Gaussian Multiple and Random Access in the Finite Blocklength Regime

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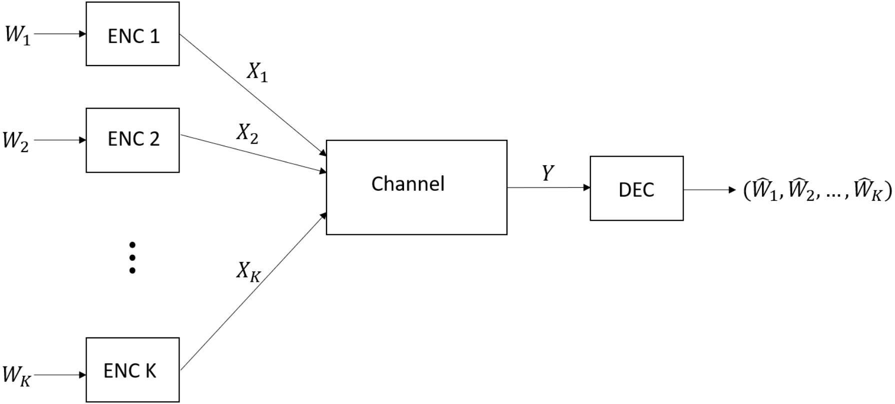
Talk Plan

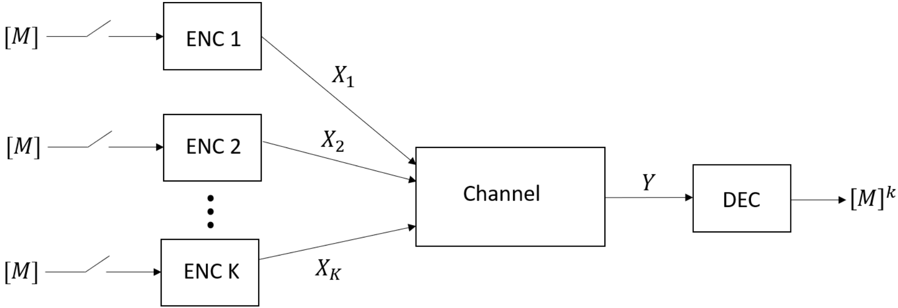
1. Gaussian Multiple Access Channel (MAC)
    1. 2-transmitter MAC
    2. K-transmitter MAC
2. Gaussian Random Access Channel (RAC)
3. Some discussions

Multiple Access Channel (MAC)



K transmitters are communicating to a single receiver. Both the transmitters and the receiver know K.

Random Access Channel (RAC)



There are K transmitters in total. A subset of those with size k are active. Nobody knows the active transmitters.

Main Goal

We propose a communication strategy that is

- **grant-free** (not required to decode transmitters' identities),
- does not require the knowledge of transmitter activity at the encoders and the decoder,
- and still performs as good as the nonasymptotic information-theoretical limits for the k-transmitter MAC as if k is known a priori.

Most strategies in use such as ALOHA require grant for the transmitters to use the channel, and do not handle that multiple transmitters communicate simultaneously. This causes a huge performance loss.

For the Gaussian MAC, orthogonalization methods such as TDMA does not perform well in terms of the second-order term.

Prior work on MAC/RAC

Multi-transmitter Communication

- **All active (MAC)** $k = K$
- **Some active (RAC)** $k < K$
  - **Active set known**
  - **Active set unknown**

All active (MAC):

Classical: Ahlswede 73'-Liao 72'
Orthogonal: TDMA, FDMA, CDMA
Finite blocklength:
MolavianJazi-Laneman 12'
Tan-Kosut 14'
Huang-Moulin 12'
Scarlett-Fong-Fabregas 15'
Many access: Chen-Guo 14'

Active set known:

Mathys 90'
Tse-Hanly 99'
Bassalygo-Pinsker

Active set unknown:

Classical (collison avoidance):
ALOHA (Abramson 70')
Collision-resolution (Capetanakis 79')
D'yachkov-Rykov 81'
Coded Slotted ALOHA (Liva 11')
Finite K: This talk,
Minero-Franceschetti-Tse 12'
Many access: Chen-Chen-Guo 17'
Thrampoulidis-Zalik-Polyanskiy 18'

Gaussian MAC

The channel model

- A Gaussian channel with K transmitters outputs

$$Y_K = \sum_{i=1}^{K} X_i + Z,$$

where $Z \sim \mathcal{N}(0,1)$, equivalently

$$P_{Y_K|X_{[K]}}(y_K|x_{[K]}) = \frac{1}{\sqrt{2\pi}} \exp\left\{-(y - x_{\langle [K] \rangle})^2\right\}$$

Notation:

$$\begin{aligned} [K] &= \{1, \ldots, K\} \\ x_{\mathcal{C}} &= (x_c : c \in \mathcal{C}) \\ x_{\langle \mathcal{C} \rangle} &= \sum_{c \in \mathcal{C}} x_c \end{aligned}$$

Point to point Channel

- Maximum message size $M^*(n, \epsilon, P)$ for P2P channel (with blocklength n, maximal power constraint P, and average error probability ϵ) satisfies

$$\log M^*(n, \epsilon, P) = nC(P) - \sqrt{nV(P)}Q^{-1}(\epsilon) + \frac{1}{2}\log n + O(1)$$

where $C(P) = \frac{1}{2}\log(1 + P)$ is the capacity, and $V(P) = \frac{P(P+2)}{2(1+P)^2}$ is the dispersion.

