

Numerical Analysis Lecture Notes

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1 Results from Real Analysis

Theorem 1. (The Mean Value Theorem) Suppose f is a real-valued function, defined and continuous on the closed interval $[a, b] \in \mathbb{R}$ and f differentiable on the open interval (a, b) . Then there exists a number $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a) \quad (1)$$

2 Solution of equations by iteration

2.1 Simple Iteration

Theorem 2. (Existence of Root) Let f be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Assume further, that $f(a)f(b) \leq 0$; then, there exists ξ in $[a, b]$ such that $f(\xi) = 0$.

Proof. The condition $f(a)f(b) \leq 0$ implies that $f(a)$ and $f(b)$ have opposite signs, or one of them is 0. If either $f(a)$ or $f(b)$ is 0, then we've found a root. Suppose that both endpoints are non-zero (in which case they have opposite signs). In this case, 0 must belong to the open interval whose endpoints are $f(a)$ and $f(b)$. The intermediate value theorem gives the existence of a root in the open interval (a, b) . Thus, in both cases, a zero is guaranteed. \square

- The converse of Theorem 2 is clearly false.

Theorem 3. (Brouwer's Fixed Point Theorem) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Then, there exists $\xi \in [a, b]$ such that $\xi = g(\xi)$. ξ is called a fixed point of the function g .

Proof. Define a function $f(x) = x - g(x)$. If we find a root ξ of f , then ξ is a fixed point of g . Then,

$$f(a)f(b) = (a - g(a))(b - g(b)) \leq 0 \quad (2)$$

By assumption, $a \leq g(a), g(b) \leq b$. Therefore, the first term is negative and the second term is positive. Therefore, $f(a)f(b) \leq 0$. By Theorem 2, there exists a $\xi \in [a, b]$ such that $f(\xi) = 0$. Then, for this ξ , $g(\xi) = \xi$. \square

Definition 1. (Simple Iteration) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Given that $x_0 \in [a, b]$, the recursion defined by

$$x_{k+1} = g(x_k) \quad (3)$$

is called simple iteration; the numbers $x_k, k \geq 0$, are referred to as iterates.

- If this sequence converges, the limit must be a fixed of g , since g is continuous on a closed interval. Note that

$$\xi = \lim_{k \rightarrow \infty} x_{k+1} = \lim_{k \rightarrow \infty} g(x_k) = g\left(\lim_{k \rightarrow \infty} x_k\right) = g(\xi) \quad (4)$$

Definition 2. (Contraction) Let g be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Then, g is said to be a contraction on $[a, b]$ if there exists a constant L such that $0 < L < 1$ and

$$|g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in [a, b] \quad (5)$$

Theorem 4. (Contraction Mapping Theorem) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose g is a contraction on $[a, b]$. Then, g has a unique fixed point ξ in the interval $[a, b]$. Moreover, the sequence (x_k) defined by simple iteration converges to ξ as $k \rightarrow \infty$ for any starting value x_0 in $[a, b]$.

Let $\epsilon > 0$ be a certain tolerance, and let $k_0(\epsilon)$ denote the smallest positive integer such that x_k is no more than ϵ away from the fixed point ξ (i.e. $|x_k - \xi| \leq \epsilon$) for all $k \geq k_0(\epsilon)$. Then,

$$k_0(\epsilon) \leq \left\lfloor \frac{\ln|x_1 - x_0| - \ln(\epsilon(1 - L))}{\ln(1/L)} \right\rfloor + 1 \quad (6)$$

Proof. Let $E_k = |x_k - \xi|$ be the error at k . Then

$$\begin{aligned} |x_{k+1} - \xi| &= |g(x_k) - g(\xi)| \\ &< L|x_k - \xi| \end{aligned}$$

Therefore

$$E_k \leq L^k E_0 \quad (7)$$

Since $L < 1$, $L^k \rightarrow 0$ as $k \rightarrow \infty$. □

Definition 3. (Stable, Unstable Fixed Point) Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$, and let ξ denote a fixed point of g . ξ is a stable fixed point of g if the sequence (x_k) defined by the iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ whenever the starting value x_0 is sufficiently close to ξ . Conversely, if no sequence (x_k) defined by this iteration converges to ξ for any starting value x_0 close to ξ , except for $x_0 = \xi$, then we say that ξ is an unstable fixed point of g .

- With this definition, a fixed point may be neither stable nor unstable.

Definition 4. (Rate of Convergence) Suppose $\xi = \lim_{k \rightarrow \infty} x_k$. Define $E_k = |x_k - \xi|$.

- The sequence (x_k) converges to ζ linearly if there exists a number $\mu \in (0, 1)$ such that

$$\lim_{k \rightarrow \infty} \frac{E_{k+1}}{E_k} = \mu \quad (8)$$

- The sequence (x_k) converges to ζ superlinearly if $\mu = 0$. That is, the sequence of μ_k generated at each step $\rightarrow 0$ as $k \rightarrow \infty$.
- The sequence (x_k) converges to ζ with order q if there exists a $\mu > 0$ such that

$$\lim_{k \rightarrow \infty} \frac{E_{k+1}}{E_k^q} = \mu \quad (9)$$

In particular, if $q = 2$, then the sequence converges quadratically.

2.2 Newton's Method

Definition 5. (Newton's Method) Newton's method for the solution of $f(x) = 0$ is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (10)$$

Geometrically, $(x_{n+1}, 0)$ is the intersection of the x -axis and the tangent of the graph of f at $(x_n, f(x_n))$.

