Definition 1. (Simple Iteration) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Given that $x_0 \in [a, b]$, the recursion defined by

$$x_{k+1} = g(x_k) \tag{1}$$

is called simple iteration; the numbers x_k, $k \geq 0$, are referred to as iterates.

Definition 2. (Contraction) Let g be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Then, g is said to be a contraction on $[a, b]$ if there exists a constant L such that $0 < L < 1$ and

$$|g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in [a, b] \tag{2}$$

Definition 3. (Stable, Unstable Fixed Point) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$, and let ξ denote a fixed point of g. ξ is a stable fixed point of g if the sequence (x_k) defined by the iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ whenever the starting value x_0 is sufficiently close to ξ. Conversely, if no sequence (x_k) defined by this iteration converges to ξ for any starting value x_0 close to ξ, except for $x_0 = \xi$, then we say that ξ is an unstable fixed point of g.

Definition 4. (Rate of Convergence) Suppose $\xi = \lim_{k\to\infty} x_k$. Define $E_k = |x_k - \xi|$.

Definition 5. (Newton's Method) Newton's method for the solution of $f(x) = 0$ is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{3}$$

Definition 6. (Secant Method) The secant method is defined by

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \tag{4}$$

Definition 7. (Unitary Matrix) A matrix $Q = [q_1 \dots q_n] \in \mathbb{R}^{m \times n}$ is unitary if and only if $\langle q_i, q_j \rangle = \delta_{ij}$.

Definition 8. (Norm) Suppose that $\mathcal{V}$ is a linear space over the field $\mathbb{R}$. The *nonnegative* real-valued function $\|\cdot\|$ is a norm on $\mathcal{V}$ if the following axioms are satisfied: Fix $v \in \mathcal{V}$

1. Positivity: $\|v\| = 0$ if and only if $v = 0$
2. Scale Preservation: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{R}$
3. Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Definition 9. (Operator Norm) Let A be an $m \times n$ matrix. That is, A is a linear transformation form $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the operator norm (or subordinate matrix norm) of A is

$$\|A\|_{p,q} = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}. \tag{5}$$

Definition 10. (Absolute Condition Number)

$$Cond(f) = \sup_{x,y \in D, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \tag{6}$$

Definition 11. (Absolute Local Condition Number)

$$Cond_x(f) = \sup_{x + \delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|} \tag{7}$$

Definition 12. (Relative Local Condition Number)

$$cond_x(f) = \sup_{x + \delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|/\|f(x)\|}{\|\delta x\|/\|x\|} \tag{8}$$

Definition 13. (Condition Number of a Nonsingular Matrix) The condition number of a nonsingular matrix A is defined by

$$\kappa(A) = \|A\|\|A^{-1}\| \tag{9}$$

If $\kappa(A) \gg 1$, the matrix is said to be ill-conditioned.

Definition 14. (Symmetric, Positive Definite, spd) The real matrix A is said to be symmetric if $A = A^T$. A square $n \times n$ matrix is called positive definite if

$$\boldsymbol{x}^T A \boldsymbol{x} > 0 \tag{10}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \neq 0$.

Algorithm 1 Cholesky Factorization

Require: $A \in \mathbb{R}^{n \times n}$, SPD

$L_1 \leftarrow \sqrt{a_{11}}$
for $k \leftarrow 2, 3, \ldots, n$ **do**
  Solve $L_{k-1} l_k = a_k$ for l_k
  $l_{kk} \leftarrow \sqrt{a_{kk} - l_k^T l_k}$
  $L_k \leftarrow \begin{pmatrix} L_{k-1} & 0 \\ l_k^T & l_{kk} \end{pmatrix}$
end for

Definition 15. (Cauchy Sequence) A sequence $(\boldsymbol{x}^{(k)}) \subset \mathbb{R}^n$ is called a Cauchy sequence in $\mathbb{R}^n$ if for any $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$ such that

$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(m)}\|_\infty < \epsilon \quad \forall k, m \geq k_0(\epsilon) \tag{11}$$

Definition 16. (Continuous function) Let $D \subset \mathbb{R}^n$ be nonempty and $f : D \to \mathbb{R}^n$. Given $\boldsymbol{\xi} \in D$, f is continuous at $\boldsymbol{\xi}$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $\boldsymbol{x} \in B(\boldsymbol{\xi}; \delta) \cap D$

$$\|f(\boldsymbol{x}) - f(\boldsymbol{\xi})\|_\infty < \epsilon \tag{12}$$

Definition 17. (Lipschitz condition, constant, and contraction) Let D be a closed subset of $\mathbb{R}^n$ and $g : D \to D$. If there exists a positive constant L such that

$$\|g(x) - g(y)\|_\infty \leq L\|x - y\|_\infty \tag{13}$$

for all $x, y \in D$, then g satisfies the Lipschitz condition on D in the ∞-norm. L is called the Lipschitz constant. If $L \in (0, 1)$, then g is called a contraction on D in the ∞-norm.

Definition 18. (Jacobian) Let $g = (g_1, \ldots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined and continuous in an (open) neighborhood of $\boldsymbol{\xi} \in \mathbb{R}^n$. Suppose the first partial derivatives of each g_i exist at $\boldsymbol{\xi}$. The Jacobian matrix $J_g(\boldsymbol{\xi})$ of g at $\boldsymbol{\xi}$ is the $n \times n$ matrix with elements

$$J_g(\boldsymbol{\xi})_{ij} = \frac{\partial g_i}{\partial x_j}(\boldsymbol{\xi}) \tag{14}$$

Definition 19. (Newton's Method) The sequence defined by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - [J_f(\boldsymbol{x}^{(k)})]^{-1} f(\boldsymbol{x}^{(k)}) \tag{15}$$

where $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$, is called Newton's method.

Algorithm 2 Power Iteration

Require: $v^{(0)} =$ some vector with $\|v^{(0)}\| = 1$

for $k \leftarrow 1, 2, \ldots$ **do**
  $w \leftarrow Av^{(k-1)}$ ▷ Apply A
  $v^{(k)} \leftarrow w / \|w\|$ ▷ Normalize
  $\lambda^{(k)} \leftarrow (v^{(k)})^T A v^{(k)} = \langle v^{(k)}, Av^{(k)} \rangle$ ▷ Rayleigh Quotient
end for

Algorithm 3 Simultaneous Iteration

Require: $Q^{(0)} = V = I$, a list of vectors V, which we choose to be the identity

for $k \leftarrow 1, 2, \ldots$ **do**
  $Z \leftarrow A\underline{Q}^{(k-1)}$ ▷ Apply A
  $Z \leftarrow \underline{Q}^{(k)} R^{(k)}$ ▷ QR factorization of Z
  $A^{(k)} \leftarrow (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}$
end for

Algorithm 4 QR Algorithm (without shifts)

Require: $A^{(0)} = A$

for $k \leftarrow 1, 2, \ldots$ **do**
  $Q^{(k)} R^{(k)} \leftarrow A^{(k-1)}$ ▷ QR factorization of $A^{(k-1)}$
  $A^{(k)} \leftarrow R^{(k)} Q^{(k)}$ ▷ Recombine factors in reverse order
end for