Lecture 15: STDP revisited and synchronization in neural networks.

Objectives

1. Further explain the STDP learning rule
2. Understand synchronization dynamics in neurons
3. Learn how to measure the degree of synchronization
4. Determine whether a synchronized state is stable or unstable

Further explanation of STDP learning rule

- The nearest-spike Pair-based STDP rule focuses on the changes in synaptic strength driven by the precise timing of the closest spikes between the pre-synaptic and post-synaptic neurons.
- **Key features of the nearest-spike pair-based STDP rule:**
  - For each spike from the pre-synaptic neuron j, the nearest preceding or following spike from the post-synaptic neuron i is considered.
  - For each spike from the post-synaptic neuron i, the nearest preceding or following spike from the pre-synaptic neuron j is considered.
  - In the context of the nearest-spike pair-based STDP rule, "preceding or following" refers to the temporal relationship between the spikes of the pre-synaptic and post-synaptic neurons. Specifically:
    - **Preceding spike:** A spike that occurs before the current spike under consideration.
    - **Following spike:** A spike that occurs after the current spike under consideration.

Further explanation of STDP learning rule

- **Now, in other words, I explain the identifying nearest spikes:** For a given spike at time t of the pre-synaptic neuron j, the nearest post-synaptic spike of neuron i is found by identifying the spike that occurs closest in time, whether it is before or after the time t. Similarly, for a spike at time t in the post-synaptic neuron i, the nearest pre-synaptic spike of neuron j is found.
  - Numerically, in your simulations the nearest-spike pair-based STDP rule is implemented as:
    - **For each spike from the pre-synaptic neuron j:** Identify the nearest spike from the post-synaptic neuron i. This nearest spike can either be the one that occurs immediately before (preceding) or immediately after (following) the pre-synaptic spike of neuron j.
    - **For each spike from the post-synaptic neuron i:** Identify the nearest spike from the pre-synaptic neuron j. This nearest spike can either be the one that occurs immediately before (preceding) or immediately after (following) the post-synaptic spike.

Further explanation of STDP learning rule

- **Calculating Timing Differences:** Once the nearest spike pairs are identified, the timing difference $\Delta t_{ij} = t_i^{(\text{post})} - t_j^{(\text{pre})}$ is calculated for each pair. The sign of this timing difference determines whether the synaptic change will be potentiation or depression.
- **Applying STDP Rule:** Only these nearest pairs are used to update the synaptic strength $\Delta G_{ij}^{C}(t)$. The magnitude of the synaptic change is then computed using the STDP rule, that is eSTDP in Eq.(6) or iSTDP in Eq.(7) given in the slides of Lecture 14.

So keep in mind that the nearest-spike pair-based STDP rule simplifies the synaptic update mechanism by considering only the closest spikes, making it computationally efficient while capturing the core dynamics of synaptic plasticity driven by precise spike timing.

Synchronization in neurons

- Synchronization is a widespread phenomenon in complex systems including the brain.
- **Definition:** Synchronization phenomena are processes wherein many dynamical systems adjust a given property (e.g., amplitude, phase, frequency, and even membrane potential in coupled neurons) of their motion due to suitable coupling configurations.
- In the brain, they can emerge from the collaboration between neurons or neural networks and significantly affect all neurons and network functioning.
- It is well-established that synchronization of neural activity within and across brain regions promotes normal physiological functioning, such as the precise temporal coordination of processes underlying cognition, working memory, and perception. However, synchronization of neural activity is also well known to be responsible for some pathological behaviors such as epilepsy, Parkinson's, and Alzheimer's. Thus, understanding synchronization is of engineering and medical importance.
- It has been shown that changes in the strength of the synaptic coupling and the connectivity of the neurons could lead to epileptic-like synchronization behaviors.

A neural synchronization problem

- Consider the following network of N FitzHugh-Nagumo (FHN) neurons coupled via gap junctions (i.e., via electrical synapses):

$$\begin{cases} \dfrac{dv_i}{dt} & = & v_i(a - v_i)(v_i - 1) - w_i + I^{ext} + \displaystyle\sum_{j=1}^{N} G_{ij}^e(v_j - v_i) \\ \dfrac{dw_i}{dt} & = & \varepsilon(bv_i - cw_i), \end{cases} \tag{1}$$

  where $v_i = v_i(t) \in \mathbb{R}$ and $w_i = w_i(t) \in \mathbb{R}$ represent the fast membrane potential and slow recovery current variables of the FHN neuron, respectively; the index $i = 1, ..., N$ stands for neuron (nodes) in the network; $G_e > 0$ is the coupling strength between pairs of neurons in the network; $b, c > 0$ $0 < \varepsilon \ll 1$ are constant parameters, excitability parameter is $0 < a < 1$, the external inputs are $I^{ext} \geq 0$, and the synaptic coupling strength between neuron i and neuron j is $G_{ij}^e \geq 0$.

- Notice that the network is a complete graph, i.e., there is a direct connection (or edge or synapse) between every pair of nodes (neurons), but there are no self-loops.