Achievability: [Tan-Tomamichel 15'], Converse: [PPV 10']

- Achievability part uses random codewords uniformly distributed over the n-dimensional sphere with radius $\sqrt{nP}$, and maximum likelihood (ML) decoding.

MAC Code Definition

Definition (2-transmitter MAC)

An $(n, M_1, M_2, \epsilon, P_1, P_2)$ code for the two-transmitter MAC consists of two encoding functions $\mathsf{f}_i : [M_i] \to \mathbb{R}^n,\ i = 1, 2$, and a decoding function $\mathsf{g} : \mathbb{R}^n \to [M_1] \times [M_2]$, with *maximal* power constraint

$$\|\mathsf{f}_\mathsf{i}(w_i)\|^2 \leq nP_i \text{ for } w_i \in [M_i], i = 1, 2$$

and the *average* probability of error does not exceed ϵ

$$\frac{1}{M_1 M_2} \sum_{w_1=1}^{M_1} \sum_{w_2=1}^{M_2} \mathbb{P}\left[\mathsf{g}(Y^n) \neq (w_1, w_2) \mid X_1^n = \mathsf{f}_1(w_1), X_2^n = \mathsf{f}_2(w_2)\right] \leq \epsilon,$$

We only consider the case with average error and maximal power constraints. The performance of the code changes significantly if we change average to maximal or vice versa.

Main Result

Theorem

For any $\epsilon \in (0,1)$ and any $P_1, P_2 > 0$, an $(n, M_1, M_2, \epsilon, P_1, P_2)$ code for the two-transmitter Gaussian MAC exists provided that

$$\begin{bmatrix} \log M_1 \\ \log M_2 \\ \log M_1 M_2 \end{bmatrix} \in n\mathbf{C}(P_1, P_2) - \sqrt{n} Q_{\mathrm{inv}}(\mathsf{V}(P_1, P_2), \epsilon) + \frac{1}{2} \log n \mathbf{1} + O(1)\mathbf{1}.$$

- $Q_{\mathrm{inv}}(\mathsf{V}, \epsilon)$ is the multidimensional counterpart of inverse Q-function

$$Q_{\mathrm{inv}}(\mathsf{V}, \epsilon) = \left\{ \mathbf{z} \in \mathbb{R}^d : \mathbb{P}\left[\mathbf{Z} \le \mathbf{z}\right] \ge 1 - \epsilon \right\}$$

  where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathsf{V})$

- $d = 1$: $Q_{\mathrm{inv}}(v, \epsilon) = \sqrt{v}\{x : x \ge Q^{-1}(\epsilon)\}$

Capacity vector and Dispersion matrix

- Capacity vector and dispersion matrix

$$\mathbf{C}(P_1, P_2) = \begin{bmatrix} C(P_1) \\ C(P_2) \\ C(P_{\langle [2] \rangle}) \end{bmatrix}$$

$$\mathsf{V}(P_1, P_2) = \begin{bmatrix} V(P_1) & V_{1,2}(P_1, P_2) & V_{1,12}(P_1, P_2) \\ V_{1,2}(P_1, P_2) & V(P_2) & V_{2,12}(P_1, P_2) \\ V_{1,12}(P_1, P_2) & V_{2,12}(P_1, P_2) & V(P_{\langle [2] \rangle}) + V_{12}(P_1, P_2) \end{bmatrix}$$

where $V(P)$ is the dispersion and

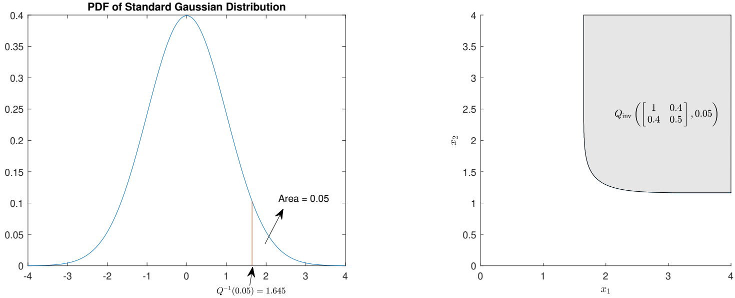
$$V_{1,2}(P_1, P_2) = \frac{1}{2} \frac{P_1 P_2}{(1 + P_1)(1 + P_2)},$$

$$V_{i,12}(P_1, P_2) = \frac{1}{2} \frac{P_i (2 + P_{\langle [2] \rangle})}{(1 + P_i)(1 + P_{\langle [2] \rangle})}, \quad i \in \{1, 2\},$$

$$V_{12}(P_1, P_2) = \frac{P_1 P_2}{(1 + P_{\langle [2] \rangle})^2}.$$

- The capacity vector and the dispersion matrix are the mean and covariance matrix of the mutual information random vector, respectively.

