EE 131A | Homework 9
Probability and Statistics | Thursday, March 07, 2019
Instructor: Lara Dolecek | Due: Thursday, March 14, 2019
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Reading: Chapter 5, 7 & 8 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. The random variables X and Y have the joint pdf

$$f_{X,Y}(x,y) = 8xy \quad \text{for } 0 \le y \le x \le 1.$$

Find the pdf of $Z = X + Y$.
Solution:

$$\begin{aligned} F_Z(z) &= P[Z \le z] \\ &= P[X + Y \le z] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x,y)dydx \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) \\ &= \int_{-\infty}^{\infty} \left[\frac{d}{dz}\int_{-\infty}^{z-x} f(x,y)dy\right] dx \\ &= \int_{-\infty}^{\infty} f(x, z-x)dx \end{aligned}$$

For $0 \le z \le 1$,

$$f_Z(z) = \int_{\frac{z}{2}}^{z} 8x(z-x)dx = \frac{2z^3}{3}$$

For $1 \le z \le 2$,

$$f_Z(z) = \int_{\frac{z}{2}}^{1} 8x(z-x)dx = 4z - \frac{8}{3} - \frac{2z^3}{3}$$

2. Let X and Y be jointly Gaussian random variables with pdf

$$f_{X,Y}(x,y) = \frac{\exp\{-2x^2 - \frac{y^2}{2}\}}{2\pi c} \quad \text{for all } x, y.$$

Find $VAR[X], VAR[Y]$, and $COV(X,Y)$.
Solution:
X and Y are jointly Gaussian random variable with PDF:

$$f_{X,Y}(x,y) = \frac{e^{(-2x^2 - y^2/2)}}{2\pi c}$$

Comparing the coefficients in jointly Gaussian distribution, we have:

$$\rho_{X,Y} = 0 \qquad m_1 = 0 \qquad m_2 = 0 \qquad -2\sigma_1^2 = -1/2 \qquad 2\sigma_2^2 = 2$$

Therefore, we have $\sigma_1^2 = 1/4$ and $\sigma_2^2 = 1$.

$$VAR(X) = \frac{1}{4} \qquad VAR(Y) = 1 \qquad COV(X,Y) = 0$$

3. Use the fact that $E[(tX+Y)^2] \geq 0$ for all t to prove the Cauchy-Schwarz inequality:

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Hint: Consider the discriminant of the quadratic equation in t that results from the above inequality.
Solution:
For all t's, we have:

$$0 \leq E[(tX+Y)^2] = t^2E[X^2] + 2tE[XY] + E[Y^2] = f(t)$$

We can view this as a quadratic equation on t. Since $f(t)$ is non-negative, $f(t) = 0$ has either 0 or imaginary roots. Thus, we can write the discriminant as:

$$(2E[XY])^2 - 4E[X^2]E[Y^2] \leq 0$$

Then, we reach $(E[XY])^2 \leq E[X^2]E[Y^2]$.

4. The random variables X and Y have joint pdf:

$$f_{X,Y}(x,y) = c\,\sin(x+y) \quad 0 \leq x \leq \frac{\pi}{2}, 0 \leq x \leq \frac{\pi}{2}.$$

(a) Find the value of the constant c.
Solution:
To find the value of the constant c, we have:

$$\int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} c\sin(x+y)\mathrm{d}y\mathrm{d}x = 1$$

Solving for c, we can find:

$$c = \frac{1}{\int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2}\sin(x+y)\mathrm{d}y\mathrm{d}x} = \frac{1}{2}$$

(b) Find the joint cdf of X and Y.
Solution:
The joint CDF of X and Y is given by:

$$F_{XY}(x,y) = \int\int f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x = \frac{1}{2}\int_{x=0}^{x}\int_{y=0}^{y}\sin(x+y)\mathrm{d}y\mathrm{d}x$$
$$= \frac{1}{2}[\sin x + \sin y - \sin(x+y)],\ 0 \leq x \leq \pi/2,\ 0 \leq y \leq \pi/2$$

Moreover, $F_{XY}(x, y) = 1$ for $x > \pi/2,\ y > \pi/2$. Additionally, $F_{XY}(x, y) = 0$ for $x < 0,\ y < 0$.

