(The Mean Value Theorem) Suppose f is a real-valued function, defined and continuous on the closed interval $[a, b] \in \mathbb{R}$ and f differentiable on the open interval (a, b). Then there exists a number $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a) \tag{1}$$

[Taylor's Theorem] Suppose that n is a nonnegative integer, and f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, such that the derivatives of f of order up to and including n are defined and continuous on the closed interval $[a, b]$. Suppose further that $f^{(n)}$ is differentiable on the open interval (a, b). Then, for each value of $x \in [a, b]$, there exists a number $\xi = \xi(x)$ in the open interval (a, b) such that

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \tag{2}$$

(Existence of Root) Let f be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Assume further, that $f(a)f(b) \leq 0$; then, there exists ξ in $[a, b]$ such that $f(\xi) = 0$.

[Brouwer's Fixed Point Theorem] Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Then, there exists $\xi \in [a, b]$ such that $\xi = g(\xi)$. ξ is called a fixed point of the function g.

[Contraction Mapping Theorem] Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose g is a contraction on $[a, b]$. Then, g has a unique fixed point ξ in the interval $[a, b]$. Moreover, the sequence (x_k) defined by simple iteration converges to ξ as $k \to \infty$ for any starting value x_0 in $[a, b]$.

Let $\epsilon > 0$ be a certain tolerance, and let $k_0(\epsilon)$ denote the smallest positive integer such that x_k is no more than ϵ away from the fixed point ξ (i.e. $|x_k - \xi| \leq \epsilon$) for all $k \geq k_0(\epsilon)$. Then,

$$k_0(\epsilon) \leq \left\lfloor \frac{\ln|x_1 - x_0| - \ln(\epsilon(1-L))}{\ln(1/L)} \right\rfloor + 1 \tag{3}$$

[Contraction Mapping Theorem when Differentiable] Suppose that g is a real-valued function, defined and continuous on a bounded closed interval

$[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $\xi = g(\xi) \in [a, b]$ be a fixed point of g (the existence of this point is guaranteed by Brouwer's fixed point theorem). Assume g has a continuous derivative in some neighborhood of ξ with $|g'(\xi)| < 1$. Then the sequence (x_k) defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ as $k \to \infty$, provided that x_0 is close to ξ.

[Unstable Fixed Points] Suppose that $\xi = g(\xi)$, where the function g has a continuous derivative in some neighborhood of ξ, and let $|g'(\xi)| > 1$ (thus ξ is an unstable fixed point). Then the sequence (x_k) defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, does not converge to ξ from any starting value x_0, $x_0 \neq \xi$.

[Convergence of Newton's Method] Suppose that f is a continuous real-valued function with continuous second derivative f'' defined on the closed interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, such that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Additionally suppose that there exists a positive constant A such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall x, y, \in I_\delta \tag{4}$$

If initially

$$|\xi - x_0| \leq h = \min(\delta, \frac{1}{A}) \tag{5}$$

then the sequence (x_k) defined by Newton's method converges quadratically to ξ.

[Convergence of Secant Method] Suppose that f is a real-valued function, defined and continuously differentiable on an interval $I = [\xi - h, \xi + h]$, $h > 0$, with center point ξ. Suppose further that $f(\xi) = 0$, $f'(\xi) \neq 0$. Then, the sequence (x_k) defined by the secant method converges at least linearly to ξ provided that x_0 and x_1 are sufficiently close to ξ.

[The 1-norm of a matrix is the largest absolute-value column sum] Let $A \in \mathbb{R}^{m \times n}$ and denote the columns of A by a_j, $j = 1, \ldots, n$. Then $\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}| = \max_{j=1,\ldots,n} \|a_j\|$.

[The ∞-norm of a matrix is the largest absolute-value row sum] Let $A \in \mathbb{R}^{m \times n}$ and denote the rows of A by b_i, $i = 1, \ldots, m$. Then $\|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| = \max_{i=1,\ldots,m} \|b_i\|$.

[The 2-norm of a symmetric positive definite matrix is the maximum absolute value of its eigenvalues] Let A be a positive definite $n \times n$ matrix. Then

$$\|A\|_2 = \max_{i=1,\ldots,n} |\lambda_i| \tag{6}$$

(The 2-norm of a matrix $A_{m\times n}$ equals its largest singular value) Let A be an $m \times n$ matrix and denote the eigenvalues of the matrix $B = A^T A$ by λ_i, $i = 1, \dots, n$. Then

$$\|A\|_2 = \max_i \sqrt{\lambda_i} \tag{7}$$

The square roots of the (nonnegative) eigenvalues of $A^T A$ are referred to as the singular values of A.

