Trustworthy Boolean Reasoning
1B: Unsatisfiability Proofs

Randal E. Bryant

Carnegie Mellon University

June, 2022

Important Ideas for These Lectures

- SAT solvers are useful tools
  - Many practical problems reducible to SAT
  - Need to learn effective encoding techniques
- For many applications, formulas should be unsatisfiable
  - **Program should generate a checkable proof**
  - **There is a well-developed proof infrastructure**
- Binary Decision Diagrams (BDDs) can play important role
  - In supplementing current SAT algorithms
  - In proof generation

Example Formula

DIMACS Format

- Standard for all solvers
- Positive integers for variables
- Negative integers for their negations
- Lists terminated with `0`

ID	Clause	DIMACS Encoding
		`p cnf 4 6`
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	`-1 -2 -3 0`
2	$\overline{a} \vee \overline{b} \vee c$	`-1 -2 3 0`
3	$a \vee \overline{d}$	`1 -4 0`
4	$a \vee d$	`1 4 0`
5	$b \vee \overline{d}$	`2 -4 0`
6	$b \vee d$	`2 4 0`

Example Proof

- Derive empty clause $\perp$ through set of resolution steps

$$\frac{a \lor d \qquad a \lor \overline{d}}{a \qquad\qquad a}$$

$$\cfrac{\cfrac{\cfrac{\overline{a} \lor \overline{b} \lor c \qquad a}{\overline{b} \lor c} \qquad \cfrac{a \qquad \overline{a} \lor \overline{b} \lor \overline{c}}{\overline{b} \lor \overline{c}}}{\overline{b}} \qquad \cfrac{b \lor \overline{d} \qquad b \lor d}{b}}{\perp}$$

But how can a program find such a proof?

Unit Propagation

Unit clauses

- If formula contains clause (x), then x must be assigned 1.
- If formula contains clause $(\overline{x})$, then x must be assigned 0.

Propagating Unit Literal ℓ

- If $\ell \in C$ and $\ell = 0$, then $C \leftarrow C - \{\ell\}$
- If $\ell \in C$ and $\ell = 1$, then C satisfied
- If any clause becomes unit, then iterate

Step	Formula	Units
1	$(\boldsymbol{a} \vee \overline{\boldsymbol{c}}) \wedge (\overline{\boldsymbol{a}} \vee \overline{\boldsymbol{b}} \vee \overline{\boldsymbol{c}}) \wedge (\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c}) \wedge (\boldsymbol{c})$	$c = 1$
2	$(\boldsymbol{a} \vee \overline{\boldsymbol{c}}) \wedge (\overline{\boldsymbol{a}} \vee \overline{\boldsymbol{b}} \vee \overline{\boldsymbol{c}}) \wedge (\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c}) \wedge (\boldsymbol{c})$	$a = 1$
3	$(\boldsymbol{a} \vee \overline{\boldsymbol{c}}) \wedge (\overline{\boldsymbol{a}} \vee \overline{\boldsymbol{b}} \vee \overline{\boldsymbol{c}}) \wedge (\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c}) \wedge (\boldsymbol{c})$	$b = 0$
–	$(\boldsymbol{a} \vee \overline{\boldsymbol{c}}) \wedge (\overline{\boldsymbol{a}} \vee \overline{\boldsymbol{b}} \vee \overline{\boldsymbol{c}}) \wedge (\boldsymbol{a} \vee \boldsymbol{b} \vee \boldsymbol{c}) \wedge (\boldsymbol{c})$	

Basic CDCL Operation

Conflict-Driven Clause Learning

- Algorithm in state-of-the art solvers
- Search, but learn from dead ends

```
while(True):
    depth ⟵ 0
    while(True):
        UnitPropagate()
        if all clauses satisfied
            return solution
        if ConflictDetected():
            Generate conflict clause
            break
        Choose variable and assign 0 or 1
        depth ⟵ depth + 1
    if depth = 0:
        return UNSAT
```

