Complex Analysis MAT656.

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These problems are from the pdf received from Prof. Blackmore. The course textbook is Ablowitz and Fokas [1].

This LaTeX document will contain the main points of the exposition, please see the handwritten work following for more details on the calculations and computations.

1 1 - Prove that if $f = u+iv$ is analytic on a domain D and $|f|^2 = u^2+v^2 = c$ is constant, then f is constant.

We are given that $|f|^2 = u^2 + v^2 = c$, for some constant c. This implies for $\sqrt{c} = k$, for some other constant k, we have that:

$$\sqrt{u^2+v^2} = \sqrt{c} = k \implies \sqrt{|f|^2} = k \implies |f| = k.$$

Thus $|f| = \sqrt{u^2+v^2} = k$ is constant. From here, we reuse the same argument used on a previous work this semester course using Cauchy Riemann equations.

We have that $f(x,y) = u(x,y) + iv(x,y)$ is analytic and therefore holomorphic on domain D, as well as $|f| = c$ in the same domain. From the Cauchy Riemann equations, we also have that $u_x = v_y$ and $u_y = -v_x$.

$|f| = c \implies |f| = \sqrt{u^2+v^2} = c$. Then $u^2 + v^2 = c^2$.

We take partial derivatives:

$2uu_x + 2vv_x = 0$. $2uu_y + 2vv_y = 0 \implies uu_x + vv_x = uu_y + vv_y = 0$.

Then $u^2u_x + uvv_x = 0$ and $uvu_y + v^2v_y = 0$.

We substitute $u_y = -v_x$ in the second equation, and add these two to get:

$u^2u_x + v^2v_y = 0$; which given $u_x = v_y \implies u^2v_y + v^2v_y = (u^2+v^2)v_y = 0$.

We know $u^2 + v^2 = c^2$, so the equation $u^2 + v^2)v_y = 0$ admits only two cases. Either $u^2 + v^2 = c^2 \implies c = 0 \implies$ f is constant on domain D, or $v_y = 0 \implies u_x = 0$.

Via similar computation and argument, we have that $(u^2+v^2)u_y = 0$ which, again in the same manner, only admits either $u^2 + v^2 = c^2 \implies c = 0 \implies$ f is constant on domain D, or $u_y = 0 \implies -v_x = 0$.

Thus, it must be that either f is constant on domain D or $-v_x = u_x = v_y = u_y = 0$, which implies that $v = u = v = u = c_i$ for some constants c_i and thus, $f(u(x,y), v(x,y))$ is constant on domain D.

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2 2 - Is it possible for a nonconstant entire function to be zero at every rational point along the real axis in the complex plane?

3 Conclusion

Thank you to Prof. Blackmore for his instruction, lectures and office hours effort. I look forward to any feedback and learning more of the material in this course.

References

[1] Mark J Ablowitz, Athanassios S Fokas, and AS Fokas. *Complex variables: introduction and applications*. Cambridge University Press, 2003.