

Statistics: Mean: 72.2 (out of 100), Standard deviation: 20.3, 25th percentile: 56.5, Median: 74.5, 75th percentile: 90, Maximum score: 103.5, Number of exams: 100

Comments:

- If you have grading questions, please submit through Gradescope. Regrades should be submitted if you believe we applied the rubric mistakenly.
- The exam was longer than we intended. We will use this information to calibrate the length and difficulty of the final.
- Given the length of the midterm, I will make the following offer: if you do better on the final exam than the midterm, I will count the final exam score as your midterm score (i.e., your final will count as 60% of your grade; with the HW being 40%).
- Please be reminded that the absolute grading scale may be relaxed. The median final grade in this class will not be lower than a B+.
- Please don't hesitate to contact Prof. Kao if you received a poor score and would like to talk further, or have overall course grade questions.

ECE 102, Fall 2018

Department of Electrical and Computer Engineering
University of California, Los Angeles

Midterm

Prof. J.C. Kao
TAs: H. Salami, S. Shahshavari

UCLA True Bruin academic integrity principles apply.

Open: Two pages of cheat sheet allowed.

Closed: Book, computer, internet.

2:00-3:50pm.

Wednesday, 14 Nov 2018.

State your assumptions and reasoning.

No credit without reasoning.

Show all work on these pages.

Name: _____

Signature: _____

ID#: _____

Problem 1 _____ / 19

Problem 2 _____ / 17

Problem 3 _____ / 16

Problem 4 _____ / 20

Problem 5 _____ / 28

BONUS _____ / 6 bonus points

Total _____ / 100 points + 6 bonus points

Problem 1 (19 points)

(a) (9 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

- i. (3 points) If $x(t)$ is an energy signal, then $y(t) = x(t) + 1$ is also an energy signal.

Solution: False

$x(t) + 1$ is not an energy signal, adding a constant to a signal will in general make its energy go to infinity. Consider for instance the finite energy signal: $x(t) = e^{-t}u(t)$, then

$$\int_{-\infty}^{+\infty} |x(t) + 1|^2 dt = \int_{-\infty}^{+\infty} (x^2(t) + 2x(t)) dt + \int_{-\infty}^{+\infty} 1 dt = \frac{1}{2} + 2 + \infty$$

Therefore, $x(t) + 1$ is not an energy signal.

- ii. (3 points) If $x(t)$ is an even signal, then $y(t) = x(t - 1)$ is also an even signal.

Solution: False

Consider for instance the unit triangle $x(t) = \Delta(t)$, which is an even function. However, shifting the unit triangle to the right by one will make it defined only over $t \geq 0$, we then obtain $x(t - 1) = \Delta(t - 1)$, which is not even.

- iii. (3 points) If the input to an LTI system is periodic, then its output is also periodic.

Solution: True

If the input is periodic, then it can be written as:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

where c_k 's are the Fourier series coefficients of $x(t)$. Then using the eigenfunction property, we obtain the corresponding output:

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha_k c_k e^{j\omega_0 kt}$$

where α_k is the eigenvalue that corresponds to $e^{j\omega_0 kt}$. Therefore, $y(t)$ is also periodic.

- (b) (10 points) Is the following system linear? Is it time invariant? (Check both properties). Explain your answer.

$$y(t) = \begin{cases} x(t-1), & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Solutions: We can equivalently write the system as follows:

$$y(t) = x(t-1)u(t-1)$$

Linearity: Suppose to inputs $x_1(t)$ and $x_2(t)$, we respectively get $y_1(t)$ and $y_2(t)$ as outputs. Now if we consider the following input $x_3(t) = ax_1(t) + bx_2(t)$, then its output:

$$y_3(t) = x_3(t-1)u(t-1) = (ax_1(t-1)u(t-1) + bx_2(t-1)u(t-1)) = ay_1(t) + by_2(t)$$

The system is then linear.

Time Invariant:

If we delay the input by τ , i.e. $x_\tau(t) = x(t-\tau)$, the output is then:

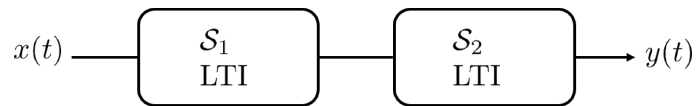
$$y_\tau(t) = x_\tau(t-1)u(t-1) = x(t-1-\tau)u(t-1)$$

Now if we delay the output, we get:

$$y(t-\tau) = x(t-\tau-1)u(t-\tau-1)$$

Since $y(t-\tau) \neq y_\tau(t)$, the system is not time-invariant.

