

Theorem 1 (The Mean Value Theorem). Suppose f is a real-valued function, defined and continuous on the closed interval $[a, b] \in \mathbb{R}$ and f differentiable on the open interval (a, b) . Then there exists a number $\xi \in (a, b)$ such that

$$f(b) - f(a) = f'(\xi)(b - a) \quad (1)$$

Theorem 2 (Taylor's Theorem). Suppose that n is a nonnegative integer, and f is a real-valued function, defined and continuous on the closed interval $[a, b]$ of \mathbb{R} , such that the derivatives of f of order up to and including n are defined and continuous on the closed interval $[a, b]$. Suppose further that $f^{(n)}$ is differentiable on the open interval (a, b) . Then, for each value of $x \in [a, b]$, there exists a number $\xi = \xi(x)$ in the open interval (a, b) such that

$$f(x) = f(a) + (x - a)f'(a) + \cdots + \frac{(x - a)^n}{n!}f^{(n)}(a) + \frac{(x - a)^{n+1}}{(n + 1)!}f^{(n+1)}(\xi) \quad (2)$$

Theorem 3 (Existence of Root). Let f be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Assume further, that $f(a)f(b) \leq 0$; then, there exists ξ in $[a, b]$ such that $f(\xi) = 0$.

Theorem 4 (Brouwer's Fixed Point Theorem). Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Then, there exists $\xi \in [a, b]$ such that $\xi = g(\xi)$. ξ is called a fixed point of the function g .

Theorem 5 (Contraction Mapping Theorem). Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose g is a contraction on $[a, b]$. Then, g has a unique fixed point ξ in the interval $[a, b]$. Moreover, the sequence (x_k) defined by simple iteration converges to ξ as $k \rightarrow \infty$ for any starting value x_0 in $[a, b]$.

Let $\epsilon > 0$ be a certain tolerance, and let $k_0(\epsilon)$ denote the smallest positive integer such that x_k is no more than ϵ away from the fixed point ξ (i.e. $|x_k - \xi| \leq \epsilon$) for all $k \geq k_0(\epsilon)$. Then,

$$k_0(\epsilon) \leq \left\lceil \frac{\ln |x_1 - x_0| - \ln(\epsilon(1 - L))}{\ln(1/L)} \right\rceil + 1 \quad (3)$$

Theorem 6 (Contraction Mapping Theorem when Differentiable). Suppose that g is a real-valued function, defined and continuous on a bounded closed

interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $\xi = g(\xi) \in [a, b]$ be a fixed point of g (the existence of this point is guaranteed by Brouwer's fixed point theorem). Assume g has a continuous derivative in some neighborhood of ξ with $|g'(\xi)| < 1$. Then the sequence (x_k) defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ as $k \rightarrow \infty$, provided that x_0 is close to ξ .

Theorem 7 (Unstable Fixed Points). Suppose that $\xi = g(\xi)$, where the function g has a continuous derivative in some neighborhood of ξ , and let $|g'(\xi)| > 1$ (thus ξ is an unstable fixed point). Then the sequence (x_k) defined by simple iteration $x_{k+1} = g(x_k)$, $k \geq 0$, does not converge to ξ from any starting value x_0 , $x_0 \neq \xi$.

Theorem 8 (Convergence of Newton's Method). Suppose that f is a continuous real-valued function with continuous second derivative f'' defined on the closed interval $I_\delta = [\xi - \delta, \xi + \delta]$, $\delta > 0$, such that $f(\xi) = 0$ and $f''(\xi) \neq 0$. Additionally suppose that there exists a positive constant A such that

$$\frac{|f''(x)|}{|f'(y)|} \leq A \quad \forall x, y \in I_\delta \quad (4)$$

If initially

$$|\xi - x_0| \leq h = \min(\delta, \frac{1}{A}) \quad (5)$$

then the sequence (x_k) defined by Newton's method converges quadratically to ξ .

Theorem 9 (Convergence of Secant Method). Suppose that f is a real-valued function, defined and continuously differentiable on an interval $I = [\xi - h, \xi + h]$, $h > 0$, with center point ξ . Suppose further that $f(\xi) = 0$, $f'(\xi) \neq 0$. Then, the sequence (x_k) defined by the secant method converges at least linearly to ξ provided that x_0 and x_1 are sufficiently close to ξ .

