EE 131A | Homework 8
Probability and Statistics | Thursday, February 28, 2019
Instructor: Lara Dolecek | Due: Thursday, March 07, 2019
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Reading: Chapter 5 & 6 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Prove the following statement: $|\rho_{XY}| = 1$ if and only if there exist numbers $a \neq 0$ and b such that $P(Y = aX + b) = 1$. If $\rho_{XY} = 1$, then $a > 0$, and if $\rho_{XY} = -1$, then $a < 0$. (X and Y are linear dependent if and only if $|\rho_{XY}| = 1$)
   Proof:
   Consider the function $h(t)$ defined by
   $$h(t) = E[(X - m_X)t + (Y - m_Y)]^2 = t^2\sigma_X^2 + 2t\text{COV}(X, Y) + \sigma_Y^2.$$
   Since $h(t) \geq 0$ and it is quadratic function, the discriminant of this quadratic function satisfies the following:
   $$(2\text{COV}(X, Y))^2 - 4\sigma_X^2\sigma_Y^2 \leq 0.$$
   This is equivalent to
   $$-\sigma_X\sigma_Y \leq \text{COV}(X, Y) \leq \sigma_X\sigma_Y.$$
   That is,
   $$-1 \leq \rho_{XY} \leq 1.$$
   $|\rho_{XY}| = 1$ if and only if the discriminant is equal to 0, that is, if and only if h(t) has a single root. But since $[(X - m_X)t + (Y - m_Y)]^2 \geq 0$, $h(t) = 0$ if and only if
   $$P[(X - m_X)t + (Y - m_Y) = 0] = 1.$$
   This $P[Y = aX + b] = 1$ with $a = -t$ and $b = m_Xt + m_Y$, where t is the root of $h(t)$. Using the quadratic formula, we see that this root is $t = -\frac{\text{COV}(X,Y)}{\sigma_X^2}$. Thus $a = -t$ has the same sign as ρ_{XY}, proving the final assertion.

2. Suppose X and Y are independent exponential random variables with common parameter λ, and $Z = \frac{X}{X+Y}$. Find the PDF $f_Z(z)$.
   Solution:
   We have $Z = \frac{X}{X+Y}$, $x \geq 0$, and $y \geq 0$. Thus, the range of z (the value Z can take) is between 0 and 1.
   We start by getting the CDF $F_Z(z)$.
   $$F_Z(z) = P[Z \leq z] = P\left[\frac{X}{X+Y} \leq z\right] = P\left[Y \geq \frac{1-z}{z}x\right].$$
   The area we need to integrate upon is shown in Figure 1.

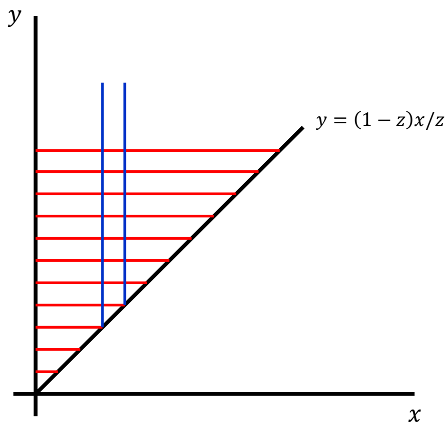

Figure 1: Area of integration for the CDF of $X/(X+Y)$.

Thus, the CDF can be derived as follows:

$$\begin{aligned}
F_Z(z) &= \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} f_{X,Y}(x,y)dydx = \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} f_X(x)f_Y(y)dydx \\
&= \int_{x=0}^{\infty}\int_{y=\frac{1-z}{z}x}^{\infty} \lambda^2 e^{-\lambda x}e^{-\lambda y}dydx \\
&= \int_{x=0}^{\infty}\lambda^2 e^{-\lambda x}\int_{y=\frac{1-z}{z}x}^{\infty} e^{-\lambda y}dydx \\
&= \int_{x=0}^{\infty}\lambda^2 e^{-\lambda x}\left[\frac{e^{-\lambda y}}{\lambda}\right]_{y=\infty}^{\frac{1-z}{z}x}.
\end{aligned}$$

Thus, we can see that:

$$\begin{aligned}
F_Z(z) &= \lambda\int_{x=0}^{\infty} e^{-\lambda x}e^{-\lambda(\frac{1}{z}-1)x}dx \\
&= \lambda\int_{x=0}^{\infty} e^{-\frac{\lambda}{z}x}dx = \lambda\left[\frac{z}{\lambda}e^{-\frac{\lambda}{z}x}\right]_{x=\infty}^{0}.
\end{aligned}$$

This means that $F_Z(z) = z$, where $0 \le z \le 1$.

