FIRING TABLE COMPUTATIONS ON THE ENIAC

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Introduction

One of the major missions of the Computing Laboratory of the Ballistic Research Laboratories is the computation of firing tables to allow new weapons to be put to use and to supplement changes in existing weapons. Having been originally designed with this mission in mind, the Eniac still has 25 per cent of its computing computing time devoted to artillery and bomb ballistics computation. While the consideration of bomb ballistics introduces some concepts not encountered in firing table work, much of the computation is similar to that performed in the preparation of a firing table, and therefore, although this paper considers the latter, much can be carried over to the computation of bombing tables.

The computation of firing tables on the Eniac is of interest for three reasons. First the problem represents the type in which full, but not overtaxing use is made of the twenty high speed registers available on the Eniac and as such typifies problems which can be economically handled on a machine with this limitation. Second, in adapting the problem to this limitation, many methods have been tried, and the experience gained could easily be applied to putting much larger problems on larger machines. Third, the problem of obtaining a realistic solution requires a close union of physical observation, mathematical theory, and human judgement.

The preparation of a firing table is accomplished in three phases. The first phase is the reduction of empirical data to parameters which may be used in a mathematical model, simple enough to allow numerical computations to be made on the Eniac. The second phase consists of the computation of trajectories with this model for both nominal (or normal) conditions and for various perturbations of the nominal conditions. The third phase involves high order interpolations on the data obtained from the second phase into information usable in the field.

Reductions

The equations for particle trajectory theory are:

$$x'' = -E(x' - w_x) + 2\,\Omega\,\cos L \sin \alpha\; y'$$

$$y'' = -E\,y' - g - 2\,\Omega\,\cos L \sin \alpha\; x'$$

$$z'' = -E(z' - w_z) + 2\,\Omega\,\sin L\, x' + 2\,\Omega\,\cos L \cos \alpha\; y'$$

where

$$x' = \frac{dx}{dt}$$

x = distance down range

y = altitude

z = distance cross range in the right hand sense

w_x = wind down range

w_z = wind across range

Ω = angular velocity of the earth

L = latitude

α = azimuth

$$E = \frac{a(y)H(y)G\left(\frac{v}{a(y)}\right)}{C}$$

a = relative velocity of sound

H = relative air density

G = drag function

$$c = \frac{m}{id^2}$$

m = mass

d = diameter

i = form factor

The drag function for a particular missile can be obtained by using that for a missile with a similar shape, or if no such comparison can be made, a special table can be obtained by wind tunnel measurements, theoretical considerations of aerodynamics, and short firings in free flight ranges. In using the latter, a linearized theory is fitted to photographic data by least square techniques. The general shape of the graph of a G function is given in figure 1.

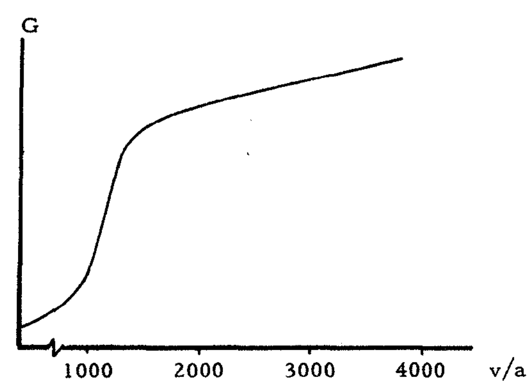

Figure 1

It will be noted that the curve is particularly steep in the region of $v/a = 1080$. In this region great care must be exercised in choosing an interval of integration if the numerical solutions of the differential equations are to be smooth.