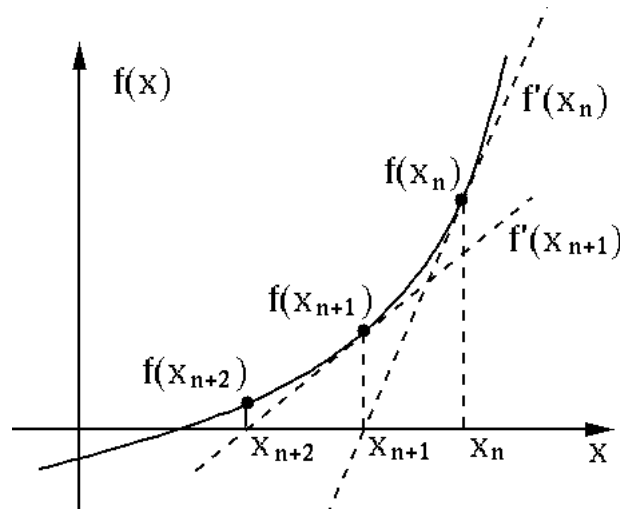


Figure 1: Geometric Interpretation of Newton's Method in \mathbb{R}

2.3 Secant Method

Observe that Newton's method requires us to know the first derivative f' of f . In applications, we might not know f' or it could be expensive to calculate. This motivates

approximating the $f'(x_k)$ in Newton's method with

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad (11)$$

Definition 6. (Secant Method) The secant method is defined by

$$x_{k+1} = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad (12)$$

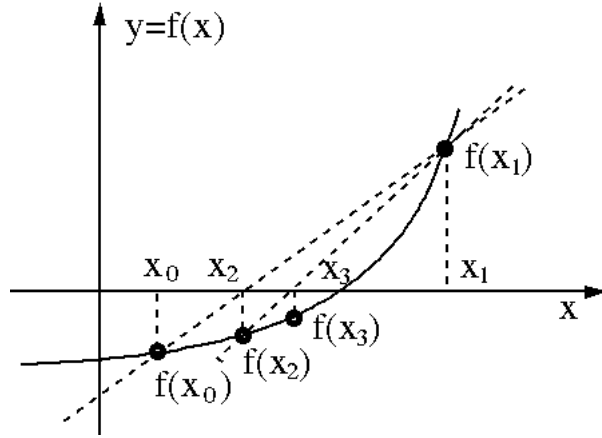


Figure 2: Geometric Interpretation of Secant Method in \mathbb{R}

Theorem 5. (Convergence of Secant Method) Suppose that f is a real-valued function, defined and continuously differentiable on an interval $I = [\xi - h, \xi + h]$, $h > 0$, with center point ξ . Suppose further that $f(\xi) = 0$, $f'(\xi) \neq 0$. Then, the sequence (x_k) defined by the secant method converges at least linearly to ξ provided that x_0 and x_1 are sufficiently close to ξ .

Proof.

□

3 Solution of systems of linear equations

3.1 LU Decomposition

3.2 Least Squares

Given a system of equations $Ax = b$, the least squares problem is

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2^2 \quad (13)$$

We can expand the objective function out as

$$\begin{aligned}\|Ax - b\|_2^2 &= (Ax - b)^T (Ax - b) \\ &= x^T A^T A x - 2b^T A x + b^T b\end{aligned}$$

To find the x that minimizes this expression we find the x that satisfies $\nabla_x F = 0$. That is

$$\nabla_x F = 0 = 2A^T A x - 2A^T b \quad (14)$$

Therefore the minimizer is $x = (A^T A)^{-1} A^T b$. $(A^T A)^{-1} A^T$ is called the pseudo-inverse of A . If A is square and invertible, then the pseudo-inverse equals A^{-1} .

3.3 Gram-Schmidt Orthogonalization

Algorithm: Denote the columns of A by a_i .

1. $q_1 = a_1$. Then normalized by $q_1 = \frac{q_1}{\|q_1\|}$.
2. $q_2 = a_2 - \langle q_1, a_2 \rangle q_1$. Then normalize by $q_2 = \frac{q_2}{\|q_2\|}$. It's simple to verify that $q_2 \perp q_1$.
3. For an arbitrary k , $q_k = a_k - \langle a_k, q_1 \rangle q_1 - \langle a_k, q_2 \rangle q_2 - \dots - \langle a_k, q_{k-1} \rangle q_{k-1}$. Then normalize by $q_k = \frac{q_k}{\|q_k\|}$.

We can observe the following properties:

1. $\|q_i\| = 1$ (this follows directly)
2. $q_i \perp q_j$ for all $i \neq j$
3. $q_k \in \text{span}(a_1, \dots, a_k)$ and $a_k \in \text{span}(q_1, \dots, q_k)$ so that $\text{span}(a_1, \dots, a_k) = \text{span}(q_1, \dots, q_k)$.

[[Write proof for 2]].

3.4 QR Factorization

Definition 7. (Unitary Matrix) A matrix $Q = [q_1 \dots q_n] \in \mathbb{R}^{m \times n}$ is unitary if and only if $\langle q_i, q_j \rangle = \delta_{ij}$.

Observations about this definition:

1. $Q^T Q = I$
2. If Q is square, then $Q^T = Q^{-1}$.

3.4.1 Application to Least Squares

Suppose that we can write $A = QR$, where $A \in \mathbb{R}^{m \times n}$, $Q \in \mathbb{R}^{m \times n}$ and unitary, and $R \in \mathbb{R}^{n \times n}$ and upper triangular. Then the least squares solution to $Ax = b$ is given by

$$\begin{aligned}
 x &= (A^T A)^{-1} A^T b \\
 &= (R^T Q^T Q R)^{-1} R^T Q^T b \\
 &= (R^T R)^{-1} R^T Q^T b \\
 \implies (R^T R)x &= R^T Q^T b \\
 Rx &= Q^T b \quad (\text{assume } R \text{ is invertible (i.e. no zeros on the diagonal)})
 \end{aligned}$$

We can then solve for x using back substitution, which is $\mathcal{O}(n^2)$.

