EE 131A
Probability and Statistics
Instructor: Lara Dolecek
TA: Ruiyi (John) Wu

Homework 4 Solution
Tuesday, January 29, 2019
Due: Tuesday, February 05, 2019
ruiyiwu@g.ucla.edu

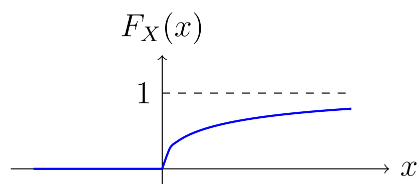
Reading: Chapter 4 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. For $\beta > 0$ and $\lambda > 0$, the Weibull random variable X has cdf:

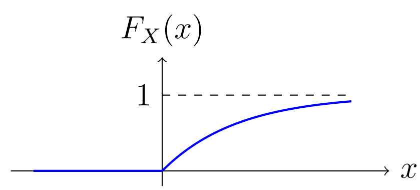
$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-(x/\lambda)^\beta} & \text{for } x \geq 0. \end{cases}$$

(a) Plot the cdf of X for $\beta = 0.5$, 1, and 2. Take $\lambda = 1$ for all the three plots.
Solution:
The plots are shown below:

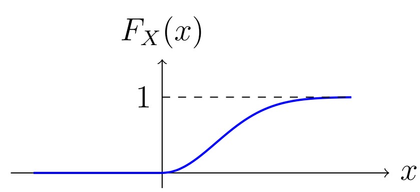
- $\beta = 0.5$



- $\beta = 1$



- $\beta = 2$



(b) Find the probability $P[k\lambda < X < (k+1)\lambda]$ and $P[X > k\lambda]$ for positive integer k.
Solution:

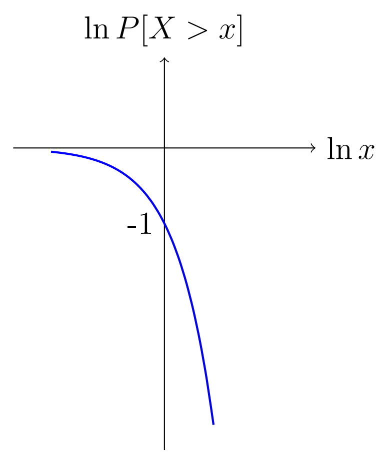
$$\begin{aligned} P[k\lambda < X < (k+1)\lambda] &= F_X((k+1)\lambda) - F_X(k\lambda) \\ &= (1 - e^{-((k+1)\lambda/\lambda)^\beta}) - (1 - e^{-(k\lambda/\lambda)^\beta}) \\ &= e^{-k^\beta} - e^{-(k+1)^\beta} \end{aligned}$$

$$P[X > k\lambda] = 1 - P[X \leq k\lambda] = 1 - F_X(k\lambda) = e^{-k^\beta}$$

(c) Plot $\ln P[X > x]$ vs. $\ln x$. Assume $\beta = 2$ and $\lambda = 1$.
Solution:
For $x \geq 0$,

$$\begin{aligned}\ln P[X > x] &= \ln(1 - P[X \leq x]) \\ &= \ln(1 - F_X(x)) \\ &= \ln e^{-(x/\lambda)^\beta} \\ &= -(x/\lambda)^\beta \\ &= -(e^{\ln x}/\lambda)^\beta\end{aligned}$$

For the following plot, $\beta = 2$, $\lambda = 1$, and $\ln P[X > x] = -e^{2\ln x}$.

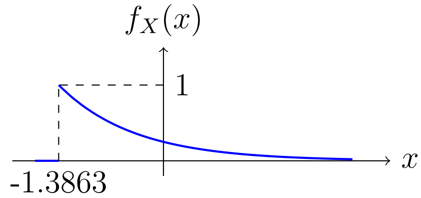


2. Find and plot the pdf of $X = -\ln(4 - 4U)$, where U is a continuous random variable, uniformly distributed on the $[0, 1]$ interval.
Solution:
First, we get the cdf. For $x < -\ln(4)$, $F_X(x) = 0$. For $x \geq -\ln(4)$,

$$\begin{aligned}F_X(x) &= P(X \leq x) \\ &= P(-\ln(4 - 4U) \leq x) = P(4 - 4U \geq e^{-x}) \\ &= P(4U \leq 4 - e^{-x}) = P\left(U \leq 1 - \frac{e^{-x}}{4}\right) \\ &= 1 - \frac{e^{-x}}{4}.\end{aligned}$$