Having chosen a drag function, one has only to decide upon the value of the ballistic coefficient to be used. Actually the form factor is the only undetermined factor, but it is more con-

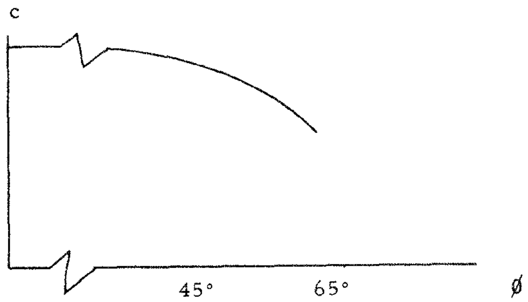

Figure 2

venient to consider th ballistic coefficient to be the parameter. In the determination of this coefficient an attempt is made to eliminate the major defect of the particle trajectory theory. This defect is that for large quadrant angles of elevation (ϕ) the axis of the missile does not turn rapidly enough to remain even approximately tangent to the path of the center of gravity of the missile, and hence a large value of yaw is developed. Once developed this yaw is sustained for the descending branch of the trajectory and causes a larger drag to act on the missile than particle trajectory theory predicts.

In considering this effect the assumption is made that the effect of yaw can be averaged over the entire trajectory, and that by making the ballistic coefficient a function of ϕ (see figure 2) this average can be effected. This method is only partially successful in that a value of c which gives a computed range which agrees with the observed range gives a computed time of flight shorter than those observed and causes a deformation exemplified in figure 3.

To obtain the ballistic coefficient a number of firings are made and the meteorological conditions and final values of ranges and times of flight are observed. These firings are made in groups with each group's being called a point. For each point the firings are made with the same quadrant angle of elevation and charge. The final results of each point are averaged and the probable error in range is computed. The range and time thus obtained are called the observed values.

The meteorological conditions for each point are used to compute trajectories on the Eniac with various values for the ballistic coefficient until two values are found which give ranges bracketing the observed range within the allowable error. From these two pairs of range and ballistic coefficient the change in range for a one per cent change in c and the value of c which corresponds to the observed range are computed. A polynomial is now fitted to the table of ballistic coefficient versus quadrant angle of elevation by a least square method which uses the change in range for change in c as a weighting factor. Polynomials of increasing degree are tried until one is found which gives values of ballistic coefficient corresponding to ranges within four probable errors of the observed values.

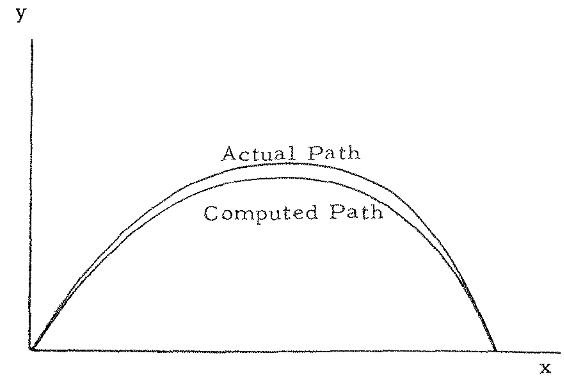

Figure 3

Numerical Integration of Trajectories

The method of integration used to compute trajectories on the Eniac is one due to Heun. Essentially it consists of an application of the trapezoidal rule of integration. A recent modification of this method is used to obtain the displacements by the use of the next term in the Euler-MacLaurin scheme for numerical integration

$$f_1 - f_o = 1/2(f'_o + f'_1)h + 1/12(f''_o - f''_1)h^2 + 0(h^5)$$

The algorithm for trajectories is

$$\bar{x}'_1 = x'_o + x''_o \Delta t$$

$$\bar{x}_1 = x_o + x'_o \Delta t$$

$$x'_1 = x'_o + (x''_o + \bar{x}''_1) \frac{\Delta t}{2}$$

$$x_1 = x_o + (x'_o + \bar{x}'_1) \frac{\Delta t}{2} + (x''_o - \bar{x}''_1) \frac{(\Delta t)^2}{12}$$

The error in this method is of the third order in Δt. Other methods using higher order approximations have been considered, but the above is quite well adapted to the Eniac's storage capacity and has sufficient accuracy for the allowable error under suitable choice of Δt.

Determination of ground values is effected by the following scheme: Computation is continued until y becomes negative. An interval of integration $\Delta t = \frac{-y}{y'_1}$ is now used to approach closer to ground. If y is still too large a new $\Delta t = \frac{-y}{y'_1}$ is chosen and another integration performed. This is continued until the desired accuracy is obtained (two iterations usually being sufficient).

