

# Analysis Notes

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March 12, 2019

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# 1 Sequence and Series of Functions

## 1.1 Uniform Convergence

**Definition 1.1 (Pointwise Convergence).** A sequence of functions  $(f_n)_n$  **converges pointwise** to a function  $f$  on  $E$  if for every  $\varepsilon > 0$  and for all  $x \in E$  there is an integer  $N$  (which depends on  $x$ ) such that for all  $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon$$

**Definition 1.2 (Uniform Convergence).** A sequence of functions  $(f_n)_n$  **converges uniformly** to a function  $f$  on  $E$  if for every  $\varepsilon > 0$  there is an integer  $N$  such that for all  $n \geq N$  and for all  $x \in E$  we have

$$|f_n(x) - f(x)| < \varepsilon$$

**Theorem 1.1 (Limit of uniformly convergent, bounded sequence of functions is bounded.).** If  $(f_n)_n$  converges uniformly to  $f$  and each  $f_n$  is bounded, then  $f$  is bounded.

*Proof.* Fix  $\varepsilon > 0$ . Uniform convergence implies that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have  $|f_n(x) - f(x)| < \varepsilon$ . Further, boundedness of the sequence of functions implies that  $\forall n \exists M_n \in [0, \infty)$  such that  $|f_n(x)| \leq M_n \forall x \in X$ . Fix an  $n \geq N$ . Then  $\forall x \in X$

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| && \text{(triangle inequality)} \\ &< \varepsilon + M_n && \text{(uniform convergence and boundedness)} \end{aligned}$$

Thus  $|f(x)| < \varepsilon + M_n \forall x \in X$ , so  $f(x)$  is bounded.  $\square$

**Theorem 1.2 (Limit of uniformly convergent, continuous sequence of functions is continuous.).** If  $(f_n)_n$  converges uniformly to  $f$  and each  $f_n$  is continuous, then  $f$  is continuous.

*Proof.* Fix  $\varepsilon > 0$ . Uniform convergence implies that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have  $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ . Continuity of each function  $f_n(x)$  at  $x \in X$  in the sequence implies that  $\exists \delta > 0$  such that  $\forall y$  for which  $|x - y| < \delta$  we have that  $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ . Then  $\forall y$  for which  $|x - y| < \delta$  we have that

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| && (\Delta) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && \text{(uniform convergence and continuity of } f_n \text{ at } x) \\ &= \varepsilon \end{aligned}$$

Therefore  $f$  is continuous at  $x$ ,  $\forall x \in X$ .  $\square$

**Definition 1.3 (Uniformly Cauchy).** A sequence of functions  $(f_n)$  is **uniformly Cauchy** if  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  such that  $\forall n, m \geq N$  and  $\forall x \in X$ , we have  $|f_n(x) - f_m(x)| < \varepsilon$ .

**Theorem 1.3 (Uniform convergence iff uniform Cauchy).** A sequence  $(f_n)_n$  of functions on a metric space  $X$  converges uniformly if and only if it is uniformly Cauchy.

*Proof.* Fix  $\varepsilon > 0$ .

$\Rightarrow$  Uniform convergence implies  $\exists N$  such that  $\forall n \geq N$  and  $\forall x \in X$  we have that  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ . Then  $\forall n, m \geq N$  and  $\forall x \in X$  we have that

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore  $(f_n)_n$  is uniformly cauchy.

$\Leftarrow$  Notice that  $(f_n)_n$  is a cauchy sequence in a complete metric space  $(\mathbb{R})$ , therefore it converges pointwise. We need to show uniform convergence. **[[Incomplete]]**  $\square$

**Theorem 1.4 (Weierstrass M-test).** Let  $(f_n)_n$  be a sequence of functions on a metric space  $X$  such that there exists a sequence of non-negative real numbers  $(M_n)_n$  such that  $\forall n$  we have that

$$|f_n(x)| \leq M_n \tag{1.1}$$

If  $\sum_{n=1}^{\infty} M_n$  converges, then the series  $\sum_{n=1}^{\infty} f_n$  converges uniformly. That is, the sequence of partial sums  $(\sum_{n=1}^m f_n)_m$  converges uniformly.

*Proof.* We will show that the sequence of partial sums is uniformly cauchy, which then implies that sequence of partial sums converges uniformly. Fix  $\varepsilon > 0$ . The Cauchy criterion (adapted for series: sequences of partial sums) implies that since  $\sum_{n=1}^{\infty} M_n$  converges, there exists an  $N$  such that  $\forall m, n, m \geq n \geq N$  we have that

$$\left| \sum_{k=n}^m M_k \right| = \sum_{k=n}^m M_k < \varepsilon$$

Therefore  $\forall m, n, m \geq n \geq N$  and  $\forall x \in X$  we have that

$$\begin{aligned} \left| \sum_{k=n}^m f_k(x) \right| &\leq \sum_{k=n}^m |f_k(x)| \\ &\leq \sum_{k=n}^m M_k \\ &\leq \varepsilon \end{aligned} \tag{\Delta}$$

Therefore  $(\sum_{n=1}^m f_n)_m$  is uniformly cauchy, so the series converges uniformly.  $\square$

## 1.2 Power Series

**Definition 1.4 (Power Series).** A **power series** is a function of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \tag{1.2}$$

where  $c_n \in \mathbb{C}$  are complex coefficients.

**Definition 1.5 (Radius of convergence).** To a power series we can associate a number  $R \in [0, \infty]$  (thus  $R$  is an *extended* real number) called its **radius of convergence** such that

- (i)  $\sum_{n=0}^{\infty} c_n x^n$  converges for every  $|x| < R$ .
- (ii)  $\sum_{n=0}^{\infty} c_n x^n$  diverges for every  $|x| > R$ .

**Theorem 1.5 (Power series continuous on interval of convergence).** A power series with radius of convergence  $R$  converges uniformly on  $[-R + \varepsilon, R - \varepsilon]$  for every  $0 < \varepsilon < R$ . Therefore, power series are continuous on  $(-R, R)$ .

**Theorem 1.6 (Abel summation (summation by parts)).**

$$\sum_{n=0}^N (a_n - a_{n-1})b_n = a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \quad (1.3)$$

*Proof.* Assume that  $a_{-1} = 0$ . We can derive this formula by reordering terms:

$$\begin{aligned} \sum_{n=0}^N (a_n - a_{n-1})b_n &= a_0 b_0 + a_1 b_1 - a_0 b_1 + a_2 b_2 - a_1 b_2 + \cdots + a_N b_N - a_{N-1} b_N \\ &= a_0(b_0 - b_1) + a_1(b_1 - b_2) + \cdots + a_{N-1}(b_{N-1} - b_N) + a_N b_N \\ &= a_N b_N + \sum_{n=0}^{N-1} a_n (b_n - b_{n+1}) \end{aligned}$$

□

**Theorem 1.7 (Abel).** Let  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  be a power series with radius of convergence  $R = 1$ . Assume  $\sum_{n=0}^{\infty} c_n$  converges. Then

$$\lim_{x \rightarrow 1^-} f(x) = \sum_{n=0}^{\infty} c_n \quad (1.4)$$

*Proof.* We use summation by parts. Set  $s_n = \sum_{k=0}^n c_k$  and by convention assume  $s_{-1} = 0$ . □

## 1.3 ?

**Definition 1.6 (Equicontinuous).** A family  $\mathcal{F}$  of complex functions  $f$  defined on a set  $E$  in a metric space  $X$  is said to be **equicontinuous** on  $E$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon$$

whenever  $d(x, y) < \delta$ ,  $x \in E$ ,  $y \in E$ , and  $f \in \mathcal{F}$ . (Notice: every member of an equicontinuous family is uniformly continuous)

