Topology Notes

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Contents

1 Set Theory

Definition 1.1 (Set cardinality $\leq$)**.** Let A, B be sets. A has **cardinality less than or equal to** B (write $|A| \leq |B|$) if there exists an injection from A to B. In notation,

$$|A| \leq |B| \iff \exists f : A \to B \text{ injective} \tag{1.1}$$

Theorem 1.1 (Cantor)**.** *For all sets X (including infinite), $X \not\geq \mathcal{P}(X)$. That is, there does not exist an injection from $\mathcal{P}(X)$ to X.*

Proof. The proof contains 2 steps:

(i) Show that there is no surjection from X to $\mathcal{P}(X)$.

(ii) Show that (i) implies that there is no injection from $\mathcal{P}(X)$ to X.

We start by proving (ii) through the following lemma:

Lemma 1.2. *Let C, D be sets, $C \neq \emptyset$. If there is an injection $i : C \to D$, then there exists a surjection $j : D \to C$.*

Proof. $\square$

The contrapositive of this lemma gives that no surjection from $D \to C$ implies no injection from $C \to D$. $\square$

Theorem 1.3 (Informal statement of the axiom of choice)**.** *Given a family $\mathscr{F}$ of nonempty sets, it is possible to pick out an element from each set in the family.*

Definition 1.2 (Partial order)**.** A **partial order** is a pair $\mathcal{A} = (A, \triangleright)$ where $A \neq \emptyset$ such that for $a, b, c \in A$

(i) Antireflexivity: $a \triangleright a$ never happens.

(ii) Transitivity: $a \triangleright b, b \triangleright c \Rightarrow a \triangleright c$

Remark. With a partial order, you *can* have incomparable elements.

Example 1.1 (Partial order)**.** For any set X, a partial order is $(\mathcal{P}(X), \subsetneq)$. For example, if $X = \{1, 2\}$, then $\{1\}$ and $\{2\}$ are incomparable.

Definition 1.3 (Maximal)**.** Let $(A, \triangleright)$ be a partial order. Then $m \in A$ is maximal if and only if no $a \triangleright m$.

Example 1.2 (Maximal elements)**.** The following are examples of posets and their maximal elements:

(i) $(\mathbb{N}, <)$ has no maximal element (there is no largest natural number).

(ii) $(\{\{1\}, \{2\}\}, \subsetneq)$ has 2 maximal elements, since the two elements of the set are not comparable.

Definition 1.4 (Chain)**.** A **chain** in a partial order $(A, \triangleright)$ is a $C \subseteq A$ such that $\forall a, b \in C$, $a = b$ or $a \triangleright b$ or $b \triangleright a$. (One interpretation in words, "C is linear")

Theorem 1.4 (Zorn's Lemma)**.** *Let $(A, \triangleright)$ be a partial order such that the following condition is satisfied:*

($\mathcal{Z}$) *In words, if every chain has an upper bound then there is a maximal element. More precisely, ff C is a chain, then $\exists x$ such that $x \triangle$*

2 Topological Spaces

Definition 2.1 (Topology, Topological Space)**.** A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a set of subsets of X (called a **topology**) having the following properties:

(i) $\varnothing$ and X are in $\mathcal{T}$.

$$\varnothing \in \mathcal{T},\ X \in \mathcal{T} \tag{2.1}$$

(ii) The union of *arbitrarily* many sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_\eta \in \mathcal{T}\ \forall \eta \in H \Rightarrow \bigcup_{\eta \in H} A_\eta = \{a \in X \text{ for some (that is, } a \in A_\eta)\} \in \mathcal{T} \tag{2.2}$$

(iii) The intersection a *finite* number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_1, \ldots, A_n \in \mathcal{T} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathcal{T} \tag{2.3}$$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**, that is, the pair $(X, \mathcal{T})$.

Example 2.1 (Examples of topologies)**.** The following are examples of topologies/topological spaces:

(i) The collection consisting of X and $\varnothing$ is called the **trivial topology** or **indiscrete topology**.

  (a) $\mathcal{T} = \{\varnothing, X\}$

(ii) If X is any set, the collection of all subsets of X is a topology on X, called the **discrete topology**.

  (a) $\mathcal{T} = \mathcal{P}(X)$

(iii) Let $X = \{1\}$. Then $\mathcal{T} = \{\varnothing, \{1\}\}$ is a topology.

