Definition 1 (Simple Iteration)**.** Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Given that $x_0 \in [a, b]$, the recursion defined by

$$x_{k+1} = g(x_k) \tag{1}$$

is called simple iteration; the numbers x_k, $k \geq 0$, are referred to as iterates.

Definition 2 (Contraction)**.** Let g be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Then, g is said to be a contraction on $[a, b]$ if there exists a constant L such that $0 < L < 1$ and

$$|g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in [a, b] \tag{2}$$

Definition 3 (Stable, Unstable Fixed Point)**.** Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$, and let ξ denote a fixed point of g. ξ is a stable fixed point of g if the sequence (x_k) defined by the iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ whenever the starting value x_0 is sufficiently close to ξ. Conversely, if no sequence (x_k) defined by this iteration converges to ξ for any starting value x_0 close to ξ, except for $x_0 = \xi$, then we say that ξ is an unstable fixed point of g.

Definition 4 (Rate of Convergence)**.** Suppose $\xi = \lim_{k\to\infty} x_k$. Define $E_k = |x_k - \xi|$.

Definition 5 (Newton's Method)**.** Newton's method for the solution of $f(x) = 0$ is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{3}$$

Definition 6 (Secant Method)**.** The secant method is defined by

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \tag{4}$$

Definition 7. (Unitary Matrix) A matrix $Q = [q_1 \dots q_n] \in \mathbb{R}^{m\times n}$ is unitary if and only if $\langle q_i, q_j\rangle = \delta_{ij}$.

Definition 8. (Norm) Suppose that $\mathcal{V}$ is a linear space over the field $\mathbb{R}$. The *nonnegative* real-valued function $\|\cdot\|$ is a norm on $\mathcal{V}$ if the following axioms are satisfied: Fix $v \in \mathcal{V}$

1. Positivity: $\|v\| = 0$ if and only if $v = 0$
2. Scale Preservation: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{R}$
3. Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Definition 9 (Operator Norm)**.** Let A be an $m \times n$ matrix. That is, A is a linear transformation form $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the operator norm (or subordinate matrix norm) of A is

$$\|A\|_{p,q} = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}. \tag{5}$$

Definition 10 (Absolute Condition Number)**.**

$$Cond(f) = \sup_{x,y \in D, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \tag{6}$$

Definition 11 (Absolute Local Condition Number)**.**

$$Cond_x(f) = \sup_{x+\delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|} \tag{7}$$

Definition 12 (Relative Local Condition Number)**.**

$$cond_x(f) = \sup_{x+\delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|/\|f(x)\|}{\|\delta x\|/\|x\|} \tag{8}$$

Definition 13 (Condition Number of a Nonsingular Matrix)**.** The condition number of a nonsingular matrix A is defined by

$$\kappa(A) = \|A\|\|A^{-1}\| \tag{9}$$

If $\kappa(A) \gg 1$, the matrix is said to be ill-conditioned.

Definition 14 (Symmetric, Positive Definite, spd)**.** The real matrix A is said to be symmetric if $A = A^T$. A square $n \times n$ matrix is called positive definite if

$$\boldsymbol{x}^T A \boldsymbol{x} > 0 \tag{10}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \neq 0$.

Algorithm 1 Cholesky Factorization

Require: $A \in \mathbb{R}^{n\times n}$, SPD

$L_1 \leftarrow \sqrt{a_{11}}$

for $k \leftarrow 2, 3, \ldots, n$ **do**

  Solve $L_{k-1} l_k = a_k$ for l_k

  $l_{kk} \leftarrow \sqrt{a_{kk} - l_k^T l_k}$

  $L_k \leftarrow \begin{pmatrix} L_{k-1} & 0 \\ l_k^T & l_{kk} \end{pmatrix}$

end for

Definition 15 (Cauchy Sequence)**.** A sequence $(\boldsymbol{x}^{(k)}) \subset \mathbb{R}^n$ is called a Cauchy sequence in $\mathbb{R}^n$ if for any $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$ such that

$$\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(m)}\|_\infty < \epsilon \quad \forall k, m \geq k_0(\epsilon) \tag{11}$$

Definition 16 (Continuous function)**.** Let $D \subset \mathbb{R}^n$ be nonempty and $f : D \to \mathbb{R}^n$. Given $\boldsymbol{\xi} \in D$, f is continuous at $\boldsymbol{\xi}$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $\boldsymbol{x} \in B(\boldsymbol{\xi}; \delta) \cap D$

$$\|f(\boldsymbol{x}) - f(\boldsymbol{\xi})\|_\infty < \epsilon \tag{12}$$

Definition 17 (Lipschitz condition, constant, and contraction)**.** Let D be a closed subset of $\mathbb{R}^n$ and $g : D \to D$. If there exists a positive constant L such that

$$\|g(x) - g(y)\|_\infty \leq L\|x - y\|_\infty \tag{13}$$

for all $x, y \in D$, then g satisfies the Lipschitz condition on D in the ∞-norm. L is called the Lipschitz constant. If $L \in (0, 1)$, then g is called a contraction on D in the ∞-norm.

