**Theorem 1** (The Mean Value Theorem)**.** Suppose $f$ is a real-valued function, defined and continuous on the closed interval $[a, b] \in \mathbb{R}$ and $f$ differentiable on the open interval $(a, b)$. Then there exists a number $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a) \tag{1}$$

**Theorem 2** (Taylor's Theorem)**.** Suppose that $n$ is a nonnegative integer, and $f$ is a real-valued function, defined and continuous on the closed interval $[a, b]$ of $\mathbb{R}$, such that the derivatives of $f$ of order up to and including $n$ are defined and continuous on the closed interval $[a, b]$. Suppose further that $f^{(n)}$ is differentiable on the open interval $(a, b)$. Then, for each value of $x \in [a, b]$, there exists a number $\xi = \xi(x)$ in the open interval $(a, b)$ such that

$$f(x) = f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \tag{2}$$

**Theorem 3** (Existence of Root)**.** Let $f$ be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Assume further, that $f(a)f(b) \leq 0$; then, there exists $\xi$ in $[a, b]$ such that $f(\xi) = 0$.

**Theorem 4** (Brouwer's Fixed Point Theorem)**.** Suppose that $g$ is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Then, there exists $\xi \in [a, b]$ such that $\xi = g(\xi)$. $\xi$ is called a fixed point of the function $g$.

**Theorem 5** (Contraction Mapping Theorem)**.** Suppose that $g$ is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose $g$ is a contraction on $[a, b]$. Then, $g$ has a unique fixed point $\xi$ in the interval $[a, b]$. Moreover, the sequence $(x_k)$ defined by simple iteration converges to $\xi$ as $k \to \infty$ for any starting value $x_0$ in $[a, b]$.

Let $\epsilon > 0$ be a certain tolerance, and let $k_0(\epsilon)$ denote the smallest positive integer such that $x_k$ is no more than $\epsilon$ away from the fixed point $\xi$ (i.e. $|x_k - \xi| \leq \epsilon$) for all $k \geq k_0(\epsilon)$. Then,

$$k_0(\epsilon) \leq \left\lfloor \frac{\ln|x_1 - x_0| - \ln(\epsilon(1-L))}{\ln(1/L)} \right\rfloor + 1 \tag{3}$$

**Theorem 6** (Contraction Mapping Theorem when Differentiable)**.** Suppose that $g$ is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $\xi = g(\xi) \in [a, b]$ be a fixed point of $g$ (the existence of this point is guaranteed by Brouwer's fixed point theorem). Assume $g$ has a continuous derivative in some neighborhood of $\xi$ with $|g'(\xi)| < 1$. Then the sequence $(x_k)$ defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to $\xi$ as $k \to \infty$, provided that $x_0$ is close to $\xi$.

**Theorem 7** (Unstable Fixed Points). Suppose that $\xi = g(\xi)$, where the function $g$ has a continuous derivative in some neighborhood of $\xi$, and let $|g'(\xi)| > 1$ (thus $\xi$ is an unstable fixed point). Then the sequence $(x_k)$ defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, does not converge to $\xi$ from any starting value $x_0$, $x_0 \neq \xi$.

**Theorem 8** (Convergence of Newton's Method). Suppose that $f$ is a continuous real-valued function with continuous second derivative $f''$ defined on the closed interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, such that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Additionally suppose that there exists a positive constant $A$ such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall x, y, \in I_\delta \tag{4}$$

If initially

$$|\xi - x_0| \leq h = \min(\delta, \frac{1}{A}) \tag{5}$$

then the sequence $(x_k)$ defined by Newton's method converges quadratically to $\xi$.

**Theorem 9** (Convergence of Secant Method). Suppose that $f$ is a real-valued function, defined and continuously differentiable on an interval $I = [\xi - h, \xi + h]$, $h > 0$, with center point $\xi$. Suppose further that $f(\xi) = 0$, $f'(\xi) \neq 0$. Then, the sequence $(x_k)$ defined by the secant method converges at least linearly to $\xi$ provided that $x_0$ and $x_1$ are sufficiently close to $\xi$.

**Theorem 10** (The 1-norm of a matrix is the largest absolute-value column sum). Let $A \in \mathbb{R}^{m \times n}$ and denote the columns of $A$ by $a_j$, $j = 1, \ldots, n$. Then $\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}| = \max_{j=1,\ldots,n} \|a_j\|$.

