x_n \mid n \in \mathbb{N}\}$). Note that A has to be an infinite (if A were finite, then there would be a constant subsequence, which is convergent).

Further, A cannot have any limit points (by definition, a point is a limit point of a sequence if there exists a subsequence converging to it). Therefore for each x_n , we can find an open ball centered at x_n that only intersects A at x_n : $B_n \cap A = \{x_n\}$. A is also closed (since it has no limit

points), so that $X - A$ is open. We can construct an open cover of X as:

$$X \subseteq \bigcup_n B_n \cup \{X - A\} \quad (2.13)$$

Compactness of A implies this open cover has a finite subcover. Therefore the finite subcover can only contain finitely many of the sets in $\bigcup_n B_n$, so can only contain finitely many of the points in A , which is a contradiction to the covering. \square

Proof of X is sequentially compact $\Rightarrow X$ is totally bounded and complete. Suppose X is sequentially compact.

Complete: Fix $\varepsilon > 0$. Let $(x_n)_n \subset X$ be a Cauchy sequence. Sequential compactness implies $(x_n)_n$ has a convergent subsequence. Call this subsequence (x_{n_k}) and its limit x . Then there exists an N such that $d(x_{n_k}, x) < \varepsilon$ for all $n_k \geq N$. Since $(x_n)_n$ is Cauchy, there exists an N such that $n, m \geq N$ implies that $d(x_n, x_m) < \varepsilon$. But then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < 2\varepsilon \quad (2.14)$$

for all $n, n_k \geq \max\{N, M\}$. Therefore $(x_n)_n$ converges, so X is complete. **In words, Cauchy sequences with convergence subsequences also converge.**

Totally Bounded: We argue by contradiction. Suppose X is not totally bounded. Then there exists an $\varepsilon > 0$ such that X cannot be covered by finitely many balls of radius ε . We know that 1 ball of radius ε cannot cover X . Therefore there must be a point of X outside of this ball: call it x_1 . Similarly, 2 balls of radius ε cannot cover X , so pick a point outside of these two balls and call it x_2 . Proceeding in this manner generates an infinite sequence. Now consider the n -th term of this sequence. When we choose $n + 1$, we must be able to choose a point such that $d(x_i, x_{n+1}) \geq \varepsilon$ for all $i \in \{1, \dots, n\}$. Otherwise, we would have that $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$, which would be a contradiction to X not being totally bounded. Thus we can construct such a sequence.

Next, since X is sequentially compact, we must be able to find a convergence subsequence. However, the sequence we've constructed has $d(x_i, x_j) \geq \varepsilon$ for all $i, j \in \mathbb{N}$ (hence no subsequence can be Cauchy, and we know that convergence sequences are Cauchy). Therefore we have reached a contradiction and it holds that X is totally bounded. \square

Proof of X is totally bounded and complete $\Rightarrow X$ is sequentially compact. Suppose X is totally bounded and complete. Let $(x_n)_n \subset X$ be a sequence. **We will construct an convergent subsequence.** By the definition of total boundedness, for all $\varepsilon > 0$, X can be covered by finitely many balls of radius ε . **Observation:** One of these balls must contain infinitely many points of $(x_n)_n$. This inspires the following process:

- (i) **Step 1:** Cover X with balls of radius 1. One of these balls must contain infinitely many points of $(x_n)_n$. These infinitely many points form a subsequence, call it $(x_n^{(0)})_n$.
- (ii) **Step 2:** Cover X with balls of radius $\frac{1}{2}$. One of these balls must contain infinitely many points of $(x_n^{(0)})_n$. These infinitely many points form a subsequence, call it $(x_n^{(1)})_n$.
- (iii) **Step n :** Cover X with balls of radius $\frac{1}{2^n}$. One of these balls must contain infinitely many points of $(x_n^{(n-1)})_n$. These infinitely many points form a subsequence, call it $(x_n^{(n)})_n$.

