ECE102, Fall 2018 **Homework #3**
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Due Wednesday, 24 Oct 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 6.
100 points total.

1. (20 points) **Linear systems**

   Determine whether each of the following systems is linear or not. Explain your answer.

   (a) $y(t) = |x(t)| + x(t)$

   Solution:
   Linearity: We can check homogeneity. If we scale the input by a, the output is:

   $$y_a(t) = |(ax(t))| + (ax(t)) = |a|\,|x(t)| + ax(t) \neq a\,(|x(t)| + x(t))$$

   The system is then non-linear.

   (b) $y(t) = 1 + x(t)\cos(\omega t)$

   Solution:
   Linearity: We can check homogeneity. If we scale the input by a, the output is:

   $$y_a(t) = 1 + (ax(t))\cos(\omega t) \neq a\,(1 + x(t)\cos(\omega t))$$

   The system is then non-linear.

   (c) $y(t) = \cos(\omega t + x(t))$

   Solution:
   Linearity: We can check homogeneity. If we scale the input by a, the output is:

   $$y_a(t) = \cos(\omega t + ax(t)) \neq a\,(\cos(\omega t + x(t)))$$

   The system is then non-linear.

   (d) $y(t) = (x(t) + x(-t))\,u(t)$

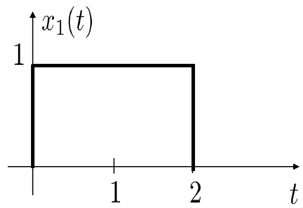
   Solution:
   We will check here homogeneity and superposition:

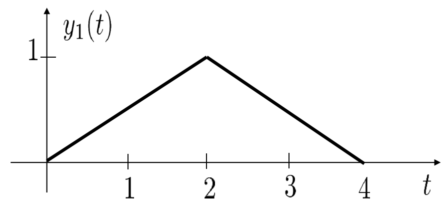
   $$\begin{aligned}(ax_1(\lambda) + bx_2(\lambda) + ax_1(-\lambda) + bx_2(-\lambda))\,u(t) &= \\ (ax_1(\lambda) + ax_1(-\lambda) + bx_2(\lambda) + bx_2(-\lambda))\,u(t) &= \\ a\,(x_1(\lambda) + x_1(-\lambda))\,u(t) + b\,(x_2(\lambda) + x_2(-\lambda))\,u(t) &= \\ ay_1(t) + by_2(t)&\end{aligned}$$

   The system is then linear.

2. (13 points) **LTI systems**

(a) (7 points) Consider an LTI (linear time-invariant) system whose response to $x_1(t)$ is $y_1(t)$, where $x_1(t)$ and $y_1(t)$ are illustrated as follows:





Sketch the response of the system to the input $x_2(t)$.

Solution:

We can express $x_2(t)$ in terms of $x_1(t)$ as follows:

$$x_2(t) = x_1(t+1) + x_1(t)$$

Since the system is LTI, the response to $x_2(t)$ is:

$$y_2(t) = y_1(t+1) + y_1(t)$$

(b) (6 points) Assume we have a linear system with the following input-output pairs:

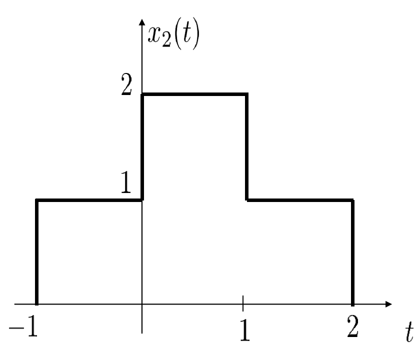
- the output is $y_1(t) = e^{-t}u(t)$ when the input is $x_1(t) = u(t)$;
- the output is $y_2(t) = e^{-t}(u(t) - u(t-1))$ when the input is $x_2(t) = \text{rect}(t - \frac{1}{2})$.

Is the system time-invariant?

