Topology Notes

Rebekah Dix

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Contents

Todo list

1 Set Theory

Definition 1.1 (Set cardinality $\leq$)**.** Let A, B be sets. A has **cardinality less than or equal to** B (write $|A| \leq |B|$) if there exists an injection from A to B. In notation,

$$|A| \leq |B| \iff \exists f : A \to B \text{ injective} \tag{1.1}$$

Theorem 1.1 (Cantor)**.** For all sets X (including infinite), $X \ngeq \mathcal{P}(X)$. That is, there does not exist an injection from $\mathcal{P}(X)$ to X.

Proof. The proof contains 2 steps:

(i) Show that there is no surjection from X to $\mathcal{P}(X)$.

(ii) Show that (i) implies that there is no injection from $\mathcal{P}(X)$ to X.

We start by proving (ii) through the following lemma:

Lemma 1.2. Let C, D be sets, $C \neq \emptyset$. If there is an injection $i : C \to D$, then there exists a surjection $j : D \to C$.

Proof. $\square$

The contrapositive of this lemma gives that no surjection from $D \to C$ implies no injection from $C \to D$. $\square$

Theorem 1.3 (Informal statement of the axiom of choice)**.** Given a family $\mathcal{F}$ of nonempty sets, it is possible to pick out an element from each set in the family.

Definition 1.2 (Partial order)**.** A **partial order** is a pair $\mathcal{A} = (A, \triangleright)$ where $A \neq \emptyset$ such that for $a, b, c \in A$

(i) Antireflexivity: $a \triangleright a$ never happens.

(ii) Transitivity: $a \triangleright b, b \triangleright c \Rightarrow a \triangleright c$

Remark. With a partial order, you *can* have incomparable elements.

Example 1.1 (Partial order)**.** For any set X, a partial order is $(\mathcal{P}(X), \subsetneq)$. For example, if $X = \{1, 2\}$, then $\{1\}$ and $\{2\}$ are incomparable.

Definition 1.3 (Maximal)**.** Let $(A, \triangleright)$ be a partial order. Then $m \in A$ is maximal if and only if no $a \triangleright m$.

Example 1.2 (Maximal elements)**.** The following are examples of posets and their maximal elements:

(i) $(\mathbb{N}, <)$ has no maximal element (there is no largest natural number).

(ii) $(\{\{1\}, \{2\}\}, \subsetneq)$ has 2 maximal elements, since the two elements of the set are not comparable.

Definition 1.4 (Chain)**.** A **chain** in a partial order $(A, \triangleright)$ is a $C \subseteq A$ such that $\forall a, b \in C$, $a = b$ or $a \triangleright b$ or $b \triangleright a$. (One interpretation in words, "C is linear")

Theorem 1.4 (Zorn's Lemma)**.** Let $(A, \triangleright)$ be a partial order such that the following condition is satisfied:

($\mathcal{Z}$) In words, if every chain has an upper bound then there is a maximal element. More precisely, ff C is a chain, then $\exists x$ such that $x \triangle$

2 Topological Spaces

Definition 2.1 (Topology, Topological Space)**.** A **topological space** is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a set of subsets of X (called a **topology**) having the following properties:

(i) $\emptyset$ and X are in $\mathcal{T}$.

$$\emptyset \in \mathcal{T},\ X \in \mathcal{T} \tag{2.1}$$

(ii) The union of *arbitrarily* many sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_\eta \in \mathcal{T}\ \forall \eta \in H \Rightarrow \bigcup_{\eta \in H} A_\eta = \{a \in X \text{ for some (that is, } a \in A_\eta)\} \in \mathcal{T} \tag{2.2}$$

(iii) The intersection a *finite* number of sets in $\mathcal{T}$ is in $\mathcal{T}$.

$$A_1, \dots, A_n \in \mathcal{T} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathcal{T} \tag{2.3}$$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**, that is, the pair $(X, \mathcal{T})$.

Example 2.1 (Examples of topologies)**.** The following are examples of topologies/topological spaces:

(i) The collection consisting of X and $\emptyset$ is called the **trivial topology** or **indiscrete topology**.

   (a) $\mathcal{T} = \{\emptyset, X\}$

(ii) If X is any set, the collection of all subsets of X is a topology on X, called the **discrete topology**.

(a) $\mathcal{T} = \mathcal{P}(X)$

(iii) Let $X = \{1\}$. Then $\mathcal{T} = \{\emptyset, \{1\}\}$ is a topology.

(iv) Sierpinski: Let $X = \{a, b\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}\}$. In this topology, b is glued to a. That is, we can't have a set with b and without a. However, a is *not* glued to a.

Remark. Observe that from these examples,

$$\text{indiscrete} \subsetneq \text{Sierpinski} \subsetneq \text{discrete}$$

(v) $X = \mathbb{R}$ and $\mathcal{T} = \{\emptyset, X\} \cup \{[a, \infty) \mid a \in \mathbb{R}\}$.

Definition 2.2 ((Strictly) Finer, (Strictly) Coarser, Comparable)**.** Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$. If $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. For these respective situations, we say that $\mathcal{T}$ is **coarser** or **strictly coarser** than $\mathcal{T}'$. We say that $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

Example 2.2 (Finest and coarsest topologies)**.** For any set X, the finest topology is the discrete topology and the coarsest topology is the trivial topology.

3 Basis for a Topology

Definition 3.1 (Basis, Basis Elements, Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If X is a set, a **basis** for a topology on X is a collection $\mathcal{B} \subset \mathcal{P}(X)$ of subsets of X (called **basis elements**) such that

(i) Every element $x \in X$ belongs to some set in $\mathcal{B}$. In symbols

$$\forall x \in X \, \exists B \in \mathcal{B} \text{ s.t. } x \in B \tag{3.1}$$

(ii) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$. More generally, in symbols

$$\forall B_1, \ldots, B_n \in \mathcal{B} \, \forall x \in \bigcap_{i=1}^{n} B_i \, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset \bigcap_{i=1}^{n} B_i \tag{3.2}$$

Example 3.1 (Bases)**.** The following are example of bases of topologies:

(i) $X = \mathbb{R}$ and $\mathcal{B} = \{(a, b) \mid b > a\}$. We can cover $\mathbb{R}$ with open intervals. Further, a real number x is contained in two intervals B_1 and B_2, then there will be an open interval B_3 contained in the intersection of the two intervals. In this example, we can actually set $B_3 = B_1 \cap B_2$.

(ii) $X = \mathbb{R}^2$ and $\mathcal{B} = \{\text{interiors of circles}\}$. We can cover $\mathbb{R}^2$ with circles. If x is in the intersection of two circles B_1 and B_2, then we can construct a circle B_3 contained in the intersection $B_1 \cap B_2$. Note in this example, we can just take the actual intersection, as it's not a circle. This example can also be extended to other polygons.

Definition 3.2 (Topology $\mathcal{T}$ generated by $\mathcal{B}$)**.** If $\mathcal{B}$ is a basis for a topology on X, then we define the **topology $\mathcal{T}$ generated by $\mathcal{B}$** as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \subset U$. Note that each basis element is itself an element of $\mathcal{T}$. More succinctly:

$$U \subset X : U \in \mathcal{T} \iff \forall x \in U\ \exists B_x \in \mathcal{B}\ s.t.\ x \in B_x \subset U$$

Theorem 3.1 (Collection $\mathcal{T}$ generated by a basis $\mathcal{B}$ is a topology on X)**.**

Proof. We check the three conditions:

(i) The empty set is vacuously open (since it has no elements), so $\emptyset \in \mathcal{T}$. Further, $X \in \mathcal{T}$ since for each $x \in X$, there must be at least one basis element B containing x, which itself is contained in X.

Finish Proof

□

Lemma 3.2 (Every open set in X can be expressed as a union of basis elements (not unique))**.** Let X be a set and $\mathcal{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Proof. We need to show two inclusions:

(i) Collection of elements of $\mathcal{B}$ in $\mathcal{T}$: In the topology $\mathcal{T}$ generated by $\mathcal{B}$, each basis element is itself an element of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, their union is also in $\mathcal{T}$.

(ii) Element of $\mathcal{T}$ in collection of all unions of elements of $\mathcal{B}$: Take $U \in \mathcal{T}$. Then we know $\forall x \in U$ $\exists B_x \in \mathcal{B}$ such that $x \in B_x \in U$. Then we claim that $U = \bigcup_{x \in U} B_x$, so that U equals a union of elements of $\mathcal{B}$. Indeed, "$\subset$" follows since $x \in U \implies x \in B_x$. And, "$\supset$" follows since $B_x \subset U$, so that the union of all such B_x is in U.

