Crk3 formulation

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1 Fracture storage

$$\zeta_f = \frac{\delta a_f}{a_f} + c_f \delta p - \beta_f \delta T \ ,$$

and we need the term

$$(\tilde{\psi}, a_f \dot{\zeta}_f)_{\tilde{\Omega}} = (\tilde{\psi}, \delta \dot{a}_f) + (\tilde{\psi}, a_f c_f \delta \dot{p}) - (\tilde{\psi}, a_f \beta_f \delta \dot{T}) \ , \tag{1}$$

where

$$a_f = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}^+ \ = (\boldsymbol{u}^+ - \boldsymbol{u}^-) \cdot \boldsymbol{n}^+ = \sum_{d=1}^{2} u_k^d \ n_k^d \ = u_k^d \ n_k^d \ .$$

where $d = 0$ for the element in the $+$ side of the fracture and $d = 1$ for the element in the $-$ side.

Opening the terms in (1) into coefficients (assuming p to be a step constant):

$$\begin{aligned}
+(\tilde{\psi}, \delta \dot{a}_f)\Delta t : & \\
\boldsymbol{K}^{\beta\gamma}_{\substack{pd \\ 0k}} &= \tilde{\psi}^\beta \quad (-1)^d \quad \phi^\gamma_d \ \ n^+_k \\
\boldsymbol{F}^{\beta}_{\substack{p \\ 0}} &= \tilde{\psi}^\beta \quad (-1)^d \quad \phi^\gamma_d \ \ n^+_k \ \ \check{u}^{\gamma}_{\substack{d \\ k}} \ , \\
+(\tilde{\psi}, a_f c_f \delta \dot{p})\Delta t : & \\
\boldsymbol{K}^{\beta\gamma}_{\substack{pd \\ 0k}} &= c_f \quad \tilde{\psi}^\beta \quad (-1)^d \quad \phi^\gamma_d \ \ n^+_k \ \ \Delta p \ , \\
-(\tilde{\psi}, a_f \beta_f \delta \dot{T})\Delta t : & \\
\boldsymbol{K}^{\beta\gamma}_{\substack{pd \\ 0k}} &= \beta_f \quad \tilde{\psi}^\beta \quad (-1)^{d+1} \quad \phi^\gamma_d \ \ n^+_k \ \ \Delta T \ .
\end{aligned}$$

where $\check{u}$ refers to the (known) solution in the previous timestep, and $c_f = \frac{1}{K_f}$. Δp and ΔT are the delta of p and T from the previous timestep to the current.

2 Derivatives for nonlinear solution

Both pressure and displacements are variables of the system. We need to derive some non-linear term for Newton's method. Let:

$$f = \left(\tilde{\psi}, \quad a_f \; c_f \; \delta\dot{p}\right) \Delta t = \sum_{d=1}^{2} \left(\tilde{\psi}, \quad c_f \; n_k^d \; u_k^d \; \delta\dot{p}\right) \Delta t \quad .$$

where $d = 0$ for the element in the + side of the fracture and $d = 1$ for the element in the $-$ side. Then,the partial derivatives are

$$\frac{\partial f}{\partial p} = \sum_{d=1}^{2} \left(\tilde{\psi}, \quad c_f \; n_k^d \; u_k^d\right) \Delta t \quad ,$$

and

$$\frac{\partial f}{\partial u_k^d} = \left(\tilde{\psi}, \quad c_f \; n_k^d \; \delta\dot{p} \; \right) \Delta t \quad .$$

3 Discontinuous galerkin for pressure

Reference: (Ruijie Liu, 2004) - PhD dissertation

Our equation:

$$\alpha \nabla \cdot \dot{u} + S_\epsilon \, \dot{p} - \nabla \cdot (\breve{\kappa} \nabla p) = 0$$

Weak formulation:

$$(w, \alpha \nabla \cdot \dot{u}) + (w, S_\epsilon \, \dot{p}) - (w, \nabla \cdot (\breve{\kappa} \nabla p)) = 0 \qquad \forall w$$

Lets focus our attention to the last term. Integrate by parts:

$$-(w, \nabla \cdot (\breve{\kappa} \nabla p)) = (\nabla w, \breve{\kappa} \nabla p) - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_\Gamma$$

When we sum element by element, the last term in Γ results in the edge skeleton Γ_i and the outside boundary Γ_p.

The full equation becomes a summation of the following:

$$(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon \, \dot{p})_{\Omega_E} + (\nabla w, \breve{\kappa} \nabla p)_{\Omega_E} - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_i} - (w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_p} = 0 \qquad \forall w$$

where Ω_E is the element interiors

We observe the following identity:

$$-(w, \breve{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_i} = -([\![w]\!], \breve{\kappa}\{\nabla p\})_{\Gamma_i} - (\breve{\kappa}[\![\nabla p]\!], \{w\})_{\Gamma_i} \tag{2}$$

where

$$[\![w]\!] = w^+ \boldsymbol{n}^+ + w^- \boldsymbol{n}^-$$
$$\{w\} = 0.5 \times (w^+ + w^-)$$

and similarly

$$[\![\nabla p]\!] = \nabla p^+ \cdot \boldsymbol{n}^+ + \nabla p^- \cdot \boldsymbol{n}^-$$
$$\{\nabla p\} = 0.5 \times (\nabla p^+ + \nabla p^-)$$

Define $d = 0$ for $+$ and $d = 1$ for $-$ and similarly $e = 0$ for $+$ and $e = 1$ for $-$. The above can be written as summations in d :

$$[\![w]\!] = \sum_d w^d n_k^d$$
$$\{w\} = 0.5 \times \sum_d w^d$$
$$[\![\nabla p]\!] = \sum_e p_{,k}^e n_k^e$$
$$\{\nabla p\} = 0.5 \times \sum_e p_{,k}^e$$

