Numerical Analysis Previous Qual - August 2018.

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This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references.

1 Problem August 2018 6 - Show the degree of precision is less than or equal to $2n-1$, and then that it is no more than the same.

Suppose inner product with weight is defined by:

$$< f, g >= \int_a^b f(x)g(x)w(x)dx.$$

Consider quadrature formula:

$$I(f) = \int_a^b f(x)w(x)dx \approx Q(f) = \sum_{j=1}^{n} w_j f(x_j).$$

We have that:

$$w_j = \int_a^b l_j(x)w(x)dx.$$

And

$$l_j(x) = \prod_{i=1,2,3,\ldots,n---i\neq j} \frac{(x-x_j)}{(x_i-x_j)}.$$

1.1 Solution, proof:

Degree of precision is defined as the highest order polynomial that the quadrature will return an exact answer for. Study note - for Simpsons rule this is three, for trapezoid rule this is one.

The key idea of this proof is division of the polynomial f. We define polynomials such that:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n-1, n, n-1, n-1$ and by construction and the interpolation of I(f), $\star$ *we have that $r(x)$ is exact and there is no further remainder.*

$\star \implies$ ASIDE note to the group - I went to Prof. Hamfeldt's office hour to talk about this problem, she mentioned this claim in italics is not clear or obvious. I asked her how I could formulate my argument better, she suggested 'expressing the remainder function $r(x)$ in terms of basis functions for the problem, and it should follow.' Still puzzling over this.

END ASIDE.

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n-1$, thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x)dx + \int_a^b r(x)w(x)dx = 0 + \int_a^b r(x)w(x)dx = Q(r).$$

Thus we find that:

$$I(f) = Q(r) = \sum w_j r(x_j) = \sum w_j(p_n(x_j)r(x_j) + r(x_j)) = \sum 0 + w_j r(x_j)) = Q(f).$$

Need to show the remainder polynomials are exact.

Thus, we have that the precision of this method (Gaussian quadrature) is $\leq 2n-1$.

1.2 Show that the precision is no greater than $2n-1$.

We assume the same construction as in the previous argument, except for the degrees of the polynomials:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n, n, n, n-1$ and by construction and the interpolation of I(f), we have that $r(x)$ is still exact and there is no further remainder.

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n-1$ but not orthogonal to all polynomials of degree n, thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x)dx + \int_a^b r(x)w(x)dx = 0 + \int_a^b r(x)w(x)dx = Q(r) + < q, p_n > .$$

Thus,$< q, p_n > \neq 0$.

We still have the same zeros of our p_n, so thus we find that:

$$Q(f) = \sum w_j r(x_j) = Q(qp_n + r) \sum w_j (p_n(x_j) r(x_j) + r(x_j)) = \sum 0 + r(x_j)) = Q(r).$$

Thus,

$$Q(f) \neq Q(r)$$

Thus, for all polynomials f of degree 2n or higher, the quadrature formula is not precise.

2 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, for reading this work, for her instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis*. John wiley & sons, 2008.