3.5 Norms and Condition Numbers

Definition 8. (Norm) Suppose that \mathcal{V} is a linear space over the field \mathbb{R} . The *nonnegative* real-valued function $\|\cdot\|$ is a norm on \mathcal{V} if the following axioms are satisfied: Fix $v \in \mathcal{V}$

1. Positivity: $\|v\| = 0$ if and only if $v = 0$
2. Scale Preservation: $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in \mathbb{R}$
3. Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Example 1. (Examples of Norms)

1. 1-norm:

$$\|v\|_1 = \sum_{i=1}^n |v_i| = |v_1| + \cdots + |v_n| \quad (15)$$

2. 2-norm:

$$\|v\|_2 = \left(\sum_{i=1}^n v_i^2 \right)^{\frac{1}{2}} = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{v^T v} \quad (16)$$

3. ∞ -norm

$$\|x\|_\infty = \max_{i=1, \dots, n} |v_i| \quad (17)$$

4. p -norm

$$\|v\|_p = \left(\sum_{i=1}^n |v_i|^p \right)^{\frac{1}{p}} \quad (18)$$

For the p -norm, proving the triangle inequality follows from the Minkowski's inequality.

Definition 9. (Operator Norm) Let A be an $m \times n$ matrix. That is, A is a linear transformation from \mathbb{R}^n to \mathbb{R}^m . Then the operator norm (or subordinate matrix norm) of A is

$$\|A\|_{p,q} = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}. \quad (19)$$

Observations about this definition:

1. It's easy to check that this definition of the operator norm satisfies the properties of a norm given in Definition 8. For the triangle inequality, observe that

$$\begin{aligned} \|(A+B)x\|_p &\leq \|Ax\|_p + \|Bx\|_p && \text{(from Minkowski's inequality)} \\ \implies \frac{\|(A+B)x\|_p}{\|x\|_p} &\leq \frac{\|Ax\|_p}{\|x\|_p} + \frac{\|Bx\|_p}{\|x\|_p} \end{aligned}$$

Taking the supremum of both sides over x shows that $\|A+B\|_p \leq \|A\|_p + \|B\|_p$.

2. The definition immediately implies that for an arbitrary $x \in \mathbb{R}^n, x \neq 0$,

$$\|Ax\|_q \leq \|A\|_{p,q} \|x\|_p \quad (20)$$

We can generalize this inequality to claim that

$$\|AB\| \leq \|A\| \|B\| \quad (21)$$

for conformable matrices A, B . Indeed, fix $0 \neq x \in \mathbb{R}^n$. Then

$$\|ABx\| \leq \|A\| \|Bx\| \leq \|A\| \|B\| \|x\| \quad (22)$$

We can divide all inequalities by $\|x\|$ to see that for all $x \neq 0$,

$$\frac{\|ABx\|}{\|x\|} \leq \|A\| \|B\| \quad (23)$$

Taking the supremum over x on the left hand side shows that $\|AB\| \leq \|A\| \|B\|$.

Theorem 6. (The 1-norm of a matrix is the largest absolute-value column sum) Let $A \in \mathbb{R}^{m \times n}$ and denote the columns of A by $a_j, j = 1, \dots, n$. Then $\|A\|_1 = \max_{j=1, \dots, n} \sum_{i=1}^m |a_{ij}| = \max_{j=1, \dots, n} \|a_j\|$.

Proof. Fix $x \in \mathbb{R}^n$. Let $C = \max_{j=1, \dots, n} \sum_{i=1}^m |a_{ij}|$. First consider the product $A \cdot x$. The i th

element is $\sum_{j=1}^n a_{ij}x_j$. Then

$$\begin{aligned}
\|Ax\|_1 &= \sum_{i=1}^m |(Ax)_i| = \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij}x_j \right| \\
&\leq \sum_{i=1}^m \sum_{j=1}^n |a_{ij}| |x_j| && \text{(triangle inequality)} \\
&= \sum_{j=1}^n |x_j| \left(\sum_{i=1}^m |a_{ij}| \right) && \text{(interchange order of summation, assumed finite)} \\
&\leq C \|x\|_1
\end{aligned}$$

Therefore $\frac{\|Ax\|_1}{\|x\|_1} \leq C$ for all x . Next, we find an x such we achieve equality with C . Call index J the index such that $\|a_J\|_1 = C = \max_{j=1,\dots,n} \sum_{i=1}^m |a_{ij}|$. Then let e_J be the n -vector of zeros with a 1 in the J th entry. Clearly $\|e_J\|_1 = 1$. But then

$$\|Ae_J\|_1 = \|a_J\|_1 = C \quad (24)$$

In sum, we first showed that for all $x \in \mathbb{R}^n$

$$\frac{\|Ax\|_1}{\|x\|_1} \leq C \quad (25)$$

We then found an $x \in \mathbb{R}^n$ such that $\frac{\|Ax\|_1}{\|x\|_1} = C$. Therefore

$$\|A\|_1 = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = C = \max_{j=1,\dots,n} \sum_{i=1}^m |a_{ij}| = \max_{j=1,\dots,n} \|a_j\|_1 \quad (26)$$

□

Theorem 7. (The ∞ -norm of a matrix is the largest absolute-value row sum) Let $A \in \mathbb{R}^{m \times n}$ and denote the rows of A by b_i , $i = 1, \dots, m$. Then $\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}| = \max_{i=1,\dots,m} \|b_i\|_1$.

Proof. Fix $x \in \mathbb{R}^n$. Let $C = \max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}|$.

$$\begin{aligned}
\|Ax\|_\infty &= \max_{i=1,\dots,m} \left| \sum_{j=1}^n a_{ij}x_j \right| \\
&\leq \max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}| |x_j| && \text{(by the triangle inequality)} \\
&\leq \max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}| \|x\|_\infty && \text{(since } |x_j| \leq \|x\|_\infty \text{ for all } j) \\
&= C \|x\|_\infty
\end{aligned}$$

Next, we find an x such we achieve equality with C . Call I the index for which $\|b_I\|_\infty = C$. Define

$$x_j = \begin{cases} 1 & a_{Ij} > 0 \\ -1 & a_{Ij} < 0 \end{cases} \quad (27)$$

Observe that $\|x\|_\infty = 1$. Then

$$\begin{aligned}
|A \cdot x|_I &= |b_I^T \cdot x| \\
&= \left| \sum_{j=1}^m a_{Ij}x_j \right| \\
&= \left| \sum_{j=1}^m |a_{Ij}| \right| \\
&= C
\end{aligned}$$

