Topology Notes

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Contents

1 Set Theory Review

Definition 1.1 (Difference)**.** The **difference** of two sets, denoted $A - B$, is the set consisting of those elements of A that are not in B. In notation

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

Theorem 1.1 (Set-Theoretic Rules)**.** *We have that, for any sets A, B, C,*

(i) First distributive law for the operations $\cap$ and $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1}$$

(ii) Second distributive law for the operations $\cap$ and $\cup$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{2}$$

(iii) DeMorgan's laws:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{3}$$

"The complement of the union equals the intersection of the complements."

$$A - (B \cap C) = (A - B) \cup (A - C) \tag{4}$$

"The complement of the intersection equals the union of the complements."

2 Topological Spaces

Definition 2.1 (Topology, Topological Space)**.** A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(i) $\varnothing$ and X are in $\mathcal{T}$.

(ii) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(iii) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.

Definition 2.2 (Open set)**.** If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition 2.3 (Discrete Topology, Trivial Topology)**.** If X is any set, the collection of all subsets of X is a topology on X, called the **discrete topology**. The collection consisting of X and $\varnothing$ is called the **trivial topology**.

Example 2.1 (Finite Complement Topology)**.** Let X be a set. Let $\mathcal{T}_f$ be the collection of all subsets of U of X such that $X - U$ is either finite or all of X. This is topology. We check the three conditions.

(i) $X \in \mathcal{T}_f$ since $X - X$ is the empty set, and hence finite. $\emptyset \in \mathcal{T}_f$ since $X - \emptyset = X$ is all of X.

(ii) Let $\{U_\alpha\}$ be an arbitrary of elements of $\mathcal{T}_f$. Then

$$\begin{aligned} X - \bigcup U_\alpha &= X - (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \cdots) \\ &= (X - U_{\alpha_1}) \cap \cdots \cap (X - U_{\alpha_n}) \cdots \\ &= \bigcap (X - U_\alpha) \end{aligned}$$

Since each $U_\alpha \in \mathcal{T}_f$, we know that each $X - U_\alpha$ is finite. The intersection of finite sets is finite, so $\{U_\alpha\} \in \mathcal{T}_f$.

(iii) Let $\{U_i\}$ be a finite collection of sets in $\mathcal{T}_f$. Then

$$\begin{aligned} X - \bigcap U_i &= X - (U_1 \cap \cdots \cap U_n) \\ &= (X - U_1) \cup \cdots \cup (X - U_n) \\ &= \bigcup (X - U_i) \end{aligned}$$

This is a finite union of finite sets, which is also finite.

Definition 2.4 ((Strictly) Finer, (Strictly) Coarser, Comparable)**.** Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. For these respective situations, we say that $\mathcal{T}$ is **coarser** or **strictly coarser** than $\mathcal{T}'$. We say that $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

3 Basis for a Topology

Definition 3.1 (Basis, Basis Elements, Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

(i) For each $x \in X$, there is at least one basis element B containing x.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Theorem 3.1 (Collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ is a topology on X)**.**

Proof. We check the three conditions:

(i) The empty set is vacuously open (since it has no elements), so $\varnothing \in \mathcal{T}$. Further, $X \in \mathcal{T}$ since for each $x \in X$, there must be at least one basis element B containing x, which itself is contained in X.

[[Incomplete]] □

Definition 3.2 (Subbasis)**.** A **subbasis** $\mathscr{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathscr{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathscr{S}$.

4 The Order Topology

Definition 4.1 (Order Topology)**.** Let X be a set with a simple order relation, and assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(i) All open intervals (a, b) in X.

(ii) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(iii) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the **order topology**.

Proof that $\mathcal{B}$ satisfies the requirements for a basis. We check the two conditions required to be a basis:

(i) Each element contained in a basis element:

(ii)

□

5 The Product Topology on $X \times Y$

Definition 5.1 (Product Topology)**.** Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

6 The Subspace Topology

7 Closed Sets and Limit Points

Definition 7.1 (Closed)**.** A subset A of a topological space X is said to be **closed** if the set $X - A$ is open.

Definition 7.2 (Interior)**.** Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A. Denoted by $\operatorname{Int} A$.

Definition 7.3 (Closure)**.** Given a subset A of a topological space X, the **closure** of A is defined as the intersection of all closed sets containing A. Denoted by $\mathrm{Cl}\,A$ or $\bar{A}$.

Definition 7.4 (Limit Point)**.** If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A is every neighborhood of x intersects A in some point other than x itself. Equivalently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. Note: The point x may lie in A or not.

Definition 7.5 (Hausdorff Space)**.** A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

8 Continuous Functions

Definition 8.1 (Continuous)**.** Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

Notes:

- $f^{-1}(V)$ is the set of all points $x \in X$ for which $f(x) \in V$. It is empty if V does not intersect the image set $f(X)$ of f.

9 The Product Topology

10 The Metric Topology

Definition 10.1 (Metric, Distance)**.** A **metric** on a set X is a function

$$d : X \times X \to R \tag{5}$$

having the following properties:

(i) (Positive) $d(x, y) \geq 0$ for all $x, y \in X$. Equality holds if and only if $x = y$.

