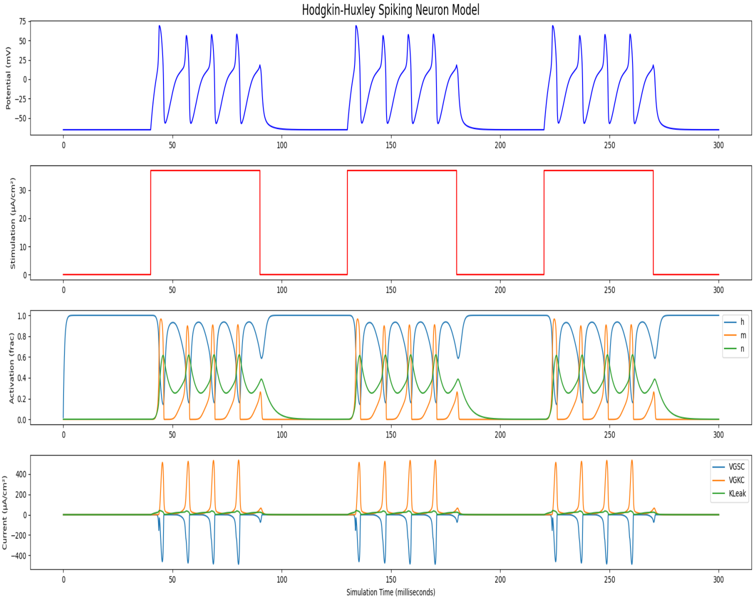
Lecture 4: Reduction of Hodgkin-Huxley Model to 2D

Objective

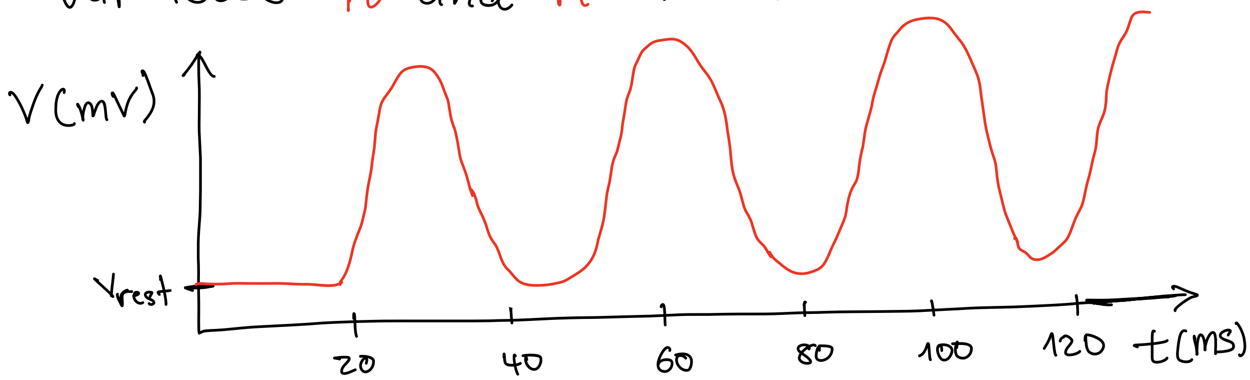
- Use timescale separation and similarity analysis of the voltage V and gating variables m, h, n of the 4D Hodgkin-Huxley neuron model to reduce it into a 2D neuron model.

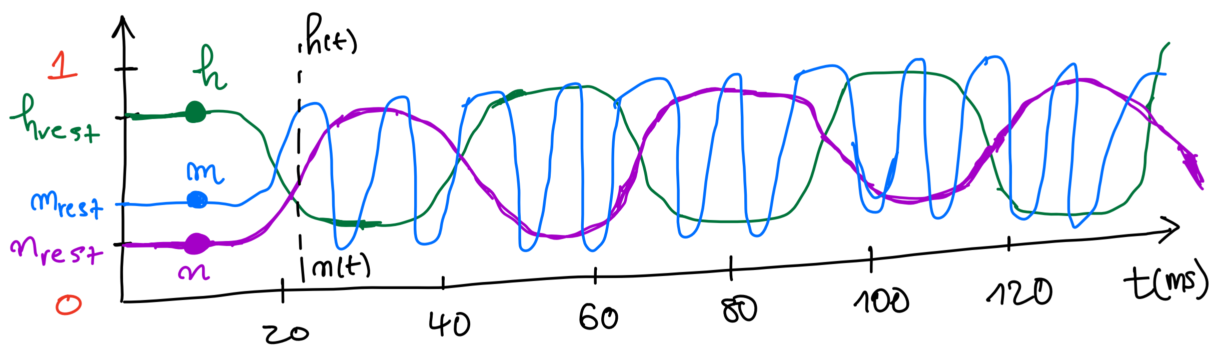
1. The Hodgkin-Huxley (HH) Neuron Model



1. Timescales separation in the HH model.

* Notice that the dynamics of the gating variables h and n are similar.