Problem 2 (17 points) Consider the series cascade of the following two systems:



The system \mathcal{S}_1 is LTI with impulse response

$$h_1(t) = \int_{-\infty}^t u(\tau) \delta(\tau - 2) d\tau$$

The system \mathcal{S}_2 is also LTI, with unknown impulse response $h_2(t)$ that we need to find. We are also given that, when the input $x(t)$ is $\delta(t)$, the output $y(t)$ is $r(t - 3) + u(t - 2)$.

Note: $r(t - 3)$ is the ramp function delayed by 3.

This question continues on the next page.

- (a) (11 points) Find the impulse response $h_2(t)$ of the system S_2 **and** determine if the system S_2 is causal.

Solution:

We first simplify the impulse response of the first system:

$$h_1(t) = \int_{-\infty}^t u(\tau)\delta(\tau-2)d\tau = \int_{-\infty}^t u(2)\delta(\tau-2)d\tau = \int_{-\infty}^t \delta(\tau-2)d\tau = u(t-2)$$

The impulse response of overall system is given by:

$$h_{eq}(t) = r(t-3) + u(t-2)$$

This is because it is given as the output of the overall system when the input is $\delta(t)$.

When the input is $x(t) = \delta(t)$, the intermediate signal between the two systems is the output of S_1 to the input $\delta(t)$. Therefore, the intermediate signal in this case is: $h_1(t) = u(t-2)$. Therefore, we have the following for system S_2 :

$$\text{Input: } u(t-2) \xrightarrow{S_2} \text{output: } r(t-3) + u(t-2)$$

Since S_2 is LTI, we can deduce its step response (by shifting the output to the left by 2):

$$\text{Input: } u(t) \xrightarrow{S_2} \text{output: } r(t-1) + u(t)$$

Therefore, the step response of S_2 is:

$$r(t-1) + u(t)$$

Thus, the impulse response of S_2 is:

$$h_2(t) = \frac{d}{dt}(r(t-1) + u(t)) = u(t-1) + \delta(t)$$

Since $h_2(t) = 0$ for $t < 0$, the LTI system S_2 is causal.

Note: We received answers like this: because in $h_2(t)$ we have $t-1$ in $u(t-1)$ and t in $\delta(t)$, the system then depends on past and present values of the input, then it is causal. This is not a right justification, because $h_2(t)$ is not the input-output relationship of the system, we used that justification when we have the mapping from the input to output. However, We can using $h_2(t)$ represent the system in terms of its input-output mapping through convolution:

$$y(t) = h_2(t) * z(t) = \int_{-\infty}^{\infty} h_2(\tau)z(t-\tau)d\tau$$

where $z(t)$ is the input to the second system. Now we check if $y(t)$ depends on past values of input by checking the arguments of z . We have $y(t)$ depends on $z(t-\tau)$ and because $h_2(\tau)$ is zero for $\tau < 0$, τ will always be positive so that $z(t-\tau)$ will always be a past value of input for $y(t)$. This is why we can say that the system is causal.

(b) (6 points) Find the output $y(t)$ to the following input:

$$x(t) = (1 + e^{-t})\delta(t + 1)$$

Solution:

Using the sampling property, we can simplify $x(t)$ as follows:

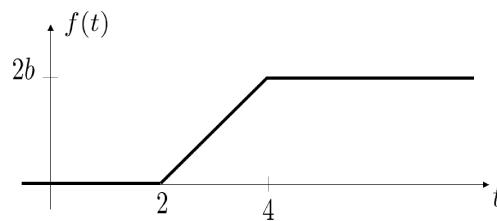
$$x(t) = (1 + e^{-t})\delta(t + 1) = (1 + e)\delta(t + 1)$$

Then,

$$\begin{aligned} y(t) &= h_{eq}(t) * x(t) \\ &= (r(t - 3) + u(t - 2)) * ((1 + e)\delta(t + 1)) = (1 + e)(r(t - 3 + 1) + u(t - 2 + 1)) \\ &= (1 + e)(r(t - 2) + u(t - 1)) \end{aligned}$$

Problem 3 (16 points)

(a) (8 points) Consider the signal $f(t)$ shown below:



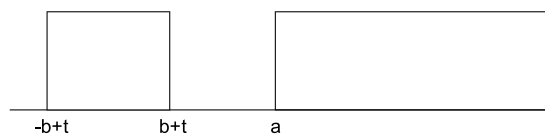
This signal can be written as

$$u(t - a) * \text{rect}\left(\frac{t}{2b}\right)$$

where $a > 0$ and $b > 0$. Find a and b . (Hint: use the flip and drag technique.)