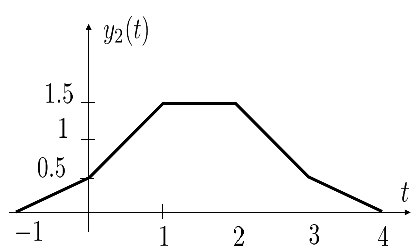
Solution:

The signal $x_2(t)$ can be written as: $x_2(t) = u(t) - u(t-1)$. Let,

$$x_3(t) = x_1(t) - x_2(t) = u(t) - u(t) + u(t-1) = u(t-1)$$



(a) $x_2(t)$



(b) $y_2(t)$

We see that $x_3(t) = x_1(t-1)$. Let us now use the properties of linear system to get the output $y_3(t)$ to input $x_3(t)$, we then compare $y_3(t)$ to $y_1(t-1)$. Since, $x_3(t) = x_1(t) - x_2(t)$, the output is then

$$y_3(t) = y_1(t) - y_2(t) = e^{-t}u(t) - e^{-t}(u(t) - u(t-1)) = e^{-t}u(t-1)$$

On the other hand,

$$y_1(t-1) = e^{-(t-1)}u(t-1)$$

Since $y_3(t) \neq y_1(t-1)$, the system is not time-invariant.

3. (42 points) **Convolution**

   (a) (10 points) For each pair of the signals given below, compute their convolution using the flip-and-drag technique.

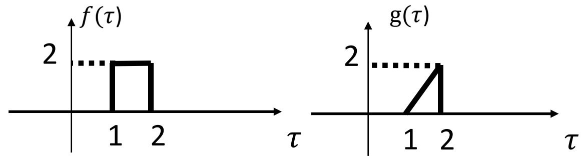
      i. $f(t) = 2\,\mathrm{rect}(t - \frac{3}{2})$, $\qquad g(t) = 2\,r(t-1)\mathrm{rect}(t - \frac{3}{2})$

      Solution:
      Let,

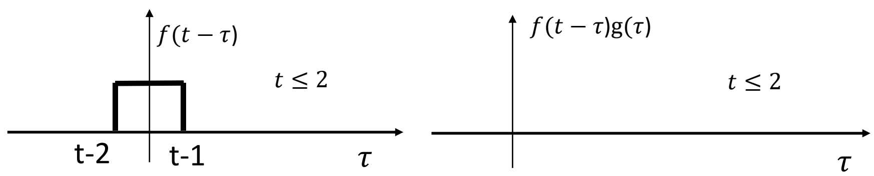
$$y(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$$

      We will flip and drag the rect function. As we can see in the figure below, for $t \leq 2$, there is no overlap between the two plots, therefore $y(t) = 0$ for for $t \leq 2$. For $2 < t \leq 3$, the rect function starts to overlap with the triangle, the convolution integral in this case is equal to the overlapped area. For $3 < t \leq 4$, the rect function starts to go out from the triangle, the convolution integral is also equal to the overlapped area. For $t > 4$, there is no overlap between the two plots, then $y(t) = 0$ for $t > 4$. Therefore, $y(t)$ is given as:

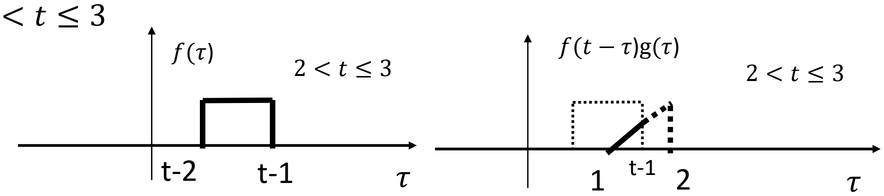
$$y(t) = \begin{cases} 0, & t \leq 2 \\ \int_1^{t-1} 4(\tau - 1)\, d\tau = 2(t-2)^2, & 2 < t \leq 3 \\ \int_{t-2}^{2} 4(\tau - 1)\, d\tau, = -2(t-4)(t-2) & 3 < t \leq 4 \\ 0, & t > 4 \end{cases}$$