Replace the above in (2) suppressing the summation sign in d and e:

$$-(w, \breve{\kappa} \; \nabla p \cdot \boldsymbol{n})_{\Gamma_i} = -0.5 \times \left[\left(w^d \; n_k^d, \quad \breve{\kappa} \; p_{,k}^e \right)_{\Gamma_i} \quad + \quad \left(\breve{\kappa} \; p_{,k}^e \; n_k^e, \quad w^d \right)_{\Gamma_i} \right]$$

Finally, we add an interior penalty term:

$$\frac{\delta_p}{|s|} \left([\![p]\!], [\![w]\!] \right)_{\Gamma_i}$$

The final expression is:

$$
\begin{aligned}
&(w, \alpha \nabla \cdot \dot{u})_{\Omega_E} + (w, S_\epsilon \, \dot{p})_{\Omega_E} + (\nabla w, \check{\kappa} \nabla p)_{\Omega_E} - (w, \check{\kappa} \nabla p \cdot \boldsymbol{n})_{\Gamma_p} \\
&- ([\![w]\!], \check{\kappa} \{\nabla p\})_{\Gamma_i} - (\check{\kappa} [\![\nabla p]\!], \{w\})_{\Gamma_i} \\
&+ \frac{\delta_p}{|s|} \, ([\![p]\!], [\![w]\!])_{\Gamma_i} = 0 \qquad \forall w
\end{aligned}
$$

4 An identity

$$
\begin{aligned}
(w, \boldsymbol{u}\cdot\boldsymbol{n})_{\Gamma_i} &= (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}_+)_{\Gamma_i} + (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-)_{\Gamma_i} \\
&= ([\![w]\!], \{\boldsymbol{u}\}) + ([\![\boldsymbol{u}]\!], \{w\}) \\
&= 0.5\times(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^E)_{\Gamma_i} && +0.5\times(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^E)_{\Gamma_i} \\
&= 0.5\times(-1)^E(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} && +0.5\times(-1)^E(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^+)_{\Gamma_i} \\
&= 0.5\times(-1)^E(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} && +0.5\times(-1)^D(w^D, \quad \boldsymbol{u}^E\ \boldsymbol{n}^+)_{\Gamma_i} \\
&= 0.5\times\left[(-1)^E+(-1)^D\right](w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= \delta^{DE}(-1)^D(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= (-1)^D(w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^+)_{\Gamma_i} \\
&= (w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^D)_{\Gamma_i}
\end{aligned}
$$

An alternative derivation:

$$
\begin{aligned}
&([\![w]\!], \{\boldsymbol{u}\}) + ([\![\boldsymbol{u}]\!], \{w\}) = \\
&= 0.5\times(w^D, \quad \boldsymbol{u}^E\cdot\boldsymbol{n}^E)_{\Gamma_i} + 0.5\times(\boldsymbol{u}^D, \quad w^E\ \boldsymbol{n}^E)_{\Gamma_i} \\
&= \\
&\qquad (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}^+) \quad + \quad (\boldsymbol{u}^+, w^+\boldsymbol{n}^+) \\
&\qquad {\color{red}(w^-, \boldsymbol{u}^+\cdot\boldsymbol{n}^+)} \quad + \quad {\color{red}(\boldsymbol{u}^+, w^-\boldsymbol{n}^-)} \\
&\qquad {\color{blue}(w^+, \boldsymbol{u}^-\cdot\boldsymbol{n}^-)} \quad + \quad {\color{blue}(\boldsymbol{u}^-, w^+\boldsymbol{n}^+)} \\
&\qquad (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-) \quad + \quad (\boldsymbol{u}^-, w^-\boldsymbol{n}^-) \\
&= \\
&\qquad (w^+, \boldsymbol{u}^+\cdot\boldsymbol{n}^+) \quad + \quad (\boldsymbol{u}^+, w^+\boldsymbol{n}^+) \\
&\qquad (w^-, \boldsymbol{u}^-\cdot\boldsymbol{n}^-) \quad + \quad (\boldsymbol{u}^-, w^-\boldsymbol{n}^-) \\
&= \\
&\qquad (w^D, \quad \boldsymbol{u}^D\cdot\boldsymbol{n}^D)_{\Gamma_i}
\end{aligned}
$$

5 Viscoelasticity (Creep)

We follow the workflow by Kumar et al. (2021) for the formulation of the viscoelastic model for salt. According to (Carter et al., 1993), the strain of a solid is

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^t + \boldsymbol{\varepsilon}^s + \boldsymbol{\varepsilon}^a \quad , \tag{3}$$

where the superscripts stand for elastic, plastic, transient (or primary or decelerating creep), steady-state (or secondary creep) and accelerating (or tertiary creep), respectively.

5.1 Multiaxial creep - development

$$\begin{aligned}\sigma_e^2 &= \frac{3}{2}\tau_{ij}\tau_{ij} \\ \tau_{ij} &= \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\end{aligned}$$

Now we want to develop the equality:

$$\dot{\varepsilon}_{ij}^s = \frac{\partial \sigma_e}{\partial \sigma_{ij}}\dot{\bar{\varepsilon}} = \frac{3}{2}\frac{\tau_{ij}}{\sigma_e}\ \dot{\bar{\varepsilon}}^s$$

Hence:

$$\begin{aligned}2\sigma_e\ \partial\sigma_e &= 3\tau_{ij}\ \frac{\partial \tau_{ij}}{\partial \sigma_{kl}}\partial\sigma_{kl} \\ &= 3\tau_{ij}\ \left[\frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3}\frac{\partial \sigma_{mm}}{\partial \sigma_{kl}}\right]\ \partial\sigma_{kl} \\ &= 3\tau_{ij}\ \left[\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right]\ \partial\sigma_{kl} \\ &= 3\tau_{kl}\partial\sigma_{kl} - \tau_{ii}\partial\sigma_{kl}\delta_{kl} \\ &= 3\tau_{kl}\partial\sigma_{kl}\end{aligned}$$

because $\tau_{ii} = 0$. Hence:

$$\frac{\partial \sigma_e}{\partial \sigma_{kl}} = \frac{3}{2}\frac{\tau_{kl}}{\sigma_e}$$

5.2 Carter model

The starting point is Carter's model (Carter et al., 1993). We write in the normalized way so that the constants have rational units (Dusseault & Fordham, 1993), and dimensionless parameters. The expression for the creep model is the following power-law for steady-state creep is obtained from the lab for uniaxial strain as

$$\dot{\bar{\varepsilon}}^s = \varepsilon_0\ \exp\left(-\frac{Q}{RT}\right)\ \left(\frac{\sigma_e}{\sigma_0}\right)^n \tag{4}$$

where σ_0 is a normalization stress arbitrary in some sense as it is a fitting parameter together with ε_0. It can be seen as a critical stress for transition of regimes (when the quotient approaches 1, we can continuously change the exponent n). See (Dusseault et al., 1987; Poiate, 2012) . Table 1 describes the variables.