Now we get the pdf. For $x < -\ln(4)$, $f_X(x) = 0$, and for $x \geq -\ln(4)$,

$$\begin{aligned}f_X(x) &= \frac{d}{dx}F_X(x) \\ &= \frac{e^{-x}}{4},\ x \geq -\ln(4) = -1.3863.\end{aligned}$$



3. Given a random variable U uniformly distributed on the $[0, 1]$ interval, i.e. $U[0, 1]$, in each case find the function g, such that for $Y = g(U)$,

   (a) $Y \sim U[5, 10]$.
   Solution:
   For $5 \leq y \leq 10$, $F_Y(y) = P(Y \leq y) = \frac{y-5}{5}$. On the other hand, $P(Y \leq y) = P(g(U) \leq y) = P(U \leq g^{-1}(y)) = g^{-1}(y)$. So, $g^{-1}(y) = \frac{y-5}{5}$ and $g(y) = 5y + 5$. Thus, $g(U) = 5U + 5$.

   (b) $f_Y(y) = \lambda e^{-\lambda y}$.
   Solution:
   $F_Y(y) = 1 - P(Y > y) = 1 - \int_y^\infty f_Y(u)du = 1 - \int_y^\infty \lambda e^{-\lambda u}du = 1 - e^{-\lambda y}$. On the other hand, $P(Y \leq y) = P(g(U) \leq y) = P(U \leq g^{-1}(y)) = g^{-1}(y)$. So, $g^{-1}(y) = 1 - e^{-\lambda y}$ and $g(y) = -\frac{1}{\lambda}\ln(1 - y)$. Thus, $g(U) = -\frac{1}{\lambda}\ln(1 - U)$.

4. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.
   Solution:
   An interpretation of this statement is that a point is picked randomly on a line segment of length L would be that the point "X" is selected from a uniform distribution over the interval $[0, L]$. Then the question asks us to find

$$P\left\{\frac{\min(X, L - X)}{\max(X, L - X)} < \frac{1}{4}\right\}.$$

   This probability can be evaluated by integrating over the appropriate region. Formally we have the above equal to

$$\int_E p(x)dx$$

   where $p(x)$ is the uniform probability density for our problem, i.e. $\frac{1}{L}$ and the set "E" is $x \in [0, L]$ and satisfying the inequality above, i.e.

$$\min(x, L - x) \leq \frac{1}{4}\max(x, L - x).$$

   Plotting the functions $\max(x, L - x)/4$, and $\min(x, L - x)$ in Figure 1, we see that the regions of X where we should compute the integral above are restricted to the two ends of the segment. Specifically, the integral above becomes,

$$\int_0^{l_1} p(x)dx + \int_{l_2}^L p(x)dx.$$

since the region $\min(x, L-x) < \frac{1}{4}\max(x, L-x)$ in satisfied in the region $[0, l_1]$ and $[l_2, L]$ only. Here l_1 is the solution to

$$\min(x, L-x) = \frac{1}{4}\max(x, L-x) \quad \text{when} \quad x < L-x,$$

i.e. we need to solve

$$x = \frac{1}{4}(L-x)$$

which has as its solution $x = \frac{L}{5}$. For l_2 we must solve

$$\min(x, L-x) = \frac{1}{4}\max(x, L-x) \quad \text{when} \quad L-x < x,$$

i.e. we need to solve

$$(L-x) = \frac{1}{4}x,$$

which has as its solution $x = \frac{4L}{5}$. With these two limits we have for our probability

$$\int_0^{\frac{L}{5}} \frac{1}{L}dx + \int_{\frac{4L}{5}}^{L} \frac{1}{L}dx = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$

