Numerical Analysis August 2019 Previous Qual Question 6

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This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references. This is for the May 2018 qualifying exam.

1 Problem August 2019 3:

1.1 a - Show that the Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

The Chebyshev polynomials are of the form:

$$T_n(x) = \cos\big(n \arccos(x)\big).$$

We consider the weighted inner product for positive integers $m, n \in \mathbf{N}$:

$$< T_n(x), T_m(x) >= \int_{-1}^{1} T_n(x) T_m(x) w(x) dx.$$

We define $\theta = \arccos(x) \implies d\theta = -\frac{1}{\sqrt{1-x^2}} dx$, and substitute into our integral:

$$\int_{-1}^{1} T_n(x) T_m(x) w(x) dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta) d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta) d\theta.$$

The integrand functions are an orthogonal family of functions which will equal to zero unless n equals to m.

$$\frac{1}{2}\int_{0}^{\pi} \cos\big((n+m)\theta\big) + \cos\big((n-m)\theta\big) d\theta = \frac{1}{2}\Big[\frac{1}{(n+m)} \sin\big((n+m)\theta\big) + \frac{1}{(n-m)} \sin\big((n-m)\theta\big)\Big]\Big|_{0}^{\pi}.$$

Thus we have that:

$$\frac{1}{2}[\frac{1}{(n+m)}\sin\big((n+m)\theta\big) + \frac{1}{(n-m)}\sin\big((n-m)\theta\big)]|_0^\pi = \begin{cases} \pi, \text{ for } m = n = 0 \\ \frac{1}{2}\pi, \text{ for } m = n \neq 0 \\ 0, \text{ otherwise }. \end{cases}$$

1.2 b - Show that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n \geq 2$.

We again define $\theta = \arccos(x) \implies \cos\theta = x$

The left hand side is:

$$T_{n+1}(x) = \cos\big((n+1)\arccos(x)\big) \implies T_{n+1}(\theta) = \cos\big((n+1)\theta\big).$$

The right hand side we have is:

$$2xT_n(x) - T_{n-1}(x) \implies 2\cos\theta\cos(n\theta) - \cos\big((n-1)\theta\big).$$

Then:

$$2\cos\theta\cos(n\theta) = \cos{(n+1)\theta} - \cos{(n-1)\theta}.$$

Thus:

$$2\cos\theta\cos(n\theta) - \cos\big((n-1)\theta\big) = \cos{(n+1)\theta}.$$

$$\therefore T_{n+1}(\theta) = \cos\big((n+1)\theta\big) = 2\cos\theta\cos(n\theta) - \cos\big((n-1)\theta\big) \implies$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)\checkmark.$$

1.3 c - Derive the three point Chebyshev Gauss quadrature formula:

$$I(f) = \int_{-1}^{1} f(x)\frac{1}{\sqrt{1-x^2}}dx \approx \sum_{i=1}^{3} w_i f(x_i).$$

We begin by computing the three roots of our $n = 3$ Chebyshev polynomial:

$$T_n(x) = \cos\big(n\arccos(x)\big) \text{ or } T_n(\theta) = \cos(n\theta).$$

Via the previous identities derived in the previous parts of this problem:

$$T_3(x) = \cos\big(3\arccos(x)\big) \implies 2xT_2(x) - T_1(x) \implies 2x\cos\big(2\arccos(x)\big) - \cos\big(1\arccos(x)\big).$$

We have that $\cos\big(1 \arccos(x)\big) = x$. We apply the same recursion formulate to our term $\cos\big(2 \arccos(x)\big)$:

$$\cos\big(2 \arccos(x)\big) = 2x \cos\big(1 \arccos(x)\big) - \cos\big(0 \arccos(x)\big).$$

Thus, we have that:

$$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 2x - x = 4x^3 - 3x.$$

This gives us our collocation points:

$$x_1 = -\frac{1}{2}\sqrt{3}, x_2 = 0, x_3 = -\frac{1}{2}\sqrt{3}.$$

We form our system of equations with our monomial basis $1, x, x^2$:

For $f(x) = 1$ we have that:

$$\int_{-1}^{1} 1 \frac{1}{\sqrt{1-x^2}} dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) = w_1 + w_2 + w_3.$$

$$\int_{-1}^{1} 1 \frac{1}{\sqrt{1-x^2}} dx = \arcsin 1 - \arcsin -1 = \frac{1}{1}\pi - (-\frac{1}{1}\pi) = \pi$$

. Thus:

$$w_1 + w_2 + w_3 = \pi$$

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For $f(x) = x$ we have that:

$$\int_{-1}^{1} x \frac{1}{\sqrt{1-x^2}} dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) = w_1 - \frac{1}{2}\sqrt{3} + w_2 0 + w_3 \frac{1}{2}\sqrt{3}.$$

$$\int_{-1}^{1} x \frac{1}{\sqrt{1-x^2}} dx = \arcsin 1 - \arcsin -1 = \text{ An odd integrand over a symmetric interval } = 0$$

. Thus:

$$0 = w_1 - \frac{1}{2}\sqrt{3} + w_2 0 + w_3 \frac{1}{2}\sqrt{3}.$$

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We multiply this equation by $2\sqrt{3}$ on both sides to find:

$$-3w_1 + 0w_2 + 3w_3 = 0.$$

For $f(x) = x^2$ we have that:

$$\int_{-1}^{1} x^2 \frac{1}{\sqrt{1-x^2}} dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) = w_1(-\frac{1}{2}\sqrt{3})^2 + w_2 0 + w_3(\frac{1}{2}\sqrt{3})^2.$$

Integrating by parts we have that:
$\int x^2 \frac{1}{\sqrt{1-x^2}} dx = \frac{1}{2}\pi$.
Thus:

$$\frac{1}{2}\pi = w_1(-\frac{1}{2}\sqrt{3})^2 + w_2 0 + w_3(\frac{1}{2}\sqrt{3})^2 = w_1(\frac{1}{4}3) + 0 + w_3(\frac{1}{4}3)$$

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Equivalently, we multiply this resulting equation by 4 on both sides, we have that:

$$2\pi = 3w_1 + 0 + 3w_3.$$

Thus we have our systems of equations:

$$w_1 + w_2 + w_3 = \pi$$

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$$-3w_1 + 0 + 3w_3 = 0.$$

$$3w_1 + 0 + 3w_3 = 2\pi.$$

We solve this system of equations to find that our weights are:

$$w_1 = w_2 = w_3 = \frac{1}{3}\pi.$$

1.4 d - Approximate $I(f) = e^{-2x^2}$.

Thus, by our previous result, we know that:

$$I(f) \approx \sum_{i=1}^{3} w_i f(x_i) = \frac{1}{3}\pi f(x_1) + \frac{1}{3}\pi f(x_2) + \frac{1}{3}\pi f(x_3) = \frac{1}{3}\pi e^{-2(-\sqrt{3}\frac{1}{2})^2} + \frac{1}{3}\pi e^{-2(0)^2} + \frac{1}{3}\pi e^{-2(\sqrt{3}\frac{1}{2})^2}.$$

Then we have that:

$$\frac{1}{3}\pi e^{-2(-\sqrt{3}\frac{1}{2})^2} + \frac{1}{3}\pi e^{-2(0)^2} + \frac{1}{3}\pi e^{-2(\sqrt{3}\frac{1}{2})^2} = \frac{1}{3}\pi(1 + e^{-6(\frac{1}{4})} + e^{-6(\frac{1}{4})}) = \frac{1}{3}\pi(2e^{-3\frac{1}{2}} + 1).$$

1.5 Show the area of A stability.

2 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, for reading this work, for her instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis.* John wiley & sons, 2008.