Numerical Analysis Previous Exam August 2018.

José L. Pabón

Spring 2020

This work is based on the course textbook [1], the material discussed in lectures and office hours related to our course MAT614 and additional references.

1 Problem August 2018 3 - Solve this system of ODE IVP:

$$y^{'} = 5y_1 - 6y_2.$$

$$y^{'} = 3y_1 - 4y_2.$$

$$y_1(0) = 4, y_2(0) = 1$$

.

1.1 Study notes:

For IVP problems of this type, we have that our standard solutions relative to eigenvalues $\lambda = \mu + i\nu$ are:

$$y = e^{\mu t}(\alpha \sin(\nu t) + \beta \cos(\nu t)).$$

1.2 Solution proof

We have that our corresponding matrix representation for this system is:

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We calculate the eigenvalues of A by solving $\det(A - \lambda I) = 0 \implies (5-\lambda)(-4-\lambda) - (3(-6)) = 0 \implies \lambda^2 - \lambda - 2 = 0$. We use these two eigenvalues to solve $(A - \lambda I)v = 0$ to find the corresponding eigenvalues:

$$\text{For } \lambda_1 = 2, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_2 = -1, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We thus have our general solution:

$$y_{gen} = \alpha e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We find our particular solution to this IVP by using the initial condition provided:

$$y_{part} = \alpha e^{20} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \beta e^{-0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

We solve this system of equations to determine that $\alpha = 3, \beta = -2$.
Thus, our final particular solution to this particular initial value problem is:

$$y_{(part.final)} = 3e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Or, equivalently:

$$\therefore y_{(part.final)} = y_1 + y_2, y_1 = 6e^{2t} - -2e^{-t}, y_2 = 3e^{2t} - -2e^{-t}. \checkmark$$

2 Problem August 2018 4:

2.1 a - Will Newtons' method converge quadratically to a root of $g(x) = x^2$?

2.2 Solution:

We have that Newton's method algorithm is of the form :

$$x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n).$$

We insert our $g(x)$:

$$x_{n+1} = x_n - \frac{1}{2(x_n)}(x_n^2) = x_n - \frac{1}{2}(x_n) = \frac{1}{2}(x_n).$$

Thus, the iterations of Newton's method would yield for us that:

$$x_{n+1} = (\frac{1}{2})^n x_0.$$

We note that we do have linear convergence:

$$\frac{1}{(x_n)}(\frac{1}{2}(x_n)) \leq 1 \implies (\frac{1}{2}) \leq 1.$$

Quadratic convergence would imply that:

$$\frac{1}{(x_n - \alpha)^2}(x_{n+1} - \alpha) \leq k.$$

$$\implies \frac{1}{(x_n - \alpha)^2}(\frac{1}{2}(x_n) - \alpha) \leq k.$$

Given our root is $\alpha = 0$, we have that:

$$\frac{1}{(x_n)^2}(\frac{1}{2}(x_n)) \leq k \implies \frac{1}{(x_n)}(\frac{1}{2}) \leq k.$$

If Newton's method were to converge for this function, we'd have $\lim_{n \longrightarrow \infty} x_n = 0$, thus:

$$\therefore \lim_{n \longrightarrow \infty} \frac{1}{(x_n)}(\frac{1}{2}) = \infty \implies \text{ no finite constant k such that :} \frac{1}{(x_n)^2}(\frac{1}{2}(x_n)) \leq k. \checkmark$$

Thanks for the advice pointing me in the right direction!

2.3 4b - $f(x) \in \mathbf{C^3}$ q has degree less than or equal to two interpolating f at $x_0, x_1, x_2,$. Let $h = \max{(x_1 - x_0), (x_2 - x_1)}$, K be the max over the x's in the interval of $\left|f^{'''}(x)\right|$. Show that:

$$\max_{x \in [x_0, x_2]} \left|f^{''}(x) - q^{''}(x)\right| = Ch^{\alpha}.$$

2.4 Solution proof:

Our general Newton's polynomial of second degree for the given points is:

$$N(x) = f(x_0) + [f(x_0), f(x_1)](x - x_0) + [f(x_0), f(x_1), f(x_2)](x - x_0)(x - x_1).$$