(c) Find the marginal pdf's of X and of Y.
Solution:
The marginal PDFs of X and Y are given by:

$$f_X(x) = \int_{y=0}^{\pi/2} \frac{1}{2}\sin(x+y)\mathrm{d}y = \frac{1}{2}(\sin x + \cos x),\ 0 \le x \le \pi/2$$

$$f_Y(y) = \int_{x=0}^{\pi/2} \frac{1}{2}\sin(x+y)\mathrm{d}x = \frac{1}{2}(\sin y + \cos y),\ 0 \le y \le \pi/2$$

(d) Find the mean, variance, and covariance of X and Y.
Solution:
The mean, variance, and covariance of X and Y are given as follows. Notice that all the below integrals are performed by parts.
Part (d) is a wake-up call for those who have not revised math basics until now.

$$E(X) = \int_{x=0}^{\pi/2} x\frac{1}{2}(\sin x + \cos x)\mathrm{d}x = \frac{\pi}{4}$$

$$E(Y) = \int_{y=0}^{\pi/2} y\frac{1}{2}(\sin y + \cos y)\mathrm{d}y = \frac{\pi}{4}$$

$$E(X^2) = \int_{x=0}^{\pi/2} x^2\frac{1}{2}(\sin x + \cos x)\mathrm{d}x = \frac{\pi^2}{8} + \frac{\pi}{2} - 2$$

$$E(Y^2) = \int_{y=0}^{\pi/2} y^2\frac{1}{2}(\sin y + \cos y)\mathrm{d}y = \frac{\pi^2}{8} + \frac{\pi}{2} - 2$$

$$VAR(X) = E(X^2) - E(X)^2 = \frac{\pi^2}{16} + \frac{\pi}{2} - 2$$

$$VAR(Y) = E(Y^2) - E(Y)^2 = \frac{\pi^2}{16} + \frac{\pi}{2} - 2$$

$$E(XY) = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} xy\frac{1}{2}\sin(x+y)\mathrm{d}y\mathrm{d}x = \frac{\pi}{2} - 1$$

$$COV(X, Y) = E(XY) - E(X)E(Y) = \frac{\pi}{2} - 1 - \frac{\pi^2}{16}$$

5. Let $X_1, \ldots, X_n$ be random variables with the same mean and with covariance function

$$COV(X_i, X_j) = \sigma^2\rho^{|i-j|},$$

where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.
Assume that $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ for $i = \{1, 2, \ldots, n\}$.

Solution:
The mean and variance of $S_n = X_1 + X_2 + \cdots + X_n$ are given by:

$$E(S_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = n\mu$$

$$\begin{aligned} VAR(S_n) &= \sum_{i=1}^{n} VAR(X_i) + \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} COV(X_i, X_j) \\ &= n\sigma^2 + 2\sigma^2 \sum_{i=2}^{n} \sum_{j=1}^{i-1} \rho^{i-j} \end{aligned}$$

Now, we use a new index, $k = i - j$, in our summation to make it easier:

$$VAR(S_n) = n\sigma^2 + 2\sigma^2 \sum_{i=2}^{n} \sum_{k=1}^{i-1} \rho^{k}$$

Thus, we can see that:

$$\begin{aligned} VAR(S_n) &= n\sigma^2 + 2\sigma^2 \sum_{i=2}^{n} \left[\frac{\rho(1-\rho^{i-1})}{1-\rho}\right] \\ &= n\sigma^2 + 2\sigma^2\rho \sum_{i=2}^{n} \left[\frac{1}{1-\rho} - \frac{\rho^{i-1}}{1-\rho}\right] \\ &= n\sigma^2 + 2\sigma^2\rho \left[\frac{n-1}{1-\rho} - \frac{\rho(1-\rho^{n-1})}{(1-\rho)^2}\right] \end{aligned}$$

6. A student uses pens whose lifetime is an exponential random variable with mean 1 week. Use the central limit theorem to determine the minimum number of pens he should buy at the beginning of a 15-week semester, so that with probability .99 he does not run out of pens during the semester.
Solution:
Based on the Central Limit Theorem, S_n is approximately Gaussian, then we have:

$$\begin{aligned} P[S_n > 15] &= P\left[\frac{S_n - \mu n}{\sigma\sqrt{n}} > \frac{15 - \mu n}{\sigma\sqrt{n}}\right] \\ &= P\left[\frac{S_n - n}{\sqrt{n}} > \frac{15 - n}{\sqrt{n}}\right] \approx Q\left(\frac{15-n}{\sqrt{n}}\right) = 0.99 \end{aligned}$$

From the Table 4.3 in the textbook, we have:

$$\frac{15-n}{\sqrt{n}} = -2.3263$$

Solving for n, we get $n = 27.04$. Therefore, the student should buy 28 pens at the beginning of the semester.