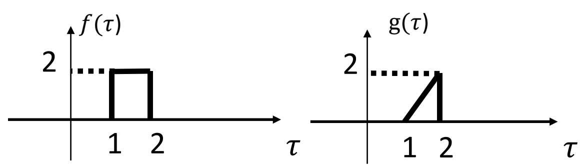


Case 1: $t \leq 2$

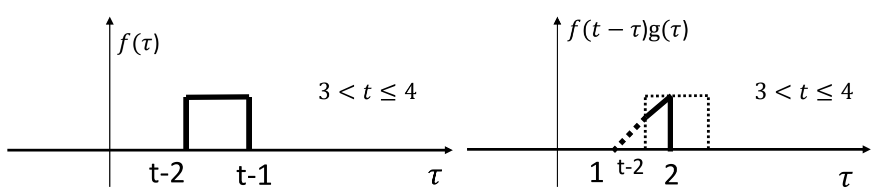


Case 2: $2 < t \leq 3$

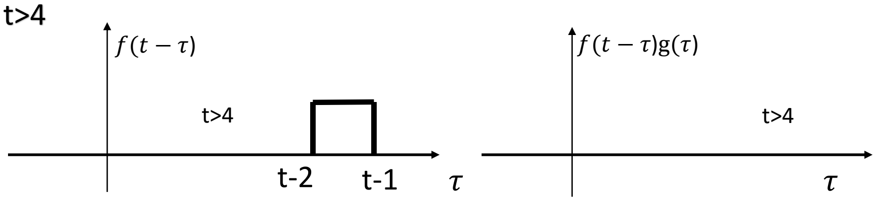




Case 3: $3 < t \leq 4$



Case 4: t>4



ii. $f(t) = u(-t-1), \qquad g(t) = e^{-t}u(t)$

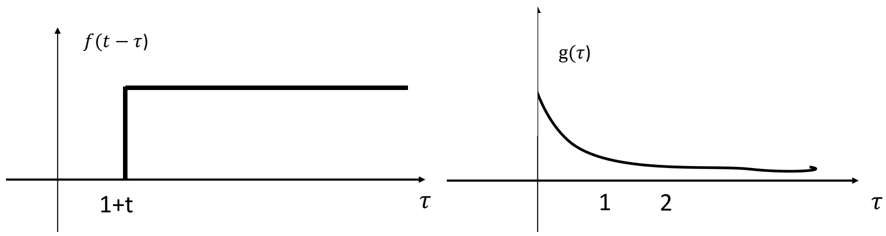
Solution:

Let,

$$y(t) = f(t) \star g(t) = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

Then,

$$y(t) = \begin{cases} \int_0^{+\infty} e^{-\tau}\, d\tau = 1, & t \leq -1 \\ \int_{1+t}^{+\infty} e^{-\tau}\, d\tau = e^{-t-1}, & -1 < t \end{cases}$$



(b) (12 points) For each of the following, find a function $h(t)$ such that $y(t) = x(t) \star h(t)$.

i. $y(t) = \int_{t-T}^{t} x(\tau)d\tau$

Solution:
We can think about $h(t)$ as the impulse response of the give LTI system, therefore

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau = u(t) - u(t-T)$$

ii. $y(t) = x(t-1)$

Solution:
We know from the shifting property that: $x(t) \star \delta(t-1) = x(t-1)$, therefore

$$h(t) = \delta(t-1)$$

iii. $y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$

Solution:
Also by applying the shifting property,

$$h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

(c) (12 points) Simplify the following expressions:

i. $\left[\int_{-\infty}^{t} u(-\tau+3)\delta(\tau-1)d\tau\right] \star (\delta(t-2) + \delta(2t-8))$

Solution:

First we have:

$$\int_{-\infty}^{t} u(-\tau+3)\delta(\tau-1)d\tau = \int_{-\infty}^{t} \delta(\tau-1)d\tau = u(t-1)$$

.

