

Crk3 formulation

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1 Fracture storage

$$\zeta_f = \frac{\delta a_f}{a_f} + c_f \delta p - \beta_f \delta T ,$$

and we need the term

$$(\tilde{\psi}, a_f \dot{\zeta}_f)_{\tilde{\Omega}} = (\tilde{\psi}, \delta \dot{a}_f) + (\tilde{\psi}, a_f c_f \delta \dot{p}) - (\tilde{\psi}, a_f \beta_f \delta \dot{T}) , \quad (1)$$

where

$$a_f = \llbracket \mathbf{u} \rrbracket \cdot \mathbf{n}^+ = (\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{n}^+ = \sum_{d=1}^2 u_k^d n_k^d = u_k^d n_k^d .$$

where $d = 0$ for the element in the $+$ side of the fracture and $d = 1$ for the element in the $-$ side.

Opening the terms in (1) into coefficients (assuming p to be a step constant):

$$\begin{aligned} & +(\tilde{\psi}, \delta \dot{a}_f) \Delta t : \\ & \quad \mathbf{K}_{pd}^{\beta\gamma} = \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \quad n_k^+ \\ & \quad \mathbf{F}_0^\beta = \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \quad n_k^+ \quad \check{u}_k^\gamma , \\ & +(\tilde{\psi}, a_f c_f \delta \dot{p}) \Delta t : \\ & \quad \mathbf{K}_{pd}^{\beta\gamma} = c_f \quad \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \quad n_k^+ \quad \Delta p , \\ & -(\tilde{\psi}, a_f \beta_f \delta \dot{T}) \Delta t : \\ & \quad \mathbf{K}_{pd}^{\beta\gamma} = \beta_f \quad \tilde{\psi}^\beta \quad (-1)^{d+1} \quad \phi_d^\gamma \quad n_k^+ \quad \Delta T . \end{aligned}$$

where \check{u} refers to the (known) solution in the previous timestep, and $c_f = \frac{1}{K_f}$. Δp and ΔT are the delta of p and T from the previous timestep to the current.

2 Derivatives for nonlinear solution

Both pressure and displacements are variables of the system. We need to derive some non-linear term for Newton's method. Let:

$$f = \left(\tilde{\psi}, \quad a_f \ c_f \ \delta \dot{p} \right) \Delta t = \sum_{d=1}^2 \left(\tilde{\psi}, \quad c_f \ n_k^d \ u_k^d \ \delta \dot{p} \right) \Delta t \quad .$$

where $d = 0$ for the element in the $+$ side of the fracture and $d = 1$ for the element in the $-$ side. Then, the partial derivatives are

$$\frac{\partial f}{\partial p} = \sum_{d=1}^2 \left(\tilde{\psi}, \quad c_f \ n_k^d \ u_k^d \right) \Delta t \quad ,$$

and

$$\frac{\partial f}{\partial u_k^d} = \left(\tilde{\psi}, \quad c_f \ n_k^d \ \delta \dot{p} \right) \Delta t \quad .$$

3 Discontinuous galerkin for pressure

Reference: (Ruijie Liu, 2004) - PhD dissertation

Our equation:

$$\alpha \nabla \cdot \dot{u} + S_\epsilon \dot{p} - \nabla \cdot (\kappa \nabla p) = 0$$

Weak formulation:

$$(w, \alpha \nabla \cdot \dot{u}) + (w, S_\epsilon \dot{p}) - (w, \nabla \cdot (\kappa \nabla p)) = 0 \quad \forall w$$

Lets focus our attention to the last term. Integrate by parts:

$$-(w, \nabla \cdot (\kappa \nabla p)) = (\nabla w, \kappa \nabla p) - (w, \kappa \nabla p \cdot \mathbf{n})_\Gamma$$

When we sum element by element, the last term in Γ results in the edge skeleton Γ_i and the outside boundary Γ_p .

The full equation becomes a summation of the following:

$$(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon \dot{p})_{\Omega_E} + (\nabla w, \kappa \nabla p)_{\Omega_E} - (w, \kappa \nabla p \cdot \mathbf{n})_{\Gamma_i} - (w, \kappa \nabla p \cdot \mathbf{n})_{\Gamma_p} = 0 \quad \forall w$$

where Ω_E is the element interiors

We observe the following identity:

$$-(w, \kappa \nabla p \cdot \mathbf{n})_{\Gamma_i} = -(\llbracket w \rrbracket, \kappa \{\nabla p\})_{\Gamma_i} - (\kappa \llbracket \nabla p \rrbracket, \{w\})_{\Gamma_i} \quad (2)$$

where

$$\begin{aligned} \llbracket w \rrbracket &= w^+ \mathbf{n}^+ + w^- \mathbf{n}^- \\ \{w\} &= 0.5 \times (w^+ + w^-) \end{aligned}$$

and similarly

$$\begin{aligned} \llbracket \nabla p \rrbracket &= \nabla p^+ \cdot \mathbf{n}^+ + \nabla p^- \cdot \mathbf{n}^- \\ \{\nabla p\} &= 0.5 \times (\nabla p^+ + \nabla p^-) \end{aligned}$$

Define $d = 0$ for $+$ and $d = 1$ for $-$ and similarly $e = 0$ for $+$ and $e = 1$ for $-$. The above can be written as summations in d :

$$\begin{aligned} \llbracket w \rrbracket &= \sum_d w^d n_k^d \\ \{w\} &= 0.5 \times \sum_d w^d \\ \llbracket \nabla p \rrbracket &= \sum_e p_{,k}^e n_k^e \\ \{\nabla p\} &= 0.5 \times \sum_e p_{,k}^e \end{aligned}$$

