

Numerical Analysis Previous Qual

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This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references. This is for the May 2018 qualifying exam.

1 Problem May 2018 1 -

1.1 1a

Let A be a real symmetric 3×3 matrix with eigenvalues 0,3,5 and corresponding eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1.1.1 Find a basis for the nullspace of A and a basis for the column space of A .

By definition, the nullspace of A is spanned by $c\mathbf{u}$, c any arbitrary scalar. Similarly, the column space is spanned by $d\mathbf{v}, e\mathbf{w}$ d, e any arbitrary scalars.

1.1.2 If possible find solution for $Ax = u$

Given that u spans the nullspace, there is no solution for $Ax = u$.

1.1.3 Is A Invertible?

Given that A is rank deficient, since it has one eigenvalue of 0, A is singular and not invertible, it is not full rank.

1.2 1b - A and B are similar matrices.

1.2.1 Show they have the same determinant

We have that:

$$A = SBS^{-1}.$$

$$AS = SB.$$

$$\det(AS) = \det(SB).$$

$$\det(A) \det(S) = \det(S) \det(B).$$

$$\det(A) = \det(B).$$

1.2.2 Show they have the same characteristic polynomial and eigenvalues.

We have that:

$$A = SBS^{-1}.$$

$$A - \lambda I = SBS^{-1} - \lambda I.$$

$$AS - \lambda IS = SB - \lambda IS.$$

$$\det((A - \lambda I)S) = \det(S(B - \lambda I)).$$

$$\det(A - \lambda I) \det(S) = \det(S) \det(B - \lambda I).$$

$$\det(A - \lambda I) = \det(B - \lambda I).$$

Thus, the characteristic polynomial of A, $\det(A - \lambda I) = 0$ is the same as $\det(B - \lambda I) = 0$.

2 Problem May 2018 4:

2.1 a - Let x_0, \dots, x_n be distinct nodes and f a given real valued function with $n + 1$ continuous derivatives. Let $P_n(x)$ be the lagrange polynomial of degree n that interpolates as $P_n(x_i) = f(x_i)$.

We are asked to show that:

$$\forall x \in \text{domain of } f \quad \exists \xi \in \mathbf{R} \mid f(x) - P_n(x) = \frac{1}{(n+1)!} (x-x_0)(x-x_1)\dots(x-x_n) f^{(n+1)}(\xi).$$

We notice the right hand side term looks suspiciously like a Taylor Remainder / end of a Taylor expansion.

We calculate the Taylor expansion around x_0 , i.e., for some $\xi \in [x, x_0]$:

$$f(x) = f(x_0) + \frac{1}{1!} (x-x_0) f'(x_0) + \frac{1}{2!} (x-x_0)^2 f''(x_0) + \dots + \frac{1}{(n+1)!} (x-x_0)^{(n+1)} f^{(n+1)}(\xi).$$

2.2 b- Prove that the function $g(x) = \frac{1}{(2+x)}$ has a unique fixed point in $x^* \in [0, 1]$, describe the convergence of the iteration $x_{n+1} = g(x_n)$, $x_0 = 1$. José L. Pabón

Similarly, we calculate the Taylor expansion around x_1 , i.e., for some $\xi \in [x, x_1]$:

$$f(x) = f(x_1) + \frac{1}{1!}(x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2f''(x_1) + \dots + \frac{1}{(n+1)!}(x - x_1)^{(n+1)}f^{(n+1)}(\xi).$$

And so on; finally we calculate the Taylor expansion around x_n , i.e., for some $\xi \in [x, x_n]$:

$$f(x) = f(x_n) + \frac{1}{1!}(x - x_n)f'(x_n) + \frac{1}{2!}(x - x_n)^2f''(x_n) + \dots + \frac{1}{(n+1)!}(x - x_n)^{(n+1)}f^{(n+1)}(\xi).$$

It is at this point I realize that I cannot bootstrap or formulate a proof in this direction. Checking our lecture notes, I found a correct argument:

Sketch of proof:

- Construct a function $\phi(x)$ which is the collection of products of $(x - x_0)(x - x_1)\dots(x - x_n)$.
- Formulate the error function $E(x) = f(x) - P_n(x)$
- Construct another function G in the form of $G(x) = E(x) - \frac{1}{\phi(x)}\phi(x)E(x)$.
- Show that $G(x)$ has $n + 2$ zeroes, and has $n + 1$ derivatives.
- Invoke higher order mean value theorem to arrive at precise result requested.

2.2 b- Prove that the function $g(x) = \frac{1}{(2+x)}$ has a unique fixed point in $x^* \in [0, 1]$, describe the convergence of the iteration $x_{n+1} = g(x_n)$, $x_0 = 1$.

