Theorem 1 (The unit element is unique)**.** Let G be a group and $\star$ its binary operation. Suppose that $e_1, e_2 \in G$ are both units elements. Then, $e_1 = e_2$.

Theorem 2 (Cancellation Law)**.** For every group G and $a, b, c \in G$ that satisfy $ab = ac$, we have $b = c$.

Theorem 3 (The inverse of a group element is unique)**.** Let G be a group and let $a \in G$. If b and c are inverses of a, then $b = c$.

Theorem 4. The order of a k-cycle is k.

Theorem 5. Disjoint cycles commute.

Theorem 6 (Basic facts about homomorphisms)**.** Let $\phi : G \to H$ be a homomorphism. Then

1. $\phi(1_G) = 1_H$ (the identity of G is mapped to the identity of H).
2. $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$.

Theorem 7. Let $\phi : G \to H$ be a homomorphism. Then $Im(\phi) = \{\phi(g) | g \in G\} \leq H$.

Theorem 8. Let $\phi : G \to H$ be a homomorphism. Then $\ker(\phi) \leq G$. That is, the kernel of ϕ is a subgroup of G.

Theorem 9 (All left cosets of H have the same size)**.** Let $H \leq G$ be groups and let $a \in G$. Then $|[a]| = |aH| = |H|$

Theorem 10 (Lagrange)**.** Let G be a finite group and let $H \leq G$. Then $|H|$ divides $|G|$.

Theorem 11 (Equivalent conditions to be a normal subgroup)**.** Let $N \leq G$. Then $N \trianglelefteq G$ if one of the following holds:

1. $\forall g \in G$, $gN = Ng$
2. $\forall g \in G$, $gNG^{-1} = N$
3. $\forall g \in G$, $gNg^{-1} \subseteq N$
4. $\forall g \in G$ and $\forall n \in N$, $gng^{-1} \in N$

Theorem 12 (The kernel of a Homomorphism is a Normal Subgroup)**.** Let $\phi : G \to H$ be homomorphism. Then $\ker(\phi) \trianglelefteq G$.

Theorem 13 (Every cyclic group is isomorphic to either $\mathbb{Z}$ or to $\mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$.)**.** For every group H for which there exists an $x \in H$ such that $H = \langle x \rangle$, there exists a bijective homomorphism (i.e. an isomorphism) $\phi : H \to C$ where $C = \mathbb{Z}$ or $C = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 1$.

Theorem 14. Let G be a finite cyclic group of order n. For every $m|n$ (m that divides n) there exists a unique subgroup H of G with $|H| = m$. Furthermore, H is cyclic.

Theorem 15 (Important Identity for Dihedral Groups)**.** $\rho\epsilon = \epsilon\rho^{-1}$.

Theorem 16. $\rho^i\epsilon = \epsilon\rho^{-i}$

Theorem 17. In the above definition, G/N is a group.

Theorem 18 (The First Isomorphism Theorem)**.** If $\phi : G \to H$ is a homomorphism of groups, then $G/\ker(\phi) \cong Im\phi$.

Theorem 19 (The Second or Diamond Isomorphism Theorem)**.** Let $H \leq G$ and $K \trianglelefteq G$. Then $HK/K \cong H/H \cap K$.

Theorem 20 (The stabilizer of a group element is a subgroup)**.** $G_x \leq G$

Theorem 21 (Orbit-Stabilizer Theorem)**.** There is a bijection

$$f : G/G_x \to O_x \tag{1}$$

In words, there is a bijection between the collection of all cosets of the stabilizer and the orbit. In particular,

$$[G : G_x] = |O_x| \tag{2}$$

(Recall we defined $|G/G_x|$ to be $[G : G_x]$).

Theorem 22. An action is transitive if and only if there exists an $x \in X$ such that $O_x = X$. That is, all elements of X have the same equivalence class.

Theorem 23 (Burnside)**.** Let G act on X. Suppose that G, X are finite. Then,

$$N = \text{ \# of orbits (equivalence classes) } = \frac{1}{|G|}\sum_{g\in G}|X^g| \tag{3}$$

Theorem 24. $(\mathbb{Z}/n\mathbb{Z})^\times$ is a group.

Theorem 25 (Fermat's Little Theorem)**.** For a prime number p and $1 \leq x \leq p-1$, we have $x^{p-1}-1$ is divisible by p.

Theorem 26. Let $p \neq q$ be odd primes. Then

$$(\mathbb{Z}/pq\mathbb{Z})^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/q\mathbb{Z})^\times \tag{4}$$

Theorem 27. If $\lambda : G \to S_n$ is a homomorphism, we can define an action of G on $\{1, \ldots, n\}$ by

$$g(i) = \lambda(g)(i) \tag{5}$$

Theorem 28. Under the above assumptions: λ is injective if and only if the action of G on X (which we can think of as $\{1, \ldots, n\}$) is faithful.

Theorem 29 (Cayley)**.** Let G be a group of order n. Then, there exists an injective homomorphism $\phi : G \to S_n$.

Theorem 30. If $|G| = n$, then G is isomorphic to a subgroup of S_n. Indeed $\phi : G \to Im(\phi) \subset S_n$.

