ECE102, Fall 2018 **Homework #1**
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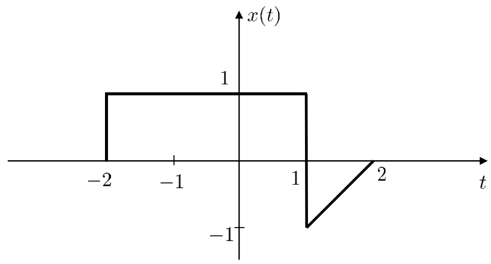
Due Wednesday, 10 Oct 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 2.
100 points total.

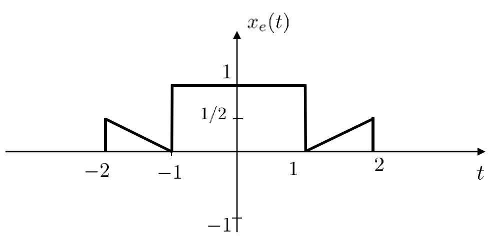
1. (10 points) **Even and odd parts.**
Sketch the even and odd components of the following signal:

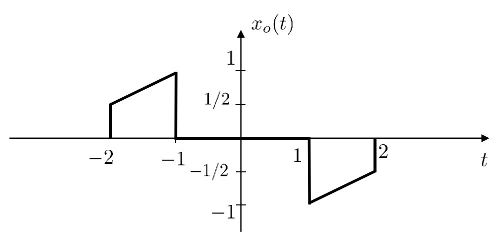


Solutions:
Using the expressions of the even and odd parts,

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$
$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

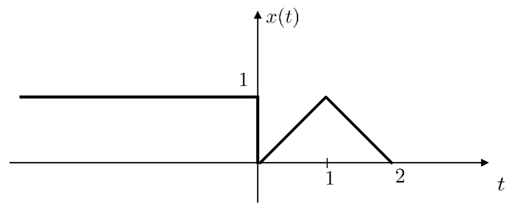
we can construct the even and odd components of $x(t)$.





2. (15 points) **Time scaling and shifting.**

   (a) (10 points) Consider the following signal.



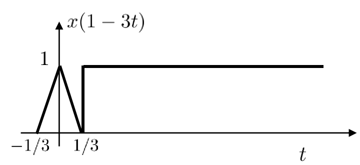
Sketch the following:

i. $x(1-3t)$

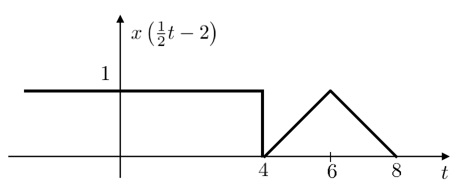
ii. $x(\frac{t}{2}-2)$

Solutions:

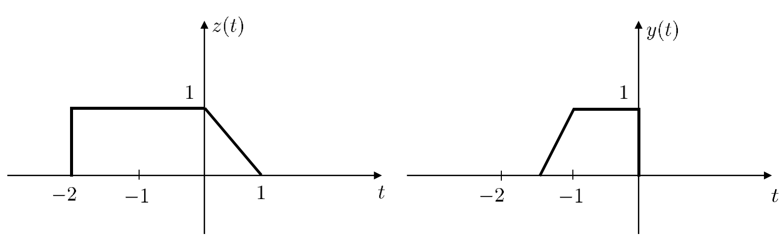
i. This is shifted left by 1, and then reversed and compressed by a factor of three.



ii. This is shifted right by 2, and then expanded by a factor of two.



   (b) (5 points) The figure below shows two signals: $z(t)$ and $y(t)$. Can you express $y(t)$ in terms of $z(t)$?