REMEMBER HAVE TO HAVE $G(X)$ CONTINUOUS.

Via theorems from class, existence of fixed points occur when for any $x \in [0, 1]$, $g(x) \in [0, 1]$. Furthermore, if $|g'(x)| < 1$, then the fixed point in that interval is unique.

Also, due to a handy corollary to the contraction mapping theorem, we have that $|g'(x)| < 1 \implies$ iterations of $x_{n+1} = g(x_n)$ for a starting $x_0 \in [0, 1]$ will converge.

We have that for $\forall x \in [0, 1]$, $\frac{1}{3} < g(x) < \frac{1}{2} \implies g(x) \in [0, 1]$. Then:

$$g'(x) = -\frac{1}{(2+x)^2}.$$

In the interval $x \in [0, 1]$, we have that $\frac{1}{9} < |g'(x)| < \frac{1}{4} \implies |g'(x)| < 1$, thus, via the theorems discussed in class the fixed point for this function exists, is unique and converges to some $x^* \forall x \in [0, 1]$.

2.2 b- Prove that the function $g(x) = \frac{1}{(2+x)}$ has a unique fixed point in $x^* \in [0, 1]$, describe the convergence of the iteration $x_{n+1} = g(x_n)$, $x_0 = 1$. José L. Pabón

We will describe the convergence in more detail, we want to calculate how fast $|x_n - x^*|$ converges to zero. We have that:

$$x_{n+1} = g(x_n).$$

$$x^* = g(x^*).$$

We consider the absolute value of the difference of these two equations:

$$|x_n - x^*| = |g(x_{n-1}) - g(x^*)| \leq \frac{1}{4}|x_{n-1} - x^*| \implies |x_n - x^*| \leq \left(\frac{1}{4}\right)^n |x_0 - x^*|.$$

Thus,

$$\implies \frac{1}{|x_0 - x^*|} |x_n - x^*| \leq \left(\frac{1}{4}\right)^n.$$

Thus the convergence is linear and will converge as expected.

2.2.1 The shortest, most succinct proof:

In the interest of preparing for the limited time nature of our qualifying doctoral exams, we wonder if this proof could be as short as:

Via theorems from class, since $\forall x \in [0, 1], \frac{1}{3} < g(x) < \frac{1}{2} \implies g(x) \in [0, 1]$, and:

$$g'(x) = -\frac{1}{(2+x)^2}.$$

such that $\frac{1}{9} < |g'(x)| < \frac{1}{4} \implies |g'(x)| < 1$, the fixed point is unique and will converge with:

$$|x_n - x^*| \approx \left(\frac{1}{4}\right)^n.$$

New stuff above here.

Old stuff below here.

2.3 Solution:

We have that Newton's method algorithm is of the form :

$$x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n).$$

We insert our $g(x)$:

$$x_{n+1} = x_n - \frac{1}{2(x_n)}(x_n^2) = x_n - \frac{1}{2}(x_n) = \frac{1}{2}(x_n).$$

Thus, the iterations of Newton's method would yield for us that:

$$x_{n+1} = \left(\frac{1}{2}\right)^n x_0.$$

Quadratic convergence would imply that:

$$\begin{aligned} \frac{1}{(x_n - \alpha)^2} (x_{n+1} - \alpha) &\leq k. \\ \implies \frac{1}{(x_n - \alpha)^2} \left(\frac{1}{2}(x_n) - \alpha\right) &\leq k. \end{aligned}$$

Given our root is $\alpha = 0$, we have that:

$$\frac{1}{(x_n)^2} \left(\frac{1}{2}(x_n)\right) \leq k \implies \frac{1}{(x_n)} \left(\frac{1}{2}\right) \leq k.$$

We note that we do have linear convergence:

$$\frac{1}{(x_n)} \left(\frac{1}{2}(x_n)\right) \leq 1 \implies \left(\frac{1}{2}\right) \leq 1.$$

2.4 4b - $f(x) \in C^3$ q has degree less than or equal to two interpolating f at x_0, x_1, x_2 . Let $h = \max(x_1 - x_0, (x_2 - x_1)$, K be the max over the x 's in the interval of $|f'''(x)|$. Show that:

$$\max_{x \in [x_0, x_2]} |f''(x) - q''(x)| = Ch^\alpha.$$

2.5 Solution proof:

Our general Newton's polynomial of second degree for the given points is:

$$N(x) = f(x_0) + [f(x_0), f(x_1)](x - x_0) + [f(x_0), f(x_1), f(x_2)](x - x_0)(x - x_1).$$

We have that:

$$[f(x_0), f(x_1)] = \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0)).$$

and:

$$[f(x_0), f(x_1), f(x_2)] = \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)).$$

We put everything together to get our form for $q(x)$:

$$q(x) = f(x_0) + \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0))(x - x_0) + \left\{ \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)) \right\} (x - x_0)(x - x_1).$$

We compute the second derivative of $q(x)$:

$$q''(x) = 2 \left(\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)) \right).$$

We follow the provided hint and consider the integral:

$$\int_{\eta}^x f'''(\xi) d\xi = f''(x) - f''(\eta).$$

3 Problem 5 - Consider the scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

.