□

Lemma 3.3. Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathcal{C}$ is a collection of open sets of X (i.e., $\mathcal{C} \subset \mathcal{T}$) such that for each open set U of X and each $x \in U$, there is an element $V \in \mathcal{C}$ such that $x \in V \subset U$. Then $\mathcal{C}$ is a basis for the topology of X (that is, $\mathcal{C}$ is a basis *and* $\mathcal{C}$ generates $\mathcal{T}$). In notation;

$$\forall U \in \mathcal{T}\ \forall a \in U\ \exists V \in \mathcal{C}\ s.t.\ a \in V, V \subset U$$

Proof. We first show that $\mathcal{C}$ is indeed a basis.

We then show $\mathcal{C}$ generates $\mathcal{T}$.

Incomplete.

□

Example 3.2 (Countable bases). Let $X = \mathbb{R}$ and

- τ_1 is the usual topology.
- τ_2 is the discrete topology.

Claim 3.1. τ_1 has a countable basis.

Proof. We use the fact that $\mathbb{Q}$ is countable. We will show that

$$\mathcal{B} = \{(a,b) \mid a < b,\ a,b \in \mathbb{Q}\} \tag{3.3}$$

generates τ_1. Let $U \in \mathcal{T}$ nonempty and take $a \in U$. Since U is open, there exists an open interval (c,d) with $a \in (c,d) \subseteq U$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Therefore we can pick rationals p,q with $c < p < a < q < d$. Then $a \in (p,q) \subseteq (c,d) \subseteq U$. Therefore $(p,q) \in \mathcal{B}$ and $\mathcal{B}$ is a basis for τ_1. $\mathcal{B}$ is countable since $\mathbb{Q}^2$ is countable. $\square$

Claim 3.2. τ_2 doesn't have a countable basis.

Proof. Suppose $\mathcal{B}$ is a basis for τ_2. Let $a \in \mathbb{R}$. We have that $\{a\} \in \tau_2$. Since $\mathcal{B}$ generates τ_2, there must exist some $U \in \mathcal{B}$ such that $a \in U \subset \{a\}$. But then $U = \{a\}$. Therefore we have found an injection from $\mathbb{R} \to \mathcal{B} : a \mapsto \{a\}$. Therefore $\mathcal{B}$ is not countable.

Clarify.

$\square$

Lemma 3.4 (When is one topology finer than another?). Suppose $\mathcal{B}$ is a bis s for a topology τ on X and $\mathcal{B}'$ is a basis for a topology τ' on X. The following are equivalent:

(i) τ' is finer than τ ($\tau' \supset \tau$).

(ii) In symbols,

$$\forall x \in X\ \forall U \in \tau : x \in U\ \exists V \in \tau'\ s.t.\ x \in V \subseteq U \tag{3.4}$$

equivalently

$$\forall x \in X\ \forall B \in \mathcal{B} : x \in B\ \exists B' \in \mathcal{B}' : x \in B' \subset B \tag{3.5}$$

Definition 3.3 (Subbasis). A **subbasis** for a topology on X is a collection of subsets $\mathcal{S} \subset \mathcal{P}(X)$ of X whose union equals X (that is, $\bigcup \mathcal{S} = X$). The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

4 Continuous Functions

Definition 4.1 (Closed). A subset A of a topological space $(X, \mathcal{T})$ is said to be **closed** if the set $X - A \in \mathcal{T}$. In words, a subset of a topological space is closed if its complement (in the space) is open.

Example 4.1 (Sets can be both closed and open). Let $(X, \mathcal{T})$ be a topological space. Then $X - X = \emptyset \in \mathcal{T}$ and $X - \emptyset = X \in \mathcal{T}$. Therefore $X, \emptyset$ are both closed and open. We call this type of set **clopen**. Further

$$\text{Closed} \neq \text{Not Open} \tag{4.1}$$

Example 4.2 (Sets can be neither closed nor open). Consider $\mathbb{Q}$ in the usual topology on $\mathbb{R}$.

Definition 4.2 (Continuous). Let $(X, \mathcal{T})$ and (Y, σ) be topological spaces. A function $f : X \to Y$ is said to be **continuous** with respect to $\mathcal{T}$ and σ if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. In symbols, $\forall S \in \sigma$, we have that $f^{-1}(S) \in \mathcal{T}$ (where $f^{-1}(S) = \{a \in X \mid f(a) \in S\}$). In words, the preimage of an open set is open.

Definition 4.3 (Homeomorphic). The topological spaces $(X, \mathcal{T})$ and (Y, σ) are **homeomorphic** if there exists a function $f : X \to Y$ such that

(i) f is bijective.

(ii) f is continuous.

(iii) f^{-1} is also continuous.

We write $(X, \mathcal{T}) \cong (Y, \sigma)$ or $f : (X, \mathcal{T}) \cong (Y, \sigma)$.

Remark. This definition states that we can find a *single* bijection that's continuous in both directions.

Example 4.3 (Continuous functions). Let X be a set with more than one element. Let $\mathcal{T}_{disc} = \mathcal{P}(X)$ and $\mathcal{T}_{ind} = \{\emptyset, X\}$ (that is, the discrete and indiscrete topologies. We require X to have more than one element, else these topologies would be the same). Let $f = id$ (that is $f(x) = x$). Then

(i) $f : (X, \mathcal{T}_{disc}) \to (X, \mathcal{T}_{ind})$ is **continuous**. Indeed, if $S \subseteq X$, $S \in \mathcal{T}_{ind}$, then $f^{-1}(S) \in \mathcal{T}_{disc}$, since $\mathcal{T}_{disc} \supset \mathcal{T}_{ind}$.

(ii) $f : (X, \mathcal{T}_{ind}) \to (X, \mathcal{T}_{disc})$ is **not continuous**. For example, suppose $X = \{1, 2\}$. Let $S = \{1\} \in \mathcal{T}_{disc}$. Then $f^{-1}(\{1\}) = \{1\} \notin \mathcal{T}_{ind}$.

Remark. This example motivates a more general claim: Any map into the indiscrete topology or out of the discrete topology is continuous. Further, the identity map from a finer topology to a coarser topology is continuous.

Definition 4.4 (Open map). $f : (X, \mathcal{T}) \to (Y, \sigma)$ is an **open map** if $\forall S \in \mathcal{T}$ we have that $f(S) \in \sigma$ (recall $f(S) = \{f(s) \mid s \in S\}$). In words: open sets map to open sets.

Definition 4.5 (Closed map). $f : (X, \mathcal{T}) \to (Y, \sigma)$ is a **closed map** if $\forall S \subset X$ such that $X - S \in \mathcal{T}$ we have that $Y - f(S) \in \sigma$. In words: closed sets map to closed sets.

Continuous, open, and closed maps don't have clean relationships.
To see this, let $X = \{1, 2\}$.

Example 4.4 (Open, closed, continuous functions). (i) Continuous, not open, not closed map: Let $f : (X, \mathcal{P}(X)) \to (X, \{\emptyset, X\})$ be the identity map (discrete to indiscrete).

(a) Continuous: Previous example.

(b) Not open: $\{1\} \in \mathcal{P}(X)$ maps to $\{1\} \notin \{\emptyset, X\}$. Thus the map is not open.

(c) Not closed: $X - \{1\} = \{2\} \in \mathcal{P}(X)$, so $\{1\}$ is closed in $(X, \mathcal{P}(X))$. But $\{2\} \notin \{\emptyset, X\}$. Thus $\{1\}$ is closed on the left, but not on the right.

(ii) Open, closed, not continuous map: Let $f : (X, \{\emptyset, X\}) \to (X, \mathcal{P}(X))$ be the identity map (indiscrete to discrete).

(a) Not continuous: Previous example.

(b) Open: Since $\mathcal{T}_{ind} \subset \mathcal{T}_{disc}$

(c) Closed: Since $\mathcal{T}_{ind} \subset \mathcal{T}_{disc}$.

(iii) Continuous, closed, not open: Let $X = \mathbb{R}$ and $\tau = \tau_e$. Let $Y = \mathbb{R}^2$ and $\tau = \tau_{e_2}$ (basis is set of open balls). Define $f : X \to Y : r \mapsto (r, 0)$.

(a) Continuous: Clear.

(b) Not open: $\mathbb{R}$ is sent to the x-axis, which is not open in the plane.