Now we express the creep flow rule in tensor form under multiaxial stress conditions (Hyde et al., 2014; Kraus, 1980; Xu et al., 2018), as:

$$\dot{\varepsilon}_{ij}^s = \frac{\partial \sigma_e}{\partial \sigma_{ij}}\dot{\bar{\varepsilon}} = \frac{3}{2}\frac{\tau_{ij}}{\sigma_e}\ \dot{\bar{\varepsilon}}^s \tag{5}$$

where σ_e is an effective stress measure (here assumed Von Misses) and τ_{ij} is the stress deviator tensor.

$$\sigma_e^2 = \frac{3}{2}\,\tau_{ij}\tau_{ij} \tag{6}$$

$$\tau_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{7}$$

Equivalent expressions for σ_e are

$$\sigma_e^2 = 3\ J_2 \qquad J_2 = 0.5 \times \tau_{ij}\tau_{ij} \tag{8}$$

$$\sigma_e^2 = 0.5 \times \left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6\ (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)\right] \tag{9}$$

$$\sigma_e^2 = \frac{3}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}(\sigma_{kk}^2) \tag{10}$$

where J_2 is the second invariant of the stress deviator tensor. We can use Carter's law to write

$$\dot{\varepsilon}_{ij}^s = \frac{3}{2}\ \varepsilon_0\ \exp\left(-\frac{Q}{RT}\right)\ \left(\frac{\sigma_e}{\sigma_0}\right)^n\ \left(\frac{\tau_{ij}}{\sigma_e}\right) \tag{11}$$

An alternative writing for this equation, as e.g. in (Kumar et al., 2021), is

$$\dot{\varepsilon}_{ij}^s = \frac{3}{2}\ A\ \exp\left(-\frac{Q}{RT}\right)\ \sigma_e^{n-1}\ \tau_{ij} \tag{12}$$

where $A = \varepsilon_0/\sigma_0^n$. However, the units of A are MPa^{-n} and A and n are mutually dependent. The use of σ_0 is convenient for decoupling and different creep modes can be calibrated using only n, for example. Alternativelly, we can use the strategy by (Dusseault et al., 1987), with a critical stress and sharp transitions between modes..

$$A = \frac{\varepsilon_0}{\sigma_0^n}\exp\left(\frac{Q}{RT}\right) \tag{13}$$

Variable	Unit	Description
R	J mol^{-1} K^{-1}	Universal gas constant ($R = 8.3144$)
T	K	Temperature
Q	J mol^{-1}	Apparent activation energy
n	$-$	Material constant
ε_0	$-$	Material constant
σ_0	MPa	Material constant
σ_e	MPa	Differential stress ($\sigma_1 - \sigma_3$), or von mises, or Tresca.

Table 1: Symbols in Carter model

5.3 Creep in small strains

Let $\boldsymbol{\varepsilon}^s$ be the steady state creep history, so that

$$\boldsymbol{\varepsilon}^s(t+\Delta t) = \boldsymbol{\varepsilon}^s(t) + \Delta\boldsymbol{\varepsilon}^s(t+\theta) \quad ,$$
$$\Delta\boldsymbol{\varepsilon}^s = \dot{\boldsymbol{\varepsilon}}(t)\ \Delta t$$

is an explicit timetepping. We can calculate $\dot{\boldsymbol{\varepsilon}}^s$ from the previous section.

The problem statement is:

Variable	Value
Q	51.6×10^{-3} J mol^{-1}
R	8.3144 J mol^{-1} K^{-1}
A	8.1×10^{-5} MPa^{-n} s^{-1}

Table 2: Values for initial testcase, to compare with results from Carter et al. (1993) and Kumar et al. (2021).

Find $\boldsymbol{u} \in \mathscr{V}$ such that $\forall \boldsymbol{w} \in \mathscr{W}$

$$
\begin{aligned}
&(w_{i,j},\ \sigma_{ij}) - (w_i,\ f_i) - (w_i,\ h_i)_{\Gamma_h} = 0 \\
&\sigma_{ij} = \mathbb{C}_{ijkl}\ \varepsilon^e_{kl} \\
&\varepsilon^e_{kl} = \varepsilon_{kl} - \varepsilon^s_{kl} \\
&\varepsilon_{kl} = 0.5 \times (u_{k,l} + u_{l,k}) \\
&u_i = g_i && \text{on } \Gamma_g \\
&\sigma_{ij}\ n_j = h_i && \text{on } \Gamma_h
\end{aligned}
$$

where Ω is the domain of the the bilinear operators whenever ommitted. u_i agregates the unknown and known (Dirichlet) degrees of freedom, as usual.

The creep term is integrated in time and added to the equilibrium equation as

$$(w_{i,j},\ \mathbb{C}_{ijkl}\ u_{k,l}) - (w_{i,j},\ \mathbb{C}_{ijkl}\ \varepsilon^s_{kl}) - (w_i,\ f_i) - (w_i,\ h_i)_{\Gamma_h} = 0$$

5.4 Munson Dawson - transient creep

This section is based on (Cheng, 2024; Munson, 1999; Reedlunn, 2018), that follow previous work at Sandia (Munson & Dawson, 1982; Munson et al., 1989).

