EE 131A
Probability and Statistics
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Homework 2 Solution
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Reading: Chapter 2 & 3 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Suppose that each child born to a couple is equally likely to be a boy or a girl, independently of the gender distribution of the other children in the family. For a couple having 5 children, compute the probabilities of the following events:

   (a) All children are of the same gender.
   Solution:
   To have all of the same gender of children means that they are all girls or all boys and will happen with a probability

   $$\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{1}{16}.$$

   (b) The 3 eldest are boys and the others girls.
   Solution:
   To first have 3 boys and then 2 girls will happen with probability

   $$\left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

   (c) Exactly 3 are boys.
   Solution:
   To have exactly 3 boys (independent of their ordering) will happen with probability

   $$\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}$$

   .

   (d) The 2 oldest are girls.
   Solution:
   To have the first 2 children be girls (independent of what the other children are) will happen with probability

   $$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

   .

   (e) There is at least 1 girl.
   Solution:

To have at least one girl (not all boys) is the complement of the even of having no girls, or of having all boys. Thus

$$1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

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2. Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Show that the events A, B, and C are pairwise independent—that is, A and B are independent, A and C are independent, and B and C are independent—but not independent.

   Solution:

   Let A be the event that the first toss lands heads and let B be the event that the second toss lands heads, and finally let C be the event that both lands on the same side. Now $P(A,B) = \frac{1}{4}$, and $P(A) = P(B) = \frac{1}{2}$, so A and B are independent. Now

$$P(A,C) = P(C|A)P(A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

   but $P(C) = \frac{1}{2}$ so $P(A,C) = P(A)P(C)$ and A and C are independent. Finally

$$P(B,C) = P(C|B)P(B) = \frac{1}{4},$$

   so again B and C are independent. Thus A, B, and C are pairwise independent but for three sets to be fully independent we must have in addition that

$$P(A,B,C) = P(A)P(B)P(C).$$

   The right hand sides of this expression is $\left(\frac{1}{2}\right)^3$ while the left hand side is the event that both tosses land heads and so $P(A,B,C) = \frac{1}{4} \neq P(A)P(B)P(C)$ and the three sets are not independent.

3. In a bolt factory machines A, B, C manifacture, respectively 25, 35 and 40 per cent of the total. Of their product 5, 4, and 2 per cent are defective bolts. A bolt is drawn at random from the produce and is found defective. What are the probabilities that it was manufactured by machines A, B and C?

   Solution:

   Let D denote the event that a bolt randomly drawn from the produce is defective and A, B, C denote the events that it was manifactured by machines A, B and C respectively. We are interested in the probabilities $P(A|D)$, $P(B|D)$, $P(C|D)$. We have,

$$\begin{aligned} P(D) &= P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C) \\ &= 0.05 \cdot 0.25 + 0.04 \cdot 0.35 + 0.02 \cdot 0.4 \qquad (1) \\ &= 0.0345 \end{aligned}$$

By the Bayesian rule, we get

$$P(A|D) = \frac{P(D|A)P(A)}{P(D)} = \frac{0.05 \cdot 0.25}{0.0345} = 0.3623,$$

$$P(B|D) = \frac{P(D|B)P(B)}{P(D)} = \frac{0.04 \cdot 0.35}{0.0345} = 0.4058,$$

and

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{0.02 \cdot 0.40}{0.0345} = 0.2319.$$

4. For the following problem, ignore the year of birth while comparing two birthdays. Moreover, assume that the year is exactly 365 days (ignore the 29th of February). Note that matching birthdays means the birthdays are the same.

   (a) You are a member of a class room that has $n+1$ students including you. What is the probability that you find at least one student, other than you, who has a birthday that matches yours?
   Solution:
   The probability that a student does not have a birthday that matches yours is fixed for any student (other than yourself), which is $\left(1 - \frac{1}{365}\right)$. Thus, the probability that no student in this room has a birthday that matches yours is:

   $$p_{nm,1} = \prod_{i=1}^{n} \left(1 - \frac{1}{365}\right) = \left(1 - \frac{1}{365}\right)^{n}. \tag{2}$$

   This means that the probability of at least one match for you is:

   $$p_{m,1} = 1 - p_{nm,1} = 1 - \left(\frac{364}{365}\right)^{n}.$$

   (b) In another classroom that has m students ($m << 365$), what is the probability that at least two students have matching birthdays?
   Solution:
   This part is exactly the last problem in Discussion 1. Consequently, no need to write the solution in detail. The probability of no matching birthdays in this room is:

   $$\begin{aligned} p_{nm,2} &= \prod_{i=1}^{m-1} \left(1 - \frac{i}{365}\right) \qquad (3) \\ &= \frac{365(364)\ldots(365-m+1)}{365^m} = \frac{P_m^{365}}{365^m} = \frac{\binom{365}{m} m!}{365^m}. \end{aligned}$$

