Definition 0.1 (Set cardinality $\leq$). Let A, B be sets. A has **cardinality less than or equal to** B (write $|A| \leq |B|$) if there exists an injection from A to B. In notation,

$$|A| \leq |B| \iff \exists f : A \to B \text{ injective} \tag{0.1}$$

Definition 0.2 (Partial order). A **partial order** is a pair $\mathcal{A} = (A, \triangleright)$ where $A \neq \varnothing$ such that for $a, b, c \in A$

(i) Antireflexivity: $a \triangleright a$ never happens.

(ii) Transitivity: $a \triangleright b, b \triangleright c \Rightarrow a \triangleright c$

Definition 0.3 (Maximal). Let $(A, \triangleright)$ be a partial order. Then $m \in A$ is maximal if and only if no $a \triangleright m$.

Definition 0.4 (Chain). A **chain** in a partial order $(A, \triangleright)$ is a $C \subseteq A$ such that $\forall a, b \in C$, $a = b$ or $a \triangleright b$ or $b \triangleright a$. (One interpretation in words, "C is linear")

Definition 0.5 (Topology, Topological Space). A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a set of subsets of X (called a **topology**) having the following properties:

(i) $\varnothing$ and X are in $\mathcal{T}$.

$$\varnothing \in \mathcal{T}, \; X \in \mathcal{T} \tag{0.2}$$

(ii) The union of *arbitrarily* many sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_\eta \in \mathcal{T} \; \forall \eta \in H \Rightarrow \bigcup_{\eta \in H} A_\eta = \left\{ a \in X \text{ for some (that is, } a \in A_\eta) \right\} \in \mathcal{T} \tag{0.3}$$

(iii) The intersection a *finite* number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_1, \ldots, A_n \in \mathcal{T} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathcal{T} \tag{0.4}$$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**, that is, the pair $(X, \mathcal{T})$.

Definition 0.6 ((Strictly) Finer, (Strictly) Coarser, Comparable). Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. For these respective situations, we say that $\mathcal{T}$ is **coarser** or **strictly coarser** than $\mathcal{T}'$. We say that $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Definition 0.7 (Basis, Basis Elements, Topology $\mathcal{T}$ generated by $\mathcal{B}$). If X is a set, a **basis** for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X (called **basis elements**) such that

(i) Every element $x \in X$ belongs to some set in $\mathcal{B}$. In symbols

$$\forall x \in X \; \exists B \in \mathcal{B} \; s.t. \; x \in B \tag{0.5}$$

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. More generally, in symbols

$$\forall B_1, \ldots, B_n \in \mathcal{B} \; \forall x \in \bigcap_{i=1}^{n} B_i \; \exists B \in \mathcal{B} \; s.t. \; x \in B \subset \bigcap_{i=1}^{n} B_i \tag{0.6}$$

Definition 0.8 (Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If $\mathcal{B}$ is a basis for a topology on X, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Note that each basis element is itself an element of $\mathcal{T}$. More succinctly:

$$U \subset X : U \in \mathcal{T} \iff \forall x \in U\ \exists B_x \in \mathcal{B}\ s.t.\ x \in B_x \subset U$$

Definition 0.9 (Subbasis)**.** A **subbasis** for a topology on X is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ of X whose union equals X (that is, $\bigcup \mathcal{S} = X$). The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 0.10 (Closed)**.** A subset A of a topological space $(X, \mathcal{T})$ is said to be **closed** if the set $X - A \in \mathcal{T}$. In words, a subset of a topological space is closed if its complement (in the space) is open.

Definition 0.11 (Continuous)**.** Let $(X, \mathcal{T})$ and (Y, σ) be topological spaces. A function $f : X \to Y$ is said to be **continuous** with respect to $\mathcal{T}$ and σ if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. In symbols, $\forall S \in \sigma$, we have that $f^{-1}(S) \in \mathcal{T}$ (where $f^{-1}(S) = \{a \in X \mid f(a) \in S\}$). In words, the preimage of an open set is open.

Definition 0.12 (Homeomorphic)**.** The topological spaces $(X, \mathcal{T})$ and (Y, σ) are **homeomorphic** if there exists a function $f : X \to Y$ such that

(i) f is bijective.

(ii) f is continuous.

(iii) f^{-1} is also continuous.

We write $(X, \mathcal{T}) \cong (Y, \sigma)$ or $f : (X, \mathcal{T}) \cong (Y, \sigma)$.

