EE 131A — Homework 5

Probability and Statistics — Tuesday, February 05, 2019

Instructor: Lara Dolecek — Due: Thursday, February 14, 2019

TA: Ruiyi (John) Wu — ruiyiwu@g.ucla.edu

Reading: Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Let $X_1, \cdots, X_n$ be iid random variables where $X_i \sim \exp(\lambda)$. Find the PDF of random variable

$$Y = \min\{X_1, \cdots, X_n\}.$$

Solution:

$$\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= 1 - P[Y > y] \\
&= 1 - P[X_1 > y, \cdots, X_n > y] \\
&= 1 - P[X_1 > y] \cdot \cdots \cdot P[X_n > y] \\
&= 1 - (1 - F_X(y))^n \\
&= 1 - e^{-\lambda n y}, \ y \geq 0.
\end{aligned}$$

So,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \lambda n e^{-\lambda n y}, \ y \geq 0.$$

2. Let X be a Gaussian random variable with mean 2 and variance 4. The reward in a system is given by

$$Y = (X)^+ = \begin{cases} X & \text{if } X \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the PDF of Y.

Solution:

Clearly, $Y \geq 0$. For $y > 0$,

$$\begin{aligned}
F_Y(y) &= P[Y \leq y] \\
&= P[X \leq y] \\
&= F_X(y)
\end{aligned}$$

Therefore, for $y > 0$,

$$f_Y(y) = f_X(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(y-2)^2}{8}}.$$

For $y = 0$,

$$\begin{aligned} F_Y(0) &= P[Y \le 0] = P[Y = 0] \\ &= P[X \le 0] \\ &= F_X(0) = F_{X_s}\left(\frac{0-2}{\sqrt{4}}\right) \\ &= 1 - Q(-1) = Q(1) \end{aligned}$$

So,

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{2\pi}} e^{-\frac{(y-2)^2}{8}} + Q(1)\delta(y) & \text{if } y \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Is Y a continuous random variable? Justify your answer.
Solution:
Since the CDF of the random variable Y has a discontinuity at the zero, Y is not a continuous random variable. Instead, Y is a mixed random variable.

3. Use the result that, for a nonnegative random variable Y,

$$E[Y] = \int_0^\infty P\{Y > t\} dt$$

to show that, for a nonnegative random variable X,

$$E[X^n] = \int_0^\infty n x^{n-1} P\{X > x\} dx$$

Solution:
Compute $E[X^n]$ by using the given identity i.e.

$$E[X^n] = \int_0^\infty P\{X^n > t\} dt.$$

To evaluate this let $t = x^n$ then $dt = nx^{n-1}dx$ and we have

$$E[X^n] = \int_0^\infty P\{X^n > x^n\} n x^{n-1} dx = \int_0^\infty n x^{n-1} P\{X > x\} dx,$$

Using the fact that $P\{X^n > x^n\} = P\{X > x\}$ when X is a non-negative random variable.

4. Compute the hazard rate function of X when X is uniformly distributed over $(0, a)$. The hazard rate function $\lambda(t)$ is defined by $\lambda(t) = \frac{f(t)}{1-F(t)}$.
Solution:
For a uniform random variable distributed between (0, a) we have:

$$f(t) = \begin{cases} \frac{1}{a} & 0 \le t \le a \\ 0 & \text{otherwise.} \end{cases}$$

and

$$F(t) = \int_0^t f(t')dt' = \int_0^t \frac{dt'}{a} = \frac{t}{a},$$

so the hazard rate function then is

$$\lambda(t) = \frac{(1/a)}{1 - \frac{t}{a}} = \frac{1}{a - t},$$

for $0 \leq t \leq a$.

5. If X has hazard rate function $\lambda_X(t)$, compute the hazard rate function of aX where a is a positive constant.

Solution:

For this problem if we are told that X has a hazard rate function $\lambda_X(t)$ we desire to compute the hazard rate function for $Y = aX$, with $a > 0$. When $Y = aX$ the probability density function of Y is given by $f_Y(y) = f_X(\frac{y}{a})(\frac{1}{a})$ and its distribution function is given by

$$F_Y(c) = P\{Y \leq c\} = P\{aX \leq c\} = P\{X \leq \frac{c}{a}\} = F_X(\frac{c}{a}),$$

so the hazard rate for Y is given by

$$\lambda(t) = \frac{f_Y(t)}{1 - F_Y(t)} = \frac{f_X(\frac{t}{a})(\frac{1}{a})}{1 - F_X(\frac{t}{a})} = \left(\frac{1}{a}\right)\left(\frac{f_X(\frac{t}{a})}{1 - F_X(\frac{t}{a})}\right) = \left(\frac{1}{a}\right)\lambda_X(\frac{t}{a}).$$

6. Consider the communication system shown below. The transmitter transmits X that can take one of two values, either 1 with probability p or -1 with probability $1 - p$, over a noisy channel. The receiver observes $Y = X + N$, and based on Y, it decides the value of Z (an estimate of X), which can only be either 1 or -1. Suppose that the random variable N, which represents the noise in this communication system, follows a Laplacian distribution that has the PDF:

$$f_N(n) = \frac{\alpha}{2}e^{-\alpha|n|}, \ -\infty < n < \infty, \, \alpha > 0.$$

(a) Find the expected value and the variance of N.

