

Definition 1 (Simple Iteration). Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$. Given that $x_0 \in [a, b]$, the recursion defined by

$$x_{k+1} = g(x_k) \quad (1)$$

is called simple iteration; the numbers x_k , $k \geq 0$, are referred to as iterates.

Definition 2. (Contraction) Let g be a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line. Then, g is said to be a contraction on $[a, b]$ if there exists a constant L such that $0 < L < 1$ and

$$|g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in [a, b] \quad (2)$$

Definition 3 (Stable, Unstable Fixed Point). Suppose that g is a real-valued function, defined and continuous on a bounded closed interval $[a, b]$ of the real line, and let $g(x) \in [a, b]$ for all $x \in [a, b]$, and let ξ denote a fixed point of g . ξ is a stable fixed point of g if the sequence (x_k) defined by the iteration $x_{k+1} = g(x_k)$, $k \geq 0$, converges to ξ whenever the starting value x_0 is sufficiently close to ξ . Conversely, if no sequence (x_k) defined by this iteration converges to ξ for any starting value x_0 close to ξ , except for $x_0 = \xi$, then we say that ξ is an unstable fixed point of g .

Definition 4 (Rate of Convergence). Suppose $\xi = \lim_{k \rightarrow \infty} x_k$. Define $E_k = |x_k - \xi|$.

Definition 5 (Newton's Method). Newton's method for the solution of $f(x) = 0$ is defined by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (3)$$

Definition 6 (Secant Method). The secant method is defined by

$$x_{k+1} = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \quad (4)$$

Definition 7. (Unitary Matrix) A matrix $Q = [q_1 \dots q_n] \in \mathbb{R}^{m \times n}$ is unitary if and only if $\langle q_i, q_j \rangle = \delta_{ij}$.

Definition 8. (Norm) Suppose that \mathcal{V} is a linear space over the field \mathbb{R} . The *nonnegative* real-valued function $\|\cdot\|$ is a norm on \mathcal{V} if the following axioms are satisfied: Fix $v \in \mathcal{V}$

1. Positivity: $\|v\| = 0$ if and only if $v = 0$
2. Scale Preservation: $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in \mathbb{R}$
3. Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$.

Definition 9 (Operator Norm). Let A be an $m \times n$ matrix. That is, A is a linear transformation from \mathbb{R}^n to \mathbb{R}^m . Then the operator norm (or subordinate matrix norm) of A is

$$\|A\|_{p,q} = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_q}{\|x\|_p}. \quad (5)$$

Definition 10 (Absolute Condition Number).

$$Cond(f) = \sup_{x, y \in D, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \quad (6)$$

Definition 11 (Absolute Local Condition Number).

$$Cond_x(f) = \sup_{x + \delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\|}{\|\delta x\|} \quad (7)$$

Definition 12 (Relative Local Condition Number).

$$cond_x(f) = \sup_{x + \delta x \in D, \delta x \neq 0} \frac{\|f(x + \delta x) - f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|} \quad (8)$$

Definition 13 (Condition Number of a Nonsingular Matrix). The condition number of a nonsingular matrix A is defined by

$$\kappa(A) = \|A\| \|A^{-1}\| \quad (9)$$

If $\kappa(A) \gg 1$, the matrix is said to be ill-conditioned.

Definition 14 (Symmetric, Positive Definite, spd). The real matrix A is said to be symmetric if $A = A^T$. A square $n \times n$ matrix is called positive definite if

$$\mathbf{x}^T A \mathbf{x} > 0 \quad (10)$$

for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$.

Algorithm 1 Cholesky Factorization

Require: $A \in \mathbb{R}^{n \times n}$, SPD

$L_1 \leftarrow \sqrt{a_{11}}$
for $k \leftarrow 2, 3, \dots, n$ **do**
 Solve $L_{k-1}l_k = a_k$ for l_k
 $l_{kk} \leftarrow \sqrt{a_{kk} - l_k^T l_k}$
 $L_k \leftarrow \begin{pmatrix} L_{k-1} & 0 \\ l_k^T & l_{kk} \end{pmatrix}$
end for

Definition 15 (Cauchy Sequence). A sequence $(\mathbf{x}^{(k)}) \subset \mathbb{R}^n$ is called a Cauchy sequence in \mathbb{R}^n if for any $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}^{(m)}\|_\infty < \epsilon \quad \forall k, m \geq k_0(\epsilon) \quad (11)$$

Definition 16 (Continuous function). Let $D \subset \mathbb{R}^n$ be nonempty and $f : D \rightarrow \mathbb{R}^n$. Given $\boldsymbol{\xi} \in D$, f is continuous at $\boldsymbol{\xi}$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every $\mathbf{x} \in B(\boldsymbol{\xi}; \delta) \cap D$

$$\|f(\mathbf{x}) - f(\boldsymbol{\xi})\|_\infty < \epsilon \quad (12)$$

Definition 17 (Lipschitz condition, constant, and contraction). Let D be a closed subset of \mathbb{R}^n and $g : D \rightarrow D$. If there exists a positive constant L such that

$$\|g(x) - g(y)\|_\infty \leq L\|x - y\|_\infty \quad (13)$$

for all $x, y \in D$, then g satisfies the Lipschitz condition on D in the ∞ -norm. L is called the Lipschitz constant. If $L \in (0, 1)$, then g is called a contraction on D in the ∞ -norm.