Replace the above in (2) suppressing the summation sign in d and e :

$$-(w, \kappa \nabla p \cdot \mathbf{n})_{\Gamma_i} = -0.5 \times \left[\left(w^d n_k^d, \kappa p_{,k}^e \right)_{\Gamma_i} + \left(\kappa p_{,k}^e n_k^e, w^d \right)_{\Gamma_i} \right]$$

Finally, we add an interior penalty term:

$$\frac{\delta_p}{|s|} (\llbracket p \rrbracket, \llbracket w \rrbracket)_{\Gamma_i}$$

The final expression is:

$$\begin{aligned} &(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon \dot{p})_{\Omega_E} + (\nabla w, \kappa \nabla p)_{\Omega_E} - (w, \kappa \nabla p \cdot \mathbf{n})_{\Gamma_p} \\ &- (\llbracket w \rrbracket, \kappa \{\nabla p\})_{\Gamma_i} - (\kappa \llbracket \nabla p \rrbracket, \{w\})_{\Gamma_i} \\ &+ \frac{\delta_p}{|s|} (\llbracket p \rrbracket, \llbracket w \rrbracket)_{\Gamma_i} = 0 \quad \forall w \end{aligned}$$

4 An identity

$$\begin{aligned}
(w, \mathbf{u} \cdot \mathbf{n})_{\Gamma_i} &= (w^+, \mathbf{u}^+ \cdot \mathbf{n}_+)_{\Gamma_i} + (w^-, \mathbf{u}^- \cdot \mathbf{n}^-)_{\Gamma_i} \\
&= (\llbracket w \rrbracket, \{\mathbf{u}\}) + (\llbracket \mathbf{u} \rrbracket, \{w\}) \\
&= 0.5 \times (w^D, \mathbf{u}^E \cdot \mathbf{n}^E)_{\Gamma_i} + 0.5 \times (\mathbf{u}^D, w^E \mathbf{n}^E)_{\Gamma_i} \\
&= 0.5 \times (-1)^E (w^D, \mathbf{u}^E \cdot \mathbf{n}^+)_{\Gamma_i} + 0.5 \times (-1)^E (\mathbf{u}^D, w^E \mathbf{n}^+)_{\Gamma_i} \\
&= 0.5 \times (-1)^E (w^D, \mathbf{u}^E \cdot \mathbf{n}^+)_{\Gamma_i} + 0.5 \times (-1)^D (w^D, \mathbf{u}^E \mathbf{n}^+)_{\Gamma_i} \\
&= 0.5 \times [(-1)^E + (-1)^D] (w^D, \mathbf{u}^E \cdot \mathbf{n}^+)_{\Gamma_i} \\
&= \delta^{DE} (-1)^D (w^D, \mathbf{u}^E \cdot \mathbf{n}^+)_{\Gamma_i} \\
&= (-1)^D (w^D, \mathbf{u}^D \cdot \mathbf{n}^+)_{\Gamma_i} \\
&= (w^D, \mathbf{u}^D \cdot \mathbf{n}^D)_{\Gamma_i}
\end{aligned}$$

An alternative derivation:

$$\begin{aligned}
&(\llbracket w \rrbracket, \{\mathbf{u}\}) + (\llbracket \mathbf{u} \rrbracket, \{w\}) = \\
&= 0.5 \times (w^D, \mathbf{u}^E \cdot \mathbf{n}^E)_{\Gamma_i} + 0.5 \times (\mathbf{u}^D, w^E \mathbf{n}^E)_{\Gamma_i} \\
&= \\
&\quad (w^+, \mathbf{u}^+ \cdot \mathbf{n}^+) + (\mathbf{u}^+, w^+ \mathbf{n}^+) \\
&\quad (\textcolor{red}{w}^-, \textcolor{red}{u}^+ \cdot \textcolor{red}{n}^+) + (\textcolor{red}{u}^+, \textcolor{red}{w}^- \textcolor{red}{n}^-) \\
&\quad (\textcolor{blue}{w}^+, \textcolor{blue}{u}^- \cdot \textcolor{blue}{n}^-) + (\textcolor{blue}{u}^-, \textcolor{blue}{w}^+ \textcolor{blue}{n}^+) \\
&\quad (w^-, \mathbf{u}^- \cdot \mathbf{n}^-) + (\mathbf{u}^-, w^- \mathbf{n}^-) \\
&= \\
&\quad (w^+, \mathbf{u}^+ \cdot \mathbf{n}^+) + (\mathbf{u}^+, w^+ \mathbf{n}^+) \\
&\quad (w^-, \mathbf{u}^- \cdot \mathbf{n}^-) + (\mathbf{u}^-, w^- \mathbf{n}^-) \\
&= \\
&\quad (w^D, \mathbf{u}^D \cdot \mathbf{n}^D)_{\Gamma_i}
\end{aligned}$$

5 Viscoelasticity (Creep)

We follow the workflow by Kumar et al. (2021) for the formulation of the viscoelastic model for salt. According to (Carter et al., 1993), the strain of a solid is

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^s + \boldsymbol{\varepsilon}^a \quad , \quad (3)$$

where the superscripts stand for elastic, plastic, transient (or primary or decelerating creep), steady-state (or secondary creep) and accelerating (or tertiary creep), respectively.