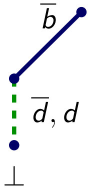
CDCL Execution Example

- No initial unit propagations

ID	Clause	UProp?
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
2	$\overline{a} \vee \overline{b} \vee c$	
3	$a \vee \overline{d}$	
4	$a \vee d$	
5	$b \vee \overline{d}$	
6	$b \vee d$	

CDCL Execution Example

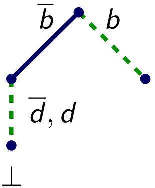
- Setting $b = 0$ causes conflict. Learn clause b.



ID	Clause	UProp?
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
2	$\overline{a} \vee \overline{b} \vee c$	
3	$a \vee \overline{d}$	
4	$a \vee d$	
5	$b \vee \overline{d}$	*
6	$b \vee d$	*
7	b	

CDCL Execution Example

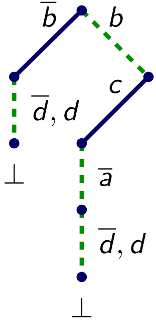
- Unit propagate b. No conflict



ID	Clause	UProp?
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
2	$\overline{a} \vee \overline{b} \vee c$	
3	$a \vee \overline{d}$	
4	$a \vee d$	
5	$b \vee \overline{d}$	
6	$b \vee d$	
7	b	

CDCL Execution Example

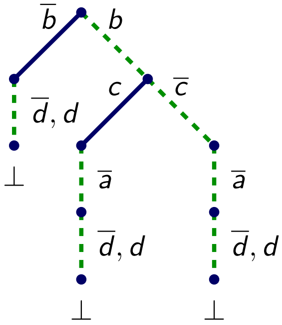
- Setting $c = 1$ causes conflict. Learn clause $\overline{c}$.



ID	Clause	UProp?
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	*
2	$\overline{a} \vee \overline{b} \vee c$	
3	$a \vee \overline{d}$	*
4	$a \vee d$	*
5	$b \vee \overline{d}$	
6	$b \vee d$	
7	b	*
8	$\overline{c}$	

CDCL Execution Example

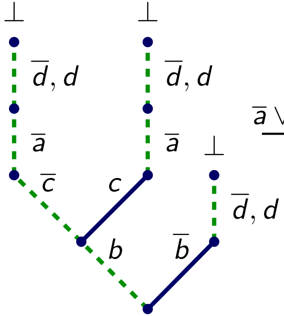
- Unit propagate b and $\overline{c}$. Causes conflict. UNSAT!



ID	Clause	UProp?
1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
2	$\overline{a} \vee \overline{b} \vee c$	*
3	$a \vee \overline{d}$	*
4	$a \vee d$	*
5	$b \vee \overline{d}$	
6	$b \vee d$	
7	b	*
8	$\overline{c}$	*
9	$\bot$	

Proof from CDCL Run

Proof Follows Branching Structure of CDCL



$$\dfrac{\dfrac{\dfrac{\overline{a} \lor \overline{b} \lor c \quad \dfrac{a \lor d \quad a \lor \overline{d}}{a}}{\overline{b} \lor c} \quad \dfrac{a \quad \overline{a} \lor \overline{b} \lor \overline{c}}{\overline{b} \lor \overline{c}}}{\overline{b}} \quad \dfrac{b \lor \overline{d} \quad b \lor d}{b}}{\perp}$$

Reverse Unit Propagation (RUP)

Purpose

- Simple and efficient rule for use by proof checkers
- Good match to operation of CDCL solvers

Operation

- Each RUP application forms one step of unsatisfiability proof
- Performs a linear sequence of resolutions steps + subsumption

Objective

- $C = \{\ell_1, \ell_2, \ldots, \ell_m\}$ Clause to be added to proof
- $D_1, D_2, \ldots, D_k$ Previous clauses ("Antecedents")
- Prove: $D_1 \wedge D_2 \wedge \cdots \wedge D_k \rightarrow C$