## 2 Compactness in Metric Spaces

### 2.1 Review of Basic Topology

**Definition 2.1 (Open, open relative to).** Let  $E \subset U \subset X$ , where  $X$  is a metric space.

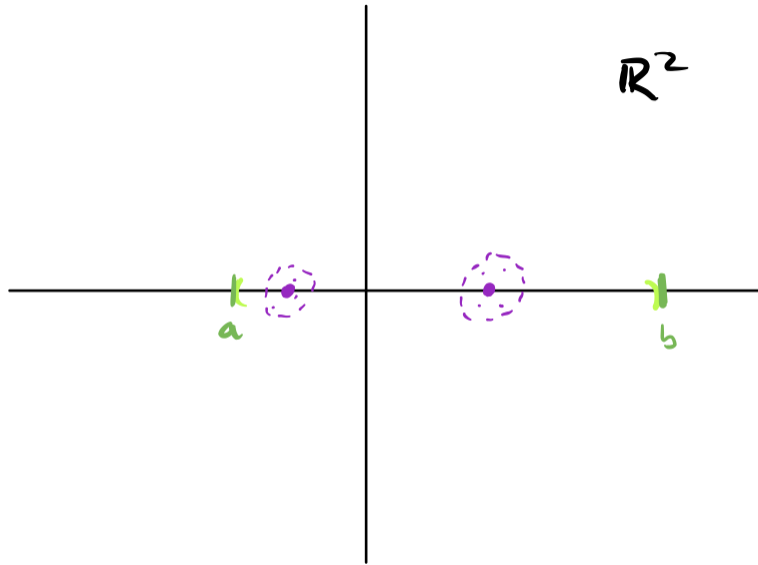
- (i)  $E$  is an **open** subset of  $X$  if for each point  $p \in E$  there exists an  $r > 0$  such that for all  $q \in X$  for which  $d(p, q) < r$  we have that  $q \in E$ .
- (ii)  $E$  is **open relative to**  $Y$  if for each point  $p \in E$  there exists an  $r > 0$  such that for all  $q \in Y$  for which  $d(p, q) < r$  we have that  $q \in E$ .

**Example 2.1 (Relative open sets).** Let  $(a, b)$  be an interval on the real line. Notice that  $(a, b) \subset \mathbb{R} \subset \mathbb{R}^2$ .

- (i)  $(a, b)$  is an open subset of (or open relative to)  $\mathbb{R}$ .



- (ii)  $(a, b)$  is *not* an open subset of  $\mathbb{R}^2$ . Indeed, any ball around a point  $x \in (a, b) \subset \mathbb{R}$  will leave the  $x$ -axis and intersect with points in the second dimension. Thus no point of  $(a, b)$  is interior relative to  $\mathbb{R}^2$ .



**Theorem 2.1 (Relative open sets).** Let  $Y \subset X$ . A subset  $E$  of  $Y$  is open relative to  $Y$  if and only if  $E = Y \cap G$  for some open subset  $G$  of  $X$ .

*Proof.* **[[Todo]]**

□

**Example 2.2 (Relative open sets).** Let  $X = \mathbb{R}$  and  $A = [0, 1]$ . Observe that  $B = [0, \frac{1}{2}) \subset A \subset X$  is open in  $A$ , but not open in  $X$ . However, there exists a  $C$  open in  $X$  such that  $B = C \cap A$ . One example is  $C = (-\frac{1}{2}, \frac{1}{2})$ .

**Definition 2.2 (Continuous).** Suppose  $X$  and  $Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Then  $f$  is **continuous** at  $p$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $d_X(x, p) < \delta$ , we have that  $d_Y(f(x), f(p)) < \varepsilon$ .

**Definition 2.3 (Uniformly Continuous).** Let  $f$  be a mapping of a metric space  $X$  into a metric space  $Y$ . We say that  $f$  is **uniformly continuous** on  $X$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $p, q \in X$  for which  $d_X(p, q) < \delta$ , we have  $d_Y(f(p), f(q)) < \varepsilon$ .

**Definition 2.4 (Dense).** TFAE:  $E$  is **dense** in  $X$  if

- (i) Every point of  $X$  is a limit point of  $E$  or a point of  $E$  (or both).
- (ii)  $\bar{E} = X$ .
- (iii)  $\forall \varepsilon > 0$  and  $\forall x \in X$  we have that  $B(x, \varepsilon) \cap E \neq \emptyset$ .

**Theorem 2.2 (Functions, inverses, and subsets).** Let  $f : X \rightarrow Y$

- (i)  $E \subset Y \Rightarrow f(f^{-1}(E)) \subset E$
- (ii)  $E \subset X \Rightarrow f(f^{-1}(E)) \supset E$

## 2.2 Basic Definitions

**Definition 2.5 (Open Cover).** Let  $I$  be an arbitrary index set. A collection  $(G_i)_{i \in I}$  of open sets  $G_i \subset X$  is called an **open cover** of  $X$  if  $X \subset \cup_{i \in I} G_i$ .

**Definition 2.6 (Compact).**  $X$  is **compact** if every open cover of  $X$  contains a finite subcover. More explicitly, for every open cover  $(G_i)_{i \in I}$  there exists  $m \in \mathbb{N}$  and  $i_1, i_2, \dots, i_m \in I$  such that  $X \subset \cup_{j=1}^m G_{i_j}$ .

**Remark.** This is also called the Heine-Borel property.

**Definition 2.7 (Compact subset).** A subset  $A \subset X$  is called **compact subset** if  $(A, d|_{A \times A})$  is a compact metric space.  $d|_{A \times A}$  is the restriction of  $d$  to  $A \times A$ .

**Theorem 2.3 (Heine-Borel).** A subset  $A \subset \mathbb{R}^n$  is compact if and only if  $A$  is closed and bounded.

**Definition 2.8 (Relatively compact or precompact).** A subset  $A \subset X$  is called **relatively compact** or **precompact** if the closure  $\bar{A} \subset X$  is compact.

**Example 2.3 (Any finite metric space is compact).** Let  $X = \{x_i\}_{i=1}^n$  be a finite metric space. Let  $J$  be an arbitrary index set, and let  $(G_j)_{j \in J}$  be an open cover of  $X$ . Therefore, each  $x_i$  must be in at least one  $G_j$ . Suppose that  $x_i \in G_{j_i}$ ,  $j_i \in J$ . Then  $\cup_{i=1}^n G_{j_i}$  is a finite subcover.

**Example 2.4 ( $K = \{0\} \cup \{1/n\}_{n=1}^\infty \subset \mathbb{R}$  is compact).** Let  $J$  be an arbitrary index set, and let  $(G_j)_{j \in J}$  be an open cover of  $K$ . First note that  $0$  must be contained in some element of the open cover; call it  $G_i$ . Since  $G_i$  is open, each element of  $G_i$  is an interior element, so there exists a ball around  $0$  of radius  $\varepsilon > 0$  contained in  $G_i$ . The ball also contains all the elements of  $K$  for which  $n > \frac{1}{\varepsilon}$ . Then, each for each of the finitely many  $n \leq \frac{1}{\varepsilon}$  there exists a  $G_{j_n}$  that contains  $\frac{1}{n}$ . Therefore every open cover has a finite subcover.

**Example 2.5 (Compactness and relative compactness in  $\mathbb{R}$ ).** Any closed and bounded interval  $[a, b]$  in  $\mathbb{R}$  is compact. Half open intervals  $[a, b)$ ,  $(a, b]$  and open intervals  $(a, b)$  in  $\mathbb{R}$  are relatively compact, since their closures are closed and bounded intervals (assuming  $b$  finite).