Theorem 10 (The 1-norm of a matrix is the largest absolute-value column sum). Let $A \in \mathbb{R}^{m \times n}$ and denote the columns of A by a_j , $j = 1, \dots, n$. Then $\|A\|_1 = \max_{j=1, \dots, n} \sum_{i=1}^m |a_{ij}| = \max_{j=1, \dots, n} \|a_j\|$.

Theorem 11 (The ∞ -norm of a matrix is the largest absolute-value row sum). Let $A \in \mathbb{R}^{m \times n}$ and denote the rows of A by b_i , $i = 1, \dots, m$. Then $\|A\|_\infty = \max_{i=1, \dots, m} \sum_{j=1}^n |a_{ij}| = \max_{i=1, \dots, m} \|b_i\|$.

Theorem 12 (The 2-norm of a symmetric positive definite matrix is the maximum absolute value of its eigenvalues). Let A be a positive definite $n \times n$ matrix. Then

$$\|A\|_2 = \max_{i=1,\dots,n} |\lambda_i| \quad (6)$$

Theorem 13 (The 2-norm of a matrix $A_{m \times n}$ equals its largest singular value). Let A be an $m \times n$ matrix and denote the eigenvalues of the matrix $B = A^T A$ by λ_i , $i = 1, \dots, n$. Then

$$\|A\|_2 = \max_i \sqrt{\lambda_i} \quad (7)$$

The square roots of the (nonnegative) eigenvalues of $A^T A$ are referred to as the singular values of A .

Theorem 14 (Properties of spd matrices). Let A be an $n \times n$ real, spd matrix. Then

1. $a_{ii} > 0$ for all $i = 1, \dots, n$ (the diagonal elements of A are positive).
2. $Ax_i = \lambda_i x_i \implies \lambda_i \in \mathbb{R}_{>0}, x_i \in \mathbb{R}^n \setminus \{0\}$ (the eigenvalues of A are real and positive, and the eigenvectors of A belong to $\mathbb{R}^n \setminus \{0\}$).
3. $x_i \perp x_j$ if $\lambda_i \neq \lambda_j$ (the eigenvectors of distinct eigenvalues of A are orthogonal)
4. $\det(A) > 0$ (the determinant of A is positive)
5. Every submatrix B of A obtained by deleting any set of rows and the corresponding set of columns from A is symmetric and positive definite (in particular, every principal submatrix is positive definite).

Theorem 15 (Cholesky). If A is spd, then there exists a lower diagonal matrix L such that $A = LL^T$. This is called the Cholesky decomposition.

Theorem 16 (Contraction Mapping Theorem in \mathbb{R}^n). Suppose D is a closed subset of \mathbb{R}^n and $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined on D , and $g(D) \subset D$. Suppose further that g is a contraction on D in the ∞ -norm. Then,

1. g has a unique fixed point $\xi \in D$
2. The sequence $(x^{(k)})$ defined by $x^{(k+1)} = g(x^{(k)})$ converges to ξ for any starting value $x^{(0)} \in D$.

Theorem 17 (Jacobian and Fixed Point Stability). Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function defined and continuous on a closed set $D \subset \mathbb{R}^n$. Let $\xi \in D$ be a fixed point of g . Suppose the first partial derivatives of each g_i are defined and continuous in some (open) neighborhood $N(\xi) \in D$ of ξ , with

$$\|J_g(\xi)\|_\infty < 1 \quad (8)$$

Then there exists $\epsilon > 0$ such that $g(\bar{B}_\epsilon(\xi)) \subset \bar{B}_\epsilon(\xi)$, and the sequence $\mathbf{x}^{(k+1)} = g(\mathbf{x}^{(k)})$ converges to ξ for all $\mathbf{x}^{(0)} \in \bar{B}_\epsilon(\xi)$ (in other words, the sequence converges to ξ as long as $\mathbf{x}^{(0)}$ is close enough to ξ).