Solution:

Using the flip and drag technique, we have:



$f(t) = 0$ when there is no overlap, i.e., when $b + t < a$ or $t < a - b$. We have $f(t) = 0$ for $t < 2$, therefore

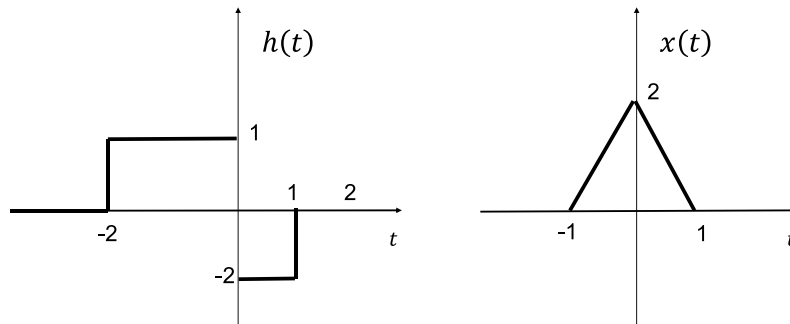
$$a - b = 2$$

The total overlap happens when $-b + t > a$ or $t > a + b$. The function $f(t)$ reaches its maximum at $2b$ and stays at this value for $t > 4$, thus

$$a + b = 4$$

Solving two equation, we get $a = 3$ and $b = 1$.

- (b) (8 points) An input, $x(t)$, is given to an LTI system with impulse response $h(t)$. Both $x(t)$ and $h(t)$ are shown below.



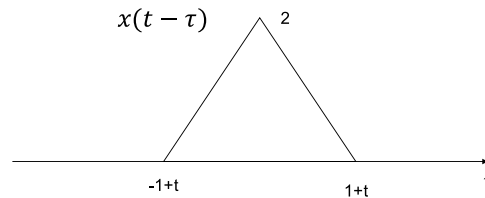
Let $y(t)$ denote the output of the system, i.e., $y(t) = x(t) * h(t)$. Find the value of t at which the output $y(t)$ reaches its maximum value. Determine this maximum value.

*Note: to answer this question, you do **not** need to find $y(t)$ for all t .*

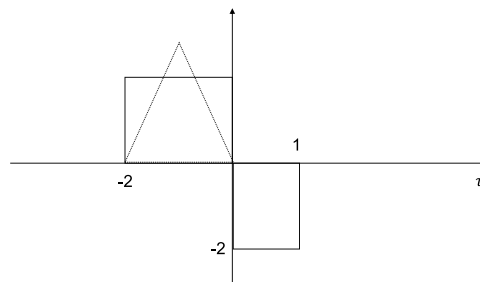
Solution:

We know that

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau)d\tau$$



The maximum value of $y(t)$ happens when the triangle totally overlaps with the rectangle part of $h(t)$ that only has positive values, as shown here:



This happens when $1 + t = 0$ therefore, $t = -1$. In this case, the maximum value is the area of the triangle is:

$$y(-1) = \frac{2 \times 2}{2} = 2$$

Problem 4 (20 points)

Consider the following two periodic signals $f(t)$ and $g(t)$. They both have the same period T_0 . Let f_k and g_k respectively denote the Fourier series coefficients of $f(t)$ and $g(t)$.

- (a) (6 points) If $f(t) = -g\left(t + \frac{T_0}{2}\right)$, how is f_k related to g_k ?

Solution: We have:

$$f(t) = \sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 k t} \quad \text{and,} \quad g(t) = \sum_{k=-\infty}^{+\infty} g_k e^{j\omega_0 k t}$$

Now, if $f(t) = -g\left(t + \frac{T_0}{2}\right)$, then

$$\begin{aligned} f(t) &= -g\left(t + \frac{T_0}{2}\right) \\ &= \sum_{k=-\infty}^{+\infty} -g_k e^{j\omega_0 k \left(t + \frac{T_0}{2}\right)} \\ &= \sum_{k=-\infty}^{+\infty} -g_k e^{j\omega_0 k \frac{T_0}{2}} e^{j\omega_0 k t} \\ &= \sum_{k=-\infty}^{+\infty} -g_k e^{j \frac{2\pi}{T_0} k \frac{T_0}{2}} e^{j\omega_0 k t} \\ &= \sum_{k=-\infty}^{+\infty} -g_k e^{j\pi k} e^{j\omega_0 k t} = \sum_{k=-\infty}^{+\infty} -g_k (-1)^k e^{j\omega_0 k t} \\ &= \sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 k t} \end{aligned}$$

Therefore,

$$f_k = -(-1)^k g_k$$

(b) (6 points) If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, for what k are the coefficients f_k zero?

