Lecture 8: Stability Analysis of the Rest States of a Neuron (Continuation of Lecture 7)

Objective

- Learn how to determine the nature (stable or unstable) of a neuron model's rest state (fixed point) and their bifurcations.

For the moment, let us return to the general fixed point (v_e, w_e) and study its stability. In order to determine the stability of such a fixed point, we need to study the linearized matrix equation:

$$\begin{pmatrix} \frac{dv}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{pmatrix} \partial F/\partial v & \partial F/\partial w \\ \partial G/\partial v & \partial G/\partial w \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \quad --- \quad (7)$$

$$d\vec{z} = J(v_e, w_e)\vec{z}.$$

The stability of the fixed point (v_e, w_e) will depend on the signs of the trace $(\mathrm{Tr}\, J)$ and determinant $(\det J)$ of the Jacobian matrix $J(v_e, w_e)$. For a fixed point (v_e, w_e) to be stable, it suffices to show that $\mathrm{Tr}\, J < 0$ and $\det J > 0$. We calculate $J(v_e, w_e)$:

$$J(v_e, w_e) = \begin{pmatrix} -3v_e^2 + 2(a+1)v_e - a & -1 \\ \varepsilon b & -\varepsilon c \end{pmatrix}$$

$$\mathrm{Tr}\, J(v_e, w_e) = -3v_e^2 + 2(a+1)v_e - a - \varepsilon c$$

$$\det J(v_e, w_e) = 3c\varepsilon v_e^2 - 2\varepsilon c(a+1)v_e + \varepsilon ac + \varepsilon b.$$

Now let's determine the nature (i.e., stable or unstable) of one of the fixed points (v_{e_1}, w_{e_1}), (v_{e_2}, w_{e_2}), (v_{e_3}, w_{e_3}). I choose the simplest of the fixed points, i.e., $(v_{e_1}, w_{e_1}) = (0,0)$. So, we have:

$$\begin{cases} Tr\, J(v_{e_1}, w_{e_1}) = -(a+\varepsilon c) \\ \det J(v_{e_1}, w_{e_1}) = \varepsilon(ac+b) \end{cases}$$

Since $0<a<1$, $0<\varepsilon<<1$, $c>0$, and $b>0$, we have:

$$\begin{cases} Tr\, J(v_{e_1}, w_{e_1}) < 0 \\ \det J(v_{e_1}, w_{e_1}) > 0 \end{cases} \Longrightarrow (v_{e_1}, w_{e_1}) = (0,0) \text{ is a fixed point.}$$

* Digression:

Let J be an arbitrary 2×2 matrix given by $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then, we have that $\begin{cases} Tr\, J := a+d \\ Det\, J := ad-bc \end{cases}$

Also, the characteristic equation associated to the matrix J is given by:

$Det[J - \lambda \mathbb{I}] = |J - \lambda \mathbb{I}| = 0$,

where $\mathbb{I}$ is the 2×2 identity matrix and λ the eigenvalue of the matrix J.

We have that: $\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$

$$\Rightarrow \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow (a-\lambda)(d-\lambda) - cb = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - cb = 0.$$

Using the quadratic formula, we get the roots of above quadratic polynomial equation as:

$$\begin{cases} \lambda_1 = \dfrac{a+d}{2} - \dfrac{1}{2}\sqrt{(a+d)^2 - 4(ad-cb)} \\ \lambda_2 = \dfrac{a+d}{2} + \dfrac{1}{2}\sqrt{(a+d)^2 - 4(ad-cd)} \end{cases}$$

Which can then be written in terms of the trace and determinant of the matrix J as:

$$\begin{cases} \lambda_1 = \dfrac{Tr\,J}{2} - \dfrac{1}{2}\sqrt{(Tr\,J)^2 - 4\,Det\,J} \\ \lambda_2 = \dfrac{Tr\,J}{2} + \dfrac{1}{2}\sqrt{(Tr\,J)^2 - 4\,Det\,J} \end{cases}$$

We notice that the eigenvalues λ_1 and λ_2 of the matrix J are in fact solutions of the secular equation $\lambda^2 - Tr\,J + Det\,J = 0$, where $Tr\,J$ and $Det\,J$ are

the trace and determinant of the matrix J, respectively.
There exist three topological equivalence classes of hyperbolic fixed points in the plane (i.e, in $\mathbb{R}^2$), namely:
(i) Stable foci or stable nodes (also called attractors and are attractive in all directions), (ii) unstable foci or unstable nodes (also called repellers and are repulsive in all direction); and (iii) Saddle (attractive in one direction and repulsive in the other).
Their classification according to the type of eigenvalue is given by the table below.

Eigenvalues	Name and stability type
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_{1,2} < 0$	Stable node (attractive)
$\lambda_{1,2} \in \mathbb{C}$, $Re(\lambda_{1,2}) < 0$	Stable focus (attractive)
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 > 0$ & $\lambda_2 < 0$	Saddle (repulsive in one direction and attractive in the other).
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 < 0$ & $\lambda_2 > 0$	Saddle (attractive in one dire. and repulsive in the other).
$\lambda_{1,2} \in \mathbb{C}$, $Re(\lambda_{1,2}) > 0$	unstable focus (repulsive)
$\lambda_{1,2} \in \mathbb{R}$, $\lambda_1 > 0$, $\lambda_2 > 0$	unstable node (repulsive)
$\lambda_1 = \lambda_2 = 0$	No conclusion can be made. Further analysis needed!