5.1 Multiaxial creep - development

$$\begin{aligned} \sigma_e^2 &= \frac{3}{2} \tau_{ij} \tau_{ij} \\ \tau_{ij} &= \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \end{aligned}$$

Now we want to develop the equality:

$$\dot{\varepsilon}_{ij}^s = \frac{\partial \sigma_e}{\partial \sigma_{ij}} \dot{\varepsilon} = \frac{3}{2} \frac{\tau_{ij}}{\sigma_e} \dot{\varepsilon}^s$$

Hence:

$$\begin{aligned} 2\sigma_e \partial \sigma_e &= 3\tau_{ij} \frac{\partial \tau_{ij}}{\partial \sigma_{kl}} \partial \sigma_{kl} \\ &= 3\tau_{ij} \left[\frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3} \frac{\partial \sigma_{mm}}{\partial \sigma_{kl}} \right] \partial \sigma_{kl} \\ &= 3\tau_{ij} \left[\delta_{ik} \delta_{jl} - \frac{1}{3} \delta_{kl} \delta_{ij} \right] \partial \sigma_{kl} \\ &= 3\tau_{kl} \partial \sigma_{kl} - \tau_{ii} \partial \sigma_{kl} \delta_{kl} \\ &= 3\tau_{kl} \partial \sigma_{kl} \end{aligned}$$

because $\tau_{ii} = 0$. Hence:

$$\frac{\partial \sigma_e}{\partial \sigma_{kl}} = \frac{3}{2} \frac{\tau_{kl}}{\sigma_e}$$

5.2 Carter model

The starting point is Carter's model (Carter et al., 1993). We write in the normalized way so that the constants have rational units (Dusseault & Fordham, 1993), and dimensionless parameters. The expression for the creep model is the following power-law for steady-state creep is obtained from the lab for uniaxial strain as

$$\dot{\varepsilon}^s = \varepsilon_0 \exp\left(-\frac{Q}{RT}\right) \left(\frac{\sigma_e}{\sigma_0}\right)^n \quad (4)$$

where σ_0 is a normalization stress arbitrary in some sense as it is a fitting parameter together with ε_0 . It can be seen as a critical stress for transition of regimes (when the quotient approaches 1, we can continuously change the exponent n). See (Dusseault et al., 1987; Poiate, 2012). Table 1 describes the variables.

Now we express the creep flow rule in tensor form under multiaxial stress conditions (Hyde et al., 2014; Kraus, 1980; Xu et al., 2018), as:

$$\dot{\varepsilon}_{ij}^s = \frac{\partial \sigma_e}{\partial \sigma_{ij}} \dot{\varepsilon} = \frac{3}{2} \frac{\tau_{ij}}{\sigma_e} \dot{\varepsilon}^s \quad (5)$$

where σ_e is an effective stress measure (here assumed Von Misses) and τ_{ij} is the stress deviator tensor.

$$\sigma_e^2 = \frac{3}{2} \tau_{ij} \tau_{ij} \quad (6)$$

$$\tau_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \quad (7)$$

Equivalent expressions for σ_e are

$$\sigma_e^2 = 3 J_2 \quad J_2 = 0.5 \times \tau_{ij} \tau_{ij} \quad (8)$$

$$\sigma_e^2 = 0.5 \times [(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6 (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)] \quad (9)$$

$$\sigma_e^2 = \frac{3}{2} \sigma_{ij} \sigma_{ij} - \frac{1}{2} (\sigma_{kk}^2) \quad (10)$$

where J_2 is the second invariant of the stress deviator tensor. We can use Carter's law to write

$$\dot{\epsilon}_{ij}^s = \frac{3}{2} \varepsilon_0 \exp\left(-\frac{Q}{RT}\right) \left(\frac{\sigma_e}{\sigma_0}\right)^n \left(\frac{\tau_{ij}}{\sigma_e}\right) \quad (11)$$

An alternative writing for this equation, as e.g. in (Kumar et al., 2021), is

$$\dot{\epsilon}_{ij}^s = \frac{3}{2} A \exp\left(-\frac{Q}{RT}\right) \sigma_e^{n-1} \tau_{ij} \quad (12)$$

where $A = \varepsilon_0 / \sigma_0^n$. However, the units of A are MPa^{-n} and A and n are mutually dependent. The use of σ_0 is convenient for decoupling and different creep modes can be calibrated using only n , for example. Alternatively, we can use the strategy by (Dusseault et al., 1987), with a critical stress and sharp transitions between modes..

$$A = \frac{\varepsilon_0}{\sigma_0^n} \exp\left(\frac{Q}{RT}\right) \quad (13)$$

Variable	Unit	Description
R	$\text{J mol}^{-1} \text{K}^{-1}$	Universal gas constant ($R = 8.3144$)
T	K	Temperature
Q	J mol^{-1}	Apparent activation energy
n	—	Material constant
ε_0	—	Material constant
σ_0	MPa	Material constant
σ_e	MPa	Differential stress ($\sigma_1 - \sigma_3$), or von mises, or Tresca.

Table 1: Symbols in Carter model

Variable	Value
Q	$51.6 \times 10^{-3} \text{ J mol}^{-1}$
R	$8.3144 \text{ J mol}^{-1} \text{K}^{-1}$
A	$8.1 \times 10^{-5} \text{ MPa}^{-n} \text{ s}^{-1}$

Table 2: Values for initial testcase, to compare with results from Carter et al. (1993) and Kumar et al. (2021).

5.3 Creep in small strains

Let ε^s be the steady state creep history, so that

$$\begin{aligned} \varepsilon^s(t + \Delta t) &= \varepsilon^s(t) + \Delta \varepsilon^s(t + \theta) \quad , \\ \Delta \varepsilon^s &= \dot{\varepsilon}(t) \Delta t \end{aligned}$$

is an explicit timetepping. We can calculate $\dot{\epsilon}^s$ from the previous section.

