EE 131A | Homework 6
Probability and Statistics | Thursday, February 14, 2019
Instructor: Lara Dolecek | Due: Thursday, February 21, 2019
TA: Ruiyi (John) Wu | ruiyiwu@g.ucla.edu

Reading: Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Find the characteristic function of a normal distribution with mean m and variance σ^2.
   Hint: Let $k = m + j\omega\sigma^2$, then $j\omega x - \frac{(x-m)^2}{2\sigma^2} = \frac{-(x-k)^2+2mj\omega\sigma^2-\omega^2\sigma^4}{2\sigma^2}$, $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.
   Solution:
   The characteristic function of a random variable X is defined as following:
   $$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx$$
   Thus, the characteristic function of a normal distribution with mean m and variance σ^2 is:
   $$\begin{aligned}
   \Phi_X(\omega) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} e^{j\omega x}dx \\
   &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{j\omega x-\frac{(x-m)^2}{2\sigma^2}}dx \\
   &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-(x-k)^2+2mj\omega\sigma^2-\omega^2\sigma^4}{2\sigma^2}}dx \quad \text{(from hint)} \\
   &= e^{\frac{2mj\omega\sigma^2-\omega^2\sigma^4}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-k)^2}{2\sigma^2}}dx \\
   &= e^{mj\omega-\frac{1}{2}\omega^2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-k}{\sqrt{2\sigma^2}}\right)^2}dx \\
   &= e^{mj\omega-\frac{1}{2}\omega^2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-z^2}\sqrt{2\sigma^2}dz \\
   &= e^{mj\omega-\frac{1}{2}\omega^2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\sigma^2} \int_{-\infty}^{\infty} e^{-z^2}dz \\
   &= e^{mj\omega-\frac{1}{2}\omega^2\sigma^2} \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2} \quad \text{(from hint)} \\
   &= e^{mj\omega-\frac{1}{2}\omega^2\sigma^2}
   \end{aligned}$$

2. Find the characteristic function of $Y = aX + b$ where X is a Gaussian random variable.
   Solution:
   The characteristic function of $Y = aX + b$ is given by
   $$\begin{aligned}
   \Phi_Y(\omega) &= E[e^{j\omega Y}] \\
   &= E[e^{j\omega(aX+b)}] \\
   &= e^{j\omega b}E[e^{ja\omega X}] \\
   &= e^{j\omega b}e^{jam\omega-a^2\sigma^2\omega^2/2}
   \end{aligned}$$

which is the characteristic function of Gaussian random variable with mean of $am + b$ and variance of $a^2\sigma^2$

3. Suppose Y is a Laplacian random variable, which is defined by the PDF:

$$f_Y(y) = he^{-4\alpha|y|}, \; -\infty < y < \infty, \, \alpha > 0.$$

   (a) Determine the value of h as a function of α.
   Solution:

$$f_Y(y) = he^{-4\alpha|y|} \begin{cases} he^{4\alpha y}, & -\infty < y < 0 \\ he^{-4\alpha y}, & 0 \leq y < \infty \end{cases}$$

$$\int_{-\infty}^{\infty} f_Y(y)dy = 1$$
$$\int_{-\infty}^{0} he^{4\alpha y}dy + \int_{0}^{\infty} he^{-4\alpha y}dy = 1$$
$$2\int_{0}^{\infty} he^{-4\alpha y}dy = 1$$
$$2h\frac{e^{-4\alpha y}}{4\alpha}\Big|_{\infty}^{0} = 1$$
$$\frac{h}{2\alpha} = 1$$

   Thus, $h = 2\alpha$

   (b) Determine the CDF of the random variable Y.
   Solution:
   Region 1, $-\infty < y < 0$

$$F_Y(y) = \int_{-\infty}^{y} 2\alpha e^{4\alpha y}dy = 2\alpha\frac{e^{4\alpha y}}{4\alpha}\Big|_{-\infty}^{y} = \frac{1}{2}e^{4\alpha y}$$

   Region 2, $0 \leq y < \infty$

$$F_Y(y) = \int_{-\infty}^{0} 2\alpha e^{4\alpha y}dy + \int_{0}^{y} 2\alpha e^{-4\alpha y}dy = \frac{1}{2} + 2\alpha\frac{e^{-4\alpha y}}{4\alpha}\Big|_{y}^{0} = \frac{1}{2} + \frac{1}{2}(1 - e^{-4\alpha y}) = 1 - \frac{1}{2}e^{-4\alpha y}$$

4. Let X be a Laplacian random variable, which is defined by the PDF:

$$f_X(x) = \frac{\alpha}{2}e^{-\alpha|x|}, \; -\infty < x < \infty, \, \alpha > 0.$$

   (a) Find the characteristic function of the Laplacian random variable.
   Solution:
   The characteristic function of a random variable X is defined as following:

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x)e^{j\omega x}dx$$

The characteristic function of a Laplacian random variable X is defined as following:

$$\begin{aligned}
\Phi_X(\omega) &= E[e^{j\omega X}] \\
&= \int_{-\infty}^{\infty} \left(\frac{\alpha}{2}e^{-\alpha|x|}\right) e^{j\omega x} dx \\
&= \int_{-\infty}^{\infty} \left(\frac{\alpha}{2}e^{-\alpha|x|}\right) (\cos(\omega x) + j\sin(\omega x)) dx \\
&= \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|}\cos(\omega x) dx + j\frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|}\sin(\omega x) dx
\end{aligned}$$

Since, $\sin(-\omega x) = -\sin(\omega x)$ and, $\cos(-\omega x) = \cos(\omega x)$. Thus,

$$\begin{aligned}
\Phi_X(\omega) &= \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|}\cos(\omega x) dx + j\frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|}\sin(\omega x) dx \\
&= \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|}\cos(\omega x) dx \\
&= \frac{\alpha}{2} \left[\int_{-\infty}^{0} e^{\alpha x}\cos(\omega x) dx + \int_{0}^{\infty} e^{-\alpha x}\cos(\omega x) dx\right] \\
&= \frac{\alpha}{2} \left[\int_{0}^{\infty} e^{-\alpha x}\cos(-\omega x) dx + \int_{0}^{\infty} e^{-\alpha x}\cos(\omega x) dx\right] \\
&= 2\frac{\alpha}{2} \int_{0}^{\infty} e^{-\alpha x}\cos(\omega x) dx \\
&= \alpha \frac{e^{-\alpha x}(\omega\sin(\omega x) - \alpha\cos(\omega x))}{\alpha^2 + \omega^2}\bigg|_0^\infty \\
&= \frac{\alpha^2}{\alpha^2 + \omega^2}
\end{aligned}$$

(b) Find the mean and variance of X by applying the moment theorem.
Solution:

$$\begin{aligned}
E[X] &= \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)\bigg|_{\omega=0} \\
&= \frac{1}{j}\frac{d}{d\omega}\left(\frac{\alpha^2}{\alpha^2 + \omega^2}\right)\bigg|_{\omega=0} \\
&= \frac{1}{j}\frac{2\alpha^2\omega}{(\omega^2 + \alpha^2)^2}\bigg|_{\omega=0} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= E[X^2] - 0 \\
&= \frac{1}{j^2}\frac{d^2}{d\omega^2}\Phi_X(\omega)\Big|_{\omega=0} \\
&= \frac{1}{j^2}\frac{d^2}{d\omega^2}\left(\frac{\alpha^2}{\alpha^2+\omega^2}\right)\Big|_{\omega=0} \\
&= \frac{1}{j^2}\left(\frac{8\alpha^2\omega^2}{(\omega^2+\alpha^2)^3} - \frac{2\alpha^2}{(\omega^2+\alpha^2)^2}\right)\Big|_{\omega=0} \\
&= \frac{2}{\alpha^2}
\end{aligned}
$$

5. Compare the Chebyshev inequality and the exact probability for the event $\{|X-m| > c\}$ as a function of c for X is a continuous uniform random variable in the interval $[-b, b]$.
Solution:
Chebyshev inequality states that,

$$P\{|X-m| > c\} \leq \frac{\sigma^2}{c^2}$$

When X is a uniform random variable in the interval $[-b, b]$, $P[X \leq c] = \frac{c-a}{b-a}$ for c in $[-b, b]$. $E[X] = 0$ and $Var[X] = \frac{b^2}{3}$. For $0 \leq c \leq b$:

$$
\begin{aligned}
P\{|X-m| > c\} &= P(X-m > c) + P(X-m < -c) \\
&= [1 - P(X-m < c)] + P(X-m < -c) \\
&= [1 - P(X < c+m)] + P(X < -c+m) \\
&= [1 - \frac{c+b}{b+b}] + \frac{-c+b}{b+b} \\
&= 1 - \frac{c}{b}
\end{aligned}
$$

Plug in the respective values to the Chebyshev inequality:

$$P\{|X-m| > c\} \leq \frac{\sigma^2}{c^2}$$

When X is a uniform random variable in the interval $[-b, b]$, $P[X \leq c] = \frac{c-a}{b-a}$ for c in $[-b, b]$. $E[X] = 0$ and $Var[X] = \frac{b^2}{3}$. For $0 \leq c \leq b$:

$$
\begin{aligned}
P\{|X-m| > c\} &\leq \frac{\sigma^2}{c^2} \\
1 - \frac{c}{b} &\leq \frac{b^2}{3} \times \frac{1}{c^2} \\
\frac{b-c}{b} &\leq \frac{b^2}{3c^2} \\
0 &\leq 3c^3 + b^3 - 3bc^2
\end{aligned}
$$

The inequality always holds for $0 \leq c \leq b$.