ECE102, Fall 2018 **Homework #6**

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Due Thursday, 29 Nov 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 12.
100 points total.

1. (15 points) **More properties of Fourier transform**

   (a) (10 points) Use Parseval's theorem to evaluate the following integrals: $\int_{-\infty}^{\infty} \mathrm{sinc}^2(t)\cos(2\pi t)dt$

   Solution: Using the hint,

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{sinc}^2 t.\cos(2\pi t)dt &= \int_{-\infty}^{+\infty} \mathrm{sinc}^2 t.(2\cos^2(\pi t) - 1)dt \\ &= 2\int_{-\infty}^{+\infty} \mathrm{sinc}^2 t.\cos^2(\pi t)dt - \int_{-\infty}^{+\infty} \mathrm{sinc}^2 t dt\end{aligned}$$

   We can solve the last term as follows:

$$\int_{-\infty}^{+\infty} \mathrm{sinc}^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{rect}^2(\omega/2\pi)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\omega = \frac{2\pi}{2\pi} = 1$$

   This is because:

$$\mathrm{sinc}(t) \longrightarrow \mathrm{rect}(\omega/2\pi)$$

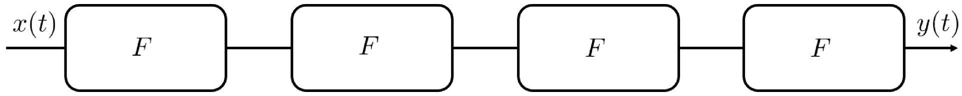
   The first integral is:

$$\begin{aligned}2\int_{-\infty}^{+\infty} \mathrm{sinc}^2(t).\cos^2(\pi t)dt &= 2\int_{-\infty}^{+\infty} (\mathrm{sinc}(t).\cos(\pi t))^2 dt \\ &= \frac{2}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{2}\left[\mathrm{rect}(\frac{\omega-\pi}{2\pi}) + \mathrm{rect}(\frac{\omega+\pi}{2\pi})\right]\right)^2 d\omega \\ &= \frac{1}{4\pi}\int_{-\infty}^{+\infty}[\mathrm{rect}(\frac{\omega-\pi}{2\pi}) + \mathrm{rect}(\frac{\omega+\pi}{2\pi})]^2 d\omega \\ &= \frac{1}{4\pi}(\int_{0}^{2\pi} d\omega + \int_{2\pi}^{0} d\omega) \\ &= \frac{1}{4\pi}(2\pi + 2\pi) \\ &= 1\end{aligned}$$

   Therefore,

$$\int_{-\infty}^{+\infty} \mathrm{sinc}^2(t).\cos^2(\pi t)dt = 2\int_{-\infty}^{+\infty} \mathrm{sinc}^2 t.\cos^2(\pi t)dt - \int_{-\infty}^{+\infty} \mathrm{sinc}^2 t dt = 1 - 1 = 0$$

(b) (5 points) Suppose we apply Fourier transform four times to signal $x(t)$ as shown below:



How is $y(t)$ related to $x(t)$?

Solution: Let $z(t)$ denote the intermediate signal that we obtain after applying Fourier transform twice. We know that:

$$x(t) \longrightarrow X(j\omega)$$

and by duality

$$X(t) \longrightarrow 2\pi x(-j\omega)$$

Therefore, $z(t) = 2\pi x(-t)$. Similarly, $y(t) = 2\pi z(-t)$, therefore, $y(t) = 4\pi^2 x(t)$.

2. (31 points) **Frequency Response**

(a) (18 points) Consider the LTI system depicted in figure 1 whose response to an unknown input, $x(t)$, is

$$y(t) = \left(4e^{-t} - 4e^{-4t}\right) u(t)$$

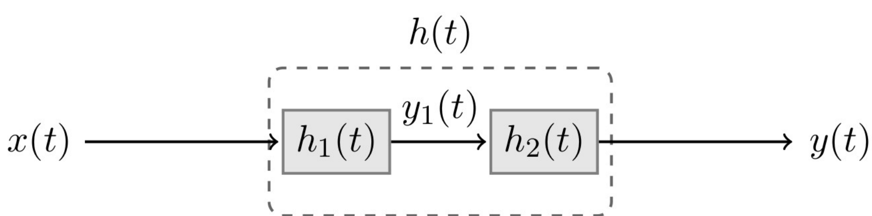


Figure 1: System for Problem 2.