Theorem 18. Suppose $f(\xi) = 0$, that in some (open) neighborhood $N(\xi)$ of ξ , where f is defined and continuous, all the second-order partial derivatives of f are defined and continuous, and that the Jacobian matrix $J_f(\mathbf{x}^{(k)})$ of f at the point ξ is nonsingular. Then the sequence defined by Newton's method converges to ξ provided that $\mathbf{x}^{(0)}$ is sufficiently close to ξ .

Theorem 19 (Abel(-Ruffini) Theorem, or “No-go Theorem”). There is no algebraic solution (that is, a solution expressed in terms of radicals) to general polynomial equations of degree five or higher with arbitrary coefficients.

Theorem 20 (Convergence of Power Iteration). Suppose $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$ and $q_1^T v^{(0)} \neq 0$. Then the iterates of power iteration satisfy

$$\begin{aligned} \|v^{(k)} - (\pm q_1)\| &= \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) && \text{(error of eigenvector)} \\ |\lambda^{(k)} - \lambda_1| &= \mathcal{O}\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2k}\right) && \text{(error of eigenvalue)} \end{aligned}$$

Theorem 21 (Error of Rayleigh Quotient). Let x_1 be the eigenvector that corresponds to the largest (in absolute value) eigenvalue. If $\|x - x_1\| = \mathcal{O}(\epsilon)$, then

$$\left| \frac{\langle x, Ax \rangle}{\langle x, x \rangle} - \lambda_1 \right| = \mathcal{O}(\epsilon^2) \quad (9)$$

Theorem 22 (Equivalence of Simultaneous Iteration and the QR Algorithm). Simultaneous Iteration and the QR Algorithm generate identical sequences of matrices $\underline{R}^{(k)}, \underline{Q}^{(k)}, A^{(k)}$. Both give

$$\begin{aligned} (a) : A^{(k)} &= \underline{Q}^{(k)} \underline{R}^{(k)} && \text{(QR factorization of the } k\text{th power of } A) \\ (b) : A^{(k)} &= (\underline{Q}^{(k)})^T A \underline{Q}^{(k)} && \text{(projection)} \end{aligned}$$

Theorem 23 (Error of Lagrange interpolation polynomial). Suppose that $n \geq 0$ and the f is a real-valued function, defined and continuous on the closed real interval $[a, b]$, such that derivative of f of order $n + 1$ exists and is continuous on $[a, b]$. Then, with $x \in [a, b]$, there exists $\xi = \xi(x)$ in (a, b) such that

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{k=0}^n (x - x_k) \quad (10)$$

is the interpolation error, where $p(x)$ is n -th order.

Theorem 24 (Chebyshev grid to minimize polynomial interpolation error). The solution to

$$\min_{\{x_i\}} \sup_{t \in [a, b]} \left| \prod_{k=0}^n (x - x_k) \right| \quad (11)$$

is given by a Chebyshev grid:

$$x_i = \cos(\theta_i), \quad \theta_i = \frac{i\pi}{n} \quad (12)$$

Theorem 25 (Orthogonal polynomials form a basis for the space of polynomials).

$$\mathbb{P}_k = \text{span}(\phi_0, \dots, \phi_k) \quad (13)$$

Theorem 26 (OP Recurrence Relation). A set of orthogonal polynomials $\{\phi\}_{i=0}^\infty$ satisfies

$$\phi_{n+1} = (\alpha_n x + \beta_n) \phi_n + \gamma_n \phi_{n-1} \quad (14)$$

Theorem 27 (Roots of Orthogonal Polynomials). If $\{\phi\}_{i=0}^\infty$, then $\phi_n(x)$ has n real roots, called Gaussian quadratures.

