Lecture 9: Hopf Bifurcation Analysis in a 2D Neuron Model

Objectives

1. Understand one of the processes via which neurons change the stability of their quiescent and spiking states.
2. Understand what a Hopf bifurcation is.
3. Show the existence and calculate the direction of a Hopf bifurcation in a 2D neuron model.

Andronov-Hopf Bifurcation in a 2D neuron model

- Consider the differential system given by Eq.(1) with a parameter $\mu \in \mathbb{R}$ such that

$$\frac{dx}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \mu) \\ f_2(x_1, x_2, \mu) \end{pmatrix} = f(x, \mu), \qquad (1)$$

- The parameter μ can modify the behavior of the solutions of the system. Thus, it should be useful to represent the evolution of the fixed point x_e according to the variation of the parameter μ_0.
- For a differential system $dx/dt = f(x, \mu_0)$, μ_0 is called a *bifurcation parameter* if a topological change of a solution occurs when the parameter μ_0 changes.
- It is called a bifurcation parameter because it can completely change the behavior of the solutions.
- There are many types of bifurcations but in this course, only the *Andronov-Hopf bifurcation* (or Hopf bifurcation for short) is discussed in detail, as it is crucial for the spiking phenomenon we are interested in.

Andronov-Hopf Bifurcation in a 2D neuron model

- For a fixed point x_e of a differential system $dx/dt = f(x, \mu)$, the pair (x_e, μ_0) is called a *Andronov-Hopf bifurcation point* when
  1. $\mathrm{Tr} D_x f(x_e, \mu_0) = 0$,
  2. $\mathrm{Det} D_x f(x_e, \mu_0) = \omega(\mu) > 0$,
  3. $\dfrac{d}{d\mu} \mathrm{Tr} D_x f(x_e, \mu)|_{\mu=\mu_0} \neq 0$.
- The first Lyapunov coefficient $L_1 \neq 0$ defined as:

$$L_1 = \frac{1}{16(\omega(\mu_0))}\left[\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\left(\frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial^2 f_1}{\partial x_2^2}\right) + \frac{\partial^2 f_2}{\partial x_1 \partial x_2}\left(\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2}\right) - \frac{\partial^2 f_1}{\partial x_1^2}\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial^2 f_1}{\partial x_2^2}\frac{\partial^2 f_2}{\partial x_2^2}\right]$$
$$+ \frac{1}{16}\left[\frac{\partial^3 f_1}{\partial x_1^3} + \frac{\partial^3 f_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 f_2}{\partial x_1^2 \partial x_2} + \frac{\partial^3 f_2}{\partial x_2^3}\right], \tag{2}$$

  determines the criticality (direction) of the Andronov-Hopf bifurcation. The Andronov-Hopf bifurcation is *super-critical* if $L_1 < 0$ and *sub-critical* if $L_1 > 0$. When conditions $1, 2$, and 3 above hold, a unique isolated periodic solution (limit cycle — self-sustained spiking activity) bifurcates when the value of the parameter μ passes μ_0.

Andronov-Hopf Bifurcation in a 2D neuron model

In the theory of bifurcations, a Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises.

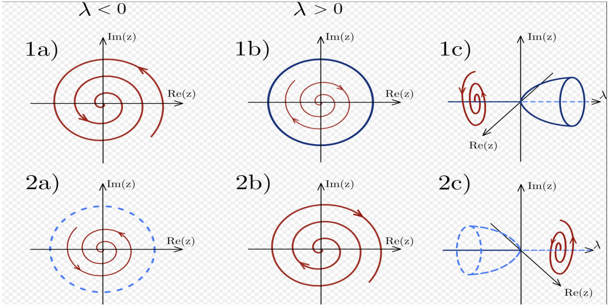

Figure: Dynamics of the Hopf bifurcation is governed by eigenvalue value $z = \lambda + i\omega$. Possible trajectories are in red, stable structures are in dark blue, and unstable structures are in dashed light blue. Super-critical Hopf bifurcation ($L_1 < 0$): 1a) stable fixed point 1b) unstable fixed point, stable limit cycle 1c) phase space dynamics. Subcritical Hopf bifurcation ($L_1 > 0$): 2a) stable fixed point, unstable limit cycle 2b) unstable fixed point 2c) phase space dynamics.

Andronov-Hopf Bifurcation in a 2D neuron model

- It is helpful to draw the bifurcation diagram (one of the variables versus the bifurcation parameter) to understand how the system is modified when the parameter μ changes.
- To illustrate this, we consider equation

$$\begin{cases} dv &= (v - \frac{v^3}{3} - w + I)dt, \\ dw &= \varepsilon(d + v)dt, \end{cases} \tag{3}$$

with $0 < \varepsilon \ll 1$. The nullclines of the system are:

$$\begin{cases} v_{null} : w = v - \frac{v^3}{3} + I, \\ w_{null} : v = -d, \end{cases} \tag{4}$$

and therefore, the fixed point is

$$(v_e, w_e) = (-d, -d + \frac{d^3}{3} + I). \tag{5}$$

- Here, d is a bifurcation parameter that can drastically change the neuron's behavior.
- To see this, we study the behavior of the linearized system in the neighborhood of the fixed point by looking at the Jacobian matrix at (v_e, w_e):

$$J(v_e, w_e) = \begin{pmatrix} 1 - v_e^2 & -1 \\ \varepsilon & 0 \end{pmatrix},$$

where $\det J = \varepsilon$ and $\operatorname{tr} J = 1 - v_e^2$. The eigenvalues of the Jacobian at (v_e, w_e) are

$$\lambda_{1,2} = \frac{1}{2}\left[(1 - d^2) \pm \sqrt{(1 - d^2)^2 - 4\varepsilon}\right]. \tag{6}$$

