

Due Thursday, 1 Nov 2018, by 11:59pm to Gradescope.

Covers material up to Lecture 8.

100 points total.

1. (28 points) **Fourier Series**

(a) (18 points) Find the Fourier series coefficients for each of the following periodic signals:

i. $f(t) = \cos(3\pi t) + \frac{1}{2} \sin(4\pi t)$

Solution:

We first find the period of $f(t)$. The first term $\cos(3\pi t)$ is periodic with period $T_1 = \frac{2\pi}{3\pi} = \frac{2}{3}$. The second term $\sin(4\pi t)$ is periodic with period $T_2 = \frac{2\pi}{4\pi} = \frac{1}{2}$. Since $\frac{T_1}{T_2} = \frac{4}{3}$, $f(t)$ is then periodic with fundamental period $T_0 = 3T_1 = 4T_2 = 2$ sec, and fundamental frequency $\omega_0 = \frac{2\pi}{T_0} = \pi$ rad/s.

Using Euler's identity, $f(t)$ can be equivalently written as:

$$\begin{aligned} f(t) &= \cos(3\pi t) + \frac{1}{2} \sin(4\pi t) = \frac{1}{2} (e^{j3\pi t} + e^{-j3\pi t}) + \frac{1}{4j} (e^{j4\pi t} - e^{-j4\pi t}) \\ &= \frac{1}{2} e^{j3\pi t} + \frac{1}{2} e^{-j3\pi t} + \frac{-j}{4} e^{j4\pi t} + \frac{j}{4} e^{-j4\pi t} \end{aligned}$$

The fundamental frequency of $f(t)$ is $\omega_0 = \pi$, and since any periodic signal can be written as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 k t}$$

we deduce for $f(t)$ the following Fourier series coefficients:

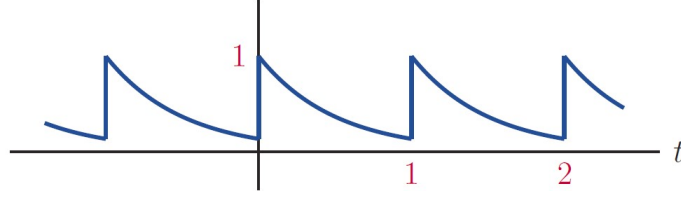
$$c_k = \begin{cases} \frac{1}{2}, & \text{if } k = -3, 3 \\ \frac{-j}{4}, & \text{if } k = 4 \\ \frac{j}{4}, & \text{if } k = -4 \\ 0, & \text{otherwise} \end{cases}$$

ii. $f(t)$ is a periodic signal with period $T = 1$ s, where one period of the signal is defined as e^{-2t} for $0 < t < 1$ s, as shown below.

Solution:

Since $f(t)$ is periodic with period $T_0 = 1$ s, we can rewrite it as:

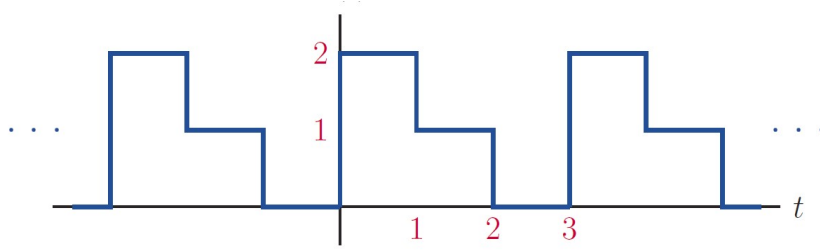
$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$



where $\omega_0 = \frac{2\pi}{T_0} = 2\pi$ rad/s and the coefficients c_k 's are as follows:

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T f(t) e^{-jk\omega_0 t} dt = \int_0^1 e^{-2t} e^{-j2k\pi t} dt \\ &= \frac{1 - e^{-(2+j2\pi k)}}{2 + j2\pi k} = \frac{1 - e^{-2}}{2 + j2\pi k} \end{aligned}$$

iii. $f(t)$ is the periodic signal shown below:



Solution:

Since $f(t)$ is periodic with period $T_0 = 3$ s, we can rewrite it as:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{3}$ rad/s and the coefficients c_k 's are as follows:

$$c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{3} \left(\int_0^1 2 dt + \int_1^2 1 dt \right) = 1$$

and for $k \neq 0$, we have:

$$\begin{aligned} c_k &= \frac{1}{T} \int_0^T f(t) e^{-jk\omega_0 t} dt = \frac{1}{3} \left(\int_0^1 2 e^{-j(2\pi/3)kt} dt + \int_1^2 e^{-j(2\pi/3)kt} dt \right) \\ &= \frac{1}{3} \left(2 \frac{1 - e^{-j(2\pi/3)k}}{j(2\pi/3)k} + \frac{e^{-j(2\pi/3)k} - e^{-j(4\pi/3)k}}{j(2\pi/3)k} \right) = \frac{2 - e^{-j(2\pi/3)k} - e^{-j(4\pi/3)k}}{j2\pi k} \\ &= \frac{2 - e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}}{j2\pi k} = \frac{2 - 2 \cos\left(\frac{2\pi k}{3}\right)}{j2\pi k} = \frac{1 - \cos\left(\frac{2\pi k}{3}\right)}{j\pi k} \end{aligned}$$

- (b) (10 points) Suppose you have two periodic signals $x(t)$ and $y(t)$, of periods T_1 and T_2 respectively. Let x_k and y_k be the Fourier series coefficients of $x(t)$ and $y(t)$.

- i. If $T_1 = T_2$, express the Fourier series coefficients of $z(t) = x(t) + y(t)$ in terms of x_k and y_k .

Solution:

If $T_1 = T_2$, then $y(t)$ is also periodic with period $T_0 = T_1 = T_2$. If $\omega_0 = \frac{2\pi}{T_0}$, then

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}$$

and

$$y(t) = \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t}$$

Therefore,

$$z(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} y_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} (x_k + y_k) e^{jk\omega_0 t}$$

Therefore, the Fourier series coefficients of $z(t)$ are:

$$z_k = x_k + y_k$$

- ii. If $T_1 = 2T_2$, express the Fourier series coefficients of $w(t) = x(t) + y(t)$ in terms of x_k and y_k .

Solution: First of all, $w(t)$ is periodic with period $T_0 = 2T_2 = T_1$, and frequency $\omega_0 = \omega_1 = \frac{1}{2}\omega_2$. Let,

$$x(t) = \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_1 t} = \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_0 t}$$

and

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_2 t} = \sum_{n=-\infty}^{\infty} y_n e^{j2n\omega_0 t}$$

Therefore, $w(t)$ can be written as:

$$w(t) = x(t) + y(t) = \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_0 t} + \sum_{n=-\infty}^{\infty} y_n e^{j2n\omega_0 t}$$

Let $m' = 2n$, then

$$\begin{aligned} w(t) &= \sum_{m=-\infty}^{\infty} x_m e^{jm\omega_0 t} + \sum_{\text{even } m'} y_{\frac{m'}{2}} e^{jm'\omega_0 t} \\ &= \sum_{\text{even } m} x_m e^{jm\omega_0 t} + \sum_{\text{odd } m} x_m e^{jm\omega_0 t} + \sum_{\text{even } m'} y_{\frac{m'}{2}} e^{jm'\omega_0 t} \end{aligned}$$

Therefore,

$$w_k = \begin{cases} x_k, & \text{for } k \text{ odd} \\ x_k + y_{\frac{k}{2}}, & \text{for } k \text{ even} \end{cases}$$

2. (20 points) **Fourier series of transformation of signals**

Suppose that $f(t)$ is a periodic signal with period T_0 , with the following Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Determine the period of each of the following signals, then express its Fourier series in terms of c_k :