Solution: If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, Then using the previous conclusion, we have:

$$f_k = -(-1)^k f_k$$

Therefore, for even k :

$$f_k = -f_k \rightarrow f_k = 0$$

(c) (8 points) This question has two parts. *Note: part (c) is independent of parts (a) and (b).*

- i. (4 points) Let $f_e(t)$ denote the even part of $f(t)$. Express the Fourier series coefficients of $f_e(t)$ in terms of f_k .

Solution:

The even part of the signal is: $f_e(t) = \frac{f(t) + f(-t)}{2}$, $f(-t)$ is also periodic we thus have:

$$\begin{aligned} f_e(t) &= \frac{f(t) + f(-t)}{2} \\ &= \frac{\sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 k t} + \sum_{k=-\infty}^{+\infty} f_k e^{-j\omega_0 k t}}{2} \\ &= \frac{1}{2} \left(\sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 k t} + \sum_{k=-\infty}^{+\infty} f_{-k} e^{j\omega_0 k t} \right) \\ &= \frac{1}{2} \sum_{k=-\infty}^{+\infty} (f_k + f_{-k}) e^{j\omega_0 k t} \\ &= \sum_{k=-\infty}^{+\infty} \frac{1}{2} (f_k + f_{-k}) e^{j\omega_0 k t} \end{aligned}$$

Therefore, the Fourier series coefficients of $f_e(t)$ is $\frac{1}{2}(f_k + f_{-k})$.

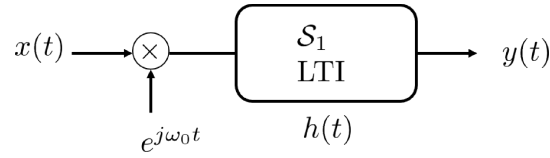
- ii. (4 points) Determine the DC component of $f_o(t)$, the odd part of $f(t)$.

Solution: The Fourier coefficients of the odd part of the signal is $f_{o,k} = \frac{1}{2}(f_k - f_{-k})$.
Therefore at $k = 0$, $f_{o,0} = \frac{1}{2}(f_0 - f_{-0}) = 0$.

In fact, any odd signal has zero DC component.

Problem 5 (28 points)

Consider the following system ($\omega_0 > 0$):



The system \mathcal{S}_1 is LTI and $h(t)$ represents its impulse response.

- (a) (10 points) Show that the overall system, with input $x(t)$ and output $y(t)$, is not time-invariant.

Solution: The input-output relationship of the system is given by:

$$y(t) = [e^{j\omega_0 t} x(t)] * h(t) = \int_{-\infty}^{+\infty} h(\tau) e^{j\omega_0(t-\tau)} x(t-\tau) d\tau = e^{j\omega_0 t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega_0 \tau} x(t-\tau) d\tau$$

If we delay the input, i.e., $x_\alpha(t) = x(t - \alpha)$, the corresponding output is:

$$y_\alpha(t) = e^{j\omega_0 t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega_0 \tau} x_\alpha(t-\tau) d\tau = e^{j\omega_0 t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega_0 \tau} x(t-\alpha-\tau) d\tau$$

On the other hand, if we shift the output, we have:

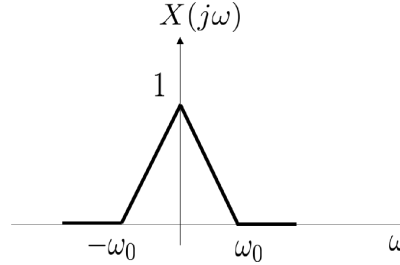
$$y(t - \alpha) = e^{j\omega_0(t-\alpha)} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega_0 \tau} x(t-\alpha-\tau) d\tau$$

Since $y(t - \alpha) \neq y_\alpha(t)$, the system is not TI.

