Crk3 formulation

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1 Fracture storage

$$\zeta_f = \frac{\delta a_f}{a_f} + c_f \delta p - \beta_f \delta T \ ,$$

and we need the term

$$(\tilde{\psi}, a_f \dot{\zeta}_f)_{\tilde{\Omega}} = (\tilde{\psi}, \delta \dot{a}_f) + (\tilde{\psi}, a_f c_f \delta \dot{p}) - (\tilde{\psi}, a_f \beta_f \delta \dot{T}) \ , \qquad (1)$$

where

$$a_f = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}^+ \ = (\boldsymbol{u}^+ - \boldsymbol{u}^-) \cdot \boldsymbol{n}^+ = \sum_{d=1}^{2} u_k^d \ n_k^d \ = u_k^d \ n_k^d \ .$$

where $d = 0$ for the element in the $+$ side of the fracture and $d = 1$ for the element in the $-$ side.

Opening the terms in (1) into coefficients (assuming p to be a step constant):

$$\begin{aligned}
+(\tilde{\psi}, \delta \dot{a}_f)\Delta t :& \\
\underset{0k}{\overset{\beta\gamma}{\boldsymbol{K}_{pd}}} &= \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \ \ n_k^+ \\
\underset{0}{\overset{\beta}{\boldsymbol{F}_{p}}} &= \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \ \ n_k^+ \ \ \underset{k}{\overset{\gamma}{\breve{u}_d}} \ , \\
+(\tilde{\psi}, a_f c_f \delta \dot{p})\Delta t :& \\
\underset{0k}{\overset{\beta\gamma}{\boldsymbol{K}_{pd}}} &= c_f \quad \tilde{\psi}^\beta \quad (-1)^d \quad \phi_d^\gamma \ \ n_k^+ \ \ \Delta p \ , \\
-(\tilde{\psi}, a_f \beta_f \delta \dot{T})\Delta t :& \\
\underset{0k}{\overset{\beta\gamma}{\boldsymbol{K}_{pd}}} &= \beta_f \quad \tilde{\psi}^\beta \quad (-1)^{d+1} \quad \phi_d^\gamma \ \ n_k^+ \ \ \Delta T \ .
\end{aligned}$$

where $\breve{u}$ refers to the (known) solution in the previous timestep, and $c_f = \frac{1}{K_f}$. Δp and ΔT are the delta of p and T from the previous timestep to the current.

2 Derivatives for nonlinear solution

Both pressure and displacements are variables of the system. We need to derive some non-linear term for Newton's method. Let:

$$f = \left(\tilde{\psi}, \quad a_f \; c_f \; \delta\dot{p}\right) \Delta t = \sum_{d=1}^{2} \left(\tilde{\psi}, \quad c_f \; n_k^d \; u_k^d \; \delta\dot{p}\right) \Delta t \quad .$$

where $d = 0$ for the element in the + side of the fracture and $d = 1$ for the element in the $-$ side. Then,the partial derivatives are

$$\frac{\partial f}{\partial p} = \sum_{d=1}^{2} \left(\tilde{\psi}, \quad c_f \; n_k^d \; u_k^d\right) \Delta t \quad ,$$

and

$$\frac{\partial f}{\partial u_k^d} = \left(\tilde{\psi}, \quad c_f \; n_k^d \; \delta\dot{p} \;\right) \Delta t \quad .$$

3 Discontinuous galerkin for pressure

Reference: (Ruijie Liu, 2004) - PhD dissertation

Our equation:

$$\alpha \nabla \cdot \dot{u} + S_\epsilon \, \dot{p} - \nabla \cdot (\breve{\kappa} \nabla p) = 0$$

Weak formulation:

$$(w, \alpha \nabla \cdot \dot{u}) + (w, S_\epsilon \, \dot{p}) - (w, \nabla \cdot (\breve{\kappa} \nabla p)) = 0 \qquad \forall w$$

Lets focus our attention to the last term. Integrate by parts:

$$-(w, \nabla \cdot (\breve{\kappa} \nabla p)) = (\nabla w, \breve{\kappa} \nabla p) - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_\Gamma$$

When we sum element by element, the last term in Γ results in the edge skeleton Γ_i and the outside boundary Γ_p.

The full equation becomes a summation of the following:

$$(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon \, \dot{p})_{\Omega_E} + (\nabla w, \breve{\kappa} \nabla p)_{\Omega_E} - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_i} - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_p} = 0 \qquad \forall w$$

where Ω_E is the element interiors

We observe the following identity:

$$-(w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_i} = -([\![w]\!], \breve{\kappa} \{\nabla p\})_{\Gamma_i} - (\breve{\kappa} [\![\nabla p]\!], \{w\})_{\Gamma_i} \tag{2}$$

where

$$[\![w]\!] = w^+ \boldsymbol{n}^+ + w^- \boldsymbol{n}^-$$
$$\{w\} = 0.5 \times (w^+ + w^-)$$

and similarly

$$[\![\nabla p]\!] = \nabla p^+ \cdot \boldsymbol{n}^+ + \nabla p^- \cdot \boldsymbol{n}^-$$
$$\{\nabla p\} = 0.5 \times (\nabla p^+ + \nabla p^-)$$

Define $d = 0$ for $+$ and $d = 1$ for $-$ and similarly $e = 0$ for $+$ and $e = 1$ for $-$. The above can be written as summations in d :

$$[\![w]\!] = \sum_d w^d n_k^d$$
$$\{w\} = 0.5 \times \sum_d w^d$$
$$[\![\nabla p]\!] = \sum_e p_{,k}^e n_k^e$$
$$\{\nabla p\} = 0.5 \times \sum_e p_{,k}^e$$

Replace the above in (2) suppressing the summation sign in d and e:

$$-(w, \breve{\kappa} \; \nabla p \cdot \boldsymbol{n})_{\Gamma_i} = -0.5 \times \left[\left(w^d \; n_k^d, \quad \breve{\kappa} \; p_{,k}^e \right)_{\Gamma_i} \quad + \quad \left(\breve{\kappa} \; p_{,k}^e \; n_k^e, \quad w^d \right)_{\Gamma_i} \right]$$

Finally, we add an interior penalty term:

$$\frac{\delta_p}{|s|} \left([\![p]\!], [\![w]\!] \right)_{\Gamma_i}$$

The final expression is:

$$
\begin{aligned}
&(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon\ \dot{p})_{\Omega_E} + (\nabla w, \check{\kappa} \nabla p)_{\Omega_E} - (w, \check{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_p} \\
&- (\llbracket w \rrbracket, \check{\kappa} \{\nabla p\})_{\Gamma_i} - (\check{\kappa} \llbracket \nabla p \rrbracket, \{w\})_{\Gamma_i} \\
&+ \frac{\delta_p}{|s|} (\llbracket p \rrbracket, \llbracket w \rrbracket)_{\Gamma_i} = 0 \qquad \forall w
\end{aligned}
$$