[Properties of spd matrices] Let A be an $n \times n$ real, spd matrix. Then

1. $a_{ii} > 0$ for all $i = 1, \dots, n$ (the diagonal elements of A are positive).
2. $Ax_i \ = \lambda_i x_i \implies \lambda_i \in \mathbb{R}_{>0}, \boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}$ (the eigenvalues of A are real and positive, and the eigenvectors of A belong to $\mathbb{R}^n \setminus \{0\}$).
3. $x_i \perp x_j$ if $\lambda_i \neq \lambda_j$ (the eigenvectors of distinct eigenvalues of A are orthogonal)
4. $\det(A) > 0$ (the determinant of A is positive)
5. Every submatrix B of A obtained by deleting any set of rows and the corresponding set of columns from A is symmetric and positive definite (in particular, every principal submatrix is positive definite).

If A is spd, then there exists a lower diagonal matrix L such that $A = LL^T$. This is called the Cholesky decomposition.

[Contraction Mapping Theorem in $\mathbb{R}^n$] Suppose D is a closed subset of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined on D, and $g(D) \subset D$. Suppose further that g is a contraction on D in the ∞-norm. Then,

1. g has a unique fixed point $\boldsymbol{\xi} \in D$
2. The sequence $(\boldsymbol{x}^{(k)})$ defined by $\boldsymbol{x}^{(k+1)} = g(\boldsymbol{x}^k)$ converges to $\boldsymbol{\xi}$ for any starting value $x^{(0)} \in D$.

Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined and continuous on a closed set $D \subset \mathbb{R}^n$. Let $\boldsymbol{\xi} \in D$ be a fixed point of g. Suppose the first partial derivatives of each g_i are defined and continuous in some (open) neighborhood $N(\boldsymbol{\xi}) \in D$ of $\boldsymbol{\xi}$, with

$$\|J_g(\boldsymbol{\xi})\|_\infty < 1 \tag{8}$$

Then there exists $\epsilon > 0$ such that $g(\bar{B}_\epsilon(\boldsymbol{\xi})) \subset \bar{B}_\epsilon(\boldsymbol{\xi})$, and the sequence $\boldsymbol{x}^{(k+1)} = g(\boldsymbol{x}^k)$ converges to $\boldsymbol{\xi}$ for all $\boldsymbol{x}^{(0)} \in \bar{B}_\epsilon(\boldsymbol{\xi})$ (in other words, the sequence converges to $\boldsymbol{\xi}$ as long as $\boldsymbol{x}^{(0)}$ is close enough to $\boldsymbol{\xi}$).

Suppose $f(\boldsymbol{\xi}) = 0$, that in some (open) neighborhood $N(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$, where f is defined and continuous, all the second-order partial derivatives of f are defined and continuous, and that the Jacobian matrix $J_f(\boldsymbol{x}^{(k)})$ of f at the point $\boldsymbol{\xi}$ is nonsingular. Then the sequence defined by Newton's method converges to $\boldsymbol{\xi}$ provided that $\boldsymbol{x}^{(0)}$ is sufficiently close to $\boldsymbol{\xi}$.

(Abel(-Ruffini) Theorem, or "No-go Theorem") There is no algebraic solution (that is, a solution expressed in terms of radicals) to general polynomial equations of degree five or higher with arbitrary coefficients.

[Convergence of Power Iteration] Suppose $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and $q_1^T v^{(0)} \neq 0$. Then the iterates of power iteration satisfy

$$\|v^{(k)} - (\pm q_1)\| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \qquad \text{(error of eigenvector)}$$

$$|\lambda^{(k)} - \lambda_1| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) \qquad \text{(error of eigenvalue)}$$

[Error of Rayleigh Quotient] Let x_1 be the eigenvector that corresponds to the largest (in absolute value) eigenvalue. If $\|x - x_1\| = \mathcal{O}(\epsilon)$, then

$$\left|\frac{\langle x, Ax\rangle}{\langle x, x\rangle} - \lambda_1\right| = \mathcal{O}(\epsilon^2) \tag{9}$$