Reverse Unit Propagation (RUP)

Objective

- $C = \{\ell_1, \ell_2, \ldots, \ell_m\}$ Clause to be added to proof
- $D_1, D_2, \ldots, D_k$ Previous clauses ("Antecedents")
- Prove: $D_1 \wedge D_2 \wedge \cdots \wedge D_k \rightarrow C$

Method

- Assume $\neg C = \overline{\ell}_1 \wedge \overline{\ell}_2 \wedge \cdots \wedge \overline{\ell}_m$
  - m unit clauses!
- Show contradiction with $D_1 \wedge D_2 \wedge \cdots \wedge D_k$
  - Accumulate unit clauses, starting with those for $\neg C$.
  - Accrue more unit clauses from $D_1, D_2, \ldots, D_{k-1}$.
  - When encounter D_k, should have contradiction

RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	

RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	
C_7	b	C_5, C_6

$$
\begin{array}{c|cccccc}
 & \overline{b} & & \overline{d} & & \perp \\
b & & C_5 & & C_6 &
\end{array}
$$

RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	
C_7	b	C_5, C_6
C_8	$\overline{c}$	C_7, C_1, C_3, C_4

$$\begin{array}{c|ccccccccc} & c & & b & & \overline{a} & & \overline{d} & & \perp \\ \overline{c} & & C_7 & & C_1 & & C_3 & & C_4 & \end{array}$$

RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	
C_7	b	C_5, C_6
C_8	$\overline{c}$	C_7, C_1, C_3, C_4
C_9	$\perp$	C_7, C_8, C_2, C_3, C_4

			b		$\overline{c}$		$\overline{a}$		$\overline{d}$		$\perp$
$\perp$		C_7		C_8		C_2		C_3		C_4	

Proof File Examples

Proof

C_7	b	C_5, C_6
C_8	$\overline{c}$	C_7, C_1, C_3, C_4
C_9	$\perp$	C_7, C_8, C_2, C_3, C_4

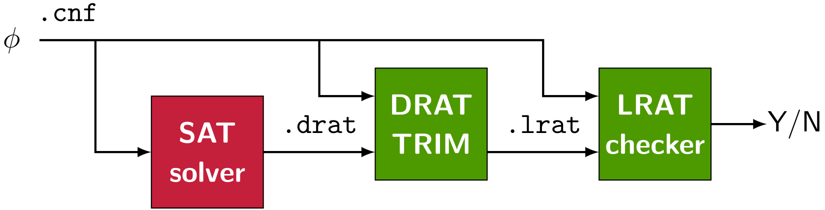
DRAT Proof File

```
2 0
-3 0
0
```

LRAT Proof File

7	2 0	5 6 0
8	-3 0	7 1 3 4 0
9	0	7 8 2 3 4 0

Proof Checking Infrastructure



Operation:

- DRAT-TRIM adds antecedents to proof steps
- LRAT checker simply checks each proof step

LRAT Checkers:

- LRAT-CHECK written in C.
  - Fast and high capacity
  - Designed to be simple enough to easily understand
- Formally verified ones. Built on ACL2, Coq, HOL, ...
  - Integrity not compromised if solver or DRAT-TRIM has bug

Resolution and CDCL

CDCL $\approx$ Resolution

- *Strength:* CDCL solver can readily generate resolution proofs
- *Weakness:* Lower bound on performance

Example: Pigeonhole Principle(PHP)

- Problem:
  - n holes, $n+1$ pigeons
  - Assign pigeons to holes:
    - Each pigeon is assigned to some hole
    - Each hole has at most one pigeon
- SAT Encoding:
  - Variables: $p_{i,j}$: Pigeon j in hole i. $1 \leq i \leq n$, $1 \leq j \leq n+1$.
  - $n+1$ at-least-one constraints
  - n at-most-one constraints
  - $O(n^3)$ total clauses
- PHP(n) resolution proofs are exponential in n [Haken, 1985]