Definition 18 (Jacobian). Let $g = (g_1, \dots, g_n)^T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function defined and continuous in an (open) neighborhood of $\boldsymbol{\xi} \in \mathbb{R}^n$. Suppose the first partial derivatives of each g_i exist at $\boldsymbol{\xi}$. The Jacobian matrix $J_g(\boldsymbol{\xi})$ of g at $\boldsymbol{\xi}$ is the $n \times n$ matrix with elements

$$J_g(\boldsymbol{\xi})_{ij} = \frac{\partial g_i}{\partial x_j}(\boldsymbol{\xi}) \quad (14)$$

Definition 19 (Newton's Method). The sequence defined by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [J_f(\mathbf{x}^{(k)})]^{-1} f(\mathbf{x}^{(k)}) \quad (15)$$

where $\mathbf{x}^{(0)} \in \mathbb{R}^n$, is called Newton's method.

Algorithm 2 Power Iteration

Require: $v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$

- 1: **for** $k \leftarrow 1, 2, \dots$ **do**
 - 2: $w \leftarrow Av^{(k-1)}$ ▷ Apply A
 - 3: $v^{(k)} \leftarrow w/\|w\|$ ▷ Normalize
 - 4: $\lambda^{(k)} \leftarrow (v^{(k)})^T Av^{(k)} = \langle v^{(k)}, Av^{(k)} \rangle$ ▷ Rayleigh Quotient
 - 5: **end for**
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Algorithm 3 Simultaneous Iteration

Require: $Q^{(0)} = V = I$, a list of vectors V , which we choose to be the identity

- 1: **for** $k \leftarrow 1, 2, \dots$ **do**
 - 2: $Z \leftarrow AQ^{(k-1)}$ ▷ Apply A
 - 3: $Z \leftarrow \underline{Q}^{(k)} R^{(k)}$ ▷ QR factorization of Z
 - 4: $A^{(k)} \leftarrow (\underline{Q}^{(k)})^T A \underline{Q}^{(k)}$ ▷ $A_{ii}^{(k)} = \langle q_i^{(k)}, Aq_i^{(k)} \rangle$
 - 5: **end for**
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Algorithm 4 QR Algorithm (without shifts)

Require: $A^{(0)} = A$

- 1: **for** $k \leftarrow 1, 2, \dots$ **do**
 - 2: $Q^{(k)} R^{(k)} \leftarrow A^{(k-1)}$ ▷ QR factorization of $A^{(k-1)}$
 - 3: $A^{(k)} \leftarrow R^{(k)} Q^{(k)}$ ▷ Recombine factors in reverse order
 - 4: **end for**
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Definition 20 (Lagrange basis polynomial). Given the data $\{x_i\}_{i=0}^n$, define

$$l_j(x) = \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)} \quad (16)$$

which satisfies

$$l_j(x_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (17)$$

(note that $\prod_{i \neq j} (x - x_i)$ is an n th order polynomial (1 less degree than the number of data points) and $\prod_{i \neq j} (x_j - x_i)$ is a constant).

Definition 21 (Lagrange interpolation polynomial). Given the data $\{x_i\}_{i=0}^n$ and corresponding function values $\{f(x_i)\}_{i=0}^n$ the Lagrange interpolation polynomial is

$$p(x) = \sum_{i=0}^n f(x_i) l_i(x) \quad (18)$$

Definition 22 (Orthogonal polynomials). Given a domain $[a, b]$ and a weight function $w(x)$ on the domain, a set of orthogonal polynomials is a list of polynomials $\phi_0, \phi_1, \dots, \phi_N, \dots$ such that

$$\langle \phi_i, \phi_j \rangle = \int_a^b \phi_i(x) \phi_j(x) w(x) dx = \delta_{ij} \quad (19)$$

Definition 23 (Autonomous). If the force \mathbf{f} has no explicit dependence on t , then we call the ODE (system) autonomous.

Definition 24 (Lipshitz continuous). If

$$|f(u) - f(u^*)| \leq L|u - u^*| \quad (20)$$

for u in a small neighborhood of u^* , then f is Lipshitz continuous at u^* . Note that if f' exists, then

$$L = |f'(u^*)| \quad (21)$$

Definition 25 (Uniformly Lipshitz continuous). If L_u has an upper bound in the domain of f , then f is uniformly Lipshitz continuous.