Solutions:

$y(t) = z(-2t-2)$

3. (22 points) **Periodic signals.**

(a) (14 points) For each of the following signals, determine whether it is periodic or not. If the signal is periodic, determine the fundamental period and frequency.

i. $x_1(t) = \sin(2t + \pi/3)$
ii. $x_2(t) = \cos(\sqrt{2}\pi t)$
iii. $x_3(t) = \sin^2(3\pi t + 3)$
iv. $x_4(t) = x_1(t) + x_2(t)$
v. $x_5(t) = x_1(\pi t) + x_3(t)$
vi. $x_6(t) = e^{-t}x_1(t)$
vii. $x_7(t) = e^{j(\pi t+1)}x_2(t)$

Solutions:

i. The signal is periodic with period is $2\pi/2 = \pi$ sec and the frequency is $1/\pi$ Hz.

ii. The signal is periodic with period is $2\pi/(\sqrt{2}\pi) = \sqrt{2}$ sec and the frequency is $1/\sqrt{2}$ Hz.

iii. $x_3(t) = \sin^2(3\pi t + 3) = \frac{1}{2}\left(1 - \cos(2(3\pi t + 3))\right) = \frac{1}{2}(1 - \cos(6\pi t + 6))$ sec, therefore the signal is periodic with period is $2\pi/(6\pi) = 1/3$ and the frequency is 3 Hz.

iv. $x_4(t) = x_1(t) + x_2(t)$: let T_1 denote the period of $x_1(t)$ and T_2 the period of T_2. If we can find integers m and n such that $mT_1 = nT_2$, $x_4(t)$ will then be periodic with period $T_4 = mT_1 = nT_2$. In other words, the ratio

$$\frac{T_1}{T_2} = \frac{n}{m}$$

need to be rational for $x_4(t)$ to be periodic. However, we have from part (i) $T_1 = \pi$ and from part (ii) $T_2 = \sqrt{2}$, so that

$$\frac{T_1}{T_2} = \frac{\pi}{\sqrt{2}}$$

The ration is not rational. Hence, $x_4(t)$ is not periodic.

v. $x_5(t) = x_1(\pi t) + x_3(t) = \sin(2\pi t + \pi/3) + x_3(t)$. The signal $x_1(\pi t) = \sin(2\pi t + \pi/3)$ is periodic with period $T_1' = 2\pi/(2\pi) = 1$ sec. From part (iii), we have $x_3(t)$ periodic with periodic $T_3 = 1/3$ sec. Now,

$$\frac{T_1'}{T_3} = \frac{1}{1/3} = 3$$

The ratio is rational, therefore $x_5(t)$ is periodic with period $T_5 = T_1' = 3T_3 = 1$ sec. The frequency is 1 Hz.

vi. $x_6(t) = e^{-t}x_1(t)$: this signal is not periodic since its magnitude decreases exponentially.

vii. $x_7(t) = e^{j(\pi t+1)}x_2(t) = e^{j(\pi t+1)} \times \cos(\sqrt{2}\pi t) = e^{j(\pi t+1)} \times \frac{1}{2}\left(e^{j\sqrt{2}\pi t} + e^{-j\sqrt{2}\pi t}\right)$. Therefore, $x_7(t)$ can be equivalently written as:

$$x_7(t) = \frac{1}{2}e^{j}\left(e^{j(\sqrt{2}+1)\pi t} + e^{j(-\sqrt{2}+1)\pi t}\right)$$

The term $e^{j(\sqrt{2}+1)\pi t}$ is periodic with period $2\pi/((\sqrt{2}+1)\pi) = 2/(\sqrt{2}+1)$. The second term $e^{j(-\sqrt{2}+1)\pi t}$ is periodic with period $2\pi/((-\sqrt{2}+1)\pi) = 2/(-\sqrt{2}+1)$. Since the ratio of these two periods is not rational, $x_7(t)$ is not periodic.

(b) (4 points) Assume that the signal $x(t)$ is periodic with period T_0, and that $x(t)$ is odd (i.e. $x(t) = -x(-t)$). What is the value of $x(T_0)$?

Solutions:

If $x(t)$ is odd, then $x(0) = 0$, since $x(0) = -x(0)$ for an odd function. Since $x(t)$ is periodic, it will also be zero at multiples of one period from $t = 0$, i.e.,

$$x(nT_0) = 0$$

In particular, $x(T_0) = 0$.

