Complex Analysis MAT656 August 2019.

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These problems are from the July 2020 preliminary qualifying examination. The course MAT656 was received from Prof. Blackmore. The course textbook is Ablowitz and Fokas [1].

1 July 2020 Problem 1. - Consider the function:

1.1 a - Show that if $f = u+iv$ is conformal on D, i.e f' is nonvanishing on D, the level curves $u =$ constant and $v =$ constant are orthogonal wherever they intersect.

We make note of the fact that the gradients of u and v are perpendicular to their level curves. In two dimensions, if the level curves $u(x,y) = c_1$, $v(x,y) = c_2$ are perpendicular then the gradients of u and v would also be perpendicular. Thus, it is the case that if the $\nabla u, \nabla v$ are perpendicular, then so must the level curves $u(x,y) = c_1$, $v(x,y) = c_2$.

We consider the dot product $\nabla u \cdot \nabla v$:

$$\nabla u(x,y,) = (\frac{du}{dx}, (\frac{du}{dy}), \nabla v(x,y,) = (\frac{dv}{dx}, (\frac{dv}{dy}) \implies \nabla u \cdot \nabla v = (\frac{du}{dx} \cdot \frac{dv}{dx}) + (\frac{du}{dy} \cdot \frac{dv}{dy}).$$

Via Cauchy Riemann equations, we know that:

$$\frac{du}{dx} = \frac{dv}{dy}, \frac{du}{dy} = -\frac{dv}{dx}.$$

We substitute these in to find that:

$$\nabla u \cdot \nabla v = (\frac{du}{dx} \cdot -\frac{du}{dy}) + (\frac{du}{dy} \cdot \frac{du}{dx}) \equiv 0 \quad \forall \quad \mathbf{f} \text{ analytic.}$$

Similarly, for v:

$$\nabla u \cdot \nabla v = (\frac{dv}{dx} \cdot \frac{dv}{dy}) + (\frac{dv}{dy} \cdot -\frac{dv}{dx}) \equiv 0 \quad \forall \quad \mathbf{f} \text{ analytic.}$$

Thus, within our domain D of analyticity, the level curves u = constant and v = constant are orthogonal wherever they intersect if f is conformal.

1.2 1b - Prove using the transformation of variables formula for integral from advanced calculus that if $\left|f^{'}(z)\right| = 1$ for all $z \in D$, the planar map $F(x, y) := (u(x, y), v(x, y))$ is area preserving.

1.2.1 Proposed solution:

We note that $\left|f^{'}(z)\right| = 1 \implies f(z) = e^{i\theta}z + c$ for some $c \in \mathbf{C}$. Thus, we have that f is a linear mapping that is a translation by c after a rotation by θ. Note that under this mapping for any two arbitrary values $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, the quantities $|x_2 - x_1| = \left|f(x_2) - f(x_1)\right|$ are identical, as well as $|y_2 - y_1| = \left|f(y_2) - f(y_1)\right|$.

We calculate the Jacobian:

$$f(z) = e^{i\theta}z + c = (\cos\theta + i\sin\theta)(x + iy) + (c_x + ic_y.$$

Note that $\text{Re}\{c\} = c_x$ and $\text{Im}\{c\} = c_y$.

We simplify the above expression:

$$F(x, y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix}.$$

We find that the map F is an isometry also, given that:

$$\det\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\} = 1.$$

Thus we have that:

$$|x_2 - x_1| = \left|F((x_2)) - F((x_1))\right|, |y_2 - y_1| = \left|F((y_2)) - F((y_1))\right|$$

Given we have that areas in our complex plane are defined by the product between the lengths in their two dimensional coordinates (x, y), we then must have that:

$$|x_2 - x_1| \cdot |y_2 - y_1| = \left|F((x_2)) - F((x_1))\right| \cdot \left|F((y_2)) - F((y_1))\right| \qquad \forall \qquad (x, y) \in D.$$

There must an equivalent statement to these that equates the in variance of dx, dy, du, dv and dA under the transformation of variables, but I do not presently have an exposition for this part. Any advice will be appreciated!

2 July 2020 Problem 2. - Consider the function:

2.1 a - Riemann Zeta Function.

Prove that the Riemann Zeta Function:

$$\xi(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is absolutely convergent for $x = \text{Re}\{z\} \geq 1 + \epsilon$ whenever $\epsilon > 0$ and explain why this function is analytic for $\text{Re}\{z\} > 1$.