(Equivalence of Simultaneous Iteration and the QR Algorithm) Simultaneous Iteration and the QR Algorithm generate identical sequences of matrices $\underline{R}^{(k)}, \underline{Q}^{(k)}, A^{(k)}$. Both give

$$(a) : A^{(k)} = \underline{Q}^{(k)}\underline{R}^{(k)} \qquad (QR \text{ factorization of the } k\text{th power of } A)$$

$$(b) : A^{(k)} = (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} \qquad \text{(projection)}$$

[Error of Lagrange interpolation polynomial] Suppose that $n \geq 0$ and the f is a real-valued function, defined and continuous on the closed real interval $[a, b]$, such that derivative of f or order $n + 1$ exists and is continuous on

$[a, b]$. Then, with $x \in [a, b]$, there exists $\xi = \xi(x)$ in (a, b) such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{n!} \prod_{k=0}^{n} (x - x_k) \tag{10}$$

[Chebyshev grid]

[OP Recurrence Relation] A set of orthogonal polynomials $\{\phi\}_{i=0}^{\infty}$ satisfies

$$\phi_{n+1} = (\alpha_n x + \beta_n)\phi_n + \gamma_n \phi_{n-1} \tag{11}$$

If $\{\phi\}_{i=0}^{\infty}$, then $\phi_n(x)$ has n real roots, called Gaussian quadratures.

[Locations of Gaussian Quadratures from Recurrence Relation] Give the recurrence relation

$$\phi_{n+1} = (\alpha_n x + \beta_n)\phi_n + \gamma_n \phi_{n-1} \tag{12}$$

we can rewrite this as

$$\alpha_n x \phi_n = \phi_{n+1} - \beta_n \phi_n - \gamma_n \phi_{n-1} \tag{13}$$

Thus for constants a_n, b_n, c_n we have that

$$x\phi_n = \phi_{n-1} + b_n \phi_n + c_n \phi_{n+1} \tag{14}$$

where this equality holds for all x in the domain. We can write this system in matrix form as follows

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & & \\ & a_2 & b_2 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ c_n \phi_{n+1} \end{pmatrix} \tag{15}$$

where A is the matrix of coefficients. We want to find the roots $\phi_{n+1}(x_i) = 0$, where $i = 1, \ldots, n+1$. Then the eigenvalues of A are the zeros of ϕ_{n+1}. In sum

$$\text{GQ of } \phi_{n+1} = eig(A) \tag{16}$$

Suppose. $f(x) \in \mathbb{P}_{2N+1}$. Then

$$\int_a^b f(x)w(x)dx = \sum_{i=0}^{N} f(x_k)w_k \tag{17}$$

if $\{x_0, \ldots, x_N\}$ are the GQ (roots) of ϕ_{N+1}, where

$$w_k = \int_a^b l_k(x)w(x)dx \tag{18}$$

where $l_k(x)$ is a Lagrange polynomial.

Let $f(x) \in \mathbb{P}_{N+1}$. Then

$$\alpha_i = \langle f, \phi_i \rangle = \int_a^b f(x)\phi_i(x)w(x)dx = \sum_{k=0}^{N} f(x_k)\phi_i(x_k)w_k = c_i \tag{19}$$

That is the projection coefficients c_i are equal to the numerical representation α_i, where the grid points are the GQ of ϕ_{N+1}.

We interpolate f as follows:

$$p(x) = \sum_{n=0}^{N} c_n\phi_n(x) \tag{20}$$

such that $p(x_i) = f(x_i)$ where the x_i are the GQ of ϕ_{N+1}. Then

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \ldots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \ldots & \phi_N(x_1) \\ \vdots & \vdots & & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \ldots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_N) \end{bmatrix} \tag{21}$$

Then A, the matrix above, is almost unitary. In particular,

$$A^T \cdot W \cdot A = I \tag{22}$$

where W is a diagonal matrix with elements $w_0, w_1, \ldots, w_N$.

[Projection the best approximation in the L^2-norm:] $p_N(x)$ is the best approximation in the L^2-norm:

$$\|f - p_N(x)\|_2^2 \leq \|f - q(x)\|_2^2 \tag{23}$$

for all $q \in \mathbb{P}_{\mathbb{N}}$.

If $f \in \mathbb{P}_{2N+1}$

$$\int f(x)w(x)dx = \sum_{i=0}^{N} f(x_i)w_i \tag{24}$$

[First mean value theorem for definite integrals] If $f : [a,b] \to \mathbb{R}$ is continuous and g is an integrable function that does not change sign on $[a,b]$, then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \tag{25}$$