3.1 Find the order of the scheme.

We use Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4)$$

$$y''_{n+1} = y''_n + hy'''_n + \frac{1}{2}h^2y''''_n + O(h^3)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1}) \implies$$

$$\implies y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5) = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

We group and simplify:

$$\frac{1}{2}h(y'_{n+1} + y'_n) = \frac{1}{2}h(y'_n + y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4))$$

We group and simplify:

$$\frac{1}{12}h^2(y''_n - y''_{n+1}) = \frac{1}{12}h^2(y''_n - y''_n - hy'''_n - \frac{1}{2}h^2y''''_n - O(h^3))$$

We group like terms:

$$y_n(1 - 1) = 0$$

$$y'_n(h - h) = 0$$

$$y''_nh^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y'''_nh^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = y'''_nh^3(0)$$

$$y''''_nh^4(\frac{1}{24} - \frac{1}{12} + \frac{1}{24}) = y''''_nh^4(0)$$

At this point we realize that we haven't expanded far enough so we need to Taylor expand farther. Incoming:

We use Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + \frac{1}{120}h^5y^V_n + O(h^6)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + \frac{1}{24}h^4y^V_n + O(h^5)$$

$$y_{n+1}'' = y_n'' + hy_n''' + \frac{1}{2}h^2y_n'''' + \frac{1}{6}h^3y_n^V + O(h^4)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y_{n+1}' + y_n') + \frac{1}{12}h^2(y_n'' - y_{n+1}'') \implies$$

$$\implies y_n + hy_n' + \frac{1}{2}h^2y_n'' + \frac{1}{6}h^3y_n''' + \frac{1}{24}h^4y_n'''' + \frac{1}{120}h^5y_n^V + O(h^6) = y_n + \frac{1}{2}h(y_{n+1}' + y_n') + \frac{1}{12}h^2(y_n'' - y_{n+1}'')$$

We group and simplify:

$$\frac{1}{2}h(y_{n+1}' + y_n') = \frac{1}{2}h(y_n' + y_n' + hy_n'' + \frac{1}{2}h^2y_n''' + \frac{1}{6}h^3y_n'''' + \frac{1}{24}h^4y_n^V + O(h^5))$$

We group and simplify:

$$\frac{1}{12}h^2(y_n'' - y_{n+1}'') = \frac{1}{12}h^2(y_n'' - y_n'' - hy_n''' - \frac{1}{2}h^2y_n'''' - \frac{1}{6}h^3y_n^V + O(h^4))$$

We group like terms:

$$y_n(1 - 1) = 0$$

$$y_n'(h - h) = 0$$

$$y_n''h^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y_n'''h^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = (0)y_n'''h^3$$

$$y_n''''h^4(\frac{1}{24} - \frac{1}{12} + \frac{1}{24}) = (0)y_n''''h^4$$

$$y_n^Vh^5(\frac{1}{120} - \frac{1}{48} + \frac{1}{72}) = (\frac{1}{720})y_n^Vh^5.$$

We have that $48 = 2^4(3)$, $72 = 2^3(3^2)$, $120 = 2^3(3)(5)$, the least common multiple for the denominators in the fractions involved is $720 = 2^4(3^2)5$.

Thus the computed coefficient is:

$$\frac{6}{720} - \frac{15}{720} + \frac{10}{720} = \frac{1}{720}.$$

Thus, the lowest term of nonzero coefficients are $(\frac{1}{720})y_n^Vh^5$, i.e. of order $O(h^5)$ in our

computation of $y_{n+1} - y_n$. Now our local truncation error is:

$$\tau_{n+1} = \frac{1}{h}(y_{n+1} - y_n) \implies \tau_{n+1} = \frac{1}{h}(\frac{1}{720})y_n^V h^5$$

Thus, the method is of fourth order $O(h^4)$.