We know that for the same unknown input $x(t)$, the intermediate signal, $y_1(t)$, is given by:

$$y_1(t) = 2e^{-t}u(t)$$

The overall LTI system is described by the following differential equation:

$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 3x(t)$$

i. Find the frequency response, $H(j\omega)$, of the overall system $h(t)$.

ii. Find the frequency responses $H_1(j\omega)$ of the first LTI system and $H_2(j\omega)$ of the second LTI system.

iii. Find the impulse responses $h(t)$, $h_1(t)$ and $h_2(t)$.

Solutions

i. Since the system is represented by the following differential equation:

$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 8y(t) = 3x(t)$$

Then,

$$(j\omega)^2 Y(j\omega) + 6(j\omega)Y(j\omega) + 8Y(j\omega) = 3X(j\omega)$$

Therefore

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{3}{(j\omega)^2 + 6(j\omega) + 8} = \frac{3}{(j\omega+4)(j\omega+2)}$$

ii. Let us first find the Fourier transform of the input $x(t)$ that corresponds to the output $y(t) = \left(4e^{-t} - 4e^{-4t}\right)u(t)$.
We have

$$Y(j\omega) = \frac{4}{j\omega+1} - \frac{4}{j\omega+4} = \frac{12}{(j\omega+1)(j\omega+4)}$$

Then,

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = \frac{\frac{12}{(j\omega+1)(j\omega+4)}}{\frac{3}{(j\omega+4)(j\omega+2)}} = 4\frac{j\omega+2}{j\omega+1}$$

Moreover, we know that the intermediate output $y_1(t) = 2e^{-t}u(t)$, then $Y_1(j\omega) = \frac{2}{j\omega+1}$. Therefore,

$$H_1(j\omega) = \frac{Y_1(j\omega)}{X(j\omega)} = \frac{1}{2}\frac{1}{j\omega+2}$$

Now we know that $h(t) = h_1(t) \star h_2(t)$, therefore,

$$H(j\omega) = H_1(j\omega)H_2(j\omega)$$

which implies

$$H_2(j\omega) = \frac{H(\omega)}{H_1(\omega)} = \frac{6}{j\omega+4}$$

$H_1(j\omega) = \frac{1}{2}\frac{1}{j\omega+2}$ then $h_1(t) = \frac{1}{2}e^{-2t}u(t)$.
$H_2(j\omega) = \frac{6}{j\omega+4}$, then $h_2(t) = 6e^{-4t}u(t)$.
$H(j\omega) = \frac{3}{(j\omega+4)(j\omega+2)} = \frac{A}{j\omega+4} + \frac{B}{j\omega+2}$, where $A = \frac{3}{-4+2} = -\frac{3}{2}$ and $B = \frac{3}{-2+4} = \frac{3}{2}$ then, $h(t) = \frac{3}{2}\left(-e^{-4t} + e^{-2t}\right)u(t)$

(b) (6 points) Assume $x(t)$ a real signal that is baseband, i.e., its Fourier transform $X(j\omega)$ is non-zero for $|\omega| \leq \omega_0$. We process this signal through an LTI system. Let $y(t)$ denote the corresponding output and let $Y(j\omega)$ denote the Fourier transform of $y(t)$. Does $y(t)$ have frequency components different than those of $x(t)$? i.e., is $Y(j\omega) = 0$ for some $|\omega| > \omega_0$? What if we process $x(t)$ through a non-LTI system?

Solution: When the system is LTI, we have:

$$y(t) = h(t) * x(t) \rightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

Therefore, if $X(j\omega) = 0$ for $|\omega| > \omega_0$, then $Y(j\omega) = 0$ for $|\omega| > \omega_0$. Therefore, processing a signal through an LTI system does not introduce any new frequency components in it.