Definition 0.13 (Open map)**.** $f : (X, \mathcal{T}) \to (Y, \sigma)$ is an **open map** if $\forall S \in \mathcal{T}$ we have that $f(S) \in \sigma$ (recall $f(S) = \{f(s) \mid s \in S\}$). In words: open sets map to open sets.

Definition 0.14 (Closed map)**.** $f : (X, \mathcal{T}) \to (Y, \sigma)$ is a **closed map** if $\forall S \subset X$ such that $X - S \in \mathcal{T}$ we have that $Y - f(S) \in \sigma$. In words: closed sets map to closed sets.

Definition 0.15 (Subspace topology)**.** Given a topological space $(X, \mathcal{T})$ and a non-empty set $A \subseteq X$, the **subspace topology on A induced (or given) by $\mathcal{T}$** is (A, τ_A) where $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$.

Definition 0.16 (Connected)**.** A space (X, τ) is **connected** if whenever sets V, W are

(i) Nonempty

(ii) Open

(iii) $V \cup W = X$

we have $V \cap W \neq \emptyset$.

Definition 0.17 (Metric space)**.** A **metric space** is a nonempty set X together with a binary function $d : X \times X \to \mathbb{R}$ which satisfies the following properties: For all $x, y, z \in X$ we have that

(i) Positivity: $d(x, y) \geq 0$

(ii) Definiteness: $d(x, y) = 0$ if and only if $x = y$

(iii) Symmetry: $d(x, y) = d(y, x)$

(iv) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$

Definition 0.18 (Metric topology)**.** Given a metric space (X, d), the set $\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon > 0, x \in X\}$ is a basis for a topology on X (where $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$) called the **metric topology**.

Definition 0.19 (T_2, Hausdorff)**.** (X, τ) is **T_2 (Hausdorff)** if for every distinct $a, b \in X$, there exist open sets $V, W \in \tau$ such that $a \in V$, $b \in W$, and $V \cap W = \emptyset$.

Definition 0.20 (Converges)**.** Fix a topological space $(X, \mathcal{T})$. A sequence of points $(a_i)_{i \in \mathbb{N}} \subset X$ **converges** to $b \in X$ if for every open set W containing b, all but finitely many of the terms of the sequence are in W. In symbols

$$(a_i)_{i \in \mathbb{N}} \to b \iff \forall W \in \mathcal{T} \text{ s.t. } b \in W, \exists N \in \mathbb{N} \text{ s.t. } \forall m > n, a_m \in W$$

Definition 0.21 (Sequentially closed)**.** A set $S \subset X$ is **sequentially closed** if for every sequence $(a_i)_{i \in \mathbb{N}}$ of points in S converging to some $b \in X$, we have $b \in S$.

Definition 0.22 (Product topology, two sets)**.** Let (X, τ) and (Y, σ) be topological spaces. The **product topology** on $X \times Y$ is the topology having as *basis* the collection $\mathcal{B}$ of all set of the form $U \times V$ where U is an open subset of X and V is an open subset of Y. In symbols,

$$\mathcal{B}_{\tau \times \sigma} = \{U \times V \mid U \in \tau, V \in \sigma\} \tag{0.7}$$

Definition 0.23 (Product space/topology for finitely many spaces)**.** Let $(X_1, \tau_1), \ldots, (X_n, \tau_n)$ be topological spaces. The set of points of the **product space** is $X_1 \times \cdots \times X_n$. The basis for the **product topology** is

$$\mathcal{B}_{\tau_1 \times \cdots \times \tau_n} = \{W_1 \times \cdots \times W_n \mid W_i \in \tau_i\} \tag{0.8}$$

Definition 0.24 (Projection Maps)**.** Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Let

$$\pi_1 : X_1 \times X_2 \to X_1 : (x_1, x_2) \mapsto x_1$$
$$\pi_2 : X_1 \times X_2 \to X_2 : (x_1, x_2) \mapsto x_2$$

then π_1 and π_2 are **projection maps**.

Definition 0.25 (Open cover)**.** An **open cover** of (X, τ) is a family of τ-open sets $\mathcal{C} \subset \tau$ such that $\bigcup \mathcal{C} = X$.

Definition 0.26 (Subcover)**.** $\mathcal{D}$ is a **subcover** of $\mathcal{C}$ if

(i) $\mathcal{D} \subset \mathcal{C}$.

(ii) $\mathcal{D}$ is an open cover.

A **finite subcover** is a subcover which is finite.

Definition 0.27 (Compact)**.** A topological space (X, τ) is **compact** if every open cover has a finite subcover.

Definition 0.28 (Difference)**.** The **difference** of two sets, denoted $A - B$, is the set consisting of those elements of A that are not in B. In notation

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$