4 An identity

$$
\begin{aligned}
(w, \boldsymbol{u}\cdot\boldsymbol{n})_{\Gamma_i} &= (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}_+)_{\Gamma_i} + (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-)_{\Gamma_i} \\
&= ([\![w]\!], \{\boldsymbol{u}\}) + ([\![\boldsymbol{u}]\!], \{w\}) \\
&= 0.5\times(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^E)_{\Gamma_i} && +0.5\times(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^E)_{\Gamma_i} \\
&= 0.5\times(-1)^E(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} && +0.5\times(-1)^E(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^+)_{\Gamma_i} \\
&= 0.5\times(-1)^E(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} && +0.5\times(-1)^D(w^D, \quad \boldsymbol{u}^E\ \boldsymbol{n}^+)_{\Gamma_i} \\
&= 0.5\times\left[(-1)^E+(-1)^D\right](w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= \delta^{DE}(-1)^D(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= (-1)^D(w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= (w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^D)_{\Gamma_i}
\end{aligned}
$$

An alternative derivation:

$$
\begin{aligned}
&([\![w]\!], \{\boldsymbol{u}\}) + ([\![\boldsymbol{u}]\!], \{w\}) = \\
&= 0.5\times(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^E)_{\Gamma_i} + 0.5\times(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^E)_{\Gamma_i} \\
&= \\
&\qquad (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}^+) \quad + \quad (\boldsymbol{u}^+, w^+\boldsymbol{n}^+) \\
&\qquad {\color{red}(w^-, \boldsymbol{u}^+\cdot\boldsymbol{n}^+) \quad + \quad (\boldsymbol{u}^+, w^-\boldsymbol{n}^-)} \\
&\qquad {\color{blue}(w^+, \boldsymbol{u}^-\cdot\boldsymbol{n}^-) \quad + \quad (\boldsymbol{u}^-, w^+\boldsymbol{n}^+)} \\
&\qquad (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-) \quad + \quad (\boldsymbol{u}^-, w^-\boldsymbol{n}^-) \\
&= \\
&\qquad (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}^+) \quad + \quad (\boldsymbol{u}^+, w^+\boldsymbol{n}^+) \\
&\qquad (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-) \quad + \quad (\boldsymbol{u}^-, w^-\boldsymbol{n}^-) \\
&= \\
&\qquad (w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^D)_{\Gamma_i}
\end{aligned}
$$

5 Viscoelasticity (Creep)

We follow the workflow by Kumar et al. (2021) for the formulation of the viscoelastic model for salt. According to (Carter et al., 1993), the strain of a solid is

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^s + \boldsymbol{\varepsilon}^a \quad , \tag{3}$$

where the superscripts stand for elastic, plastic, transient (or primary or decelerating creep), steady-state (or secondary creep) and accelerating (or tertiary creep), respectively.

5.1 Multiaxial creep - development

$$\sigma_e^2 = \frac{3}{2}\tau_{ij}\tau_{ij}$$
$$\tau_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Now we want to develop the equality:

$$\dot{\varepsilon}^s_{ij} = \frac{\partial \sigma_e}{\partial \sigma_{ij}}\dot{\bar{\varepsilon}} = \frac{3}{2}\frac{\tau_{ij}}{\sigma_e}\ \dot{\bar{\varepsilon}}^s$$

Hence:

$$\begin{aligned}
2\sigma_e\ \partial\sigma_e &= 3\tau_{ij}\ \frac{\partial \tau_{ij}}{\partial \sigma_{kl}}\partial\sigma_{kl} \\
&= 3\tau_{ij}\ \left[\frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3}\frac{\partial \sigma_{mm}}{\partial \sigma_{kl}}\right]\ \partial\sigma_{kl} \\
&= 3\tau_{ij}\ \left[\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right]\ \partial\sigma_{kl} \\
&= 3\tau_{kl}\partial\sigma_{kl} - \tau_{ii}\partial\sigma_{kl}\delta_{kl} \\
&= 3\tau_{kl}\partial\sigma_{kl}
\end{aligned}$$

because $\tau_{ii} = 0$. Hence:

$$\frac{\partial \sigma_e}{\partial \sigma_{kl}} = \frac{3}{2}\frac{\tau_{kl}}{\sigma_e}$$

5.2 Carter model

The starting point is Carter's model (Carter et al., 1993). We write in the normalized way so that the constants have rational units (Dusseault & Fordham, 1993), and dimensionless parameters. The expression for the creep model is the following power-law for steady-state creep is obtained from the lab for uniaxial strain as

$$\dot{\bar{\varepsilon}}^s = \varepsilon_0\ \exp\left(-\frac{Q}{RT}\right)\ \left(\frac{\sigma_e}{\sigma_0}\right)^n \tag{4}$$

where σ_0 is a normalization stress arbitrary in some sense as it is a fitting parameter together with ε_0. It can be seen as a critical stress for transition of regimes (when the quotient approaches 1, we can continuously change the exponent n). See (Dusseault et al., 1987; Poiate, 2012) . Table 1 describes the variables.

Now we express the creep flow rule in tensor form under multiaxial stress conditions (Hyde et al., 2014; Kraus, 1980; Xu et al., 2018), as:

$$\dot{\varepsilon}^s_{ij} = \frac{\partial \sigma_e}{\partial \sigma_{ij}}\dot{\bar{\varepsilon}} = \frac{3}{2}\frac{\tau_{ij}}{\sigma_e}\ \dot{\bar{\varepsilon}}^s \tag{5}$$

where σ_e is an effective stress measure (here assumed Von Misses) and τ_{ij} is the stress deviator tensor.

$$\sigma_e^2 = \frac{3}{2}\,\tau_{ij}\tau_{ij} \tag{6}$$

$$\tau_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{7}$$

Equivalent expressions for σ_e are

$$\sigma_e^2 = 3\ J_2 \qquad J_2 = 0.5 \times \tau_{ij}\tau_{ij} \tag{8}$$

$$\sigma_e^2 = 0.5 \times \left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\ (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)\right] \tag{9}$$

$$\sigma_e^2 = \frac{3}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}(\sigma_{kk}^2) \tag{10}$$

where J_2 is the second invariant of the stress deviator tensor. We can use Carter's law to write

$$\dot{\varepsilon}_{ij}^s = \frac{3}{2}\ \varepsilon_0\ \exp\left(-\frac{Q}{RT}\right)\ \left(\frac{\sigma_e}{\sigma_0}\right)^n\ \left(\frac{\tau_{ij}}{\sigma_e}\right) \tag{11}$$

An alternative writing for this equation, as e.g. in (Kumar et al., 2021), is

$$\dot{\varepsilon}_{ij}^s = \frac{3}{2}\ A\ \exp\left(-\frac{Q}{RT}\right)\ \sigma_e^{n-1}\ \tau_{ij} \tag{12}$$

where $A = \varepsilon_0/\sigma_0^n$. However, the units of A are MPa^{-n} and A and n are mutually dependent. The use of σ_0 is convenient for decoupling and different creep modes can be calibrated using only n, for example. Alternativelly, we can use the strategy by (Dusseault et al., 1987), with a critical stress and sharp transitions between modes..

$$A = \frac{\varepsilon_0}{\sigma_0^n}\exp\left(\frac{Q}{RT}\right) \tag{13}$$

Variable	Unit	Description
R	J mol^{-1} K^{-1}	Universal gas constant ($R = 8.3144$)
T	K	Temperature
Q	J mol^{-1}	Apparent activation energy
n	–	Material constant
ε_0	–	Material constant
σ_0	MPa	Material constant
σ_e	MPa	Differential stress ($\sigma_1 - \sigma_3$), or von mises, or Tresca.