On the other hand, consider the following non-LTI system:

$$y(t) = x(2t)$$

In this case,

$$Y(j\omega) = \frac{1}{2}X\left(j\frac{\omega}{2}\right)$$

Therefore, this non-LTI system expands the signal in the frequency domain, so that $Y(j\omega)$ is non zero for $\omega_0 \leq |\omega| \leq 2\omega_0$.

(c) (7 points) Consider the following two LTI systems with impulse responses:

$$h_1(t) = \text{sinc}\left(\frac{t}{2}\right)\cos(\pi t)$$

and

$$h_2(t) = 2\ \text{sinc}\,(2t)$$

Find the output of each system to the following input $x(t) = \cos(3\pi t)\cos(4\pi t)$. If we are given an input-output pair of an unknown LTI system, can we always identify this system?

Solution: The signal $x(t)$ is given by:

$$x(t) = \cos(3\pi t)\cos(4\pi t) = \frac{1}{2}\left(\cos(7\pi t) + \cos(\pi t)\right)$$

To compute the output of each system, we are going to use the eigenfunction property:

$$\text{Input: } e^{j\omega_0 t} \rightarrow \text{Output: } H(j\omega_0)e^{j\omega_0 t} = |H(j\omega_0)|e^{j(\omega_0 t + \angle H(j\omega_0))}$$

where $H(j\omega)$ is the frequency response of the system. Now when the input is $\cos(\omega_0 t)$ and when $h(t)$ is real, this same property reduces to the following:

$$\text{Input: } \cos(\omega_0 t) \rightarrow \text{Output: } |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

You can find the proof of this property in practice problems 6.

The frequency response of the first system is:

$$H_1(j\omega) = \text{rect}\left(\frac{\omega - \pi}{\pi}\right) + \text{rect}\left(\frac{\omega + \pi}{\pi}\right)$$

Since $H_1(7\pi) = 0$ and $H_1(\pi) = 1$, the output to $x(t)$ is:

$$y_1(t) = \frac{1}{2}\cos(\pi t)$$

The frequency response of the first system is:

$$H_2(j\omega) = \text{rect}\left(\frac{\omega}{4\pi}\right)$$

Since $H_2(7\pi) = 0$ and $H_2(\pi) = 1$, the output to $x(t)$ is:

$$y_2(t) = \frac{1}{2}\cos(\pi t)$$

We see that $y_1(t) = y_2(t)$. Therefore, it is not always possible to identify an LTI system from some of its input-output pair, because for the same periodic signal, different LTI systems can exhibit the same response.

3. (17 points) **Filters**

   (a) (5 points) Consider an ideal low-pass filter $h_{LP,1}(t)$ with frequency response $H_{LP,1}(j\omega)$ depicted below in figure 2.

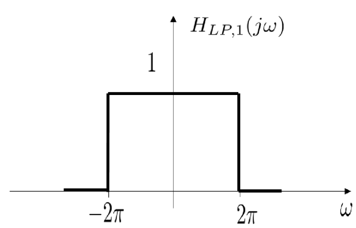


Figure 2: An ideal low pass filter

Using this filter, we construct the following new system:

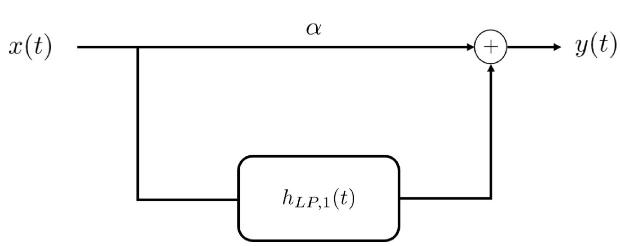


Figure 3: New system

We are given two choices for α: 1 or -1. Which value should we choose so that the new system is a high-pass filter? Does the new filter have any phase in its frequency response?

Solution: The equivalent system has the following impulse response:

$$h_{eq}(t) = \alpha\delta(t) + h_{LP,1}(t)$$

Therefore,

$$H_{eq}(j\omega) = \alpha + H_{LP,1}(j\omega)$$

To obtain a high-pass filter, α should be chosen as -1. In this case,

$$H_{eq}(j\omega) = -1 + H_{LP,1}(j\omega) = \begin{cases} -1, & |\omega| \geq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

It has a phase of π, because it has a negative value for all ω.