The problem statement is:

Find $\mathbf{u} \in \mathcal{V}$ such that $\forall \mathbf{w} \in \mathcal{W}$

$$\begin{aligned} (w_{i,j}, \sigma_{ij}) - (w_i, f_i) - (w_i, h_i)_{\Gamma_h} &= 0 \\ \sigma_{ij} &= \mathbb{C}_{ijkl} \varepsilon_{kl}^e \\ \varepsilon_{kl}^e &= \varepsilon_{kl} - \varepsilon_{kl}^s \\ \varepsilon_{kl} &= 0.5 \times (u_{k,l} + u_{l,k}) \\ u_i &= g_i && \text{on } \Gamma_g \\ \sigma_{ij} n_j &= h_i && \text{on } \Gamma_h \end{aligned}$$

where Ω is the domain of the the bilinear operators whenever ommitted. u_i agregates the unknown and known (Dirichlet) degrees of freedom, as usual.

The creep term is integrated in time and added to the equilibrium equation as

$$(w_{i,j}, \mathbb{C}_{ijkl} u_{k,l}) - (w_{i,j}, \mathbb{C}_{ijkl} \varepsilon_{kl}^s) - (w_i, f_i) - (w_i, h_i)_{\Gamma_h} = 0$$

5.4 Creep integration with large strain elasticity

For small timesteps, we can determine the creep strain variation explicitley:

$$\varepsilon^s(t + \Delta t) = \varepsilon^s(t) + \Delta \varepsilon^s(t + \theta)$$

where θ indicates the time when the increment in creep strain is measured. We start using an explicit measurement, $\theta = 0$ (stable for small Δt), so that

$$\Delta \varepsilon^s = \dot{\varepsilon}^s(t) \Delta t$$

Before moving to the next timestep, recalculate the stresses using the above creep strain. Check for a threshold and reduce the time increment if necessary.

Rewrite the mechanical equilibrium equation with the creep increments. We know how to solve elastic stresses, so:

$$\begin{aligned} \mathcal{F} &= (W_{i,I}, P_{iI}^e)_{\Omega_0} - (W_i, \tilde{T}_i)_{\Gamma_H} = 0 \\ P_{iI}^e &= P_{iI} - P_{iI}^s \end{aligned}$$

We need to move the creep contribution P_{iI}^s to the current configuration:

$$\begin{aligned} P_{iI}^s &= J \sigma_{ij}^s F_{Ij}^{-1} \\ \mathcal{F} &= (W_{i,I}, P_{iI})_{\Omega_0} - (W_{i,I}, J \sigma_{ij}^s F_{Ij}^{-1})_{\Omega_0} - (W_i, \tilde{T}_i)_{\Gamma_H} = 0 \end{aligned}$$

Now we must calculate σ_{ij}^s :

$$\begin{aligned} \sigma_{ij}^s &= \mathbb{C}_{ijkl} \varepsilon_{kl}^s \\ \mathbb{C}_{ijkl} &= J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL} \mathbb{C}_{IJKL} \end{aligned}$$

$$\begin{aligned} P_{iI}^e &= F_{iJ} S_{JI} \\ S_{JI}^e &= \mathbb{C}_{JIKL} E_{KL}^e \\ E_{KL}^e &= E_{KL} - E_{KL}^s \end{aligned}$$

where E_{KL}^e is the elastic behavior of the Green-Lagrange strain and E_{KL}^s is the GL strain for the secondary creep. We now need to compute E_{KL}^s from ε^s :

$$\varepsilon_{ij}^e$$

6 Large Strains formulation for elasticity

6.1 Some definitions

The deformation gradient \mathbf{F} :

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \mathbf{I} \quad (14)$$

or, equivalently

$$F_{iJ} = u_{i,J} + \delta_{iJ} \quad (15)$$

and

$$J = \det \mathbf{F} \quad . \quad (16)$$

First and second Piola Kirchhoff

$$P_{iI} = F_{iJ} S_{JI} \quad (17)$$

$$\mathbb{C}_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} \quad (18)$$

$$\mathbb{C}_{ijkl} = J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL} \mathbb{C}_{IJKL} \quad (19)$$

6.2 Elasticity derivation

Articulating things in the initial configuration:

$$\nabla \cdot \mathbf{P} = 0 \quad \text{on } \Omega_0 \quad (20)$$

$$\mathbf{U} = \tilde{\mathbf{U}} \quad \text{on } \Gamma_G \quad (21)$$

$$\mathbf{T} = \mathbf{P} \quad \mathbf{N} = \tilde{\mathbf{T}} \quad \text{on } \Gamma_H \quad (22)$$

The weak form is

$$\mathcal{F} = (W_{i,I}, P_{iI})_{\Omega_0} - (W_i, \tilde{T}_i)_{\Gamma_H} = 0 \quad (23)$$

We want to linearize \mathcal{F} .

$$D\mathcal{F}(\varphi) \cdot \Delta U_i = \frac{d}{d\varepsilon} \left(W_{i,I}, P_{iI}(\varphi + \varepsilon \Delta U_i) \right) \Big|_{\Omega_0} \Big|_{\varepsilon \rightarrow 0} \quad (24)$$

Let's work on P and derive wrt ε .

$$P = F_{iJ} S_{JI} = \frac{\partial \varphi_i}{\partial X_J} S_{JI}(\varphi_i) \quad (25)$$

$$P(\varphi_i + \varepsilon \Delta U_i) = \frac{\partial(\varphi_i + \varepsilon \Delta U_i)}{\partial X_J} S_{JI}(\varphi_i + \varepsilon \Delta U_i) \quad (26)$$

$$(27)$$

$$\frac{\partial P(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} = \frac{\partial \Delta U_i}{\partial X_J} S_{JI} + \frac{\partial \varphi_i}{\partial X_J} \frac{\partial S_{JI}}{\partial \varepsilon} \quad (28)$$

$$= \Delta U_{i,J} S_{JI} + F_{iJ} \frac{\partial S_{JI}}{\partial E_{KL}} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \quad (29)$$

$$= \Delta U_{i,J} S_{JI} + F_{iJ} \mathbb{C}_{JIKL} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \quad (30)$$