(c) Closed: Fix $A \subset \mathbb{R}$ closed (in τ_e-sense). We need to show that $f(A) = \{f(a) \mid a \in A\}$ is closed (in τ_{e_2} sense). Thus we must show $\mathbb{R}^2 - f(A)$ is open, which is equivalent to showing that for all $b \in \mathbb{R}^2$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(b) \subset \mathbb{R}^2 - f(A)$. We prove by cases:

i. b not on x-axis: Let $\varepsilon =$ distance from b to x-axis. Thus $B_\varepsilon(b) \cap (\text{x-axis}) = \emptyset$, and $f(A) \subset (\text{x-axis})$. Thus $B_\varepsilon(A) \cap f(A) = \emptyset$, and $B_\varepsilon(b) \subset \mathbb{R}^2 - f(A)$.

ii. b is on x-axis: Look at $a = f^{-1}(b)$ (note: a exists and is unique. f is injective and hits all of x-axis. Thus $b \notin f(A) \Rightarrow a \notin A$). Since A is closed in $\mathbb{R}$, if $a \notin A$, then $\exists \varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq \mathbb{R} - A$. Then

Claim 4.1. $B_\varepsilon(b) \subset \mathbb{R}^2 - f(A)$

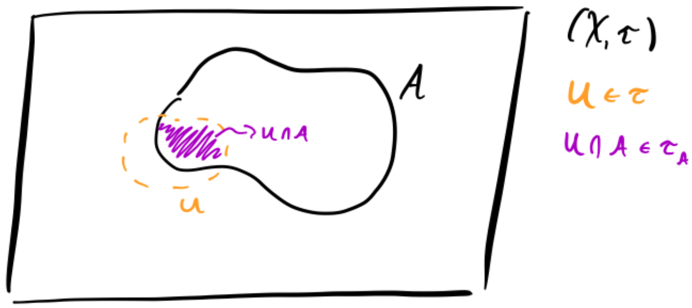
Proof. $B_\varepsilon(b) \cap (\text{x-axis}) = f((a - \varepsilon, a + \varepsilon))$. But $f((a - \varepsilon, a + \varepsilon)) \cap f(A) = \emptyset$, since $(a - \varepsilon, a + \varepsilon) \cap A = \emptyset$. Thus

$$\begin{aligned} B_\varepsilon(b) \cap f(A) &= B_\varepsilon(b) \cap f(A) \cap (\text{x-axis}) \\ &= f((a - \varepsilon, a + \varepsilon)) \cap f(A) \\ &= \emptyset \end{aligned}$$

□

5 Subspace Topology

Definition 5.1 (Subspace topology)**.** Given a topological space $(X, \mathcal{T})$ and a non-empty set $A \subseteq X$, the **subspace topology on A induced (or given) by $\mathcal{T}$** is (A, τ_A) where $\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$.



Remark. Intuitively, $U \cap A$ "should" be open in the sense of A, since it's the "all of the open set U" as far as A knows.

Proof that (A, τ_A) is a topological space. We check the axioms:

(i) $\emptyset \in \mathcal{T}_A$: $\emptyset \in \mathcal{T}$, and $\emptyset \cap A = \emptyset$, so $\emptyset \in \mathcal{T}_A$.

(ii) $A \in \mathcal{T}_A$: $X \in \mathcal{T}$, and $X \cap A = A$, so $A \in \mathcal{T}_A$.

(iii) Closure under finite intersections [2 for simplicity]: Suppose $B_1, B_2 \in \tau_A$. We want to show that $B_1 \cap B_2 \in \tau_A$. We know that there exist $C_1, C_2 \in \mathcal{T}$ such that $B_1 = C_1 \cap A$ and $B_2 = C_2 \cap A$. Then

$$B_1 \cap B_2 = (C_1 \cap A) \cap (C_2 \cap A) = (C_1 \cap C_2) \cap A \tag{5.1}$$

But $C_1 \cap C_2 \in \mathcal{T}$ since $\mathcal{T}$ is a topology, therefore $B_1 \cap B_2$ can be written as the intersection of a set in $\mathcal{T}$ and A, so that $B_1 \cap B_2 \in \tau_A$.

□

Claim 5.1 (Inclusion is Continuous)**.** If (X, τ) is a topological space and $A \subseteq X$, then $f : A \to X : a \mapsto a$ is continuous (with respect to τ_A and τ).

Proof. Let $U \in \tau$. Then $f^{-1}(U) = U \cap A$. Therefore $f^{-1}(U) \in \tau_A$ by the definition of the subspace topology. □

Claim 5.2. Let (X, τ) and (Y, σ) be topological spaces and $f : X \to Y$ be a continuous function (with respect to τ and σ). Let $A \subset X$ be nonempty and τ_A its subspace topology. Let $B \subset Y$ be nonempty and σ_B its subspace topology. Suppose further that $f(A) \subseteq B$ (that is, for $x \in A$, we have that $f(x) \in B$). Define $\hat{f} : A \to B : x \mapsto f(x)$ (that is, the restriction of f). **Then** $\hat{f} : A \to B$ is continuous (with respect to τ_A and τ_B).

Proof. Let $U \subset B$ be σ_B-open. Then there exists a $W \in \sigma$ such that $U = B \cap W$. Since f is continuous, $f^{-1}(W) \in \tau$. But then $f^{-1}(W) \cap A \in \tau_A$ and $f^{-1}(W) \cap A = f^{-1}(U)$.

Incomplete

□

5.1 Connectedness

Definition 5.2 (Connected). A space (X, τ) is **connected** if whenever sets V, W are

(i) Nonempty

(ii) Open

(iii) $V \cup W = X$

we have $V \cap W \neq \emptyset$.

Remark. Equivalently, a set is connected if and only the only clopen sets are $\emptyset, X$.

Example 5.1 (Connected Set). Let $(X, \tau) = (\mathbb{R}, \tau_e)$ and $A = (0,1) \cup (1,2)$. Notice that

(i) $(0,1) \in \tau_A$ since $(0,1) = (0,1) \cap A$ and $(0,1) \in \tau_e$).

(ii) Similarly, $(1,2) \in \tau_A$.

Then notice that $(0,1) = A - (1,2)$, so that the complements of both $(0,1)$ and $(1,2)$ are both open. Therefore each set is clopen. Thus A is not connected.

6 Metric Spaces

Definition 6.1 (Metric space). A **metric space** is a nonempty set X together with a binary function $d : X \times X \to \mathbb{R}$ which satisfies the following properties: For all $x, y, z \in X$ we have that

(i) Positivity: $d(x,y) \geq 0$

(ii) Definiteness: $d(x,y) = 0$ if and only if $x = y$

(iii) Symmetry: $d(x,y) = d(y,x)$

(iv) Triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$

Definition 6.2 (Metric topology). Given a metric space (X, d), the set $\mathcal{B} = \{B_\varepsilon(x) \mid \varepsilon > 0, x \in X\}$ is a basis for a topology on X (where $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$) called the **metric topology**.

Definition 6.3 (Open Set in Metric Topology). A set is **open** in the metric topology induced by d if and only if for each $y \in U$ there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

Theorem 6.1 ("Continuous" = "Continuous"). If (X_1, d_1) and (X_2, d_2) are metric spaces with induced topologies τ_{d_1} and τ_{d_2}, then for $f : X_1 \to X_2$, the following are equivalent:

(i) f is continuous with respect to τ_{d_1} and τ_{d_2}.

(ii) For all $a \in X_1$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $b \in X_1$ for which $d_1(a, b) < \delta$, we have that $d_2(f(a), f(b)) < \varepsilon$.

Proof. $(i) \Rightarrow (ii)$:

Incomplete.

□

6.1 Special Properties and Maps

Definition 6.4 (T_2, Hausdorff). (X, τ) is **T_2 (Hausdorff)** if for every distinct $a, b \in X$, there exist open sets $V, W \in \tau$ such that $a \in V$, $b \in W$, and $V \cap W = \emptyset$.

Theorem 6.2 (Separation Axiom). Metric spaces are always T_2.

Proof. Let $\varepsilon = \frac{d(a,b)}{2}$. Then let $V = B_\varepsilon(a)$ and $W = B_\varepsilon(b)$. □

Remark. Not all topological spaces are T_2.

7 Sequences

Definition 7.1 (Converges). Fix a topological space $(X, \mathcal{T})$. A sequence of points $(a_i)_{i \in \mathbb{N}} \subset X$ **converges** to $b \in X$ if for every open set W containing b, all but finitely many of the terms of the sequence are in W. In symbols

$$(a_i)_{i \in \mathbb{N}} \to b \iff \forall W \in \mathcal{T} \text{ s.t. } b \in W, \exists N \in \mathbb{N} \text{ s.t. } \forall m > n, a_m \in W$$

Definition 7.2 (Sequentially closed). A set $S \subset X$ is **sequentially closed** if for every sequence $(a_i)_{i \in \mathbb{N}}$ of points in S converging to some $b \in X$, we have $b \in S$.

How do sequentially closed and closed sets relate?

- In a metric space, a set is closed if and only if it is sequentially closed.
- In general topological spaces, this need not be true.

How bad can convergence be in an arbitrary topological space? Pretty bad.

Example 7.1 (Every sequence converges to every point)**.** Let X be a set with at least 2 points and $\mathcal{T}$ be the indiscrete (or trivial) topology (note: $|X| = 1$ isn't that interesting since every sequence in X would then be constant and hence convergent). Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of points in X and fix a point $b \in X$.