Normals and Effects

The backbone of a firing table is a group of trajectories which are computed using normal atmospheric conditions, no rotation of earth, and the ballistic coefficients given by the curve obtained from the reductions. Normal atmospheric conditions do not share the same place in physics as do the weightless beam and frictionless pulley. Rather, they represent a conventional standard which is a fairly realistic average of the temperature and density structure of the air as a function of altitude. Up to altitudes which are too large for consideration in gun ballistics, these averages may be approximated closely by assuming that density and the velocity of sound decrease exponentially. On the Eniac the approximation $e^{y} \sim \left(\frac{2N - y}{2N + y}\right)^N$ is used in this connection with $N = 16$.

Values along the normal trajectories are usually printed for every second of time of flight and for ground. Among the values printed at the ground is the summital value of altitude which has been obtained previously in the trajectory computation in a manner similar to that used for obtaining ground values.

To allow the results of the normal trajectories to be used for non-standard atmospheric conditions a set of trajectories are computed for modified normal conditions. These modifications are introduced separately for fairly large variations and include: Range wind, percentage change in density, percentage change in velocity of sound (temperature variation), muzzle velocity variation, weight variation, and addition of rotation of earth effects. From these the variation in range is computed for small increments.

Site

Up to this point data has been obtained which gives range and altitude as functions of quadrant angle of elevation and time of flight. As is often the case in describing physical events the natural parameters, here the initial values of the differential equations and the independent variable t, are not of much value in practice. The artillery man has as parameters the range of the target and the angle of site (ϵ) which is the angle made with the horizontal by the line joining gun and target. From this information it is desired to determine the quadrant angle of elevation (ϕ) which will initiate a trajectory that passes through the target. (See figure 4)

In putting this information into a firing table . two conventions are in vogue. Each of these involve the quantity ϕ_o which is the quadrant angle of elevation for the range under consideration for ground (ϵ = o). The two conventional quantities are complementary angle of site ($\phi - \phi_o - \epsilon$) and site ($\phi - \phi_o$).

Due to the requirement for accurate interpolation there is usually a variation in method for determining ϕ when $\phi \leq 45°$ and $\phi > 45°$. For quadrant angles of elevation below 45 degrees, $\frac{dy}{dx}$ is at most slightly over unity and the best procedure is to determine y for a given x and ϕ and then interpolate on these to obtain ϕ for $y = x \tan \epsilon$ with x and ϵ in even intervals. To accomplish the former interpolation either x may be used as independent variable in integrating the original differential equations or osculating interpolation may be used in trajectories computed with t as independent variable. Although x has often been used as independent variable there seems to be difficulty in obtaining smooth results therewith, and the use of osculating interpolation appears to be the better approach. For the second interpolation the method due to Aitken is generally used.

For quadrant angles of elevation above 45°, $\frac{dy}{dx}$ is greater than unity for values of y which are of interest and the best method is to first interpolate by osculating methods to obtain x for even ϵ and ϕ, and then to interpolate for ϕ for given ϵ and x using Aitken's method.

Osculating interpolation involves the fitting of a polynomial which assumes not only the functional values, but also the values of some of the derivatives of the function to be matched, at certain values of the argument. The method used in connection with obtaining site is of the fifth order and fits the functional values and first two derivatives thereof at two points: For example

$$y \sim \sum_{i=o}^{1} \sum_{j=o}^{2} Pij(x, x_o, x_1)\left(\frac{d^j y}{dx^j}\right)_i$$

where

$$\frac{d^m}{dx^m}\left[Pij(x_k, x_o, x_1)\right] = \delta_{mj}\, \delta_{ik}.$$

The P i j's are fifth order polynomials in x.