3.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

We define $y' = \lambda y$, $y(0) = y_0$, and we consider the method applied to this function. We will use notation of $f_n \equiv f^n$, which will not denote the n th derivative. We have that:

$$y_n'' = \frac{df(x_n, y_n)}{dx} + f(x_n, y_n) \frac{df(x_n, y_n)}{dx}.$$

$$f^n = \lambda y_n.$$

$$f_x^n = \lambda y'(x_n) = \lambda f^n = \lambda^2 y_n.$$

$$f_y^n = \lambda.$$

We examine the method within this framework:

$$y_{n+1} = y_n + \frac{h}{2}[y_n' + y_{n+1}'] + \frac{h^2}{12}[y_n'' - y_{n+1}''].$$

$$y_{n+1} = y_n + \frac{h}{2}[f^n + f^{n+1}] + \frac{h^2}{12}[f_x^n + f^n f_y^n - f_x^{n+1} + f^{n+1} f_y^{n+1}].$$

$$y_{n+1} = y_n + \frac{h}{2}[\lambda y_n + \lambda y_{n+1}] + \frac{h^2}{12}[\lambda^2 y_n + \lambda^2 y_n - \lambda^2 y_{n+1} - \lambda^2 y_{n+1}].$$

$$y_{n+1} = y_n + \frac{h\lambda}{2}[y_n + y_{n+1}] + \frac{h^2\lambda^2}{6}[y_n - y_{n+1}].$$

$$\implies (1 - \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2)y_{n+1} = (1 + \frac{1}{2}h\lambda + \frac{1}{6}h^2\lambda^2)y_n \implies (6 - 3h\lambda + h^2\lambda^2)y_{n+1} = (6 + 3h\lambda + h^2\lambda^2)y_n.$$

Thus we have our relation:

$$y_{n+1} = \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)}y_n.$$

We verify the stability of the method for this function by studying:

$$y_n = \left(\frac{6 + 3h\lambda + h^2\lambda^2}{6 - 3h\lambda + h^2\lambda^2}\right)^n y_0.$$

By definition and convention, we have step size of $h > 0$ and we define $\lambda < 0 \implies h\lambda < 0$. For the method to be stable, we find we need the condition:

$$\left| \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} \right| < 1.$$

$$h\lambda < 0 \implies (6 - 3h\lambda + h^2\lambda^2) > 0$$

So we have that

$$-1 < \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} < 1.$$

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2).$$

We examine these inequalities and have that:

$$-1(6 - 3h\lambda + h^2\lambda^2) < (6 + 3h\lambda + h^2\lambda^2) \implies -6 < (6 + 2h^2\lambda^2). \checkmark$$

This inequality holds for all values of h, λ . For the second set of inequalities we find that:

$$(6 + 3h\lambda + h^2\lambda^2) < (6 - 3h\lambda + h^2\lambda^2) \implies 3h\lambda < -3h\lambda. \checkmark$$

This inequality holds for $h\lambda < 0$.

Thus

$$\left| \frac{(6 + 3h\lambda + h^2\lambda^2)}{(6 - 3h\lambda + h^2\lambda^2)} \right| < 1.$$

When $h > 0, \lambda < 0 \implies h\lambda < 0$ giving us our region of stability.

\therefore The region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

4 Problem 6

4.1 a - Show that $T_{\frac{1}{2}h}(f) = \frac{1}{2}(T_h(f) + M_h(f))$.

4.2 b - Given $T_h(f) = I(f) + k_2h^2 + k_4h^4 + O(h^6)$, **find similar rule for** $M_h(f)$.

From part a we know that $2T_{\frac{1}{2}h}(f) = (T_h(f) + M_h(f)) \implies M_h(f) = 2T_{\frac{1}{2}h}(f) - T_h(f)$.

We are given:

$$T_h(f) = I(f) + k_2h^2 + k_4h^4 + O(h^6).$$

We compute:

$$T_{\frac{1}{2}h}(f) = I(f) + k_2 \frac{1}{4}h^2 + k_4 \frac{1}{16}h^4 + O(h^6).$$

$$2T_{\frac{1}{2}h}(f) - T_h(f) = 2I(f) + 2k_2 \frac{1}{4}h^2 + 2k_4 \frac{1}{16}h^4 + O(h^6) - I(f) - k_2h^2 - k_4h^4 - O(h^6)$$

.

$$\therefore 2T_{\frac{1}{2}h}(f) - T_h(f) = I(f) - k_2 \frac{1}{2}h^2 - k_4 \frac{7}{8}h^4 + O(h^6) = M_h. \checkmark$$

4.3 c -

5 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, for reading this work, for her instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course.

References

- [1] Kendall E Atkinson. *An introduction to numerical analysis*. John wiley & sons, 2008.