Solution:

The expected value of N is obtained as follows:

$$E(N) = \int_{n=-\infty}^{\infty} n f_N(n)dn = \int_{n=-\infty}^{0} n\left(\frac{\alpha}{2}e^{\alpha n}\right)dn + \int_{n=0}^{\infty} n\left(\frac{\alpha}{2}e^{-\alpha n}\right)dn = 0.$$

This result can be reached directly from the symmetry of the Laplacian distribution around the zero. Moreover, The variance of N is obtained as follows:

$$E(N^2) = \int_{n=-\infty}^{\infty} n^2 f_N(n)dn = \int_{n=-\infty}^{0} n^2\left(\frac{\alpha}{2}e^{\alpha n}\right)dn + \int_{n=0}^{\infty} n^2\left(\frac{\alpha}{2}e^{-\alpha n}\right)dn = 2I,$$

$$\begin{aligned} I &= \int_{n=0}^{\infty} n^2\left(\frac{\alpha}{2}e^{-\alpha n}\right)dn = \frac{\alpha}{2}\int_{n=0}^{\infty} \frac{-n^2}{\alpha} d\left(e^{-\alpha n}\right) = 0 + \int_{n=0}^{\infty} n\left(e^{-\alpha n}\right)dn \\ &= \int_{n=0}^{\infty} \frac{-n}{\alpha} d\left(e^{-\alpha n}\right)dn = 0 + \int_{n=0}^{\infty} \frac{1}{\alpha}\left(e^{-\alpha n}\right)dn = \frac{1}{\alpha^2}. \end{aligned}$$

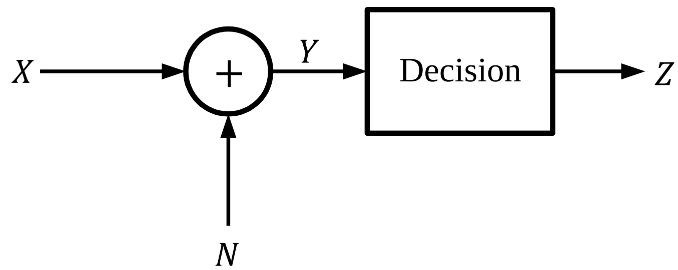

Figure 1: The communication system of problem 7.

Thus, we conclude the following:

$$E(N^2) = 2I = \frac{2}{\alpha^2},$$
$$V(N) = E\left(N^2\right) - \left(E(N)\right)^2 = \frac{2}{\alpha^2}.$$

(b) Suppose that the receiver decides $Z = 1$ if $Y = y > T$, and $Z = -1$ if $Y = y \leq T$. Assuming that $-1 < T < 1$, find the decision threshold T in this system as a function of α and p.

Solution:

We let the PDF of Y which is conditioned on $X = 1$ be $f_{Y_1}(y)$, and the PDF of Y which is conditioned on $X = -1$ be $f_{Y_{-1}}(y)$. To get the decision threshold, we need the conditional probabilities:

$$\begin{aligned}
P[X = 1|y < Y < y + h] &= \frac{P[y < Y < y + h|X = 1]P[X = 1]}{P[y < Y < y + h]} \\
&= \frac{f_{Y_1}(y)hp}{f_{Y_1}(y)hp + f_{Y_{-1}}(y)h(1-p)}, \\
P[X = -1|y < Y < y + h] &= \frac{P[y < Y < y + h|X = -1]P[X = -1]}{P[y < Y < y + h]} \\
&= \frac{f_{Y_{-1}}(y)h(1-p)}{f_{Y_1}(y)hp + f_{Y_{-1}}(y)h(1-p)},
\end{aligned}$$

where h is infinitesimal, i.e., we can approximate $P[y < Y < y + h|X = 1]$ as simply $f_{Y_1}(y)h$. Then, around the decision threshold $y = T$, we should have:

$$\begin{aligned}
P[X = 1|y < Y < y + h] &= P[X = -1|y < Y < y + h] \\
f_{Y_1}(y)hp &= f_{Y_{-1}}(y)h(1-p) \\
\frac{\alpha}{2}e^{\alpha(T-1)}p &= \frac{\alpha}{2}e^{-\alpha(T+1)}(1-p),
\end{aligned}$$

where the last equality is due to the given that $-1 < T < 1$. Thus, we reach:

$$\begin{aligned} e^{2\alpha T} &= \frac{1-p}{p} \\ T &= \frac{1}{2\alpha} ln\left(\frac{1-p}{p}\right). \end{aligned}$$

(c) Define the probability of error in this system to be $P[Z \neq X]$. If $p = 0.5$, what is the probability of error?

Hint 1: If you cannot find explicitly the decision threshold T of part (b), for this part try to intuitively find T for $p = 0.5$. If you cannot do that, assume T is given when you solve this part.

Solution:

At $p = 0.5$, $T = 0$ (can be found even intuitively). Thus,

$$\begin{aligned} P[error] &= P[error|X=1]P[X=1] + P[error|X=-1]P[X=-1] \\ &= \frac{1}{2}\int_{y=-\infty}^{0} f_{Y_1}(y)dy + \frac{1}{2}\int_{y=0}^{\infty} f_{Y_{-1}}(y)dy. \end{aligned}$$

From the symmetry, we conclude that:

$$\begin{aligned} P[error] &= \frac{\alpha}{2}\int_{y=0}^{\infty} e^{-\alpha(y+1)}dy = \frac{\alpha}{2}e^{-\alpha}\int_{y=0}^{\infty} e^{-\alpha y}dy \\ &= \frac{\alpha}{2}e^{-\alpha}\left[\frac{e^{-\alpha y}}{\alpha}\right]_{\infty}^{0} = \frac{1}{2}e^{-\alpha}. \end{aligned}$$

(d) Explain the effect of α on the probability of error.

Hint 2: You can answer this part even without reaching the result of part (c).

Solution: As α increases, the variance of the noise N decreases, and in response, the probability of error $P[error]$ decreases. This is consistent with the relation we reached in (c), in which $P[error]$ decreases exponentially with α.