Methods of applied mathematics and Modeling.

José Pabón

July 2020

These problems are from past quals mostly. Thanks for reading.

1 Modeling practice:

The chemical master equation is:

$$\frac{dp_n}{dt} = (n-1)r_b p_{n-1} + (n+1)r_d p_{n+1} - (n-1) + (-r_b - r_d)np_n.$$

We collect our r terms:

$$\frac{dp_n}{dt} = r_b((n-1)p_{n-1} - np_n) + r_d((n+1)p_{n+1} - np_n).$$

We multiply the whole equation by z^n:

$$z^n \frac{dp_n}{dt} = z^n\{r_b((n-1)p_{n-1} - np_n) + r_d((n+1)p_{n+1} - np_n)\}.$$

Via class theorems we know that:

$$\frac{d}{dt}F(z,t) = \sum_{n=0}^{\infty} z^n \frac{dp_n}{dt} = \sum_{n=0}^{\infty} z^n\{r_b((n-1)p_{n-1} - np_n) + r_d((n+1)p_{n+1} - np_n)\}.$$

We want to get this whole expression in terms of p_n:

For $z^n(n-1)p(n-1)$, we let $m = n-1$, which would make this summation $z^{m+1}(m)p(m)$, which we can reindex to n abd then will leave us with:

$$\frac{d}{dt}F(z,t) = \sum_{n=0}^{\infty} z^n\{r_b(z(n)p_n - np_n) + r_d((n+1)p_{n+1} - np_n)\}.$$

which we can simplify to:

$$\frac{d}{dt}F(z,t) = \sum_{n=0}^{\infty} z^n\{(n)p_n r_b(z-1) + r_d((n+1)p_{n+1} - np_n)\}.$$

Methods of applied mathematics outline:

2 First topic (ODE)

Methods to solve ODEs:

- Exact equations
- Substitutions or change of variables, such as $y^{'} = F\frac{1}{x}y$.
- Variation of parameters for second order ODEs, catch all. Pro's always works, con's - takes longer, have to do integrals.
- Undetermined coefficients - Useful until its not. Many times may be faster. Keep in mind that different right hand side nonhomogeneous terms will require different starting points to determine the particular solution.
- Reduction of order - seems useful, but we haven't seen it on many qual problems.
- Hamiltonian systems
- Frobenius series solutions
- Integrating factor.
- Sturm Lioville

3 Frobenius

Step 1 Write everything in terms of Frobenius form.

$$R(x)x^2y + P(x)xy + Q(x)y = 0.$$

Where R, P, Q are analytic functions of x.
You should get coefficients for as far as the order of the ODE provided.
To determine if it is a regular singular point:
$\lim \frac{P}{R}$ and $\lim \frac{Q}{R}$.
We check the indicial equation, we check the difference between the roots $S_+ - S_-$, if it is not equal a nonnegative integer, then there exists two linearly independent solutions.
(Other cases exist)
Then find the recurrence relationship.

4 Phase portraits

- I think maybe Hamiltonian systems go here.
- This is two
- So on

5 Partial Differential equations

5.1 General PDEs, linear

- Homogeneous PDE, homogeneous boundary conditions, simplest case.
- Some problems have reference temperature distributions, which may be found applying the steady state.
- So on

5.2 Heat equation

- Method of Separation of variables, which has different flavors in terms of the sources (homogeneous terms), boundary conditions, etc.
- This is two
- So on

5.3 Wave equation

- Method of characteristics?
- There's three types of wave equations, linear, quasi-linear, semi-linear.
- So on

5.4 Laplace equation

- Generally, Laplace is the same process as previously detailed linear PDEs but you have to follow the procedure on Page 72, section 2.5 in Haberman, which includes separation of variables.
- This is two
- So on

5.5 Algorithm for Method of Characteristics

Algorithm to do Method of Characteristics general PDE style, not Matveev Modeling style A du/dx + B du/dy + C(f(

6 Practicing - MAT651 Final exam

6.1 First ODE:

a- $x^2y^{''} - 3xy^{'} + 4y = x^2$.

We find an Euler form ODE, we use ansatz of $y = x^s$ into our homogeneous ODE, we find that:

$$x^s((s)(s-1) - 3s + 4) = 0.$$

We discard the trivial solution and find that:

$$s^2 - 4s + 4 = 0 \implies s_+ = 2 = s_-.$$

We thus have that we have two solutions:

$$y_1 = Cx^2, y_2 = Dx^2 \ln x$$

.

For any constants $C, D \in \mathbf{C}$.

6.1.1 Undetermined coefficients method for finding the nonhomogenous solution.

$a - A(2x(\ln x)^2 + 2x \ln x$.

$$y_1 = Cx^2, y_2 = Dx^2 \ln x$$

.

We find that one of our solutions are linearly dependent on the nonhomogeneous term x^2, thus we have that we ansatz for our particular solution:

$$y_p = Ax^2(\ln x)^2; y_p^{'} = A(2x(\ln x)^2 + 2x \ln x) = A(2x \ln x((\ln x) + 1)).$$

$$y_p^{''} = A(2(\ln x)^2 + 6 \ln x + 2).$$

We plug this particular solution into our ODE a- $x^2y^{''} - 3xy^{'} + 4y = x^2$.

$$x^2A(2(\ln x)^2 + 6\ln x + 2) - 3x(A(2x(\ln x)^2 + 2x\ln x)) + 4Ax^2(\ln x)^2 = x^2.$$

$$2A(\ln x)^2x^2 + 6Ax^2\ln x + 2Ax^2 - 6Ax^2(\ln x)^2 - 6Ax^2(\ln x) + 4Ax^2(\ln x)^2 = x^2.$$

$$2Ax^2 = x^2 \implies A = \frac{1}{2}.$$

6.2 Second ODE:

$y^{'''} - 3y^{''} + 3y^{'} - y = \cos x$.

We solve the homogenous equation, we ansatz $y = e^{rx}$ and find that:
$r^3 - 3y^2 + 3r - 1 = 0 \implies (r-1)^3 = 0 \implies r = 1$ with a multiplicity of 3.

We form our homogeneous solution:
$y_{homogeneous} = e^x + xe^x + x^2e^x$.

We ansats our particular solution $y_{particular} = A(\sin(x)) + B\cos(x)$, take its derivatives, plug it into the ODE, to find that.

$$\sin(x)(B + 3A - 3B - A) \implies 2A - 2B = 0 \implies A = B$$

.

$$\cos(x)(-A + 3B + 3A - B) \implies 2A + 2B = 1 \implies A = B = \frac{1}{4}$$

.

Full solution is thus:

$$y = Ce^x + Dxe^x + Ex^2e^x + \frac{1}{4}(\sin(x)) + \frac{1}{4}\cos(x)$$

.

References