**Example 2.6 (Unit circle in  $\mathbb{R}^n$  compact).** The set  $C = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i|^2 = 1\right\} \subset \mathbb{R}^n$  is compact. To show closedness, consider the map  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f(\mathbf{x}) = \sum_{i=1}^n x_i^2$ ,  $\mathbf{x} \in \mathbb{R}^n$ . This map is continuous (since it is the sum of continuous functions). Then  $C = f^{-1}(\{1\})$ , and the singleton  $\{1\}$  is a closed set in  $\mathbb{R}$ . To show boundedness, note that no single element of a vector can have an absolute value greater than one. Therefore  $C \subset [-1, 1]^n$ .  $C$  is closed and bounded, and by Heine-Borel, compact.

**Example 2.7 (Orthogonal matrices).** The set of orthogonal  $n \times n$  matrices with real entries (call this  $O(n, \mathbb{R})$ ) is compact as a subset in  $\mathbb{R}^2$ . To see this, let  $M_n$  be the set of all  $n \times n$  matrices with real entries. Define a function  $f : M_n \rightarrow M_n$  where  $f(A) = A^T A$ . This mapping is continuous. To see this, first note that  $f : A \rightarrow A$  is the identity map and hence continuous. Also  $f : A \rightarrow A^T$  is continuous: this follows since  $\|A\| = \|A^t\|$ , so

$$\begin{aligned}\|f(A) - f(B)\| &= \|B^t - A^t\| \\ &= \|(B - A)^t\| \\ &= \|B - A\|\end{aligned}$$

Thus whenever  $\|B - A\| < \varepsilon$ , we have  $\|f(A) - f(B)\| < \varepsilon$  (hence we set  $\delta = \varepsilon$ ). The product of two continuous functions is continuous.

Since an orthogonal matrix  $O$  has inverse  $O^T$ , we have that  $f^{-1}(I) = O(n, \mathbb{R})$ . The continuity of  $f$  implies that  $O(n, \mathbb{R})$  is closed since it is the preimage of a singleton (which is a closed set).

To show boundedness, recall that the columns of orthogonal matrices are orthonormal. Therefore, the absolute value of an element cannot be larger than 1 in absolute value. Thus  $O(n, \mathbb{R}) \subset [-1, 1]^{n^2}$ . Then by Heine-Borel,  $O(n, \mathbb{R})$  is compact.

**Theorem 2.4 (Compact subsets of metric spaces are closed.).**

*Proof.* Let  $X$  be a metric space and  $K$  a compact subset of  $X$ . To show  $K$  is closed, we will show that its complement  $K^c$  is open. To do this, we must show that for all  $p \in K^c$ , there exists a neighborhood of  $p$  completely contained in  $K^c$  (and hence does *not* intersect  $K$ ).

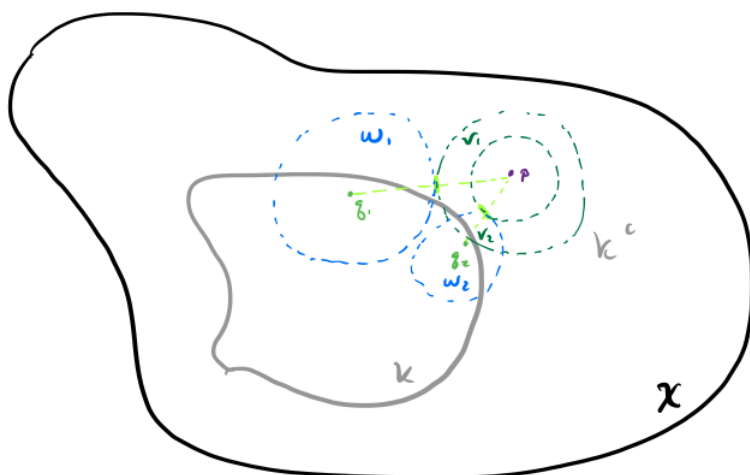
Let  $q \in K$ . We will construct two types of neighborhoods:

$$\begin{aligned}W_q &= \left\{x \in X \mid d(x, q) < \frac{1}{2}d(p, q)\right\} && \text{(neighborhood of } q) \\ V_q &= \left\{x \in X \mid d(x, p) < \frac{1}{2}d(p, q)\right\} && \text{(neighborhood of } p)\end{aligned}$$

Notice that the union of all  $W_q$  forms an open cover of  $K$ . Since  $K$  is compact, this open cover must have a finite subcover,  $W = \bigcup_{i=1}^n W_{q_i}$ . Define  $V = \bigcap_{i=1}^n V_{q_i}$ , which is still a neighborhood of  $p$ . By construction (since we have used open balls),  $V_{q_i} \cap W_{q_i} = \emptyset$ . Since  $V \subseteq V_{q_i}$ ,  $V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \emptyset$ . Therefore

$$V \cap W = (V \cap W_1) \cup \dots \cup (V \cap W_n) = \emptyset$$

Since  $K \subseteq W$ , we have that  $V \subseteq K = \emptyset$ . Thus  $V$  is a neighborhood of  $p$  completely contained in  $K^c$ . Since  $p$  was arbitrary,  $K^c$  is open, and  $K$  is closed.

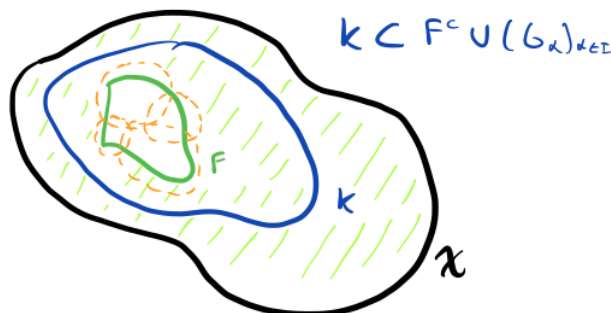


$$\begin{aligned} V &:= \bigcap_{i=1}^{\infty} V_i \\ W &:= \bigcap_{i=1}^{\infty} W_i \\ V_i \cap W_i &= \emptyset \\ \Rightarrow V \cap W &= \emptyset. \end{aligned}$$

□

**Theorem 2.5** (Closed subsets of compact sets are compact.).

*Proof with compactness.* Let  $F \subset K \subset X$ , where  $F$  is closed (relative to  $X$ ) and  $K$  is compact. Let  $(G_\alpha)_{\alpha \in I}$  be an open cover of  $F$ . Since  $F$  is closed,  $F^c$  is open and  $F^c \cup (G_\alpha)_{\alpha \in I}$  is an open cover of  $K$ .  $K$  is compact, so every open cover has a finite subcover:  $K \subset F^c \cup (G_{\alpha_i})_{i=1}^n$ . Since  $F \subset K$ , this finite open cover also covers  $F$ , but clearly we don't need  $F^c$  in the cover. Therefore  $F$  has a finite subcover:  $F \subset (G_{\alpha_i})_{i=1}^n$ .



□

*Proof with sequential compactness.* Let  $F \subset K \subset X$ , where  $F$  is closed (relative to  $X$ ) and  $K$  is compact. Let  $(x_n)$  be a sequence of points in  $F$  (so is also in  $X$ ). Since  $X$  is sequentially compact, there is a subsequence  $(x_{n_k})$  converging to some point  $x \in X$ . Since  $F$  is closed,  $x \in F$ . Therefore  $F$  is sequentially compact since every sequence in  $F$  has a convergent subsequence. □

**Theorem 2.6** (The intersection of a nested sequence of compact nonempty sets is compact and nonempty.).



*Proof.* Let  $(A_n)$  be a sequence of compact nonempty sets. **Compact:** We know that each  $A_n$  is closed. The intersection of closed sets is closed, so  $\cap A_n$  is closed. Notice that  $\cap A_n \subset A_1$  is a closed subset of a compact set, so is also compact.

