ECE102, Fall 2018 — **Homework #5**

Signals & Systems — Prof. J.C. Kao

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Due Thursday, 8 Nov 2018, by 11:59pm to Gradescope.
Covers material up to Lecture 10.
100 points total.

1. (18 points) **Fourier Series**

(a) (7 points) When the periodic signal $f(t)$ is real, you have seen in class some properties of symmetry for the Fourier series coefficients of $f(t)$ (handout 8, slide 41). How do these properties of symmetry change when $f(t)$ is pure imaginary?

Solution: Since $f(t)$ is pure imaginary, it can equivalently written as $f(t) = jg(t)$, where $g(t)$ is real. Using the equations in slide 8-40,

$$\begin{aligned}
c_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\left[\cos\left(\frac{2\pi k}{T_0}t\right) - j\sin\left(\frac{2\pi k}{T_0}t\right)\right] dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} jg(t)\cos\left(\frac{2\pi k}{T_0}t\right) + g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
&= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt + j\frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

Now, because $g(t)$ is real:

$$\begin{aligned}
Re(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\sin\left(\frac{2\pi k}{T_0}t\right) dt \\
Im(c_k) &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} g(t)\cos\left(\frac{2\pi k}{T_0}t\right) dt
\end{aligned}$$

Therefore,

$$\begin{aligned}
Re(c_k) &= -Re(c_{-k}) \\
Im(c_k) &= Im(c_{-k}) \\
c_k^* &= -c_{-k} \\
|c_k| &= |c_{-k}| \\
-\angle c_k &= \angle c_{-k} \pm \pi
\end{aligned}$$

(b) (7 points) A *real* and *even* signal $x(t)$ has the following properties:

- it is a periodic signal with period 1 s;

- it has a DC component of 1 and one positive frequency component;
- it has a power of 9.

What is $x(t)$?

Solution:

The signal has a fundamental period of 1 s, and its frequency is: $\omega_0 = 2\pi$. Therefore, it can be written as:

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi kt}$$

The DC component of $x(t)$ is 1, hence $c_0 = 1$. Moreover, since $x(t)$ is real and even, the coefficients $c_k's$ are real and even. Since $x(t)$ has one positive frequency component, it can be reduced to the following:

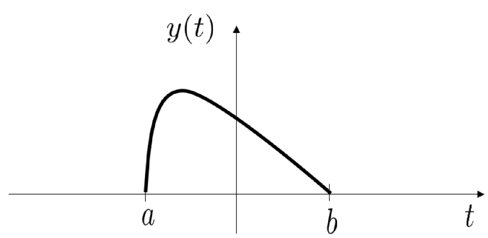
$$x(t) = c_{-1}e^{-j2\pi t} + 1 + c_1 e^{j2\pi t} = 1 + c_1\left(e^{j2\pi t} + e^{-j2\pi t}\right) = 1 + 2c_1\cos(2\pi t)$$

Using Parseval's relation, we have:

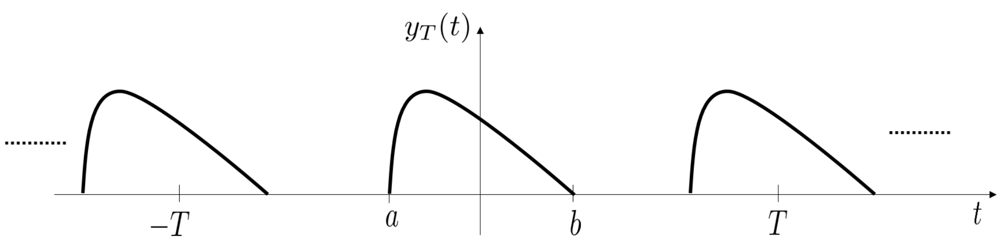
$$1 + 2c_1^2 = 9 \implies c_1 = \pm 2$$

Therefore, $x(t) = 1 + 4\cos(2\pi t)$ or $x(t) = 1 - 4\cos(2\pi t)$

(c) (4 points) Consider the signal $y(t)$ shown below and let $Y(j\omega)$ denote its Fourier transform.



Let $Y_T(t)$ denote its periodic extension:



How the Fourier series coefficients of $y_T(t)$ can be obtained from the Fourier transform $Y(j\omega)$ of $y(t)$? (Note that the figures given in this problem are for illustrative purposes, the question is for any arbitrary $y(t)$).