Claim 7.1. $(a_i)_{i\in\mathbb{N}} \to b$

Proof. Let $U \subset X$ such that $b \in U$ and $U \in \mathcal{T}$. Since $b \in U$, we have that $U \neq \emptyset$, so that the only possibility is that $U = X$ (since $\mathcal{T} = \{\emptyset, X\}$). But then U contains all elements of the sequence $(a_i)_{i\in\mathbb{N}}$. Thus $(a_i)_{i\in\mathbb{N}}$ converges to b. Since b was arbitrary, $(a_i)_{i\in\mathbb{N}}$ converges to every point of X. □

Example 7.2 (Every sequence converges to exactly one point or doesn't converge, or converges to everything)**.** Let $(X, \mathcal{T})$ be the cofinite topology (on an infinite set X). For simplicity, let $X = \mathbb{N}$. Let $(a_i)_{i\in\mathbb{N}}$ be a sequence of points in X. We can divide the possible forms of $(a_i)_{i\in\mathbb{N}}$ into 3 cases:

(i) No infinite repetition of any terms (ex. $(1, 2, 3, 4, \ldots)$).

(ii) Exactly one value gets repeated infinitely often (ex. $(1, 2, 1, 3, 1, 4, 1, \ldots)$).

(iii) At least 2 values get repeated infinitely often.

We will show that the respective outcomes for these cases are

(i) Converges to every point.

(ii) Converges to point repeated infinitely often.

(iii) Doesn't converge.

Claim 7.2. A sequence with no infinite repetition converges to every point.

Proof. Let $a = (a_i)_{i\in\mathbb{N}}$ be a sequence in X with no infinite repetition and let $b \in X$. Let $b \in U$ where $U \in \mathcal{T}$ (U open). Note that $U \neq \emptyset$. Thus U is cofinite (that is, $X - U$ is finite, so that finitely many points of X are *not* in U). Therefore each point not in U only appears finitely many times in the sequence (since there is no infinite repetition in a). Therefore only finitely many of the terms of a are not in U (finite $\times$ finite = finite). Therefore the sequence converges to b. □

Claim 7.3 (Metric space: closed $\iff$ sequentially closed)**.** In a metric space, a set is closed if and only if it is sequentially closed. More formally: If (X, d) is a metric space and $\mathcal{T}_d$ is the induced topology on X, then $S \subset X$ is sequentially closed if and only if S is closed (with respect to $\mathcal{T}$, that is $X - S \in \mathcal{T}$).

Closed $\Rightarrow$ sequentially closed. Suppose (for contradiction) that $A \subset X$, A closed, but A not sequentially closed. Since A is not sequentially closed, there is a sequence of points $(a_i)_{i\in\mathbb{N}}$ from A and a point $b \in X - A$ such that $(a_i)_{i\in\mathbb{N}} \to b$.

A closed means $X - A$ is open. So since $b \in X - A$, there is some $U \in \mathcal{T}_d$ with $b \in U$ such that $U \cap A = \emptyset$. Of course, $U = X - A$ works. But, we can find a basic open (i.e., a set in the basis we're using, here the set of open balls). Since $b \in X - A$ and $X - A$ open, there exists $\varepsilon > 0$ such that $B_\varepsilon(b) \subset X - A$. Thus we have that

(i) $B_\varepsilon(b)$ is open.

(ii) $b \in B_\varepsilon(b)$.

(iii) $B_\varepsilon(b)$ contains none of the terms of $(a_i)_{i\in\mathbb{N}}$ (since $a_i \in A$ for all i).

But then $(a_i)_{i\in\mathbb{N}} \not\to b$, a contradiction. □

8 Product Topology

Definition 8.1 (Product topology, two sets). Let (X, τ) and (Y, σ) be topological spaces. The **product topology** on $X \times Y$ is the topology having as *basis* the collection $\mathcal{B}$ of all set of the form $U \times V$ where U is an open subset of X and V is an open subset of Y. In symbols,

$$\mathcal{B}_{\tau\times\sigma} = \{U \times V \mid U \in \tau, V \in \sigma\} \tag{8.1}$$

Remark. Note, open sets of $X \times Y$ need not be of the form open set in X $\times$ open set in Y.

Proof that $\mathcal{B}_{\tau\times\sigma}$ is indeed a basis for a topology on $X \times Y$. We check the two conditions required to be a basis:

(i) $\mathcal{B}$ "covers" X: Note that $X \in \tau$ and $Y \in \sigma$ (since they are each topologies). Therefore $X \times Y \in \mathcal{B}$. Thus for any $(x, y) \in X \times Y$, we have that $(x, y) \in X \times Y \in \mathcal{B}$.

(ii) Intersection Property: Take two basis elements $U_1 \times V_1$ and $U_2 \times V_2$. Notice that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \tag{8.2}$$

But then $U_1 \cap U_2 \in \tau$ and $V_1 \times V_2 \in \sigma$, so that $(U_1 \times V_1) \cap (U_2 \times V_2) \in \mathcal{B}$, so the intersection of two basis elements is again a basis element. This is stronger than the property required to be a basis, yet of course sufficient.

□

Definition 8.2 (Product space/topology for finitely many spaces). Let $(X_1, \tau_1), \ldots, (X_n, \tau_n)$ be topological spaces. The set of points of the **product space** is $X_1 \times \cdots \times X_n$. The basis for the **product topology** is

$$\mathcal{B}_{\tau_1\times\cdots\times\tau_n} = \{W_1 \times \cdots \times W_n \mid W_i \in \tau_i\} \tag{8.3}$$

Remark. Again, $\mathcal{B}_{\tau_1\times\cdots\times\tau_n}$ is indeed a basis since the first condition is trivially satisfied ($X_1 \times \cdots \times X_n$ is a basis element) and the intersection of products is a product of intersections, and hence again a basis element.

Definition 8.3 (Projection Maps). Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Let

$$\pi_1 : X_1 \times X_2 \to X_1 : (x_1, x_2) \mapsto x_1$$
$$\pi_2 : X_1 \times X_2 \to X_2 : (x_1, x_2) \mapsto x_2$$

then π_1 and π_2 are **projection maps**.

Claim 8.1. Let π_1 be π_2 projection maps (as above). Then

(i) π_1 is continuous with respect to $\tau_1 \otimes \tau_2$ and τ_1.

(ii) π_2 is continuous with respect to $\tau_1 \otimes \tau_2$ and τ_2.

Proof. We show π_1 is continuous. Suppose $S \subset X_1$ is open (i.e., $\in \tau_1$). We want to show that $\pi^{-1}(S) \in \tau_1 \otimes \tau_2$. We have that

$$\begin{aligned} \pi_1^{-1}(S) &= \{p \in X_1 \times X_2 \mid \pi_1(p) \in S\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 \mid \pi_1((x_1, x_2)) \in S\} \\ &= \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in S\} \\ &= S \times X_2 \end{aligned}$$

Thus we have that

- S is τ_1-open.
- X_2 is τ_2-open.

so that $S \times X_2$ is in our basis for $\tau_1 \otimes \tau_2$. Thus, $S \times X_2 \in \tau_1 \otimes \tau_2$.

□

Example 8.1 (Projection Maps)**.** Take $(X_1, \tau_1) = (X_2, \tau_2) = (\mathbb{R}, \tau_e)$. Take $S \subset \mathbb{R}^2$ to be the unit ball: $S = \{(x, y) \mid x^2 + y^2 < 1\}$. Observe that

$$\pi_1(S) = \pi_2(S) = (-1, 1) \tag{8.4}$$

9 Compactness

Definition 9.1 (Open cover)**.** An **open cover** of (X, τ) is a family of τ-open sets $\mathcal{C} \subset \tau$ such that $\bigcup \mathcal{C} = X$.

Remark. The notation $\bigcup \mathcal{C} = X$ means X is the union of "stuff" in $\mathcal{C}$.

Example 9.1 (Open covers)**.** The following are examples of open covers of (X, τ):

(i) Any basis for τ is an open cover of (X, τ).

(ii) Any subbasis for τ is an open cover of (X, τ).

(iii) $\{X\}$.

(iv) τ.

(v) Let $X = \mathbb{R}$ and $\tau = \tau_e$. Then $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$ is an open cover of X (call each individual interval U_n). (If $\mathcal{W} \subset \mathcal{U}$ is finite, let $N = \max\{n : U_n \in \mathcal{W}\}$. Then $N \notin \bigcup \mathcal{W}$)

Definition 9.2 (Subcover)**.** $\mathcal{D}$ is a **subcover** of $\mathcal{C}$ if

(i) $\mathcal{D} \subset \mathcal{C}$.

(ii) $\mathcal{D}$ is an open cover.

A **finite subcover** is a subcover which is finite.