We then found an $x \in \mathbb{R}^n$ such that $\frac{\|Ax\|_\infty}{\|x\|_\infty} = C$. Therefore

$$\|A\|_\infty = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = C = \max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}| = \max_{i=1,\dots,m} \|b_i\| \quad (28)$$

□

Theorem 8. (The 2-norm of a symmetric positive definite matrix is the maximum absolute value of its eigenvalues) Let A be a positive definite $n \times n$ matrix. Then

$$\|A\|_2 = \max_{i=1,\dots,n} |\lambda_i| \quad (29)$$

Proof. Since A is positive definite, A has n distinct eigenvalues, which implies that it has n linearly independent eigenvectors. Therefore, for an arbitrary $x \in \mathbb{R}^n$, we can write x

as a linearly combination of the eigenvectors x_1, \dots, x_n . Then

$$\begin{aligned} x &= c_1 x_1 + \dots + c_n x_n \\ Ax &= c_1 A x_1 + \dots + c_n A x_n \\ &= c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n \end{aligned}$$

We can normalize the eigenvectors of A so that $x_i^T x_i = 1$. Then $\|Ax\|_2 = \sqrt{\sum_{i=1}^n c_i^2 \lambda_i^2}$ and $\|x\|_2 = \sqrt{\sum_{i=1}^n c_i^2}$. Therefore

$$\frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\frac{\sum_{i=1}^n c_i^2 \lambda_i^2}{\sum_{i=1}^n c_i^2}} \leq \max_i |\lambda_i| = |\lambda_I| \quad (30)$$

Now we'll find an x such that we actually achieve equality. Call I the index of the maximum absolute value of an eigenvalue. Then, consider the eigenvector associated with this eigenvalue, called x_I . Then

$$\frac{\|Ax_I\|_2}{\|x_I\|_2} = \frac{|\lambda_I| \|x_I\|}{\|x_I\|} = |\lambda_I| \quad (31)$$

This shows that $\|A\|_2 = \max_i |\lambda_i|$. □

Theorem 9. (The 2-norm of a matrix $A_{m \times n}$ equals its largest singular value) Let A be an $m \times n$ matrix and denote the eigenvalues of the matrix $B = A^T A$ by $\lambda_i, i = 1, \dots, n$. Then

$$\|A\|_2 = \max_i \sqrt{\lambda_i} \quad (32)$$

The square roots of the (nonnegative) eigenvalues of $A^T A$ are referred to as the singular values of A .

3.5.1 Conditioning

Conditioning helps us quantify the sensitivity of the output to perturbations of the input. In what follows, let f be a mapping from a subset D of a normed linear space \mathcal{V} to another normed linear space \mathcal{W} .

Definition 10. (Absolute Condition Number)

$$\text{Cond}(f) = \sup_{x, y \in D, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \quad (33)$$

Definition 11. (Absolute Local Condition Number)

$$\text{Cond}_x(f) = \sup_{x + \delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|} \quad (34)$$

The previous two definitions depend on the magnitudes of $f(x)$ and x . In applications, it's often better to rescale as follows

Definition 12. (Relative Local Condition Number)

$$\text{cond}_x(f) = \sup_{x+\delta x \in D, \delta x \neq 0} \frac{\|f(x+\delta x) - f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|} \quad (35)$$

In these definitions, if f is differentiable then we can replace the differences with the appropriate derivatives.

Example 2. (Example of condition numbers) Let D be a subinterval of $[0, \infty)$ and $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$.

1. If $D = [1, 2]$, then $\text{Cond}(f) = \frac{1}{2}$.
2. If $D = [0, 1]$, then $\text{Cond}(f) = \infty$.
3. If $D = (0, \infty)$, then the absolute local condition number of f at $x \in D$ is

$$\text{Cond}_x(f) = \frac{1}{2\sqrt{x}} \quad (36)$$

Thus as $x \rightarrow 0$, $\text{Cond}_x(f) \rightarrow \infty$, and as $x \rightarrow \infty$, $\text{Cond}_x(f) \rightarrow 0$.

4. If $D = (0, \infty)$, then the relative local condition number of f is $\text{cond}_x(f) = 1/2$ for all $x \in D$.

Definition 13. (Condition Number of a Nonsingular Matrix) The condition number of a nonsingular matrix A is defined by

$$\kappa(A) = \|A\| \|A^{-1}\| \quad (37)$$

If $\kappa(A) \gg 1$, the matrix is said to be ill-conditioned.

Observations about this definition:

1. $\kappa(A) = \kappa(A^{-1})$
2. For all A , $\kappa(A) \geq 1$. This follows because

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| \quad (38)$$

3. The condition number of a matrix is unaffected by scaling all its elements by multiplying by a nonzero constant.
4. There is a condition number for each norm, and the size of the condition number is strongly dependent on the choice of norm.

4 Special Matrices

4.1 Symmetric Positive Definite Matrices

Definition 14. (Symmetric, Positive Definite, spd) The real matrix A is said to be symmetric if $A = A^T$. A square $n \times n$ matrix is called positive definite if

$$\mathbf{x}^T A \mathbf{x} > 0 \quad (39)$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$.

Theorem 10. (Properties of spd matrices) Let A be an $n \times n$ real, spd matrix. Then

1. $a_{ii} > 0$ for all $i = 1, \dots, n$ (the diagonal elements of A are positive).
2. $A\mathbf{x}_i = \lambda_i \mathbf{x}_i \implies \lambda_i \in \mathbb{R}_{>0}, \mathbf{x}_i \in \mathbb{R}^n \setminus \{0\}$ (the eigenvalues of A are real and positive, and the eigenvectors of A belong to $\mathbb{R}^n \setminus \{0\}$).
3. $\mathbf{x}_i \perp \mathbf{x}_j$ if $\lambda_i \neq \lambda_j$ (the eigenvectors of distinct eigenvalues of A are orthogonal)
4. $\det(A) > 0$ (the determinant of A is positive)
5. Every submatrix B of A obtained by deleting any set of rows and the corresponding set of columns from A is symmetric and positive definite (in particular, every principal submatrix is positive definite).