Now, we apply the shifting property:

$$\begin{aligned} u(t-1) \star (\delta(t-2)+\delta(2t-8)) &= u(t-1) \star \left(\delta(t-2)+\frac{1}{2}\delta(t-4)\right) \\ &= u(t-3)+\frac{1}{2}u(t-5) \end{aligned}$$

ii. $[\delta(t-3)+\delta(t+2)] * \left[e^{3t}u(-t)+\delta(t+2)+2\right]$

Solution:
We apply the shifting property:

$$\begin{aligned} e^{3(t-3)}u(-t+3)+\delta(t-1)+2+e^{3(t+2)}u(-t-2)+\delta(t+4)+2 = \\ e^{3(t-3)}u(-t+3)+\delta(t-1)+e^{3(t+2)}u(-t-2)+\delta(t+4)+4 \end{aligned}$$

iii. $\frac{d}{dt}\left[(u(t)-u(t-1)) \star u(t-2)\right]$, *Hint: Show first that* $u(t) \star u(t) = r(t)$ *where* $r(t)$ *is the ramp function.*

Solution:
We first show that $u(t) \star u(t) = r(t)$:

$$u(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \left(\int_{0}^{t} 1 d\tau\right) u(t) = tu(t) = r(t)$$

Therefore, using the properties of convolutions,

$$(u(t)-u(t-1)) \star u(t-2) = r(t-2) - r(t-3)$$

Thus,

$$\frac{d}{dt}\left(r(t-2)-r(t-3)\right) = u(t-2)-u(t-3)$$

(d) (8 points) Explain whether each of the following statements is true or false.

i. If $x(t)$ and $h(t)$ are both odd functions, and $y(t) = x(t) \star h(t)$, then $y(t)$ is an even function.

Solution:
True: We will show this statement by applying the definition of convolution.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Let $\tau' = -\tau$, then

$$y(t) = -\int_{\infty}^{-\infty} x(-\tau)h(t+\tau')d\tau' = \int_{-\infty}^{\infty} x(-\tau')h(t+\tau')d\tau'$$

Since $x(t)$ and $h(t)$ are both odd functions, we have:

$$y(t) = \int_{-\infty}^{\infty} (-x(\tau'))(-h(-t-\tau'))d\tau' = \int_{-\infty}^{\infty} x(\tau')h(-t-\tau')d\tau' = y(-t)$$

Therefore, $y(t)$ is even.

ii. If $y(t) = x(t) \star h(t)$, then $y(2t) = h(2t) \star x(2t)$.

Solution:
False: Consider the following counter example; let $x(t) = \delta(t)$ and $h(t) = u(t)$, then $x(2t) = \delta(2t) = \frac{1}{2}\delta(t)$ and $h(2t) = u(2t) = u(t)$. Therefore, we have: $y(t) = x(t) \star h(t) = u(t)$, so $y(2t) = u(2t) = u(t)$. On the other hand, $x(2t) \star h(2t) = \frac{1}{2}u(t)$. Thus, $y(2t) \neq h(2t) \star x(2t)$.
What is true instead is that $y(2t) = 2\left(h(2t) \star x(2t)\right)$. This can be shown using the definition of convolution:

$$x(2t) \star h(2t) = \int_{-\infty}^{\infty} x(2\tau)h(2t-2\tau)d\tau$$

Let $\tau' = 2\tau$, then

$$x(2t) \star h(2t) = \frac{1}{2}\int_{-\infty}^{\infty} x(\tau')h(2t-\tau')d\tau' = \frac{1}{2}y(2t)$$

4. (9 points) **Impulse response and LTI systems**

Consider the following three LTI systems:

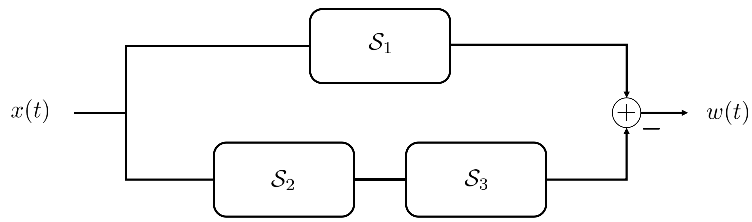
- The first system $\mathcal{S}_1$ is given by its input-output relationship: $y(t) = \int_{-\infty}^{t+t_0} x(\tau)d\tau$;
- The second system $\mathcal{S}_2$ is given by its impulse response: $h_2(t) = u(t-2)$;
- The third system $\mathcal{S}_3$ is given by its impulse response: $h_3(t) = \delta(t-3)$.

(a) (3 points) Compute the impulse responses $h_1(t)$ of system $\mathcal{S}_1$.

