EE 131A — Homework 3 Solution
Probability and Statistics — Tuesday, January 22, 2019
Instructor: Lara Dolecek — Due: Tuesday, January 29, 2019
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Reading: Chapter 3 & 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.
Solution:
There are 9 possible outcomes, as summarized in Table 1. Summing all possible ways to get the various values of X we find

$$\begin{aligned} P[X=0] &= 0.28 \\ P[X=500] &= 0.21+0.06=0.27 \\ P[X=1000] &= 0.21+0.045+0.06=0.315 \\ P[X=1500] &= 0.045+0.045=0.09 \\ P[X=2000] &= 0.045. \end{aligned}$$

Sale from Customer 1	Sales from Customer 2	X	Probability
0	0	0	$(1-0.3)(1-0.6)=0.28$
0	500	500	$(1-0.3)(0.6)(0.5)=0.21$
0	1000	1000	$(1-0.3)(0.6)(0.5)=0.21$
500	0	500	$(0.3)(0.5)(1-0.6)=0.06$
500	500	1000	$(0.3)(0.5)(0.6)(0.5)=0.045$
500	1000	1500	$(0.3)(0.5)(0.6)(0.5)=0.045$
1000	0	1000	$(0.3)(0.5)(1-0.6)=0.06$
1000	500	1500	$(0.3)(0.5)(0.6)(0.5)=0.045$
1000	1000	2000	$(0.3)(0.5)(0.6)(0.5)=0.045$

Table 1: Encyclopedia Sales.

2. An insurance company writes a policy to the effect that an amount of money A must be paid if some event E occurs within a year. If the company estimates that E will occur within a year with probability p, what should it charge the customer in order that its expected profit will be 10 percent of A?
Solution:
The company desires to make $0.1A$ of a profit. Assuming the cost charged to each

customer is C, the expected profit of the company then given by

$$C + p(-A) + (1-p)(0) = C - pA.$$

This can be seen as the fixed cost received from the paying customers minus what is lost if a claim must be paid out. For this to be $0.1A$ we should have $C - pA = 0.1A$ or solving for C we have

$$C = \left(p + \frac{1}{10}\right) A.$$

3. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.
Solution:
We can explicitly calculate the number of defective items obtained in the sample of twenty. We find that

$$\begin{aligned}
P_0 &= \frac{\binom{16}{3}\binom{4}{0}}{\binom{20}{3}} = 0.491 \\
P_1 &= \frac{\binom{16}{2}\binom{4}{1}}{\binom{20}{3}} = 0.421 \\
P_2 &= \frac{\binom{16}{1}\binom{4}{2}}{\binom{20}{3}} = 0.084 \\
P_3 &= \frac{\binom{16}{0}\binom{4}{3}}{\binom{20}{3}} = 0.0035
\end{aligned}$$

so the expected number of defective items is given by

$$3P_3 + 2P_2 + 1P_1 + 0P_0 = \frac{3}{5}.$$

4. Suppose X is a Binomial random variable with parameters $n = 4$, and p.

   (a) Express $E\left[sin\left(\pi X/2\right)\right]$ in terms of p.
   Solution:
   The PMF of X is given by:

   $$p_X(x) = \binom{4}{x} p^x (1-p)^{4-x},\ x = 0, 1, 2, \ldots, n.$$

   As a result, we conclude that:

   $$\begin{aligned}
   E\left[sin\left(\frac{\pi X}{2}\right)\right] &= \sum_{x=0}^{4} sin\left(\frac{\pi x}{2}\right)\binom{4}{x} p^x (1-p)^{4-x} \\
   &= 0 + sin\left(\frac{\pi}{2}\right)\binom{4}{1} p(1-p)^3 + 0 + sin\left(\frac{3\pi}{2}\right)\binom{4}{3} p^3(1-p) + 0 \\
   &= 4p(1-p)^3 - 4p^3(1-p) = 4p(1-p)(1-2p).
   \end{aligned}$$

(b) Express $E\left[cos\left(\pi X/2\right)\right]$ in terms of p.

Solution:

Similar to part (a), we can see that:

$$E\left[cos\left(\frac{\pi X}{2}\right)\right] = \sum_{x=0}^{4} cos\left(\frac{\pi x}{2}\right)\binom{4}{x}p^x(1-p)^{4-x}$$
$$= cos\left(0\right)\binom{4}{0}(1-p)^4 + 0 + cos\left(\pi\right)\binom{4}{2}p^2(1-p)^2$$
$$+ 0 + cos\left(2\pi\right)\binom{4}{4}p^4 = (1-p)^2(1-2p-5p^2) + p^4.$$

5. A computer reserves a path in a network for 10 minutes. To extend the reservation the computer must successfully send a "refresh" message before the expiry time. However, messages are lost with probability 1/2. Suppose that it takes 10 seconds to send a refresh request and receive an acknowledgment. When should the computer start sending refresh messages in order to have a 99% chance of successfully extending the reservation time?