* Because the gating variable m is fast (fast compared to h and n), the time constant τ_m must be smaller than the time constants τ_h and τ_n, i.e, $\tau_m << \tau_h$ and $\tau_m << \tau_n$.

* Note that the m should also be fast compared to the external stimulus I. In other words, I should not be too fast for a dimension reduction, so that we can say m follows immediately the voltage variable V.

* Consider the equation:

$$\frac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m m = \frac{m_\infty(V) - m}{\tau_m(V)};$$

If the voltage variable V is slow (which is the case if the external stimulus I is <u>not</u> too fast), and the gating variables h and n cannot be fast because they are controlled by slower (larger) time constants $\tau_h \gg \tau_m$ and $\tau_n \gg \tau_m$, then m will approach m_∞ (i.e, $m \longrightarrow m_\infty$) rapidly and fast compared to voltage variable V.

Hence we can replace the gating variable $m(t)$ by its instantaneous value $m_\infty[V(t)]$, i.e,

$m(t) = m_\infty[V(t)]$.

* Note that this technique works for all sorts of coupled differential equations whenever you observe a difference in timescale. You can always exploit these timescale differences to eliminate the fast variable.

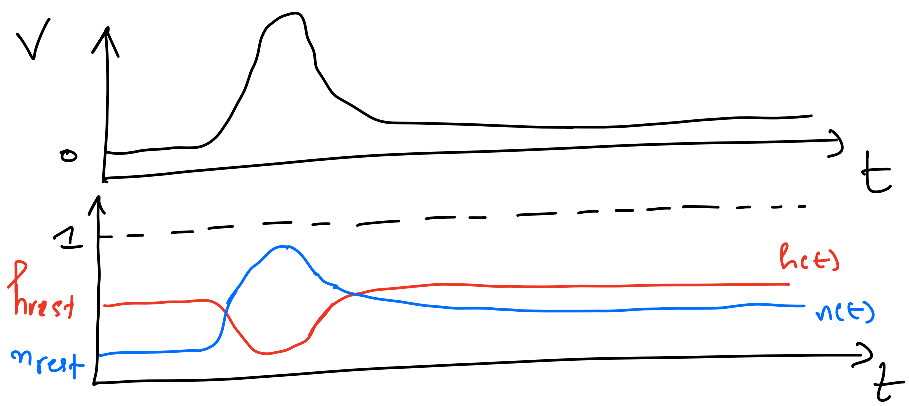
i.e, $\begin{cases} \tau_1 \frac{dx}{dt} = -x + h(y) \\ \tau_2 \frac{dy}{dt} = f(y) + g(x) \end{cases} \xrightarrow[\text{So that } x = h(y)]{\text{If } \tau_1 \ll \tau_2} \left\{ \tau_2 \frac{dy}{dt} = f(y) + g[h(y)] \right.$

* So the voltage equation of the HH neuron model becomes

$$C\frac{dV}{dt} = n^4 g_k (E_k - V) + [m_\infty(V)]^3 n\, g_{Na} (E_{Na} - V) + g_L (E_L - V).$$

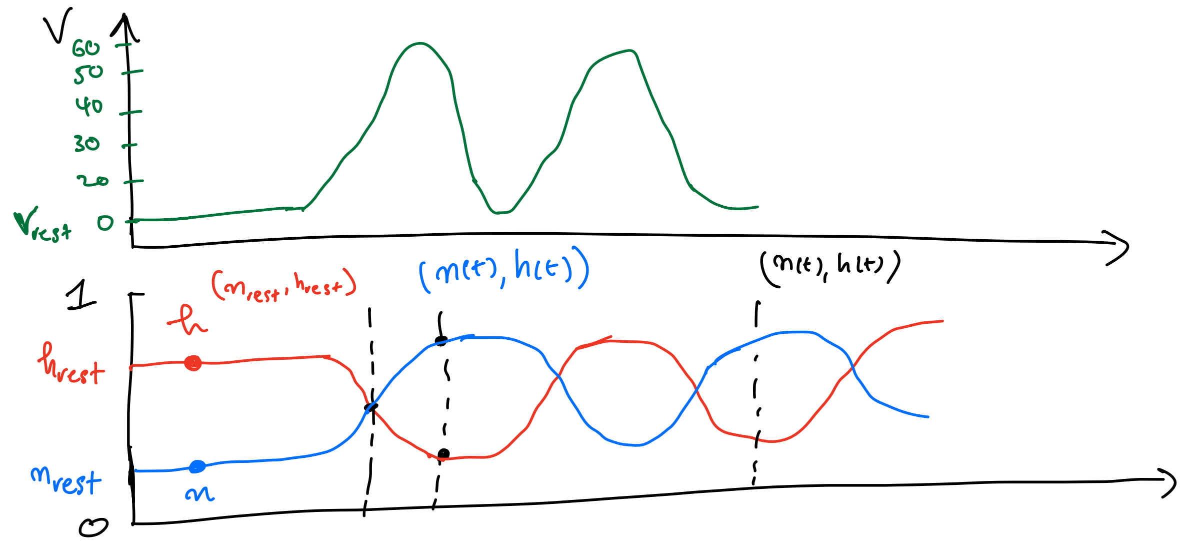
2. Exploitation of similarities/correlations

