Complex Analysis MAT656 August 2019.

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These problems are from the pdf received from Prof. Blackmore. The course textbook is Ablowitz and Fokas [1].

1 August 2019 Problem 6. - Consider the function:

$$f(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + ... + \frac{1}{n!z^n}$$

1.1 a - What is counted by the integral?

$$f(z) = \frac{1}{2\pi i} \int \frac{1}{f(z)} f^{'}(z) dz$$

1.1.1 Solution:

By the argument principle, this is the number of zeros of f minus the number of poles of f (counted with multiplicity) inside the disk of radius r

1.2 b - What is the value of the integral for large n and fixed r?

To make the notation clearer, we redefine f as f_n then we have that $f_n(z) = \sum_{k=0}^{n} z^{-k}/k!$ Note that $f_n(z) \longrightarrow e^{1/z}$ as $n \longrightarrow \infty$ for each z (this is just the Taylor series for $e^{1/z}$). By uniform convergence, the limit as $n \longrightarrow \infty$ of the integral of $\frac{1}{f_n} f_n^{'}$ is the same as the integral of the limit as $n \longrightarrow \infty$ of $\frac{1}{f_n} f_n^{'}$. Since $f_n(z) \longrightarrow e^{1/z}$, we find that $f_n^{'}(z) \longrightarrow -\frac{1}{z^2} e^{\frac{1}{z}}$, so $\frac{1}{f_n(z)} f_n^{'}(z) \longrightarrow -\frac{1}{z^2}$. In other words, $lim_{n \to \infty} \frac{1}{(2\pi i)} \int_{|z|=r} \frac{1}{f_n(z)} f_n^{'}(z) dz = \frac{1}{(2\pi i)} \int_{|z|=r} -\frac{1}{z^2} dz$. This integral can be evaluated by parameterizing the curve $|z| = r$, and we find it is 0.

1.3 What does this tell you about the zeros of $f(z)$ for large n?

For fixed r, the integral approaches 0 as n approaches infinity. Since the integral always takes integral values, we find that the integral is exactly 0 for all n sufficiently large. Thus $f(z)$ has no zeroes or poles for large n.

2 Conclusion

Thank you to Prof. Blackmore for his instruction, lectures and office hours effort. It's been an honor and a privilege to be your student. I look forward to any feedback and learning more of the material in this course and beyond.

References

[1] Mark J Ablowitz, Athanassios S Fokas, and AS Fokas. *Complex variables: introduction and applications*. Cambridge University Press, 2003.