**Theorem 11** (The $\infty$-norm of a matrix is the largest absolute-value row sum). Let $A \in \mathbb{R}^{m \times n}$ and denote the rows of $A$ by $b_i$, $i = 1, \ldots, m$. Then $\|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| = \max_{i=1,\ldots,m} \|b_i\|$.

**Theorem 12** (The 2-norm of a symmetric positive definite matrix is the maximum absolute value of its eigenvalues)**.** Let $A$ be a positive definite $n \times n$ matrix. Then

$$\|A\|_2 = \max_{i=1,\dots,n} |\lambda_i| \tag{6}$$

**Theorem 13** (The 2-norm of a matrix $A_{m\times n}$ equals its largest singular value)**.** Let $A$ be an $m \times n$ matrix and denote the eigenvalues of the matrix $B = A^T A$ by $\lambda_i$, $i = 1, \dots, n$. Then

$$\|A\|_2 = \max_i \sqrt{\lambda_i} \tag{7}$$

The square roots of the (nonnegative) eigenvalues of $A^T A$ are referred to as the singular values of $A$.

**Theorem 14** (Properties of spd matrices)**.** Let $A$ be an $n \times n$ real, spd matrix. Then

1. $a_{ii} > 0$ for all $i = 1, \dots, n$ (the diagonal elements of $A$ are positive).
2. $Ax_i = \lambda_i x_i \implies \lambda_i \in \mathbb{R}_{>0}, \boldsymbol{x} \in \mathbb{R}^n \setminus \{0\}$ (the eigenvalues of $A$ are real and positive, and the eigenvectors of $A$ belong to $\mathbb{R}^n \setminus \{0\}$).
3. $x_i \perp x_j$ if $\lambda_i \neq \lambda_j$ (the eigenvectors of distinct eigenvalues of $A$ are orthogonal)
4. $\det(A) > 0$ (the determinant of $A$ is positive)
5. Every submatrix $B$ of $A$ obtained by deleting any set of rows and the corresponding set of columns from $A$ is symmetric and positive definite (in particular, every principal submatrix is positive definite).

**Theorem 15** (Cholesky)**.** If $A$ is spd, then there exists a lower diagonal matrix $L$ such that $A = LL^T$. This is called the Cholesky decomposition.

**Theorem 16** (Contraction Mapping Theorem in $\mathbb{R}^n$)**.** Suppose $D$ is a closed subset of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is defined on $D$, and $g(D) \subset D$. Suppose further that $g$ is a contraction on $D$ in the $\infty$-norm. Then,

1. $g$ has a unique fixed point $\boldsymbol{\xi} \in D$
2. The sequence $(\boldsymbol{x}^{(k)})$ defined by $\boldsymbol{x}^{(k+1)} = g(\boldsymbol{x}^k)$ converges to $\boldsymbol{\xi}$ for any starting value $x^{(0)} \in D$.

**Theorem 17** (Jacobian and Fixed Point Stability)**.** Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined and continuous on a closed set $D \subset \mathbb{R}^n$. Let $\boldsymbol{\xi} \in D$ be a fixed point of $g$. Suppose the first partial derivatives of each $g_i$ are defined and continuous in some (open) neighborhood $N(\boldsymbol{\xi}) \in D$ of $\boldsymbol{\xi}$, with

$$\|J_g(\boldsymbol{\xi})\|_\infty < 1 \tag{8}$$

Then there exists $\epsilon > 0$ such that $g(\bar{B}_\epsilon(\boldsymbol{\xi})) \subset \bar{B}_\epsilon(\boldsymbol{\xi})$, and the sequence $\boldsymbol{x}^{(k+1)} = g(\boldsymbol{x}^k)$ converges to $\boldsymbol{\xi}$ for all $\boldsymbol{x}^{(0)} \in \bar{B}_\epsilon(\boldsymbol{\xi})$ (in other words, the sequence converges to $\boldsymbol{\xi}$ as long as $\boldsymbol{x}^{(0)}$ is close enough to $\boldsymbol{\xi}$).

**Theorem 18.** Suppose $f(\boldsymbol{\xi}) = 0$, that in some (open) neighborhood $N(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$, where $f$ is defined and continuous, all the second-order partial derivatives of $f$ are defined and continuous, and that the Jacobian matrix $J_f(\boldsymbol{x}^{(k)})$ of $f$ at the point $\boldsymbol{\xi}$ is nonsingular. Then the sequence defined by Newton's method converges to $\boldsymbol{\xi}$ provided that $\boldsymbol{x}^{(0)}$ is sufficiently close to $\boldsymbol{\xi}$.