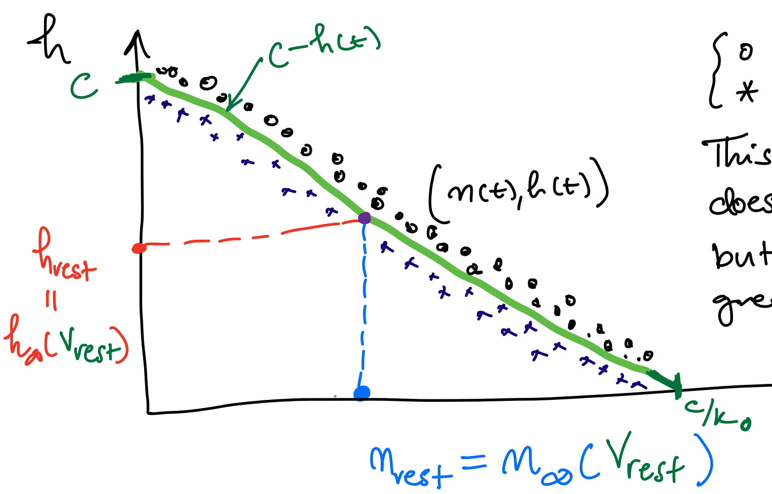
* The dynamics of the gating variables h and n are very similar in the sense that if h increases, n decreases and vice versa.



Therefore, we have a mirror symmetry which allows us to say that $1-h(t)$ is proportional to $n(t)$, i.e., $1-h(t)$ is very similar to $n(t)$ up to some constant k_0. We write: $1 - h(t) = k_0 n(t)$.

Let us make this argument more precise by looking at graph given below:



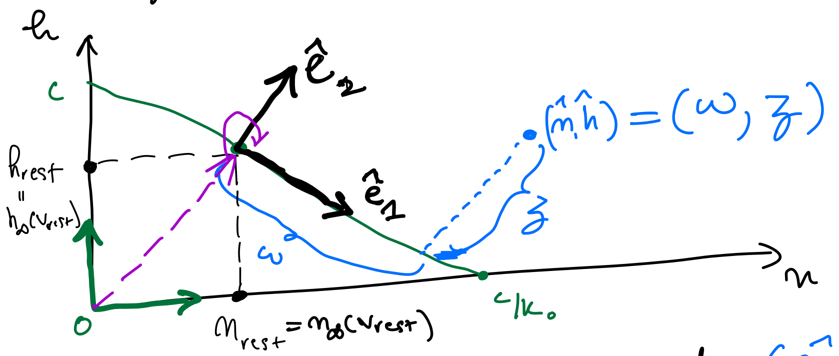


{ o For a certain stimulus I
{ * For another stimulus I'

This cloud of data points does not fill the entire n-h plan but it is elongated about the green line given by $c-h=k_0 n$

* When the voltage variable V spikes, the gating variables h and n exhibit detours which mirror images.
* Considering the green line $c-h=k_0 n$ shown in the previous figure, we notice that if $n=0$, then $h=c$ (where $c \in [0,1]$). For the sake of simplicity let us assume that $c=1$.

* If $h = 0$, then $k_0 n = 1$. Consider the diagram below:



* In this diagram, the point $(\hat{n}, \hat{h})$ can be written in terms of the original coordinate system (n, h) centered at the origin O. This same point, i.e., $(\hat{n}, \hat{h})$, can also be written in terms of a new coordinate system $(\hat{e}_1, \hat{e}_2)$ centered at the rest point $(n_\infty(V_{rest}), h_\infty(V_{rest}))$.