Definition 18 (Jacobian)**.** Let $g = (g_1, \ldots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined and continuous in an (open) neighborhood of $\boldsymbol{\xi} \in \mathbb{R}^n$. Suppose the first partial derivatives of each g_i exist at $\boldsymbol{\xi}$. The Jacobian matrix $J_g(\boldsymbol{\xi})$ of g at $\boldsymbol{\xi}$ is the $n \times n$ matrix with elements

$$J_g(\boldsymbol{\xi})_{ij} = \frac{\partial g_i}{\partial x_j}(\boldsymbol{\xi}) \tag{14}$$

Definition 19 (Newton's Method)**.** The sequence defined by

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - [J_f(\boldsymbol{x}^{(k)})]^{-1} f(\boldsymbol{x}^{(k)}) \tag{15}$$

where $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$, is called Newton's method.

Algorithm 2 Power Iteration

Require: $v^{(0)} =$ some vector with $\|v^{(0)}\| = 1$

```
for k ← 1, 2, ... do
    w ← Av^(k−1)                                  ▷ Apply A
    v^(k) ← w/‖w‖                                 ▷ Normalize
    λ^(k) ← (v^(k))^T Av^(k) = ⟨v^(k), Av^(k)⟩    ▷ Rayleigh Quotient
end for
```

- **for** $k \leftarrow 1, 2, \ldots$ **do**
  - $w \leftarrow Av^{(k-1)}$ ▷ Apply A
  - $v^{(k)} \leftarrow w/\|w\|$ ▷ Normalize
  - $\lambda^{(k)} \leftarrow (v^{(k)})^T A v^{(k)} = \langle v^{(k)}, Av^{(k)} \rangle$ ▷ Rayleigh Quotient
- **end for**

Algorithm 3 Simultaneous Iteration

Require: $Q^{(0)} = V = I$, a list of vectors V, which we choose to be the identity

- **for** $k \leftarrow 1, 2, \ldots$ **do**
  - $Z \leftarrow A\underline{Q}^{(k-1)}$ ▷ Apply A
  - $Z \leftarrow \underline{Q}^{(k)} R^{(k)}$ ▷ QR factorization of Z
  - $A^{(k)} \leftarrow (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}$ ▷ $A_{ii}^{(k)} = \langle q_i^{(k)}, Aq_i^{(k)} \rangle$
- **end for**

Algorithm 4 QR Algorithm (without shifts)

Require: $A^{(0)} = A$

- **for** $k \leftarrow 1, 2, \ldots$ **do**
  - $Q^{(k)} R^{(k)} \leftarrow A^{(k-1)}$ ▷ QR factorization of $A^{(k-1)}$
  - $A^{(k)} \leftarrow R^{(k)} Q^{(k)}$ ▷ Recombine factors in reverse order
- **end for**

Definition 20 (Lagrange basis polynomial)**.** Given the data $\{x_i\}_{i=0}^n$, define

$$l_j(x) = \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)} \tag{16}$$

which satisfies

$$l_j(x_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{17}$$

(note that $\prod_{i \neq j}(x - x_i)$ is an nth order polynomial (1 less degree than the number of data points) and $\prod_{i \neq j}(x_j - x_i)$ is a constant).

Definition 21 (Lagrange interpolation polynomial)**.** Given the data $\{x_i\}_{i=0}^n$ and corresponding function values $\{f(x_i)\}_{i=0}^n$ the Lagrange interpolation polynomial is

$$p(x) = \sum_{i=0}^{n} f(x_i) l_i(x) \tag{18}$$

Definition 22 (Orthogonal polynomials)**.** Given a domain $[a, b]$ and a weight function $w(x)$ on the domain, a set of orthogonal polynomials is a list of polynomials $\phi_0, \phi_1, \ldots, \phi_N, \ldots$ such that

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x)\phi_j(x)w(x)dx = \delta_{ij} \tag{19}$$

Definition 23 (Autonomous)**.** If the force $\boldsymbol{f}$ has no explicit dependence on t, then we call the ODE (system) autonomous.

Definition 24 (Lipshitz continuous)**.** If

$$|f(u) - f(u^*)| \leq L|u - u^*| \tag{20}$$

for u in a small neighborhood of u^*, then f is Lipshitz continuous at u^*. Note that if f' exists, then

$$L = |f'(u^*)| \tag{21}$$

Definition 25 (Uniformly Lipshitz continuous)**.** If L_u has an upper bound in the domain of f, then f is uniformly Lipshitz continuous.

Definition 26 (Local Truncation Error (LTE))**.** The local truncation error is by how much the true solution fails to satisfy the approximation scheme, which can be written as

$$\tau_n = \frac{u_{n+1} - u_n}{\Delta t} - f(u_n) \tag{22}$$

Definition 27 (Consistency)**.** We say a method is consistent is the LTE goes to 0 as $\Delta \to 0$.