**Theorem 19** (Abel(-Ruffini) Theorem, or "No-go Theorem")**.** There is no algebraic solution (that is, a solution expressed in terms of radicals) to general polynomial equations of degree five or higher with arbitrary coefficients.

**Theorem 20** (Convergence of Power Iteration)**.** Suppose $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and $q_1^T v^{(0)} \neq 0$. Then the iterates of power iteration satisfy

$$\|v^{(k)} - (\pm q_1)\| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \qquad \text{(error of eigenvector)}$$

$$|\lambda^{(k)} - \lambda_1| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) \qquad \text{(error of eigenvalue)}$$

**Theorem 21** (Error of Rayleigh Quotient)**.** Let $x_1$ be the eigenvector that corresponds to the largest (in absolute value) eigenvalue. If $\|x - x_1\| = \mathcal{O}(\epsilon)$, then

$$\left|\frac{\langle x, Ax\rangle}{\langle x, x\rangle} - \lambda_1\right| = \mathcal{O}(\epsilon^2) \tag{9}$$

**Theorem 22** (Equivalence of Simultaneous Iteration and the QR Algorithm)**.** Simultaneous Iteration and the QR Algorithm generate identical sequences of matrices $\underline{R}^{(k)}, \underline{Q}^{(k)}, A^{(k)}$. Both give

$$(a) : A^{(k)} = \underline{Q}^{(k)}\underline{R}^{(k)} \qquad \text{($QR$ factorization of the $k$th power of $A$)}$$

$$(b) : A^{(k)} = (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} \qquad \text{(projection)}$$

**Theorem 23** (Error of Lagrange interpolation polynomial)**.** Suppose that $n \geq 0$ and the $f$ is a real-valued function, defined and continuous on the closed real interval $[a, b]$, such that derivative of $f$ or order $n+1$ exists and is continuous on $[a, b]$. Then, with $x \in [a, b]$, there exists $\xi = \xi(x)$ in $(a, b)$ such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{n!} \prod_{k=0}^{n} (x - x_k) \tag{10}$$

is the interpolation error, where $p(x)$ is $n$-th order.

**Theorem 24** (Chebyshev grid to minimize polynomial interpolation error)**.** The solution to

$$\min_{\{x_i\}} \sup_{t \in [a,b]} \left| \prod_{k=0}^{n} (x - x_k) \right| \tag{11}$$

is given by a Chebyshev grid:

$$x_i = \cos(\theta_i), \quad \theta_i = \frac{i\pi}{N} \tag{12}$$

**Theorem 25** (Orthogonal polynomials form a basis for the space of polynomials)**.**

$$\mathbb{P}_k = span(\phi_0, \ldots, \phi_k) \tag{13}$$

**Theorem 26** (OP Recurrence Relation)**.** A set of orthogonal polynomials $\{\phi\}_{i=0}^{\infty}$ satisfies

$$\phi_{n+1} = (\alpha_n x + \beta_n)\phi_n + \gamma_n \phi_{n-1} \tag{14}$$

**Theorem 27** (Roots of Orthogonal Polynomials)**.** If $\{\phi\}_{i=0}^{\infty}$, then $\phi_n(x)$ has $n$ real roots, called Gaussian quadratures.

**Theorem 28** (Locations of Gaussian Quadratures from Recurrence Relation)**.** Give the recurrence relation

$$\phi_{n+1} = (\alpha_n x + \beta_n)\phi_n + \gamma_n \phi_{n-1} \tag{15}$$

we can rewrite this as

$$\alpha_n x \phi_n = \phi_{n+1} - \beta_n \phi_n - \gamma_n \phi_{n-1} \tag{16}$$

Thus for constants $a_n, b_n, c_n$ we have that

$$x\phi_n = \phi_{n-1} + b_n \phi_n + c_n \phi_{n+1} \tag{17}$$

where this equality holds for all $x$ in the domain. We can write this system in matrix form as follows

