Lecture 10: Slow-fast analysis, singular Hopf bifurcation, phase plane analysis, and excitability in a slow-fast neuron model

Objectives

1. Using multiple timescales (slow-fast) analysis to understand the excitability
2. Understand and calculate the singular Hopf bifurcation
3. Use phase plane analysis to understand excitability and oscillatory regimes of neurons

The slow-fast and phase plane analyses of a neuron model

- Consider the general equation of the FitzHugh-Nagumo (FHN) neuron model given two coupled ODEs on either the slow timescale τ or the fast timescale t

$$\begin{cases} \varepsilon \frac{dx_\tau}{d\tau} &= F(x_\tau, y_\tau), \\ \frac{dy_\tau}{d\tau} &= G(x_\tau, y_\tau), \end{cases} \tag{1}$$

$$\begin{cases} \frac{dx_t}{dt} &= F(x_t, y_t), \\ \frac{dy_t}{dt} &= \varepsilon G(x_t, y_t), \end{cases} \tag{2}$$

where the functions F and G are two polynomials of the form:

$$\begin{cases} F(x, y) &= -ax + (a+1)x^2 + ex^3 + fy + I \\ G(x, y) &= d + bx - cy, \end{cases} \tag{3}$$

and $0 < \varepsilon := \frac{\tau}{t} \ll 1$ is the timescale separation parameter (called in slow-fast analysis the the singular parameter) between the fast voltage variable $x \in \mathbb{R}$ and the slow recovery variable $y \in \mathbb{R}$ that restores the resting state of the neuron.

- We note that Eq. (2) preserves the sense of the dynamics of trajectories of Eq. (1). The only difference is the speed of the trajectories in phase space.

- We also note that in the literature, there are three different versions of the FHN neuron model depending on the values of the parameters a, b, c, d, e and f.

The slow-fast and phase plane analyses of a neuron model

1. The simplest version is obtained with $a = -1$, $b = 1$, $c = 0$, $e = -1/3$, $f = -1$

$$\begin{cases} dx &= (x_t - x_t^3/3 - y + I)dt, \\ dy_t &= \varepsilon(x_t + d)dt, \end{cases} \tag{4}$$

2. The second version is obtained with $a = -1$, $b = 1$, $c = 0$, $e = -1/3$, $f = -1$

$$\begin{cases} dx_t &= (x_t - x_t^3/3 - y_t + I)dt, \\ dy_t &= \varepsilon(x_t + d - cy_t)dt, \end{cases} \tag{5}$$

3. The third version is obtained with $d = 0$, $e = -1$, $f = -1$

$$\begin{cases} dx_t &= [x_t(a - x_t)(x_t - 1) - y_t + I]dt, \\ dy_t &= \varepsilon(bx_t - cy_t)dt, \end{cases} \tag{6}$$

- For this lecture, we shall use the third version because it allows for richer dynamical behaviors than the two others.
- Exercise 3A
  1. Use the definition of the singular parameter (i.e., $0 < \varepsilon := \frac{\tau}{t} \ll 1$) to show how we can convert Eq.(6) from its fast timescale to it corresponding slow timescale version (i.e., where the variable are now x_τ, y_τ, and the time is τ).
  2. By setting $\varepsilon \to 0$ (also known as taking the singular limit) reduce Eq.(6) into a 1-dimensional ODE governing the evolution of the fast variable x on the slow timescale τ.
- Let us start by giving some important general definitions:

General slow-fast dynamical system: singular limits

Definition

$$\begin{cases} dv_t &= f(v_t, w_t)dt \\ dw_t &= \varepsilon g(v_t, w_t)dt \end{cases} \quad \xleftrightarrow{\varepsilon t = \tau} \quad \begin{cases} \varepsilon dv_\tau &= f(v_\tau, w_\tau)d\tau \\ dw_\tau &= g(v_\tau, w_\tau)d\tau \end{cases}$$

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Layer problem

Reduced problem

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Layer problem — Reduced problem

- $\mathcal{M}_0 := \{f = 0\}$ = critical manifold = fixed points of layer problem.

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- v slaved to w through constraint $f = 0$.

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- $\mathcal{M}_0$ is normally hyperbolic if the scalar $(d_v f)(p) \neq 0$, $\forall p \in \mathcal{M}_0$.
- $\mathcal{M}_0$ is attracting if $(d_v f)(p) < 0$, repelling if $(d_v f)(p) > 0$.

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- $\mathcal{M}_0$ is of saddle-type if ...$\leq re(\lambda_{j_k}) < 0 < re(\lambda_{j_{k+1}}) \leq$...

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- $\mathcal{M}_0$ is attracting if $(d_v f)(p) < 0$, repelling if $(d_v f)(p) > 0$.
- $\mathcal{M}_0$ is of saddle-type if $\ldots \leq re(\lambda_{j_k}) < 0 < re(\lambda_{j_{k+1}}) \leq \ldots$
- a fold point s satisfies $(d_v f)(s) = 0$, $\mathcal{M}_0$ looses hyperbolicity at s.

The slow-fast and phase plane analyses of a neuron model

- Now we consider the FHN model given by Eq.(6), where for simplicity, we assume that there is no input current, i.e., I=0:

$$\begin{cases} \varepsilon \frac{dx_\tau}{d\tau} &= F(x_\tau, y_\tau) = x_\tau(a - x_\tau)(1 - x_\tau) - y_\tau, \\ \frac{dy_\tau}{d\tau} &= G(x_\tau, y_\tau, c) = bx_\tau - cy_\tau, \end{cases} \tag{7}$$

$$\begin{cases} \frac{dx_t}{dt} &= F(x_t, y_t) = x_t(a - x_t)(1 - x_t) - y_t, \\ \frac{dy_t}{dt} &= \varepsilon G(x_t, y_t) = \varepsilon(bx_t - cy_t), \end{cases} \tag{8}$$

  where $b > 0$ and $c > 0$ is a co-dimension one Hopf bifurcation, $0 < a < 1$, and $0 < \varepsilon := \frac{\tau}{t} \ll 1$.
- We shall focus on the slow-fast analysis of Eqs.(7) and (8), which allows us to analytically and geometrically understand the mechanism behind the dynamical behaviors of the FHN neuron model during excitability and spiking.
- The first major idea to analyze Eqs.(7) and (8) is to take the singular limit (i.e., $\varepsilon \to 0$)