(a) $g(t) = f(t) + 1$

Solution:

The function $g(t)$ has the same period of $f(t)$. Adding a constant to a signal will only affect the Fourier coefficient c_k for $k = 0$. This can be seen as follows:

$$g(t) = f(t) + 1 = c_0 + 1 + \sum_{k \neq 0} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega_0 t}$$

where,

$$c'_k = \begin{cases} c_k, & \text{for } k \neq 0 \\ c_0 + 1, & \text{for } k = 0 \end{cases}$$

(b) $g(t) = f(-t)$

Solution:

$g(t)$ has the same period of $f(t)$.

$$g(t) = f(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_{-k} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c'_k e^{jk\omega_0 t}$$

Therefore $c'_k = c_{-k}$

(c) $g(t) = f(t - t_0)$

Solution:

$g(t)$ has the same period of $f(t)$.

$$g(t) = f(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} (c_k e^{-jk\omega_0 t_0}) e^{jk\omega_0 t}$$

Therefore $c'_k = c_k e^{-jk\omega_0 t_0}$

- (d) $g(t) = f(at)$, where a is positive real number

Solution:

The period of $g(t)$ is $T'_0 = \frac{T_0}{a}$, and its corresponding frequency is: $\omega'_0 = a\omega_0$. Therefore,

$$g(t) = f(at) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0(at)} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega'_0 t}$$

Therefore, the Fourier series coefficients of $f(t)$ and $g(t)$ are the same.

3. (10 points) **Eigenfunctions and LTI systems**

- (a) (5 points) Show that $f(t) = \cos(\omega_0 t)$ is not an eigenfunction of an LTI system.

Solution:

Assume that $h(t)$ is the impulse response of the system. Then the output $y(t)$ to input $f(t) = \cos(\omega_0 t) = \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})$ is as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{j\omega_0(t-\tau)}h(\tau)d\tau + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j\omega_0(t-\tau)}h(\tau)d\tau \\ &= \frac{1}{2} e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{-j\omega_0 \tau} h(\tau) d\tau}_{=a_1} + \frac{1}{2} e^{-j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} e^{j\omega_0 \tau} h(\tau) d\tau}_{=a_2} \end{aligned}$$

For $f(t)$ to be an eigenfunction for the system, its corresponding output should be of the form $af(t)$, where a is constant. The output to $\cos(\omega_0 t)$ is:

$$y(t) = \frac{1}{2} a_1 e^{j\omega_0 t} + \frac{1}{2} a_2 e^{-j\omega_0 t}$$

Since, in general $a_1 \neq a_2$, we cannot construct again $\cos(\omega_0 t)$ in $y(t)$. For instance, suppose $f(t) = \delta(t-4)$, then $a_1 = e^{-j4\omega_0}$ and $a_2 = e^{j4\omega_0}$. Therefore,

$$y(t) = \frac{1}{2} e^{j\omega_0(t-4)} + \frac{1}{2} e^{-j\omega_0(t-4)} = \cos(\omega_0(t-4))$$

We then see the output is not of the form $a \cos(\omega_0 t)$, therefore $\cos(\omega_0 t)$ is not an eigenfunction for an LTI system.

- (b) (5 points) Show that $f(t) = t$ is not an eigenfunction of an LTI system.

Solution:

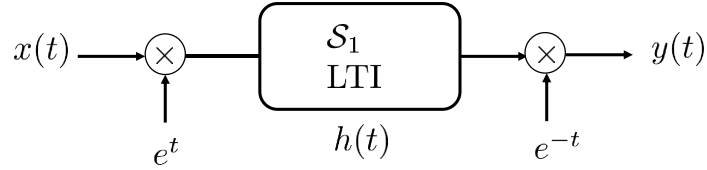
Assume that $h(t)$ is the impulse response of the system. Then the output $y(t)$ to input $f(t) = t$ is as follows:

$$y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} (t-\tau)h(\tau)d\tau = t \underbrace{\int_{-\infty}^{\infty} h(\tau)d\tau}_{=a_1} - \underbrace{\int_{-\infty}^{\infty} \tau h(\tau)d\tau}_{=a_2}$$

$y(t)$ is of the form $a_1t + a_2$, therefore the function $f(t) = t$ is not an eigenfunction of an LTI system.