**Nonempty:** Since each  $A_n$  is nonempty, fix an  $a_n \in A_n$ . The sequence  $(a_n) \subset A_1$ . Sequential compactness of  $A_1$  implies that there is a subsequence  $(a_{n_k})$  converging to some point  $a \in A_1$ . Notice that this limit must also be in  $A_2$ , since the sequence  $(a_{n_k})$  is also in  $A_2$  (except for potentially the first term, which does not affect convergence). This holds for all  $A_n$ , so  $a \in \cap A_n$ , which shows the intersection is nonempty.  $\square$

**Theorem 2.7.** *Let  $X$  be a compact metric space. Then there exists a countable, dense set  $E \subset X$ .*

*Proof.* We will show that  $\forall \varepsilon > 0$  and  $\forall x \in X$  we can find a set  $E$  such that for each open ball in  $X$  we have that  $B(x, \varepsilon) \cap E \neq \emptyset$ . Fix  $\varepsilon > 0$  and define an open cover of  $X$  as follows:

$$B_n = \bigcup_{x \in X} B(x, \frac{1}{n}) \quad (2.1)$$

$X$  is compact so the open cover  $B_n$  must have a finite subcover. Let  $E$  be the union of the centers of the balls of each finite subcover.  $E$  is the countable union of finite sets, so it is countable. Now fix  $x \in X$ . Choose  $n$  such that  $\varepsilon < \frac{1}{n}$ . Since  $B_n$  covers  $X$ , there must be some ball centered at a point of  $E$ , call it  $y$ , that contains  $x$ . Thus  $d(x, y) < \frac{1}{n} < \varepsilon$ . Thus  $y \in B(x, \varepsilon) \cap E$ .  $\square$

## 2.3 Compactness and Continuity

**Theorem 2.8 (Continuous mappings on compact sets are uniformly continuous.).** *Let  $X, Y$  be metric spaces and assume  $X$  is compact. If  $f : X \rightarrow Y$  is continuous, then it is uniformly continuous.*

*Proof.* Fix  $\varepsilon > 0$ . **Goal:** We need to find a  $\delta > 0$  such that for all  $x, y \in X$  for which  $d_X(x, y) < \delta$ , we have  $d_Y(f(x), f(y)) < \varepsilon$ .

**Use continuity to create balls of points (which cover)  $X$  and are mapped by the function:** Since  $f$  is continuous, we know that for each  $x \in X$ , there exists a number  $\delta_x > 0$  such that for all  $y \in X$  for which  $d_X(x, y) < \delta_x$ , we have that  $d_Y(f(x), f(y)) < \varepsilon/2$ . Now, let  $B_x$  be a ball of radius  $\delta_x/2$  centered at  $x$ . Formally, we can write

$$B_x = B(x, \delta_x/2) = \{y \in X \mid d_X(x, y) < \delta_x/2\}$$

Then  $(B_x)_{x \in X}$  is an open cover of  $X$ .

**Use compactness to select a finite subcover of these balls and use the smallest radius to create uniformity:** By compactness, there exists a finite subcover of  $X$ . Let  $x_1, \dots, x_m$  be the  $x$  which generate this finite subcover. Define  $\delta > 0$  as follows

$$\delta = \frac{1}{2} \min(\delta_{x_1}, \dots, \delta_{x_m}) \quad (2.2)$$

(notice that  $\delta$  is indeed positive, since we are taking the minimum of finitely many positive numbers).

**Show that our choice of  $\delta$  works:** Fix  $x, y \in X$  such that  $d_X(x, y) < \delta$ . There exists an index

$i \in \{1, \dots, m\}$  such that  $x \in B_{x_i}$  (i.e., an element of  $X$  is in some ball of the finite subcover). Then

$$d_X(x_i, y) \leq d_X(x_i, x) + d_X(x, y) < \frac{1}{2}\delta_{x_i} + \delta < \delta_{x_i} \quad (2.3)$$

Now, using the definition of  $\delta_{x_i}$ , we have that

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(y)) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \quad (2.4)$$

since  $d_X(x_i, x) < \delta_x/2 < \delta_x$  so continuity implies  $d_Y(f(x), f(x_i)) < \varepsilon/2$ , and  $d_X(x_i, y) < \delta_{x_i}$  so continuity implies  $d_Y(f(x_i), f(y)) < \varepsilon/2$ .  $\square$

**Theorem 2.9** (The image of a continuous function which maps from a compact set is compact). *Let  $X, Y$  be metric spaces and assume  $X$  is compact. If  $f : X \rightarrow Y$  is continuous, then  $f(X) \subset Y$  is compact.*

*Proof with compactness.* Let  $(V_i)_{i \in I}$  be an open cover of  $f(X)$ . Since  $f$  is continuous, we have that  $U_i = f^{-1}(V_i) \subset X$  is open for each  $i$ . Note that  $U \subset f^{-1}(f(U))$  for every  $U \subset X$ . Therefore

$$X \subset f^{-1}(f(X)) \subset \cup_{i \in I} f^{-1}(V_i) = \cup_{i \in I} U_i \quad (2.5)$$

which shows that  $\cup_{i \in I} U_i$  is an open cover of  $X$  (the second subset relation uses that applying  $f^{-1}$  preserves unions). Since  $X$  is compact, there are finitely many indices, say up to  $m$ , such

$$X \subset \bigcup_{j=1}^m U_{i_j} \quad (2.6)$$

Applying  $f$  preserves inclusions and  $V \supset f(f^{-1}(V))$ , so we have that

$$f(X) \subset \bigcup_{j=1}^m f(U_{i_j}) \subset \bigcup_{j=1}^m V_{i_j} \quad (2.7)$$

Thus every open cover of  $f(X)$  has a finite subcover, so  $f(X)$  is compact.  $\square$

*Proof with sequential compactness.* Let  $(y_n)$  be a sequence in  $f(X) \subset Y$ . For each  $n \in \mathbb{N}$  choose a point  $x_n$  such that  $f(x_n) = y_n$ . By sequential compactness of  $X$ , there is some subsequence  $(x_{n_k})$  converging to a point  $x \in X$ . The continuity of  $f$  implies that  $f(x_{n_k})$  converges to  $f(x) \in f(X)$ . Therefore every sequence in  $f(X)$  has a convergent subsequence, so  $f(X)$  is sequentially compact.  $\square$

**Theorem 2.10** (Functions from compact sets to the reals achieve their minimum and maximum values). *Let  $X$  be a compact metric space and  $f : X \rightarrow \mathbb{R}$  a continuous function. Then there exists  $x_0 \in X$  such that  $f(x_0) = \sup_{x \in X} f(x)$ .*

*Proof.* Since  $X$  is compact, we know that the image  $f(X)$  is also compact. Since  $f(X) \subset \mathbb{R}$ , the Heine-Borel theorem tells us that  $f(X)$  is closed and bounded. Completeness of the real numbers (every nonempty subset of the real numbers that is bounded above has a supremum in  $\mathbb{R}$ ). Since  $f(X)$  is closed, this supremum must be contained in  $f(X)$ . Therefore there exists  $x_0 \in X$  such that  $f(x_0) = \sup_{x \in X} f(x)$ .  $\square$

## 2.4 Sequential Compactness and Total Boundedness

**Definition 2.9** (Sequentially compact). A metric space  $X$  is **sequentially compact** if every sequence in  $X$  has a convergent subsequence.

