Lecture 7: Stability Analysis of the Rest States of a Neuron (Continuation of Lecture 6)

Objective

- Learn how to determine the nature (stable or unstable) of a neuron model's rest state (fixed point) and their bifurcations.

* A search for solution yields

$$\vec{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \end{pmatrix} = k_0 e^{\Lambda t},$$ where $k_0 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$

are constants and $\Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ are the eigenvalue of J.

* To calculate Λ, we solve the eigenvalue problem given by:

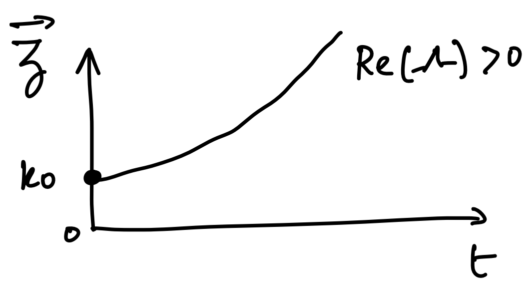
$$\det\left[J\vec{z} - \Lambda I\right] = \left|J\vec{z} - \Lambda I\right| = 0,$$ where I is a 2x2 identity matrix.

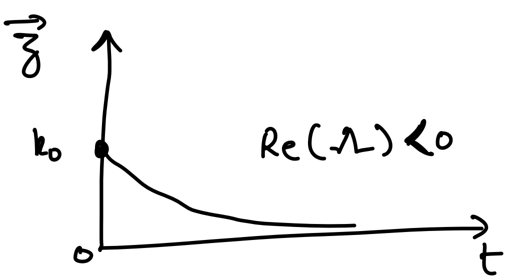
* The two solutions of $|J\vec{z} - \Lambda I| = 0$ are given λ_1 and λ_2,

where
$$\begin{cases} \lambda_1 + \lambda_2 = \partial F_1/\partial x + \partial F_2/\partial y & \text{--- (a)} \\ \lambda_1 \cdot \lambda = \partial F_1/\partial x \, \partial F_2/\partial y - \partial F_1/\partial y \, \partial F_2/\partial x & \text{--- (b)} \end{cases}$$

Equation (a) above is called the Trace of J, denoted by TrJ.
Equation (b) above is called the determinant of J, (det J).

* The solution $\vec{z}(t) = k_0 e^{\Lambda t}$ can either diverge or convergence towards the fixed point as $t \longrightarrow \infty$.





Hence the stability of the fixed point $(\vec{z}_e) = \begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ requires that $\mathrm{Re}\{\Lambda\} = \begin{pmatrix} \mathrm{Re}\{\lambda_1\} < 0 \\ \mathrm{Re}\{\lambda_2\} < 0 \end{pmatrix}$.

↳ If $Re\{\lambda_1\}>0$ and $Re\{\lambda_2\}>0$, then the fixed point $\vec{z}_e$ is unstable.

↳ If $Re\{\lambda_1\}<0$ and $Re\{\lambda_2\}>0$, then the fixed point $\vec{z}_e$ is a saddle point, i.e., stable in the x-direction and unstable in the y-direction.

↳ If $Re\{\lambda_1\}>0$ and $Re\{\lambda_2\}<0$, then the fixed point $\vec{z}_e$ is a saddle point which is stable in y-direction and unstable in the x-direction.

↳ If $Re\{\lambda_1\}=Re\{\lambda_2\}=0$, then the fixed point is "marginally stable" or simply undetermine, in which higher order terms need to be considered in the Taylor expansion.

* Note that the fixed point (V_e, W_e) of the original differential equations given by:

$$\begin{cases} \tau_V \dfrac{dV}{dt} = F_1(V,W) \\ \tau_W \dfrac{dW}{dt} = F_2(V,W) \end{cases}$$

has been translated to the origin. That is, we have used the transformations $V_e = V - x$ and $W_e = W - y$ to shift the fixed point (V_e, W_e) to $(0,0)$. In other words, We have translated the differential equations:

$$\begin{cases} \tau_v \dfrac{dv}{dt} = F_1(v,w) \\ \tau_w \dfrac{dw}{dt} = F_2(v,w) \end{cases} \xrightarrow{\text{to}} \begin{cases} \tau_v \dfrac{dx}{dt} = x\dfrac{\partial F_1}{\partial x} + y\dfrac{\partial F_1}{\partial y} \\ \tau_w \dfrac{dy}{dt} = x\dfrac{\partial F_2}{\partial x} + y\dfrac{\partial F_2}{\partial y} \end{cases}$$

See that the fixed point is at (v_e, w_e) which could be at any values.

See that the fixed point is at $(x_e, y_e) = (0,0)$.

* Note that for a 2D dynamical system with fixed point (v_e, w_e) and Jacobian matrix J, we have that $\begin{cases} Tr\, J < 0 \\ \det J > 0 \end{cases} \Longleftrightarrow (v_e, w_e)$ is stable.

* Example: Investigate the stability of one of the fixed points of the so-called FitzHugh-Nagumo (FHN) neuron model given by the following equations.

$$\begin{cases} \dfrac{dv}{dt} = v(a-v)(v-1) - w = F(v,w) \\ \dfrac{dw}{dt} = \varepsilon(bv - cw) = G(v,w) \end{cases}$$

where $(v,w) \in \mathbb{R}^2$ represent the membrane voltage and recovery current respectively. $0 < a < 1$, $b > 0$, $c > 0$, and $0 < \varepsilon << 1$ are all constant parameters.

Solution:

At a fixed point (v_e, w_e) [i.e; the rest state of the FHN neuron model), the variables $v(t)$ and $w(t)$ reach a stationary state while the set of fixed points is defined by the intersection of nullclines as:

$$(V_e, w_e) := \left\{ (v,w) \in \mathbb{R}^2 \;\middle|\; F(v,w) = G(v,w) = 0 \right\}. \quad --- \text{①}$$

From equation ①, we obtain the fixed point equations as:

$$\begin{cases} \frac{b}{c} V_e = -V_e^3 + (a+1)V_e^2 - aV_e \\ w_e = \frac{b}{c} V_e \end{cases} \quad --- \text{②}$$

which has solutions for the v-variable as:

$$\begin{cases} V_{e_1} = 0 \\ V_{e_2} = \frac{a+1}{2} - \sqrt{\frac{(a-1)^2}{4} - \frac{b}{c}} \\ V_{e_3} = \frac{a+1}{2} + \sqrt{\frac{(a-1)^2}{4} - \frac{b}{c}} \end{cases} \quad --- \text{③}$$

where V_{e_2} and V_{e_3} exist (i.e, $V_{e_2}, V_{e_3} \in \mathbb{R}$) only if we have:

$$\frac{(a-1)^2}{4} \geq \frac{b}{c} \quad --- \text{④}$$

Since we assumed that $b>0$, $c>0$, and $0<a<1$, we have the following ordering:

$$V_{e_1} < a < V_{e_{2,3}} < 1 \quad --- \text{⑤}$$

V_{e_2} and V_{e_3} coincide (i.e, $V_{e_2} = V_{e_3}$) when we have:

$$\frac{(a-1)^2}{4} = \frac{b}{c} \quad --- \text{⑥}$$