Define the total plastic strain rate as

$$\dot{\bar{\varepsilon}}_p = \dot{\bar{\varepsilon}}_{ss} + \dot{\bar{\varepsilon}}_{tr} = F\ \dot{\bar{\varepsilon}}_{ss} \tag{14}$$

where the steadystate creep is

$$\dot{\bar{\varepsilon}}_{ss} = \varepsilon_0\ \exp\frac{-Q}{RT}\ \bar{\sigma}^n \tag{15}$$

where the normalized stress is

$$\bar{\sigma} = \frac{\sigma_e}{\sigma_0} \tag{16}$$

and σ_e is the deviatoric (or equivalent, or von mises, or Hosford) stress.

The transient creep is

$$\dot{\bar{\varepsilon}}_{tr} = (1 - F)\ \dot{\bar{\varepsilon}}_{ss} \tag{17}$$

where

$$F = \begin{cases} \exp\left(\omega_w\ \zeta^2\right) & ,\quad \zeta > 0\ (\varepsilon_{tr} \leqslant \varepsilon^*_{tr}) \\ \exp\left(-\omega_r\ \zeta^2\right) & ,\quad \zeta < 0\ (\varepsilon_{tr} > \varepsilon^*_{tr}) \end{cases} \tag{18}$$

where

$$\zeta = 1 - \frac{\varepsilon_{tr}}{\varepsilon^*_{tr}} \tag{19}$$

$$\omega_w = \alpha_w + \beta_w \ \log \bar{\sigma} \tag{20}$$

$$\omega_r = \alpha_r + \beta_r \ \log \bar{\sigma} \tag{21}$$

and

$$\varepsilon^*_{tr} = \kappa \ \exp(cT) \ \bar{\sigma}^m \tag{22}$$

The above expressions are the ones given by MD. Now we want to simplify to ease our implementation and analysis.

$$\begin{cases} \alpha = \alpha_w & , \quad \beta = \beta_w \quad \text{if } \zeta > 0 \\ \alpha = \alpha_r & , \quad \beta = \beta_r \quad \text{if } \zeta \leqslant 0 \end{cases} \quad , \quad a = \text{sign}(\zeta) \qquad \Longrightarrow \qquad F = e^{a\,\alpha\,\zeta^2} \ \bar{\sigma}^{a\,\beta\,\zeta^2} \tag{23}$$

5.5 Creep integration with large strain elasticity

For small timesteps, we can determine the creep strain variation explicitely:

$$\boldsymbol{\varepsilon}^s(t + \Delta t) = \boldsymbol{\varepsilon}^s(t) + \Delta\boldsymbol{\varepsilon}^s(t + \theta)$$

where θ indicates the time when the increment in creep strain is measured. We start using an explicit measurement, $\theta = 0$ (stable for small Δt), so that

$$\Delta\boldsymbol{\varepsilon}^s = \dot{\boldsymbol{\varepsilon}}(t) \ \Delta t$$

Before moving to the next timestep, recalculate the stresses using the above creep strain. Check for a threshold and reduce the time increment if necessary.

Rewrite the mechanical equilibrium equation with the creep increments. We know how to solve elastic stresses, so:

$$\begin{aligned} \mathscr{F} &= (W_{i,I}, \ P^e_{iI})_{\Omega_0} - (W_i, \ \tilde{T}_i)_{\Gamma_H} = 0 \\ P^e_{iI} &= P_{iI} - P^s_{iI} \end{aligned}$$

We need to move the creep contribution P^s_{iI} to the current configuration:

$$\begin{aligned} P^s_{iI} &= J \ \sigma^s_{ij} \ F^{-1}_{Ij} \\ \mathscr{F} &= (W_{i,I}, \ P_{iI})_{\Omega_0} - (W_{i,I}, \ J \ \sigma^s_{ij} \ F^{-1}_{Ij})_{\Omega_0} - (W_i, \ \tilde{T}_i)_{\Gamma_H} = 0 \end{aligned}$$

Now we must calculate σ^s_{ij}:

$$\begin{aligned} \sigma^s_{ij} &= \mathbb{c}_{ijkl} \ \varepsilon^s_{kl} \\ \mathbb{c}_{ijkl} &= J^{-1} \ F_{iI} F_{jJ} F_{kK} F_{lL} \ \mathbb{C}_{IJKL} \end{aligned}$$

$$\begin{aligned} P^e_{iI} &= F_{iJ} \ S_{JI} \\ S^e_{JI} &= \mathbb{C}_{JIKL} \ E^e_{KL} \\ E^e_{KL} &= E_{KL} - E^s_{KL} \end{aligned}$$

where E^e_{KL} is the elastic behavior of the Green-Lagrange strain and E^s_{KL} is the GL strain for the secondary creep. We now need to compute E^s_{KL} from $\boldsymbol{\varepsilon}^s$:

$$\varepsilon^e_{ij}$$

6 Large Strains formulation for elasticity

6.1 Some definitions

The deformation gradient $\boldsymbol{F}$:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} + \boldsymbol{I} \tag{24}$$

or, equivalently

$$F_{iJ} = u_{i,J} + \delta_{iJ} \tag{25}$$

and

$$J = \det \boldsymbol{F} \quad . \tag{26}$$

First and second Piola Kirchhoff

$$P_{iI} = F_{iJ}\ S_{JI} \tag{27}$$

$$\mathbb{C}_{IJKL} = \frac{\partial S_{IJ}}{\partial E_{KL}} \tag{28}$$

$$\mathbb{c}_{ijkl} = J^{-1}\ F_{iI} F_{jJ} F_{kK} F_{lL}\ \mathbb{C}_{IJKL} \tag{29}$$

6.2 Elasticity derivation

Articulating things in the initial configuration:

$$\nabla \cdot P = 0 \qquad \text{on } \Omega_0 \tag{30}$$

$$U = \tilde{U} \qquad \text{on } \Gamma_G \tag{31}$$

$$T = P\ N = \tilde{T} \qquad \text{on } \Gamma_H \tag{32}$$

The weak form is

$$\mathscr{F} = (W_{i,I},\ P_{iI})_{\Omega_0} - (W_i,\ \tilde{T}_i)_{\Gamma_H} = 0 \tag{33}$$