Table 1: Symbols in Carter model

5.3 Creep in small strains

Let $\boldsymbol{\varepsilon}^s$ be the steady state creep history, so that

$$\boldsymbol{\varepsilon}^s(t+\Delta t) = \boldsymbol{\varepsilon}^s(t) + \Delta\boldsymbol{\varepsilon}^s(t+\theta) \quad ,$$
$$\Delta\boldsymbol{\varepsilon}^s = \dot{\boldsymbol{\varepsilon}}(t)\ \Delta t$$

is an explicit timetepping. We can calculate $\dot{\boldsymbol{\varepsilon}}^s$ from the previous section.

The problem statement is:

Variable	Value
Q	51.6×10^{-3} J mol^{-1}
R	8.3144 J mol^{-1} K^{-1}
A	8.1×10^{-5} MPa^{-n} s^{-1}

Table 2: Values for initial testcase, to compare with results from Carter et al. (1993) and Kumar et al. (2021).

Find $\boldsymbol{u} \in \mathscr{V}$ such that $\forall \boldsymbol{w} \in \mathscr{W}$

$$
\begin{aligned}
&(w_{i,j},\ \sigma_{ij}) - (w_i,\ f_i) - (w_i,\ h_i)_{\Gamma_h} = 0 \\
&\sigma_{ij} = \mathbb{C}_{ijkl}\ \varepsilon^e_{kl} \\
&\varepsilon^e_{kl} = \varepsilon_{kl} - \varepsilon^s_{kl} \\
&\varepsilon_{kl} = 0.5 \times (u_{k,l} + u_{l,k}) \\
&u_i = g_i && \text{on } \Gamma_g \\
&\sigma_{ij}\ n_j = h_i && \text{on } \Gamma_h
\end{aligned}
$$

where Ω is the domain of the the bilinear operators whenever ommitted. u_i agregates the unknown and known (Dirichlet) degrees of freedom, as usual.

The creep term is integrated in time and added to the equilibrium equation as

$$
(w_{i,j},\ \mathbb{C}_{ijkl}\ u_{k,l}) - (w_{i,j},\ \mathbb{C}_{ijkl}\ \varepsilon^s_{kl}) - (w_i,\ f_i) - (w_i,\ h_i)_{\Gamma_h} = 0
$$

5.4 Munson Dawson - transient creep

This section is based on (Cheng, 2024; Munson, 1999; Reedlunn, 2018), that follow previous work at Sandia (Munson & Dawson, 1982; Munson et al., 1989).

Define the total plastic strain rate as

$$
\dot{\bar{\varepsilon}}_p = \dot{\bar{\varepsilon}}_{ss} + \dot{\bar{\varepsilon}}_{tr} = F\ \dot{\bar{\varepsilon}}_{ss} \tag{14}
$$

where the steadystate creep is

$$
\dot{\bar{\varepsilon}}_{ss} = \varepsilon_0\ \exp \frac{-Q}{RT}\ \bar{\sigma}^n \tag{15}
$$

where the normalized stress is

$$
\bar{\sigma} = \frac{\sigma_e}{\sigma_0} \tag{16}
$$

and σ_e is the deviatoric (or equivalent, or von mises, or Hosford) stress.

The transient creep is

$$
\dot{\bar{\varepsilon}}_{tr} = (F - 1)\ \dot{\bar{\varepsilon}}_{ss} \tag{17}
$$

where

$$
F = \begin{cases} \exp\left(\omega_w\ \zeta^2\right) & , \quad \zeta > 0\ (\varepsilon_{tr} \leqslant \varepsilon^*_{tr}) \\ \exp\left(-\omega_r\ \zeta^2\right) & , \quad \zeta < 0\ (\varepsilon_{tr} > \varepsilon^*_{tr}) \end{cases} \tag{18}
$$

where

$$\zeta = 1 - \frac{\varepsilon_{tr}}{\varepsilon_{tr}^{*}} \tag{19}$$

$$\omega_w = \alpha_w + \beta_w \ \log_{10} \bar{\sigma} \tag{20}$$

$$\omega_r = \alpha_r + \beta_r \ \log_{10} \bar{\sigma} \tag{21}$$

and

$$\varepsilon_{tr}^{*} = K \ \exp(cT) \ \bar{\sigma}^m \tag{22}$$

The above expressions are the ones given by MD. Now we want to simplify to ease our implementation and analysis.

$$\dot{\varepsilon}_{ss} = \varepsilon_0 \ \exp \tfrac{-Q}{RT} \ \bar{\sigma}^n \qquad \varepsilon_{tr}^{*} = K \ \exp(cT) \ \bar{\sigma}^m$$

$$\zeta = 1 - \tfrac{\varepsilon_{tr}}{\varepsilon_{tr}^{*}} \qquad \rightarrow \qquad \begin{cases} \zeta > 0 & \rightarrow \quad (\alpha \ , \ \beta) = + \ (\alpha_w \ , \ \beta_w) \\ \zeta \leqslant 0 & \rightarrow \quad (\alpha \ , \ \beta) = - \ (\alpha_r \ , \ \beta_r) \end{cases}$$

$$F = e^{\alpha \, \zeta^2} \ \bar{\sigma}^{\beta \, \zeta^2} \qquad \rightarrow \qquad \dot{\varepsilon}_{tr} = (F-1) \ \dot{\varepsilon}_{ss} \qquad , \qquad \dot{\varepsilon}_p = F \ \dot{\varepsilon}_{ss}$$

$$\varepsilon_{tr} = \varepsilon_{tr}^{n} + \Delta t \ \dot{\varepsilon}_{tr} \qquad \varepsilon_p = \varepsilon_p^n + \Delta t \ \dot{\varepsilon}_p$$

Note that MD formulation uses $\log_{10}$ operator for the ω parameters. I chose to remove it, because ω is a fitting parameter. The idea is to keep the formulation clearer. Note, however, that this changes the interpretation of the material fitting parameter β.

5.5 Adapting MD to our problem

For our problem and data, the transient effect is not central, and we do not have too much data to calibrate. On the other hand, the multiple modes of the steady state solution are central. The temperature dependency of the steady state creep rate can benefit of an empirical fudge factor in form of a stretched exponential.