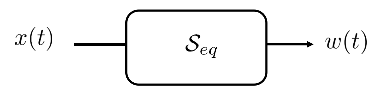
Solution:
For system $\mathcal{S}_1$, the impulse response is given by:

$$h_1(t) = \int_{-\infty}^{t+t_0} \delta(\tau)d\tau = u(t+t_0)$$

(b) (3 points) The three systems are interconnected as shown below.



Determine the impulse response $h_{eq}(t)$ of the equivalent system.



Solution:

$$h_{eq}(t) = h_1(t) - (h_2(t) \star h_3(t)) = u(t + t_0) - u(t - 5)$$

(c) (3 points) Determine the response of the overall system to the input $x(t) = \delta(t) + 2\delta(t - 3)$.

Solution:
The response is: $y(t) = x(t) \star h_{eq}(t) = u(t + t_0) - u(t - 5) + 2u(t + t_0 - 3) - 2u(t - 8)$

5. (16 points) **MATLAB**

To complete the following MATLAB tasks, we will provide you with a MATLAB function (`nconv()`), which numerically evaluates the convolution of two continuous-time functions. Make sure to download it from CCLE and save it in your working directory in order to use it.

The function syntax is as follows:

```
[y, ty] = nconv(x,tx,h,th)
```

where the inputs are:

`x` : input signal vector
`tx`: times over which `x` is defined
`h` : impulse response vector
`th`: times over which `h` is defined

and the outputs are:

`y` : output signal vector
`ty`: times over which `y` is defined.

The function is implemented with the MATLAB's `conv()` function. You are encouraged to look at the implementation of the function provided (the explanations are included as comments in the code).

(a) (7 points) **Task 1**

Use the `nconv()` function to check your results for problem 3(a)(i). Plot the output for each problem (you can consider either function to be the input). Properly label the axes of the plots. Make sure to use the same step size for `tx` and `th`.

Solution:

Code for part 3(a)i:

```
t=[1:0.001:2]; f=2*ones(1,length(t)); g=2*(t-1); [y, ty] = nconv(f,t,g,t);
plot(ty,y); grid on;
xlabel('t(sec)'); ylabel('The output of part 3(a)-i');
```

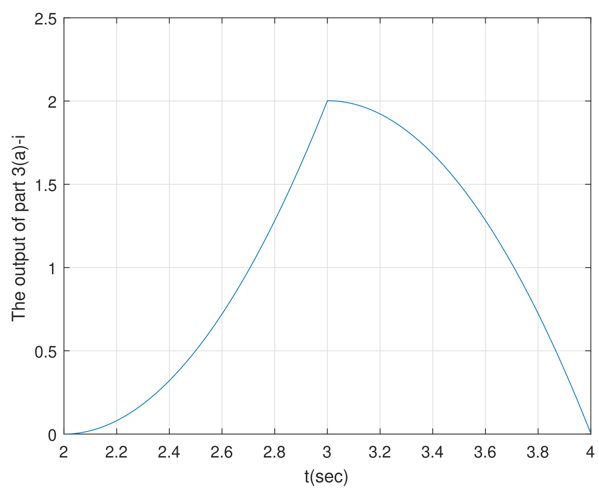


Figure 1: Task 1a

(b) (5 points) **Task 2**

Using the `nconv()` function, perform the convolution of two unit rect functions: rect(t)$\star$ rect(t). Plot and label the result.

Solution:

Code:
```
t=[-0.5:0.001:0.5];x=ones(1,length(t));[y, ty] = nconv(x,t,x,t);
plot(ty,y); grid on;
xlabel('t(sec)'); ylabel('Convolution of two rect');
```