(iv) Sierpinski: Let $X = \{a, b\}$ and $\mathcal{T} = \{\varnothing, X, \{a\}\}$. In this topology, b is glued to a. That is, we can't have a set with b and without a. However, a is *not* glued to a.

  Remark. Observe that from these examples,

$$\text{indiscrete} \subsetneq \text{Sierpinski} \subsetneq \text{discrete}$$

(v) $X = \mathbb{R}$ and $\mathcal{T} = \{\varnothing, X\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$.

Definition 2.2 ((Strictly) Finer, (Strictly) Coarser, Comparable)**.** Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. For these respective situations, we say that $\mathcal{T}$ is **coarser** or **strictly coarser** than $\mathcal{T}'$. We say that $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Example 2.2 (Finest and coarsest topologies)**.** For any set X, the finest topology is the discrete topology and the coarsest topology is the trivial topology.

3 Basis for a Topology

Definition 3.1 (Basis, Basis Elements, Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

(i) Every element $x \in X$ belongs to some set in $\mathcal{B}$.

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Example 3.1 (Bases)**.** The following are example of bases of topologies:

(i) $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) \mid b > a\}$. We can cover $\mathbb{R}$ with open intervals. Further, a real number x is contained in two intervals B_1 and B_2, then there will be an open interval B_3 contained in the intersection of the two intervals. In this example, we can actually set $B_3 = B_1 \cap B_2$.

(ii) $X = \mathbb{R}^2$ and $\mathcal{B} = \{\text{interiors of circles}\}$. We can cover $\mathbb{R}^2$ with circles. If x is in the intersection of two circles B_1 and B_2, then we can construct a circle B_3 contained in the intersection $B_1 \cap B_2$. Note in this example, we can just take the actual intersection, as it's not a circle. This example can also be extended to other polygons.

Definition 3.2 (Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If $\mathcal{B}$ is a basis for a topology on X, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Note that each basis element is itself an element of $\mathcal{T}$. More succinctly:

$$U \subset X : U \in \mathcal{T} \iff \forall x \in U \; \exists B_x \in \mathcal{B} \; s.t. \; x \in B_x \subset U$$

Theorem 3.1 (Collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ is a topology on X)**.**

Proof. We check the three conditions:

(i) The empty set is vacuously open (since it has no elements), so $\emptyset \in \mathcal{T}$. Further, $X \in \mathcal{T}$ since for each $x \in X$, there must be at least one basis element B containing x, which itself is contained in X.

[[Incomplete]] □

Lemma 3.2 (Every open set in X can be expressed as a union of basis elements (not unique))**.** *Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. We need to show two inclusions:

(i) Collection of elements of $\mathcal{B}$ in $\mathcal{T}$: In the topology $\mathcal{T}$ generated by $\mathcal{B}$, each basis element is itself an element of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, their union is also in $\mathcal{T}$.

(ii) Element of $\mathcal{T}$ in collection of all unions of elements of $\mathcal{B}$: Take $U \in \mathcal{T}$. Then we know $\forall x \in U$ $\exists B_x \in \mathcal{B}$ such that $x \in B_x \in U$. Then we claim that $U = \bigcup_{x \in U} B_x$, so that U equals a union of elements of $\mathcal{B}$. Indeed, "$\subset$" follows since $x \in U \implies x \in B_x$. And, "$\supset$" follows since $B_x \subset U$, so that the union of all such B_x is in U.

□

Lemma 3.3. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathscr{C}$ is a collection of open sets of X (i.e., $\mathscr{C} \subset \mathcal{T}$) such that for each open set U of X and each $x \in U$, there is an element $V \in \mathscr{C}$ such that $x \in V \subset U$. Then $\mathscr{C}$ is a basis for the topology of X. In notation;*

$$\forall U \in \mathcal{T}\ \forall a \in U\ \exists V \in \mathscr{C} \text{ s.t. } a \in V, V \subset U$$

Definition 3.3 (Subbasis)**.** A **subbasis** $\mathscr{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathscr{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathscr{S}$.

4 Continuous Functions

Definition 4.1 (Closed)**.** A subset A of a topological space $(X, \mathcal{T})$ is said to be **closed** if the set $X - A \in \mathcal{T}$. In words, a subset of a topological space is open if its complement (in the space) is open.