**Remark.** This is also called the Bolzano-Weierstrass property. Recall that the Bolzano-Weierstrass theorem states that every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

**Example 2.8** (Finite sets are sequentially compact). Suppose  $(x_n)$  is a sequence contained in a finite set. Then this sequence must repeat a term infinitely often. Define the subsequence which is constant at this term: this subsequence clearly converges.

**Theorem 2.11** ((Sequentially) compact sets are closed and bounded).

*Proof.* Let  $A$  be a sequentially compact subset of a metric space  $X$ .

**Closed:** Let  $p$  be a limit point of  $A$ . Then there is a sequence  $(a_n)$  in  $A$  that converges to  $p$ . Sequential compactness of  $A$  implies that there is a subsequence of  $(a_n)$  which converges to some  $q \in A$ . However every subsequence of a convergent sequence must converge to the same limit as the full sequence, which implies  $p = q \in A$ . Therefore  $A$  is closed.

**Bounded:** Fix a point  $x \in X$ . Suppose  $A$  is not bounded. Then for each  $n \in \mathbb{N}$  there must be some point  $a_n$  such that  $d(x, a_n) \geq n$ . However, sequential compactness implies that some subsequence  $(a_{n_k})$  must converge. Convergent sequences are bounded, so we cannot have that  $d(x, a_{n_k}) \rightarrow \infty$  as  $k \rightarrow \infty$ . Thus the sequence  $(a_n)$  cannot behave as assumed, and it must be that  $A$  is bounded for some  $r \in \mathbb{R}$ . □

**Definition 2.10** (Bounded metric space). A metric space  $X$  is **bounded** if it fits in a single fixed ball. More precisely, there exists some  $x_0 \in X$  and  $r > 0$  such that  $X \subseteq B(x_0, r)$ .

**Definition 2.11** (Totally bounded). A metric space  $X$  is **totally bounded** if for every  $\varepsilon > 0$  there exist finitely many balls of radius  $\varepsilon$  that cover  $X$ .

**Claim 2.1** (Totally bounded implies bounded). Let  $X$  be a totally bounded metric space. Then it is bounded.

*Proof.* Fix  $\varepsilon > 0$  and  $x, y \in X$ . Total boundedness implies there exists points  $(x_i)_{i=1}^n$  such that  $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$ . Suppose  $x \in B(x_i, \varepsilon)$  and  $y \in B(x_j, \varepsilon)$ . Then

$$d(x, y) \leq d(x, x_i) + d(x_i, x_j) + d(x_j, y) < d(x_i, x_j) + 2\varepsilon \quad (2.8)$$

There are only finitely many values of  $d(x_i, x_j)$  we can set  $M = \max_{i,j} d(x_i, x_j)$ . Therefore  $d(x, y) < M + 2\varepsilon$ . Therefore  $X$  is bounded. □

**Example 2.9** (Closed and bounded interval in  $\mathbb{R}$  is totally bounded). A closed and bounded interval  $I = [a, b] \subset \mathbb{R}$  is totally bounded.

**Example 2.10** (Bounded but not totally bounded). The following are examples of metric spaces which are bounded but not totally bounded.

- (i)  $\ell^1$ -space is defined to be the collection of all sequences  $(a_n)_n$  with  $\sum_{n=1}^{\infty} |a_n| < \infty$ . We define a distance  $d$  on  $\ell^1$  as follows:  $d((a_n)_n, (b_n)_n) = \sum_{n=1}^{\infty} |a_n - b_n|$ . The unit ball of  $\ell^1$  centered at the zero-element (call this call  $B_1$ ) (i.e., the zero sequence) is bounded but not totally bounded. More precisely,

$$B_1 = \left\{ (a_n)_n \left| \sum_{n=1}^{\infty} |a_n| \leq 1 \right. \right\} \quad (2.9)$$

Boundedness is clear. Consider the set of sequences  $A = \{a_n\}$  where each  $a_n$  is zero except for a 1 in the  $n$ -th entry. Each  $a_n \in B_1$ . However,

$$d(a_n, a_m) = \begin{cases} 2 & n \neq m \\ 0 & n = m \end{cases} \quad (2.10)$$

If we take balls of radius  $\varepsilon = \frac{1}{2}$ , then a ball can cover at most one sequence in  $A$ . Therefore we can't cover  $A$  with finitely  $\varepsilon$ -balls and hence can't cover  $B_1$  with finitely many  $\varepsilon$ -balls, so that  $B_1$  is not totally bounded.

- (ii)  $\ell^\infty$ -space is defined to be the collection of all sequences  $(a_n)_n$  with  $\sup_n |a_n| < \infty$ . We define a distance  $d$  on  $\ell^\infty$  as follows:  $d((a_n)_n, (b_n)_n) = \sup_n |a_n - b_n|$ . Using the same set of sequences as the previous example, note that for all  $n$  we have that

$$d(a_n, \{0\}_{n=1}^{\infty}) = 1 \quad (2.11)$$

so that  $A \subset B_1$  (with the sup norm). Further,

$$d(a_n, a_m) = \begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases} \quad (2.12)$$

But again, if we take balls of radius  $\varepsilon = \frac{1}{2}$ , then a ball can cover at most one sequence in  $A$ . Therefore we can't cover  $A$  with finitely  $\varepsilon$ -balls and hence can't cover  $B_1$  with finitely many  $\varepsilon$ -balls, so that  $B_1$  is not totally bounded.

**Theorem 2.12 (Characterizations of compactness).** *Let  $X$  be a metric space. The following are equivalent:*

- (i)  $X$  is compact.
- (ii)  $X$  is sequentially compact.
- (iii)  $X$  is totally bounded and complete.

*Proof of  $X$  is compact  $\Rightarrow X$  is sequentially compact.* We argue by contradiction. Suppose  $X$  is compact but not sequentially compact. Thus there must exist some sequence  $(x_n)_n \subset X$  without a convergence subsequence. Let  $A$  be the range of the sequence (more explicitly,  $A = \{x_n \mid n \in \mathbb{N}\}$ ). Note that  $A$  has to be an infinite (if  $A$  were finite, then there would be a constant subsequence, which is convergent).

Further,  $A$  cannot have any limit points (by definition, a point is a limit point of a sequence if there exists a subsequence converging to it). Therefore for each  $x_n$ , we can find an open ball centered at  $x_n$  that only intersects  $A$  at  $x_n$ :  $B_n \cap A = \{x_n\}$ .  $A$  is also closed (since it has no limit

points), so that  $X - A$  is open. We can construct an open cover of  $X$  as:

$$X \subseteq \bigcup_n B_n \cup \{X - A\} \quad (2.13)$$

Compactness of  $A$  implies this open cover has a finite subcover. Therefore the finite subcover can only contain finitely many of the sets in  $\bigcup_n B_n$ , so can only contain finitely many of the points in  $A$ , which is a contradiction to the covering.  $\square$

*Proof of  $X$  is sequentially compact  $\Rightarrow X$  is totally bounded and complete.* Suppose  $X$  is sequentially compact.