Theorem 28 (Locations of Gaussian Quadratures from Recurrence Relation). Give the recurrence relation

$$\phi_{n+1} = (\alpha_n x + \beta_n) \phi_n + \gamma_n \phi_{n-1} \quad (15)$$

we can rewrite this as

$$\alpha_n x \phi_n = \phi_{n+1} - \beta_n \phi_n - \gamma_n \phi_{n-1} \quad (16)$$

Thus for constants a_n, b_n, c_n we have that

$$x \phi_n = \phi_{n-1} + b_n \phi_n + c_n \phi_{n+1} \quad (17)$$

where this equality holds for all x in the domain. We can write this system in matrix form as follows

$$x \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} = \begin{pmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & a_2 & b_2 & \ddots & \\ & & \ddots & \ddots & \\ & & & c_{n-1} & b_n \\ & & & a_n & \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \vdots \\ \vdots \\ \phi_n(x) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ c_n \phi_{n+1} \end{pmatrix} \quad (18)$$

where A is the matrix of coefficients. We want to find the roots $\phi_{n+1}(x_i) = 0$, where $i = 1, \dots, n+1$. Then the eigenvalues of A are the zeros of ϕ_{n+1} . In sum

$$\text{GQ of } \phi_{n+1} = \text{eig}(A) \quad (19)$$

Theorem 29 (Exact integration of $f(x) \in \mathbb{P}_{2N+1}$ using $N+1$ grid points). Suppose. $f(x) \in \mathbb{P}_{2N+1}$. Then

$$\int_a^b f(x)w(x)dx = \sum_{i=0}^N f(x_k)w_k \quad (20)$$

if $\{x_0, \dots, x_N\}$ are the GQ (roots) of ϕ_{N+1} , where

$$w_k = \int_a^b l_k(x)w(x)dx \quad (21)$$

where $l_k(x)$ is a Lagrange polynomial.

Theorem 30 (Projection Coefficients Equivalent to Numerical Representation). Let $f(x) \in \mathbb{P}_{N+1}$. Then

$$\alpha_i = \langle f, \phi_i \rangle = \int_a^b f(x)\phi_i(x)w(x)dx = \sum_{k=0}^N f(x_k)\phi_i(x_k)w_k = c_i \quad (22)$$

That is the projection coefficients c_i are equal to the numerical representation α_i , where the grid points are the GQ of ϕ_{N+1} .

Theorem 31 (Interpolation with Orthogonal Polynomials (Almost Unitary Matrix)). We interpolate f as follows:

$$p(x) = \sum_{n=0}^N c_n \phi_n(x) \quad (23)$$

such that $p(x_i) = f(x_i)$ where the x_i are the GQ of ϕ_{N+1} . Then

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \dots & \phi_N(x_1) \\ \vdots & \vdots & & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \dots & \phi_N(x_N) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix} \quad (24)$$

Then A , the matrix above, is almost unitary. In particular,

$$A^T \cdot W \cdot A = I \quad (25)$$

where W is a diagonal matrix with elements w_0, w_1, \dots, w_N .

Theorem 32 (Projection the best approximation in the L^2 -norm:). $p_N(x)$ is the best approximation in the L^2 -norm:

$$\|f - p_N(x)\|_2^2 \leq \|f - q(x)\|_2^2 \quad (26)$$

for all $q \in \mathbb{P}_N$.

Theorem 33 (Error from Approximation by Projection). Suppose $f \in \mathcal{C}^\infty$ and $\{\phi_i\}_{i=0}^\infty$ is a set orthogonal polynomials. We can write

$$f(x) = \sum_{k=0}^{\infty} c_k \phi_k(x) \quad (27)$$

and define the projection

$$p_N(x) = \sum_{k=0}^N \alpha_k \phi_k(x) \quad (28)$$

Then the error of this approximation is

$$error = \sum_{k=N+1}^{\infty} \alpha_k \phi_k(x) \quad (29)$$

which depends on $\{\alpha_{N+1}, \alpha_{N+2}, \dots\}$. In particular, if $f(x) \in \mathcal{C}^\gamma$, then

$$\alpha_n = \mathcal{O}(n^{-\gamma}) \quad (30)$$

for $n > N$ and

$$\alpha_n = \mathcal{O}\left(\frac{1}{N^\gamma}\right) \quad (31)$$

for $n < N$.

Theorem 34 (First mean value theorem for definite integrals). If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and g is an integrable function that does not change sign on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx \quad (32)$$

Theorem 35 (ODE reduction). Any high order, non-autonomous ODE can be reduced to a 1st order, autonomous ODE (system).

Theorem 36 (Uniqueness). If the force term $f(u)$ is uniformly Lipschitz, then the equation has a unique solution.

Theorem 37 (Forward Euler (one-step) is consistent).