Definition 9.3 (Compact)**.** A topological space (X, τ) is **compact** if every open cover has a finite subcover.

Example 9.2 (Non-compact Set)**.** Consider the topological space $(\mathbb{R}, \tau_e)$. This space has a finite subcover: $\{\mathbb{R}\}$. But does this imply that $(\mathbb{R}, \tau_e)$ is compact? No! We need to see if *all* open covers have finite subcovers. Consider the open cover

$$\mathcal{C} = \{(-n, n) \mid n \in \mathbb{N}\} \tag{9.1}$$

$\mathcal{C}$ has no finite subcover. Thus $(\mathbb{R}, \tau_e)$ is not compact.

9.1 Applications

9.1.1 Optimization

Incomplete.

Theorem 9.1 (Bounded Value Theorem (BVT))**.** If $f : [0, 1] \to \mathbb{R}$ is continuous, then $\operatorname{rng} f$ is bounded (above and below).

9.1.2 Cantor Space

Cantor space $\mathcal{C}$ has

- Points given by infinite binary sequences (ex. $(1, 0, 1, 0, 1, 0, \ldots)$ is a point). (For concreteness, can think about base-2 representation of numbers in $[0, 1]$).
- Open sets are generated by finite strings: $U \subset \mathcal{C}$ is open if and only if for all $f \in U$, there exists a finite binary string σ such that every infinite binary sequence beginning with σ is in U.

Theorem 9.2. The Cantor Space $\mathcal{C}$ is compact.

Proof. ☐

Incomplete.

9.2 Creating New Compact Spaces from Old

Theorem 9.3 ("Continuous images" of compact spaces are compact)**.** Suppose (X, τ) is compact and $f : (X, \tau) \to (Y, \sigma)$ is continuous and surjective (that is, $Y = \operatorname{im} f$). Then (Y, σ) is compact.

Proof. Let $\mathcal{U}$ be an open cover of Y. We must show there exists a finite subcover. For $U \in \mathcal{U}$ Let $W_U = f^{-1}(U)$. W_U is open since f is continuous and $U \in \sigma$ (open in Y). Then $\{W_U : U \in \mathcal{U}\}$ covers X. To see this, take $x \in X$. Then $f(x) \in Y$, so that there exists some $U \in \mathcal{U}$ containing $f(x)$. But then $x \in f^{-1}(U) = W_u$. (X, τ) is compact, so $\{W_U : U \in \mathcal{U}\}$ has a finite subcover: $\{W_{U_i}\}_{i=1}^n$. f is surjective, so

$$\bigcup_{i=1}^{n} f(W_{U_i}) = \bigcup_{i=1}^{n} U_i = Y \tag{9.2}$$

is a finite subcover of Y.

Proof summary: "Pull back (continuity), push forward (surjectivity)." □

Corollary 9.4. $[0, 1]$ is compact.

Proof. Use the map from cantor space $\mathcal{C}$ to $[0, 1]$ that is the binary expansion. That is, $\mathcal{C} \to [0, 1] : f \mapsto f(1)f(2)f(2)\ldots$ in binary. This is a continuous surjection, and $\mathcal{C}$ is compact, so that by the above theorem, $[0, 1]$ is compact. □

Theorem 9.5 (Closed subspaces of compact spaces are compact)**.** If (X, τ) is compact and $A \subseteq X$ is closed, then (A, τ_A) is compact.

Proof. Let $\mathcal{U}$ be an open cover of A. Using the definition of the subspace topology, we can write each $U \in \mathcal{U}$ as $U = V_U \cap A$ where $V_U \in \tau$. Since A is closed, we have that $X - A$. Therefore we can form an open cover of X by

$$\hat{\mathcal{U}} = \{V_U : U \in \mathcal{U}\} \cup \{X - A\} \tag{9.3}$$

$\hat{\mathcal{U}}$ must have a finite subcover since X is compact, and therefore this finite subcover must use finitely many elements from the set $\{V_U : U \in \mathcal{U}\}$ (say $V_{U_1}, \ldots, V_{U_n}$). But then

$$A \subseteq \bigcup_{i=1}^{n} U_i \tag{9.4}$$

is a finite subcover of A. Therefore every open cover of A has a finite subcover, so A is compact. □

Theorem 9.6 (Finite Tychonoff)**.** Suppose (X, τ) and (Y, σ) are compact spaces. Then their product $(X \times Y, \tau \otimes \sigma)$ is compact. This also holds for n spaces with $n < \infty$.

Proof. We prove the case of $n = 2$.

Incomplete.

□

Example 9.3 (Example of Finite Tychonoff)**.** The closed unit cube $[0, 1] \times [0, 1] \times [0, 1]$ is compact.

9.3 Tychonoff's Theorem

Definition 9.4 (Cartesian Product (Possibly Infinite)). Let $\{X_i\}_{i\in I}$ be a family of sets, where I is an arbitrary index set. The **Cartesian product** $\prod_{i\in I} X_i$ is the set of all functions f such that

(i) $f : I \to \cup_{i\in I} X_i$

(ii) $f(i) \in X_i$

In words, f picks out a point from each set. In notation

$$\prod_{i\in I} X_i = \{f : I \to \cup_{i\in I} X_i \mid f(i) \in X_i)\forall i \in I\} \tag{9.5}$$

Figure.

Example 9.4 (Finite Cartesian Product). Let's check that this definition agrees with the standard notion of a finite Cartesian product.

Let $I = \{1, 2\}$. The point $p = (a, b) \in X_1 \times X_2$ corresponds to the function

$$f : \{1, 2\} \to X_1 \cup X_2 : \begin{array}{ll} 1 \mapsto a & \text{(1st coordinate of } p \text{ is } a) \\ 2 \mapsto b & \text{(2nd coordinate of } p \text{ is } b) \end{array} \tag{9.6}$$

Definition 9.5 (Product Space). Let I be an arbitrary index set. Suppose for each $i \in I$ we have that (X_i, τ_i) is a topological space. Their **product space** has

- Underlying set:

$$\prod_{i\in I} X_i \tag{9.7}$$

- Topology:
  - Informally: $\bigotimes_{i\in I} \tau_i$ generated by the subbasis of "wedges": that is, the topology generated by the subbasis

$$\mathcal{B} = \text{“all sets of the form”} \quad \cdots \times X_i \times X_j \times \cdots \times \underbrace{U}_{n\text{th term}} \times X_k \times \cdots \tag{9.8}$$

  for U open in τ_n.

  Figure.

  - Formally: $\bigotimes_{i\in I} \tau_i$ generated by the subbasis $\mathcal{B}$, which is sets of the form

$$\left\{ f \in \prod_{i\in I} X_i \,\middle|\, f(k) \in W \right\} \quad k \in I, W \in \tau_k \tag{9.9}$$

  - Alternative: $\bigotimes_{i\in I} \tau_i$ is the coarsest topology making all projection maps continuous (where $\pi_i : f \mapsto f(i)$).

$$\pi_i^{-1}(U) \in \mathcal{B} \tag{9.10}$$

What does this mean?

Theorem 9.7 (Full Tychonoff). Suppose (X_i, τ_i) is compact for all $i \in I$. Then the space

$$\left(\prod_{i\in I} X_i, \bigotimes_{i\in I} \tau_i\right) \tag{9.11}$$

is compact.

9.3.1 Ultrafilters

Definition 9.6 (Ultrafilter). Suppose I is an set (wlog infinite). An **ultrafilter** $\mathbb{U}$ on I is a family of subsets of I such that: $\mathbb{U}$ is a filter:

(i) **Contains I, not $\emptyset$**: $I \in \mathbb{U}, \emptyset \notin \mathbb{U}$.

(ii) **Closed upwards**: $A \in \mathbb{U},\ A \subseteq B \Rightarrow B \in \mathbb{U}$.

(iii) **Closed under finite intersections:** $A_1, \ldots, A_n \in \mathbb{U} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathbb{U}$.

and $\mathbb{U}$ satisfies the additional "ultra" condition:

(iv) $\forall A \subseteq I$, $A \in \mathbb{U}$ or $I - A \in \mathbb{U}$ (but not both, by (i) and (iii))

Example 9.5 (Frechet Filter). Recall that a set is cofinite if its complement is finite. Then

$$\mathcal{F} = \{\text{cofinite subsets of } I\} \tag{9.12}$$

is a filter on I. Note that

- Each cofinite set is "most" of I
- This is not a topology (doesn't have $\emptyset$)

We check the conditions required to be a filter:

(i) $I - I = \emptyset$, which is finite, so $I \in \mathcal{F}$.
$I - \emptyset = I$, which is assumed infinite, so $\emptyset \notin \mathcal{F}$.

