Lecture 16: Synchronization in neural networks
(Continuation of Lecture 15)

A neural complete synchronization problem

- Furthermore, it appears that the stability of the synchronized states in Eq. (8) (see the slides of Lecture 15) depends on the coupling strength G^e. To determine the stability of the synchronized state of the two coupled neurons with respect to G^e (or any other parameter of the neuron model), we can use the Krasovskii-Lyapunov stability theory.

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Definition

The Lyapunov function of a given dynamical system $de_i/dt = f_i(e_i)$ with a fixed point at e_i^* (i=1,2,..,n) is a real-valued function $\mathbb{V}$ which is defined over a region Ω of the phase space $\mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$) that contains the fixed point e_i^* and satisfies the following requirements:

1. V is continuously differentiable and positive definite (i.e., $\mathbb{V} \in C^1$, $\mathbb{V}(e_i) > 0 \; \forall e_i \neq e_i^*$, $\mathbb{V}(e_i^*) = 0$)
2. $\mathbb{V}(e_i)$ has a unique minimum with respect to the neighbourhood of Ω (i.e., $\exists! e_i^* \in \Omega$ such that $\mathbb{V}(e_i^*) \leq \mathbb{V}(e_i) \; \forall e_i \in \Omega$)
3. Along any trajectory of error dynamical $de_i/dt = f_i(e_i)$, contained in Ω, the value of $\mathbb{V}(e_i)$ never increases (i.e., $d\mathbb{V}(e_i)/dt = \nabla\mathbb{V}(e_i) \cdot f_i(e_i) < 0, \forall e_i \in \Omega \backslash \{e_i^*\}$).

If $\dfrac{d\mathbb{V}}{dt} \leq 0, \forall e_i \in \Omega \subset \mathbb{R}^n \backslash \{e_i^*\}$, then e_i^* is stable.
If $\dfrac{d\mathbb{V}}{dt} < 0, \forall e_i \in \Omega \subset \mathbb{R}^n \backslash \{e_i^*\}$, then e_i^* is locally asymptotically stable.
If $\dfrac{d\mathbb{V}}{dt} < 0, \forall e_i \in \mathbb{R}^n \backslash \{e_i^*\}$, then e_i^* is globally asymptotically stable.

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- Note that the Lyapunov function of a dynamical system is not unique, however, the challenge of the Lyapunov approach to stability is that constructing a proper Lyapunov function is generally not easy. But once it is successfully constructed, stability analysis becomes relatively easier.
- Following the Krasovskii-Lyapunov theory which has been widely used in identifying the stability of synchronized states we define a continuous, positive-definite Lyapunov function with a continuous first partial derivative of the form:

$$\mathbb{V}(e_v, e_w) = e_v(t)^2 + e_w(t)^2, \tag{9}$$

which has a unique minimum at the fixed point $(e_v^*, e_w^*) = (0, 0)$. The derivative of the function V along a trajectory of the error dynamical system in Eq. (8) is given by:

$$\frac{d\mathbb{V}}{dt} = 2e_v\frac{de_v}{dt} + 2e_w\frac{de_w}{dt}. \tag{10}$$

Substituting Eq. (8) (see slides of Lecture 15) in Eq. (10) yields:

$$\frac{d\mathbb{V}}{dt} = -[6v^2 - 4(1+a)v + 2a + 4G_e]e_v^2 - 2e_ve_w + 2\varepsilon(be_ve_w - ce_w^2). \tag{11}$$

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- Thus, the sufficient condition for a
  1. stable,
  2. locally asymptotically stable,
  3. globally asymptotically stable,

synchronized states, with respect to the coupling strength G^e provided that it is fulfilled at all points of the attractor of Eq. (4) (see slides in Lecture 15), is that the time derivative of the Lyapunov function $d\mathbb{V}/dt$ satisfies

1. $$\frac{d\mathbb{V}}{dt} \leq 0, \forall (e_v, e_w) \in \Omega \subset \mathbb{R}^2 \backslash \{(0,0)\},$$
2. $$\frac{d\mathbb{V}}{dt} < 0, \forall (e_v, e_w) \in \Omega \subset \mathbb{R}^2 \backslash \{(0,0)\},$$
3. $$\frac{d\mathbb{V}}{dt} < 0, \forall (e_v, e_w) \in \mathbb{R}^2 \backslash \{(0,0)\}$$

respectively, where Ω is some neighborhood containing the fixed point $(e_v^*, e_w^*) = (0,0)$ of the synchronized error dynamical system given in Eq. (8). Otherwise, i.e., if $d\mathbb{V}/dt > 0$, then the synchronized state is unstable.