Recall that

$$2E_{KL} = F_{lK} F_{lL} - \delta_{KL} \quad (31)$$

$$= \frac{\partial \varphi_l}{\partial X_K} \frac{\partial \varphi_l}{\partial X_L} - \delta_{KL} \quad (32)$$

and

$$2 \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = \partial_\varepsilon F_{lK} F_{lL} + \partial_\varepsilon F_{lL} F_{lK} \quad (33)$$

$$= \Delta U_{l,K} F_{lL} + \Delta U_{l,L} F_{lK} \quad (34)$$

Observing the K - L symmetry in C_{JIKL} , we can say that

$$C_{JIKL} \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = C_{JIKL} F_{lK} \Delta U_{l,L} \quad (35)$$

Subs back into the (30) to obtain

$$\partial_\varepsilon P = \Delta U_{i,J} S_{JI} + F_{iJ} F_{lK} C_{JIKL} \Delta U_{l,L} \quad (36)$$

The inner product becomes:

$$D\mathcal{F}(\varphi_l) \cdot \Delta U_l = (W_{i,I} \quad , \quad U_{i,J} S_{JI} + F_{iJ} F_{lK} C_{JIKL} \Delta U_{l,L}) \quad (37)$$

And the linearized stiffness matrix:

$$K_{il}^{\beta\gamma} = \int_{\Omega} \left[\phi_{,I}^\beta \phi_{,J}^\gamma S_{JI} \delta_{il} + \phi_{,I}^\beta \phi_{,L}^\gamma F_{iJ} F_{lK} C_{JIKL} \right] d\Omega \quad (38)$$

6.3 Pressure loading - calculations using F (initial configuration).

In the current configuration, write:

$$h_i = -p n_i \quad \text{on } \Gamma_h \quad (39)$$

$$\int_{\Gamma_h} w_i h_i d\Gamma = - \int_{\Gamma_h} p \mathbf{w} \cdot \mathbf{n} d\Gamma \quad \text{on } \Gamma_h \quad (40)$$

$$(41)$$

where p is the constant pressure (normal force) at the boundary. Note that the coordinate system is not constant, so the force direction changes with time.

Recall crammers rule:

$$\text{cof } \mathbf{F}^T = \mathbf{F}^{-1} \det(\mathbf{F}) \quad (42)$$

$$\text{cof } \mathbf{F} = \mathbf{F}^{-T} \det(\mathbf{F}) = J \mathbf{F}^{-T} \quad (43)$$

$$[\text{cof } \mathbf{F}]_{iI} = J (F^{-1})_{Ii} \quad (44)$$

We know that

$$\mathbf{n} da = J \mathbf{F}^{-T} \mathbf{N} dA = \text{cof } (\mathbf{F}) \mathbf{N} dA \quad (45)$$

$$n_i da = J [\mathbf{F}^{-1}]_{Ii} N_I dA = [\text{cof } (\mathbf{F})]_{iI} N_I dA \quad (46)$$

where

$$\text{cof } (\mathbf{F})_{iI} = (-1)^{i+I} M_{iI} \quad (47)$$

and M_{iI} is the minor of the element iI . We can now write

$$-\int_{\Gamma_h} p n_i w_i d\Gamma = -\int_{\Gamma_H} p \mathbf{W} \cdot J \mathbf{F}^{-T} \mathbf{N} d\Gamma = \quad (48)$$

$$-\int_{\Gamma_H} p \mathbf{W} \cdot [\text{cof}(\mathbf{F}) \mathbf{N}] d\Gamma \quad (49)$$

$$-\int_{\Gamma_H} p W_i (J \mathbf{F}^{-1})_{Ii} N_I d\Gamma = \quad (50)$$

Now we need to linearize the above in the initial configuration. Preliminary derivation (Jacobi formula):

$$\partial J = \partial \det \mathbf{F} = \frac{d[\det \mathbf{F}]}{dF_{jJ}} \partial F_{jJ} \quad (51)$$

$$= \det \mathbf{F} [\mathbf{F}^{-T}]_{jJ} \partial F_{jJ} \quad (52)$$

$$= J [\mathbf{F}^{-T}]_{jJ} \partial F_{jJ} \quad (53)$$

$$= J [\mathbf{F}^{-1}]_{Jj} \partial F_{jJ} \quad (54)$$

$$= \det \mathbf{F} \text{tr} \{ \mathbf{F}^{-1} \partial \mathbf{F} \} \quad (55)$$

$$= J [\mathbf{F}^{-1} \partial \mathbf{F}]_{kk} \quad (56)$$

$$= [J \mathbf{F}^{-1} \partial \mathbf{F}]_{kk} \quad (57)$$

$$= [\text{cof} \mathbf{F}^T \partial \mathbf{F}]_{kk} \quad (58)$$

$$= [\text{adj} \mathbf{F} \partial \mathbf{F}]_{kk} \quad (59)$$

The derivative of an inverse matrix:

$$\mathbf{F} \mathbf{F}^{-1} = \mathbf{I} \quad (60)$$

$$\rightarrow \partial [\mathbf{F} \mathbf{F}^{-1}] = 0 \quad (61)$$

$$\rightarrow \mathbf{F} \partial [\mathbf{F}^{-1}] + \partial \mathbf{F} \mathbf{F}^{-1} = 0 \quad (62)$$

$$\rightarrow \mathbf{F} \partial [\mathbf{F}^{-1}] = -\partial \mathbf{F} \mathbf{F}^{-1} \quad (63)$$

$$\rightarrow \partial [\mathbf{F}^{-1}] = -\mathbf{F}^{-1} \partial \mathbf{F} \mathbf{F}^{-1} \quad (64)$$

$$\rightarrow \partial [\mathbf{F}^{-1}]_{Ii} = -[\mathbf{F}^{-1}]_{Ij} [\partial \mathbf{F}]_{jJ} [\mathbf{F}^{-1}]_{Ji} \quad (65)$$