Solution:

Let X be the number of refresh messages sent by the computer till receiving the first successful acknowledgment. For $k \geq 1$, $P(X = k) = (\frac{1}{2})^k$. We are looking for the minimum number of messages sent by the computer, k_{min}, such that $P(X \leq k_{min}) \geq 0.99$.

$P(X \leq k_{min}) = \sum_{i=1}^{k_{min}}(\frac{1}{2})^i = \frac{\frac{1}{2}(1-(\frac{1}{2})^{k_{min}})}{1-\frac{1}{2}} = 1 - (\frac{1}{2})^{k_{min}} \geq 0.99$, therefore, $k_{min} \geq 7$. It means the computer needs to send at least 7 refresh messages to make sure that it would extend the reservation with probability of 0.99. Since every transmission takes 10 seconds, the computer has to start sending refresh messages 70 seconds before its 10 minutes ends, i.e, 8 minutes and 50 seconds after starting the session.

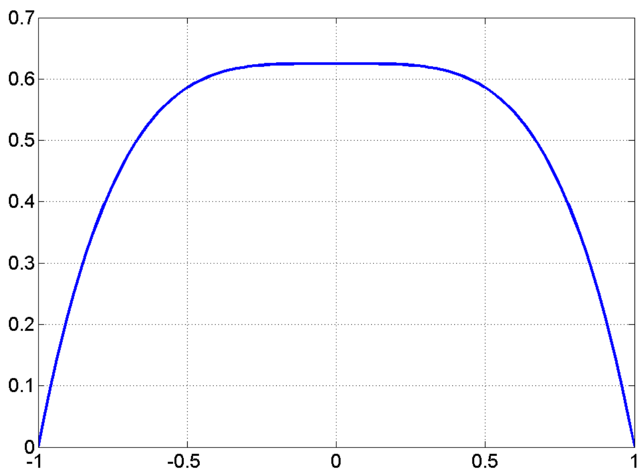
6. Consider a random variable X with pdf given by

$$f_X(x) = \begin{cases} c(1-x^4) & -1 \le x \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) What is c? Plot the pdf.
Solution:
The pdf must have area 1 under its curve, so we have that:

$$\int_{-1}^{1} f_X(x)dx = \int_{-1}^{1} c(1-x^4)dx = 1.$$

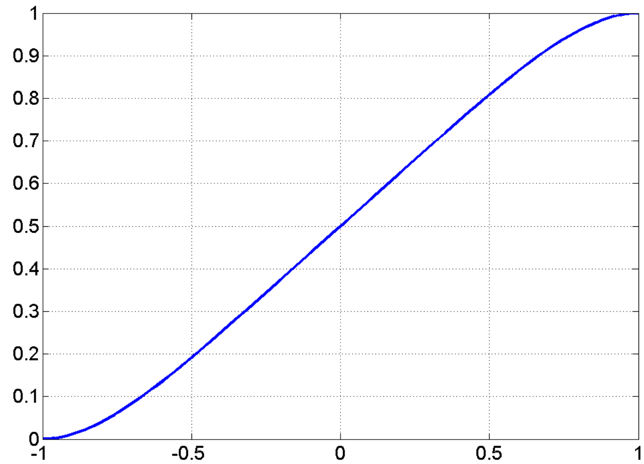
Performing the definite integral, we get that $(cx - cx^5/5)|_{-1}^{1} = 1$, or, $c(2-2/5) = 1$. Thus, $c = 1/(2-2/5) = 1/(8/5) = 5/8$. The pdf is $f_X(x) = \frac{5}{8}(1-x^4)$. The plot:



(b) Plot the cdf of X.
Solution:
Now we use the integral of the pdf to get the cdf.
We have that for $-1 \le x \le 1$, $F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{-1}^{x} \frac{5}{8}(1-t^4)dt = \left[\frac{5}{8}t - \frac{1}{8}t^5\right]_{-1}^{x}$. This is just $\frac{5}{8}x - \frac{1}{8}x^5 - (-\frac{5}{8} + \frac{1}{8}) = \frac{5}{8}x - \frac{1}{8}x^5 + \frac{1}{2}$. For the other two cases, we have $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 1$. Thus,

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{5}{8}x - \frac{1}{8}x^5 + \frac{1}{2} & -1 \le x \le 1 \\ 1 & x > 1. \end{cases}$$

The plot is below:



(c) Find $P(|X - 0.5| < 0.3)$.
Solution:
$P(|X - 0.5| < 0.3) = P(-0.3 < X - 0.5 < 0.3) = P(0.2 < X < 0.8) = F_X(0.8) - F_X(0.2) = 0.3341$.