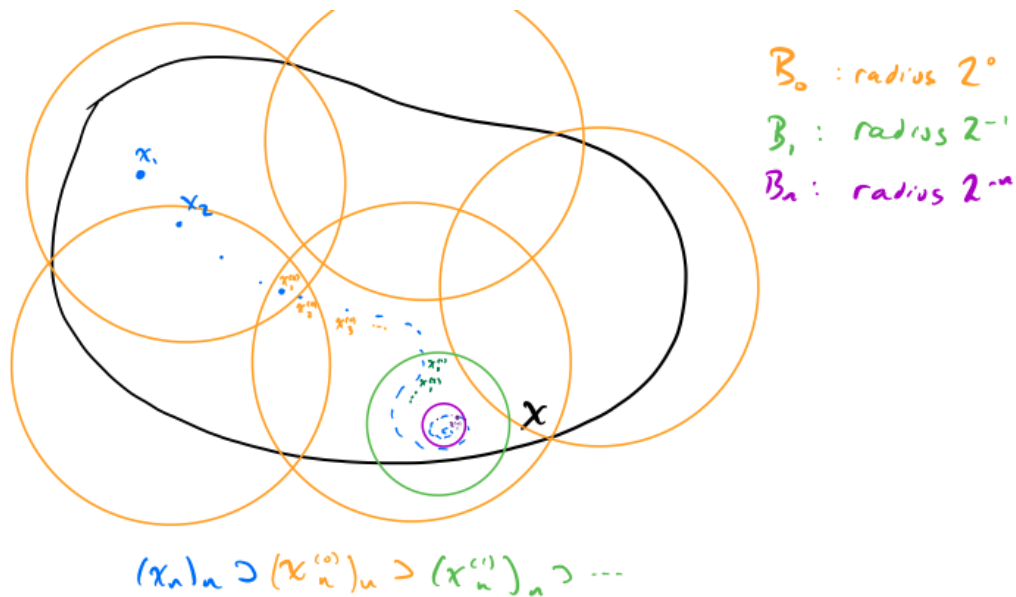
Therefore we have founded nested sequences: $(x_n^{(0)})_n \subset (x_n^{(1)})_n \subset \dots \subset (x_n^{(n-1)})_n \dots$. Set $a_n = x_n^{(n)}$ (which is a subsequence of $(x_n)_n$).

Now we show that a_n is a Cauchy sequence. Fix $\varepsilon > 0$ and choose N such that $2^{-N+1} < \varepsilon$. Now for $m > n \geq N$ we have that

$$d(a_m, a_n) < 2 \cdot 2^{-n} < 2^{-N+1} < \varepsilon \quad (2.15)$$

since a_m and a_n are contained in the same ball of radius 2^{-n} . Therefore $(a_n)_n$ is a Cauchy sequence, and completeness implies it converges.

We have found a convergence subsequence, so X is sequentially compact.



□

2.5 Equicontinuity and the Arzela-Ascoli Theorem

[[To prove relatively compact, we construct a sequence of functions that converges uniformly.]]

3 Approximation Theory and Fourier Series

3.1 Orthonormal Systems

Theorem 3.1 (Best L^2 -approximation).

Theorem 3.2 (Bessel's Inequality). *If $(\phi_n)_n$ is an orthonormal system on $[a, b]$ and $f \in pc([a, b])$, then*

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2 \quad (3.1)$$

Corollary 3.3 (Riemann-Lebesgue Lemma). *Let $(\phi_n)_n$ is an orthonormal system and $f \in pc([a, b])$, then*

$$\lim_{n \rightarrow \infty} \langle f, \phi_n \rangle = 0 \quad (3.2)$$

Definition 3.1 (Complete Orthonormal System). An orthonormal system $(\phi_n)_n$ is called **complete** if

$$\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 = \|f\|_2^2 \quad \forall f \in pc([a, b]) \quad (3.3)$$

3.2 Trigonometric Polynomials

Definition 3.2 (Trigonometric Polynomial, Degree). A **trigonometric polynomial** is a function of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad x \in \mathbb{R} \quad (3.4)$$

where $N \in \mathbb{N}$ and $c_n \in \mathbb{C}$. The largest N for which either c_N or c_{-N} is non-zero is called the **degree** of f .