Proof. We prove each claim in the theorem as follows

1. Let \mathbf{e}_i be the i th canonical basis vector in \mathbb{R}^n . Then

$$a_{ii} = \mathbf{e}_i^T A \mathbf{e}_i > 0 \quad (40)$$

since A is pd. A few observations: this only relies on A being pd. $\mathbf{e}_i^T A$ picks out the i th row of A . $A \mathbf{e}_i$ picks out the i th column of A .

2. We'll first show that the eigenvalues of A are real. Suppose λ, \mathbf{x} are an eigenvalue/vector pair of A . Thus $A\mathbf{x} = \lambda\mathbf{x}$. We can conjugate this equation to find that $\bar{A}\bar{\mathbf{x}} = A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}$ (thus complex eigenvalues of real valued matrices come in conjugate pairs). Then

$$\begin{aligned} \mathbf{x}^T A \bar{\mathbf{x}} &= \bar{\lambda} \mathbf{x}^T \bar{\mathbf{x}} \\ \mathbf{x}^T A^T \bar{\mathbf{x}} &= (A\mathbf{x})^T \bar{\mathbf{x}} = \lambda \mathbf{x}^T \bar{\mathbf{x}} \end{aligned}$$

Since $A = A^T$, we know that $\lambda \mathbf{x}^T \bar{\mathbf{x}} = \bar{\lambda} \mathbf{x}^T \bar{\mathbf{x}}$. As long as $\mathbf{x} \neq 0$, then $\mathbf{x}^T \bar{\mathbf{x}} \neq 0$. Therefore $\bar{\lambda} = \lambda$, which shows $\lambda \in \mathbb{R}$.

The fact that the eigenvector associated with λ has real elements follows by noting that all elements of the singular matrix $A - \lambda I$ are real numbers. Therefore, the

columns of $A - \lambda I$ are linearly dependent in \mathbb{R}^n . Hence there exists an $\mathbf{x} \in \mathbb{R}^n$ such that $(A - \lambda I)\mathbf{x} = 0$.

This proof only requires that A is symmetric – therefore any real, symmetric matrix has real eigenvalues/vectors.

Next we'll show the eigenvalues of A are positive. Suppose λ, \mathbf{x} are an eigenvalue/vector pair of A . Then

$$0 < \mathbf{x}^T A \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} \quad (41)$$

Since $\mathbf{x} \neq 0$ and $\mathbf{x}^T \mathbf{x}$ is positive (it's actually the squared 2-norm of \mathbf{x}), then $\lambda > 0$. Note that this part of the proof requires A be pd.

3. Let λ_i, λ_j be distinct eigenvalues of A , and $\mathbf{x}_i, \mathbf{x}_j$ the corresponding eigenvectors. Then

$$\begin{aligned} \mathbf{x}_i^T A \mathbf{x}_j &= \lambda_j \mathbf{x}_i^T \mathbf{x}_j \\ \mathbf{x}_i^T A^T \mathbf{x}_j &= (A \mathbf{x}_i)^T \mathbf{x}_j = \lambda_i \mathbf{x}_i^T \mathbf{x}_j \end{aligned}$$

Since A is symmetric, these two string of equalities must be equal. We can subtract them to find that

$$(\lambda_i - \lambda_j) \mathbf{x}_i^T \mathbf{x}_j = 0 \quad (42)$$

Since we assumed $\lambda_i \neq \lambda_j$, then it must be that $\mathbf{x}_i^T \mathbf{x}_j = 0$. Therefore $\mathbf{x}_i \perp \mathbf{x}_j$. This proof again only relies on the symmetry of A .

4. This follows from the fact that the determinant of A is equal to the product of its eigenvalues.

5.

□

4.2 Banded Matrices and Differential Equations

4.3 Cholesky Factorization

Notation:

- L_{k-1} : the first $k - 1 \times k - 1$ upper left corner of L
- a_k : the first $k - 1$ entries in column k of A
- l_k : the first $k - 1$ entries in column k of L^T [??]
- a_{kk}, l_{kk} : the kk entries of A and L , respectively

Algorithm 1 Cholesky Factorization

Require: $A \in \mathbb{R}^{n \times n}$, SPD

```
 $L_1 \leftarrow \sqrt{a_{11}}$   
for  $k \leftarrow 2, 3, \dots, n$  do  
  Solve  $L_{k-1}l_k = a_k$  for  $l_k$   
   $l_{kk} \leftarrow \sqrt{a_{kk} - l_k^T l_k}$   
   $L_k \leftarrow \begin{pmatrix} L_{k-1} & 0 \\ l_k^T & l_{kk} \end{pmatrix}$   
end for
```

5 Simultaneous nonlinear equations

5.1 Analysis Preliminaries

Definition 15. (Cauchy Sequence) A sequence $(x^{(k)}) \subset \mathbb{R}^n$ is called a Cauchy sequence in \mathbb{R}^n if for any $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$ such that

$$\|x^{(k)} - x^{(m)}\|_\infty < \epsilon \quad \forall k, m \geq k_0(\epsilon) \quad (43)$$

Remark 1. \mathbb{R}^n is **complete** in the sense that every Cauchy sequence $(x^{(k)})$ converges to some $\xi \in \mathbb{R}^n$.

Definition 16. (Continuous function) Let $D \subset \mathbb{R}^n$ be nonempty and $f : D \rightarrow \mathbb{R}^n$. Given $\xi \in D$, f is continuous at ξ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $x \in B(\xi; \delta) \cap D$

$$\|f(x) - f(\xi)\|_\infty < \epsilon \quad (44)$$

Lemma 1. Let $D \subset \mathbb{R}^n$ be nonempty and $f : D \rightarrow \mathbb{R}^n$ be defined and continuous on D . If $(x^{(k)}) \subset D$ converges in \mathbb{R}^n to $\xi \in D$, then $f(x^{(k)})$ also converges to $f(\xi)$.