Solution:
The Fourier transform of $y(t)$ is given by:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{a}^{b} y(t)e^{-j\omega t}dt \tag{1}$$

The coefficients of the Fourier series for $y_T(t)$ are given by:

$$Y_k = \frac{1}{T}\int_{a}^{b} y(t)e^{-j(2\pi k/T)t}\, dt \tag{2}$$

for any integer k. Therefore, by comparing (2) to (1), we conclude:

$$Y_k = \frac{1}{T}Y(j\omega)|_{\omega=\frac{2\pi k}{T}}$$

2. (32 points) **Symmetry properties of Fourier transform**

   (a) (16 points) Determine which of the signals, whose Fourier transforms are depicted in Fig. 1, satisfy each of the following:

   i. $x(t)$ is even

   Solution: If $x(t)$ is even, then its Fourier transform should be even. Since $X(j\omega)$ in (a), (d) and (e) are even, signals (a), (d) and (e) are all even in the time domain.

   ii. $x(t)$ is odd

   Solution: If $x(t)$ is odd, then its Fourier transform should be odd. Since $X(j\omega)$ in (f) is odd, signal in (f) is odd in the time domain.

   iii. $x(t)$ is real

   Solution: If $x(t)$ is real, then $X(j\omega)$ is Hermitian, i.e., $X(-j\omega) = X^*(j\omega)$. This means the real part of $X(j\omega)$ is even and the imaginary part of $X(j\omega)$ is odd. It also means that the magnitude of $X(j\omega)$ is even and the phase of $X(j\omega)$ is odd. Since $X(j\omega)$ in (c) and (e) are both Hermitian, signals (c) and (e) are real in the time domain.

   iv. $x(t)$ is complex (neither real, nor pure imaginary)

   Solution: For $x(t)$ to be complex (not real neither pure imaginary), $X(j\omega)$ should not be Hermitian or anti-Hermitian. We know from the previous part that $X(j\omega)$ in (c) and (e) are Hermitian. Signals in (d) and (f) are anti-Hermitian. Therefore, signals in (a) and (b) are both complex in the time domain.

   v. $x(t)$ is real and even

   Solution: If $x(t)$ is real and even, then $X(j\omega)$ is real and even. Therefore, it is (e).

vi. $x(t)$ is imaginary and odd

Solution: If $x(t)$ is imaginary and odd, then $X(j\omega)$ is real and odd. Therefore, it is (f).

vii. $x(t)$ is imaginary and even

Solution: If $x(t)$ is imaginary and even, then $X(j\omega)$ is imaginary and even. Therefore, it is (d).

viii. There exists a non-zero ω_0 such that $e^{j\omega_0 t}x(t)$ is real and even

Solution: If $e^{j\omega_0 t}x(t)$ is real and even, then $X(j(\omega - \omega_0))$ is real and even. none of signals has this property. However:
Corrections:
In (b), if the signal was starting from 1, then it was symmetric and if we shift $X(j\omega)$ to the left by 2 (i.e., $\omega_0 = -2$), we obtain a real and even Fourier transform.