Assumptions:

- Our data suggest the use of a stretched exponential of the form $\dot{\varepsilon}_{ss} = \exp(A^p)\sigma^n$, where p is an empirical fudge parameter used for fitting
- I need at least two mechanisms to map the steadystate creep
- Transient creep can be modeled with a single-mechanism
- Transient creep depends only on the state variable ζ so that F does not depend on σ ($\beta = 0$ in the MD formulation)
- $\alpha = 0$ when $\zeta \leqslant 0$. This means that the transient result is only relevant in small ε_{tr} (that is, $\varepsilon_{tr} \leqslant \varepsilon_{tr}^{*}$)
- The coefficients ε_0 and σ_0 are redundant. I merged them into ${}_i\sigma_0$.

The new set of equations is

$$\dot{\varepsilon}^{ss} = \sum \dot{\varepsilon}_i^{ss} \qquad \rightarrow \qquad \dot{\varepsilon}_i^{ss} = \exp\left[-\left(\tfrac{Q_i}{RT}\right)^{p_i}\right] \ \left(\tfrac{\sigma}{\sigma_i^{ss}}\right)^{n_i}$$

$$\varepsilon_{*}^{tr} = K \ \exp(cT) \ \left(\tfrac{\sigma}{\sigma^{tr}}\right)^m \qquad , \qquad \zeta = 1 - \tfrac{\varepsilon^{tr}}{\varepsilon_{*}^{tr}} \quad \begin{cases} \zeta > 0 & \rightarrow \quad \alpha = \alpha_w \\ \zeta \leqslant 0 & \rightarrow \quad \alpha = 0 \end{cases}$$

$$F = \exp\left(\alpha \, \zeta^2\right) \qquad \rightarrow \qquad \dot{\varepsilon}^{tr} = (F-1) \ \dot{\varepsilon}^{ss} \qquad , \qquad \dot{\varepsilon}^{p} = F \ \dot{\varepsilon}^{ss}$$

$$\varepsilon^{tr} = \varepsilon_n^{tr} + \Delta t \ \dot{\varepsilon}^{tr} \qquad \varepsilon^{p} = \varepsilon_n^{p} + \Delta t \ \dot{\varepsilon}^{p}$$

5.6 Creep integration with large strain elasticity

For small timesteps, we can determine the creep strain variation explicitely:

$$\boldsymbol{\varepsilon}^s(t+\Delta t) = \boldsymbol{\varepsilon}^s(t) + \Delta\boldsymbol{\varepsilon}^s(t+\theta)$$

where θ indicates the time when the increment in creep strain is measured. We start using an explicit measurement, $\theta = 0$ (stable for small Δt), so that

$$\Delta\boldsymbol{\varepsilon}^s = \dot{\boldsymbol{\varepsilon}}(t)\ \Delta t$$

Before moving to the next timestep, recalculate the stresses using the above creep strain. Check for a threshold and reduce the time increment if necessary.

Rewrite the mechanical equilibrium equation with the creep increments. We know how to solve elastic stresses, so:

$$\begin{aligned}
\mathscr{F} &= (W_{i,I},\ P^e_{iI})_{\Omega_0} - (W_i,\ \tilde{T}_i)_{\Gamma_H} = 0 \\
P^e_{iI} &= P_{iI} - P^s_{iI}
\end{aligned}$$

We need to move the creep contribution P^s_{iI} to the current configuration:

$$\begin{aligned}
P^s_{iI} &= J\ \sigma^s_{ij}\ F^{-1}_{Ij} \\
\mathscr{F} &= (W_{i,I},\ P_{iI})_{\Omega_0} - (W_{i,I},\ J\ \sigma^s_{ij}\ F^{-1}_{Ij})_{\Omega_0} - (W_i,\ \tilde{T}_i)_{\Gamma_H} = 0
\end{aligned}$$

Now we must calculate σ^s_{ij}:

$$\begin{aligned}
\sigma^s_{ij} &= \mathbb{c}_{ijkl}\ \varepsilon^s_{kl} \\
\mathbb{c}_{ijkl} &= J^{-1}\ F_{iI}F_{jJ}F_{kK}F_{lL}\ \mathbb{C}_{IJKL}
\end{aligned}$$

$$\begin{aligned}
P^e_{iI} &= F_{iJ}\ S_{JI} \\
S^e_{JI} &= \mathbb{C}_{JIKL}\ E^e_{KL} \\
E^e_{KL} &= E_{KL} - E^s_{KL}
\end{aligned}$$

where E^e_{KL} is the elastic behavior of the Green-Lagrange strain and E^s_{KL} is the GL strain for the secondary creep. We now need to compute E^s_{KL} from $\boldsymbol{\varepsilon}^s$:

$$\varepsilon^e_{ij}$$

6 Large Strains formulation for elasticity

6.1 Some definitions

The deformation gradient $\boldsymbol{F}$:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \boldsymbol{I} \tag{23}$$

or, equivalently

$$F_{iJ} = u_{i,J} + \delta_{iJ} \tag{24}$$

and

$$J = \det \boldsymbol{F} \quad . \tag{25}$$

First and second Piola Kirchhoff

$$P_{iI} = F_{iJ}\ S_{JI} \tag{26}$$

$$\mathbb{C}_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} \tag{27}$$

$$\mathbb{c}_{ijkl} = J^{-1}\ F_{iI} F_{jJ} F_{kK} F_{lL}\ \mathbb{C}_{IJKL} \tag{28}$$

6.2 Elasticity derivation

Articulating things in the initial configuration:

$$\nabla \cdot P = 0 \qquad \text{on } \Omega_0 \tag{29}$$

$$U = \tilde{U} \qquad \text{on } \Gamma_G \tag{30}$$

$$T = P\ N = \tilde{T} \qquad \text{on } \Gamma_H \tag{31}$$

The weak form is

$$\mathscr{F} = (W_{i,I},\ P_{iI})_{\Omega_0} - (W_i,\ \tilde{T}_i)_{\Gamma_H} = 0 \tag{32}$$

We want to linearize $\mathscr{F}$.

$$D\mathscr{F}(\varphi) \cdot \Delta U_i = \frac{d}{d\varepsilon} \Big(W_{i,I},\ P_{iI}(\varphi + \varepsilon \Delta U_i) \Big)_{\Omega_0} \Big|_{\varepsilon \to 0} \tag{33}$$

Let's work on P and derive wrt ε.

$$P = F_{iJ}\ S_{JI} = \frac{\partial \varphi_i}{\partial X_J}\ S_{JI}(\varphi_i) \tag{34}$$

$$P(\varphi_i + \varepsilon \Delta U_i) = \frac{\partial (\varphi_i + \varepsilon \Delta U_i)}{\partial X_J}\ S_{JI}(\varphi_i + \varepsilon \Delta U_i) \tag{35}$$

$$\tag{36}$$

$$\frac{\partial P(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} = \frac{\partial \Delta U_i}{\partial X_J}\ S_{JI} + \frac{\partial \varphi_i}{\partial X_J}\ \frac{\partial S_{JI}}{\partial \varepsilon} \tag{37}$$

$$= \Delta U_{i,J}\ S_{JI} + F_{iJ} \frac{\partial S_{JI}}{\partial E_{KL}} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \tag{38}$$

$$= \Delta U_{i,J}\ S_{JI} + F_{iJ}\ \mathbb{C}_{JIKL} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \tag{39}$$