We have that:

$$[f(x_0), f(x_1)] = \frac{1}{(x_1 - x_0)}(f(x_1 - f(x_0)).$$

and:

$$[f(x_0), f(x_1), f(x_2)] = \frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0)).$$

We put everything together to get our form for $q(x)$:

$$q(x) = f(x_0) + \frac{1}{(x_1 - x_0)}(f(x_1) - f(x_0))(x - x_0)$$

$$+\{\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0))\}(x - x_0)(x - x_1).$$

We compute the second derivative of $q(x)$:

$$q^{''}(x) = 2(\frac{1}{(x_2 - x_1)(x_2 - x_0)}(f(x_2) - f(x_1)) - \frac{1}{(x_1 - x_0)(x_2 - x_0)}(f(x_1) - f(x_0))).$$

Thus, we know that the second derivative of q is just a number. Perhaps computing all of the before is not a judicious use of time.

We will now show that the expression of $f^{''}(x) - q^{''}(x)$ has at least one zero in the domain for this problem. We use a clever new function suggested by Prof. Siegel, $m(x) = f(x) - q(x)$. By construction, $m(x) = 0$ at $x = x_0, x_1, x_2$. Via mean value theorem (for derivatives?), there exists some $\eta_1 \in [x_0, x_1]$ such that $m^{'}(\eta_1) = 0$, $\eta_2 \in [x_1, x_2]$ such that $m^{'}(\eta_2) = 0$. Thus, we have that $m^{'}(x)$ has at least two zeroes in the interval $\in [x_0, x_2]$.

Using the same, or, you know, extremely similar mean value theorem argument, there exists some $\eta \in [x_0, x_2]$ such that $m''(\eta) = 0$. Thus, we know $f''(\eta) - q''(\eta) = m''(\eta) = 0$.

We follow the provided hint and consider the integral:

$$\int_\eta^x f'''(\xi) - q'''(\xi) d\xi = \int_\eta^x (f'''(\xi) - 0) d\xi = \left| f''(x) - q''(x) - (f''(\eta) - q''(\eta) \right| = \left| f''(x) - q''(x) \right|.$$

Now we apply the constants provided in the problem K, h:

$$\int_\eta^x f'''(\xi) d\xi \leq (x - \eta) \max \left| f'''(x) \right|.$$

Given the x that maximizes this could be 'on the other side' of x_1 with respect to η, we put our inequalities together and have that $(x - \eta) \leq 2h$, $C = \frac{1}{2}K$, such that:

This equation:

$$\int_\eta^x f'''(\xi) - q'''(\xi) d\xi = \int_\eta^x (f'''(\xi) - 0) d\xi = \left| f''(x) - q''(x) \right|$$

Combined with this inequality:

$$\int_\eta^x f'''(\xi) d\xi \leq (x - \eta) \max \left| f'''(x) \right| \leq 2hK.$$

$$\implies \text{for } \alpha = 1, (x - \eta) \leq 2h, C = \frac{1}{2}K \text{ provides the requested shown :}$$

$$\therefore \max_{x \in [x_0, x_2]} \left| f''(x) - q''(x) \right| \leq Ch^\alpha . \checkmark$$

I'm unsure if the inequalities here can be considered equality for some x in the interval as requested by the original enunciation of the problem.

3 Problem 5 - Consider the scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

.

3.1 Find the order of the scheme.

We use Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4)$$

$$y''_{n+1} = y''_n + hy'''_n + \frac{1}{2}h^2y''''_n + O(h^3)$$

We plug this into our scheme:

$$y_{n+1} = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1}) \implies$$

$$\implies y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{1}{24}h^4y''''_n + O(h^5) = y_n + \frac{1}{2}h(y'_{n+1} + y'_n) + \frac{1}{12}h^2(y''_n - y''_{n+1})$$

We group and simplify:

$$\frac{1}{2}h(y'_{n+1} + y'_n) = \frac{1}{2}h(y'_n + y'_n + hy''_n + \frac{1}{2}h^2y'''_n + \frac{1}{6}h^3y''''_n + O(h^4))$$