We want to linearize $\mathscr{F}$.

$$D\mathscr{F}(\varphi) \cdot \Delta U_i = \frac{d}{d\varepsilon}\Big(W_{i,I},\ P_{iI}(\varphi + \varepsilon \Delta U_i)\Big)_{\Omega_0}\Big|_{\varepsilon \to 0} \tag{34}$$

Let's work on P and derive wrt ε.

$$P = F_{iJ}\ S_{JI} = \frac{\partial \varphi_i}{\partial X_J}\ S_{JI}(\varphi_i) \tag{35}$$

$$P(\varphi_i + \varepsilon \Delta U_i) = \frac{\partial(\varphi_i + \varepsilon \Delta U_i)}{\partial X_J}\ S_{JI}(\varphi_i + \varepsilon \Delta U_i) \tag{36}$$

$$\tag{37}$$

$$\frac{\partial P(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} = \frac{\partial \Delta U_i}{\partial X_J}\ S_{JI} + \frac{\partial \varphi_i}{\partial X_J}\ \frac{\partial S_{JI}}{\partial \varepsilon} \tag{38}$$

$$= \Delta U_{i,J}\ S_{JI} + F_{iJ} \frac{\partial S_{JI}}{\partial E_{KL}} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \tag{39}$$

$$= \Delta U_{i,J}\ S_{JI} + F_{iJ}\ \mathbb{C}_{JIKL} \frac{\partial E_{KL}(\varphi_i + \varepsilon \Delta U_i)}{\partial \varepsilon} \tag{40}$$

Recall that

$$2E_{KL} = F_{lK}\ F_{lL} - \delta_{KL} \tag{41}$$
$$= \frac{\partial \varphi_l}{\partial X_K}\frac{\partial \varphi_l}{\partial X_L} - \delta_{KL} \tag{42}$$

and

$$2\lim_{\varepsilon\to 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = \partial_\varepsilon F_{lK}\ F_{lL} + \partial_\varepsilon F_{lL}\ F_{lK} \tag{43}$$
$$= \Delta U_{l,K}\ F_{lL} + \Delta U_{l,L}\ F_{lK} \tag{44}$$

Observing the K-L simmetry in C_{JIKL}, we can say that

$$C_{JIKL}\ \lim_{\varepsilon\to 0} \partial_\varepsilon E_{KL}(\varphi_l + \varepsilon \Delta U_l) = C_{JIKL}\ F_{lK}\ \Delta U_{l,L} \tag{45}$$

Subs back into the (40) to obtain

$$\partial_\varepsilon P = \Delta U_{i,J}\ S_{JI} + F_{iJ}\ F_{lK}\ \mathbb{C}_{JIKL}\ \Delta U_{l,L} \tag{46}$$

The inner product becomes:

$$D\mathscr{F}(\varphi_l)\cdot \Delta U_l = (W_{i,I} \quad , \quad U_{i,J}\ S_{JI} + F_{iJ}\ F_{lK}\ \mathbb{C}_{JIKL}\ \Delta U_{l,L}) \tag{47}$$

And the linearized stiffness matrix:

$$K^{\beta\gamma}_{il} = \int_\Omega \left[\phi^\beta_{,I}\ \phi^\gamma_{,J}\ S_{JI}\ \delta_{il} + \phi^\beta_{,I}\ \phi^\gamma_{,L}\ F_{iJ}\ F_{lK}\mathbb{C}_{JIKL} \right] d\Omega \tag{48}$$

6.3 Pressure loading - calculations using F (initial configuration).

In the current configuration, write:

$$h_i = -p\ n_i \qquad \text{on } \Gamma_h \tag{49}$$
$$\int_{\Gamma_h} w_i\ h_i\ d\Gamma = -\int_{\Gamma_h} p\ \boldsymbol{w}\cdot\boldsymbol{n}\ d\Gamma \qquad \text{on } \Gamma_h \tag{50}$$
$$\tag{51}$$

where p is the constant pressure (normal force) at the boundary. Note that the coordinate system is not constant, so the force direction changes with time.

Recall cramers rule:

$$\text{cof } \boldsymbol{F}^T = \boldsymbol{F}^{-1}\det(\boldsymbol{F}) \tag{52}$$
$$\text{cof } \boldsymbol{F} = \boldsymbol{F}^{-T}\det(\boldsymbol{F}) = J\ \boldsymbol{F}^{-T} \tag{53}$$
$$[\text{cof } \boldsymbol{F}]_{iI} = J\ (F^{-1})_{Ii} \tag{54}$$

We know that

$$\boldsymbol{n}\ da = J\boldsymbol{F}^{-T}\ \boldsymbol{N}\ dA = \text{cof }(\boldsymbol{F})\ \boldsymbol{N}\ dA \tag{55}$$
$$n_i\ da = J[\boldsymbol{F}^{-1}]_{Ii}\ N_I\ dA = [\text{cof }(\boldsymbol{F})]_{iI}\ N_I\ dA \tag{56}$$

where

$$\text{cof }(\boldsymbol{F})_{iI} = (-1)^{i+I} M_{iI} \tag{57}$$

and M_{iI} is the minor of the element iI. We can now write

$$-\int_{\Gamma_h} p\ n_i\ w_i\ d\Gamma = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot JF^{-T}\boldsymbol{N}\ d\Gamma = \tag{58}$$

$$-\int_{\Gamma_H} p\ \boldsymbol{W} \cdot [\mathrm{cof}\,(\boldsymbol{F})\ \boldsymbol{N}]\ d\Gamma \tag{59}$$

$$-\int_{\Gamma_H} p\ W_i(J\ F^{-1})_{Ii}\ N_I\ d\Gamma = \tag{60}$$

Now we need to linearize the above in the initial configuration. Preliminary derivation (Jacobi formula):

$$\partial J = \partial \det \boldsymbol{F} = \frac{d[\det \boldsymbol{F}]}{dF_{jJ}}\ \partial F_{jJ} \tag{61}$$