Now, we are ready to get the PDF:

$$f_Z(z) = \frac{d}{dz}F_Z(z) = 1,\ 0 \le z \le 1.$$

Therefore, the random variable $Z = \frac{X}{X+Y}$ is uniform in the interval $[0, 1]$.

3. Consider the jointly Gaussian random variables X and Y that have the following joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right].$$

(a) Prove that Y is a Gaussian random variable by deriving its marginal PDF, $f_Y(y)$. Find the mean and variance of Y.

Solution:

The marginal PDF of Y, $f_Y(y)$ is derived as follows:

$$\begin{aligned}
f_Y(y) &= \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)dx \\
&= \int_{x=-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right)\right] dx.
\end{aligned}$$

To perform this integral, we need to complete a square inside the argument of the exponential.

$$\begin{aligned}
\text{Exp_arg} &= -\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right) \\
&= -\frac{1}{2(1-\rho^2)}\left(\left[\frac{x}{\sigma_X}-\frac{\rho y}{\sigma_Y}\right]^2-\frac{\rho^2y^2}{\sigma_Y^2}+\frac{y^2}{\sigma_Y^2}\right) \\
&= -\frac{1}{2(1-\rho^2)}\left[\frac{x}{\sigma_X}-\frac{\rho y}{\sigma_Y}\right]^2-\frac{1}{2(1-\rho^2)}\frac{(1-\rho^2)y^2}{\sigma_Y^2} \\
&= -\frac{1}{2(1-\rho^2)\sigma_X^2}\left[x-\frac{\rho\sigma_X y}{\sigma_Y}\right]^2-\frac{y^2}{2\sigma_Y^2}.
\end{aligned}$$

Substituting this exponential argument in the integral of $f_Y(y)$ gives us:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{y^2}{2\sigma_Y^2}\right]\int_{x=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left[-\frac{\left[x-\frac{\rho\sigma_X y}{\sigma_Y}\right]^2}{2\sigma_X^2(1-\rho^2)}\right]dx$$

The value of this integral is 1. Thus,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{y^2}{2\sigma_Y^2}\right], \quad -\infty < y < -\infty,$$

which proves that Y is a Gaussian random variable with mean 0 and variance σ_Y^2.

(b) Prove that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable, then find its mean and variance.

Solution:

The conditional PDF $f_{X|Y}(x|y)$ is derived as follows:

$$\begin{aligned}
f_{X|Y}(x|y) &= f_{X,Y}(x,y)/f_Y(y) \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right)+\frac{y^2}{2\sigma_Y^2}\right].
\end{aligned}$$

One more time, we operate on the exponential argument:

$$\begin{aligned}
\text{Exp_arg} &= -\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}-\frac{2\rho xy}{\sigma_X\sigma_Y}\right)+\frac{y^2}{2\sigma_Y^2} \\
&= -\frac{1}{2(1-\rho^2)}\left[\frac{x}{\sigma_X}-\frac{\rho y}{\sigma_Y}\right]^2+\frac{1}{2(1-\rho^2)}\left[\frac{\rho^2y^2}{\sigma_Y^2}-\frac{y^2}{\sigma_Y^2}\right]+\frac{y^2}{2\sigma_Y^2} \\
&= -\frac{1}{2(1-\rho^2)}\left[\frac{x}{\sigma_X}-\frac{\rho y}{\sigma_Y}\right]^2-\frac{y^2}{2\sigma_Y^2}+\frac{y^2}{2\sigma_Y^2} \\
&= -\frac{1}{2\sigma_X^2(1-\rho^2)}\left[x-\frac{\rho\sigma_X y}{\sigma_Y}\right]^2.
\end{aligned}$$

Consequently, we conclude that:

$$f_{X|Y}(x|y)=\frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2\sigma_X^2(1-\rho^2)}\left[x-\frac{\rho\sigma_X y}{\sigma_Y}\right]^2\right],$$

where $-\infty < x < \infty$. This proves that $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable with mean $\rho\sigma_X y/\sigma_Y$, and variance $\sigma_X^2(1-\rho^2)$.

4. Suppose X and Y are two independent Gaussian random variables, each with mean zero and variance σ^2. Let $Z=\sqrt{X^2+Y^2}$.