The value of this scheme for the Eniac lies in the fact that two IBM cards (each containing values for a particular t and ϕ) give sufficient information for the computation. From the first card only y_o, x_o, y'_o, x'_o, and E_o need be stored in the high speed memory. The corresponding values from the second card may be allowed to

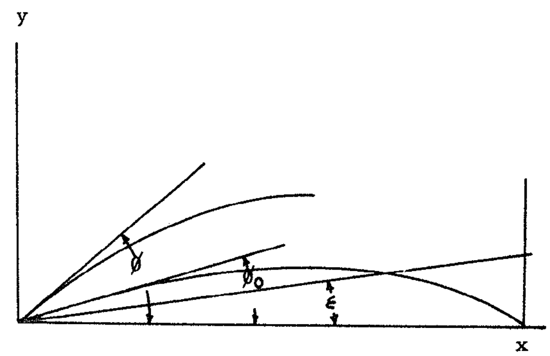

Figure 4

remain in the Constant Transmitter (the relays associated with the IBM reader).

Aitken's interpolation is based on the iterative scheme

$$\phi(y, y_o, \ldots, y_n) = \frac{\begin{vmatrix} \phi(y, y_o, \ldots, y_{n-1}) & (y_{n-1} - y) \\ \phi(y, y_o, \ldots, y_{n-2}, y_n) & (y_n - y) \end{vmatrix}}{y_n - y_{n-1}}$$

and is used since $\frac{\partial \phi}{\partial y}$ is not readily obtainable.

Firing Table Elements

For ground values (y = o) it is desired to find $v = \sqrt{x^2 + y^2}$, y_s = summit, ω = arc tan $\left(-\frac{dy}{dx}\right)$, ϕ and t for even values of x. The values of v, y_s, ω, t, and x are printed for each trajectory in the manner described previously and these values are then subjected to a fourth order interpolation using Lagrange's formula

$$f(x) = \sum_{i=o}^{4} \prod_{\substack{i \neq j \\ j=o}}^{4} \left(\frac{x - x_j}{x_i - x_j}\right) f(x_i).$$

This method is used in preference to that of Aitken since the coefficients $\prod \left(\frac{x - x_j}{x_i - x_j}\right)$ may be used for more than one function. The saving obtained in this manner justifies the considerably more complicated programming required by the high demands on storage(by storing two quantities in one register three functions may be considered at once) and the necessity of carrying a floating decimal point to maintain sufficient accuracy in the computation of the coefficients.

After the functions have been computed by interpolation they are used in the computation of extra quantities which are of value to the gunner. These include fuze settings (F. S. = polynomial in t), cross wind effect in mils (W - D = 497.977 $[T/x - 1/v_o \cos \emptyset]$), $\frac{\partial \emptyset}{\partial R}$, $\frac{\partial R}{\partial \emptyset}$, and corrected time of flight (T = polynomial in t and $\emptyset$). These elements plus v, ω, $\emptyset$, and y_s, together with site constitute the major portion of a firing table.

The corrected time of flight mentioned above is another of the measures taken to alleviate the difficulties in using particle trajectory theory. In the reductions only ranges are matched. Times of flight are in error usually on the short side and to compensate for this a correcting factor Δt ($\emptyset$) is fitted to the results of the reductions so that

$$T = t + \Delta T(\emptyset) = \text{observed time of flight.}$$

T is then called the observed or corrected time of flight and t is called the computed time of flight.

Graphical Firing Table

The final series of computations on the normal trajectories is the compilation of data which is used to construct a slide rule known as a graphical firing table. For this, subtabulations of range for even values of the following functions: $\emptyset$, $\frac{\partial \emptyset}{\partial R}$, W - D, F. S., T, $\frac{\partial \emptyset}{\partial \varepsilon}$ (change in $\emptyset$ for 10 mils change in angular height), and fork (change in $\emptyset$ for four probable errors change in range) are required. Each of these interpolations is performed by using Aitken's method.

For the Future

While the present methods give quite satisfactory results for the most part, there is at least an aesthetic basis for the desire to develop a more complete mathematical picture for the expression of the freely flying missile. Such a model would consider the moment equations to allow a more realistic representation of the effects of yaw. However, such considerations require the addition of two additional second order equations bringing the total to five second order differential equations. The Eniac, at present, has not been able to handle this problem and little is known as to whether such an approach is practical or even desirable. In order to determine this, some empirical work along these lines will be performed on either the Edvac or Ordvac.