(b) (12 points) Consider the following impulse response for system S_1 :

$$h(t) = e^{j\frac{\omega_0}{2}t} \text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$$

We give the system an input $x(t)$, where $x(t)$ has the following Fourier transform $X(j\omega)$:



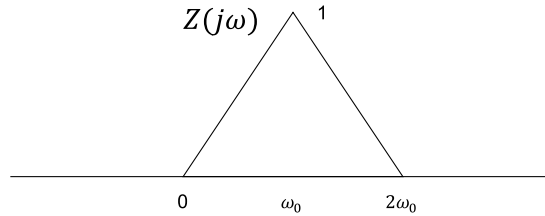
Find and sketch the Fourier transform $Y(j\omega)$ of the corresponding output $y(t)$. After this, determine (i) if $y(t)$ is real and (ii) if $y(t)$ is even. *Note: you do not need to give an expression for $Y(j\omega)$, a sketch of it is enough. There is some space on the next page if needed.*

Solution:

If $z(t) = x(t)e^{j\omega_0 t}$, then using the Fourier transform properties, we have:

$$Z(j\omega) = X(j(\omega - \omega_0))$$

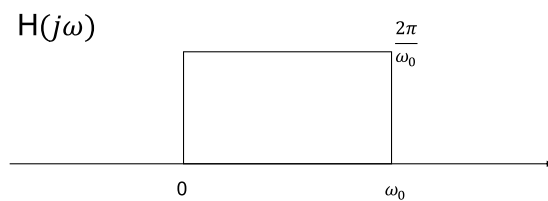
Here is a sketch of $Z(j\omega)$:



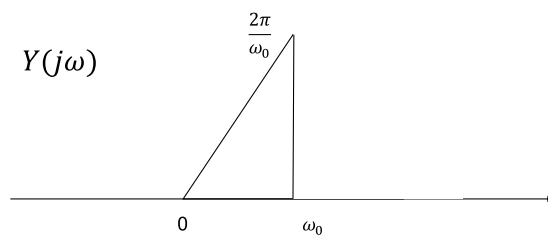
Now using the properties we find the Fourier transform of $h(t)$,

$$\begin{aligned} \text{sinc}(t) &\longleftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right) \\ \text{sinc}\left(\frac{\omega_0}{2\pi}t\right) &\longleftrightarrow \frac{2\pi}{\omega_0} \text{rect}\left(\frac{\omega}{2\pi} \cdot \frac{2\pi}{\omega_0}\right) \\ e^{j\frac{\omega_0}{2}t} \text{Sinc}\left(\frac{\omega_0}{2\pi}t\right) &\longleftrightarrow \frac{2\pi}{\omega_0} \text{rect}\left(\frac{\omega - \frac{\omega_0}{2}}{\omega_0}\right) \end{aligned}$$

Therefore, $H(j\omega)$ is as follows:



Since $y(t) = h(t) * z(t)$, therefore $Y(j\omega) = H(j\omega)X(j\omega)$. Therefore,



Since $Y^*(j\omega) \neq Y(-j\omega)$, $y(t)$ is not real.

Since $Y(j\omega)$ is not even, $y(t)$ is not even.

(c) (6 points) Suppose

$$z(t) = y(3t - 2)$$

Express $Z(j\omega)$ in terms of $Y(j\omega)$. *Note: part (c) is independent of parts (a) and (b).*

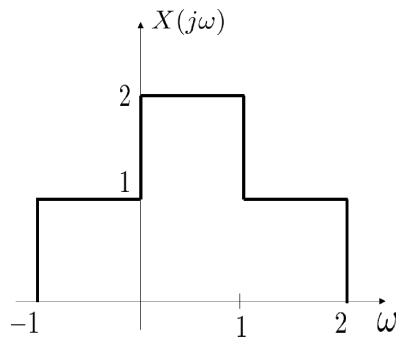
Solution:

Using the properties:

$$\begin{aligned}y(t) &\longleftrightarrow Y(j\omega) \\y(3t) &\longleftrightarrow \frac{1}{3}Y\left(\frac{j\omega}{3}\right) \\y\left(3\left(t - \frac{2}{3}\right)\right) &\longleftrightarrow \frac{1}{3}e^{-j\frac{2}{3}\omega}Y\left(\frac{j\omega}{3}\right)\end{aligned}$$

BONUS (6 points)

(a) (4 points) The Fourier transform $X(j\omega)$ of a signal $x(t)$ is given as follows:



Find the phase of $x^2(t)$.

Solution:

Let $F(j\omega) = X(j(\omega + \frac{1}{2}))$. Thus, $F(j\omega)$ is real and even. Therefore,

$$f(t) = e^{-jt\frac{1}{2}}x(t)$$

is a real function. Thus,

$$x(t) = e^{jt\frac{1}{2}}f(t)$$

and,

$$x^2(t) = e^{jt}f^2(t)$$

Therefore the phase of $x^2(t)$ is t .