A neural complete synchronization problem

- For the sake of simplicity, let's study the phenomenon of complete synchronization within the smallest network, i.e., when the network size in Eq. (1) is given by $N = 2$. In this case, we rewrite Eq. (1) as

$$\begin{cases} \dfrac{dv_1}{dt} &= v_1(a - v_1)(v_1 - 1) - w_1 + I^{ext} + G^e_{12}(v_2 - v_1) \\ \dfrac{dw_1}{dt} &= \varepsilon(bv_1 - cw_1), \\ \dfrac{dv_1}{dt} &= v_2(a - v_2)(v_2 - 1) - w_2 + I^{ext} - G^e_{21}(v_2 - v_1) \\ \dfrac{dw_2}{dt} &= \varepsilon(bv_2 - cw_2), \end{cases} \tag{2}$$

- Notice that the coupling strength between neuron 1 and neuron 2 is symmetric, i.e., $G^e_{12} = G^e_{21}$. Hence, in the sequel, for the sake of simpler notation, we will write $G^e_{12} = G^e_{21} = G^e$
- A typical synchronization problem would be to (i) find the threshold synaptic strength G_e for which the two coupled FHN neurons may exhibit complete synchronization and (ii) determine the stability of this complete synchronization state once it is achieved.

A neural compete synchronization problem

Definition

Two systems $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ are said to have achieved complete synchronization if there exists a set of initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ such that the trajectories of the systems satisfy $\lim_{t\to\infty} \|x(t) - y(t)\| = 0$.

where $\|\cdot\|$ denotes the Euclidean norm, for initial conditions from some neighborhood of the synchronization manifold $\mathcal{S}$ such that we have:

$$\begin{cases} v_1(t) = v_2(t) = v(t), \\ w_1(t) = w_2(t) = w(t). \end{cases} \tag{3}$$

A neural compete synchronization problem

- This means that the synchronization solution in Eq. (3) is always a solution of the coupled system, and hence when the neurons are synchronized (i.e., when all the terms $G_e(v_2 - v_1)$ in Eq. (2) becomes zero) their common dynamics behave like a single neuron, i.e., like:

$$\begin{cases} \dfrac{dv}{dt} = v(a-v)(v-1) - w + I^{ext}, \\ \dfrac{dw}{dt} = \varepsilon(bv - cw). \end{cases} \tag{4}$$

- However, this synchronization solution might be unstable under some conditions. Thus, the necessity of studying the stability of the synchronized states which we will do with the use of the Lyapunov function criteria for stability.

A neural compete synchronization problem

- It should be noted that achieving complete (ideal, perfect) synchronization in real-world systems, where the trajectories of interacting systems converge exactly over time (i.e., $\lim_{t\to\infty} \|x(t) - y(t)\| = 0$), is often challenging or even impossible. This difficulty can stem from factors such as parameter mismatches, uncertainties, noise, different initial conditions, differing governing equations of the interacting systems, and numerical integration errors during computation.
- Consequently, synchronization in real-world systems is frequently pursued in a practical sense, aiming for a sufficient level of coordination tailored to specific applications to ensure the effective operation of systems, even if perfect alignment is not attainable. This concept is crucial in fields such as control theory, communications, and network synchronization, where exact alignment may be impracticable or unnecessary, yet effective coordination remains essential.
- In such practical cases, we talk of practical synchronization which is formal defined as follows:

A neural compete synchronization problem

Definition

Two systems $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^n$ are said to have achieved practical synchronization if there exists a set of initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ and $0 < \delta \ll 1$ such that the trajectories of the systems satisfy $\lim\limits_{t\to\infty} \|x(t) - y(t)\| \leq \delta$.

Now let's study the synchronization of the coupled system in Eq. (2). By introducing coordinates transformation to the synchronization manifold defined by the errors e_v and e_w, defined as:

$$\begin{cases} e_v = v_2 - v_1, \\ e_w = w_2 - w_1, \end{cases} \tag{5}$$

we obtain the following set of error dynamical systems governing the dynamics of the synchronization errors:

$$\begin{cases} \dfrac{de_v}{dt} = -e_v^3 - (3v_1 - a - 1)e_v^2 - [3v_1^2 - 2(1+a)v_1 + a + 2G^e]e_v - e_w, \\ \dfrac{de_w}{dt} = \varepsilon(be_v - ce_w). \end{cases} \tag{6}$$

A neural compete synchronization problem

- Notice that the fixed point $(e_v, e_w) = (0, 0)$ of the error dynamical system in Eq. (6) lies on synchronization manifold $\mathcal{S}$, i.e., where the errors e_v and e_w are both equal to zero.
- If the two neurons achieve complete synchronization, i.e., if we have:

$$\begin{cases} \lim_{t\to\infty} \|v_2(t) - v_1(t)\| = 0, \\ \lim_{t\to\infty} \|w_2(t) - w_1(t)\| = 0, \end{cases} \tag{7}$$

  then in the error dynamical system given in Eq.(6), we can neglect the second and higher order of the errors, so that Eq.(6) becomes:

$$\begin{cases} \dfrac{de_v}{dt} = -[3v^2 - 2(1+a)v + a + 2G^e]e_v - e_w, \\ \dfrac{de_w}{dt} = \varepsilon(be_v - ce_w). \end{cases} \tag{8}$$

- Notice that in Eq. (8), the subscript "1" on variable v_1 has been dropped. This is because, on the synchronization manifold, the two coupled neurons behave identically, i.e., just like one neuron given by Eq.(4), where we have just v instead of v_1.