We group and simplify:

$$\frac{1}{12}h^2(y''_n - y''_{n+1}) = \frac{1}{12}h^2(y''_n - y''_n - hy'''_n - \frac{1}{2}h^2y''''_n - O(h^3))$$

We group like terms:

$$y_n(1-1) = 0$$

$$y'_n(h-h) = 0$$

$$y''_nh^2(\frac{1}{2} - \frac{1}{2}) = 0$$

$$y'''_nh^3(\frac{1}{6} - \frac{1}{4} + \frac{1}{12}) = y'''_nh^3(0)$$

$$y_n^{''''}h^4(\frac{1}{24}-\frac{1}{12}+\frac{1}{24})=y_n^{''''}h^4(0)$$

At this point we realize that we haven't expanded far enough so we need to Taylor expand farther. Incoming:

We use Taylor expansions:

$$y_{n+1}=y_n+hy_n^{'}+\frac{1}{2}h^2y_n^{''}+\frac{1}{6}h^3y_n^{'''}+\frac{1}{24}h^4y_n^{''''}+\frac{1}{120}h^5y_n^V+O(h^6)$$

$$y_{n+1}^{'}=y_n^{'}+hy_n^{''}+\frac{1}{2}h^2y_n^{'''}+\frac{1}{6}h^3y_n^{''''}+\frac{1}{24}h^4y_n^V+O(h^5)$$

$$y_{n+1}^{''}=y_n^{''}+hy_n^{'''}+\frac{1}{2}h^2y_n^{''''}+\frac{1}{6}h^3y_n^V+O(h^4)$$

We plug this into our scheme:

$$y_{n+1}=y_n+\frac{1}{2}h(y_{n+1}^{'}+y_n^{'})+\frac{1}{12}h^2(y_n^{''}-y_{n+1}^{''})\implies$$

$$\implies y_n+hy_n^{'}+\frac{1}{2}h^2y_n^{''}+\frac{1}{6}h^3y_n^{'''}+\frac{1}{24}h^4y_n^{''''}+\frac{1}{120}h^5y_n^V+O(h^6)=y_n+\frac{1}{2}h(y_{n+1}^{'}+y_n^{'})+\frac{1}{12}h^2(y_n^{''}-y_{n+1}^{''})$$

We group and simplify:

$$\frac{1}{2}h(y_{n+1}^{'}+y_n^{'})=\frac{1}{2}h(y_n^{'}+y_n^{'}+hy_n^{''}+\frac{1}{2}h^2y_n^{'''}+\frac{1}{6}h^3y_n^{''''}+\frac{1}{24}h^4y_n^V+O(h^5))$$

We group and simplify:

$$\frac{1}{12}h^2(y_n^{''}-y_{n+1}^{''})=\frac{1}{12}h^2(y_n^{''}-y_n^{''}-hy_n^{'''}-\frac{1}{2}h^2y_n^{''''}-\frac{1}{6}h^3y_n^V+O(h^4)))$$

We group like terms:

$$y_n(1-1)=0$$

$$y_n^{'}(h-h)=0$$

$$y_n^{''}h^2(\frac{1}{2}-\frac{1}{2})=0$$

$$y_n^{'''}h^3(\frac{1}{6}-\frac{1}{4}+\frac{1}{12})=(0)y_n^{'''}h^3$$

$$y_n'''' h^4 \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{24}\right) = (0) y_n'''' h^4$$

$$y_n^V h^5 \left(\frac{1}{120} - \frac{1}{48} + \frac{1}{72}\right) = \left(\frac{1}{720}\right) y_n^V h^5.$$

We have that $48 = 2^4(3), 72 = 2^3(3^2), 120 = 2^3(3)(5)$, the least common multiple for the denominators in the fractions involved is $720 = 2^4(3^2)5$.

