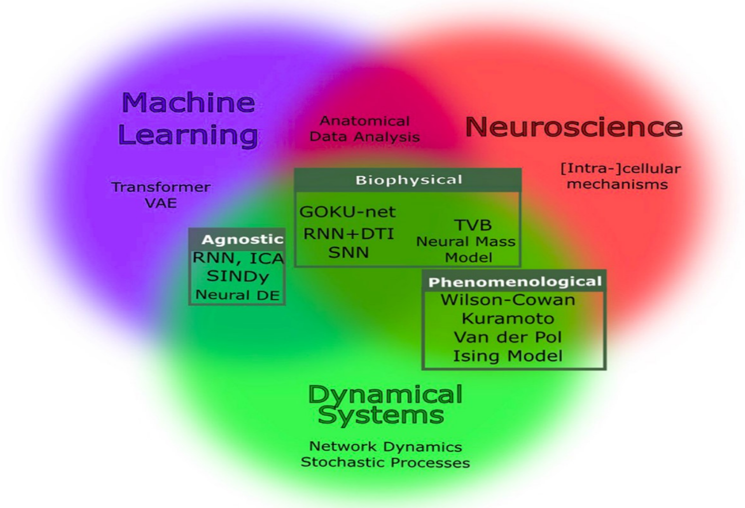
Lecture 6: Elements of Dynamical Systems Theory

Objective

- Present the basic notions of dynamical system theory, which are essential in analyzing the basic dynamics of the neurons.



* Consider the dynamical equations given by:

$$\begin{cases} \tau_v \dfrac{dv}{dt} = F_1[v(t), w(t)] \\ \tau_w \dfrac{dw}{dt} = F_2[v(t), w(t)] \end{cases} \quad --- \quad (1)$$

where $(v,w) \in \mathbb{R}^2$, $F_1, F_2 \in \mathcal{C}^k(\mathbb{R})$, $k>2$ (i.e, F_1 & F_2 are at least 2 times continuously differentiable functions), $\tau_v << \tau_w$.

* We define the following invariant sets:

1. The set of fixed points of equation (1) defined by:

$$(v_e, w_e) := \{(v,w) \in \mathbb{R}^2 \mid F_1(v,w) = F_2(v,w) = 0\}. \quad --- \quad (2)$$

2. The v-nullcline and the w-nullcline of Equation (1) defined by:

$$\text{v-nullcline} := \{(v,w) \in \mathbb{R}^2 \mid F_1(v,w) = 0\} \quad --- \quad (3)$$
$$\text{w-nullcline} := \{(v,w) \in \mathbb{R}^2 \mid F_2(v,w) = 0\}$$

* <u>Example</u>: Consider a specific example of Equation (1) where the vector fields are given by:

$$\begin{cases} \tau_v \dfrac{dv}{dt} = av - w + I_0 \\ \tau_w \dfrac{dw}{dt} = cv - w \end{cases}$$

↳ The set of fixed points of equation (1) are given by:

$$(v_e, w_e) := \{(v,w) \in \mathbb{R}^2 \mid av - w + I_0 = 0, \; cv - w = 0\}.$$

$$\Rightarrow \begin{cases} av_e - w_e + I_0 = 0 \\ cv_e - w_e = 0 \end{cases} \Rightarrow \begin{cases} (a-c)V_e + I_0 = 0 \\ w_e = cV_e \end{cases} \Rightarrow \begin{cases} V_e = \dfrac{I_0}{c-a} \\ w_e = c\left(\dfrac{I_0}{c-a}\right) \end{cases}$$

So

$$(V_e, w_e) := \{(V,w) \in \mathbb{R}^2 \mid aV - w + I_0 = cV - w = 0\} = \left(\frac{I_0}{c-a}, c\left(\frac{I_0}{c-a}\right)\right)$$