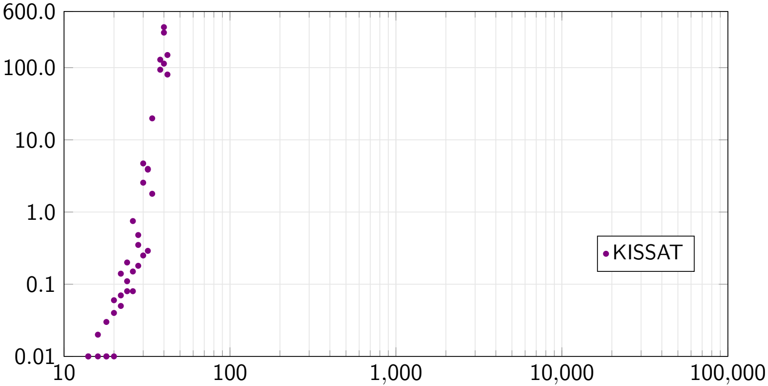
Parity Benchmark

- Chew and Heule, SAT 2020
- For random permtuation π:

$$\begin{array}{ccccccccccc} x_1 & \oplus & x_2 & \oplus & \cdots & \oplus & x_n & = & 1 & & \text{Odd parity} \\ x_{\pi(1)} & \oplus & x_{\pi(2)} & \oplus & \cdots & \oplus & x_{\pi(n)} & = & 0 & & \text{Even parity} \end{array}$$

- Conjunction unsatisfiable
- Very challenging for CDCL solvers
- Unit propagation of limited value
  - k-way parity constraint
  - Only propagate when $k-1$ variables assigned

Parity Benchmark Runtime



- KISSAT: State-of-the-art CDCL solver
- Tried 3-different seeds for each value of n
- Limited to $n \leq 42$ within 600 seconds

Extended Resolution

- Tseitin, 1967

Can introduce extension variables

- Variable z that has not yet occurred in proof
- Must add *defining* clauses
  - Encode constraint of form $z \leftrightarrow F$
  - Boolean formula z over input and earlier extension variables

Extension variable z becomes shorthand for formula F

- Repeated use can yield exponentially smaller proof

Similar to use of encoding variables in SAT formulas

- That's why they're called "Tseitin variables"
- But here they become part of proof, not of input formula

Extended RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	

Strategy

- Use z to encode $a \wedge b$.
- E.g., C_1 becomes $\overline{z} \vee \overline{c}$.

Extended RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	
C_7	$\overline{z} \vee a$	Defining Clauses
C_8	$\overline{z} \vee b$	
C_9	$z \vee \overline{a} \vee \overline{b}$	

Extended RUP Proof Example

ID	Clause	Antecedents
C_1	$\overline{a} \vee \overline{b} \vee \overline{c}$	
C_2	$\overline{a} \vee \overline{b} \vee c$	
C_3	$a \vee \overline{d}$	
C_4	$a \vee d$	
C_5	$b \vee \overline{d}$	
C_6	$b \vee d$	
C_7	$\overline{z} \vee a$	Defining Clauses
C_8	$\overline{z} \vee b$	
C_9	$z \vee \overline{a} \vee \overline{b}$	
C_{10}	$\overline{z} \vee \overline{c}$	C_7, C_8, C_1
C_{11}	$\overline{z}$	C_7, C_8, C_2, C_{10}
C_{12}	d	C_4, C_6, C_9, C_{11}
C_{13}	$\perp$	C_{12}, C_3, C_5, C_9, C_{11}

Can Extended Resolution Yield Faster SAT Solvers?

PHP Proof Complexity

- Exponential for ordinary resolution
- $O(n^4)$ for extended resolution [Cook, 1976]

Use in SAT?

- No clear way to choose which formulas to abbreviate
- No clear way to shorten search by using abbreviations