Claim 3.1 (Alternative form of trigonometric polynomial). *Every trigonometric polynomial can also be written in the form*

$$f(x) = a_0 + \sum_{n=1}^N a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \quad (3.5)$$

for coefficients a_n, b_n .

Proof. Using Euler's identity, recall that

$$e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x) \quad (3.6)$$

Then, for $x \in \mathbb{R}$,

$$\begin{aligned} f(x) &= \sum_{n=-N}^N c_n e^{2\pi i n x} \\ &= \sum_{n=-N}^N [\cos(2\pi i n x) + i \sin(2\pi i n x)] \\ &= c_0 + \sum_{n=1}^N [c_n (\cos(2\pi i n x) + i \sin(2\pi i n x)) + c_{-n} (\cos(2\pi i (-n)x) + i \sin(2\pi i (-n)x))] \end{aligned}$$

$$= c_0 + \sum_{n=1}^N [(c_n + c_{-n}) \cos(2\pi i n x) + i(c_n - c_{-n}) \sin(2\pi i n x)]$$

Therefore the claim alternative holds for $(1 \leq n \leq N)$

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned}$$

□

Definition 3.3 (Partial sums). For a 1-periodic function $f \in \text{pc}$ we define the **partial sums**

$$S_N f(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x} \quad (3.7)$$

Definition 3.4 (Convolution). For two 1-periodic functions $f, g \in \text{pc}$ we define their **convolution** by

$$f * g(x) = \int_0^1 f(t) g(x - t) dt \quad (3.8)$$

Definition 3.5 (Approximation of Unity). A sequence of 1-periodic continuous functions $(k_n)_n$ is called **approximation of unity** if for all 1-periodic continuous functions f we have that $f * k_n$ converges uniformly to f on \mathbb{R} . That is

$$\sup_{x \in \mathbb{R}} |f * k_n(x) - f(x)| \rightarrow 0 \quad (3.9)$$

as $n \rightarrow \infty$

Theorem 3.4 (Sufficient conditions for approximation of unity). Let $(k_n)_n$ be a sequence of 1-periodic continuous functions such that

- (i) Non-negative: $k_n(x) \geq 0$.
- (ii) Integrates to 1: $\int_{-1/2}^{1/2} k_n(t) dt = 1$.
- (iii) "Mass" of k_n concentrated near the origin: For all $1/2 \geq \delta > 0$ we have

$$\int_{-\delta}^{\delta} k_n(t) dt \rightarrow 1 \text{ as } n \rightarrow \infty \quad (3.10)$$

Proof. Let f be a function that is 1-periodic and continuous. **What can we say about f ?** Using **continuity**: Consider the interval $[-1/2, 1/2]$. f is continuous on this compact set so that on $[-1/2, 1/2]$ it is

- (i) Bounded
- (ii) Uniformly continuous

Using periodicity: by periodicity, on all of \mathbb{R} , f is

- (i) Bounded

(ii) Uniformly continuous

Thus there exists a $\delta > 0$ such that

$$|f(x-t) - f(x)| \leq \varepsilon/2 \text{ for all } |t| < \delta, x \in \mathbb{R} \quad (3.11)$$

Then we can write

$$f * k_n(x) - f(x) = \int_{-1/2}^{1/2} (f(x-t) - f(x)) k_n(t) dt \quad (3.12)$$

Let's break this integral up into two pieces:

$$\begin{aligned} A &= \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t) dt \\ B &= \int_{1/2 \geq \delta > 0} (f(x-t) - f(x)) k_n(t) dt \end{aligned}$$

To bound A , we use uniform continuity and (ii):

$$|A| = \left| \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t) dt \right|$$

Incomplete.

□

Theorem 3.5 (Fejér). *For every 1-periodic, continuous function f we have*

$$\sigma_N f \rightarrow f \text{ as } N \rightarrow \infty \quad (3.13)$$

uniformly on \mathbb{R} .

