Lecture 3: Timescale separation analysis and dimension reduction

Objective

- Perform timescale separation analysis of ordinary differential equations (ODEs) and dimension reduction.

Timescale separation analysis of one ODE

- Consider a 1D linear ODE characterized by a timescale $\varepsilon > 0$:

$$\varepsilon \frac{dx(t)}{dt} = -x(t) + \ell(t), \tag{1}$$

where the driving term (also called forcing term) $\ell(t)$ could be (1) constant (2) piece-wise constant, i.e., it may change its value after some time intervals (3) changes continuously but slowly (4) changes continuously and rapidly.

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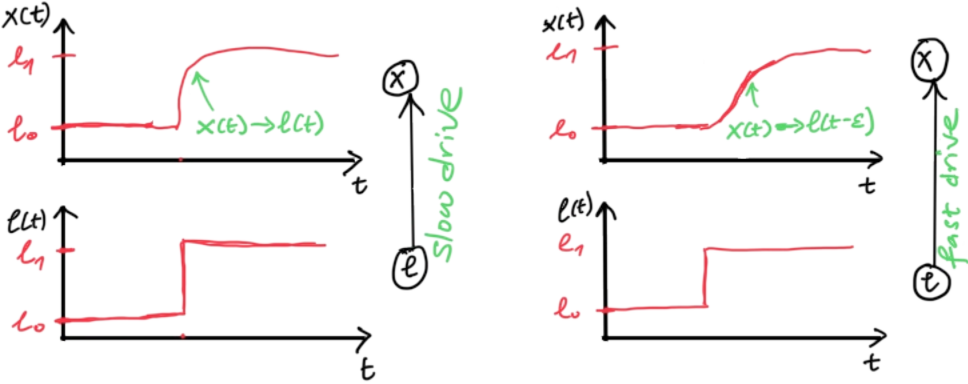
- The solution $x(t)$ of this ODE will approach the driving term $\ell(t)$ with some time constant ε which can be

  1. short, i.e., has a small value ($0 < \varepsilon \ll 1$), in which case we talk of slow drive.
  2. long, i.e., has a large value ($\varepsilon \ggg 1$), in which case we talk of fast drive.

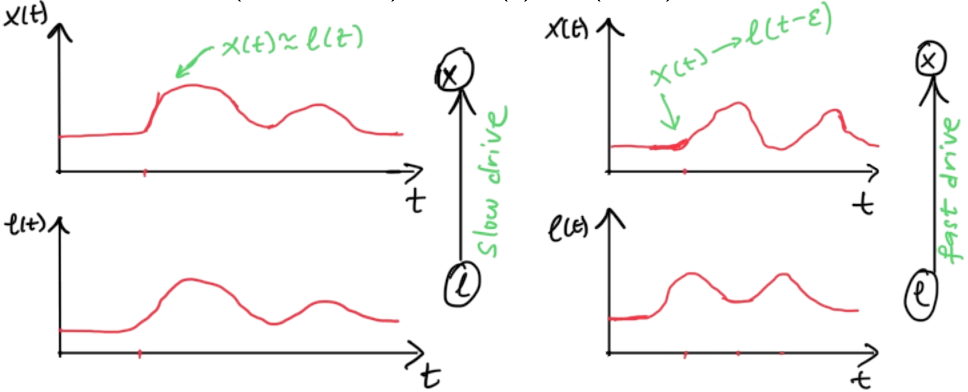
Timescale separation analysis of one ODE

- Piece-wise drive and short timescale: $x(t)$ will approach the value of $\ell(t)$ rapidly (almost instantaneously). That is, if $\varepsilon \to 0$ (i.e., $0 < \varepsilon \ll 1$), then $x(t) \to \ell(t)$.
- Piece-wise drive and long timescale: $x(t)$ will approach the value of $\ell(t)$ exponentially (i.e., with some delay). That is, if $\varepsilon \to \infty$ (i.e., $\varepsilon \ggg 1$), then $x(t) \to \ell(t - \varepsilon)$.