$$= \det \boldsymbol{F}\ [\boldsymbol{F}^{-T}]_{jJ}\ \partial F_{jJ} \tag{62}$$

$$= J\ [\boldsymbol{F}^{-T}]_{jJ}\ \partial F_{jJ} \tag{63}$$

$$= J\ [\boldsymbol{F}^{-1}]_{Jj}\ \partial F_{jJ} \tag{64}$$

$$= \det \boldsymbol{F}\,\mathrm{tr}\left\{\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right\} \tag{65}$$

$$= J\left[\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right]_{kk} \tag{66}$$

$$= \left[J\ \boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\right]_{kk} \tag{67}$$

$$= \left[\mathrm{cof}\ \boldsymbol{F}^T\ \partial \boldsymbol{F}\right]_{kk} \tag{68}$$

$$= [\mathrm{adj}\ \boldsymbol{F}\ \partial \boldsymbol{F}]_{kk} \tag{69}$$

The derivative of an inverse matrix:

$$\boldsymbol{F}\ \boldsymbol{F}^{-1} = \boldsymbol{I} \tag{70}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}\boldsymbol{F}^{-1}\right] = 0 \tag{71}$$

$$\rightarrow\quad \boldsymbol{F}\ \partial\left[\boldsymbol{F}^{-1}\right] + \partial \boldsymbol{F}\ \boldsymbol{F}^{-1} = 0 \tag{72}$$

$$\rightarrow\quad \boldsymbol{F}\ \partial\left[\boldsymbol{F}^{-1}\right] = -\partial \boldsymbol{F}\ \boldsymbol{F}^{-1} \tag{73}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}^{-1}\right] = -\boldsymbol{F}^{-1}\ \partial \boldsymbol{F}\ \boldsymbol{F}^{-1} \tag{74}$$

$$\rightarrow\quad \partial\left[\boldsymbol{F}^{-1}\right]_{Ii} = -\left[\boldsymbol{F}^{-1}\right]_{Ij}[\partial \boldsymbol{F}]_{jJ}\left[\boldsymbol{F}^{-1}\right]_{Ji} \tag{75}$$

Now we can derive the derivative of the cofactors:

$$\partial\,[\mathrm{cof}\ \boldsymbol{F}] = \partial\,[\mathrm{cof}\ \boldsymbol{F}]_{iI} = \partial\left[J\boldsymbol{F}^{-T}\right] \tag{76}$$

$$= \left[\boldsymbol{F}^{-T}\right]\partial J + J\ \partial\left[\boldsymbol{F}^{-T}\right] \tag{77}$$

$$= \left[\boldsymbol{F}^{-T}\right]_{iI}\partial J + J\ \partial\left[\boldsymbol{F}^{-T}\right]_{iI} \tag{78}$$

$$= \left[\boldsymbol{F}^{-1}\right]_{Ii}\partial J + J\ \partial\left[\boldsymbol{F}^{-1}\right]_{Ii} \tag{79}$$

$$= \left[\boldsymbol{F}^{-1}\right]_{Ii}\ J\ [\boldsymbol{F}^{-1}]_{Jj}\ \partial\,[\boldsymbol{F}]_{jJ} - J\left[\boldsymbol{F}^{-1}\right]_{Ij}[\partial \boldsymbol{F}]_{jJ}\left[\boldsymbol{F}^{-1}\right]_{Ji} \tag{80}$$

$$= J\partial\,[\boldsymbol{F}]_{jJ}\left\{\left[\boldsymbol{F}^{-1}\right]_{Ii}\ [\boldsymbol{F}^{-1}]_{Jj}\ -\ \left[\boldsymbol{F}^{-1}\right]_{Ij}\ \left[\boldsymbol{F}^{-1}\right]_{Ji}\right\} \tag{81}$$

and the directional derivative becomes

$$\partial\,[\mathrm{cof}\ \boldsymbol{F}(\varphi_i + \varepsilon\Delta u_i)]_{iI} = \left\{\left[\boldsymbol{F}^{-1}\right]_{Ii}\ [\boldsymbol{F}^{-1}]_{Jj}\ -\left[\boldsymbol{F}^{-1}\right]_{Ij}\left[\boldsymbol{F}^{-1}\right]_{Ji}\right\}\ J\ \Delta u_{j,J} \tag{82}$$

$$\tag{83}$$

And finally we transport ths restult to the linearization of the pressure loading. Let

$$\mathscr{F}(\varphi_i) = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot \left[\text{cof}\left(\boldsymbol{F}(\varphi) \right)\ \boldsymbol{N} \right] d\Gamma \tag{84}$$

$$= -\int_{\Gamma_H} p\ W_i\ \left[\text{cof}\left(\boldsymbol{F}(\varphi) \right] _{iI}\ N_I\ d\Gamma \tag{85}$$

$$\boldsymbol{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}} \tag{86}$$

$$F_{iI} = \frac{dx_i}{dX_I} = \frac{d\varphi_i}{dX_I} \tag{87}$$

$$F_{iI}(\varphi_i + \varepsilon\Delta u_i) = \frac{d\left(\varphi_i + \varepsilon\Delta u_i\right)}{dX_I} \tag{88}$$

$$\lim_{\varepsilon\to 0} \partial_\varepsilon F_{jJ}(\varphi_k + \varepsilon\Delta u_k) = \frac{d\Delta u_k}{dX_K} = \Delta u_{j,J} \tag{89}$$