In words, the sequence σ_n of Cesàro means of the sequence (s_n) of partial sums of the Fourier series of f converges uniformly to f on $[0, 1]$.

Corollary 3.6. *Every 1-periodic continuous function can be uniformly approximated by trigonometric polynomials.*

Proof.

□

Corollary 3.7 (Fejér kernel an approximation of unity).

Proof. We must verify the hypotheses of the above theorem.

Incomplete.

□

Theorem 3.8 (Partial sum convoluted with 1-periodic, continuous function converges in 2-norm). *Let f be a 1-periodic and continuous function. Then*

$$\lim_{N \rightarrow \infty} \|S_N f - f\|_2 = 0 \quad (3.14)$$

Proof. Let $\varepsilon > 0$.

Incomplete.

□

Corollary 3.9 (Parseval's Theorem). *If f, g are 1-periodic, continuous functions, then*

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \quad (3.15)$$

Theorem 3.10. *Let f be a 1-periodic continuous function and let $x \in \mathbb{R}$. Suppose f is differentiable at x . Then*

$$S_N f(x) \rightarrow f(x) \text{ as } N \rightarrow \infty \quad (3.16)$$

As a special case

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \quad (3.17)$$

3.3 Weierstrass Theorem

Theorem 3.11 (Weierstrass Theorem). *For every $f \in C([a, b])$ there exists a sequence of polynomials that converges uniformly to f .*

Proof. Let $\varepsilon > 0$.

Incomplete.

□

Remark. This shows that polynomials are dense in $C([a, b])$.

3.4 Stone-Weierstrass Theorem

Theorem 3.12 (Stone-Weierstrass Theorem (Sufficient conditions for a subset of $C(K)$, K compact, to be dense)). *Suppose*

- K is a compact metric space.
- $\mathcal{A} \subset C(K)$ such that

(i) \mathcal{A} is a self-adjoint algebra: for $f, g \in \mathcal{A}$, $c \in \mathbb{C}$, we have

$$f + g \in \mathcal{A}, f \cdot g \in \mathcal{A}, c \cdot f \in \mathcal{A}, \bar{f} \in \mathcal{A} \quad (3.18)$$

(ii) \mathcal{A} separates points: for all $x, y \in K$ with $x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

(iii) \mathcal{A} vanishes nowhere: for all $x \in K$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Then \mathcal{A} is dense in $C(K)$ (that is, $\bar{\mathcal{A}} = C(K)$).

Remark. Polynomials and trigonometric polynomials satisfy the conditions of the Stone-Weierstrass Theorem (Theorem 3.12).

Show this.

We prove a sequence of lemmas before proving the theorem.

Lemma 3.13 (Sequence of polynomials uniformly converging to absolute value function). *For every $a > 0$ there exists a sequence of polynomials $(p_n)_n$ with real coefficients such that $p_n(0) = 0 \forall n$ and*

$$\sup_{x \in [-a, a]} |p_n(x) - |x|| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.19)$$

Proof. $f(x) = |x|$ is a continuous function on $[-a, a]$. By Weierstrass' theorem, there exists a sequence of polynomials $(q_n)_n$ converging uniformly to $f(x) = |x|$ on $[-a, a]$. Now set $p_n(x) = q_n(x) - q_n(0)$. It is clear that $p_n(0) = 0$. Further, $p_n(x)$ converges uniformly to $|x|$, since $q_n(x)$ uniformly converges to $|x|$ and $q_n(0)$ converges to 0.

More formally: fix $\varepsilon > 0$. Then $\exists N$ such that $|q_n(x) - |x|| < \varepsilon$ for all $n > N$ and $x \in [-a, a]$. Thus, for this N , $|q_n(0)| < \frac{\varepsilon}{2}$ and $|q_n(x)| < \frac{\varepsilon}{2}$ for all $x \neq 0$. Therefore

$$|p_n(x)| = |q_n(x) - q_n(0)| \leq |q_n(x)| + |q_n(0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad (3.20)$$

□

Lemma 3.14. *If $f \in \bar{A}$, then $|f| \in \bar{A}$.*

4 Linear Operators and Derivatives

5 Differential Calculus in \mathbb{R}^n

6 The Baire category theorem

A Review From Elementary Analysis

A.1 The Real and Complex Number System

Definition A.1 (Supremum, Infimum). Let S be an ordered set, $E \subset S$, and E be bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- (i) α is an upper bound of E .
- (ii) If $\gamma < \alpha$ then γ is not an upper bound of E .