(ii) Let $A \in \mathcal{F}$ and $B \subseteq I$ such that $A \subseteq B$. Then $I - B \subseteq I - A$, and $I - A$ is finite, so $I - B$ must be finite. Therefore $B \in \mathbb{U}$.

(iii) (For simplicity, check with two sets) Let $A_1, A_2 \in \mathbb{U}$. Then

$$I - (A_1 \cap A_2) = (I - A_1) \cup (I - A_2) \tag{9.13}$$

Each of $(I - A_1)$ and $(I - A_2)$ is finite, so that their union is also finite. Therefore $A_1 \cap A_2 \in \mathbb{U}$.

Example 9.6 (Principal filter). Suppose $I = \mathbb{N}$. The set

$$\mathcal{F} = \{\text{all sets containing } 7\} \tag{9.14}$$

is a filter. Note:

- We can interpret this filter like a dictatorship in voting.

  Complete intuition.

We check the conditions:

(i) $12 \in \mathbb{N}$, so $I = \mathbb{N} \in \mathcal{F}$.

$12 \notin \emptyset$, so $\emptyset \notin \mathcal{F}$.

(ii) Clear.

(iii) Clear.

Definition 9.7 (Principal ultrafilter)**. Principal ultrafilters** take the form

$$\langle a \rangle = \{A \subseteq I : a \in A\} \qquad \text{for some } a \in I \tag{9.15}$$

Example 9.7 (Frechet filter is not an ultrafilter)**.** For a concrete example, let $I = \mathbb{N}$, $A = \{\text{evens}\} \subset I$. But $A \notin \mathbb{U}$ and $I - A \notin \mathbb{U}$.

Example 9.8 (Principal filters are ultrafilters)**.** Consider $I = \mathbb{N}$. Let $A \subseteq \mathbb{N}$. Then $7 \in A$ or $7 \in \mathbb{N} - A$.

Example 9.9 (Interpretation of non-principal ultrafilter)**.** Think of a game of ∞-questions. The premise of the game: I have $n \in \mathbb{N}$, and you want to find it. Example questions:

Q	A
Even	No
> 3	Yes
Prime	No
> 500	Yes
Div 13	No

A non-principal ultrafilter is a cheating strategy where you never get caught (all my answers are consistent). The strategy is:

$$Q : \text{“Is } n \in A\text{”}$$
$$A : \begin{cases} A \in \mathbb{U} & \text{yes} \\ \mathbb{N} - A \in \mathbb{U} & \text{no} \end{cases}$$

This is an ultrafilter:

(i) The number needs to be in $\mathbb{N}$.

(ii) We need closure upwards (so we don't get caught with a finite set of consistent answers remaining).

(iii) We need to answer compound questions correctly.

This ultrafilter is non-principal: We never say yes to a specific number.

Definition 9.8 (Finite intersection property (FIP)). A collection of subsets $\mathcal{F}$ of I has **FIP** if whenever $A_1, \ldots, A_n \in \mathcal{F}$ we have $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

Corollary 9.8. There exist non-principal ultrafilers.

Proof. Let I be an infinite set and let $\mathcal{F}$ be the Frechet filter on I. Filters have FIP by properties (i) (contains the entire set and *doesn't* contain the empty set) and (iii) (closed under finite intersections). Indeed, since the finite intersection of any elements of $\mathcal{F}$ must be in $\mathcal{F}$, and the empty set is not in $\mathcal{F}$, filters have FIP. By Theorem 9.9, there exists an ultrafilter $\mathbb{U}$ such that $\mathbb{U} \supseteq \mathcal{F}$.

Claim 9.1. $\mathbb{U}$ is non-principal.

Proof. Suppose that $\mathbb{U}$ were principal. Then by definition $\{a\} \in \mathbb{U}$ for some $a \in I$. But $I - \{a\}$ is cofinite. Therefore $I - \{a\} \in \mathcal{F} \subseteq \mathbb{U}$. This already contradicts property (iv) (the ultra condition). We also have a contradiction to (i) (doesn't contain the empty set). Indeed, $(I - \{a\}) \cap \{a\} = \emptyset \in \mathbb{U}$, since $\mathbb{U}$ is closed under finite intersections. □

Therefore, extending the Frechet filter gives us a non-principal ultrafilter. □

Theorem 9.9 (FIP $\Rightarrow$ ultrafilter extension). If $\mathcal{F}$ has FIP, then there exists an ultrafilter $\mathbb{U}$ extending $\mathcal{F}$. That is, $\exists \mathbb{U}$ such that $\mathbb{U} \supseteq \mathcal{F}$.

Proof. Define a partial order as follows: let

$$\mathbb{P} = \{ \text{ set of subsets of I which (i) have FIP and (ii) are supersets of } \mathcal{F}\}$$

Formally

$$\mathbb{P} = \{A \subseteq \mathcal{P}(I) \mid A \supseteq \mathcal{F} \text{ and } A \text{ has FIP}\} \tag{9.16}$$

The partial order $\triangleleft$ is $\subsetneq$.

Example 9.10. Even/Odd

Don't understand this example.

Claim 9.2. Chains in $\mathbb{P}$ have upper bounds.

Remark. With this claim, we can apply Zorn's lemma. We will showing the the maximal elements of $\mathbb{P}$ (which exist by Zorn) are ultrafilters containing $\mathcal{F}$.

Proof. Let $\mathcal{C} \subseteq \mathbb{P}$ be a chain (thus any two elements are comparable). Define

$$\mathcal{D} = \bigcup \mathcal{C} \tag{9.17}$$

Informally, $\mathcal{D} = \{A \subseteq I : A \text{ in some element of } \mathcal{C}\}$. We show that $\mathcal{D}$ is an upper bound for $\mathcal{C}$ in $\mathbb{P}$ (notice, $\mathcal{D}$ must indeed be **in** $\mathbb{P}$).

(i) $\mathcal{D}$ is an upper bound: clear. By definition, $\mathcal{D}$ contains each element of $\mathcal{C}$.

(ii) $\mathcal{D} \in \mathbb{P}$: We must show the two conditions $\mathbb{P}$ requires.

  (a) $\mathcal{D} \supseteq \mathcal{F}$: Since each element of $\mathcal{C}$ is a superset of $\mathcal{F}$, we have that $\mathcal{D} \supseteq \mathcal{F}$.

  (b) $\mathcal{D}$ has FIP: Let $A_1, \ldots, A_n \in \mathcal{D}$. These sets must come from elements of $\mathcal{C}$. Thus there exist $C_1, \ldots, C_n \in \mathcal{C}$ such that $A_1 \in C_1, \ldots, A_n \in C_n$. Since $\mathcal{C}$ is a chain, one of $C_1, \ldots, C_n$ must contain the others. WLOG say C_n contains the others. Then $A_1, \ldots, A_n \in C_n$. $C_n \in \mathbb{P}$, so C_n has FIP. Therefore $A_1 \cap \cdots \cap A_n \neq \emptyset$.

□

By Zorn's lemma, we get a $\mathbb{U} \in \mathbb{P}$ maximal element. We show that $\mathbb{U}$ is an ultrafilter. First show $\mathbb{U}$ is a filter:

(i)

(ii)

(iii)

Show filter properties.

Now show the ultra property. Let $A \subseteq I$. We must show that $A \in \mathbb{U}$ or $I - A \in \mathbb{U}$. For the sake of contradiction, suppose neither. Then

(i) $\mathbb{U} \cap \{A\} \supsetneq \mathbb{U}$

(ii) $\mathbb{U} \cap \{I - A\} \supsetneq \mathbb{U}$

Since $\mathbb{U}$ is $\mathbb{P}$-maximal, we must have that

(i) $\mathbb{U} \cap \{A\} \notin \mathbb{P}$

(ii) $\mathbb{U} \cap \{I - A\} \notin \mathbb{P}$

Since these sets are not in $\mathbb{P}$, they must violate one of the two properties elements of $\mathbb{P}$ must have. We have that both are supersets of $\mathcal{F}$. Thus the sets must violate FIP. We get that

(i) $X_1, \ldots, X_m \in \mathbb{U}$ such that $X_1 \cap \cdots \cap X_m \cap A = \emptyset$.

(ii) $Y_1, \ldots, Y_n \in \mathbb{U}$ such that $Y_1 \cap \cdots \cap Y_n \cap (I - A) = \emptyset$.

But then

$$Z = X_1 \cap \cdots \cap X_m \cap Y_1 \cap \cdots \cap Y_n = \emptyset \tag{9.18}$$

This shows that $\mathbb{U}$ doesn't have FIP, which is a contradiction. □

9.4 Ultrafilters in Topology

Definition 9.9 (Ultrafilter convergence)**.** Suppose (X, τ) is a topological space and $\mathbb{U}$ is an ultrafilter on X. Then $\mathbb{U}$ **converges** to α (and we write $\mathbb{U} \to \alpha$) for $\alpha \in X$ if every open set containing α is in $\mathbb{U}$. In symbols

$$\mathbb{U} \to \alpha \iff \forall V_\alpha \in \tau : \alpha \in V_\alpha, V_\alpha \in \mathbb{U} \tag{9.19}$$

Theorem 9.10. TFAE

(i) (X, τ) is compact.

