Numerical Analysis Previous Qual - May 2018.

José L. Pabón

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This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references.

1 Problem May 2018 1 -

1.1 1a

Let A be a real symmetric 3×3 matrix with eigenvalues 0,3,5 and corresponding eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1.1.1 Find a basis for the nullspace of A and a basis for the column space of A.

By definition, the nullspace of A is spanned by $c\mathbf{u}$, c any arbitrary scalar. Similarly, the column space is spanned by $d\mathbf{v}, e\mathbf{w}$ d, e any arbitrary scalars.

1.1.2 If possible find solution for $Ax = u$

Given that u spans the nullspace, there is no solution for $Ax = u$.

1.1.3 Is A Invertible?

Given that A is rank deficient, since it has one eigenvalue of 0, A is singular and not invertible, it is not full rank.

1.2 1b - A and B are similar matrices.

1.2.1 Show they have the same determinant

We have that:

$$A = SBS^{-1}.$$

$$AS = SB.$$

$$\det(AS) = \det(SB).$$

$$\det(A)\det(S) = \det(S)\det(B).$$

$$\det(A) = \det(B).$$

1.2.2 Show they have the same characteristic polynomial and eigenvalues.

We have that:

$$A = SBS^{-1}.$$

$$A - \lambda I = SBS^{-1} - \lambda I.$$

$$AS - \lambda IS = SB - \lambda IS.$$

$$\det\big((A - \lambda I)S\big) = \det\big(S(B - \lambda I)\big).$$

$$\det(A - \lambda I)\det(S) = \det(S)\det(B - \lambda I)).$$

$$\det(A - \lambda I) = \det(B - \lambda I)).$$

Thus, the characteristic polynomial of A, $\det(A - \lambda I) = 0$ is the same as $\det(B - \lambda I) = 0$.

2 Problem 2

2.1 2a

2.2 2b

3 Problem August 2018 3 - Solve this system of ODE IVP:

$$y^{'} = 5y_1 - 6y_2.$$

$$y^{'} = 3y_1 - 4y_2.$$

$$y_1(0) = 4, y_2(0) = 1$$

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3.1 Study notes:

For IVP problems of this type, we have that our standard solutions relative to eigenvalues $\lambda = \mu + i\nu$ are:

$$y = e^{\mu t}(\alpha \sin(\nu t) + \beta \cos(\nu t)).$$

3.2 Solution proof

We have that our corresponding matrix representation for this system is:

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We calculate the eigenvalues of A by solving $\det(A - \lambda I) = 0 \implies (5-\lambda)(-4-\lambda) - (3(-6)) = 0 \implies \lambda^2 - \lambda - 2 = 0$. We use these two eigenvalues to solve $(A - \lambda I)v = 0$ to find the corresponding eigenvalues:

$$\text{For } \lambda_1 = 2, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_2 = -1, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We thus have our general solution:

$$y_{gen} = \alpha e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find our particular solution to this IVP by using the initial condition provided:

$$y_{part} = \alpha e^{20} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

We solve this system of equations to determine that $\alpha = 3, \beta = -2$.
Thus, our final particular solution to this particular initial value problem is:

$$y_{(part.final)} = 3e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Or, equivalently:

$$\therefore y_{(part.final)} = y_1 + y_2, y_1 = 6e^{2t} - -2e^{-t}, y_2 = 3e^{2t} - -2e^{-t}. \checkmark$$

4 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, for reading this work, for her instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis.* John wiley & sons, 2008.