Theorem 31 (Cardinality of set of fixed points of action of set on p-group equals cardinality of set mod p)**.** Let G be a p-group that acts on a finite set X. Let $X^G = \cap_{g\in G} X^g$ where $X^g = \{x \in X | gx = x\}$, that is those $x \in X$ such that for all $g \in G$, $gx = x$. Then p divides $|X| - |X^G|$, that is

$$|X^G| \equiv |X| \quad (modp) \tag{6}$$

Theorem 32 (A p-group has a non-trivial center)**.** Let G be a p-group. Then $Z(G) \neq \{1\}$. In words, there has to be a non-trivial element of the group that commutes with everything else.

Corollary 1. Let p be a prime number and let G be a group of order p^2. Then G is abelian.

Theorem 33 (Cauchy)**.** Let G be a finite group and suppose that $p \big| |G|$ for some prime p. Then there exists an element of order p in G.

Theorem 34 (Correspondence Theorem)**.** Let G, H be groups, and let $\phi : G \to H$ be a group homomorphism. Then there exists a correspondence (i.e. a bijection)

$$\{\text{Subgroups } K \text{ of } G \text{ containing } \ker\phi\} \iff \{\text{Subgroups } L \text{ of } H \text{ contained in } Im(\phi)\}$$

given by $K \mapsto \phi(K)$ and $L \mapsto \phi^{-1}(L)$. In addition, let K_1 and K_2 be subgroups of G containing $\ker(\phi)$ and L_1 and L_2 subgroups L of H contained in $Im(\phi)$.

1. $K_1 \leq K_2 \implies \phi(K_1) \leq \phi(K_2)$
2. $L_1 \leq L_2 \implies \phi^{-1}(L_1) \leq \phi^{-1}(L_2)$

and

1. $K_1 \leq K_2 \implies [K_2 : K_1] = [\phi(K_2) : \phi(K_1)]$
2. $L_1 \leq L_2 \implies [L_2 : L_1] = [\phi(L_2) : \phi(L_1)]$

Theorem 35 (Corollary of Correspondence Theorem)**.** Let G be a group and let $N \trianglelefteq G$. Then the subgroups of G/N are all of the form R/N for some $N \leq R \leq G$. Moreover,

$$[G : R] = [G/N : R/N] \tag{7}$$

Theorem 36 (Sylow's Theorem)**.** Let p be a prime number, let G be a finite group, and let p^n be the largest power of p that divides $|G|$. Then G contains a subgroup P of order p^n. P is called a p-Sylow subgroup of G.

Theorem 37 (p-Sylow subgroups are conjugate)**.** Let G be a finite group and let P, Q be p-Sylow subgroups of G. Then there exists $g \in G$ such that $gPg^{-1} = Q$.

Corollary 2 (p-Sylow subgroup unique if and only if normal subgroup.)**.** Let G be a finite group and let P be a p-Sylow subgroup of G. Then P is a unique p-Sylow subgroup if and only if $P \trianglelefteq G$.

Theorem 38 (Sylow's Theorem (General))**.** Let G be a finite group and p a prime. Suppose that p^r divides $|G|$. Then G has a subgroup H of order p^r. Moreover, every subgroup of order p^r is contained in a Sylow subgroup.

Theorem 39 ($0 \cdot a = 0$)**.**

$$\begin{aligned} 0 \cdot a &= (0 + 0) \cdot a && \text{(0 additive identity)} \\ &= 0 \cdot a + 0 \cdot a && \text{(distributivity)} \end{aligned}$$

Then cancellation gives $0 = 0 \cdot a$.

Theorem 40 ($-a = (-1) \cdot a$)**.** We want to show that $(-1) \cdot a$ is the additive inverse of a. To that end

$$\begin{aligned} a + (-1) \cdot a &= 1 \cdot a + (-1) \cdot a && \text{(1 multiplicative identity)} \\ &= (1 + -1) \cdot a && \text{(distributivity)} \\ &= 0 \cdot a \\ &= 0 \end{aligned}$$

Theorem 41 (Homomorphism of rings injective injective if and only its kernel is trivial)**.** Suppose ϕ is a homomorphism of rings. Then ϕ is injective if and only if $\ker(\phi) = \{0\}$.

Theorem 42 (Field has only 2 ideals: the trivial ideal and the field itself.)**.** If R is a field, then its ideals are R and $\{0\}$.

Theorem 43 (There are only 2 ideals in $M_{n\times n}(\mathbb{R})$)**.**

Theorem 44 (Kernel of homomorphism of rings is an ideal of the ring which is the domain of the homomorphism)**.** Suppose A, B are rings and let $\phi : A \to B$ be a ring homomorphism. Then $\ker(\phi)$ is an ideal of A.

Theorem 45 (Ideal is an additive normal subgroup)**.** Suppose A is a ring and I is an ideal of A. Then I is an an additive normal subgroup A.

Theorem 46. Suppose $\phi : A \to A/I$ is a homomorphism of rings where $\phi(a) = I + a$. Then

$$\ker(\phi) = I \tag{8}$$