Example 4.1 (Sets can be both closed and open)**.** Let $(X, \mathcal{T})$ be a topological space. Then $X - X = \emptyset \in \mathcal{T}$ and $X - \emptyset = X \in \mathcal{T}$. Therefore $X, \emptyset$ are both closed and open. We call this type of set **clopen**. Further

$$\text{Closed} \neq \text{Not Open} \tag{4.1}$$

Example 4.2 (Sets can be neither closed nor open)**.** Consider $\mathbb{Q}$ in the usual topology on $\mathbb{R}$.

Definition 4.2 (Continuous)**.** Let $(X, \mathcal{T})$ and (Y, σ) be topological spaces. A function $f : X \to Y$ is said to be **continuous** with respect to $\mathcal{T}$ and σ if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. In symbols, $\forall S \in \sigma$, we have that $f^{-1}(S) \in \mathcal{T}$ (where $f^{-1}(S) = \{a \in X \mid f(a) \in S\}$). In words, the preimage of an open set is open.

Definition 4.3 (Homeomorphic)**.** The topological spaces $(X, \mathcal{T})$ and (Y, σ) are **homeomorphic** if there exists a function $f : X \to Y$ such that

(i) f is bijective.

(ii) f is continuous.

(iii) f^{-1} is also continuous.

We write $(X, \mathcal{T}) \cong (Y, \sigma)$ or $f : (X, \mathcal{T}) \cong (Y, \sigma)$.

Remark. This definition states that we can find a *single* bijection that's continuous in both directions.

Example 4.3 (Continuous functions)**.** Let X be a set with more than one element. Let $\mathcal{T}_{disc} = \mathcal{P}(X)$ and $\mathcal{T}_{ind} = \{\emptyset, X\}$ (that is, the discrete and indiscrete topologies. We require X to have more than one element, else these topologies would be the same). Let $f = id$. Then

(i) $f : (X, \mathcal{T}_{disc}) \to (X, \mathcal{T}_{ind})$ is **continuous**. Indeed, if $S \subseteq X$, $S \in \mathcal{T}_{ind}$, then $f^{-1}(S) \in \mathcal{T}_{disc}$, since $\mathcal{T}_{disc} \supset \mathcal{T}_{ind}$.

(ii) $f : (X, \mathcal{T}_{ind}) \to (X, \mathcal{T}_{disc})$ is **not continuous**. For example, suppose $X = \{1, 2\}$. Let $S = \{1\} \in \mathcal{T}_{disc}$. Then $f^{-1}(\{1\}) = \{1\} \notin \mathcal{T}_{ind}$.

Example 4.4 (Open, closed, continuous functions)**.**

Definition 4.4 (Open map)**.** $f : (X, \mathcal{T}) \to (Y, \sigma)$ is an **open map** if $\forall S \in \mathcal{T}$ we have that $f(S) \in \sigma$ (recall $f(S) = \{f(s) \mid s \in S\}$). In words: open sets map to open sets.

Definition 4.5 (Closed map)**.** $f : (X, \mathcal{T}) \to (Y, \sigma)$ is a **closed map** if $\forall S \subset X$ such that $X - S \in \mathcal{T}$ we have that $Y - f(S) \in \sigma$. In words: closed sets map to closed sets.

Continuous, open, and closed maps don't have clean relationships: Let $X = \{1, 2\}$.

(i) Continuous, not open, not closed map: Let $f : (X, \mathcal{P}(X)) \to (X, \{\emptyset, X\})$ be the identity map (discrete to indiscrete).

  (a) Continuous: Previous example.
  (b) Not open: $\{1\} \in \mathcal{P}(X)$ maps to $\{1\} \notin \{\emptyset, X\}$. Thus the map is not open.
  (c) Not closed: $X - \{1\} = \{2\} \in \mathcal{P}(X)$, so $\{1\}$ is closed in $(X, \mathcal{P}(X))$. But $\{2\} \notin \{\emptyset, X\}$. Thus $\{1\}$ is closed on the left, but not on the right.

(ii) Open, closed, not continuous map: Let $f : (X, \{\emptyset, X\}) \to (X, \mathcal{P}(X))$ be the identity map (indiscrete to discrete).

  (a) Not continuous: Previous example.
  (b) Open: Since $\mathcal{T}_{ind} \subset \mathcal{T}_{disc}$
  (c) Closed: Since $\mathcal{T}_{ind} \subset \mathcal{T}_{disc}$.