4. (29 points) **LTI systems**

Consider the following system:



The system takes as input $x(t)$, it first multiplies the input with e^t , then sends it through an LTI system. The output of the LTI system gets multiplied by e^{-t} to form the output $y(t)$.

(a) Show that we can write $y(t)$ as follows:

$$y(t) = [(e^t x(t)) * h(t)] e^{-t} \quad (1)$$

Solution:

The input $x(t)$ gets first multiplied by e^t and forms the intermediate signal:

$$y_1(t) = e^t x(t)$$

Next, $y_1(t)$ is fed to the LTI system, the output $y_2(t)$ is then the convolution of $y_1(t)$ with $h(t)$:

$$y_2(t) = y_1(t) * h(t) = (e^t x(t)) * h(t)$$

Finally, $y_2(t)$ gets finally multiplied by e^{-t} :

$$y(t) = e^{-t} y_2(t) = [(e^t x(t)) * h(t)] e^{-t}$$

(b) Use the definition of convolution to show that (1) can be equivalently written as:

$$y(t) = \int_{-\infty}^{\infty} h'(\tau) x(t - \tau) d\tau \quad (2)$$

where $h'(t)$ is a function to define in terms of $h(t)$.

Solution:

By applying the definition of convolution, we obtain:

$$\begin{aligned}
y(t) &= [(e^t x(t)) * h(t)] e^{-t} \\
&= e^{-t} \int_{-\infty}^{\infty} h(\tau) e^{t-\tau} x(t-\tau) d\tau \\
&= e^{-t} e^t \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) e^{-\tau} x(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h'(\tau) x(t-\tau) d\tau
\end{aligned}$$

where $h'(\tau) = h(\tau) e^{-\tau}$.

- (c) Equation (2) represents a description of the equivalent system that maps $x(t)$ to $y(t)$. Show using (2) that the equivalent system is LTI and determine its impulse response $h_{eq}(t)$ in terms of $h(t)$.

Solution:

Linearity:

Suppose that for inputs $x_1(t)$ and $x_2(t)$, we have respectively the corresponding outputs $y_1(t)$ and $y_2(t)$ outputs. Now, let $x(t) = ax_1(t) + bx_2(t)$, we then have the following:

Method 1: Using the equation from part b:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h'(\tau) x(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} h'(\tau) (ax_1(t-\tau) + bx_2(t-\tau)) d\tau \\
&= \int_{-\infty}^{\infty} (ah'(\tau) x_1(t-\tau) + bh'(\tau) x_2(t-\tau)) d\tau \\
&= \int_{-\infty}^{\infty} ah'(\tau) x_1(t-\tau) d\tau + \int_{-\infty}^{\infty} bh'(\tau) x_2(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} ah'(\tau) x_1(t-\tau) d\tau + \int_{-\infty}^{\infty} bh'(\tau) x_2(t-\tau) d\tau \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Method 2:

$$\begin{aligned}
y(t) &= [(e^t x(t)) * h(t)] e^{-t} \\
&= [(e^t (ax_1(t) + bx_2(t))) * h(t)] e^{-t} \\
&= [(ae^t x_1(t) + be^t x_2(t)) * h(t)] e^{-t} \\
&= [(ae^t x_1(t)) * h(t) + (be^t x_2(t)) * h(t)] e^{-t} \\
&= [(ae^t x_1(t)) * h(t)] e^{-t} + [(be^t x_2(t)) * h(t)] e^{-t} \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

Therefore system is linear.