2.1.1 Solution - prove the absolute convergence.:

For this exposition we will use the notation of $\exp\{z\} = e^z$.

We have that:

$$\frac{1}{n^z} = \frac{1}{n^{x+iy}} = \frac{1}{n^x}\frac{1}{n^{iy}}$$

.

We know that $n^z = \exp\{\log n^z\} = \exp\{z \log n\}$, thus we have that:

$$\frac{1}{n^z} = \exp\{-z \log n\} = \exp\{-x \log n\} \exp\{-iy \log n\}$$

.

We verify the absolute convergence of our whole infinite sums by considering the infinite sum of the above expression. We know that $\left|\exp\{-iy \log n\}\right| = \left|e^{ix}\right| = 1 \ \forall$ x, y $\in \mathbf{R}$. Thus:

$$\left|\xi(z)\right| := \left|\sum_{n=1}^{\infty} \frac{1}{n^z}\right| = \sum_{n=1}^{\infty} \left|\exp\{-x \log n\} \exp\{-iy \log n\}\right| = \sum_{n=1}^{\infty} \left|\exp\{-x \log n\}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{n^x}\right|.$$

Via our familiar 'p series' test, the series sum converges for exponent $p > 1$, thus,

$$\xi(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is absolutely convergent for $x = \text{Re}\{z\} \geq 1 + \epsilon$ whenever $\epsilon > 0$.

2.1.2 Solution - Explain why the function is analytic.

We examine the analyticity of the series by computing its derivative,

Given :

$$\xi(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \exp\{-z \log n\}.$$

We compute derivative

$$\xi'(z) := \sum_{n=1}^{\infty} -\log n \exp\{-z \log n\}.$$

We apply an nth term test with

$$a_n = -\log n \exp\{-z \log n\}, a_{n+1} = -\log(n+1) \exp\{-z \log(n+1)\}$$

.

We then have that the ratio $\frac{a_{n+1}}{a_n} = \frac{\log(n+1)}{n \log(n)}$, which in the limit as n goes to infinity gives us an indeterminate form of $\frac{\infty}{\infty}$.

We apply L'Hopital's Rule to this limit to find that:

$$\lim_{n \longrightarrow \infty} \frac{1}{n + (n+1)(\log n)} = 0.$$

for

$\therefore$Thus the series representation for $\xi'(z) := \sum_{n=1}^{\infty} -\log n \exp\{-z \log n\}$ converges and is well defined, thus, $\xi(z)$ is analytic wherever the series is convergent, which we found in an earlier result is valid for $\mathrm{Re}\{z\} > 1$ ✓.

2.2 2b - Contour integral.

We skip this integral at this time, the problem is well in line with other contour integral problems discussed in class.

2.3 2c - Explain why an entire function f that vanishes at the points $z_n = \frac{1}{n}$ for all positive integers n must be identically zero.

The author realizes this exposition is perhaps attempting to be more rigorous than may be necessary; a succinct proof for this problem may simply invoke the isolated zero theorem and pronounce that any arbitrary neighborhood centered at zero inside of $|z| \leq 1$ *contains an infinite number of elements of* z_n.

From the Isolated Zero Theorem we know that the zeros of an analytic function are isolated. The definition of an entire function is that it is analytic over the entire complex plane.

We claim that for some ϵ greater than zero, there is a neighborhood $|z| \leq \epsilon$ such that there are an infinite number of elements of z_n in that neighborhood. We can prove this by constructing $N_\epsilon = \lceil \frac{1}{\epsilon} \rceil$, such that for every n greater than N_ϵ we have that $\frac{1}{n} \leq \epsilon$ and thus there is a neighborhood $|z| \leq \epsilon$ such that there are an infinite number of elements of z_n in that neighborhood.

∴ Thus, in this neighborhood we have that our entire, analytic function f has an infinite amount of zeroes. For the analyticity of this function to hold, all of the limits the values of the function in all directions must agree, and therefore the function must be identically zero in this neighborhood. Given the arbitrary nature of the selected ϵ and the fact that our function f is entire (and thus analytic everywhere on the complex plane), our function f is therefore proven to have to necessarily be identically zero everywhere in the complex plane. ✓

3 Conclusion

Thank you to Prof. Blackmore for his instruction, lectures and office hours effort. It's been an honor and a privilege to be your student. I look forward to any feedback and learning more of the material in this course and beyond.

References

[1] Mark J Ablowitz, Athanassios S Fokas, and AS Fokas. *Complex variables: introduction and applications*. Cambridge University Press, 2003.