(iii) Continuous, closed, not open:

5 Subspace Topology

Definition 5.1 (Subspace topology)**.** Given a topological space $(X, \mathcal{T})$ and a non-empty set $A \subseteq X$, the **subspace topology on A induced (or given) by $\mathcal{T}$** is $(A, \mathcal{T}_A)$ where $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$.

Proof that $(A, \mathcal{T}_A)$ is a topological space. We check the axioms:

(i) $\emptyset \in \mathcal{T}_A$: $\emptyset \in \mathcal{T}$, and $\emptyset \cap A = \emptyset$, so $\emptyset \in \mathcal{T}_A$.

(ii) $A \in \mathcal{T}_A$: $X \in \mathcal{T}$, and $X \cap A = A$, so $A \in \mathcal{T}_A$.

(iii) Closure under finite intersections:

□

6 Metric Spaces

Definition 6.1 (Metric space)**.** A **metric space** is a nonempty set X together with a binary function $d : X \times X \to \mathbb{R}$ which satisfies the following properties: For all $x, y, z \in X$ we have that

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$

Definition 6.2 (Metric topology)**.** Given a metric space (X, d), the set $\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon > 0, x \in X\}$ is a basis for a topology on X (where $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$) called the **metric topology**.

A Set Theory Review

Definition A.1 (Difference)**.** The **difference** of two sets, denoted $A - B$, is the set consisting of those elements of A that are not in B. In notation

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

Theorem A.1 (Set-Theoretic Rules)**.** *We have that, for any sets A, B, C,*

(i) First distributive law for the operations $\cap$ and $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{A.1}$$

(ii) Second distributive law for the operations $\cap$ and $\cup$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{A.2}$$

(iii) DeMorgan's laws:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{A.3}$$

"The complement of the union equals the intersection of the complements."

$$A - (B \cap C) = (A - B) \cup (A - C) \tag{A.4}$$

"The complement of the intersection equals the union of the complements."

A.1 Functions

Exercise A.1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$. Then

(i) $A_0 \subset f^{-1}(f(A_0))$ and equality holds if f is injective.

(ii) $f(f^{-1}(B_0)) \subset B_0$ and equality holds if f is surjective.

Solution. We prove each item in turn:

(i) Let $a \in A_0$. Then $f(a) \in f(A_0)$. We have that $f^{-1}(f(A_0)) = \{a \,|\, f(a) \in f(A_0)\}$. Then $f(a) \in f^{-1}(f(A_0))$, so that $A_0 \subset f^{-1}(f(A_0))$. We can actually show equality holds if and only if f is injective.

  (a) $\Leftarrow$ Suppose f is injective. Let $a \in f^{-1}(f(A_0))$. Then $f(a) \in f(A_0)$. Therefore there exists some $b \in f(A_0)$ such that $f(a) = f(b)$. Injectivity implies $a = b \in A_0$.

  (b) $\Rightarrow$ We will prove the contrapositive. Suppose f is *not* injective. Then $f(a) = f(b)$ for some $a \neq b$. Therefore $\{a, b\} \subset f^{-1}(f(\{a\}))$. Thus $f^{-1}(f(\{a\})) \not\subset \{a\}$.

(ii) Let $x \in f(f^{-1}(B_0))$. Then there is some $b \in f^{-1}(B_0)$ such that $f(b) = x$. But $f(b) \in B_0$, so $x \in B_0$.

  (a) $\Leftarrow$ Suppose f is surjective. Take $b \in B_0$, then there exists some $a \in A_0$ such that $f(a) = b$, so that $a \in f^{-1}(B_0)$, and $b = f(a) \in f(f^{-1}(B_0))$.

Exercise A.2.

Solution. (i) Let $B_0 \subset B_1$. Fix $x \in f^{-1}(B_0)$. Then $f(x) \in B_0$, which implies $f(x) \in B_1$. Thus $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(ii) We show two inclusions:

(a) $\supset$: We can use (i), since $B_i \subset B_0 \cup B_1$, so $f^{-1}(B_0) \subset f^{-1}(B_0 \cup B_1)$ and $f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$, so $f^{-1}(B_0) \cup f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$.

(b) $\subset$: Let $x \in f^{-1}(B_0 \cup B_1)$. Thus there exists some $b \in B_0 \cup B_1$ such that $f(x) = b$. Therefore $x \in f^{-1}(B_0)$ or $x \in f^{-1}(B_1)$, so that $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_1)$.

(iii)