**Complete:** Fix  $\varepsilon > 0$ . Let  $(x_n)_n \subset X$  be a Cauchy sequence. Sequential compactness implies  $(x_n)_n$  has a convergent subsequence. Call this subsequence  $(x_{n_k})$  and its limit  $x$ . Then there exists an  $N$  such that  $d(x_{n_k}, x) < \varepsilon$  for all  $n_k \geq N$ . Since  $(x_n)_n$  is Cauchy, there exists an  $N$  such that  $n, m \geq N$  implies that  $d(x_n, x_m) < \varepsilon$ . But then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < 2\varepsilon \quad (2.14)$$

for all  $n, n_k \geq \max\{N, M\}$ . Therefore  $(x_n)_n$  converges, so  $X$  is complete. **In words, Cauchy sequences with convergence subsequences also converge.**

**Totally Bounded:** We argue by contradiction. Suppose  $X$  is not totally bounded. Then there exists an  $\varepsilon > 0$  such that  $X$  cannot be covered by finitely many balls of radius  $\varepsilon$ . We know that 1 ball of radius  $\varepsilon$  cannot cover  $X$ . Therefore there must be a point of  $X$  outside of this ball: call it  $x_1$ . Similarly, 2 balls of radius  $\varepsilon$  cannot cover  $X$ , so pick a point outside of these two balls and call it  $x_2$ . Proceeding in this manner generates an infinite sequence. Now consider the  $n$ -th term of this sequence. When we choose  $n + 1$ , we must be able to choose a point such that  $d(x_i, x_{n+1}) \geq \varepsilon$  for all  $i \in \{1, \dots, n\}$ . Otherwise, we would have that  $X \subseteq \bigcup_{i=1}^n B(x_i, \varepsilon)$ , which would be a contradiction to  $X$  not being totally bounded. Thus we can construct such a sequence.

Next, since  $X$  is sequentially compact, we must be able to find a convergence subsequence. However, the sequence we've constructed has  $d(x_i, x_j) \geq \varepsilon$  for all  $i, j \in \mathbb{N}$  (hence no subsequence can be Cauchy, and we know that convergence sequences are Cauchy). Therefore we have reached a contradiction and it holds that  $X$  is totally bounded.  $\square$

*Proof of  $X$  is totally bounded and complete  $\Rightarrow X$  is sequentially compact.* Suppose  $X$  is totally bounded and complete. Let  $(x_n)_n \subset X$  be a sequence. **We will construct an convergent subsequence.** By the definition of total boundedness, for all  $\varepsilon > 0$ ,  $X$  can be covered by finitely many balls of radius  $\varepsilon$ . **Observation:** One of these balls must contain infinitely many points of  $(x_n)_n$ . This inspires the following process:

- (i) **Step 1:** Cover  $X$  with balls of radius 1. One of these balls must contain infinitely many points of  $(x_n)_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(0)})_n$ .
- (ii) **Step 2:** Cover  $X$  with balls of radius  $\frac{1}{2}$ . One of these balls must contain infinitely many points of  $(x_n^{(0)})_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(1)})_n$ .
- (iii) **Step  $n$ :** Cover  $X$  with balls of radius  $\frac{1}{2^n}$ . One of these balls must contain infinitely many points of  $(x_n^{(n-1)})_n$ . These infinitely many points form a subsequence, call it  $(x_n^{(n)})_n$ .

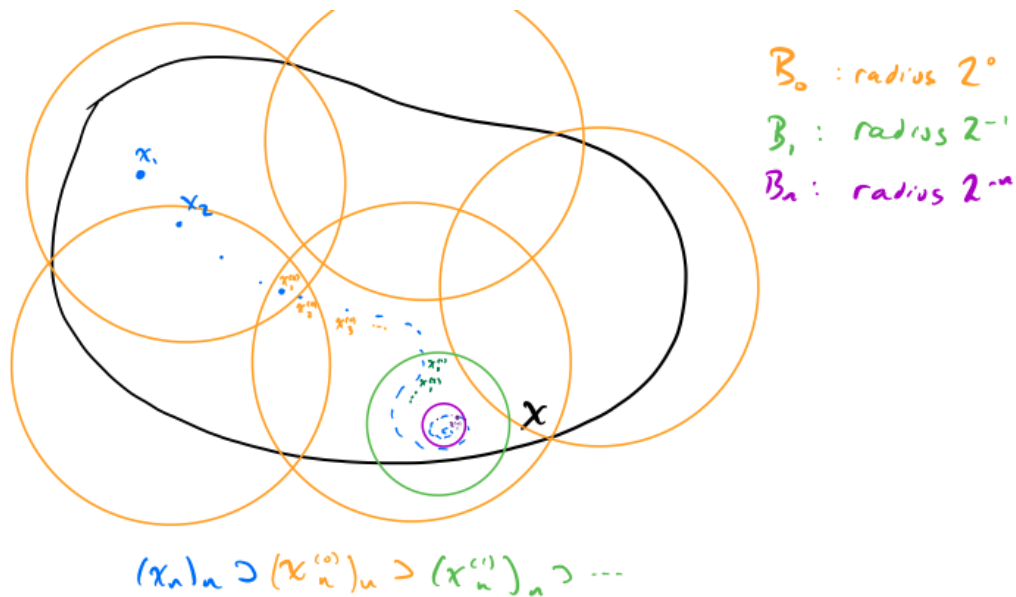
Therefore we have founded nested sequences:  $(x_n^{(0)})_n \subset (x_n^{(1)})_n \subset \dots \subset (x_n^{(n-1)})_n \dots$ . Set  $a_n = x_n^{(n)}$  (which is a subsequence of  $(x_n)_n$ ).

**Now we show that  $a_n$  is a Cauchy sequence.** Fix  $\varepsilon > 0$  and choose  $N$  such that  $2^{-N+1} < \varepsilon$ . Now for  $m > n \geq N$  we have that

$$d(a_m, a_n) < 2 \cdot 2^{-n} < 2^{-N+1} < \varepsilon \quad (2.15)$$

since  $a_m$  and  $a_n$  are contained in the same ball of radius  $2^{-n}$ . Therefore  $(a_n)_n$  is a Cauchy sequence, and completeness implies it converges.

**We have found a convergence subsequence, so  $X$  is sequentially compact.**



□

## 2.5 Equicontinuity and the Arzela-Ascoli Theorem

[[To prove relatively compact, we construct a sequence of functions that converges uniformly.]]

### 3 Approximation Theory and Fourier Series

#### 3.1 Trigonometric Polynomials

**Definition 3.1** (Trigonometric Polynomial, Degree). A **trigonometric polynomial** is a function of the form

$$f(x) = \sum_{n=-N}^N c_n e^{2\pi i n x} \quad x \in \mathbb{R} \quad (3.1)$$

where  $N \in \mathbb{N}$  and  $c_n \in \mathbb{C}$ . The largest  $N$  for which either  $c_N$  or  $c_{-N}$  is non-zero is called the **degree** of  $f$ .

**Claim 3.1** (Alternative form of trigonometric polynomial). *Every trigonometric polynomial can also be written in the form*

$$f(x) = a_0 + \sum_{n=1}^N a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \quad (3.2)$$

for coefficients  $a_n, b_n$ .