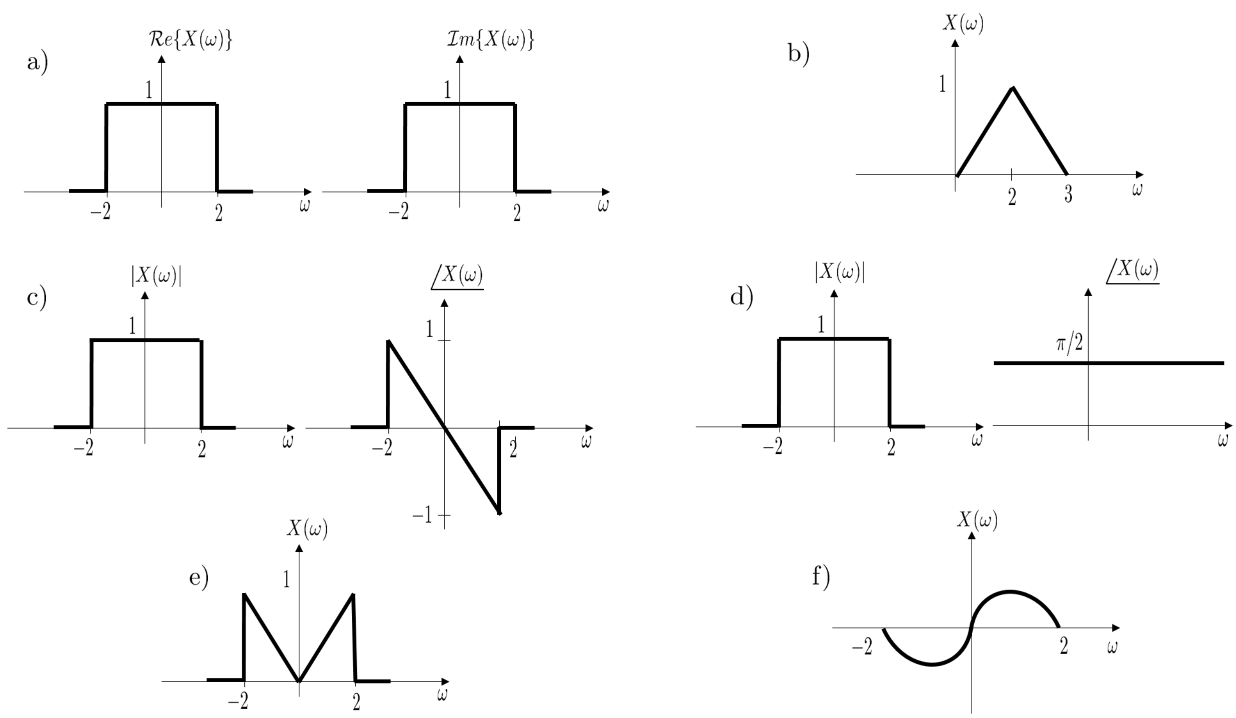


Figure 1: P2.a

(b) (8 points) Using the properties of Fourier transform, determine whether the assertions are true or false.

i. The convolution of a real and even signal and a real and odd signal is odd.

Solution: Let $f(t)$ be a real and even signal, and $g(t)$ be a real and odd signal. Then $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd. The convolution

$h(t) = (f * g)(t)$ has the Fourier transform

$$H(j\omega) = F(j\omega)G(j\omega)$$

If $F(j\omega)$ is real and even, and $G(j\omega)$ is imaginary and odd, then $H(j\omega)$ is imaginary and odd, and $h(t)$ is real and odd. The assertion is true.

ii. The convolution of a signal and the same signal reversed is an even signal.

Solution: Let $f(t)$ be a signal, and $f_R(t) = f(-t)$. Let $h(t) = (f * f_R)(t)$. Then

$$H(j\omega) = F(j\omega)F_R(j\omega) = F(j\omega)F(-j\omega)$$

which is even (replacing ω by $-\omega$ results in the same expression). The assertion is true.

(c) (8 points) Show the following statements:

i. If $x(t) = x^*(-t)$, then $X(j\omega)$ is real.

Solution:

$$\begin{aligned}
X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* \\
&= \int_{-\infty}^{+\infty} \left[x(t)e^{-j\omega t}\right]^* dt \\
&= \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt \\
&= \int_{-\infty}^{+\infty} x^*(-\tau)^{-j\omega\tau}\,d\tau, \text{ here we did the variable change } \tau = -t \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}\,d\tau, \text{ here we used the fact that } x(\tau) = x^*(-\tau) \\
&= X(j\omega),
\end{aligned}$$

Since $X^*(j\omega) = X(j\omega)$, we conclude that $X(j\omega)$ is real.

ii. If $x(t)$ is a real signal with $X(j\omega)$ its Fourier transform, then the Fourier transforms $X_e(j\omega)$ and $X_o(j\omega)$ of the even and odd components of $x(t)$ satisfy the following:

$$X_e(j\omega) = Re\{X(j\omega)\}$$

and

$$X_o(j\omega) = jIm\{X(j\omega)\}$$

Solution:
Since $x(t) = x_e(t) + x_o(t)$, the Fourier transform of $x(t)$ is given by:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \underbrace{\int_{-\infty}^{\infty} x_e(t)e^{-j\omega t}dt}_{X_e(j\omega)} + \underbrace{\int_{-\infty}^{\infty} x_o(t)e^{-j\omega t}dt}_{X_o(j\omega)}$$

Now using Euler,

$$X_e(j\omega) = \int_{-\infty}^{+\infty} x_e(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt$$