Then α is called the least upper bound of E or **supremum** of E and we write $\alpha = \sup E$. Similarly, α is the greatest lower bound of E or **infimum** of E if

- (i) α is a lower bound of E .
- (ii) If $\beta > \alpha$ then β is not a lower bound of E .

and we write $\alpha = \inf E$.

Definition A.2 (Limit Superior, Inferior). Let (x_n) be a sequence of real numbers.

- (i) The **limit superior** of the sequence is defined by

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup_{m \geq n} x_m \right) \quad (\text{A.1})$$

or

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \left(\sup_{m \geq n} x_m \right) = \inf \{ \sup \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.2})$$

Alternatively, the limit superior of the sequence is the smallest $b \in \mathbb{R}$ such that $\forall \varepsilon > 0$ $\exists N$ such that $x_n < b + \varepsilon \forall n > N$. Thus, any number larger than the limit superior is an upper bound for the sequence after a finite number of terms (hence, only a finite number of elements are greater than $b + \varepsilon$).

Alternatively, the limit superior of the sequence is the supremum of the set of subsequential limits.

- (ii) The **limit inferior** of the sequence is defined by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf_{m \geq n} x_m \right) \quad (\text{A.3})$$

or

$$\liminf_{n \rightarrow \infty} x_n = \sup_{n \geq 0} \left(\inf_{m \geq n} x_m \right) = \sup \{ \inf \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.4})$$

Theorem A.1 (Properties of limit superiors). Let (x_n) and (y_n) be sequence of real numbers. Then

$$(i) \limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \text{ (as long as the RHS is not of the form } \infty - \infty)$$

Proof. We prove each item in turn.

(i) We have that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} \left(\sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} \right) \quad (\text{A.5})$$

Then

$$M_n := \sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} = \sup_{m \geq n} \{x_m + y_m\} \leq \sup_{m \geq n} \{x_m\} + \sup_{m \geq n} \{y_m\} \quad (\text{A.6})$$

Take the limit of both sides to get

$$\lim_{n \rightarrow \infty} M_n \leq \lim_{n \rightarrow \infty} \sup_{m \geq n} \{x_m\} + \lim_{n \rightarrow \infty} \sup_{m \geq n} \{y_m\} \quad (\text{A.7})$$

Thus

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \quad (\text{A.8})$$

□

A.2 Basic Topology

In what follows, assume X is a metric space.

Definition A.3 (Limit point). A point p is a **limit point** of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.

Definition A.4 (Closed). E is **closed** if every limit point of E is a point of E .

Definition A.5 (Interior). A point p is an **interior point** of E if there is a neighborhood N of p such that $N \subset E$.

Definition A.6 (Open). E is **open** if every point of E is an interior point of E .

Definition A.7 (Bounded). E is **bounded** if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

Definition A.8 (Separated, Connected). Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty (i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A). A set $E \subset X$ is said to be **connected** if E is not a union of two nonempty separated sets.

A.3 Numerical Sequences and Series

A.3.1 Sequences

Definition A.9 (Convergent Sequence). A sequence (p_n) in a metric space X is said to **converge** if there is a point $p \in X$ such that for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $d(p_n, p) < \varepsilon$.

Definition A.10 (Subsequence, Subsequential Limit). Given a sequence (p_n) , consider a sequence (n_k) of positive integers such that $n_1 < n_2 < n_3 < \dots$. Then the sequence (p_{n_i}) is called a **subsequence** of (p_n) . If (p_{n_i}) converges, its limit is called a **subsequential limit** of (p_n) .

Observations:

- (p_n) converges to p if and only if every subsequence of (p_n) converges to p .