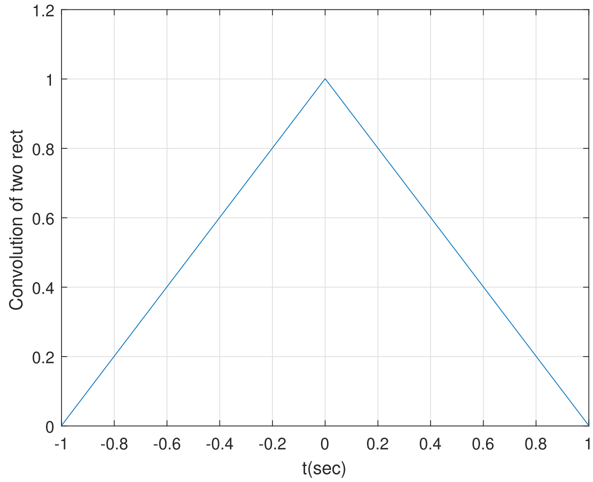


Figure 2: Task 2

(c) (4 points) **Task 3**

Using the result of task 2 and the same MATLAB function, calculate $y(t) = \text{rect(t)} \star \text{rect(t)} \star \text{rect(t)}$. Plot and label the result.

Code: we used as input to nconv the same output we obtained for task 2.

```
 [y, ty] = nconv(x,t,y,ty);
plot(ty,y); grid on; xlabel('t(sec)'); ylabel('Convolution of a rect and a
triangle');
```

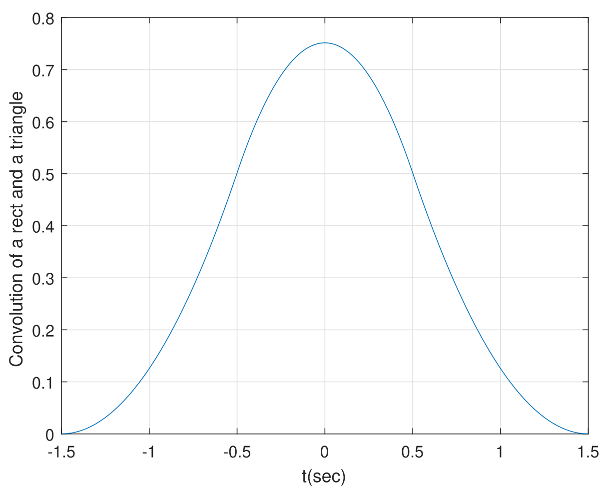


Figure 3: Task 3

(d) (Optional) **Task 4**

Now, what happens if we consider $\text{rect(t)} \star \text{rect(t)} \star \cdots \star \text{rect(t)} = \text{rect}^{(N)}(t)$? Using for loop, calculate the result of convolving N rect(t) functions together. Plot and label the results (use $N = 100$).

Code:

```
t=[-0.5:0.001:0.5]; x=ones(1,length(t)); [y, ty] = nconv(x,t,x,t);
N=100-2;
for n=1:N
[y, ty] = nconv(x,t,y,ty);
end
plot(ty,y); grid on;
xlabel('t(sec)'); ylabel('Convolution of N rect');
```

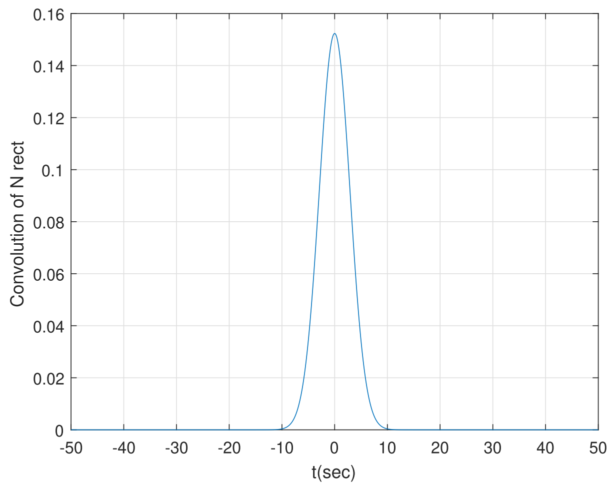


Figure 4: Task 4

Side note in case you have taken any probability course before: Convolution is an operator that is also useful in statistics. We use it to compute the pdf (probability density function) of the sum of N independent random variables. So if we have $Y = X_1 + X_2 + X_3$, the pdf of Y is the convolution of the pdfs of X_1, X_2 and X_3. In task 4, we are computing the pdf of the sum of N uniform random variables (the pdf of a uniform random variable is a rect function), by convolving N times the rect function. The resulting curve will have a bell-shape. This is related to a theorem in statistics called 'The Central Limit Theorem'.