Now we can derive the derivative of the cofactors:

$$\partial [\text{cof} \mathbf{F}] = \partial [\text{cof} \mathbf{F}]_{iI} = \partial [J \mathbf{F}^{-T}] \quad (66)$$

$$= [\mathbf{F}^{-T}] \partial J + J \partial [\mathbf{F}^{-T}] \quad (67)$$

$$= [\mathbf{F}^{-T}]_{iI} \partial J + J \partial [\mathbf{F}^{-T}]_{iI} \quad (68)$$

$$= [\mathbf{F}^{-1}]_{Ii} \partial J + J \partial [\mathbf{F}^{-1}]_{Ii} \quad (69)$$

$$= [\mathbf{F}^{-1}]_{Ii} J [\mathbf{F}^{-1}]_{Jj} \partial [\mathbf{F}]_{jJ} - J [\mathbf{F}^{-1}]_{Ij} [\partial \mathbf{F}]_{jJ} [\mathbf{F}^{-1}]_{Ji} \quad (70)$$

$$= J \partial [\mathbf{F}]_{jJ} \left\{ [\mathbf{F}^{-1}]_{Ii} [\mathbf{F}^{-1}]_{Jj} - [\mathbf{F}^{-1}]_{Ij} [\mathbf{F}^{-1}]_{Ji} \right\} \quad (71)$$

and the directional derivative becomes

$$\partial [\text{cof} \mathbf{F}(\varphi_i + \varepsilon \Delta u_i)]_{iI} = \left\{ [\mathbf{F}^{-1}]_{Ii} [\mathbf{F}^{-1}]_{Jj} - [\mathbf{F}^{-1}]_{Ij} [\mathbf{F}^{-1}]_{Ji} \right\} J \Delta u_{j,J} \quad (72)$$

$$(73)$$

And finally we transport this result to the linearization of the pressure loading. Let

$$\mathcal{F}(\varphi_i) = - \int_{\Gamma_H} p \mathbf{W} \cdot \left[\text{cof} \left(\mathbf{F}(\varphi) \right) \mathbf{N} \right] d\Gamma \quad (74)$$

$$= - \int_{\Gamma_H} p W_i [\text{cof}(\mathbf{F}(\varphi))]_{iI} N_I d\Gamma \quad (75)$$

$$\mathbf{F} = \frac{d\mathbf{x}}{d\mathbf{X}} \quad (76)$$

$$F_{iI} = \frac{dx_i}{dX_I} = \frac{d\varphi_i}{dX_I} \quad (77)$$

$$F_{iI}(\varphi_i + \varepsilon \Delta u_i) = \frac{d(\varphi_i + \varepsilon \Delta u_i)}{dX_I} \quad (78)$$

$$\lim_{\varepsilon \rightarrow 0} \partial_\varepsilon F_{jJ}(\varphi_k + \varepsilon \Delta u_k) = \frac{d\Delta u_k}{dX_K} = \Delta u_{j,J} \quad (79)$$

Then

$$\lim_{\varepsilon \rightarrow 0} \partial_\varepsilon \mathcal{F}(\varphi + \varepsilon \Delta \mathbf{u}) = - \int_{\Gamma_H} p \mathbf{W} \cdot \left\{ \partial_\varepsilon \text{cof} \left(\mathbf{F}(\varphi + \varepsilon \Delta \mathbf{u}) \right) \mathbf{N} \right\} d\Gamma \quad (80)$$

$$= - \int_{\Gamma_H} p J W_i N_I \left\{ [\mathbf{F}^{-1}]_{Ii} [\mathbf{F}^{-1}]_{Jj} - [\mathbf{F}^{-1}]_{Ij} [\mathbf{F}^{-1}]_{Ji} \right\} \Delta u_{j,J} d\Gamma \quad (81)$$

$$= - \int_{\Gamma_H} p J W_i N_I \left\{ [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} \delta_{ik} \delta_{jl} - [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} \delta_{il} \delta_{jk} \right\} \Delta u_{j,J} d\Gamma \quad (82)$$

$$= - \int_{\Gamma_H} p J W_i N_I \left\{ [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} \delta_{ik} \delta_{jl} - [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} \delta_{il} \delta_{jk} \right\} \Delta u_{j,J} d\Gamma \quad (83)$$

$$= - \int_{\Gamma_H} p J W_i N_I [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \Delta u_{j,J} d\Gamma \quad (84)$$

$$= \int_{\Gamma_H} p J W_i N_I [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) \Delta u_{j,J} d\Gamma \quad (85)$$

$$(86)$$

The element matrix becomes:

$$K_{ij}^{\beta\gamma} = \int_{\Gamma_H} \phi^\beta \phi_{,J}^\gamma p J N_I [\mathbf{F}^{-1}]_{Ik} [\mathbf{F}^{-1}]_{Jl} (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) d\Gamma \quad (87)$$

6.4 Pressure loading - calculating from the isoparametric state

Recall that

$$d\Gamma = \left| \left(\frac{\partial \hat{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \hat{\mathbf{x}}}{\partial \xi_2} \right) \right| d\Gamma^\square \quad (88)$$

where Γ^\square is the surface in the reference (isoparametric) configuration. We can calculate the normal vector in the current configuration using

$$n_i = \frac{\left(\frac{\partial \hat{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \hat{\mathbf{x}}}{\partial \xi_2} \right)_i}{\left| \left(\frac{\partial \hat{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \hat{\mathbf{x}}}{\partial \xi_2} \right) \right|} \quad (89)$$

so that

$$\int_{\Gamma_h} w_i h_i d\Gamma = - \int_{\Gamma_h^\square} p \mathbf{w} \cdot \left(\frac{\partial \hat{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \hat{\mathbf{x}}}{\partial \xi_2} \right) d\Gamma^\square$$