- Thus, to evaluate the expression of $d\mathbb{V}/dt$ and determine its sign, it suffices to solve two equations simultaneously, i.e., Eq. (4) for v, and Eq. (8) for e_v and e_w, and use the current values of v, e_v, and e_w calculated at each time step t to evaluate the the expression $d\mathbb{V}/dt$ given in Eq. (11) at time t.

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- This can be done for a range of values of G_e and determine the stability property of the synchronized state for each value of G_e. Note that Eq. (4), Eq. (8), and Eq. (11) have to be solved and evaluated simultaneously as time changes.
- Another common and important type of synchronization is known as phase synchronization. This involves sub-system properties called phases and is characterized by the 2π phase locking of two or more oscillators, even if their amplitudes are uncorrelated.
- It's important to note the difference between complete synchronization and phase synchronization. In complete synchronization, all values of the spike trains are used to compute synchronization. In contrast, phase synchronization only considers the timing of the spikes in the spike trains.
- It has been shown that synchronization of oscillatory phases between different brain regions supports both working memory and long-term memory and facilitates neural communication by promoting neural plasticity.

A neural complete synchronization problem

- How can one quantify the complete synchronization of N (and not only 2) neurons?

Answer: **Complete synchronization (CS):** The ability of the N neurons in the network to completely synchronize the actual value of their membrane potential variables $v_k(t)$ can be quantified by the **statistical index of complete synchronization** Θ, given by the standard deviation of $v_k(t)$ for these N neurons as:

$$\Theta = \langle \rho(t) \rangle_t \quad \text{with} \quad \rho(t) = \sqrt{\frac{\frac{1}{N}\sum_{k=1}^{N}\big(v_k(t)\big)^2 - \left(\frac{1}{N}\sum_{k=1}^{N} v_k(t)\right)^2}{N-1}}, \quad (12)$$

and where the angle brackets $\langle \cdot \rangle_t$ represents the average over time t, i.e., the total number of simulation time steps. $\rho(t)$ measures the degree of CS at a given time t. The value of Θ is an excellent indicator of the degree of CS and reveals different synchronization levels and related transitions. Smaller values of Θ indicate higher degrees of CS, and $\Theta = 0$ indicates the highest degree of CS.

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- How can one quantify the phase synchronization of N neurons?

Answer: **Phase synchronization (PS):** The time-averaged **Kuramoto order parameter**, $\overline{R}$, can be used to measure the degree of PS among the N-coupled neurons. It is given by:

$$\overline{R} = \frac{1}{T}\int_0^T \left| \frac{1}{N}\sum_{k=1}^{N} \exp\big(i\phi_k(t)\big)\right| dt, \tag{13}$$

where $\phi_k(t) = 2\pi n + 2\pi \dfrac{t - t_k^{(n)}}{t_k^{(n+1)} - t_k^{(n)}}$ and $t_k^{(n)} \leq t < t_k^{(n+1)}$. In the argument of the exponential function, $i = \sqrt{-1}$ and the quantity $\phi_k(t)$ approximates the phase of the kth neuron in the network and linearly increases over 2π from one spike to the next. The norm of this complex exponential function is represented by $\big|\cdot\big|$. The time at which the kth neuron exhibits its nth spike $(n = 0, 1, 2, ...)$ is represented by $t_k^{(n)}$. We determine the spike time occurrences from the instant $v_k(t)$ crosses the corresponding threshold v_{th} value from below. The time-averaged Kuramoto order parameter $\overline{R}$ ranges from 0 corresponding to no PS to 1 corresponding to full PS (i.e., all neurons fire at precisely the same times).