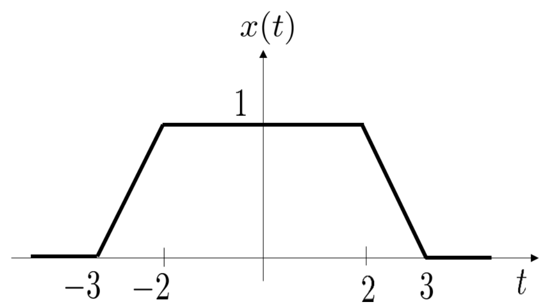
$$X_o(j\omega) = \int_{-\infty}^{+\infty} x_o(t)(\cos(\omega t) - j\sin(\omega t))dt = -j\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt$$

Since $x(t)$ is real, $x_e(t)$ and $x_o(t)$ are both real. Therefore, $X_e(j\omega)$ is real and $X_o(j\omega)$ is pure imaginary. Therefore,

$$\mathcal{R}e\{X(j\omega)\} = \int_{-\infty}^{+\infty} x_e(t)\cos(\omega t)dt = X_e(j\omega)$$
$$\mathcal{I}m\{X(j\omega)\} = -\int_{-\infty}^{+\infty} x_o(t)\sin(\omega t)dt = -jX_o(j\omega)$$

3. (15 points) **Fourier transform properties**

Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ sketched below:



Evaluate the following quantities without explicitly finding $X(j\omega)$:

(a) $\int_0^\infty X(j\omega)d\omega$

Solution:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$
$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} X(j\omega)\,d\omega$$

This is because $x(t)$ is real and even, then $X(j\omega)$ is real and even. This implies: $\int_{-\infty}^{+\infty} X(j\omega)\,d\omega = 2\int_0^{+\infty} X(j\omega)\,d\omega$. Therefore,

$$\int_0^{+\infty} X(j\omega)\,d\omega = \pi x(0) = \pi$$

(b) $X(j\omega)|_{\omega=0}$

Solution:
Since

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt$$

we then have:

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\, dt|_{\omega=0} = \int_{-\infty}^{+\infty} x(t)\, dt = \frac{(6+4)}{2} = 5$$

(c) $\underline{/X(j\omega)}$

Solution: The Fourier transform of a real and even function is real and even. Therefore the phase of $X(j\omega)$ is either 0 or π. It is zero when $X(j\omega) \geq 0$ and it is π when $X(j\omega) < 0$.

(d) $\int_{-\infty}^{\infty} e^{-j\omega} X(j\omega) d\omega$

Solution:

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega \\ x(t)|_{t=-1} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\, d\omega|_{t=-1} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega}\, d\omega \end{aligned}$$

Therefore,

$$\int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega}\, d\omega = 2\pi x(-1) = 2\pi$$

(e) Plot the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$

Solution: Let $Y(j\omega) = e^{-3j\omega}X(j\omega)$, then $y(t) = x(t-3)$. Since $y(t)$ is real,

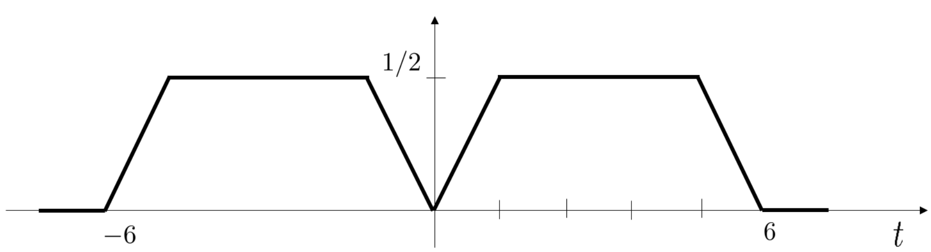


Figure 2: P3.e

$$\mathcal{R}e\{e^{-3j\omega}X(j\omega)\} = \mathcal{R}e\{Y(j\omega)\} = Y_e(j\omega)$$

where $Y_e(j\omega)$ is the Fourier transform of the even component of $y(t)$. Therefore, the inverse Fourier transform of $\mathcal{R}e\{e^{-3j\omega}X(j\omega)\}$ is the even component of $x(t-3)$.