5.2 Simultaneous iteration

Example 3.

Definition 17. (Lipschitz condition, constant, and contraction) Let D be a closed subset of \mathbb{R}^n and $g : D \rightarrow D$. If there exists a positive constant L such that

$$\|g(x) - g(y)\|_\infty \leq L\|x - y\|_\infty \quad (45)$$

for all $x, y \in D$, then g satisfies the Lipschitz condition on D in the ∞ -norm. L is called the Lipschitz constant. If $L \in (0, 1)$, then g is called a contraction on D in the ∞ -norm.

Observations about this definition:

- Any function g that satisfies the Lipschitz condition on D is continuous on D (to see this, set $\delta = \frac{\epsilon}{L}$).
- If g satisfies the Lipschitz condition on D in the ∞ -norm, then it also does in the p -norm for $p \in [1, \infty)$ and vice-versa. However the size of L depends on the choice of norm.

Theorem 11. (Contraction Mapping Theorem in \mathbb{R}^n) Suppose D is a closed subset of \mathbb{R}^n and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined on D , and $g(D) \subset D$. Suppose further that g is a contraction on D in the ∞ -norm. Then,

1. g has a unique fixed point $\xi \in D$
2. The sequence $(x^{(k)})$ defined by $x^{(k+1)} = g(x^{(k)})$ converges to ξ for any starting value $x^{(0)} \in D$.

Proof. The proof has three parts:

1. First prove uniqueness, assuming existence of a fixed point.
2. Prove the iteration generates a Cauchy sequence (then convergence to some ξ follows from the completeness of the space).
3. Show ξ is indeed the fixed point.

Uniqueness: Suppose ξ, η are both fixed points of g in D . Then,

$$\begin{aligned} \|\xi - \eta\|_{\infty} &= \|g(\xi) - g(\eta)\| && (\xi, \eta \text{ are fixed points}) \\ &\leq L\|\xi - \eta\|_{\infty} && (g \text{ is a contraction on } D) \end{aligned}$$

We can rearrange this to see that $(1 - L)\|\xi - \eta\|_{\infty} \leq 0$. By assumption, $L \in (0, 1)$, and the norm of a quantity is always weakly positive. Therefore, $\|\xi - \eta\|_{\infty} = 0$ which implies $\xi = \eta$.

Convergence: Assuming g has a fixed point $\xi \in D$, the sequence $x^{(k+1)} = g(x^{(k)})$ will converge to ξ for any $x^{(0)} \in D$. This follows because

$$\|x^{(k)} - \xi\|_{\infty} \leq L^k \frac{1}{1 - L} \|x^{(1)} - x^{(0)}\|_{\infty} \quad (46)$$

Since $L \in (0, 1)$, $\lim_{k \rightarrow \infty} L^k = 0$, and therefore

$$\lim_{k \rightarrow \infty} \|x^{(k)} - \xi\|_{\infty} = 0 \quad (47)$$

Existence: First observe that if $x^{(0)}$ belongs to D , then $x^{(k+1)} = g(x^{(k)}) \in D$ for all $k \geq 1$ since $g(D) \subset D$ (this is important since the proof relies on g being a contraction on D).

Next, consider the distance between two adjacent terms on the sequence $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$

$$\begin{aligned}\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty &= \|g(\mathbf{x}^{(k-1)}) - g(\mathbf{x}^{(k-2)})\|_\infty && \text{(definition of } g) \\ &\leq L\|\mathbf{x}^{(k-1)} - \mathbf{x}^{(k-2)}\|_\infty && (g \text{ is a contraction on } D) \\ &\leq L^{k-1}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty && \text{(induction)}\end{aligned}$$

Now, fix positive integers m, k such that $m > k$. Then

$$\begin{aligned}\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\|_\infty &= \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)} + \mathbf{x}^{(m-1)} - \mathbf{x}^{(m-2)} + \dots + \mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty \\ &\leq \|\mathbf{x}^{(m)} - \mathbf{x}^{(m-1)}\|_\infty + \dots + \|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\|_\infty && \text{(triangle inequality)} \\ &\leq (L^{m-1} + \dots + L^k)\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty && (g \text{ a contraction}) \\ &= L^k(L^{m-k-1} + \dots + 1)\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty \\ &\leq L^k \frac{1}{1-L} \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty && \text{(geometric series)}\end{aligned}$$

Since $L \in (0, 1)$, $\lim_{k \rightarrow \infty} L^k = 0$. Therefore, $\mathbf{x}^{(k)}$ is a Cauchy sequence in \mathbb{R}^n , that is for all $\epsilon > 0$, there exists a k_0 such that

$$\|\mathbf{x}^{(m)} - \mathbf{x}^{(k)}\|_\infty < \epsilon \quad \forall m, k \geq k_0 \quad (48)$$

Any Cauchy sequence in \mathbb{R}^n is convergent in \mathbb{R}^n . Thus, there exists some $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $\boldsymbol{\xi} = \lim_{k \rightarrow \infty} \mathbf{x}^{(k)}$.