(c) (4 points) If $x(t)$ is periodic, are the even and odd components of $x(t)$ also periodic?

Solutions:

The even component of $x(t)$ is defined as:

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

If $x(t)$ is periodic with period T, then $x(-t)$ is periodic with period T, because $x(-(t+T)) = x(-t-T) = x(-t)$. The signal $x_e(t)$ is then the sum of two periodic signals of the same period T, which implies that $x_e(t)$ is also periodic.

Similarly, the odd component of $x(t)$ is defined as:

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

$x(t)$ and $x(-t)$ are both periodic, and have the same period T. Therefore, $x_o(t)$ is periodic.

4. (21 points) **Energy and power signals.**

(a) (15 points) Determine whether the following signals are energy or power signals. If the signal is an energy signal, determine its energy. If the signal is a power signal, determine its power.

i. $x(t) = e^{-|t|}$

ii. $x(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

iii. $x(t) = \begin{cases} 1 + e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Solutions:

i. $x(t) = e^{-|t|}$

The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| e^{-|t|} \right|^2 dt = \int_{-\infty}^{\infty} e^{-|2t|} dt = \int_{-\infty}^{0} e^{2t} dt + \int_{0}^{\infty} e^{-2t} dt$$
$$= 2 \int_{0}^{\infty} e^{-2t} dt = \left(-e^{-2t}\right) \Big|_{t=0}^{\infty} = 1$$

Therefore it's a energy signal, its power is then 0.

ii. $x(t) = \begin{cases} \frac{1}{\sqrt{t}}, & \text{if } t \geq 1 \\ 0, & \text{otherwise} \end{cases}$

The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{1}^{\infty} \left| 1/\sqrt{t} \right|^2 dt = \int_{1}^{\infty} \frac{1}{t} dt = \log(t) \Big|_{t=1}^{\infty} = \infty$$

Therefore it's not an energy signal.

On the other hand the power is given by:

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{1}^{T} \frac{1}{t} dt = \lim_{T \to \infty} \frac{1}{2T} \log(T) = 0$$

The signal has zero power, therefore it is not a power signal.

iii. $x(t) = \begin{cases} 1 + e^{-t}, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$

The energy is given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{0}^{\infty} \left| 1 + e^{-t} \right|^2 dt = \infty$$

Therefore it's not an energy signal.

On the other hand the power is:

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} (1 + e^{-t})^2 dt$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} \left(1 + 2e^{-t} + e^{-2t}\right) dt$$
$$= \lim_{T \to \infty} \frac{1}{2T} \left(T - 2(e^{-T} + 1) - \frac{1}{2}(e^{-2T} - 1) \right)$$
$$= \lim_{T \to \infty} \frac{1}{2T} T = \frac{1}{2}$$

(b) (6 points) Show the following two properties:

- If $x(t)$ is an even signal and $y(t)$ is an odd signal, then $x(t)y(t)$ is an odd signal;
- If $z(t)$ is an odd signal, then for any $\tau > 0$ we have:

$$\int_{-\tau}^{\tau} z(t)dt = 0$$

Use these two properties to show that the energy of $x(t)$ is the sum of the energy of its even component $x_e(t)$ and the energy of its odd component $x_o(t)$, i.e.,

$$E_x = E_{x_e} + E_{x_o}$$

Assume $x(t)$ is a real signal.

Solutions:
First property: $x(-t)y(-t) = x(t)(-y(t)) = -x(t)y(t)$, therefore it's odd.
Second property:

$$\int_{-\tau}^{\tau} z(t)dt = \int_{-\tau}^{0} z(t)dt + \int_{0}^{\tau} z(t)dt$$

We apply to the first integral the following variable change: $t = -\lambda$.

$$\int_{-\tau}^{\tau} z(t)dt = -\int_{\tau}^{0} z(-\lambda)d\lambda + \int_{0}^{\tau} z(t)dt$$

We then change the order of the limits of the first integral:

$$\int_{-\tau}^{\tau} z(t)dt = \int_{0}^{\tau} z(-\lambda)d\lambda + \int_{0}^{\tau} z(t)dt$$