Thus the computed coefficient is:

$$\frac{6}{720} - \frac{15}{720} + \frac{10}{720} = \frac{1}{720}.$$

Thus, the lowest term of nonzero coefficients are $(\frac{1}{720}) y_n^V h^5$, i.e. of order $O(h^5)$ in our computation of $y_{n+1} - y_n$. Now our local truncation error is:

$$\tau_{n+1} = \frac{1}{h}(y_{n+1} - y_n) = \Longrightarrow \tau_{n+1} = \frac{1}{h}\left(\frac{1}{720}\right) y_n^V h^5$$

Thus, the method is of fourth order $O(h^4)$✓.

3.2 5b - Show that the region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

We define $y' = \lambda y, y(0) = y_0$, and we consider the method applied to this function. We will use notation of $f_n \equiv f^n$, which will not denote the nth derivative. We have that:

$$y_n'' = \frac{df(x_n, y_n)}{dx} + f(x_n, y_n)\frac{df(x_n, y_n)}{dx}.$$

$$f^n = \lambda y_n.$$

$$f_x^n = \lambda y\prime(x_n) = \lambda f^n = \lambda^2 y_n.$$

$$f_y^n = \lambda.$$

We examine the method within this framework:

$$y_{n+1} = y_n + \frac{h}{2}[y_n' + y_{n+1}'] + \frac{h^2}{12}[y_n'' - y_{n+1}''].$$

$$y_{n+1} = y_n + \frac{h}{2}[f^n + f^{n+1}] + \frac{h^2}{12}[f_x^n + f^n f_y^n - f_x^{n+1} + f^{n+1} f_y^{n+1}].$$

$$y_{n+1} = y_n + \frac{h}{2}[\lambda y_n + \lambda y_{n+1}] + \frac{h^2}{12}[\lambda^2 y_n + \lambda^2 y_n - \lambda^2 y_{n+1} - \lambda^2 y_{n+1}].$$

$$y_{n+1} = y_n + \frac{h\lambda}{2}[y_n + y_{n+1}] + \frac{h^2\lambda^2}{6}[y_n - y_{n+1}].$$

$$\implies (1-\frac{1}{2}h\lambda+\frac{1}{6}h^2\lambda^2)y_{n+1} = (1+\frac{1}{2}h\lambda+\frac{1}{6}h^2\lambda^2)y_n \implies (6-3h\lambda+h^2\lambda^2)y_{n+1} = (6+3h\lambda+h^2\lambda^2)y_n.$$

Thus we have our relation:

$$y_{n+1} = \frac{(6+3h\lambda+h^2\lambda^2)}{(6-3h\lambda+h^2\lambda^2)}y_n.$$

We verify the stability of the method for this function by studying:

$$y_n = (\frac{(6+3h\lambda+h^2\lambda^2)}{(6-3h\lambda+h^2\lambda^2)})^n y_0.$$

By definition and convention, we have step size of $h > 0$ and we define $\lambda < 0 \implies h\lambda < 0$. For the method to be stable, we find we need the condition:

$$\left|\frac{(6+3h\lambda+h^2\lambda^2)}{(6-3h\lambda+h^2\lambda^2)}\right| < 1.$$

$h\lambda < 0 \implies (6-3h\lambda+h^2\lambda^2) > 0$

So we have that

$$-1 < \frac{(6+3h\lambda+h^2\lambda^2)}{(6-3h\lambda+h^2\lambda^2)} < 1.$$

$$-1(6-3h\lambda+h^2\lambda^2) < (6+3h\lambda+h^2\lambda^2) < (6-3h\lambda+h^2\lambda^2).$$

We examine these inequalities and have that:

$$-1(6-3h\lambda+h^2\lambda^2) < (6+3h\lambda+h^2\lambda^2) \implies -6 < (6+2h^2\lambda^2).\checkmark$$

This inequality holds for all values of h, λ. For the second set of inequalities we find that:

$$(6+3h\lambda+h^2\lambda^2) < (6-3h\lambda+h^2\lambda^2) \implies 3h\lambda < -3h\lambda.\checkmark$$

This inequality holds for $h\lambda < 0$.

Thus

$$\left|\frac{(6+3h\lambda+h^2\lambda^2)}{(6-3h\lambda+h^2\lambda^2)}\right| < 1.$$

When $h > 0$, $\lambda < 0 \implies h\lambda < 0$ giving us our region of stability.