(b) (3 points) Why are the ideal filters non-realizable systems?

Solution: They are non-realizable because they are non-causal. Moreover, the filter impulse response has an infinite duration, and thus to convolve would take an infinite amount of time.

Note: Ideal filters are also unstable which make them non-realizable. (You won't be penalized if you do not mention this).

(c) (5 points) We want to design a causal non-ideal low-pass filter $h_{LP,2}(t)$, using the following frequency response:

$$H_{LP,2}(j\omega) = \frac{k}{\beta + j\omega}$$

Find k and β so that $H_{LP,2}(j\omega)$ is unity for $\omega = 0$ and its cutoff frequency is $\omega_0 = 2\pi$ rad/s, (i.e., the magnitude of $H_{LP,2}(j\omega)$ is $1/\sqrt{2}$ for $\omega = 2\pi$ rad/s).

Solution: We want:

$$H_{LP,2}(0) = 1 \implies k = \beta$$

Moreover,

$$|H_{LP,2}(j2\pi)|^2 = 1/2 \implies \frac{\beta^2}{\beta^2 + 4\pi^2} = \frac{1}{2}$$

Thus, $\beta^2 = 4\pi^2$. We chose $\beta = 2\pi$ and not -2π, because in this case: $h_2(t) = 2\pi e^{-2\pi t}u(t)$, which is a causal system as required.

Note: If we instead chose $\beta = -2\pi$, then $\dfrac{-2\pi}{-2\pi + j\omega} = \dfrac{2\pi}{2\pi - j\omega}$ which gives us in the time domain: $2\pi e^{2\pi t}u(-t)$ (non-causal impulse response $\implies$ non-causal system).

(d) (4 points) We again consider the system of part (a) where instead of the ideal low-pass filter, we are going to use the non-ideal low-pass filter of part (c).

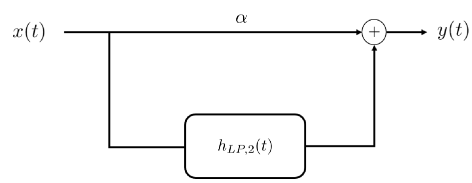


Figure 4: The system of part (a) with the non-ideal low pass filter

For the same value of α you found in part (a), find the frequency response of the equivalent system. Explain if the new system behaves as a high-pass filter.

Solution: The frequency response is given by:

$$H_{eq}(j\omega) = -1 + \frac{2\pi}{2\pi + j\omega} == \frac{-j\omega}{2\pi + j\omega}$$

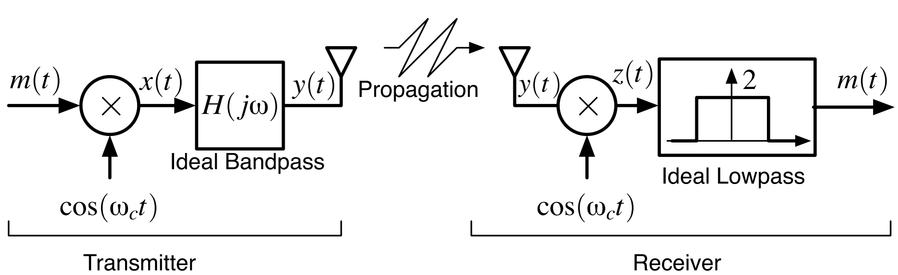
Therefore, the magnitude is given:

$$|H_{eq}(j\omega)| = \sqrt{\frac{\omega^2}{4\pi^2 + \omega^2}}$$

When $\omega \gg 2\pi$, $|H_{eq}(j\omega)| \approx 1$. When $\omega \approx 0$, $|H_{eq}(j\omega)| \approx 0$. This is why it behaves like a high pass filter.