Recall that

$$2E_{KL} = F_{lK}\ F_{lL} - \delta_{KL} \tag{40}$$

$$= \frac{\partial \varphi_l}{\partial X_K}\frac{\partial \varphi_l}{\partial X_L} - \delta_{KL} \tag{41}$$

and

$$2\lim_{\varepsilon\to 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = \partial_\varepsilon F_{lK}\ F_{lL} + \partial_\varepsilon F_{lL}\ F_{lK} \tag{42}$$

$$= \Delta U_{l,K}\ F_{lL} + \Delta U_{l,L}\ F_{lK} \tag{43}$$

Observing the K-L simmetry in C_{JIKL}, we can say that

$$C_{JIKL}\ \lim_{\varepsilon\to 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = C_{JIKL}\ F_{lK}\ \Delta U_{l,L} \tag{44}$$

Subs back into the (39) to obtain

$$\partial_\varepsilon P = \Delta U_{i,J}\ S_{JI} + F_{iJ}\ F_{lK}\ \mathbb{C}_{JIKL}\ \Delta U_{l,L} \tag{45}$$

The inner product becomes:

$$D\mathscr{F}(\varphi_l)\cdot \Delta U_l = (W_{i,I} \quad , \quad U_{i,J}\ S_{JI} + F_{iJ}\ F_{lK}\ \mathbb{C}_{JIKL}\ \Delta U_{l,L}) \tag{46}$$

And the linearized stiffness matrix:

$$K^{\beta\gamma}_{il} = \int_\Omega \left[\phi^\beta_{,I}\ \phi^\gamma_{,J}\ S_{JI}\ \delta_{il} + \phi^\beta_{,I}\ \phi^\gamma_{,L}\ F_{iJ}\ F_{lK}\mathbb{C}_{JIKL} \right] d\Omega \tag{47}$$

6.3 Pressure loading - calculations using F (initial configuration).

In the current configuration, write:

$$h_i = -p\ n_i \qquad \text{on } \Gamma_h \tag{48}$$

$$\int_{\Gamma_h} w_i\ h_i\ d\Gamma = -\int_{\Gamma_h} p\ \boldsymbol{w}\cdot\boldsymbol{n}\ d\Gamma \qquad \text{on } \Gamma_h \tag{49}$$

$$\tag{50}$$

where p is the constant pressure (normal force) at the boundary. Note that the coordinate system is not constant, so the force direction changes with time.

Recall cramers rule:

$$\text{cof}\ \boldsymbol{F}^T = \boldsymbol{F}^{-1}\det(\boldsymbol{F}) \tag{51}$$

$$\text{cof}\ \boldsymbol{F} = \boldsymbol{F}^{-T}\det(\boldsymbol{F}) = J\ \boldsymbol{F}^{-T} \tag{52}$$

$$[\text{cof}\ \boldsymbol{F}]_{iI} = J\ (F^{-1})_{Ii} \tag{53}$$

We know that

$$\boldsymbol{n}\ da = J\boldsymbol{F}^{-T}\ \boldsymbol{N}\ dA = \text{cof}\,(\boldsymbol{F})\ \boldsymbol{N}\ dA \tag{54}$$

$$n_i\ da = J[\boldsymbol{F}^{-1}]_{Ii}\ N_I\ dA = [\text{cof}\,(\boldsymbol{F})]_{iI}\ N_I\ dA \tag{55}$$

where

$$\text{cof}\,(\boldsymbol{F})_{iI} = (-1)^{i+I} M_{iI} \tag{56}$$

and M_{iI} is the minor of the element iI. We can now write

$$-\int_{\Gamma_h} p\ n_i\ w_i\ d\Gamma = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot JF^{-T}\boldsymbol{N}\ d\Gamma = \tag{57}$$

$$-\int_{\Gamma_H} p\ \boldsymbol{W} \cdot [\mathrm{cof}\,(\boldsymbol{F})\ \boldsymbol{N}]\ d\Gamma \tag{58}$$

$$-\int_{\Gamma_H} p\ W_i(J\ F^{-1})_{Ii}\ N_I\ d\Gamma = \tag{59}$$

Now we need to linearize the above in the initial configuration. Preliminary derivation (Jacobi formula):

$$\partial J = \partial \det \boldsymbol{F} = \frac{d[\det \boldsymbol{F}]}{dF_{jJ}}\ \partial F_{jJ} \tag{60}$$

$$= \det \boldsymbol{F}\ [\boldsymbol{F}^{-T}]_{jJ}\ \partial F_{jJ} \tag{61}$$

$$= J\ [\boldsymbol{F}^{-T}]_{jJ}\ \partial F_{jJ} \tag{62}$$

$$= J\ [\boldsymbol{F}^{-1}]_{Jj}\ \partial F_{jJ} \tag{63}$$

$$= \det \boldsymbol{F}\,\mathrm{tr}\left\{\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right\} \tag{64}$$

$$= J\left[\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right]_{kk} \tag{65}$$

$$= \left[J\ \boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right]_{kk} \tag{66}$$

$$= \left[\mathrm{cof}\,\boldsymbol{F}^{T}\ \partial \boldsymbol{F}\right]_{kk} \tag{67}$$

$$= [\mathrm{adj}\,\boldsymbol{F}\ \partial \boldsymbol{F}]_{kk} \tag{68}$$

The derivative of an inverse matrix:

$$\boldsymbol{F}\ \boldsymbol{F}^{-1} = \boldsymbol{I} \tag{69}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}\boldsymbol{F}^{-1}\right] = 0 \tag{70}$$

$$\rightarrow\quad \boldsymbol{F}\ \partial\left[\boldsymbol{F}^{-1}\right] + \partial \boldsymbol{F}\ \boldsymbol{F}^{-1} = 0 \tag{71}$$

$$\rightarrow\quad \boldsymbol{F}\ \partial\left[\boldsymbol{F}^{-1}\right] = -\partial \boldsymbol{F}\ \boldsymbol{F}^{-1} \tag{72}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}^{-1}\right] = -\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\ \boldsymbol{F}^{-1} \tag{73}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}^{-1}\right]_{Ii} = -\left[\boldsymbol{F}^{-1}\right]_{Ij}[\partial \boldsymbol{F}]_{jJ}\left[\boldsymbol{F}^{-1}\right]_{Ji} \tag{74}$$