   This means that the probability of at least one match is:

   $$p_{m,2} = 1 - p_{nm,2} = 1 - \frac{\binom{365}{m} m!}{365^m}.$$

(c) Now let P_1 be the probability that you do not find any student, in your classroom that has $n+1$ students, who has a birthday that matches yours. Additionally, let P_2 be the probability that there are no birthday matches in the room that has m students ($m << 365$). Derive the relation between n and m such that $P_1 = P_2^2$. *Hint: The following approximation will be useful: for small x, $1 - x \approx e^{-x}$.*
Solution:
It is clear that $P_1 = p_{nm,1}$ and $P_2 = p_{nm,2}$. Since we have already computed the probabilities of no matches, we can use (2) and (3) to conclude the following if $P_1 = P_2^2$:

$$\left(1 - \frac{1}{365}\right)^n = \left[\prod_{i=1}^{m-1}\left(1 - \frac{i}{365}\right)\right]^2. \tag{4}$$

Since $1 << 365$, and also $m << 365$, we can use the approximation in the hint to simplify (3):

$$\begin{aligned} \left(e^{-1/365}\right)^n &= \left[\prod_{i=1}^{m-1}\left(e^{-i/365}\right)\right]^2 \\ e^{-n/365} &= e^{-2(1+2+\cdots+m-1)/365}. \end{aligned} \tag{5}$$

By taking the natural base log (which is ln) of both sides, we reach:

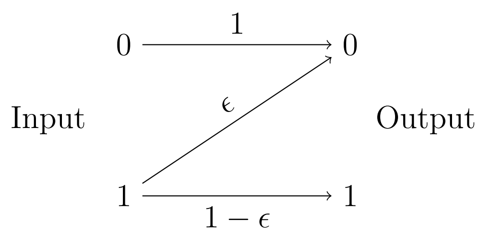
$$\begin{aligned} -\frac{n}{365} &= -2\left(\frac{1+2+\cdots+m-1}{365}\right) \\ n &= 2\frac{(m-1)m}{2} = m(m-1). \end{aligned} \tag{6}$$

5. A binary Z-channel is show in the figure. Assume the input is "0" with probability p and "1" with probability $1-p$.

(a) What can you say about the input bit if "1" is received?
Solution:
According to figure of the channel, there is no way to send "0" and receive "1". Hence, receiving "1" reveals that the input bit has been "1".

(b) Find the the probability that the input was "1" given that the output is "0".
Solution:
Define four events as $T_0 = \{$"0" is transmitted$\}$, $T_1 = \{$"1" is transmitted$\}$, $R_0 = \{$"0" is received$\}$, and $R_1 = \{$"1" is received$\}$.
$P(R_0) = P(R_0|T_0)P(T_0) + P(R_0|T_1)P(T_1) = 1 \cdot p + \epsilon \cdot (1-p)$.

$$P(T_1|R_0) = \frac{P(T_1)P(R_0|T_1)}{P(R_0)} = \frac{(1-p)\epsilon}{p + \epsilon(1-p)}$$



6. A transmitter randomly sends one of the messages in $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The receiver either receives the transmitted message with probability p, or mistakenly receives one of the other messages with equal probabilities,

   (a) What is the probability of receiving α_1 at the receiver?
   Solution:
   Let $P(R = \alpha_i)$ be the probability of α_i is received, and $P(T = \alpha_i)$ be the probability of α_i is transmitted.

$$\begin{aligned} P(R = \alpha_1) &= P(R = \alpha_1 | T = \alpha_1) P(T = \alpha_1) + P(R = \alpha_1 | T \neq \alpha_1) P(T \neq \alpha_1) \\ &= \frac{1}{n} P(R = \alpha_1 | T = \alpha_1) + \frac{n-1}{n} P(R = \alpha_1 | T \neq \alpha_1) \\ &= \frac{1}{n} p + \frac{n-1}{n} \frac{1-p}{n-1} = \frac{p}{n} + \frac{1-p}{n} = \frac{1}{n}. \end{aligned}$$

   This can also be inferred from the symmetry of the problem (the fact that the n messages do not have any advantages over each other).

   (b) If α_1 is received at the receiver, what is the probability that the transmitted message was α_1?
   Solution:

$$\begin{aligned} P(T = \alpha_1 | R = \alpha_1) &= \frac{P(R = \alpha_1 | T = \alpha_1) P(T = \alpha_1)}{P(R = \alpha_1)} \\ &= \frac{p\frac{1}{n}}{\frac{1}{n}} = p. \end{aligned}$$