4. (35 points) **Fourier transform and its inverse**

   (a) (20 points) Find the Fourier transform of each of the signals given below:
   Hint: you may use Fourier Transforms derived in class.

      i. $x_1(t) = 2\text{rect}\left(\frac{-t-3}{2}\right)\cos(10\pi t)$

      Solution:
      We know that:

$$\text{rect}(t) \longleftrightarrow \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

      Therefore,

$$\text{rect}\left(\frac{-t}{2}\right) \longleftrightarrow 2\text{sinc}\left(-2\frac{\omega}{2\pi}\right) = 2\text{sinc}\left(\frac{\omega}{\pi}\right), \text{sinc is an even function}$$

$$2\text{rect}\left(\frac{-1}{2}(t+3)\right) \longleftrightarrow 4\text{sinc}\left(\frac{\omega}{\pi}\right)e^{j3\omega}$$

$$2\text{rect}\left(\frac{-1}{2}(t+3)\right)\cos(10\pi t) \longleftrightarrow 2\text{sinc}\left(\frac{\omega}{\pi}-10\right)e^{j3(\omega-10\pi)}+2\text{sinc}\left(\frac{\omega}{\pi}+10\right)e^{j3(\omega+10\pi)}$$

      ii. $x_2(t) = e^{(2+3j)t}u(-t+1)$

      Solution:
      We can write $x_2(t)$ as follows:

$$x_2(t) = e^{j3t}e^{2t}u(-t+1) = e^{j3t}e^2e^{2(t-1)}u(-(t-1))$$

      We know:

$$e^{-2t}u(t) \longleftrightarrow \frac{1}{2+j\omega}$$
$$e^{2t}u(-t) \longleftrightarrow \frac{1}{2-j\omega}$$
$$e^{2(t-1)}u(-(t-1)) \longleftrightarrow \frac{e^{-j\omega}}{2-j\omega}$$
$$e^{j3t}e^{2(t-1)}u(-(t-1)) \longleftrightarrow \frac{e^{-j(\omega-3)}}{2-j(\omega-3)}$$

      Therefore,

$$X_2(j\omega) = e^2\frac{e^{-j(\omega-3)}}{2-j(\omega-3)}$$

iii. $x_3(t) = \begin{cases} 1 + \cos(\pi t), & |t| < 1 \\ 0, & \text{otherwise} \end{cases}$

Solution:
We can compute $X_3(j\omega)$ by applying the definition of Fourier transform:

$$\begin{aligned} X_3(j\omega) &= \int_{-1}^{1} [1 + \cos(\pi t)] e^{-j\omega t} dt \\ &= -\frac{1}{j\omega}[e^{-j\omega} - e^{j\omega}] + \frac{1}{j2(\pi - \omega)}[e^{j(\pi-\omega)} - e^{-j(\pi-\omega)}] - \frac{1}{j2(\pi + \omega)}[e^{-j(\pi+\omega)} - e^{j(\pi+\omega)}] \\ &= \frac{2\sin(\omega)}{\omega} + \frac{\sin(\pi - \omega)}{\pi - \omega} + \frac{\sin(\pi + \omega)}{\pi + \omega} = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega - \pi}{\pi}\right) + \text{sinc}\left(\frac{\omega + \pi}{\pi}\right) \end{aligned}$$

Or we can see that:

$$x_3(t) = \text{rect}\left(\frac{t}{2}\right) + \cos(\pi t)\text{rect}\left(\frac{t}{2}\right)$$

so that,

$$X_3(j\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right) + \text{sinc}\left(\frac{\omega - \pi}{\pi}\right) + \text{sinc}\left(\frac{\omega + \pi}{\pi}\right)$$

iv. $x_4(t) = te^{-t}u(t)$

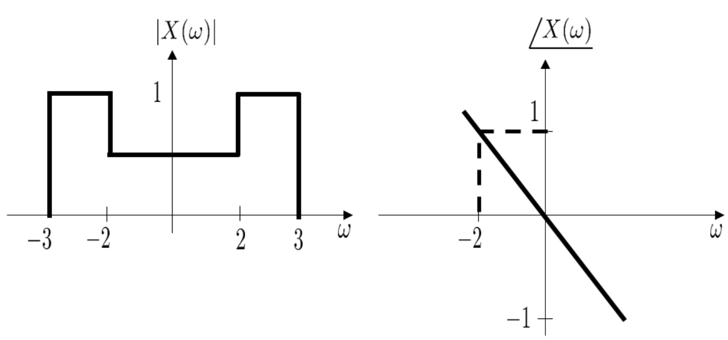
Solution:
We know that:

$$-jtf(t) \longleftrightarrow F'(j\omega)$$
$$e^{-t}u(t) \longleftrightarrow \frac{1}{1 + j\omega}$$