Now we can derive the derivative of the cofactors:

$$\partial\,[\mathrm{cof}\,\boldsymbol{F}] = \partial\,[\mathrm{cof}\,\boldsymbol{F}]_{iI} = \partial\left[J\boldsymbol{F}^{-T}\right] \tag{75}$$

$$= \left[\boldsymbol{F}^{-T}\right]\partial J + J\ \partial\left[\boldsymbol{F}^{-T}\right] \tag{76}$$

$$= \left[\boldsymbol{F}^{-T}\right]_{iI}\partial J + J\ \partial\left[\boldsymbol{F}^{-T}\right]_{iI} \tag{77}$$

$$= \left[\boldsymbol{F}^{-1}\right]_{Ii}\partial J + J\ \partial\left[\boldsymbol{F}^{-1}\right]_{Ii} \tag{78}$$

$$= \left[\boldsymbol{F}^{-1}\right]_{Ii}\ J\ [\boldsymbol{F}^{-1}]_{Jj}\ \partial\,[\boldsymbol{F}]_{jJ} - J\left[\boldsymbol{F}^{-1}\right]_{Ij}[\partial \boldsymbol{F}]_{jJ}\left[\boldsymbol{F}^{-1}\right]_{Ji} \tag{79}$$

$$= J\partial\,[\boldsymbol{F}]_{jJ}\left\{\left[\boldsymbol{F}^{-1}\right]_{Ii}\ [\boldsymbol{F}^{-1}]_{Jj}\ -\ \left[\boldsymbol{F}^{-1}\right]_{Ij}\ \left[\boldsymbol{F}^{-1}\right]_{Ji}\right\} \tag{80}$$

and the directional derivative becomes

$$\partial\,[\mathrm{cof}\,\boldsymbol{F}(\varphi_i + \varepsilon\Delta u_i)]_{iI} = \left\{\left[\boldsymbol{F}^{-1}\right]_{Ii}\ [\boldsymbol{F}^{-1}]_{Jj}\ -\left[\boldsymbol{F}^{-1}\right]_{Ij}\left[\boldsymbol{F}^{-1}\right]_{Ji}\right\}\ J\ \Delta u_{j,J} \tag{81}$$

$$\tag{82}$$

And finally we transport ths restult to the linearization of the pressure loading. Let

$$\mathscr{F}(\varphi_i) = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot \left[\operatorname{cof}\left(\boldsymbol{F}(\varphi)\right)\ \boldsymbol{N}\right] d\Gamma \tag{83}$$

$$= -\int_{\Gamma_H} p\ W_i\ \left[\operatorname{cof}\left(\boldsymbol{F}(\varphi)\right]_{iI}\ N_I\ d\Gamma \tag{84}$$

$$\boldsymbol{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \tag{85}$$

$$F_{iI} = \frac{dx_i}{dX_I} = \frac{d\varphi_i}{dX_I} \tag{86}$$

$$F_{iI}(\varphi_i + \varepsilon\Delta u_i) = \frac{d\left(\varphi_i + \varepsilon\Delta u_i\right)}{dX_I} \tag{87}$$

$$\lim_{\varepsilon\to 0} \partial_\varepsilon F_{jJ}(\varphi_k + \varepsilon\Delta u_k) = \frac{d\Delta u_k}{dX_K} = \Delta u_{j,J} \tag{88}$$

Then

$$\lim_{\varepsilon\to 0} \partial_\varepsilon \mathscr{F}(\varphi + \varepsilon\Delta\boldsymbol{u}) = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot \left\{\partial_\varepsilon \operatorname{cof}\left(\boldsymbol{F}(\varphi + \varepsilon\Delta\boldsymbol{u})\right)\ \boldsymbol{N}\right\} d\Gamma \tag{89}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left\{\left[\boldsymbol{F}^{-1}\right]_{Ii} [\boldsymbol{F}^{-1}]_{Jj} - \left[\boldsymbol{F}^{-1}\right]_{Ij} \left[\boldsymbol{F}^{-1}\right]_{Ji}\right\} \Delta u_{j,J}\ d\Gamma \tag{90}$$

$$\tag{91}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left\{\left[\boldsymbol{F}^{-1}\right]_{Ik} [\boldsymbol{F}^{-1}]_{Jl}\delta_{ik}\delta_{jl} - \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \delta_{il}\delta_{jk}\right\} \Delta u_{j,J}\ d\Gamma \tag{92}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} [\boldsymbol{F}^{-1}]_{Jl}\ \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)\ \Delta u_{j,J}\ d\Gamma \tag{93}$$

$$= \int_{\Gamma_H} p\ J\ W_i\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} [\boldsymbol{F}^{-1}]_{Jl}\ \left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)\ \Delta u_{j,J}\ d\Gamma \tag{94}$$

$$\tag{95}$$

The element matrix becomes:

$$K_{ij}^{\beta\gamma} = \int_{\Gamma_H} \phi^\beta \phi^\gamma_{,J}\ p\ J\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} [\boldsymbol{F}^{-1}]_{Jl}\ \left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)\ d\Gamma \tag{96}$$

6.4 Pressure loading - calculating from the isoparametric state

Recall that

$$d\Gamma = \left|\left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2}\right)\right| d\Gamma^\square \tag{97}$$

where $\Gamma^\square$ is the surface in the reference (isoparametric) configuration. We can calculate the normal vector in the current configuration using

$$n_i = \frac{\left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2}\right)}{\left|\left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2}\right)\right|} \tag{98}$$

so that

$$\int_{\Gamma_h} w_i\ h_i\ d\Gamma = -\int_{\Gamma_h^\square} p\ \boldsymbol{w} \cdot \left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2}\right) d\Gamma^\square$$

Now we need to compute $\frac{\partial \hat{x}}{\partial \xi_i}$. So

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_i} = \frac{\partial \phi^\gamma}{\partial \xi_i} U^\gamma = \boldsymbol{F} \ \frac{\partial \hat{\boldsymbol{X}}}{\partial \xi_i}$$

$$\frac{\partial \hat{\boldsymbol{X}}}{\partial \xi_i} = \frac{\partial \phi^\gamma}{\partial \xi_i} \ X_i^\gamma$$

We finally obtain

$$\begin{aligned}
\mathscr{F} = \int_{\Gamma_h} w_i \ h_i \ d\Gamma &= - \int_{\Gamma_h^\square} p \ \boldsymbol{W} \cdot \left(\frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_1} \times \frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_2} \right) d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p \ \boldsymbol{W} \cdot \left(\frac{\partial \boldsymbol{U}}{\partial \xi_1} \times \frac{\partial \boldsymbol{U}}{\partial \xi_2} \right) d\Gamma^\square \\
&= - \int_{\Gamma_h^\square} p \ \boldsymbol{W} \cdot \left(\frac{\partial \phi^\gamma}{\partial \xi_1} \boldsymbol{U}^\gamma \times \frac{\partial \phi^\gamma}{\partial \xi_2} \boldsymbol{U}^\gamma \right) d\Gamma^\square
\end{aligned}$$