Now we need to compute $\frac{\partial \hat{\mathbf{x}}}{\partial \xi_i}$. So

$$\begin{aligned}\frac{\partial \hat{\mathbf{x}}}{\partial \xi_i} &= \frac{\partial \phi^\gamma}{\partial \xi_i} U^\gamma = \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_i} \\ \frac{\partial \hat{\mathbf{X}}}{\partial \xi_i} &= \frac{\partial \phi^\gamma}{\partial \xi_i} X_i^\gamma\end{aligned}$$

We finally obtain

$$\begin{aligned}\mathcal{F} &= \int_{\Gamma_h} w_i h_i d\Gamma = - \int_{\Gamma_h^\square} p \mathbf{W} \cdot \left(\frac{\partial \hat{\mathbf{x}}}{\partial \xi_1} \times \frac{\partial \hat{\mathbf{x}}}{\partial \xi_2} \right) d\Gamma^\square \\ &= - \int_{\Gamma_h^\square} p \mathbf{W} \cdot \left(\frac{\partial \mathbf{U}}{\partial \xi_1} \times \frac{\partial \mathbf{U}}{\partial \xi_2} \right) d\Gamma^\square \\ &= - \int_{\Gamma_h^\square} p \mathbf{W} \cdot \left(\frac{\partial \phi^\gamma}{\partial \xi_1} \mathbf{U}^\gamma \times \frac{\partial \phi^\gamma}{\partial \xi_2} \mathbf{U}^\gamma \right) d\Gamma^\square\end{aligned}$$

Now we can linearize (see Belytschko book, pg365)

$$\begin{aligned}
D\mathcal{F} \cdot \Delta U &= \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon \mathcal{F}(\varphi + \varepsilon \Delta U) \\
&= \lim_{\varepsilon \rightarrow 0} \left[- \int_{\Gamma_h^\square} p W_i \partial_\varepsilon \left(\mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} \right)_i d\Gamma^\square \right] \\
&= \lim_{\varepsilon \rightarrow 0} \left[- \int_{\Gamma_h^\square} p W_i \left(\partial_\varepsilon \hat{\mathbf{x}}_{,\xi_1} \times \hat{\mathbf{x}}_{,\xi_2} + \hat{\mathbf{x}}_{,\xi_1} \times \partial_\varepsilon \hat{\mathbf{x}}_{,\xi_2} \right)_i d\Gamma^\square \right] \\
&= - \int_{\Gamma_h^\square} p W_i \left(\Delta U_{,\xi_1} \times \hat{\mathbf{x}}_{,\xi_2} + \hat{\mathbf{x}}_{,\xi_1} \times \Delta U_{,\xi_2} \right)_i d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_i \left(\Delta U_{,\xi_1} \times \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} + \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \Delta U_{,\xi_2} \right)_i d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_i \left(\Delta U_{j,\xi_1} \times F_{jJ} \frac{\partial \hat{X}_J}{\partial \xi_2} + F_{kK} \frac{\partial \hat{X}_K}{\partial \xi_1} \times \Delta U_{k,\xi_2} \right)_i d\Gamma^\square
\end{aligned}$$

$$\begin{aligned}
D\mathcal{F} \cdot \Delta U &= \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon \mathcal{F}(\varphi + \varepsilon \Delta U) \\
&= \lim_{\varepsilon \rightarrow 0} \left[- \int_{\Gamma_h^\square} p W_i \left(\partial_\varepsilon \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} + \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \partial_\varepsilon \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} \right) d\Gamma^\square \right] \\
&= - \int_{\Gamma_h^\square} p W_i \left(\nabla(\Delta U) \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} + \mathbf{F} \frac{\partial \hat{\mathbf{X}}}{\partial \xi_1} \times \nabla(\Delta U) \frac{\partial \hat{\mathbf{X}}}{\partial \xi_2} \right)_i d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_i \left(\Delta U_{j,I} \frac{\partial \hat{X}_I}{\partial \xi_1} F_{kJ} \frac{\partial \hat{X}_J}{\partial \xi_2} + F_{jI} \frac{\partial \hat{X}_I}{\partial \xi_1} \Delta U_{k,J} \frac{\partial \hat{X}_J}{\partial \xi_2} \right) \varepsilon_{jki} d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_k \left(\Delta U_{i,I} \frac{\partial \hat{X}_I}{\partial \xi_1} F_{jJ} \frac{\partial \hat{X}_J}{\partial \xi_2} + F_{iI} \frac{\partial \hat{X}_I}{\partial \xi_1} \Delta U_{j,J} \frac{\partial \hat{X}_J}{\partial \xi_2} \right) \varepsilon_{ijk} d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_k \left(F_{jJ} \Delta U_{i,I} + F_{iI} \Delta U_{j,J} \right) \frac{\partial \hat{X}_I}{\partial \xi_1} \frac{\partial \hat{X}_J}{\partial \xi_2} \varepsilon_{ijk} d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p W_k \Delta U_{i,I} F_{jJ} \frac{\partial \hat{X}_I}{\partial \xi_1} \frac{\partial \hat{X}_J}{\partial \xi_2} \varepsilon_{ijk} d\Gamma^\square - \int_{\Gamma_h^\square} p W_k \Delta U_{j,J} F_{iI} \frac{\partial \hat{X}_I}{\partial \xi_1} \frac{\partial \hat{X}_J}{\partial \xi_2} \varepsilon_{ijk} d\Gamma^\square \\
&= - W_k^\beta \Delta U_i^\gamma \int_{\Gamma_h^\square} p \phi^\beta \phi_{,I}^\gamma F_{jJ} \frac{\partial \hat{X}_I}{\partial \xi_1} \frac{\partial \hat{X}_J}{\partial \xi_2} \varepsilon_{ijk} d\Gamma^\square \\
&\quad - W_k^\beta \Delta U_j^\gamma \int_{\Gamma_h^\square} p \phi^\beta \phi_{,J}^\gamma F_{iI} \frac{\partial \hat{X}_I}{\partial \xi_1} \frac{\partial \hat{X}_J}{\partial \xi_2} \varepsilon_{ijk} d\Gamma^\square
\end{aligned}$$