$\boldsymbol{\xi}$ is indeed the fixed point: Since g satisfies the Lipschitz condition on D , g is continuous on D . Therefore,

$$\boldsymbol{\xi} = \lim_{k \rightarrow \infty} \mathbf{x}^{(k+1)} = \lim_{k \rightarrow \infty} g(\mathbf{x}^{(k)}) = g\left(\lim_{k \rightarrow \infty} \mathbf{x}^{(k)}\right) = g(\boldsymbol{\xi}) \quad (49)$$

therefore $\boldsymbol{\xi}$ is a fixed point of g , and observe that $\boldsymbol{\xi} \in D$ since D is closed. \square

Definition 18. (Jacobian) Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function defined and continuous in an (open) neighborhood of $\boldsymbol{\xi} \in \mathbb{R}^n$. Suppose the first partial derivatives of each g_i exist at $\boldsymbol{\xi}$. The Jacobian matrix $J_g(\boldsymbol{\xi})$ of g at $\boldsymbol{\xi}$ is the $n \times n$ matrix with elements

$$J_g(\boldsymbol{\xi})_{ij} = \frac{\partial g_i}{\partial x_j}(\boldsymbol{\xi}) \quad (50)$$

Theorem 12. Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function defined and continuous on a closed set $D \subset \mathbb{R}^n$. Let $\boldsymbol{\xi} \in D$ be a fixed point of g . Suppose the first partial derivatives of each g_i are defined and continuous in some (open) neighborhood $N(\boldsymbol{\xi}) \in D$ of $\boldsymbol{\xi}$, with

$$\|J_g(\boldsymbol{\xi})\|_\infty < 1 \quad (51)$$

Then there exists $\epsilon > 0$ such that $g(\bar{B}_\epsilon(\xi)) \subset \bar{B}_\epsilon(\xi)$, and the sequence $x^{(k+1)} = g(x^{(k)})$ converges to ξ for all $x^{(0)} \in \bar{B}_\epsilon(\xi)$ (in other words, the sequence converges to ξ as long as $x^{(0)}$ is close enough to ξ).

Example 4.

Newton's Method

Definition 19. (Newton's Method) The sequence defined by

$$x^{(k+1)} = x^{(k)} - [J_f(x^{(k)})]^{-1}f(x^{(k)}) \quad (52)$$

where $x^{(0)} \in \mathbb{R}^n$, is called Newton's method.

Theorem 13. Suppose $f(\xi) = 0$, that in some (open) neighborhood $N(\xi)$ of ξ , where f is defined and continuous, all the second-order partial derivatives of f are defined and continuous, and that the Jacobian matrix $J_f(x^{(k)})$ of f at the point ξ is nonsingular. Then the sequence defined by Newton's method converges to ξ provided that $x^{(0)}$ is sufficiently close to ξ .

6 Eigenvalues of Eigenvectors of a symmetric matrix

Another matrix decomposition is

$$A = X\Lambda X^{-1} \quad (53)$$

where X is a matrix of the eigenvectors and Λ is a diagonal matrix with the eigenvalues.

6.1 Why we use iteration to calculate eigenvalues/eigenvectors

We call λ an eigenvalue and $x \neq 0$ an eigenvector of A if $Ax = \lambda x$. Thus, $(Ax - \lambda I)x = 0$. Therefore, $x \in \text{Null}(A - \lambda I)$. Since $x \neq 0$, $A - \lambda I$ has a non-trivial nullspace, so we must have $\det(A - \lambda I) = 0$. This suggests a way to transform an eigenvalue finding problem to a root finding problem. Define

$$\rho(\lambda) = \det(A - \lambda I) \quad (54)$$

Recall that the determinant of a matrix is the product of its eigenvalues. If A is a $n \times n$ real, symmetric matrix, then $\rho(\lambda)$ is an n -th order polynomial in λ , whose roots are the eigenvalues of A .

Theorem 14. (Abel(-Ruffini) Theorem, or "No-go Theorem") There is no algebraic solution (that is, a solution expressed in terms of radicals) to general polynomial equations of degree five or higher with arbitrary coefficients.

Therefore, there is no finite-number operation procedure that provides an eigenvalue decomposition.

6.2 Power Iteration

Find the biggest eigenvalue/vector.

Algorithm 2 Power Iteration

Require: $v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$

```

1: for  $k \leftarrow 1, 2, \dots$  do
2:    $w \leftarrow Av^{(k-1)}$  ▷ Apply  $A$ 
3:    $v^{(k)} \leftarrow w / \|w\|$  ▷ Normalize
4:    $\lambda^{(k)} \leftarrow (v^{(k)})^T Av^{(k)} = \langle v^{(k)}, Av^{(k)} \rangle$  ▷ Rayleigh Quotient
5: end for

```

Theorem 15. (Convergence of Power Iteration) Suppose $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$ and $q_1^T v^{(0)} \neq 0$. Then the iterates of power iteration satisfy

$$\|v^{(k)} - (\pm q_1)\| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \quad (\text{error of eigenvector})$$

$$|\lambda^{(k)} - \lambda_1| = \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) \quad (\text{error of eigenvalue})$$

Proof. Convergence of eigenvector: Write $v^{(0)} = v$ as a linear combination of the orthonormal eigenvectors q_i :

$$v = c_1 q_1 + \dots + c_n q_n \quad (55)$$

$v^{(k)}$ is a scalar multiple of $A^k v^{(0)}$. Therefore

$$\begin{aligned} v^{(k)} &= \alpha_k A^k v^{(0)} && (\alpha_k \text{ a normalization constant}) \\ &= \alpha_k \left(\sum_{i=1}^n \lambda_i^k a_i q_i \right) \\ &= \alpha_k \lambda_1^k \left(c_1 q_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k q_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k q_n \right) \end{aligned}$$

We can choose α_k such that $\alpha_k \lambda_1^k$ is 1. Therefore, $c_1 q_1$ is dominating (as long as $c_1 \neq 0$). The other terms are of order $\mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$.

Convergence of eigenvalue: see proposition below.

□

Theorem 16. (Error of Rayleigh Quotient) Let x_1 be the eigenvector that corresponds to the largest (in absolute value) eigenvalue. If $\|x - x_1\| = \mathcal{O}(\epsilon)$, then

$$\left| \frac{\langle x, Ax \rangle}{\langle x, x \rangle} - \lambda_1 \right| = \mathcal{O}(\epsilon^2) \quad (56)$$

Proof. **TODO.**

□

6.3 Inverse Iteration

Find the smallest eigenvalue/vector.

6.4 Simultaneous Iteration

Obtain the full set of eigenvalues/vectors simultaneously.