$\therefore$ The region of absolute stability contains the entire negative real axis of the $h\lambda$ plane.

4 Problem August 2018 6 - Show the degree of precision is less than or equal to $2n-1$, and then that it is no more than the same.

Suppose inner product with weight is defined by:

$$< f, g >= \int_a^b f(x)g(x)w(x)dx.$$

Consider quadrature formula:

$$I(f) = \int_a^b f(x)w(x)dx \approx Q(f) = \sum_{j=1}^{n} w_j f(x_j).$$

We have that:

$$w_j = \int_a^b l_j(x)w(x)dx.$$

And

$$l_j(x) = \prod_{i=1,2,3,\ldots,n---i\neq j} \frac{(x-x_j)}{(x_i-x_j)}.$$

4.1 Solution, proof:

Degree of precision is defined as the highest order polynomial that the quadrature will return an exact answer for. Study note - for Simpsons rule this is three, for trapezoid rule this is one.

The key idea of this proof is division of the polynomial f. We define polynomials such that:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n-1, n, n-1, n-1$ and by construction and the interpolation of I(f), $\star$ *we have that $r(x)$ is exact and there is no further remainder.*

Note to reader - I went to Prof. Hamfeldt's office hour to talk about this problem, she mentioned this claim in italics is not clear or obvious. I thus proffer the follow explanation:

We have that $r(x)$ is exact without a further remainder because it is of degree less than or equal to $n-1$, thus there via fundamental properties of polynomials, there is an expression for $r(x)$ in terms of basis function monomials of up to n degree, thus for some constantsa_i, $r(x) = a_0 + a_1x + a_2x^2 + \ldots$ such that $r(x)$ is an exact $n-1$ degree polyomial in x.

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n-1$, thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x)dx + \int_a^b r(x)w(x)dx = 0 + \int_a^b r(x)w(x)dx = Q(r).$$

Thus we find that:

$$I(f) = Q(r) = \sum w_j r(x_j) = \sum w_j(p_n(x_j)r(x_j) + r(x_j)) = \sum 0 + w_j r(x_j)) = Q(f).$$

Need to show the remainder polynomials are exact.

Thus, we have that the precision of this method (Gaussian quadrature) is $\leq 2n-1$.

4.2 Show that the precision is no greater than $2n - 1$.

We assume the same construction as in the previous argument, except for the degrees of the polynomials:

$$\frac{f(x)}{p_n(x)} - \frac{r(x)}{p_n(x)} = q(x) \implies f(x) = p_n(x)q(x) + r(x).$$

Where rational expression is shorthand for polynomial division. We construct this with the degrees of f, p, q, r being $\leq 2n, n, n, n-1$ and by construction and the interpolation of I(f), we have that $r(x)$ is still exact and there is no further remainder.

We have that $p_n x$ is orthogonal to all polynomials of degree $\leq n-1$ but not orthogonal to all polynomials of degree n, thus we have that:

$$I(f) = I(qp_n + r) = \int_a^b q(x)p_n(x) + r(x)dx + \int_a^b r(x)w(x)dx = 0 + \int_a^b r(x)w(x)dx = Q(r) + < q, p_n > .$$

Thus,$< q, p_n > \neq 0$.

We still have the same zeros of our p_n, so thus we find that:

$$Q(f) = \sum w_j r(x_j) = Q(qp_n + r) \sum w_j(p_n(x_j)r(x_j) + r(x_j)) = \sum 0 + r(x_j)) = Q(r).$$

Thus,

$$Q(f) \neq Q(r)$$

Thus, for all polynomials f of degree 2n or higher, the quadrature formula is not precise.

5 Conclusion

Thank you to Prof. Hamfeldt, neé Froese, as well as anyone else for reading this work, as well as any instruction, lectures and future office hours efforts. I look forward to any feedback and learning more of the material in this course and qualifying exams.

References

[1] Kendall E Atkinson. *An introduction to numerical analysis.* John wiley & sons, 2008.