*Proof.* Using Euler's identity, recall that

$$e^{2\pi i n x} = \cos(2\pi n x) + i \sin(2\pi n x) \quad (3.3)$$

Then, for  $x \in \mathbb{R}$ ,

$$\begin{aligned} f(x) &= \sum_{n=-N}^N c_n e^{2\pi i n x} \\ &= \sum_{n=-N}^N [\cos(2\pi n x) + i \sin(2\pi n x)] \\ &= c_0 + \sum_{n=1}^N [c_n (\cos(2\pi n x) + i \sin(2\pi n x)) + c_{-n} (\cos(2\pi i(-n)x) + i \sin(2\pi i(-n)x))] \\ &= c_0 + \sum_{n=1}^N [(c_n + c_{-n}) \cos(2\pi n x) + i(c_n - c_{-n}) \sin(2\pi n x)] \end{aligned}$$

Therefore the claim alternative holds for  $(1 \leq n \leq N)$

$$\begin{aligned} a_0 &= c_0 \\ a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned}$$

□

**Definition 3.2** (Approximation of Unity). A sequence of 1-periodic continuous functions  $(k_n)_n$  is called **approximation of unity** if for all 1-periodic continuous functions  $f$  we have that  $f * k_n$  converges uniformly to  $f$  on  $\mathbb{R}$ . That is

$$\sup_{x \in \mathbb{R}} |f * k_n(x) - f(x)| \rightarrow 0 \quad (3.4)$$

as  $n \rightarrow \infty$

**Theorem 3.1** (Sufficient conditions for approximation of unity). Let  $(k_n)_n$  be a sequence of 1-periodic continuous functions such that

- (i) Non-negative:  $k_n(x) \geq 0$ .
- (ii) Integrates to 1:  $\int_{-1/2}^{1/2} k_n(t) dt = 1$ .
- (iii) "Mass" of  $k_n$  concentrated near the origin: For all  $1/2 \geq \delta > 0$  we have

$$\int_{-\delta}^{\delta} k_n(t) dt \rightarrow 1 \text{ as } n \rightarrow \infty \quad (3.5)$$

*Proof.* Let  $f$  be a function that is 1-periodic and continuous. **What can we say about  $f$ ? Using continuity:** Consider the interval  $[-1/2, 1/2]$ .  $f$  is continuous on this compact set so that on  $[-1/2, 1/2]$  it is

- (i) Bounded
- (ii) Uniformly continuous

**Using periodicity:** by periodicity, on all of  $\mathbb{R}$ ,  $f$  is

- (i) Bounded
- (ii) Uniformly continuous

Thus there exists a  $\delta > 0$  such that

$$|f(x-t) - f(x)| \leq \varepsilon/2 \text{ for all } |t| < \delta, x \in \mathbb{R} \quad (3.6)$$

Then we can write

$$f * k_n(x) - f(x) = \int_{-1/2}^{1/2} (f(x-t) - f(x)) k_n(t) dt \quad (3.7)$$

Let's break this integral up into two pieces:

$$\begin{aligned} A &= \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t) dt \\ B &= \int_{1/2 \geq \delta > 0} (f(x-t) - f(x)) k_n(t) dt \end{aligned}$$

To bound  $A$ , we use uniform continuity and (ii):

$$|A| = \left| \int_{|t| \leq \delta} (f(x-t) - f(x)) k_n(t) dt \right|$$

Incomplete.

□

**Corollary 3.2** (Fejér kernel an approximation of unity).



*Proof.* We must verify the hypotheses of the above theorem.

Incomplete.

□

**Theorem 3.3** (Partial sum convoluted with 1-periodic, continuous function converges in 2-norm).

Let  $f$  be a 1-periodic and continuous function. Then

$$\lim_{N \rightarrow \infty} \|S_N f - f\|_2 = 0 \quad (3.8)$$

*Proof.* Let  $\varepsilon > 0$ .

Incomplete.

□

### 3.2 Weierstrass Theorem

**Theorem 3.4** (Weierstrass Theorem). For every  $f \in C([a, b])$  there exists a sequence of polynomials that converges uniformly to  $f$ .

*Proof.* Let  $\varepsilon > 0$ .

Incomplete.

□

**Remark.** This shows that polynomials are dense in  $C([a, b])$ .

### 3.3 Stone-Weierstrass Theorem

**Theorem 3.5** (Stone-Weierstrass Theorem (Sufficient conditions for a subset of  $C(K)$ ,  $K$  compact, to be dense)). Suppose

- $K$  is a compact metric space.
- $\mathcal{A} \subset C(K)$  such that

(i)  $\mathcal{A}$  is a self-adjoint algebra: for  $f, g \in \mathcal{A}$ ,  $c \in \mathbb{C}$ , we have

$$f + g \in \mathcal{A}, f \cdot g \in \mathcal{A}, c \cdot f \in \mathcal{A}, \bar{f} \in \mathcal{A} \quad (3.9)$$

(ii)  $\mathcal{A}$  separates points: for all  $x, y \in K$  with  $x \neq y$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

(iii)  $\mathcal{A}$  vanishes nowhere: for all  $x \in K$ , there exists  $f \in \mathcal{A}$  such that  $f(x) \neq 0$ .

Then  $\mathcal{A}$  is dense in  $C(K)$  (that is,  $\bar{\mathcal{A}} = C(K)$ ).

**Remark.** Polynomials and trigonometric polynomials satisfy the conditions of the Stone-Weierstrass Theorem (Theorem 3.5).

## 4 Linear Operators and Derivatives

## 5 Differential Calculus in $\mathbb{R}^n$

## 6 The Baire category theorem

### A Review From Elementary Analysis

#### A.1 The Real and Complex Number System

**Definition A.1 (Supremum, Infimum).** Let  $S$  be an ordered set,  $E \subset S$ , and  $E$  be bounded above. Suppose there exists an  $\alpha \in S$  with the following properties:

- (i)  $\alpha$  is an upper bound of  $E$ .
- (ii) If  $\gamma < \alpha$  then  $\gamma$  is not an upper bound of  $E$ .

Then  $\alpha$  is called the least upper bound of  $E$  or **supremum** of  $E$  and we write  $\alpha = \sup E$ . Similarly,  $\alpha$  is the greatest lower bound of  $E$  or **infimum** of  $E$  if

- (i)  $\alpha$  is a lower bound of  $E$ .
- (ii) If  $\beta > \alpha$  then  $\beta$  is not a lower bound of  $E$ .

and we write  $\alpha = \inf E$ .

**Definition A.2 (Limit Superior, Inferior).** Let  $(x_n)$  be a sequence of real numbers.

- (i) The **limit superior** of the sequence is defined by

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} x_m \right) \quad (\text{A.1})$$

or

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \left( \sup_{m \geq n} x_m \right) = \inf \{ \sup \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.2})$$

**Alternatively**, the limit superior of the sequence is the smallest  $b \in \mathbb{R}$  such that  $\forall \varepsilon > 0$   $\exists N$  such that  $x_n < b + \varepsilon \forall n > N$ . Thus, any number larger than the limit superior is an upper bound for the sequence after a finite number of terms (hence, only a finite number of elements are greater than  $b + \varepsilon$ ).

**Alternatively**, the limit superior of the sequence is the supremum of the set of subsequential limits.

- (ii) The **limit inferior** of the sequence is defined by

$$\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{m \geq n} x_m \right) \quad (\text{A.3})$$

or

$$\liminf_{n \rightarrow \infty} x_n = \sup_{n \geq 0} \left( \inf_{m \geq n} x_m \right) = \sup \{ \inf \{ x_m \mid m \geq n \} \mid n \geq 0 \} \quad (\text{A.4})$$

**Theorem A.1 (Properties of limit superiors).** Let  $(x_n)$  and  $(y_n)$  be sequence of real numbers. Then

$$(i) \limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \text{ (as long as the RHS is not of the form } \infty - \infty)$$

*Proof.* We prove each item in turn.

(i) We have that

$$\limsup_{n \rightarrow \infty} (x_n + y_n) = \lim_{n \rightarrow \infty} \left( \sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} \right) \quad (A.5)$$

Then

$$M_n := \sup_{m \geq n} \{x_m + y_m, x_{m+1} + y_{m+1}, \dots\} = \sup_{m \geq n} \{x_m + y_m\} \leq \sup_{m \geq n} \{x_m\} + \sup_{m \geq n} \{y_m\} \quad (A.6)$$

Take the limit of both sides to get

$$\lim_{n \rightarrow \infty} M_n \leq \lim_{n \rightarrow \infty} \sup_{m \geq n} \{x_m\} + \lim_{n \rightarrow \infty} \sup_{m \geq n} \{y_m\} \quad (A.7)$$

Thus

$$\limsup_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \limsup_{n \rightarrow \infty} y_n \quad (A.8)$$

□

## A.2 Basic Topology

In what follows, assume  $X$  is a metric space.

