Lecture 5: Reduction of Hodgkin-Huxley into 2D model Continued

Objective

- Same as in Lecture 4.

* As we have seen before, all the data points are alongated in the direction of unit vector $\hat{e}_1$. So an arbitrary point $(\hat{n}, \hat{h})$ in the plane spanned by the units vector $(\hat{e}_1, \hat{e}_2)$ will have a very small $\hat{e}_2$ component, and especially, a zero $\hat{e}_2$ component at (n_{rest}, h_{rest}).
* Hence the point $(\hat{n}, \hat{h}) = (\omega, 0)$. So we suppress the z-coordinate because it is very small and sometimes even zero, which give the reduction of dimensionality.

* To determine the slope k_0 of the line given by $1-h(t) = k_0 n(t)$, we need at least two points on this line. One of these points is (n_{rest}, h_{rest}) which we can determine without a mistake (i.e, with high precision).

* So, at rest, we have that

$$1-h_\infty(V_{rest}) = k_0 n_\infty(V_{rest})$$

$$\Rightarrow \boxed{k_0 = \frac{1-h_\infty(V_{rest})}{n_\infty(V_{rest})}.} \quad --- \quad (*)$$

This is how we can reduce the two-dimensional description in the 2-dimensional plane to a one-dimensional coordinate that corresponds to the projection on the the green line $1-h(t) = k_0 n(t)$.

* What about the dynamics? We know that $\frac{dh}{dt} = \frac{h_\infty(V)-h}{\tau_h(V)}$ and $\frac{dn}{dt} = \frac{n_\infty(V)-n}{\tau_n(V)}$, that

is, h approaches h_∞ with some time constant τ_h, and n approaches n_∞ with some time constant τ_n. Therefore, we can reformulate the dynamics of the gating variables h and n in terms of the dynamics of a new variable w by:

(1) Translate and rotate coordinate system

(2) Suppress one coordinate

(3) Express the dynamics in new coordinate.

So we have: $1 - h(t) = k_0 n(t) = w(t)$

$$\left.\begin{aligned} \frac{dh}{dt} &= \frac{h_\infty(V) - h}{\tau_h(V)} \\ \frac{dn}{dt} &= \frac{n_\infty(V) - h}{\tau_n(V)} \end{aligned}\right\} \equiv \frac{dw}{dt} = \frac{w_\infty(V) - w}{\tau_{eff}(V)}$$

* With these analysis, we have arrived at the end of the argument.

* In the HH neuron model, we have inserted the instantaneous (momentary) value of the fast gating variable $m(t)$ in the voltage equation of the model.

$m(t) = m_\infty(V(t))$.

* From the equation $1 - h(t) = \omega(t)$, we can now replace the variable $h(t)$ in the voltage equation by $h(t) = 1 - \omega(t)$.

[Recall that this is because $h(t)$ and $n(t)$ are similar: $\underbrace{1 - h(t)}_{\omega(t)} = \underbrace{k_0 n(t)}_{\omega(t)}$.]

* Also we replace the gating variable $n(t)$ in the voltage equation by $n(t) = \dfrac{\omega(t)}{k_0}$

where the constant k_0 is given by

$$k_0 = \frac{1 - h_\infty(V_{rest})}{n_\infty(V_{rest})}. \quad (\text{see equation } ⊛ \text{ above}).$$

* So we can now write the 4D HH neuron into a 2D neuron model as;

$$C\frac{dV}{dt} = \left(\frac{\omega}{k_0}\right)^4 g_K(E_K - V) + [m_\infty(V)]^3(1-\omega)g_{Na}(E_{Na} - V) + g_L(E_L - V) + I$$

$$\frac{d\omega}{dt} = \frac{\omega_\infty(V) - \omega}{\tau_{eff}(V)}$$

where $m_\infty(V)$ is the instantaneous value of the m variable (at rest), since it is fast.

* If we multiply the first equation above by the leak resistance given by $R = \frac{1}{g_L}$

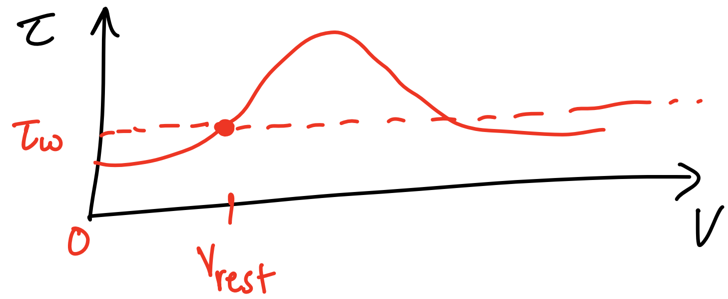
$$RC\frac{dV}{dt} = R\left(\frac{\omega}{k_0}\right)^4 g_K(E_K - V) + R[m_\infty(V)]^3(1-\omega)g_{Na}(E_{Na} - V) + (E_L - V) + RI$$

where membrane time constant is given by $\tau := RC$ that controls the dynamics of the voltage variable V.

* We do something similar to the time constant τ_{eff} (which is not trivial to calculate but possible).

We assume that the effective time constant

$\tau_{eff}(V)$ changes as:



$$\tau_w \frac{dw}{dt} = \frac{w_\infty(V) - w}{\frac{1}{\tau_w}\,\tau_{eff}(V)}$$

where τ_w is some typical value and $\frac{1}{\tau_w}\tau_{eff}(V)$ is some normalization factor. The net result is differential equation for w where τ_w could be representive value of τ (e.g. a mean value of τ between 0 and V_{rest}, etc) which controls the dynamics of w.

* So we can generically write the 2D neuron model as:

$$\begin{cases} \tau \dfrac{dV}{dt} = F(V(t), w(t)) + RI \\[2ex] \tau_w \dfrac{dw}{dt} = G(V(t), w(t)) \end{cases} \qquad (**)$$

* The 2D neuron model in equation (**) enables graphical analysis of the model and even an analytical treatment when $\tau \ll \tau_w$ ($\tau_w \ll \tau$).

* The graphical analysis enables us to discuss repetitive spiking in neurons and to distinguish between Type I and Type II neurons via their bifurcations.