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} b_0 & c_0 & & & & \\ a_1 & b_1 & c_1 & & & \\ & a_2 & b_2 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ c_n \phi_{n+1} \end{pmatrix} \tag{18}$$

where $A$ is the matrix of coefficients. We want to find the roots $\phi_{n+1}(x_i) = 0$, where $i = 1, \ldots, n+1$. Then the eigenvalues of $A$ are the zeros of $\phi_{n+1}$. In sum

$$\text{GQ of } \phi_{n+1} = eig(A) \tag{19}$$

**Theorem 29.** Suppose. $f(x) \in \mathbb{P}_{2N+1}$. Then

$$\int_a^b f(x)w(x)dx = \sum_{i=0}^{N} f(x_k)w_k \tag{20}$$

if $\{x_0, \ldots, x_N\}$ are the GQ (roots) of $\phi_{N+1}$, where

$$w_k = \int_a^b l_k(x)w(x)dx \tag{21}$$

where $l_k(x)$ is a Lagrange polynomial.

**Theorem 30** (Projection Coefficients Equivalent to Numerical Representation)**.** Let $f(x) \in \mathbb{P}_{N+1}$. Then

$$\alpha_i = \langle f, \phi_i \rangle = \int_a^b f(x)\phi_i(x)w(x)dx = \sum_{k=0}^{N} f(x_k)\phi_i(x_k)w_k = c_i \tag{22}$$

That is the projection coefficients $c_i$ are equal to the numerical representation $\alpha_i$, where the grid points are the GQ of $\phi_{N+1}$.

**Theorem 31** (Interpolation with Orthogonal Polynomials (Almost Unitary Matrix))**.** We interpolate $f$ as follows:

$$p(x) = \sum_{n=0}^{N} c_n \phi_n(x) \tag{23}$$

such that $p(x_i) = f(x_i)$ where the $x_i$ are the GQ of $\phi_{N+1}$. Then

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_N(x_1) \\ \vdots & \vdots & & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \dots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_N) \end{bmatrix} \tag{24}$$

Then $A$, the matrix above, is almost unitary. In particular,

$$A^T \cdot W \cdot A = I \tag{25}$$

where $W$ is a diagonal matrix with elements $w_0, w_1, \dots, w_N$.

**Theorem 32** (Projection the best approximation in the $L^2$-norm:)**.** $p_N(x)$ is the best approximation in the $L^2$-norm:

$$\|f - p_N(x)\|_2^2 \leq \|f - q(x)\|_2^2 \tag{26}$$

for all $q \in \mathbb{P}_{\mathbb{N}}$.

**Theorem 33** (Error from Approximation by Projection)**.** Suppose $f \in \mathcal{C}^\infty$ and $\{\phi_i\}_{i=0}^\infty$ is a set orthogonal polynomials. We can write

$$f(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \tag{27}$$

and define the projection

$$p_N(x) = \sum_{k=0}^{N} \alpha_k \phi_k(x) \tag{28}$$

Then the error of this approximation is

$$error = \sum_{k=N+1}^{\infty} \alpha_k \phi_k(x) \tag{29}$$

which depends on $\{\alpha_{N+1}, \alpha_{N+2}, \dots\}$. In particular, if $f(x) \in \mathcal{C}^\gamma$, then

$$\alpha_n = \mathcal{O}(n^{-\gamma}) \tag{30}$$

for $n > N$ and

$$\alpha_n = \mathcal{O}\left(\frac{1}{N^\gamma}\right) \tag{31}$$

for $n < N$.

**Theorem 34.** If $f \in \mathbb{P}_{2N+1}$

$$\int f(x)w(x)dx = \sum_{i=0}^{N} f(x_i)w_i \tag{32}$$

**Theorem 35** (First mean value theorem for definite integrals)**.** If $f : [a,b] \to \mathbb{R}$ is continuous and $g$ is an integrable function that does not change sign on $[a,b]$, then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \tag{33}$$

**Theorem 36** (ODE reduction)**.** Any high order, non-autonomous ODE can be reduced to a 1st order, autonomous ODE (system).

**Theorem 37** (Uniqueness)**.** If the force term $f(u)$ is uniformly Lipshitz, then the equation has a unique solution.

**Theorem 38** (Forward Euler (one-step) is consistent)**.** We take the equation for LTE

$$\tau_n = \frac{u_{n+1} - u_n}{\Delta t} - f(u_n) \tag{34}$$

and substitute in the Taylor expansion of $u_{n+1} = u(t_{n+1})$ around $u_n = u(t_n)$. This Taylor expansion is

$$u_{n+1} = u_n + \tag{35}$$

Incomplete