Now we can linearize (see Belytschko book, pg365)

$$
\begin{aligned}
D\mathscr{F}\cdot\Delta U &= \lim_{\varepsilon\to 0}\partial_\varepsilon \mathscr{F}(\varphi+\varepsilon\Delta U)\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\ \ \partial_\varepsilon\left(\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)_i d\Gamma^\square\right]\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\left(\partial_\varepsilon\hat{\boldsymbol{x}}_{,\xi_1}\times\hat{\boldsymbol{x}}_{,\xi_2}+\hat{\boldsymbol{x}}_{,\xi_1}\times\partial_\varepsilon\hat{\boldsymbol{x}}_{,\xi_2}\right)_i\right.\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta\boldsymbol{U}_{,\xi_1}\times\hat{\boldsymbol{x}}_{,\xi_2}+\hat{\boldsymbol{x}}_{,\xi_1}\times\Delta\boldsymbol{U}_{,\xi_2}\right)_i\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta\boldsymbol{U}_{,\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}+\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\Delta\boldsymbol{U}_{,\xi_2}\right)_i\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta U_{j,\xi_1}\times F_{jJ}\frac{\partial\hat{X}_J}{\partial\xi_2}+F_{kK}\frac{\partial\hat{X}_K}{\partial\xi_1}\times\Delta U_{k,\xi_2}\right)_i
\end{aligned}
$$

$$
\begin{aligned}
D\mathscr{F}\cdot\Delta U &= \lim_{\varepsilon\to 0}\partial_\varepsilon \mathscr{F}(\varphi+\varepsilon\Delta U)\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\ \left(\partial_\varepsilon\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\ +\ \boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\partial_\varepsilon\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)d\Gamma^\square\right]\\
&= -\int_{\Gamma_h^\square} p\ W_i\ \left(\nabla(\Delta\boldsymbol{U})\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\ +\ \boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\nabla(\Delta\boldsymbol{U})\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)_i d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_i\ \left(\Delta U_{j,I}\frac{\partial\hat{X}_I}{\partial\xi_1}F_{kJ}\frac{\partial\hat{X}_J}{\partial\xi_2}\ +\ F_{jI}\frac{\partial\hat{X}_I}{\partial\xi_1}\Delta U_{k,J}\frac{\partial\hat{X}_J}{\partial\xi_2}\right)\ \varepsilon_{jki}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \left(\Delta U_{i,I}\frac{\partial\hat{X}_I}{\partial\xi_1}F_{jJ}\frac{\partial\hat{X}_J}{\partial\xi_2}\ +\ F_{iI}\frac{\partial\hat{X}_I}{\partial\xi_1}\Delta U_{j,J}\frac{\partial\hat{X}_J}{\partial\xi_2}\right)\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \left(F_{jJ}\ \Delta U_{i,I}\ +\ F_{iI}\ \Delta U_{j,J}\right)\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \Delta U_{i,I}\ F_{jJ}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square-\int_{\Gamma_h^\square} p\ W_k\ \Delta U_{j,J}\ F_{iI}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\,W_k^\beta\ \Delta U_i^\gamma\int_{\Gamma_h^\square} p\ \phi^\beta\ \phi^\gamma_{,I}\ F_{jJ}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ d\Gamma^\square\\
&\quad -\ W_k^\beta\ \Delta U_j^\gamma\int_{\Gamma_h^\square} p\ \phi^\beta\ \phi^\gamma_{,J}\ F_{iI}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ d\Gamma^\square
\end{aligned}
$$

where

$$
\varepsilon_{ijk}=\begin{cases}+1 & \text{if } (ijk)=(123),(231),(312)-\text{even permutation}\\ -1 & \text{if } (ijk)=(321),(132),(213)-\text{odd permutation}\\ 0 otherwise\end{cases}\tag{99}
$$

7 Incremental formulation

Start from the residual weak formulation.

$$
\begin{aligned}
\mathscr{F} &= (W_{i,I},\ P_{iI})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0 \\
&= (W_{i,I},\ F_{iJ}\ S_{JI})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0 \\
&= (W_{i,I},\ F_{iJ}\ \mathbb{C}_{JIKL}\ E_{KL})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0
\end{aligned}
$$

We need to define the timestep of the variables. Define the states 0, 1, 2 as the initial, last solved and next to solve respectively. Then

$$
\begin{aligned}
E_{KL} &= {}^2_0E_{KL} = {}^1_0E_{KL} + {}^{1\to 2}_{\ \ \ 0}\delta E_{KL} \\
\delta E_{KL} &= {}^{1\to 2}_{\ \ \ 0}\delta E_{KL} = {}^2_0E_{KL} - {}^1_0E_{KL}
\end{aligned}
$$

$$
\begin{aligned}
S_{JI} &= {}^2_0S_{JI} \\
&= {}^1_0S_{JI} + {}^2_0\delta S_{JI}
\end{aligned}
$$

Calculate E_{KL} from the displacements:

$$
\begin{aligned}
2\delta E_{KL} &= {}^{2}_{0}\Big[U_{K,L} + U_{L,K} + U_{i,K}\ U_{i,L}\Big] - {}^{1}_{0}\Big[U_{K,L} + U_{L,K} + U_{i,K}\ U_{i,L}\Big] \\
&= \left(\delta U_{K,L} + \delta U_{L,K} + {}^1_0U_{i,K}\ \delta U_{i,L} + {}^1_0U_{i,L}\ \delta U_{i,K}\right) + \left(\delta U_{i,L}\delta U_{i,K}\right)
\end{aligned}
$$

Back to the formulation:

$$
\begin{aligned}
\mathscr{F} &= \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0E_{KL}\big) + (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta E_{KL}) - (W_i, \tilde{T}) \\
&= \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0E_{KL}\big) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{K,L}) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{L,K}) \\
&+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0U_{i,K}\ \delta U_{i,L}\big) \\
&+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0U_{i,L}\ \delta U_{i,K}\big) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{i,L}\ \delta U_{i,K}) \\
&- (W_i,\quad \tilde{T})_{\Gamma_H}
\end{aligned}
$$

Linearize:

$$
\begin{aligned}
D\mathscr{F}\cdot\delta U_i &= \lim_{\varepsilon\to 0}\partial_\varepsilon\mathscr{F}(\delta U + \varepsilon\Delta U_i) \\
&= (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{K,L}) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{L,K}) \\
&+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0U_{i,K}\ \Delta U_{i,L}\big) \\
&+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0U_{i,L}\ \Delta U_{i,K}\big) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{i,L}\ \Delta U_{i,K}) \\
&+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{i,L}\ \delta U_{i,K})
\end{aligned}
$$