Therefore,

$$X_4(j\omega) = -\frac{1}{j}\left(\frac{d}{d\omega}\frac{1}{1 + j\omega}\right) = \frac{1}{(1 + j\omega)^2}$$

(b) (7 points) Find the inverse Fourier transform of the signal shown below:



Solution: We have:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|e^{j\angle X(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{-2}^{2}\frac{1}{2}e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega + \int_{2}^{3} e^{-j\frac{1}{2}\omega}e^{j\omega t}d\omega\right) \\
&= \frac{1}{2\pi}\left(\int_{-3}^{-2} e^{j(t-\frac{1}{2})\omega}d\omega + \int_{-2}^{2}\frac{1}{2}e^{j(t-\frac{1}{2})\omega}d\omega + \int_{2}^{3} e^{j(t-\frac{1}{2})\omega}d\omega\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-j2(t-\frac{1}{2})}-e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})} + \frac{e^{j2(t-\frac{1}{2})}-e^{-j2(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})}-e^{j2(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{j2(t-\frac{1}{2})}-e^{-2j(t-\frac{1}{2})}}{j2(t-\frac{1}{2})} + \frac{e^{j3(t-\frac{1}{2})}-e^{-j3(t-\frac{1}{2})}}{j(t-\frac{1}{2})}\right) \\
&= \frac{1}{2\pi}\left(\frac{\sin\left(2(t-\frac{1}{2})\right)}{(t-\frac{1}{2})} - \frac{2\sin\left(3(t-\frac{1}{2})\right)}{(t-\frac{1}{2})}\right)
\end{aligned}
$$

(c) (8 points) Two signals $f_1(t)$ and $f_2(t)$ are defined as

$$
\begin{aligned}
f_1(t) &= \operatorname{sinc}(2t) \\
f_2(t) &= \operatorname{sinc}(t)\cos(4\pi t)
\end{aligned}
$$

Let the convolution of the two signals be

$$f(t) = (f_1 * f_2)(t)$$

i. Find $F(j\omega)$, the Fourier transform of $f(t)$.

Solution: We know that:

$$
\begin{aligned}
f_1(t) = \operatorname{sinc}(2t) &\longleftrightarrow F_1(j\omega) = \frac{1}{2}\operatorname{rect}\left(\frac{\omega}{4\pi}\right) \\
f_2(t) = \operatorname{sinc}(t)\cos(4\pi t) &\longleftrightarrow F_2(j\omega) = \frac{1}{2}\left(\operatorname{rect}\left(\frac{\omega-4\pi}{2\pi}\right) + \operatorname{rect}\left(\frac{\omega+4\pi}{2\pi}\right)\right)
\end{aligned}
$$

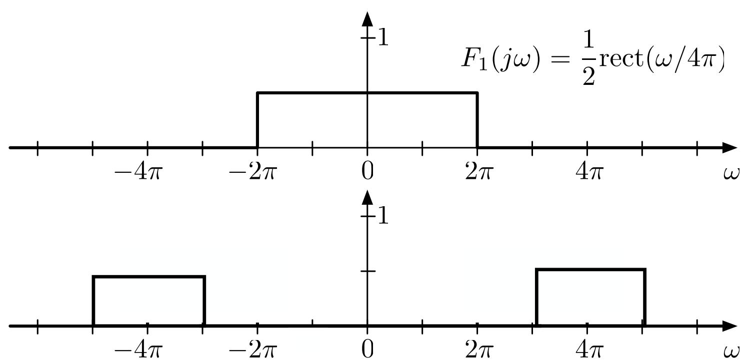
We then have:

$$f(t) = (f_1 * f_2)(t) \longleftrightarrow F(j\omega) = F_1(j\omega)F_2(j\omega)$$

To see what the multiplication of $F_1(j\omega)$ and $F_2(j\omega)$ gives us, let us first plot them. We clear see that $F_1(j\omega)$ and $F_2(j\omega)$ do not overlap, therefore $F(j\omega) = 0$.

ii. Find $f(t)$.

Solution:
Since $F(j\omega) = 0$, $f(t)$ is then 0



$$F_2(j\omega) = \frac{1}{2}\left[\mathrm{rect}\left(\frac{\omega - 4\pi}{2\pi}\right) + \mathrm{rect}\left(\frac{\omega + 4\pi}{2\pi}\right)\right]$$