(ii) (Symmetric) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is called the **distance** between x and y in the metric d.

Definition 10.2 (ε-ball centered at x)**.** Given $\varepsilon > 0$, the set

$$B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\} \tag{6}$$

consists of all points y whose distance from x is less than ε. It is called the **ε-ball centered at x**.

Definition 10.3 (Metric Topology)**.** If d is a metric on the set X, then the collection of all ε-balls $B_d(x, \varepsilon)$, where $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the **metric topology** induced by d.

11 The Quotient Topology

12 Connected Spaces

Definition 12.1 (Separation, Connected)**.** Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of x.

13 Connected Subspaces of the Real Line

Definition 13.1 (Path, Path Connected)**.** Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path connected** if every pair of points X can be joined by a path in X.

14 Compact Spaces

Definition 14.1 (Cover(ing), Open Covering)**.** A collection $\mathscr{A}$ of subsets of a space X is said to cover X or be a **covering** of X, if the union of the elements of $\mathscr{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition 14.2 (Compact)**.** A space X is said to be **compact** if every open covering $\mathscr{A}$ of X contains a finite subcollection that also covers X.

15 Exercises in Munkres Topology

16 Basis for a Topology

Exercise 16.1. Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

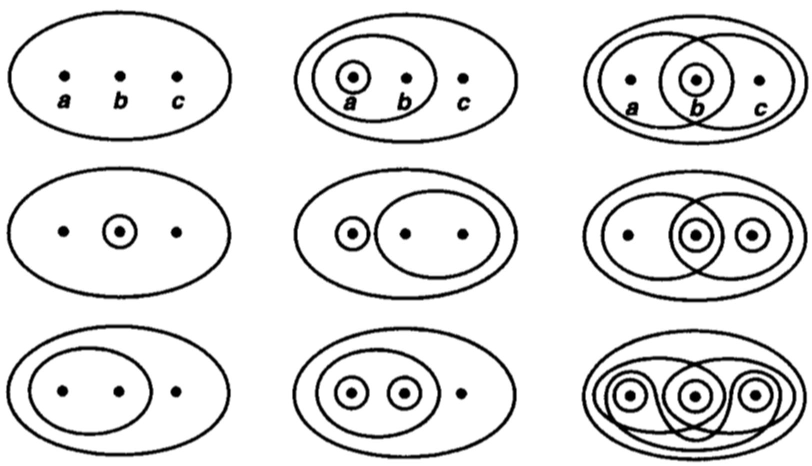
Solution. We want to show that $A \in \mathcal{T}$. For each $x \in A$ there is an open set U_x containing x such that $U_x \subset A$. We claim that $\bigcup_x U_x = A$. We show two inclusions.

(i) $\supset$: Let $y \in A$. There is an open set U_y containing y, which is in the union. Thus $y \in \bigcup_x U_x$.

(ii) $\subset$: Let $y \in \bigcup_x U_x$. Therefore there is some x such that $y \in U_x$. Then $y \in A$ since $U_x \subset A$.

Therefore A is the union of open sets, so $A \in \mathcal{T}$ is also an open set.

Exercise 16.2. Consider the nine topologies on the set $X = \{a, b, c\}$ in the figure below. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is finer.



Solution. We'll label the examples by the coordinates (i, j) where $i, j \in \{1, 2, 3\}$ correspond to the row/column number. Then, in the matrix below, we'll list which pair is finer, or "inc" if the pair is incomparable.

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	=	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,2)		=	inc	inc	inc	inc	(1,2)	(3,2)	(3,3)
(1,3)			=	(1,3)	inc	(2,3)	(1,3)	inc	(3,3)
(2,1)				=	inc	(2,3)	inc	(3,2)	(3,3)
(2,2)					=	inc	inc	inc	(3,3)
(2,3)						=	(2,3)	inc	(3,3)
(3,1)							=	(3,2)	(3,3)
(3,2)								=	(3,3)
(3,3)									=

Exercise 16.3. Show $\mathcal{T}_c$ is a topology: Let X be a set. Let $\mathcal{T}_c$ be the collection of all subsets of U of X such that $X - U$ either is countable or is all of X.

Solution. We check the three conditions:

(i) $X - \emptyset = X$ is all of X, so $\emptyset \in \mathcal{T}_c$. $X - X = \emptyset$, which is finite, so $X \in \mathcal{T}_c$.

(ii) Let $\{U_\alpha\}$ be an indexed family of nonempty elements of $\mathcal{T}_c$. Note that for each α, $X - U_\alpha$ is countable. Then

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha) \tag{7}$$

The intersection of countable sets is also countable, so $\{U_\alpha\} \in \mathcal{T}_c$.

(iii) Let $\{U_i\}_{i=1}^n$ be nonempty elements of $\mathcal{T}_c$. Then

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i) \tag{8}$$

The finite union of countable sets is also finite, so that $\{U_i\}_{i=1}^n \in \mathcal{T}_c$.

Exercise 16.4. Is the collection $\mathcal{T}_\infty = \{U | X - U$ is infinite or empty or all of $X\}$ a topology on X?

Solution. No. Counter example: