Topology Notes

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Contents

1 Set Theory

Definition 1.1 (Set cardinality $\leq$)**.** Let A, B be sets. A has **cardinality less than or equal to** B (write $|A| \leq |B|$) if there exists an injection from A to B. In notation,

$$|A| \leq |B| \iff \exists f : A \to B \text{ injective} \tag{1}$$

Theorem 1.1 (Cantor)**.** *For all sets X (including infinite), $X \ngeq \mathcal{P}(X)$. That is, there does not exist an injection from $\mathcal{P}(X)$ to X.*

Proof. The proof contains 2 steps:

(i) Show that there is no surjection from X to $\mathcal{P}(X)$.

(ii) Show that (i) implies that there is no injection from $\mathcal{P}(X)$ to X.

We start by proving (ii) through the following lemma:

Lemma 1.2. *Let C, D be sets, $C \neq \emptyset$. If there is an injection $i : C \to D$, then there exists a surjection $j : D \to C$.*

Proof. □

The contrapositive of this lemma gives that no surjection from $D \to C$ implies no injection from $C \to D$. □

Theorem 1.3 (Informal statement of the axiom of choice)**.** *Given a family $\mathscr{F}$ of nonempty sets, it is possible to pick out an element from each set in the family.*

Definition 1.2 (Partial order)**.** A **partial order** is a pair $\mathcal{A} = (A, \triangleright)$ where $A \neq \emptyset$ such that for $a, b, c \in A$

(i) Antireflexivity: $a \triangleright a$ never happens.

(ii) Transitivity: $a \triangleright b, b \triangleright c \Rightarrow a \triangleright c$

Remark. With a partial order, you *can* have incomparable elements.

Example 1.1 (Partial order)**.** For any set X, a partial order is $(\mathcal{P}(X), \subsetneq)$. For example, if $X = \{1, 2\}$, then $\{1\}$ and $\{2\}$ are incomparable.

Definition 1.3 (Maximal)**.** Let $(A, \triangleright)$ be a partial order. Then $m \in A$ is maximal if and only if no $a \triangleright m$.

Example 1.2 (Maximal elements)**.** The following are examples of posets and their maximal elements:

(i) $(\mathbb{N}, <)$ has no maximal element (there is no largest natural number).

(ii) $(\{\{1\}, \{2\}\}, \subsetneq)$ has 2 maximal elements, since the two elements of the set are not comparable.

Definition 1.4 (Chain)**.** A **chain** in a partial order $(A, \triangleright)$ is a $C \subseteq A$ such that $\forall a, b \in C$, $a = b$ or $a \triangleright b$ or $b \triangleright a$. (One interpretation in words, "C is linear")

Theorem 1.4 (Zorn's Lemma)**.** *Let $(A, \triangleright)$ be a partial order such that the following condition is satisfied:*

$(\mathcal{Z})$ *In words, if every chain has an upper bound then there is a maximal element. More precisely, ff C is a chain, then $\exists x$ such that $x\triangle$*

2 Topological Spaces

Definition 2.1 (Topology, Topological Space)**.** A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a set of subsets of X (called a **topology**) having the following properties:

(i) $\emptyset$ and X are in $\mathcal{T}$.

$$\emptyset \in \mathcal{T},\ X \in \mathcal{T} \tag{2}$$

(ii) The union of *arbitrarily* many sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_\eta \in \mathcal{T}\ \forall \eta \in H \Rightarrow \bigcup_{\eta \in H} A_\eta = \{a \in X \text{ for some (that is, } a \in A_\eta)\} \in \mathcal{T} \tag{3}$$

(iii) The intersection a *finite* number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_1, \ldots, A_n \in \mathcal{T} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathcal{T} \tag{4}$$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**, that is, the pair $(X, \mathcal{T})$.

Example 2.1 (Examples of topologies)**.** The following are examples of topologies/topological spaces:

(i) The collection consisting of X and $\emptyset$ is called the **trivial topology** or **indiscrete topology**.