- (b) (2 points) If a signal $x(t)$ is causal with $x(0) = 0$, how can we retrieve $x(t)$ from its even component $x_e(t)$?

Solution:

Since this signal is causal, therefore:

$$x(t) = 0, \text{ for } t < 0$$

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \begin{cases} \frac{x(-t)}{2}, & t < 0 \\ \frac{x(t)}{2}, & t > 0 \end{cases}$$

Therefore,

$$x(t) = 2x_e(t), \text{ for } t > 0$$

and

$$x(t) = 0, \text{ for } t \leq 0$$

Fourier Transform Tables

Property	Signal	Transform
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(j\omega) + \beta X_2(j\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(j\omega) = X(-j\omega)$ Magnitude: $ X(-j\omega) = X(j\omega) $ Phase: $\Theta(-\omega) = -\Theta(\omega)$ Real part: $X_r(-j\omega) = X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = -X_i(j\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(j\omega) = -X(-j\omega)$ Magnitude: $ X(-j\omega) = X(j\omega) $ Phase: $\Theta(-\omega) = -\Theta(\omega) \mp \pi$ Real part: $X_r(-j\omega) = -X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = X_i(j\omega)$
Even signal	$x(-t) = x(t)$	$X(j\omega)$: even
Odd signal	$x(-t) = -x(t)$	$X(j\omega)$: odd
Time shifting	$x(t - \tau)$	$X(j\omega) e^{-j\omega\tau}$
Frequency shifting	$x(t) e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
Modulation property	$x(t) \cos(\omega_0 t)$	$\frac{1}{2} [X(j(\omega - \omega_0)) + X(j(\omega + \omega_0))]$
Time and frequency scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{j\omega}{a}\right)$
Differentiation in time	$\frac{d^n}{dt^n} [x(t)]$	$(j\omega)^n X(j\omega)$
Differentiation in frequency	$(-jt)^n x(t)$	$\frac{d^n}{d\omega^n} [X(j\omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega) X_2(j\omega)$
Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda$	$\frac{X(j\omega)}{j\omega} + \pi X(0) \delta(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) ^2 d\omega$	

Table 4.4 – Fourier transform properties.

Additional properties:	$x(t)$: even and real	$X(j\omega)$: even and real
	$x(t)$: odd and real	$X(j\omega)$: odd and imaginary
	$x(t)$: even and imaginary	$X(j\omega)$: even and imaginary
	$x(t)$: odd and imaginary	$X(j\omega)$: odd and real

Name	Signal	Transform
Rectangular pulse	$x(t) = A \text{rect}(t/\tau)$	$X(j\omega) = A\tau \text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
Triangular pulse	$x(t) = A \Lambda(t/\tau)$	$X(j\omega) = A\tau \text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Right-sided exponential	$x(t) = e^{-at} u(t)$	$X(j\omega) = \frac{1}{a + j\omega}$
Two-sided exponential	$x(t) = e^{-a t }$	$X(j\omega) = \frac{2a}{a^2 + \omega^2}$
Signum function	$x(t) = \text{sgn}(t)$	$X(j\omega) = \frac{2}{j\omega}$
Unit impulse	$x(t) = \delta(t)$	$X(j\omega) = 1$
Sinc function	$x(t) = \text{sinc}(t)$	$X(j\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right)$
Constant-amplitude signal	$x(t) = 1, \text{ all } t$	$X(j\omega) = 2\pi \delta(\omega)$
	$x(t) = \frac{1}{\pi t}$	$X(j\omega) = -j \text{sgn}(\omega)$
Unit-step function	$x(t) = u(t)$	$X(j\omega) = \pi \delta(\omega) + \frac{1}{j\omega}$
Modulated pulse	$x(t) = \text{rect}\left(\frac{t}{\tau}\right) \cos(\omega_0 t)$	$X(j\omega) = \frac{\tau}{2} \text{sinc}\left(\frac{(\omega - \omega_0)\tau}{2\pi}\right) + \frac{\tau}{2} \text{sinc}\left(\frac{(\omega + \omega_0)\tau}{2\pi}\right)$

Note:

$$\text{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\text{rect}(t/\tau) = u(t + \tau/2) - u(t - \tau/2)$$

Table 4.5 – Some Fourier transform pairs.