Since $z(t)$ is an odd signal, we then have $z(-\lambda) = -z(\lambda)$. Thus,

$$\int_{-\tau}^{\tau} z(t)dt = -\int_{0}^{\tau} z(\lambda)d\lambda + \int_{0}^{\tau} z(t)dt = 0$$

The energy of signal $x(t)$ is given by:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x_e(t) + x_o(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left(x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t)\right) dt \\ &= \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt = E_e + E_o \end{aligned}$$

This is because $2x_e(t)x_o(t)$ is odd, therefore its integral is zero (according to the second property).

5. (17 points) **Euler's identity and complex numbers.**

(a) (9 points) Use Euler's formula to prove the following identities:

i. $\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$

ii. $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$

iii. $e^{j\alpha} + e^{j\beta} = 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{j\frac{\alpha+\beta}{2}}$

Solutions:

i. $\frac{d}{d\theta}\sin(\theta) = \frac{d}{d\theta}(e^{j\theta} - e^{-j\theta})/2j = (je^{j\theta} + je^{-j\theta})/2j = \cos(\theta)$

ii. $\sin^2(\theta) = \left(\frac{1}{2j}(e^{j\theta} - e^{-j\theta})\right)^2 = -\frac{1}{4}\left(-2 + e^{j2\theta} + e^{-j2\theta}\right) = -\frac{1}{4}\left(-2 + 2\cos(2\theta)\right) = \frac{1}{2}(1-\cos(2\theta))$

iii. $e^{j\alpha} + e^{j\beta} = e^{j\frac{\alpha+\beta}{2}}(e^{j\alpha - j\frac{\alpha+\beta}{2}} + e^{j\beta - j\frac{\alpha+\beta}{2}}) = e^{j\frac{\alpha+\beta}{2}}(e^{j\frac{\alpha-\beta}{2}} + e^{-j\frac{\alpha-\beta}{2}}) = 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{j\frac{\alpha+\beta}{2}}$

(b) (8 points) Let $x(t) = -(1+j)e^{j(1+2t)}$.

i. Compute the real and imaginary parts of $x(t)$.

ii. Compute the magnitude and phase of $x(t)$.

Solutions:

i. $x(t) = -(1+j)e^{j(1+2t)} = -(1+j)(\cos(1+2t) + j\sin(1+2t)) = -\cos(1+2t) + \sin(1+2t) + j(-\cos(1+2t) - \sin(1+2t))$.
Therefore, the real part is: $-\cos(1+2t) + \sin(1+2t)$. The imaginary part is:$-\cos(1+2t) - \sin(1+2t)$

ii. $x(t) = -(1+j)e^{j(1+2t)} = e^{j\pi}\sqrt{2}e^{j\pi/4}e^{j(1+2t)} = \sqrt{2}e^{j(2t+1+5\pi/4)}$. Therefore the magnitude is: $\sqrt{2}$ and the phase is $(2t+1+5\pi/4)$ rad.

6. (15 points) **MATLAB tasks**

(a) (2 points) **Task 1**

Plot the waveform

$$x(t) = e^{-t^2}\cos(2\pi t)$$

for $-5 \le t \le 5$, with a step size of 0.1. Label the time axis.

Solutions:

The code is:

```
t=-5:0.1:5;
x=exp(-t.^ 2).*cos(2*pi*t);
plot(t,x);
grid on;
title('Plot of x(t)=e^{-t^ 2}cos(2\pit)'); xlabel('t(sec)');ylabel('x(t)');
```

The code generates the plot shown in Fig. 1.