Algorithm 3 Simultaneous Iteration

Require: $Q^{(0)} = V = I$, a list of vectors V , which we choose to be the identity

- 1: **for** $k \leftarrow 1, 2, \dots$ **do**
 - 2: $Z \leftarrow A Q^{(k-1)}$ ▷ Apply A
 - 3: $Z \leftarrow \underline{Q}^{(k)} R^{(k)}$ ▷ QR factorization of Z
 - 4: $A^{(k)} \leftarrow (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}$ ▷ $A_{ii}^{(k)} = \langle q_i^{(k)}, A q_i^{(k)} \rangle$
 - 5: **end for**
-

Intuitively,

$$A^K \cdot V = [\sum_i \lambda_i^k c_{1i} \tilde{q}_i \mid \sum_i \lambda_i^k c_{2i} \tilde{q}_i \mid \dots] \quad (57)$$

The first column vector will converge to \tilde{q}_1 . The second vector will converge to $\tilde{q}_1 + \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \tilde{q}_2$.

$\underline{Q}^{(k)}$ will converge to the matrix of eigenvectors:

$$X = [x_1 \mid x_2 \mid \dots \mid x_n] \quad (58)$$

$\underline{A}^{(k)}$ will converge to a diagonal matrix containing the eigenvalues.

6.5 Shifted Power Iteration

Find the eigenvalue close to a specific number.

6.6 QR Algorithm

The QR can be viewed as a stable procedure for computing QR factorizations of the matrix powers A, A^2, A^3, \dots

Algorithm 4 QR Algorithm (without shifts)

Require: $A^{(0)} = A$

- 1: **for** $k \leftarrow 1, 2, \dots$ **do**
 - 2: $Q^{(k)} R^{(k)} \leftarrow A^{(k-1)}$ \triangleright QR factorization of $A^{(k-1)}$
 - 3: $A^{(k)} \leftarrow R^{(k)} Q^{(k)}$ \triangleright Recombine factors in reverse order
 - 4: **end for**
-

6.7 Simultaneous Iteration equivalent to QR Algorithm

The QR algorithm is equivalent to simultaneous iteration applied to a full set of initial vectors, namely, $\hat{Q}^{(0)} = I$. Summary of each algorithm:

Simultaneous Iteration

$$\begin{aligned} \underline{Q}^{(0)} &= I && \text{(initial condition)} \\ \underline{Z} &= A \underline{Q}^{(k-1)} && \text{(apply } A) \\ \underline{Z} &= \underline{Q}^{(k)} R^{(k)} && \text{(resemblance of normalization, QR factorization of } \underline{Z}) \\ A^{(k)} &= (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} && \text{(resemblance of Rayleigh quotient)} \end{aligned}$$

QR Algorithm

$$\begin{aligned} A^{(0)} &= A && \text{(initial condition)} \\ A^{(k-1)} &= Q^{(k)} R^{(k)} && \text{(compute QR factorization)} \\ A^{(k)} &= R^{(k)} Q^{(k)} && \text{(reverse order of factors)} \\ \underline{Q}^{(k)} &= Q^{(1)} Q^{(2)} \dots Q^{(k)} && \text{(definition of } \underline{Q}^{(k)}) \end{aligned}$$

and

$$\underline{R}^{(k)} = R^{(k)} R^{(k-1)} \dots R^{(1)} \quad \text{(definition of } \underline{R}^{(k)})$$

Theorem 17. (Equivalence of Simultaneous Iteration and the QR Algorithm) Simultaneous Iteration and the QR Algorithm generate identical sequences of matrices $\underline{R}^{(k)}, \underline{Q}^{(k)}, A^{(k)}$.

Both give

$$\begin{aligned} (a) : A^{(k)} &= \underline{Q}^{(k)} \underline{R}^{(k)} && \text{(QR factorization of the } k\text{th power of } A) \\ (b) : A^{(k)} &= (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} && \text{(projection)} \end{aligned}$$

Proof. By induction on k (number of iterations). The base case $k = 0$ is trivial.

1. QR gives (a): Assume $A^{(k-1)} = \underline{Q}^{(k-1)} \underline{R}^{(k-1)}$. The inductive hypothesis for (b) gives that $A^{(k-1)} = (\underline{Q}^{(k-1)})^T A \underline{Q}^{(k-1)}$ or that $\underline{Q}^{(k-1)} A^{(k-1)} = A \underline{Q}^{(k-1)}$. Then

$$\begin{aligned} A^{(k)} &= A A^{(k-1)} && \text{(decompose to use inductive hypothesis)} \\ &= A \underline{Q}^{(k-1)} \underline{R}^{(k-1)} && \text{(inductive hypothesis)} \\ &= \underline{Q}^{(k-1)} A^{(k-1)} \underline{R}^{(k-1)} && \text{(inductive hypothesis from (b))} \\ &= \underline{Q}^{(k-1)} \underline{R}^{(k-1)} \underline{Q}^{(k-1)} \underline{R}^{(k-1)} && \text{(from algorithm)} \\ &= \underline{Q}^{(k)} \underline{R}^{(k)} && \text{(from definitions of } \underline{Q}^{(k)}, \underline{R}^{(k)}) \end{aligned}$$

2. QR gives (b): Assume $A^{(k-1)} = (\underline{Q}^{(k-1)})^T A \underline{Q}^{(k-1)}$. From the relationship $A^{(k-1)} = \underline{Q}^{(k-1)} \underline{R}^{(k-1)}$ and the fact that $\underline{Q}^{(k)}$ is orthogonal, we can apply $(\underline{Q}^{(k)})^T$ to both sides (on the left) to get that $(\underline{Q}^{(k)})^T A^{(k-1)} = \underline{R}^{(k)}$. Then

$$\begin{aligned} A^{(k)} &= \underline{R}^{(k)} \underline{Q}^{(k)} \\ &= (\underline{Q}^{(k)})^T A^{(k-1)} \underline{Q}^{(k)} \\ &= (\underline{Q}^{(k)})^T (\underline{Q}^{(k-1)})^T A \underline{Q}^{(k-1)} \underline{Q}^{(k)} && \text{(inductive hypothesis)} \\ &= (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} && \text{(definition of } \underline{Q}^{(k)}) \end{aligned}$$

□