4. (25 points) **Modulation and Demodulation**

(a) (15 points) Consider the communication system shown below:

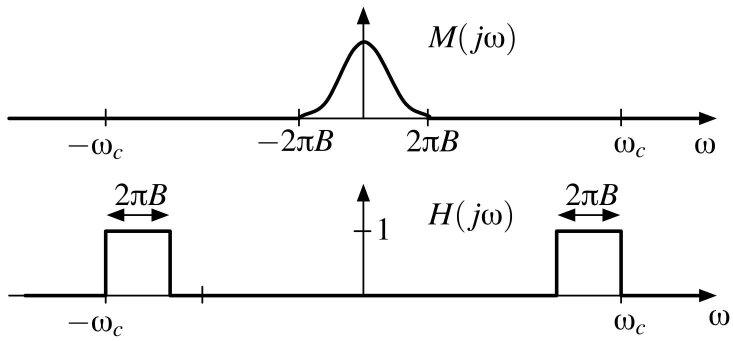


The signal $m(t)$ is first modulated by $\cos(\omega_c t)$, and then passed through an ideal bandpass filter. The spectrum of the input $M(j\omega)$ and the frequency response of the ideal bandpass filter $H(j\omega)$ are:

The modulated signal is $x(t)$, and the output of the ideal bandpass is $y(t)$. This signal is transmitted through a channel. We assume that this channel does not introduce distortion into $y(t)$. The received signal $y(t)$ is then processed by a receiver. Sketch the signal spectrum at

i. the output of the modulator, i.e., $X(j\omega)$,

ii. the output of the ideal bandpass, $Y(j\omega)$, and

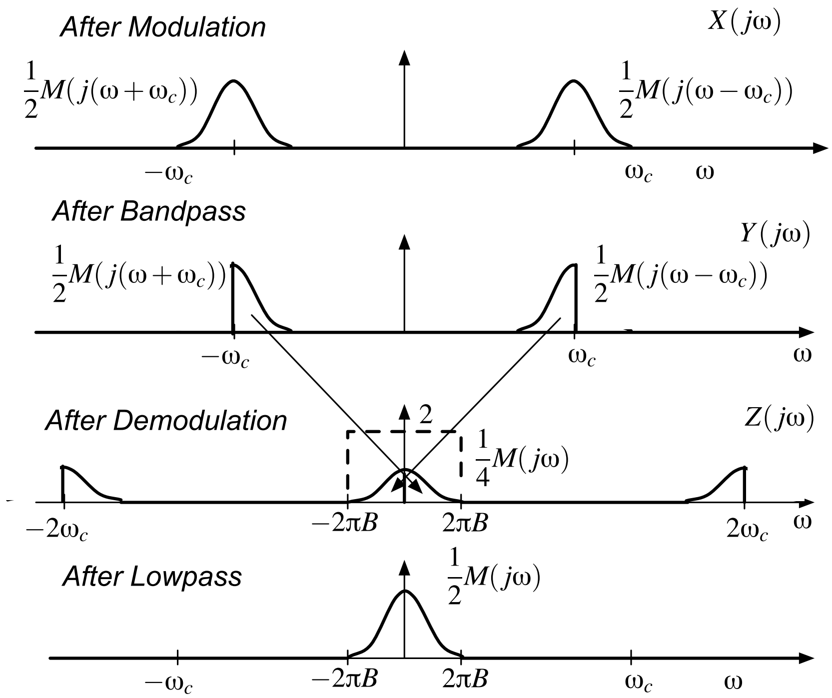


iii. the output of the demodulator, $Z(j\omega)$

Does this system recover $m(t)$?

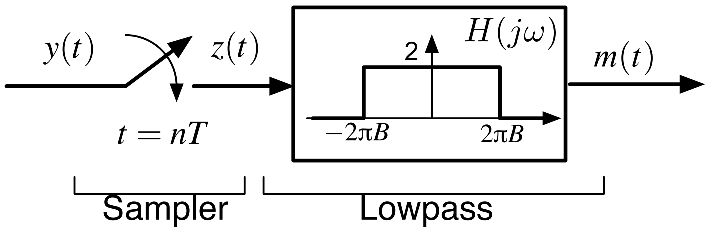
Solution:

The spectrum at the different point in the system are



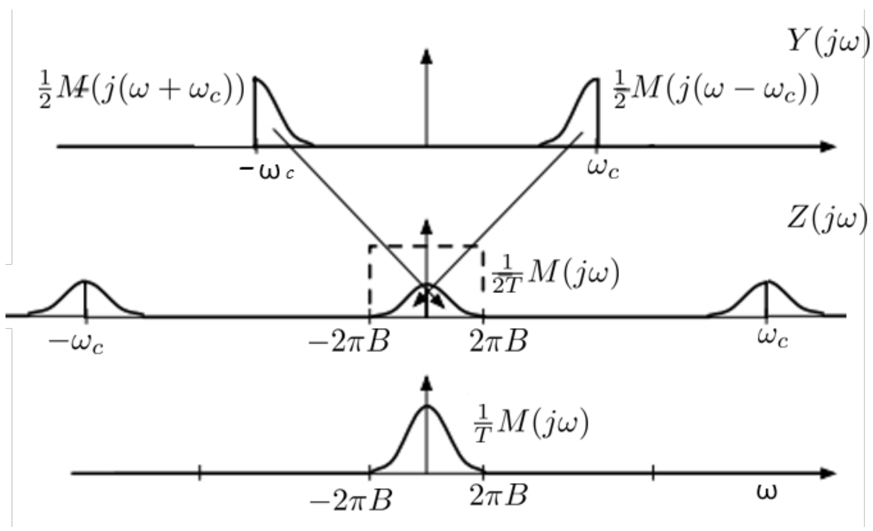
So $m(t)$ is recovered (except for a factor of 2).

(b) (10 points) In the first part of this problem, you have seen that to demodulate the received signal, we multiply $y(t)$ by $\cos(\omega_c t)$, and then to recover $m(t)$, we low-pass filter the result. In this part, you will show that you can achieve the same effect with an ideal sampler. In other words, we assume instead the following block diagram of the receiver: where the ideal sampler is drawn as a switch that closes instantaneously



every T seconds to acquire a new sample. Show that we can recover $m(t)$ if the ideal sampler operates at a frequency ω_c (i.e. samples at a rate of $\omega_c/2\pi$ samples/s). Draw the spectrum of the signal right before the lowpass filter $Z(j\omega)$.

Solution: After sampling $y(t)$, the spectrum of $Y(j\omega)$ get replicated every ω_c, as depicted below. Therefore, $m(t)$ is recovered with a factor of T.



5. (12 points) **Sampling**

(a) Assume $x(t)$ a real bandlimited signal where $X(j\omega)$ is non-zero for $|\omega| \leq 2\pi B$ rad/s. If F_s Hz is the Nyquist rate of $x(t)$, determine the Nyquist rate in Hz of the following signals in terms of B:

i. $x(t-1)$

Solution: If $x_1(t) = x(t-1)$ therefore,

$$X_1(j\omega) = e^{-j\omega}X(j\omega)$$

Therefore if $X_1(j\omega)$ is nonzero for the same ω, i.e., for $|\omega| \leq 2\pi B$. Therefore we have the same Nyquist rate which is $2B$ Hz.

ii. $\cos(2\pi Bt)x(t)$

Solution: If $x_2(t) = \cos(2\pi Bt)x(t)$, then $X_2(j\omega)$ is non-zero for:

$$-4\pi B \leq \omega \leq 4\pi B$$

Therefore the highest frequency in the new signal is $2B$ Hz. Therefore the Nyquist rate is: $2(2B) = 4B$.

iii. $x(t) + x(\frac{t}{2})$

Solution: If $x_3(t) = x(t) + x(\frac{t}{2})$ therefore,

$$X_3(j\omega) = X(j\omega) + 2X(j2\omega)$$

$X(j\omega)$ is no-zero for $|\omega| \leq 2\pi B$, and $X(j2\omega)$ is non-zero for $|\omega| \leq \pi B$. Therefore $X_3(j\omega)$ is nonzero for $|\omega| \leq 2\pi B$. Thus, we still have the sane Nyquist rate: $2B$.