Following the table given in Lecture 8, depending on the value of the parameter d, we identify the following three cases:

Andronov-Hopf Bifurcation in a 2D neuron model

1. If $|d| > 1$, then (v_e, w_e) is stable node. At this point, it is important to point out that a neuron is said to be in an *excitable state* when starting with some initial condition in the basin of attraction of a *unique stable fixed point* will result in at *most one* large non-monotonic excursion (spike) into the phase space after which the phase trajectory returns and stays at this fixed point.

2. If $|d| < 1$, then (v_e, w_e) is an unstable node and there exists a stable periodic cycle around it. The neuron is said to be in oscillatory state.

3. If $d = 1$, the fixed point at $(v_e, w_e) = (-1, -2 + I)$ is a Andronov-Hopf bifurcation point, for all $I \geq 0$. Furthermore, the value and, most importantly, the sign of the first Lyapunov coefficient is given by $L_1 = -8.0$. This implies that the fixed point at $(v_e, w_e) = (-1, -2 + I)$ undergoes a super-critical Hopf bifurcation; i.e., the stable fixed becomes unstable by creating a stable limit cycle (spiking) surrounding it.

Andronov-Hopf Bifurcation in a 2D neuron model

- The bifurcation diagram for Eq.(3) above is shown in the Figure below, where we can see the range of values of the bifurcation parameter d for which we have either a no spiking (no limit cycle) or spiking (limit cycle).

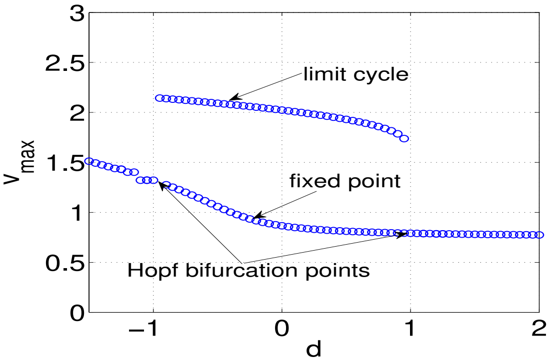

Figure: Bifurcation diagram computed with the maximum values (v_{max}) of the membrane potential variable. The blue circles represent a stable fixed point for $|d| > 1$. When $|d| = 1$ a Hopf bifurcation occurs. When $|d| < 1$ the system has an unstable fixed point and a stable limit cycle. $\varepsilon = 0.05$, $I = 0.001$.

Andronov-Hopf Bifurcation in a 2D neuron model

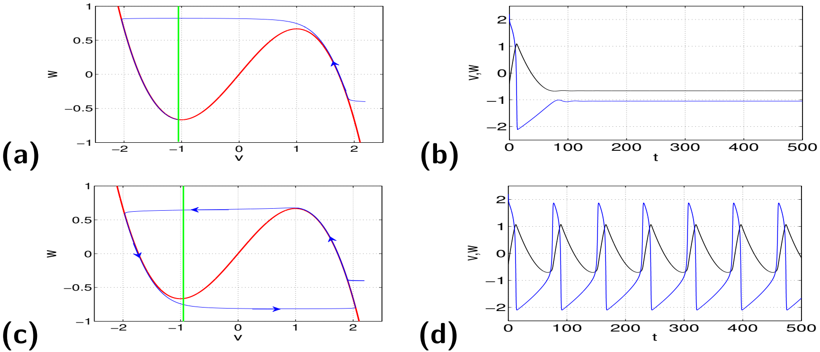

Figure: In **(a)** and **(c)** the red curve represents the cubic v-nullcline, the green vertical line is the w-nullcline which intersects the red curve at the fixed point at $(v_e, w_e) = (-d, -d + \frac{d^3}{3} + I)$. The blue curve with arrow shows the behavior of a trajectory with initial conditions at $(v_i, w_i) = (2.25, -0.35)$ for $d = 1.05$ in **(a)**, and $d = 0.95$ in **(c)**. **(b)** and **(d)** show the time series of v (in blue) and w (in black) of the phase portraits **(a)** and **(c)** respectively. In **(b)** the model is in the excitable state with no possibility of a limit cycle solution, while in **(d)**, there is a limit cycle (Spiking). $\varepsilon = 0.05$, $I = 0.001$.

Remark: The stimulus current I in Eq. (3) is not a bifurcation parameter. Fixing d and varying $I \geq 0$ only moves the red curve representing the v-nullcline upward if I is increased or downward if I is decreased. Since the w-nullcline is vertical, changing I does not change the position of the v-coordinate of the fixed point at $(v_e, w_e) = (-d, -d + \frac{d^3}{3} + I)$ and therefore, the fixed point remains stable if $|d| > 1$ or unstable if $|d| < 1$. This can also be easily seen by looking at Eq.(6) above, which determines the stability of the unique fixed point (v_e, w_e), which is independent of I.