(ii) Every ultrafilter on X converges to some point.

Proof that (i) implies (ii). Prove the **contrapositive**. Suppose (X, τ) is not compact. Let $\mathcal{C}$ be an open cover with no finite subcover. We want to demonstrate/construct a non-convergent ultrafilter on X. Clearly, this ultrafilter can't be principal (since then every set in the principal ultrafilter contains a special point, so the ultrafilter trivially converges to this point). We will use the ultrafilter extension lemma: that is, we will construct a filter $\mathcal{F}$ with FIP, so that we have an ultrafilter extending $\mathcal{F}$. Let

$$\mathcal{F} = \{X - A \mid A \in \mathcal{C}\} \tag{9.20}$$

Claim 9.3. $\mathcal{F}$ has FIP (and is a filter).

Proof. $\mathcal{F}$ is the Frechet filter. Proof of FIP by **contradiction**. Suppose there exists a finite collection of sets $\{B_i\}_{i=1}^n \subseteq \mathcal{F}$ such that $\bigcap_{i=1}^n B_i = \emptyset$. But then

$$\begin{aligned} X &= X - \bigcap_{i=1}^n B_i \\ &= \bigcup_{i=1}^n (X - B_i) \end{aligned}$$

By assumption, since $B_i \in \mathcal{F}$, we have that $X - B_i \in \mathcal{C}$. Therefore we have demonstrated a finite subcover of $\mathcal{C}$ (of X), which contradicts that $\mathcal{C}$ has no finite subcover. □

Thus by the ultrafilter extension lemma, there exists an ultrafilter $\mathbb{U}$ on X such that $\mathbb{U} \supseteq \mathcal{F}$. We want to show $\mathbb{U}$ doesn't converge, so we must show $\forall \alpha \in X$, $\exists V_\alpha \in \tau$ such that $V_\alpha \notin \mathbb{U}$. To show this, fix an $\alpha \in X$. Since $\mathcal{C}$ is a cover, $\exists A_\alpha \in \mathcal{C}$ such that $\alpha \in A_\alpha$. But then by construction $X - A_\alpha \in \mathcal{F} \subseteq \mathbb{U}$. Thus $A \notin \mathbb{U}$ (by axioms (1) and (3), since $X - A \in \mathbb{U}$).

□

Proof that (ii) implies (i). Prove the **contrapositive**. Suppose $\mathbb{U}$ is an ultrafilter with no limit. This means

$$\forall \alpha \in X, \exists V_\alpha \in \tau : \alpha \in V_\alpha, V_\alpha \notin \mathbb{U} \tag{9.21}$$

Consider

$$\mathcal{C} = \{V_\alpha : \alpha \in X\} \tag{9.22}$$

Claim 9.4. $\mathcal{C}$ is an (open) cover with no finite subcover.

Proof. In words, there is no finite subcover iff no finite union is everything. We show each property.

(i) $\mathcal{C}$ is an open cover: $\forall \alpha \in X$, $\alpha \in V_\alpha \in \mathcal{C}$, so $\bigcup \mathcal{C} = X$.

(ii) $\mathcal{C}$ does not have a finite subcover: prove by contradiction. Suppose there exist $V_{\alpha_1}, \dots, V_{\alpha_n} \in \mathcal{C}$ such that $\bigcup_{i=1}^{n} V_{\alpha_i} = X$. Then

$$X - \bigcup_{i=1}^{n} V_{\alpha_i} = \emptyset \tag{9.23}$$

But also

$$X - \bigcup_{i=1}^{n} V_{\alpha_i} = \bigcap_{i=1}^{n} (X - V_{\alpha_i}) \tag{9.24}$$

We have that, for each i, $V_{\alpha_i} \notin \mathbb{U}$, so that by the ultrafilter property, $X - V_{\alpha_i} \in \mathbb{U}$. But then $\bigcap_{i=1}^{n}(X - V_{\alpha_i}) = \emptyset$, and we have thus found a finite collection of sets in $\mathbb{U}$ with an empty intersection, so $\mathbb{U}$ doesn't have FIP. This is a contradiction.

This claim shows that (X, τ) is not compact, which proves the contrapositive. □

□

9.5 Proof of Tychonoff via ultrafilters

Proof of Tychonoff via ultrafilters. □

10 Quotient Topology

Definition 10.1 (Quotient Map)**.** Let (X, τ), (Y, σ) be topological spaces. A map $f : (X, \tau) \to (Y, \sigma)$ is a **quotient map** with respect to (τ, σ) iff

(i) $\forall A \subseteq Y : A \in \sigma \iff f^{-1}(A) \in \tau$ (in words, a subset A of Y is open in Y iff $f^{-1}(A)$ is open in X).

(ii) f surjective

Definition 10.2 (Quotient topology)**.** If (X, τ) is a topological space and $f : (X, \tau) \to Y$ ($Y \neq \emptyset$) surjective, then the **quotient topology** given by f is

$$\sigma = \left\{ A \subseteq Y \,\middle|\, f^{-1}(A) \in \tau \right\} \tag{10.1}$$

11 Practice Questions

11.1 Midterm

Exercise 11.1. Suppose

- X, Y are nonempty sets
- $\tau = \mathcal{P}(X)$
- $\sigma = \mathcal{P}(Y)$

Show (X, τ) and (Y, σ) are topological spaces and $\tau \otimes \sigma = \mathcal{P}(X \times Y)$.

Proof. (X, τ) and (Y, σ) are clearly topological spaces since the power set is all possible subsets: this immediately implies all the axioms are satisfied. $\square$

A Set Theory Review

Definition A.1 (Difference)**.** The **difference** of two sets, denoted $A - B$, is the set consisting of those elements of A that are not in B. In notation

$$A - B = \{x | x \in A \text{ and } x \notin B\}$$

Theorem A.1 (Set-Theoretic Rules)**.** We have that, for any sets A, B, C,

(i) First distributive law for the operations $\cap$ and $\cup$:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{A.1}$$

(ii) Second distributive law for the operations $\cap$ and $\cup$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{A.2}$$

(iii) DeMorgan's laws:

$$A - (B \cup C) = (A - B) \cap (A - C) \tag{A.3}$$

"The complement of the union equals the intersection of the complements."

$$A - (B \cap C) = (A - B) \cup (A - C) \tag{A.4}$$

"The complement of the intersection equals the union of the complements."

A.1 Functions

Exercise A.1. Let $f : A \to B$. Let $A_0 \subset A$ and $B_0 \subset B$. Then

(i) $A_0 \subset f^{-1}(f(A_0))$ and equality holds if f is injective.

(ii) $f(f^{-1}(B_0)) \subset B_0$ and equality holds if f is surjective.

Solution. We prove each item in turn:

(i) Let $a \in A_0$. Then $f(a) \in f(A_0)$. We have that $f^{-1}(f(A_0)) = \{a \,|\, f(a) \in f(A_0)\}$. Then $f(a) \in f^{-1}(f(A_0))$, so that $A_0 \subset f^{-1}(f(A_0))$. We can actually show equality holds if and only if f is injective.

  (a) $\Leftarrow$ Suppose f is injective. Let $a \in f^{-1}(f(A_0))$. Then $f(a) \in f(A_0)$. Therefore there exists some $b \in f(A_0)$ such that $f(a) = f(b)$. Injectivity implies $a = b \in A_0$.

  (b) $\Rightarrow$ We will prove the contrapositive. Suppose f is *not* injective. Then $f(a) = f(b)$ for some $a \neq b$. Therefore $\{a, b\} \subset f^{-1}(f(\{a\}))$. Thus $f^{-1}(f(\{a\})) \not\subset \{a\}$.

(ii) Let $x \in f(f^{-1}(B_0))$. Then there is some $b \in f^{-1}(B_0)$ such that $f(b) = x$. But $f(b) \in B_0$, so $x \in B_0$.

(a) $\Leftarrow$ Suppose f is surjective. Take $b \in B_0$, then there exists some $a \in A_0$ such that $f(a) = b$, so that $a \in f^{-1}(B_0)$, and $b = f(a) \in f(f^{-1}(B_0))$.

Exercise A.2.

Solution. (i) Let $B_0 \subset B_1$. Fix $x \in f^{-1}(B_0)$. Then $f(x) \in B_0$, which implies $f(x) \in B_1$. Thus $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(ii) We show two inclusions:

(a) $\supset$: We can use (i), since $B_i \subset B_0 \cup B_1$, so $f^{-1}(B_0) \subset f^{-1}(B_0 \cup B_1)$ and $f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$, so $f^{-1}(B_0) \cup f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$.