Then

$$\lim_{\varepsilon\to 0} \partial_\varepsilon \mathscr{F}(\varphi + \varepsilon\Delta\boldsymbol{u}) = -\int_{\Gamma_H} p\ \boldsymbol{W} \cdot \left\{ \partial_\varepsilon \,\text{cof}\left(\boldsymbol{F}(\varphi + \varepsilon\Delta\boldsymbol{u}) \right)\ \boldsymbol{N} \right\} d\Gamma \tag{90}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left\{ \left[\boldsymbol{F}^{-1}\right]_{Ii} \left[\boldsymbol{F}^{-1}\right]_{Jj} - \left[\boldsymbol{F}^{-1}\right]_{Ij} \left[\boldsymbol{F}^{-1}\right]_{Ji} \right\} \Delta u_{j,J}\ d\Gamma \tag{91}$$

$$\tag{92}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left\{ \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \delta_{ik}\delta_{jl} - \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \delta_{il}\delta_{jk} \right\} \Delta u_{j,J}\ d\Gamma \tag{93}$$

$$= -\int_{\Gamma_H} p\ J\ W_i\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \Delta u_{j,J}\ d\Gamma \tag{94}$$

$$= \int_{\Gamma_H} p\ J\ W_i\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right) \Delta u_{j,J}\ d\Gamma \tag{95}$$

$$\tag{96}$$

The element matrix becomes:

$$K_{ij}^{\beta\gamma} = \int_{\Gamma_H} \phi^\beta \phi^\gamma_{,J}\ p\ J\ N_I\ \left[\boldsymbol{F}^{-1}\right]_{Ik} \left[\boldsymbol{F}^{-1}\right]_{Jl} \left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right) d\Gamma \tag{97}$$

6.4 Pressure loading - calculating from the isoparametric state

Recall that

$$d\Gamma = \left| \left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2} \right) \right| d\Gamma^\square \tag{98}$$

where $\Gamma^\square$ is the surface in the reference (isoparametric) configuration. We can calculate the normal vector in the current configuration using

$$n_i = \frac{\left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2} \right)}{\left| \left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2} \right) \right|} \tag{99}$$

so that

$$\int_{\Gamma_h} w_i\ h_i\ d\Gamma = -\int_{\Gamma_h^\square} p\ \boldsymbol{w} \cdot \left(\frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_1} \times \frac{\partial\hat{\boldsymbol{x}}}{\partial\xi_2} \right) d\Gamma^\square$$

Now we need to compute $\frac{\partial \hat{x}}{\partial \xi_i}$. So

$$\frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_i} = \frac{\partial \phi^\gamma}{\partial \xi_i} U^\gamma = \boldsymbol{F} \; \frac{\partial \hat{\boldsymbol{X}}}{\partial \xi_i}$$

$$\frac{\partial \hat{\boldsymbol{X}}}{\partial \xi_i} = \frac{\partial \phi^\gamma}{\partial \xi_i} \; X_i^\gamma$$

We finally obtain

$$\begin{aligned}
\mathscr{F} = \int_{\Gamma_h} w_i \; h_i \; d\Gamma &= -\int_{\Gamma_h^\square} p \; \boldsymbol{W} \cdot \left(\frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_1} \times \frac{\partial \hat{\boldsymbol{x}}}{\partial \xi_2} \right) d\Gamma^\square \\
&= -\int_{\Gamma_h^\square} p \; \boldsymbol{W} \cdot \left(\frac{\partial \boldsymbol{U}}{\partial \xi_1} \times \frac{\partial \boldsymbol{U}}{\partial \xi_2} \right) d\Gamma^\square \\
&= -\int_{\Gamma_h^\square} p \; \boldsymbol{W} \cdot \left(\frac{\partial \phi^\gamma}{\partial \xi_1} \boldsymbol{U}^\gamma \times \frac{\partial \phi^\gamma}{\partial \xi_2} \boldsymbol{U}^\gamma \right) d\Gamma^\square
\end{aligned}$$

Now we can linearize (see Belytschko book, pg365)

$$
\begin{aligned}
D\mathscr{F}\cdot\Delta U &= \lim_{\varepsilon\to 0}\partial_\varepsilon \mathscr{F}(\varphi+\varepsilon\Delta U)\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\ \ \partial_\varepsilon\left(\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)_i d\Gamma^\square\right]\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\left(\partial_\varepsilon\hat{\boldsymbol{x}}_{,\xi_1}\times\hat{\boldsymbol{x}}_{,\xi_2}+\hat{\boldsymbol{x}}_{,\xi_1}\times\partial_\varepsilon\hat{\boldsymbol{x}}_{,\xi_2}\right)_i\right.\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta\boldsymbol{U}_{,\xi_1}\times\hat{\boldsymbol{x}}_{,\xi_2}+\hat{\boldsymbol{x}}_{,\xi_1}\times\Delta\boldsymbol{U}_{,\xi_2}\right)_i\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta\boldsymbol{U}_{,\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}+\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\Delta\boldsymbol{U}_{,\xi_2}\right)_i\\
&= -\int_{\Gamma_h^\square} p\ W_i\left(\Delta U_{j,\xi_1}\times F_{jJ}\frac{\partial\hat{X}_J}{\partial\xi_2}+F_{kK}\frac{\partial\hat{X}_K}{\partial\xi_1}\times\Delta U_{k,\xi_2}\right)_i
\end{aligned}
$$