Trying from the deformation gradient perspective:

$$
\begin{aligned}
\delta E_{KL} &= \left({}^2_0\boldsymbol{F}^T \; {}^2_0\boldsymbol{F} - \boldsymbol{I}\right) - \; \left({}^1_0\boldsymbol{F}^T \; {}^1_0\boldsymbol{F} - \boldsymbol{I}\right) \\
&= {}^2_0\boldsymbol{F}^T \; {}^2_0\boldsymbol{F} - \; {}^1_0\boldsymbol{F}^T \; {}^1_0\boldsymbol{F} \\
&= {}^2_0F_{iK} \; {}^2_0F_{iL} - \; {}^1_0F_{iK} \; {}^1_0F_{iL}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\mathscr{F} &= \left(W_{i,I}, \quad F_{iJ} \; C_{JIKL} \; {}^1_0E_{KL}\right) + (W_{i,I}, \quad F_{iJ} \; C_{JIKL} \; \delta E_{KL}) - (W_i, \tilde{T}) \\
&= \left(W_{i,I}, \quad F_{iJ} \; C_{JIKL} \; {}^1_0E_{KL}\right) \\
&+ \left(W_{i,I}, \quad F_{iJ} \; C_{JIKL} \; {}^2_0F_{iK} \; {}^2_0F_{iL}\right) \\
&- \left(W_{i,I}, \quad F_{iJ} \; C_{JIKL} \; {}^1_0F_{iK} \; {}^1_0F_{iL}\right) \\
&- (W_i, \tilde{T})
\end{aligned}
$$

Did not work out, because the ΔU_i does not show. Need to open the terms, and will get to the same verbosity.

8 Enriched galerkin - first try

We want to conserve

$$(w_i \ , \ \sigma_{ij,j})_\Omega \equiv \sum_E (w_i \ , \ \sigma_{ij,j})_{\Omega_E} \tag{100}$$

Integrate by parts in each element:

$$(w_i \ , \ \sigma_{ij,j})_{\Omega_E} \equiv -(w_{i,j} \ , \ \sigma_{ij})_{\Omega_E} + (w_i \ , \ \sigma_{ij} n_j)_{\Gamma_E} \tag{101}$$

We can integrate in the element domain Ω_E as usual. Let's focus on the element boundaries for conservativeness.

$$\sum (w_i \ , \ \sigma_{ij} n_j)_{\Gamma_E} \equiv (w_i \ , \ \sigma_{ij} n_j)_\Gamma + (w_i \ , \ \sigma_{ij} n_j)_{\Gamma_I} \tag{102}$$

where Γ is the outer boundary of the domain, set as boundary conditions. Let's focus on the internal skeleton Γ_I. Recall the identity:

$$(w_i \ , \ \sigma_{ij} n_j)_{\Gamma_I} \equiv \Big([\![\boldsymbol{w}]\!]_{ij} \ , \ \{\boldsymbol{\sigma}\}_{ij} \Big)_{\Gamma_I} + \Big(\{\boldsymbol{w}\}_i \ , \ [\![\boldsymbol{\sigma}]\!]_i \Big)_{\Gamma_I} \tag{103}$$

where

$$[\![\boldsymbol{w}]\!]_{ij} = w_i^+ \ n_j^+ \ + \ w_i^- \ n_j^- \qquad \text{(tensor)} \tag{104}$$

$$[\![\boldsymbol{\sigma}]\!]_i = \sigma_{ij}^+ \ n_j^+ \ + \ \sigma_{ij}^- \ n_j^- \qquad \text{(vector)} \tag{105}$$

$$\{\boldsymbol{w}\} = 0.5 \ (\ w_i^+ + \ w_i^- \ ) \qquad \text{(vector)} \tag{106}$$

$$\{\boldsymbol{\sigma}\} = 0.5 \ (\ \sigma_{ij}^+ + \ \sigma_{ij}^- \ ) \qquad \text{(tensor)} \tag{107}$$

Proof:

$$(w_i \ , \ \sigma_{ij} \ n_j)_{\Gamma_I} = \tag{108}$$

$$= \sum_e (w_i \ , \ \sigma_{ij} \ n_j)_{\Gamma_I^e} \tag{109}$$

$$= \sum_e (w_i^+ \ , \ \sigma_{ij}^+ \ n_j^+)_{\Gamma_I^e} + (w_i^- \ , \ \sigma_{ij}^- \ n_j^-)_{\Gamma_I^e} \tag{110}$$

Next we suppress the domain assignment and summation sign for clarity.

$$(w_i^+ \ , \ \sigma_{ij}^+ \ n_j^+) + (w_i^- \ , \ \sigma_{ij}^- \ n_j^-) = \tag{111}$$

$$= 0.5 \ (w_i^+ \ , \ \sigma_{ij}^+ \ n_j^+) + 0.5 \ (w_i^- \ , \ \sigma_{ij}^- \ n_j^-) \tag{112}$$

$$+ 0.5 \ (\sigma_{ij}^+ \ , \ w_i^+ \ n_j^+) + 0.5 \ (\sigma_{ij}^- \ , \ w_i^- \ n_j^-) \tag{113}$$

Next we introduce convenient terms into the equation, that are clearly zero as $\boldsymbol{n}^+ = -\boldsymbol{n}^-$:

$$0 = 0.5 \ (\sigma_{ij}^- \ , \ w_i^+ \ n_j^+) + 0.5 \ (w_i^+ \ , \ \sigma_{ij}^- \ n_j^-) \tag{114}$$

$$0 = 0.5 \ (\sigma_{ij}^+ \ , \ w_i^- \ n_j^-) + 0.5 \ (w_i^- \ , \ \sigma_{ij}^+ \ n_j^+) \tag{115}$$

By summing the above equations throughout the elements, we obtain the relation

$$(w_i \ , \ \sigma_{ij} n_j)_{\Gamma_I} \equiv \Big([\![\boldsymbol{w}]\!]_{ij} \ , \ \{\boldsymbol{\sigma}\}_{ij} \Big)_{\Gamma_I} + \Big(\{\boldsymbol{w}\}_i \ , \ [\![\boldsymbol{\sigma}]\!]_i \Big)_{\Gamma_I} \tag{116}$$

The conserved equation is:

$$(w_i \ , \ \sigma_{ij,j})_\Omega = \tag{117}$$

$$\sum_E (w_i \ , \ \sigma_{ij,j})_{\Omega_E} = \tag{118}$$

$$\sum_E -(w_{i,j} \ , \ \sigma_{ij})_{\Omega_E} + (w_i \ , \ \sigma_{ij} \ n_j)_{\Gamma_E} = \tag{119}$$

$$\sum_E -(w_{i,j} \ , \ \sigma_{ij})_{\Omega_E} + (w_i \ , \ \sigma_{ij} n_j)_\Gamma + (w_i \ , \ \sigma_{ij} n_j)_{\Gamma_I} = \tag{120}$$

$$\sum_E -(w_{i,j} \ , \ \sigma_{ij})_{\Omega_E} + (w_i \ , \ \sigma_{ij} n_j)_\Gamma + (\llbracket \boldsymbol{w} \rrbracket_{ij} \ , \ \{\boldsymbol{\sigma}\}_{ij})_{\Gamma_I} + (\{\boldsymbol{w}\}_i \ , \ \llbracket \boldsymbol{\sigma} \rrbracket_i)_{\Gamma_I} \tag{121}$$

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