Time invariance:

Using result from part b, if we delay the input for t_0 :

$$\begin{aligned}y_{t_0}(t) &= \int_{-\infty}^{\infty} h'(\tau)x(t - \tau - t_0)d\tau \\&= \int_{-\infty}^{\infty} h'(\tau)x(t - t_0 - \tau)d\tau \\&= y(t - t_0)\end{aligned}$$

Therefore system is TI. From part b, we know that $h'(t) = h(t)e^{-t}$. Therefore, the impulse response of the equivalent system is:

$$h_{eq}(t) = h(t)e^{-t}$$

- (d) Suppose that system \mathcal{S}_1 is given by its step response $s(t) = r(t - 1)$. Find the impulse response $h(t)$ of \mathcal{S}_1 . What can you say about the causality and stability of system \mathcal{S}_1 ? What can you say about the causality and stability of the overall equivalent system?

Solution:

The impulse response of system \mathcal{S}_1 is:

$$h(t) = \frac{d}{dt}s(t) = u(t - 1)$$

Since $h(t) = 0$ for $t < 0$, the system \mathcal{S}_1 is causal. However, this same system is not stable because

$$\int_{-\infty}^{\infty} |h(t)|dt \rightarrow \infty$$

The equivalent system has the following equivalent impulse response:

$$h_{eq}(t) = e^{-t}u(t - 1)$$

Since $h_{eq}(t) = 0$ for $t < 0$, the system is causal. It is also stable, because:

$$\int_{-\infty}^{\infty} |h_{eq}(t)|dt = \int_{t=1}^{\infty} e^{-t}dt = e^{-1} < \infty$$

5. (13 points) **MATLAB**

- (a) (6 points) **Task 1**

Write an m-file that takes a set of Fourier series coefficients, a fundamental frequency, and a vector of output times, and computes the truncated Fourier series evaluated at these times. The declaration and help for the m-file might be:

```
function fn = myfs(Dn,omega0,t)
%
```



```
% fn = myfs(Dn,omega0,t)
% % Evaluates the truncated Fourier Series at times t
%
% Dn -- vector of Fourier series coefficients
%
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
%
% fn -- truncated Fourier series evaluated at t
The output of the m-file should be
```

$$f_N(t) = \sum_{n=-N}^N D_n e^{j\omega_0 n t}$$

The length of the vector Dn should be $2N + 1$. You will need to calculate N from the length of Dn.

Solution:

```
function fn = myfs(Dn,omega0,t)
% fn = myfs(Dn,omega0,t)
% Evaluates the truncated Fourier Series at times t
% Dn -- vector of Fourier series coefficients
% assumed to run from -N:N, where length(Dn) is 2N+1
% omega0 -- fundamental frequency
% t -- vector of times for evaluation
% fn -- truncated Fourier series evaluated at t
N = (length(Dn)-1)/2;
fn = zeros(size(t));
for n = -N:N
    D_n = Dn(n+N+1);
    fn = fn + D_n*exp(j*omega0*n*t);
end
```

(b) (7 points) **Task 2**

Repeat the steps of Task 2 for the case of the signal from Problem 1-a-iii.

```
N=10;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);  
omega0=2*pi;t=-3.5:0.001:3.5;  
fn = myfsHs(D,omega0,t);  
subplot(3,1,1);  
plot(t,fn);  
xlabel('t(sec)'); ylabel('N=10');
```

```
N=50;n=-N:1:N;D=(1-exp(-2))./(2+2*j*pi*n);  
omega0=2*pi;t=-3.5:0.001:3.5;  
fn = myfsHs(D,omega0,t);  
subplot(3,1,2);  
plot(t,fn);  
xlabel('t(sec)'); ylabel('N=50');
```

```
N=100;n=-N:1:N; D=(1-exp(-2))./(2+2*j*pi*n);  
omega0=2*pi;t=-3.5:0.001:3.5;fn = myfsHs(D,omega0,t);  
subplot(3,1,3);  
plot(t,fn);  
xlabel('t(sec)'); ylabel('N=100');
```