$$
\begin{aligned}
D\mathscr{F}\cdot\Delta U &= \lim_{\varepsilon\to 0}\partial_\varepsilon \mathscr{F}(\varphi+\varepsilon\Delta U)\\
&= \lim_{\varepsilon\to 0}\left[-\int_{\Gamma_h^\square} p\ W_i\ \left(\partial_\varepsilon\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\ +\ \boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\partial_\varepsilon\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)d\Gamma^\square\right]\\
&= -\int_{\Gamma_h^\square} p\ W_i\ \left(\nabla(\Delta\boldsymbol{U})\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\ +\ \boldsymbol{F}\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_1}\times\nabla(\Delta\boldsymbol{U})\frac{\partial\hat{\boldsymbol{X}}}{\partial\xi_2}\right)_i d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_i\ \left(\Delta U_{j,I}\frac{\partial\hat{X}_I}{\partial\xi_1}F_{kJ}\frac{\partial\hat{X}_J}{\partial\xi_2}\ +\ F_{jI}\frac{\partial\hat{X}_I}{\partial\xi_1}\Delta U_{k,J}\frac{\partial\hat{X}_J}{\partial\xi_2}\right)\ \varepsilon_{jki}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \left(\Delta U_{i,I}\frac{\partial\hat{X}_I}{\partial\xi_1}F_{jJ}\frac{\partial\hat{X}_J}{\partial\xi_2}\ +\ F_{iI}\frac{\partial\hat{X}_I}{\partial\xi_1}\Delta U_{j,J}\frac{\partial\hat{X}_J}{\partial\xi_2}\right)\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \left(F_{jJ}\ \Delta U_{i,I}\ +\ F_{iI}\ \Delta U_{j,J}\right)\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\int_{\Gamma_h^\square} p\ W_k\ \Delta U_{i,I}\ F_{jJ}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square-\int_{\Gamma_h^\square} p\ W_k\ \Delta U_{j,J}\ F_{iI}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ \ d\Gamma^\square\\
&= -\ W_k^\beta\ \Delta U_i^\gamma\int_{\Gamma_h^\square} p\ \phi^\beta\ \phi^\gamma_{,I}\ F_{jJ}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ d\Gamma^\square\\
&\quad -\ W_k^\beta\ \Delta U_j^\gamma\int_{\Gamma_h^\square} p\ \phi^\beta\ \phi^\gamma_{,J}\ F_{iI}\ \frac{\partial\hat{X}_I}{\partial\xi_1}\ \frac{\partial\hat{X}_J}{\partial\xi_2}\ \varepsilon_{ijk}\ d\Gamma^\square
\end{aligned}
$$

where

$$
\varepsilon_{ijk}=\begin{cases}+1 & \text{if } (ijk)=(123),(231),(312)-\text{even permutation}\\ -1 & \text{if } (ijk)=(321),(132),(213)-\text{odd permutation}\\ 0otherwise\end{cases}\tag{100}
$$

7 Incremental formulation

Start from the residual weak formulation.

$$\begin{aligned}\mathscr{F} &= (W_{i,I},\ P_{iI})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0\\ &= (W_{i,I},\ F_{iJ}\ S_{JI})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0\\ &= (W_{i,I},\ F_{iJ}\ \mathbb{C}_{JIKL}\ E_{KL})_{\Omega_0} - (W_i,\ \tilde{T})_{\Gamma_H} = 0\end{aligned}$$

We need to define the timestep of the variables. Define the states 0, 1, 2 as the initial, last solved and next to solve respectively. Then

$$\begin{aligned}E_{KL} &= {}_0^2E_{KL} = {}_0^1E_{KL} + {}_{\ 0}^{1\to 2}\delta E_{KL}\\ \delta E_{KL} &= {}_{\ 0}^{1\to 2}\delta E_{KL} = {}_0^2E_{KL} - {}_0^1E_{KL}\end{aligned}$$

$$\begin{aligned}S_{JI} &= {}_0^2S_{JI}\\ &= {}_0^1S_{JI} + {}_0^2\delta S_{JI}\end{aligned}$$

Calculate E_{KL} from the displacements:

$$\begin{aligned}2\delta E_{KL} &= {}_0^2\Big[U_{K,L} + U_{L,K} + U_{i,K}\ U_{i,L}\Big] - {}_0^1\Big[U_{K,L} + U_{L,K} + U_{i,K}\ U_{i,L}\Big]\\ &= \Big(\delta U_{K,L} + \delta U_{L,K} + {}_0^1U_{i,K}\ \delta U_{i,L} + {}_0^1U_{i,L}\ \delta U_{i,K}\Big) + \Big(\delta U_{i,L}\delta U_{i,K}\Big)\end{aligned}$$

Back to the formulation:

$$\begin{aligned}\mathscr{F} &= \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1E_{KL}\big) + (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta E_{KL}) - (W_i, \tilde{T})\\ &= \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1E_{KL}\big)\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{K,L})\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{L,K})\\ &+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1U_{i,K}\ \delta U_{i,L}\big)\\ &+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1U_{i,L}\ \delta U_{i,K}\big)\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{i,L}\ \delta U_{i,K})\\ &- (W_i,\quad \tilde{T})_{\Gamma_H}\end{aligned}$$

Linearize:

$$\begin{aligned}D\mathscr{F}\cdot\delta U_i &= \lim_{\varepsilon\to 0}\partial_\varepsilon\mathscr{F}(\delta U + \varepsilon\Delta U_i)\\ &= (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{K,L})\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{L,K})\\ &+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1U_{i,K}\ \Delta U_{i,L}\big)\\ &+ \big(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}_0^1U_{i,L}\ \Delta U_{i,K}\big)\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta U_{i,L}\ \Delta U_{i,K})\\ &+ (W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \Delta U_{i,L}\ \delta U_{i,K})\end{aligned}$$

Trying from the deformation gradient perspective:

$$\begin{aligned}\delta E_{KL} &= \left({}^2_0\boldsymbol{F}^T\ {}^2_0\boldsymbol{F} - \boldsymbol{I}\right) - \left({}^1_0\boldsymbol{F}^T\ {}^1_0\boldsymbol{F} - \boldsymbol{I}\right)\\ &= {}^2_0\boldsymbol{F}^T\ {}^2_0\boldsymbol{F} - {}^1_0\boldsymbol{F}^T\ {}^1_0\boldsymbol{F}\\ &= {}^2_0F_{iK}\ {}^2_0F_{iL} - {}^1_0F_{iK}\ {}^1_0F_{iL}\end{aligned}$$

Hence:

$$\begin{aligned}\mathscr{F} &= \left(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0E_{KL}\right) + \left(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ \delta E_{KL}\right) - (W_i, \tilde{T})\\ &= \left(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0E_{KL}\right)\\ &+ \left(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^2_0F_{iK}\ {}^2_0F_{iL}\right)\\ &- \left(W_{i,I},\quad F_{iJ}\ C_{JIKL}\ {}^1_0F_{iK}\ {}^1_0F_{iL}\right)\\ &- (W_i, \tilde{T})\end{aligned}$$

Did not work out, because the ΔU_i does not show. Need to open the terms, and will get to the same verbosity.

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