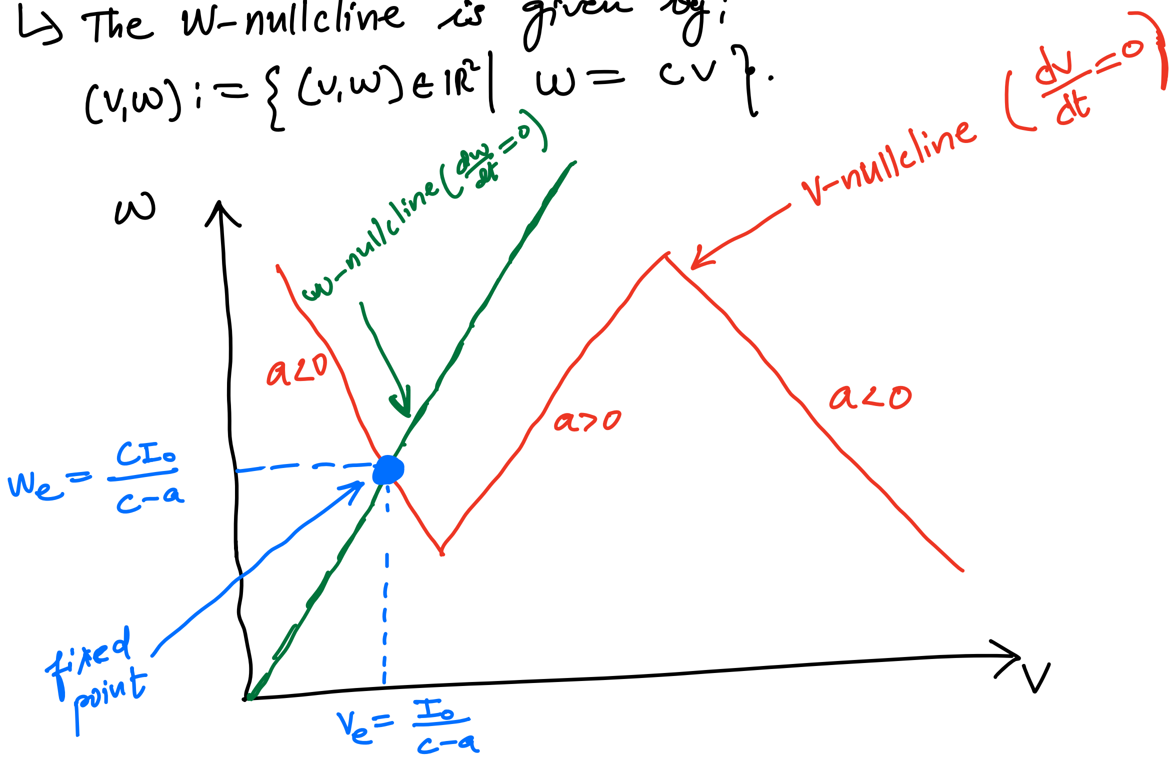
We note that the fixed points of the differentialle equation representing a neuron correspond to the rest state of the neuron, i.e., a state in which the neuron has no activity (no spiking). The rest state of the neuron is also refered to as the quiescent state.

↳ The V-nullcline is given by:

$$(V, w) := \{(V,w) \in \mathbb{R}^2 \mid w = aV + I_0\}.$$

↳ The W-nullcline is given by:

$$(V,w) := \{(V,w) \in \mathbb{R}^2 \mid w = cV\}.$$



Note that the figure of the phase space above is obtained under the assumptions that $c>0$, $I_0>0$, and $c-a>0$.

For $c<0$, $I_0>0$, $c-a>0$, we can have a different phase portrait. The phase portrait will depend on the sign & magnitudes of these parameters!