**Definition A.3 (Limit point).** A point  $p$  is a **limit point** of the set  $E$  if every neighborhood of  $p$  contains a point  $q \neq p$  such that  $q \in E$ .

**Definition A.4 (Closed).**  $E$  is **closed** if every limit point of  $E$  is a point of  $E$ .

**Definition A.5 (Interior).** A point  $p$  is an **interior point** of  $E$  if there is a neighborhood  $N$  of  $p$  such that  $N \subset E$ .

**Definition A.6 (Open).**  $E$  is **open** if every point of  $E$  is an interior point of  $E$ .

**Definition A.7 (Bounded).**  $E$  is **bounded** if there is a real number  $M$  and a point  $q \in X$  such that  $d(p, q) < M$  for all  $p \in E$ .

**Definition A.8 (Separated, Connected).** Two subsets  $A$  and  $B$  of a metric space  $X$  are said to be **separated** if both  $A \cap \bar{B}$  and  $\bar{A} \cap B$  are empty (i.e., if no point of  $A$  lies in the closure of  $B$  and no point of  $B$  lies in the closure of  $A$ ). A set  $E \subset X$  is said to be **connected** if  $E$  is not a union of two nonempty separated sets.

## A.3 Numerical Sequences and Series

### A.3.1 Sequences

**Definition A.9 (Convergent Sequence).** A sequence  $(p_n)$  in a metric space  $X$  is said to **converge** if there is a point  $p \in X$  such that for every  $\varepsilon > 0$  there is an integer  $N$  such that  $n \geq N$  implies  $d(p_n, p) < \varepsilon$ .

**Definition A.10 (Subsequence, Subsequential Limit).** Given a sequence  $(p_n)$ , consider a sequence  $(n_k)$  of positive integers such that  $n_1 < n_2 < n_3 < \dots$ . Then the sequence  $(p_{n_i})$  is called a **subsequence** of  $(p_n)$ . If  $(p_{n_i})$  converges, its limit is called a **subsequential limit** of  $(p_n)$ .

Observations:

- $(p_n)$  converges to  $p$  if and only if every subsequence of  $(p_n)$  converges to  $p$ .

**Theorem A.2 (Sequences in compact metric spaces have a convergent subsequence).** If  $(p_n)$  is a sequence in a compact metric space  $X$ , then some subsequence of  $(p_n)$  converges to a point of  $X$ .

**Theorem A.3 (Bolzano-Weierstrass).** Every bounded sequence in  $\mathbb{R}^k$  has a convergent subsequence.

**Definition A.11 (Cauchy Sequence).** A sequence  $(p_n)$  in a metric space  $X$  is said to be a **Cauchy Sequence** if for every  $\varepsilon > 0$  there is an integer  $N$  such that  $d(p_n, p_m) < \varepsilon$  if  $n \geq N$  and  $m \geq N$ .

**Definition A.12 (Diameter).** Let  $E$  be a nonempty subset of a metric space  $X$ , and let  $S$  be the set of all real numbers of the form  $d(p, q)$  with  $p \in E$  and  $q \in E$ . The sup of  $S$  is called the **diameter** of  $E$ .

**Theorem A.4 (Facts about Cauchy sequences).** We have that

- (i) In any metric space  $X$ , every convergent sequence is a Cauchy sequence.
- (ii) If  $X$  is a compact metric space and if  $(p_n)$  is a Cauchy sequence in  $X$ , then  $(p_n)$  converges to some point of  $X$ .
- (iii) In  $\mathbb{R}^k$ , every Cauchy sequence converges.

**Definition A.13 (Complete).** A metric space in which every Cauchy sequence converges is **complete**.

**Definition A.14 (Monotonically increasing, decreasing).** A sequence  $(s_n)$  of real numbers is said to be

- (i) **Monotonically increasing** if  $s_n \leq s_{n_1}$  for all  $n$ .
- (ii) **Monotonically decreasing** if  $s_n \geq s_{n_1}$  for all  $n$ .

**Theorem A.5 (Convergence of monotonic sequences).** Let  $(s_n)$  be a monotonic sequence. Then  $(s_n)$  converges if and only if it is bounded.

### A.3.2 Series

**Definition A.15 (Convergent Series).** Let  $\sum_{n=1}^{\infty} a_n$  be an infinite series. Define  $s_n = \sum_{k=1}^n a_k$  to be the  $n$ th partial sum of the series. If the sequence of partial sums  $\{s_n\}$  converges to  $s$ , we say the series **converges**.

**Theorem A.6 (Cauchy Criterion for Series).**  $\sum a_n$  converges iff

$$\forall \varepsilon > 0 \exists N \text{ s.t. } m, n \geq N \Rightarrow \left| \sum_{k=n}^m a_k \right| \leq \varepsilon \quad (\text{A.9})$$

**Theorem A.7 (Necessary condition for convergence: individual terms of series go to 0.).** If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Theorem A.8.** A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

**Theorem A.9 (Root Test).** Given  $\sum a_n$ , let  $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . Then

- (i) If  $\alpha < 1$ ,  $\sum a_n$  converges.
- (ii) If  $\alpha > 1$ ,  $\sum a_n$  diverges.
- (iii) If  $\alpha = 1$ , the test gives no information.

## A.4 Continuity

**Definition A.16 (Limit).** Let  $X$  and  $Y$  be metric spaces,  $E \subset X$ ,  $f$  map  $E$  into  $Y$ , and  $p$  be a limit point of  $E$ . We write  $f(x) \rightarrow q$  as  $x \rightarrow p$  or  $\lim_{x \rightarrow p} f(x) = q$  if there is a point  $q \in Y$  such that for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $0 < d_X(x, p) < \delta$ , we have  $d_Y(f(x), q) < \varepsilon$ .

**Definition A.17 (Continuous).** Suppose  $X$  and  $Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Then  $f$  is **continuous** at  $p$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $x \in E$  for which  $d_X(x, p) < \delta$ , we have that  $d_Y(f(x), f(p)) < \varepsilon$ .

**Definition A.18 (Uniformly Continuous).** Let  $f$  be a mapping of a metric space  $X$  into a metric space  $Y$ . We say that  $f$  is **uniformly continuous** on  $X$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $p, q \in X$  for which  $d_X(p, q) < \delta$ , we have  $d_Y(f(p), f(q)) < \varepsilon$ .

Observations:

- Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.
- If  $f$  is continuous on  $X$ , then for each  $\varepsilon > 0$  and each  $p \in X$ , we can find a  $\delta > 0$  that satisfies the condition in the definition. For uniform continuity, we can find one  $\delta > 0$  that works for all points  $p \in X$ .
- Every uniformly continuous function is continuous. The two concepts are equivalent on compact sets.

## A.5 Differentiation

**Definition A.19 (Differentiable, Derivative).** Let  $f$  be defined (and real-valued) on  $[a, b]$ . For any  $x \in [a, b]$  define

$$f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \quad (\text{A.10})$$

provided this limit exists. If  $f'$  is defined at a point  $x$ , we say that  $f$  is **differentiable** at  $x$ .  $f'$  is called the **derivative** of  $f$ .

**Theorem A.10 (Mean Value Theorem).** If  $f$  is a real continuous function on  $[a, b]$  which is differentiable on  $(a, b)$ , then there is a point  $x \in (a, b)$  at which

$$f(b) - f(a) = (b - a)f'(x) \quad (\text{A.11})$$

## References

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