Timescale separation analysis of one ODE

- Continuous drive and short timescale: $x(t)$ will approach the value of $\ell(t)$ rapidly (almost instantaneously). That is, if $\varepsilon \to 0$ (i.e., $0 < \varepsilon \ll 1$), then $x(t) \to \ell(t)$.
- Continuous drive and long timescale: $x(t)$ will approach the value of $\ell(t)$ exponentially (i.e., with some delay). That is, if $\varepsilon \to \infty$ (i.e., $\varepsilon \ggg 1$), then $x(t) \to \ell(t - \varepsilon)$.

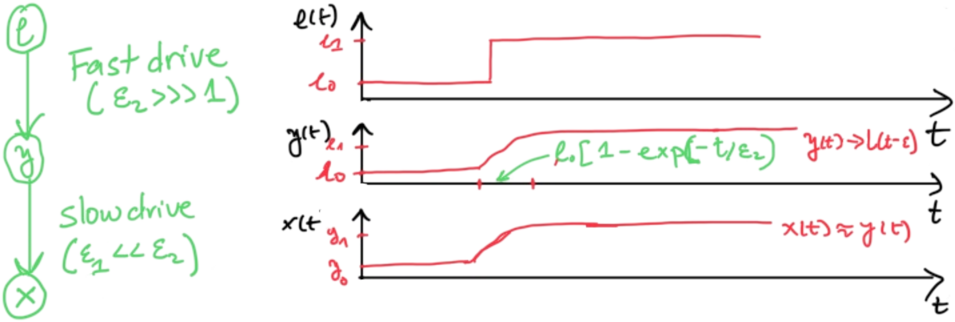


Timescale separation analysis of a pair of coupled ODEs

- Consider a pair of unidirectionally (like in chemical synapses) coupled ODEs characterized by the timescales ε_1 and ε_2.

$$\begin{cases} \varepsilon_1 \dfrac{dx}{dt} = -x(t) + y(t), \\[2ex] \varepsilon_2 \dfrac{dy}{dt} = -y(t) + \ell(t), \end{cases} \tag{2}$$

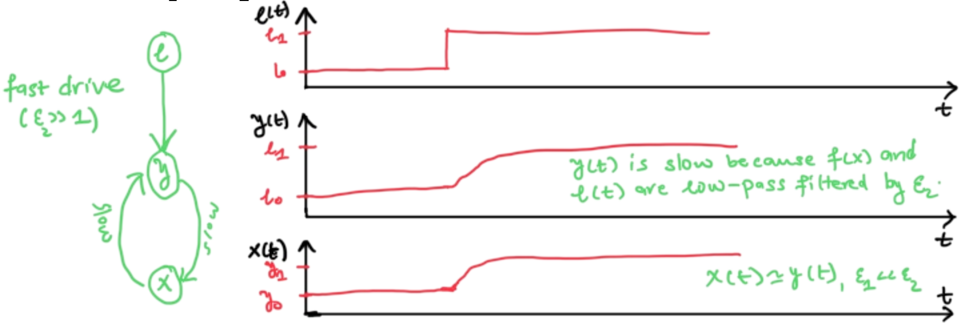
where $\varepsilon_1 \ll \varepsilon_2$.



- Let us generalize to a pair of bidirectionally (like in electrical synapses) coupled ODEs characterized by the timescales ε_1 and ε_2.

$$\begin{cases} \varepsilon_1 \dfrac{dx}{dt} = -x(t) + y(t), \\ \varepsilon_2 \dfrac{dy}{dt} = -y(t) + f(x) + \ell(t), \end{cases} \tag{3}$$

where $\varepsilon_1 \ll \varepsilon_2$.



Timescale separation and reduction of HH model to 2D

- The presence of multiple timescales in a set of coupled ODEs can be used to reduce the dimension of the system by eliminating the fast variables.

$$\begin{cases} \varepsilon_1 \dfrac{dx}{dt} = -x(t) + h(y), \\ \varepsilon_2 \dfrac{dy}{dt} = f(y) + g(x), \end{cases} \tag{4}$$

such that $\varepsilon_1 \ll \varepsilon_2 \Rightarrow x = h(y)$, then the reduced 1D ODE is given by:

$$\varepsilon_2 \frac{dy}{dt} = f(y) + g[h(y)] \tag{5}$$

- These are the types of analysis we will use to reduce the 4D HH neuron model to a 2D simpler neuron model.