Theorem A.2 (Sequences in compact metric spaces have a convergent subsequence). If (p_n) is a sequence in a compact metric space X , then some subsequence of (p_n) converges to a point of X .

Theorem A.3 (Bolzano-Weierstrass). Every bounded sequence in \mathbb{R}^k has a convergent subsequence.

Definition A.11 (Cauchy Sequence). A sequence (p_n) in a metric space X is said to be a **Cauchy Sequence** if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Definition A.12 (Diameter). Let E be a nonempty subset of a metric space X , and let S be the set of all real numbers of the form $d(p, q)$ with $p \in E$ and $q \in E$. The sup of S is called the **diameter** of E .

Theorem A.4 (Facts about Cauchy sequences). We have that

- In any metric space X , every convergent sequence is a Cauchy sequence.
- If X is a compact metric space and if (p_n) is a Cauchy sequence in X , then (p_n) converges to some point of X .
- In \mathbb{R}^k , every Cauchy sequence converges.

Definition A.13 (Complete). A metric space in which every Cauchy sequence converges is **complete**.

Definition A.14 (Monotonically increasing, decreasing). A sequence (s_n) of real numbers is said to be

- Monotonically increasing** if $s_n \leq s_{n_1}$ for all n .
- Monotonically decreasing** if $s_n \geq s_{n_1}$ for all n .

Theorem A.5 (Convergence of monotonic sequences). Let (s_n) be a monotonic sequence. Then (s_n) converges if and only if it is bounded.

A.3.2 Series

Definition A.15 (Convergent Series). Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Define $s_n = \sum_{k=1}^n a_k$ to be the n th partial sum of the series. If the sequence of partial sums $\{s_n\}$ converges to s , we say the series **converges**.

Theorem A.6 (Cauchy Criterion for Series). $\sum a_n$ converges iff

$$\forall \varepsilon > 0 \exists N \text{ s.t. } m, n \geq N \Rightarrow \left| \sum_{k=n}^m a_k \right| \leq \varepsilon \quad (\text{A.9})$$

Theorem A.7 (Necessary condition for convergence: individual terms of series go to 0.). If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem A.8. A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Theorem A.9 (Root Test). Given $\sum a_n$, let $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then

- (i) If $\alpha < 1$, $\sum a_n$ converges.
- (ii) If $\alpha > 1$, $\sum a_n$ diverges.
- (iii) If $\alpha = 1$, the test gives no information.

A.4 Continuity

Definition A.16 (Limit). Let X and Y be metric spaces, $E \subset X$, f map E into Y , and p be a limit point of E . We write $f(x) \rightarrow q$ as $x \rightarrow p$ or $\lim_{x \rightarrow p} f(x) = q$ if there is a point $q \in Y$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ for which $0 < d_X(x, p) < \delta$, we have $d_Y(f(x), q) < \varepsilon$.

Definition A.17 (Continuous). Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y . Then f is **continuous** at p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ for which $d_X(x, p) < \delta$, we have that $d_Y(f(x), f(p)) < \varepsilon$.

Definition A.18 (Uniformly Continuous). Let f be a mapping of a metric space X into a metric space Y . We say that f is **uniformly continuous** on X if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $p, q \in X$ for which $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$.

Observations:

- Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.
- If f is continuous on X , then for each $\varepsilon > 0$ and each $p \in X$, we can find a $\delta > 0$ that satisfies the condition in the definition. For uniform continuity, we can find one $\delta > 0$ that works for all points $p \in X$.
- Every uniformly continuous function is continuous. The two concepts are equivalent on compact sets.

A.5 Differentiation

Definition A.19 (Differentiable, Derivative). Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ define

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \quad (\text{A.10})$$

provided this limit exists. If f' is defined at a point x , we say that f is **differentiable** at x . f' is called the **derivative** of f .

Theorem A.10 (Mean Value Theorem). If f is a real continuous function on $[a, b]$ which is differentiable on (a, b) , then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b - a)f'(x) \quad (\text{A.11})$$

References

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