* Stability analysis of the quiescent state:

Consider again the generic dynamical system given above, i.e;

$$\begin{cases} \tau_v \dfrac{dv}{dt} = F_1(v,w) \\ \tau_w \dfrac{dw}{dt} = F_2(v,w) \end{cases} \quad --- \quad (*)$$

* Fixed point $(v_e, w_e) := \{(v,w) \in \mathbb{R}^2 \mid F_1 = F_2 = 0\}$.

$$\Rightarrow \begin{cases} 0 = F_1(v_e, w_e) + I_0 \\ 0 = F_2(v_e, w_e) \end{cases}$$

* Let us zoom into the neighborhood of the fixed point (v_e, w_e) using the transformations:

$$\begin{cases} x = v - v_e \\ y = w - w_e \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = \dfrac{d}{dt}(v - v_e) \\ \dfrac{dy}{dt} = \dfrac{d}{dt}(w - w_e) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = \dfrac{dv}{dt} - \dfrac{dv_e}{dt} = F_1(v,w) = F_1(x+v_e, y+w_e) \\ \dfrac{dy}{dt} = \dfrac{dw}{dt} - \dfrac{dw_e}{dt} = F_2(v,w) = F_2(x+v_e, y+w_e) \end{cases}$$

If we Taylor expand $F_1(x+v_e, y+w_e)$ and $F_2(x+v_e, y+w_e)$ about the fixed point (v_e, w_e), we get:

$$F_1(x+v_e, y+w_e) = F_1(v_e,w_e) + \left[(x+v_e - v_e) \left.\frac{\partial F_1}{\partial x}\right|_{v_e,w_e} + (y+w_e - w_e) \left.\frac{\partial F_1}{\partial y}\right|_{v_e,w_e} \right]$$

$$+ \frac{1}{2!}\left[(x+v_e-v_e)^2 \left.\frac{\partial^2 F_1}{\partial x^2}\right|_{v_e,w_e} + 2(x+v_e-v_e)(y+w_e-w_e)\left.\frac{\partial^2 F_1}{\partial x \partial y}\right|_{v_e,w_e} + (y+w_e-w_e)^2 \left.\frac{\partial^2 F_1}{\partial y^2}\right|_{v_e,w_e} \right] + H.O.T$$

$$\Rightarrow F_1(x+v_e, y+w_e) = \cancelto{0}{F_1(v_e,w_e)} + x \left.\frac{\partial F_1}{\partial x}\right|_{v_e,w_e} + y \left.\frac{\partial F_1}{\partial y}\right|_{v_e,w_e} + H.O.T$$

Notice that the term $F_1(v_e,w_e)$ vanishes by the definition of the fixed point i.e, $(v_e,w_e) := \{(v,w) \in \mathbb{R}^2 \mid F_1 = F_2 = 0\}$.

$$\Rightarrow F_1(x+v_e, y+w_e) \approx x \frac{\partial F_1}{\partial x}(v-v_e+v_e, w-w_e+w_e) + y \frac{\partial F_1}{\partial y}(v-v_e+v_e, w-w_e+w_e)$$

$$\Rightarrow F_1(x+v_e, y+w_e) \approx x \frac{\partial F_1}{\partial x} + y \frac{\partial F_1}{\partial y}$$

Similarly for $F_2(x+v_e, y+w_e)$ we Taylor expand and get

$$F_2(x+v_e, y+w_e) \approx x \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}$$

So our original differential equations can be written

$$\begin{cases} I_v \dfrac{dx}{dt} = x \dfrac{\partial F_1}{\partial x} + y \dfrac{\partial F_1}{\partial y} \\ I_w \dfrac{dy}{dt} = x \dfrac{\partial F_2}{\partial x} + y \dfrac{\partial F_2}{\partial y} \end{cases} \quad --- \quad **$$

* Let $\vec{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, then we can write the above equation in a compact matrix form as follows:

$$\frac{d\vec{z}}{dt} = \begin{pmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{pmatrix} \vec{z} = J \vec{z} \quad --- \quad ***$$

where the matrix J is called the Jacobian matrix of the differential equation ** given above.