   (a) Find the CDF $F_Z(z)$ and the PDF $f_Z(z)$ of the random variable Z.
   Solution:
   Observe that $z=\sqrt{x^2+y^2}$ is the equation of a circle with radius z, $z\geq 0$. Thus, we can derive the CDF, $F_Z(z)$, as follows:

$$\begin{aligned}
F_Z(z) &= P[Z\leq z]=P[\sqrt{X^2+Y^2}\leq z] \\
&= P[-\sqrt{z^2-x^2}\leq Y\leq\sqrt{z^2-x^2}] \\
&= \int_{x=-z}^{z}\int_{y=-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f_{X,Y}(x,y)dydx.
\end{aligned}$$

   Since X and Y are independent Gaussian random variables with mean zero and variance σ^2, $f_{X,Y}(x,y)=f_X(x)f_Y(y)$, which results in:

$$F_Z(z)=\int_{x=-z}^{z}\int_{y=-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}}\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right]dydx. \tag{1}$$

   In order to perform such integral easily, we transform our coordinates into the polar coordinates as follows:

$$\begin{aligned}
x=rcos\theta,\ y=rsin\theta,\ x^2+y^2=r^2, \\
dydx=dxdy=rdrd\theta.
\end{aligned} \tag{2}$$

Substituting (2) in (1) gives us:

$$\begin{aligned}
F_Z(z) &= \int_{\theta=0}^{2\pi}\int_{r=0}^{z} \frac{1}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right] r dr d\theta \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{z} -\frac{1}{2\pi} d\left(\exp\left[-\frac{r^2}{2\sigma^2}\right]\right) d\theta \\
&= \int_{\theta=0}^{2\pi} \frac{1}{2\pi}\left(\exp\left[-\frac{r^2}{2\sigma^2}\right]\right)_{r=z}^{0} d\theta \\
&= \int_{\theta=0}^{2\pi} \frac{1}{2\pi}\left(1-\exp\left[-\frac{z^2}{2\sigma^2}\right]\right) d\theta \\
&= 1-\exp\left[-\frac{z^2}{2\sigma^2}\right], \ z \geq 0.
\end{aligned}$$

By differentiating the CDF, we get the PDF as follows:

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{z}{\sigma^2}\exp\left[-\frac{z^2}{2\sigma^2}\right], \ z \geq 0.$$

(b) Prove that $f_Z(z)$ is indeed a valid PDF.
Solutions:
To prove that $f_Z(z)$ is a valid PDF, we need to check on two conditions. First, it has to be non-negative over its entire range, which is obviously the case for $f_Z(z)$, when $z \geq 0$. Second, we need to make sure that its integration over the entire range gives 1.

$$\begin{aligned}
\int_{z=0}^{\infty} f_Z(z)dz &= \int_{z=0}^{\infty} \frac{z}{\sigma^2}\exp\left[-\frac{z^2}{2\sigma^2}\right] dz = \int_{z=0}^{\infty} -d\left(\exp\left[-\frac{z^2}{2\sigma^2}\right]\right) \\
&= \left(\exp\left[-\frac{z^2}{2\sigma^2}\right]\right)_{z=\infty}^{0} = 1,
\end{aligned}$$

which means it is a valid PDF (Z is a Rayleigh random variable).

5. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \frac{x}{4}, & 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 2\}$.

(a) Find $E[X]$, $P[A]$, $f_{X|A}(x)$, and $E[X|A]$.
Solutions:

$$E[X] = \int_1^3 x\frac{x}{4}dx = \left[\frac{x^3}{12}\right]_1^3 = \frac{13}{6}$$

$$P[A] = \int_2^3 \frac{x}{4}dx = \left[\frac{x^2}{8}\right]_2^3 = \frac{5}{8}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{f_X(x)}{P[A]} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X|A}(x|A) = \begin{cases} \frac{2x}{5} & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$E[X|A] = \int_2^3 x\frac{2x}{5}dx = \left[\frac{2x^3}{15}\right]_2^3 = \frac{38}{15}$$

(b) Let $Y = X^2$. Find $E[Y]$ and $Var[Y]$.
Solutions:

$$E[Y] = E[X^2] = \int_1^3 x^2\frac{x}{4}dx = \left[\frac{x^4}{16}\right]_1^3 = 5$$

$$E[Y^2] = E[X^4] = \int_1^3 x^4\frac{x}{4}dx = \left[\frac{x^6}{24}\right]_1^3 = \frac{91}{3}$$

$$Var[Y] = E[Y^2] - E[Y]^2 = \frac{16}{3}$$