where

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (ijk) = (123), (231), (312) - \text{even permutation} \\ -1 & \text{if } (ijk) = (321), (132), (213) - \text{odd permutation} \\ 0 & \text{otherwise} \end{cases} \quad (90)$$

7 Incremental formulation

Start from the residual weak formulation.

$$\begin{aligned}\mathcal{F} &= (W_{i,I}, P_{iI})_{\Omega_0} - (W_i, \tilde{T})_{\Gamma_H} = 0 \\ &= (W_{i,I}, F_{iJ} S_{JI})_{\Omega_0} - (W_i, \tilde{T})_{\Gamma_H} = 0 \\ &= (W_{i,I}, F_{iJ} \mathbb{C}_{JIKL} E_{KL})_{\Omega_0} - (W_i, \tilde{T})_{\Gamma_H} = 0\end{aligned}$$

We need to define the timestep of the variables. Define the states 0, 1, 2 as the initial, last solved and next to solve respectively. Then

$$\begin{aligned}E_{KL} &= {}^2_0 E_{KL} = {}^1_0 E_{KL} + {}^{1 \rightarrow 2}_0 \delta E_{KL} \\ \delta E_{KL} &= {}^{1 \rightarrow 2}_0 \delta E_{KL} = {}^2_0 E_{KL} - {}^1_0 E_{KL}\end{aligned}$$

$$\begin{aligned}S_{JI} &= {}^2_0 S_{JI} \\ &= {}^1_0 S_{JI} + {}^2_0 \delta S_{JI}\end{aligned}$$

Calculate E_{KL} from the displacements:

$$\begin{aligned}2\delta E_{KL} &= {}^2_0 \left[U_{K,L} + U_{L,K} + U_{i,K} U_{i,L} \right] - {}^1_0 \left[U_{K,L} + U_{L,K} + U_{i,K} U_{i,L} \right] \\ &= \left(\delta U_{K,L} + \delta U_{L,K} + {}^1_0 U_{i,K} \delta U_{i,L} + {}^1_0 U_{i,L} \delta U_{i,K} \right) + \left(\delta U_{i,L} \delta U_{i,K} \right)\end{aligned}$$

Back to the formulation:

$$\begin{aligned}\mathcal{F} &= (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 E_{KL}) + (W_{i,I}, F_{iJ} C_{JIKL} \delta E_{KL}) - (W_i, \tilde{T}) \\ &= (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 E_{KL}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \delta U_{K,L}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \delta U_{L,K}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 U_{i,K} \delta U_{i,L}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 U_{i,L} \delta U_{i,K}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \delta U_{i,L} \delta U_{i,K}) \\ &- (W_i, \tilde{T})_{\Gamma_H}\end{aligned}$$

Linearize:

$$\begin{aligned}D\mathcal{F} \cdot \delta U_i &= \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon \mathcal{F}(\delta U + \varepsilon \Delta U_i) \\ &= (W_{i,I}, F_{iJ} C_{JIKL} \Delta U_{K,L}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \Delta U_{L,K}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 U_{i,K} \Delta U_{i,L}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} {}^1_0 U_{i,L} \Delta U_{i,K}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \delta U_{i,L} \Delta U_{i,K}) \\ &+ (W_{i,I}, F_{iJ} C_{JIKL} \Delta U_{i,L} \delta U_{i,K})\end{aligned}$$

Trying from the deformation gradient perspective:

$$\begin{aligned}\delta E_{KL} &= ({}^2_0 \mathbf{F}^T {}^2_0 \mathbf{F} - \mathbf{I}) - ({}^1_0 \mathbf{F}^T {}^1_0 \mathbf{F} - \mathbf{I}) \\ &= {}^2_0 \mathbf{F}^T {}^2_0 \mathbf{F} - {}^1_0 \mathbf{F}^T {}^1_0 \mathbf{F} \\ &= {}^2_0 F_{iK} {}^2_0 F_{iL} - {}^1_0 F_{iK} {}^1_0 F_{iL}\end{aligned}$$

Hence:

$$\begin{aligned}
\mathcal{F} &= (W_{i,I}, \quad F_{iJ} \ C_{JIKL} \ {}^1_0 E_{KL}) + (W_{i,I}, \quad F_{iJ} \ C_{JIKL} \ \delta E_{KL}) - (W_i, \tilde{T}) \\
&= (W_{i,I}, \quad F_{iJ} \ C_{JIKL} \ {}^1_0 E_{KL}) \\
&+ (W_{i,I}, \quad F_{iJ} \ C_{JIKL} \ {}^2_0 F_{iK} \ {}^2_0 F_{iL}) \\
&- (W_{i,I}, \quad F_{iJ} \ C_{JIKL} \ {}^1_0 F_{iK} \ {}^1_0 F_{iL}) \\
&- (W_i, \tilde{T})
\end{aligned}$$

Did not work out, because the ΔU_i does not show. Need to open the terms, and will get to the same verbosity.

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