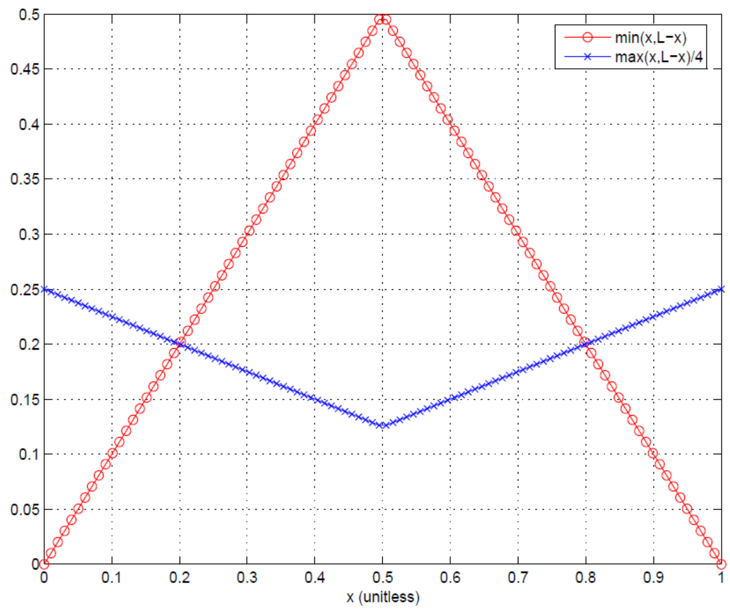


Figure 1: A graphical view of the region of x's over which the integral for this problem should be computed.

5. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} ax^2 e^{-bx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $b = m/2kT$ and k, T, and m denote, respectively, Boltzmann's constant, the absolute temperature of the gas, and the mass of the molecule. Evaluate a in terms of b.
Hint: The following will be useful: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ *and* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
Solution:
Since $f(x)$ is a probability density if must integrate to one $\int_{-\infty}^{\infty} f(x)dx = 1$. In the case here using integration by parts this becomes

$$\begin{aligned}\int_0^\infty ax^2e^{-bx^2}dx &= a\frac{xe^{-bx^2}}{(-2b)}\Big|_0^\infty - a\int_0^\infty \left(\frac{e^{-bx^2}}{(-2b)}\right)dx \\ &= 0 - 0 + \frac{a}{2b}\int_0^\infty e^{-bx^2}dx.\end{aligned}$$

To evaluate this integral let $v = bx^2$ so that $dv = 2bxdx$, $x = \pm\sqrt{\frac{v}{b}}$, $dv = 2b\sqrt{\frac{v}{b}}dx$, which gives

$$dx = \left(\frac{b^{\frac{1}{2}}}{2b}\right)v^{-\frac{1}{2}}dv = \frac{v^{-\frac{1}{2}}}{2\sqrt{b}}dv,$$

and our integral above becomes

$$1 = \frac{a}{2b}\frac{1}{2\sqrt{b}}\int_0^\infty v^{-\frac{1}{2}}e^{-v}dv.$$

Now the integral remaining can be seen to be

$$\int_0^\infty v^{-\frac{1}{2}}e^{-v}dv = \int_0^\infty v^{\frac{1}{2}-1}e^{-v}dv = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Using this we have

$$1 = \frac{a}{4b^{\frac{3}{2}}}\sqrt{\pi}.$$

Thus $a = \frac{4b^{\frac{3}{2}}}{\sqrt{\pi}}$ is the relationship between a and b.

6. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.
Solution:
We desire to calculate $E[P(D)]$, where $P(D)$ is the points scored when the distance to

the target is D. This becomes

$$\begin{aligned}
E[P(D)] &= \int_0^{10} P(D) f(D) dD \\
&= \frac{1}{10} \int_0^{10} P(D) dD \\
&= \frac{1}{10} \left(\int_0^1 10 dD + \int_1^3 5 dD + \int_3^5 3 dD + \int_5^{10} 0 dD \right) \\
&= \frac{1}{10} (10 + 5(2) + 3(2)) \\
&= \frac{26}{10} \\
&= 2.6
\end{aligned}$$