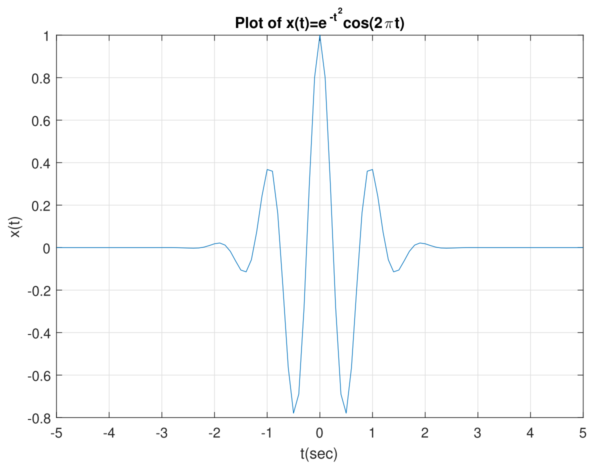


Figure 1: Task 1

(b) (3 points) **Task 2**

Create a vector x corresponding to the function given in Problem 1. Use a sample spacing of 0.01 over the range -2 to 2. Plot this vector. Properly label the time axis.

Solutions:

Code:

```
t1=-2:0.01:0.99; x1=ones(1,length(t1));
t2=1:0.01:2; x2=t2-2;
t=[t1 t2]; x=[x1 x2];
plot(t,x); grid on;
title(‘Plot of x(t) in problem 1’);
xlabel(‘t(sec)’);ylabel(’x(t)’);
axis([-2 2 -2 2]);
ax = gca; % current axes
ax.XAxisLocation = ‘origin’; %display axis lines through origin
ax.YAxisLocation = ‘origin’;
```

The code generates the plot shown in Fig. 2.

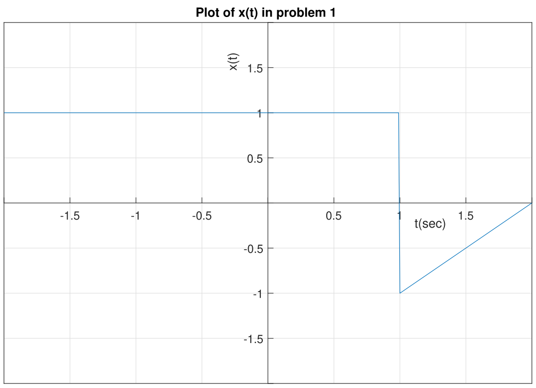


Figure 2: Task 2

(c) (4 points) **Task 3**

Create two vectors that represent the even and odd components of the vector x created in Task 2. A vector in MATLAB is reversed by

```
>> xrev = x(length(x):-1:1);
```

Plot the even and odd components of x.

Solutions:

Code:

```
t1=-2:0.01:0.99; x1=ones(1,length(t1));
t2=1:0.01:2; x2=t2-2;
t=[t1 t2]; x=[x1 x2];
xrev=x(length(x):-1:1);
xe=0.5*(x+xrev);
xo=0.5*(x-xrev);
figure(1);
plot(t,xe);grid on;
title('Plot of the even part'); xlabel('t(sec)');ylabel('x_e(t)');
axis([-2 2 -2 2]);
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
figure(2);
plot(t,xo);grid on;
title('Plot of the odd part'); xlabel('t(sec)');ylabel('x_o(t)'); axis([-2
2 -2 2]);
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```

The code generates the plot shown in Fig. 3 and 4.

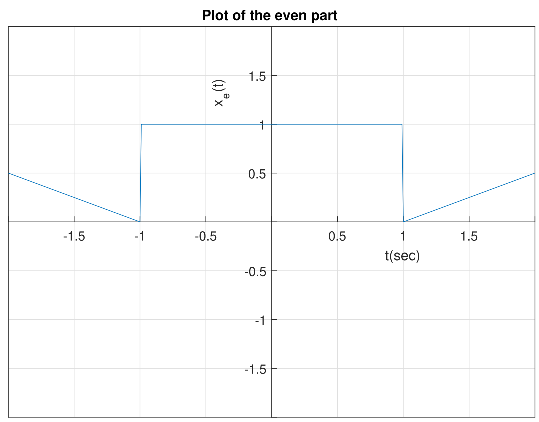


Figure 3: Task 3: Even Part

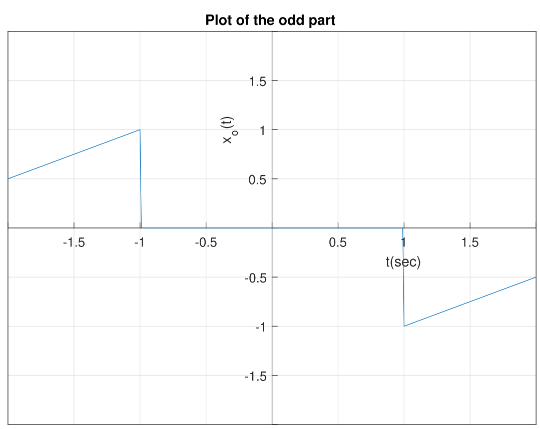


Figure 4: Task 3: Odd Part

(d) (6 points) **Task 4**

Consider the following three signals:

$$x_1(t) = \cos(2\pi t)$$
$$x_2(t) = \cos(60\pi t)$$
$$x_3(t) = x_1(t)x_2(t)$$

Plot the signals separately (you can use the function subplot) for $-3 \leq t \leq 3$, with a step size of 0.001.

Solutions:
Code:
```
t=-3:0.001:3;
x=cos(2*pi*t);y=cos(60*pi*t);z=x.*y;
subplot(3,1,1);
plot(t,x);
grid on;
title('Plot of x(t)=cos(2\pit)'); xlabel('t(sec)');ylabel('x');
subplot(3,1,2);
plot(t,y);
grid on;
title('Plot of y(t)=cos(60\pit)'); xlabel('t(sec)');ylabel('y');
subplot(3,1,3);
plot(t,z);
grid on;
title('Plot of z(t)=x(t)y(t)'); xlabel('t(sec)');ylabel('z');
```
The code generates the plot shown in Fig. 5.

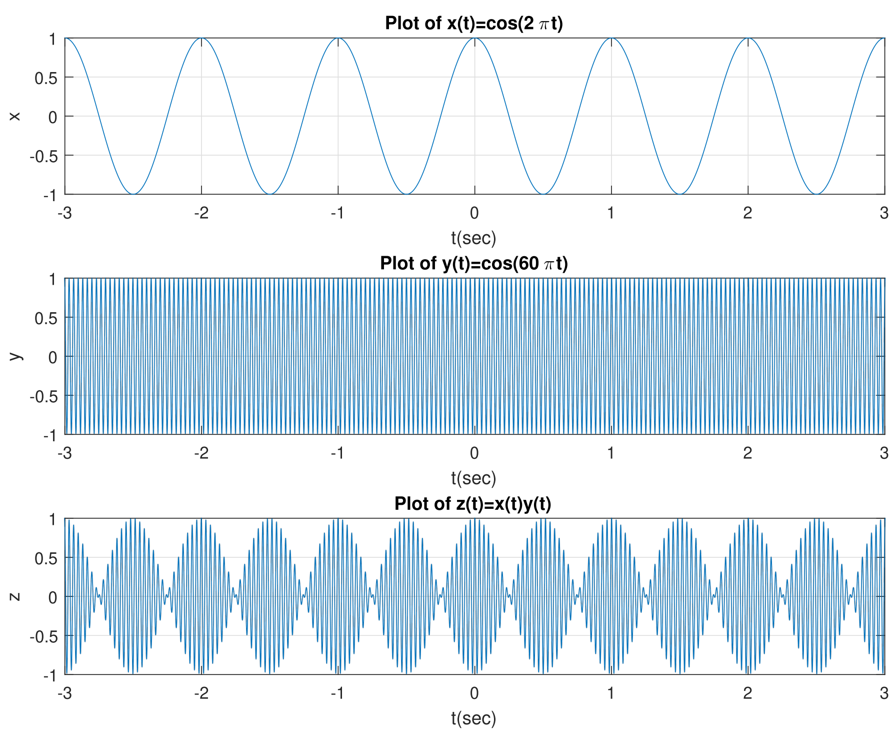


Figure 5: Task 4