(b) $\subset$: Let $x \in f^{-1}(B_0 \cup B_1)$. Thus there exists some $b \in B_0 \cup B_1$ such that $f(x) = b$. Therefore $x \in f^{-1}(B_0)$ or $x \in f^{-1}(B_1)$, so that $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_1)$.

(iii)

B Practice

Exercise B.1. Give an example of two topologies on the same set which are incomparable (that is, neither is finer than the other).

Solution. Consider $X = \{a, b\}$. Define two topologies as

$$\tau_1 = \{\emptyset, \{a\}, X\}$$
$$\tau_2 = \{\emptyset, \{b\}, X\}$$

Then τ_1 and τ_2 are both topologies but are not comparable.

Exercise B.2. Prove that for any (nonempty) set X and any family F of subsets of X, there is a smallest topology τ on X with $F \subseteq \tau$.

Solution. Define a new family of subsets of X by $F' = F \cup \{X\}$. Clearly F' is a subbasis of X (to be a subbasis, the union of the elements of F' must equal X, but this follows immediately since $X \in F'$). We know that the topology $\tau_{F'}$ generated by a subbasis F' is the coarsest topology on a set X containing F'. Now notice that any topology τ containing F must also contain X, since a topology must contain the entire set. Thus any topology τ must also contain F'. Thus $\tau_{F'}$ is the smallest topology on X containing τ.

Exercise B.3. Suppose (X, τ) and (Y, σ) are topological spaces and $f : X \to Y$ is a function. Show that each of the following implies that f is continuous:

(i) For every $A \subseteq Y$ closed in the sense of σ, $f^{-1}(A)$ is closed in the sense of τ.

(ii) σ is the indiscrete topology on Y.

(iii) τ is the discrete topology on X.

Proof of (i). We need to prove a simple result from set theory:

Claim B.1. The preimage of the complement of a set is the complement of the preimage of that set. More precisely, suppose $f : X \to Y$ is a function and $U \subseteq Y$. Then

$$f^{-1}(Y - U) = X - f^{-1}(U) \tag{B.1}$$

Proof.

$$\begin{aligned}
f^{-1}(Y - U) &= \{x \in X \mid f(x) \in Y - U\} \\
&= \{x \in X \mid f(x) \in Y \text{ and } f(x) \notin U\} \\
&= \{x \in X \mid f(x) \in Y\} \cap \{x \in X \mid f(x) \notin U\} \\
&= \{x \in X \mid f(x) \in Y\} \cap (X - \{x \in X \mid f(x) \in U\}) \\
&= f^{-1}(Y) \cap \left(X - f^{-1}(U)\right) \\
&= f^{-1}(Y) - f^{-1}(U) \\
&= X - f^{-1}(U)
\end{aligned}$$

□

Using this claim, let $U \in \sigma$. Then $Y - U$ is closed, and by assumption, $f^{-1}(Y - U)$ is also closed. By the claim, we have that $f^{-1}(Y - U) = X - f^{-1}(U)$. Thus $X - f^{-1}(U)$ is closed so that $f^{-1}(U)$ is open and f is continuous. □

Proof of (ii). Suppose σ is the indiscrete topology on Y. Let $U \in \sigma$. Then either $U = \emptyset$ or $U = X$. Then $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. But since τ is a topology, we must have that $\emptyset, X \in \tau$. Therefore in both cases, $f^{-1}(U)$ is open (in τ sense), so that f is continuous. □

Proof of (iii). Suppose τ is the discrete topology on X. Let $U \in \sigma$. But then $f^{-1}(U) \subseteq X$, so that $f^{-1}(U) \in \mathcal{P}(X)$. Therefore $f^{-1}(U) \in \tau$, so that f is continuous. □

Exercise B.4. Give an example of a metric on $\mathbb{R}$ which induces the discrete topology.

Solution. The discrete metric induces the discrete topology. Recall:

Claim B.2. A set is open in the metric topology induced by d if and only if for each $y \in U$ there is a $\delta > 0$ such that $B_d(y, \delta) \subset U$.

For $x, y \in \mathbb{R}$, the discrete metric is defined by

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \tag{B.2}$$

Fix $U \subset \mathbb{R}$ and take $y \in U$. Let $\delta = 1$ (or anything less but positive). Then $B_d(y, \delta) = \{y\}$. Therefore $B_d(y, \delta) \subset U$. This shows that the discrete metric d induces the discrete topology.

Exercise B.5. Show no metric on $\mathbb{R}$ induces the indiscrete topology.

Exercise B.6. Show that if (X, d) is a metric space, then (X, e) is also a metric space, where

$$e(x, y) = \min\{d(x, y), 1\} \tag{B.3}$$

Proof. Positivity (and definiteness) and symmetry follow immediately from properties of d and min. We need to show the triangle inequality holds. We prove this in two cases. Fix $x, y, x \in X$. We must show that $e(x,y) + e(y,z) \geq e(x,z)$.

(i) $d(x,y), d(y,z) \leq 1$. In this case,

$$\begin{aligned} e(x,y) + e(y,z) &= d(x,y) + d(y,z) \\ &\geq d(x,z) && (\triangle \text{ inequality with } d) \end{aligned}$$

If $d(x,z) \leq 1$, then $e(x,z) = \min\{d(x,z), 1\} = d(x,z)$. If $d(x,z) \geq 1$, then $d(x,z) \geq e(x,z) = \min\{d(x,z), 1\} = 1$. The triangle inequality holds in both cases.

(ii) At least one of $d(x,y), d(y,z) \geq 1$. Notice that by definition $e(x,z) \leq 1$. In this case, $e(x,y) + e(y,z) \geq 1$ or even $e(x,y) + e(y,z) \geq 2$. But then since $e(x,z) \leq 1$, the triangle inequality follows.

□

Exercise B.7. Suppose (X, τ) is a topological space, $A, B \subseteq X$, $A \cup B = X$, and the subspace topologies (A, τ_A) and (B, τ_B) are each compact. Then (X, τ) is compact.

Solution. Let $\mathcal{U}$ be an open cover X. Since $A, B \subseteq X$, we also must have that $\mathcal{U}$ is an open cover of A and B, in the sense of τ. Define

$$\mathcal{A} = \{U \cap A \mid U \in \mathcal{U}\} \tag{B.4}$$

and

$$\mathcal{B} = \{U \cap B \mid U \in \mathcal{U}\} \tag{B.5}$$

Then $\mathcal{A}$ and $\mathcal{B}$ are open covers of A and B respectively (with the subspace topologies τ_A and τ_B). (A, τ_A) and (B, τ_B) are each compact, so $\mathcal{A}$ and $\mathcal{B}$ must have finite subcovers. More explicitly, there exist finite sets $\mathcal{C}$ and $\mathcal{D} \subset \mathcal{U}$ such that

$$A \subseteq \{U \cap A \mid U \in \mathcal{C}\} \tag{B.6}$$

and

$$B \subseteq \{U \cap B \mid U \in \mathcal{D}\} \tag{B.7}$$

But then $\mathcal{C} \cup \mathcal{D}$ is finite and covers $A \cup B = X$.

Exercise B.8. The topological definition of continuity and the $\varepsilon - \delta$ definition of continuity are equivalent. That is, TFAE:

- For every $x \in \mathbb{R}$ and every $\varepsilon > 0$, there is some $\delta > 0$ such that for every $y \in (x - \delta, x + \delta)$ we have $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$.
- For every open set $U \subseteq \mathbb{R}$, the preimage $f^{-1}(U)$ is open.

Solution. We first show that $\varepsilon - \delta$ implies topological. Fix $U \subseteq \mathbb{R}$ open. We must show that $f^{-1}(U)$ is open. That is, that for all $x \in f^{-1}(U)$, we can find an open interval around x entirely contained in $f^{-1}(U)$. So fix $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists some $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq U$. By the $\varepsilon - \delta$ condition, there exists a $\delta > 0$ such that $f((x - \delta, x + \delta)) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq U$. Thus $(x - \delta, x + \delta) \subseteq f^{-1}(U)$.

Now suppose f satisfies the topological definition of continuity. Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. Let $U = (f(x) - \varepsilon, f(x) + \varepsilon)$. U is open. Thus $f^{-1}(U)$ is also open. Thus for $x \in f^{-1}(U)$, we can find an open interval (i.e., a basic open) (a, b) such that $x \in (a, b) \subseteq f^{-1}(U)$. Set $\delta_1 = x - a$ and $\delta_2 = b - x$ and $\delta = \min\{\delta_1, \delta_2\}$. Then $(x - \delta, x + \delta) \subseteq (a, b) \subset f^{-1}(U)$. So we have found a $\delta > 0$ such that $f((x - \delta, x + \delta)) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$.