  (a) $\mathcal{T} = \{\emptyset, X\}$

(ii) If X is any set, the collection of all subsets of X is a topology on X, called the **discrete topology**.

  (a) $\mathcal{T} = \mathcal{P}(X)$

(iii) Let $X = \{1\}$. Then $\mathcal{T} = \{\emptyset, \{1\}\}$ is a topology.

(iv) Sierpinski: Let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. In this topology, b is glued to a. That is, we can't have a set with b and without a. However, a is *not* glued to a.

Remark. Observe that from these examples,

$$\text{indiscrete} \subsetneq \text{Sierpinski} \subsetneq \text{discrete}$$

(v) $X = \mathbb{R}$ and $\mathcal{T} = \{\emptyset, X\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$.

A Set Theory Review

Definition A.1 (Difference)**.** The **difference** of two sets, denoted $A - B$, is the set consisting of those elements of A that are not in B. In notation

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

Theorem A.1 (Set-Theoretic Rules)**.** *We have that, for any sets A, B, C,*

(i) First distributive law for the operations $\cap$ and $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{5}$$

(ii) Second distributive law for the operations $\cap$ and $\cup$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{6}$$

(iii) DeMorgan's laws:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{7}$$

"The complement of the union equals the intersection of the complements."

$$A - (B \cap C) = (A - B) \cup (A - C) \tag{8}$$

"The complement of the intersection equals the union of the complements."

A.1 Functions

Exercise A.1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$. Then

(i) $A_0 \subset f^{-1}(f(A_0))$ and equality holds if f is injective.

(ii) $f(f^{-1}(B_0)) \subset B_0$ and equality holds if f is surjective.

Solution. We prove each item in turn:

(i) Let $a \in A_0$. Then $f(a) \in f(A_0)$. We have that $f^{-1}(f(A_0)) = \{a \mid f(a) \in f(A_0)\}$. Then $f(a) \in f^{-1}(f(A_0))$, so that $A_0 \subset f^{-1}(f(A_0))$. We can actually show equality holds if and only if f is injective.

(a) $\Leftarrow$ Suppose f is injective. Let $a \in f^{-1}(f(A_0))$. Then $f(a) \in f(A_0)$. Therefore there exists some $b \in f(A_0)$ such that $f(a) = f(b)$. Injectivity implies $a = b \in A_0$.

(b) $\Rightarrow$ We will prove the contrapositive. Suppose f is *not* injective. Then $f(a) = f(b)$ for some $a \neq b$. Therefore $\{a, b\} \subset f^{-1}(f(\{a\}))$. Thus $f^{-1}(f(\{a\})) \not\subset \{a\}$.

(ii) Let $x \in f(f^{-1}(B_0))$. Then there is some $b \in f^{-1}(B_0)$ such that $f(b) = x$. But $f(b) \in B_0$, so $x \in B_0$.

(a) $\Leftarrow$ Suppose f is surjective. Take $b \in B_0$, then there exists some $a \in A_0$ such that $f(a) = b$, so that $a \in f^{-1}(B_0)$, and $b = f(a) \in f(f^{-1}(B_0))$.

Exercise A.2.

Solution. (i) Let $B_0 \subset B_1$. Fix $x \in f^{-1}(B_0)$. Then $f(x) \in B_0$, which implies $f(x) \in B_1$. Thus $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(ii) We show two inclusions:

(a) $\supset$: We can use (i), since $B_i \subset B_0 \cup B_1$, so $f^{-1}(B_0) \subset f^{-1}(B_0 \cup B_1)$ and $f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$, so $f^{-1}(B_0) \cup f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$.

(b) $\subset$: Let $x \in f^{-1}(B_0 \cup B_1)$. Thus there exists some $b \in B_0 \cup B_1$ such that $f(x) = b$. Therefore $x \in f^{-1}(B_0)$ or $x \in f^{-1}(B_1)$, so that $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_1)$.

(iii)