From norms to metrics in non-Archimedean geometry.

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Δ	nlan:

B. Pluripotential theory in the non-Archimedean world.

C. Finite-energy spaces and geodesics.

D. Interactions with complex pluripotential theory.



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The "real world" is Archimedean. In a non-Archimedean world, this would not necessarily be possible: you could put those segments back to back infinitely many times, and still never reach a certain length.

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This is equivalent to the fact that there exists some real number $\alpha \in \mathbb{R}_{>0}$ such that, for all $n \in \mathbb{N}$,

$$|n1_{K}| = |1_{K} + 1_{K} + \dots + 1_{K}| \le \alpha.$$

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For applications to complex geometry, we will be particularly interested in $(\mathbb{C}, |\cdot|_0)$ and $\mathbb{C}((t))$.

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We want to import these objects to non-Archimedean fields and varieties.

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Problem 2: having done that, we will still not be able to perform "differential calculus": we must work *globally* as much as possible.

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 X^{an} is a space made of valuations. In the case where K is $\mathbb C$ with the trivial absolute value, X^{an} is simply the set of all valuations on function fields K(Y) of irreducible subvarieties $Y\subset X$.

We now address our second Problem: having to work globally.

If X is a complex projective manifold, and L is an ample line bundle on X, one can more generally characterize a psh metric on L as a *decreasing limit* of smooth metrics on L with positive curvature form (Demailly '92).

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This characterization is global in nature: we can use it as a definition in the non-Archimedean case, provided we have a "good" class of metrics to take decreasing limits of.

One can also work with Fubini-Study metrics, i.e. metrics of the form

$$k^{-1} \log \sum_{i=1}^{h^0(X,kL)} |s_i|^2 e^{2\lambda_i},$$

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If L o X are non-Archimedean, one can similarly define a (non-Archimedean) Fubini-Study metric on $L^{\rm an}$, as a metric of the form

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We then define a (non-Archimedean) **plurisubharmonic metric** on $L^{\rm an}$ to be a decreasing limit (of a net) of Fubini-Study metrics.

Finite-energy spaces and geodesics.

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$$E(\phi_0,\phi_1) = rac{1}{V(d+1)} \sum_{k=0}^d \int_X (\phi_0 - \phi_1) (dd^c \phi_0)^k \wedge (dd^c \phi_1)^{d-k},$$

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Because it is decreasing in the first variable, it extends naturally to all of $PSH(L^{an})$, possibly taking $-\infty$ as values. We define

$$\mathcal{E}^{1}(L^{\mathrm{an}}) := \{ \phi \in \mathrm{PSH}(L^{\mathrm{an}}), \ E(\phi, \phi_{\mathrm{ref}}) > -\infty \ \forall \phi_{\mathrm{ref}} \in \mathrm{PSH} \cap C^{0}(L^{\mathrm{an}}) \}.$$

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In the non-Archimedean case, if the endpoints are continuous, geodesics can furthermore be realized as limits of "Fubini-Study segments"

$$[0,1]
i t \mapsto k^{-1} \max_i (\log |s_i| - (1-t)\lambda_i - t\lambda_i'),$$

mirroring a result due to Berndtsson in the complex case. To prove this, we "quantize" the non-Archimedean distance via finite-dimensional distances in spaces of norms on each $H^0(X, kL)$.

More on geodesics.

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They are also detected by the Monge-Ampère energy E, in the sense that a psh segment $t\mapsto \phi_t$ is geodesic if and only if $t\mapsto E(\phi_t,\phi_{\mathrm{ref}})$ is affine for any (hence all) reference metric(s) ϕ_{ref} .

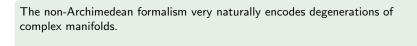
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Theorem (R.)

A non-Archimedean psh segment $t \mapsto \phi_t \in \mathcal{E}^1(L^{\mathrm{an}})$ is geodesic if and only if $t \mapsto E(\phi_t)$ is affine.



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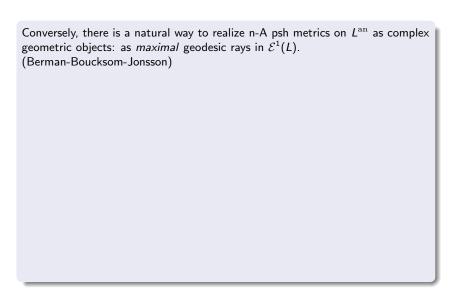
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In that sense, n-A psh metrics on $L^{\rm an}$ can be understood as generalized test configurations for (X,L).



Conversely, there is a natural way to realize n-A psh metrics on $L^{\rm an}$ as complex geometric objects: as maximal geodesic rays in $\mathcal{E}^1(L)$. (Berman-Boucksom-Jonsson)

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Given ϕ^{NA} rational Fubini-Study on L^{an} , i.e. given by a test configuration $(\mathcal{X},\mathcal{L})$, its maximal geodesic ray is the largest psh ray $t\mapsto \phi_t$ such that its associated metric Φ on $L\times\mathbb{D}^*$ extends as a *locally bounded* psh metric on \mathcal{L} .

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More generally, one constructs the maximal ray associated to $\phi^{\mathrm{NA}} \in \mathcal{E}^1(L^{\mathrm{an}})$ as the "least singular" ray Φ with singularity data given by ϕ^{NA} . Such rays have played an essential role in the variational proofs of various versions of the Yau-Tian-Donaldson conjecture (BBJ, Li, Han-Li...)

Denote by $\mathcal{R}^1(L)$ the space of finite-energy psh rays (emanating from ϕ_0). It can be metrized via the "slope at infinity"

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Theorem (Berman-Boucksom-Jonsson '15)

The mapping sending ϕ^{NA} to its associated maximal ray is an isometric (i.e. distance-preserving and injective) map from $(\mathcal{E}^{1}(L^{\mathrm{an}}), d_{1}^{\mathrm{NA}})$ to $(\mathcal{R}^{1}(L), \hat{d}_{1})$.

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On the one hand, we can realize X and L as varieties X_K and L_K over $K=\mathbb{C}((t))$. We thus naturally have a space of n-A finite-energy metrics $(\mathcal{E}^1(L_K^{\mathrm{an}}),d_1^{\mathrm{NA}})$.

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One can similarly as in \mathbb{R}^1 define:

- the "maximal family" in $\mathcal{D}^1(L)$ corresponding to a given metric $\phi^{\mathrm{NA}} \in \mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}})$, which is defined as a solution to an envelope problem again, and correspond to maximal rays in the invariant case;
- ② a distance \hat{d}_1 on this space via Lelong numbers $\nu_0(z \mapsto d_1(\phi_{0,z},\phi_{1,z}))$. (for the last point, we require additional technical hypotheses mimicking geodesicity.)

Theorem (R. '21)

The map sending ϕ^{NA} to its maximal family is an isometric map from $(\mathcal{E}^1(L_{\mathrm{K}}^{\mathrm{an}}), d_1^{\mathrm{NA}})$ to $(\mathcal{D}^1(L), \hat{d}_1)$. In particular, both spaces are complete, geodesic metric spaces.

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- 1. d_p , $p \neq 1$ structures on non-Archimedean metrics (as in Darvas, complex case).
- 2. Interpreting non-Archimedean K-stability over $\mathbb{C}((t))$?
 3. Convexity of some important functionals $(K^{\mathrm{NA}}, H^{\mathrm{NA}})$ and geometric properties of their minimizers.

