EE 131A
Probability and Statistics
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Homework 7
Thursday, February 21, 2019
Due: Thursday, February 28, 2019
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Reading: Chapter 5 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. Find the Chernoff bound for the exponential random variable with parameter λ.
Solution:
Let X be distributed binomially with parameter λ.
The Chernoff bound for the exponential random variable with parameter λ is given by:

$$P[X \geq a] \leq e^{-sa}E[e^{sX}] \text{ where } s > 0.$$

Since X is exponentially distributed, $E[e^{sX}] = \frac{\lambda}{\lambda - s}$. Therefore, the Chernoff bound is:

$$P[X \geq a] \leq e^{-sa} \times \frac{\lambda}{\lambda - s}$$

Let $g(s) = e^{-sa} \times \frac{\lambda}{\lambda - s}$.
Differentiate $g(s)$ with respect to s.

$$g'(s) = \frac{-\lambda a e^{-sa}(\lambda - s) + \lambda e^{-sa}}{(\lambda - s)^2}$$

When $g'(s) = 0$, implies $s = \lambda - \frac{1}{a}$. Since $s > 0$, from the relation $s = \lambda - \frac{1}{a}$ implies $\lambda > \frac{1}{a}$ or $a > \frac{1}{\lambda}$. Plug in the value of $s = \lambda - \frac{1}{a}$, the Chernoff bound is:

$$P[X \geq a] \leq \lambda a e^{-(\lambda a - 1)} \text{ for } a > \frac{1}{\lambda}.$$

2. The input X to a communication channel is "-1" or "1", with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$.The output of the channel Y is equal to: the corresponding input X with probability $1 - p - p_e$; $-X$ with probability p; 0 with probability p_e.

   (a) Describe the underlying space S of this random experiment and show the mapping from S to S_{XY}, the range of the pair (X, Y).
   Solution:
   Input $X \in \{-1, 1\}$ and output $Y \in \{-1, 0, 1\}$, then the mapping from S to S_{XY} is all possible combinations of (X, Y) pairs:

   $$S_{XY} = \{(-1, -1), (-1, 0), (-1, 1), (1, -1), (1, 0), (1, 1)\}$$

   (b) Find the probabilities for all values of (X, Y).
   Solution:
   The probabilities for all values of (X, Y) are given by:
   $P[X = -1, Y = -1] = (1 - p - p_e)/4$

$P[X=-1, Y=0] = p_e/4$
$P[X=-1, Y=1] = p/4$
$P[X=1, Y=-1] = 3p/4$
$P[X=1, Y=0] = 3p_e/4$
$P[X=1, Y=1] = 3(1-p-p_e)/4$

(c) Find $P[X \neq Y]$, $P[Y=0]$.
Solution:
$P[X \neq Y] = p_e/4 + p/4 + 3p/4 + 3p_e/4 = p + p_e$
$P[Y=0] = p_e/4 + 3p_e/4 = p_e$

3. The pair (X, Y) has joint cdf given by:

$$F_{X,Y}(x,y) = \begin{cases} (1-\frac{1}{x^2})(1-\frac{1}{y^2}) & \text{for } x > 1, y > 1 \\ 0 \quad \text{elsewhere.} \end{cases}$$

(a) Sketch the joint cdf.
Solution:
The joint CDF is sketched in Figure 1:

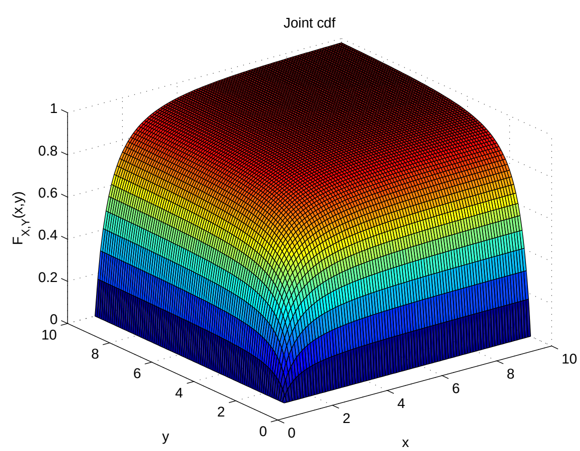

Figure 1: Joint CDF in Problem 6

(b) Find the marginal cdf of X and of Y.
Solution:
The marginal CDF of X and Y can be computed as follows:

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = 1 - \frac{1}{x^2},\ x > 1$$

$$F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y) = 1 - \frac{1}{y^2},\ y > 1$$

(c) Find the probability of the following events: $\{X < 3, Y \le 5\}, \{X > 4, Y > 3\}$.
Solution:
The probabilities of the following events are given by:

$$P\{X < 3, Y \le 5\} = F_{X,Y}(3,5) = (1 - \frac{1}{3^2})(1 - \frac{1}{5^2}) = \frac{64}{75}$$

$$\begin{aligned} P\{X > 4, Y > 3\} &= 1 - F_{X,Y}(4,\infty) - F_{X,Y}(\infty,3) + F_{X,Y}(4,3) \\ &= 1 - \frac{15}{16} - \frac{8}{9} + \frac{5}{6} = \frac{1}{144} \end{aligned}$$

4. Is the following a valid cdf? Why?

$$F_{X,Y}(x,y) = \begin{cases} (1 - \frac{1}{x^2y^2}) & \text{for } x > 1, y > 1 \\ 0 \quad \text{elsewhere.} \end{cases}$$

Solution: The joint CDF is given by:

$$F_{X,Y}(x,y) = \begin{cases} (1 - 1/x^2y^2) & \text{for } x > 1, y > 1 \\ 0 & \text{elsewhere} \end{cases}$$

Then the marginal CDF of X is given by:

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y) = 1 \text{ for all } x > 1$$

The marginal CDF of X cannot be 1 for all $x > 1$ (not right-continuous), therefore $F_{X,Y}(x,y)$ is not a valid joint CDF.

5. Let X and Y have the joint pdf:

$$f_{X,Y}(x,y) = ye^{-y(1+x)} \quad \text{for } x > 0, y > 0.$$

Find the marginal pdf of X and of Y.
Solution:
The joint PDF is given by:

$$f_{X,Y}(x,y) = ye^{-y(1+x)},\ x > 0,\ y > 0$$

Therefore the marginal PDFs of X and Y can be found as follows:

$$
\begin{aligned}
f_X(x) &= \int_0^\infty y e^{-y(1+x)} \mathrm{d}y = -\frac{1}{1+x} e^{-y(1+x)} y \mid_0^\infty + \frac{1}{1+x} \int_0^\infty e^{-y(1+x)} \mathrm{d}y \\
&= \frac{1}{(1+x)^2}, \; x > 0
\end{aligned}
$$

$$
f_Y(y) = \int_0^\infty y e^{-y(1+x)} \mathrm{d}x = y e^{-y} \int_0^\infty e^{-xy} \mathrm{d}x = e^{-y}, \; y > 0
$$