How does $Q_{\text{inv}}(\mathsf{V}, \epsilon)$ look like?



Comparison with the literature

- Our third-order term improves!

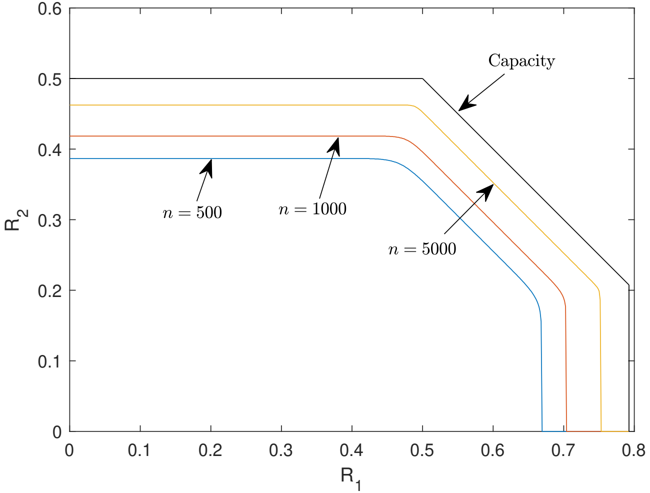
$$n\mathbf{C}(P_1, P_2) - \sqrt{n}Q_{\mathrm{inv}}(\mathsf{V}(P_1, P_2), \epsilon) + \frac{1}{2}\log n\mathbf{1} + O(1)\mathbf{1}$$

$> O\left(n^{1/4}\right)\mathbf{1}$ [MolavianJazi-Laneman 15']
$> O\left(n^{1/4}\log n\right)\mathbf{1}$ [Scarlett et al. 15']

- Proof techniques:
  - **Our bound**: Spherical inputs + Maximum-likelihood decoder
  - [MolavianJazi-Laneman 15'] : Spherical inputs + threshold decoder
  - [Scarlett et al. 15'] : Constant composition codes + Quantization of inputs

Example

Achievable region for $P_1 = 2, P_2 = 1$ and $\epsilon = 10^{-3}$. $R_i = \frac{\log_2 M_i}{n}$ are the rates in bits/channel use.



Some notation

1. $$\mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^n : \text{channel inputs}$$
$$\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathsf{I}_n) : \text{Gaussian noise}$$
$$\mathbf{Y}_2 = \mathbf{X}_{\langle[2]\rangle} + \mathbf{Z} : \text{channel output}$$

2. $$\imath_1(\mathbf{x}_1; \mathbf{y}_2|\mathbf{x}_2) = \log \frac{P_{\mathbf{Y}_2|\mathbf{X}_1\mathbf{X}_2}(\mathbf{y}|\mathbf{x}_1, \mathbf{x}_2)}{P_{\mathbf{Y}_2|\mathbf{X}_2}(\mathbf{y}|\mathbf{x}_2)},$$
$$\imath_2(\mathbf{x}_2; \mathbf{y}_2|\mathbf{x}_1) = \log \frac{P_{\mathbf{Y}_2|\mathbf{X}_1\mathbf{X}_2}(\mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2)}{P_{\mathbf{Y}_2|\mathbf{X}_1}(\mathbf{y}_2|\mathbf{x}_1)},$$
$$\imath_{1,2}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}_2) = \log \frac{P_{\mathbf{Y}_2|\mathbf{X}_1\mathbf{X}_2}(\mathbf{y}_2|\mathbf{x}_1, \mathbf{x}_2)}{P_{\mathbf{Y}_2}(\mathbf{y}_2)},$$

3. $$\boldsymbol{\imath}_2 = \begin{bmatrix} \imath_1(\mathbf{X}_1; \mathbf{Y}_2|\mathbf{X}_2) \\ \imath_2(\mathbf{X}_2; \mathbf{Y}_2|\mathbf{X}_1) \\ \imath_{1,2}(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}_2) \end{bmatrix}$$

Encoding and decoding

- **Encoding**: independently generate M_i codewords uniformly distributed over n-dimensional sphere with radius $\sqrt{nP_i}$

$$\mathbf{X}_i \sim \mathrm{Unif}(\mathbb{S}^{n-1}(\sqrt{nP_i})), \quad i = 1, 2, \quad \mathbf{X}_1, \mathbf{X}_2 \text{ are independent}$$

  [Shannon 49'] also used spherical inputs to bound error exponent of the point to point Gaussian channel.

- **Decoding**: Decode message pair (m_1, m_2) with the maximum mutual information density $\imath_{1,2}(\mathsf{f}_1(m_1), \mathsf{f}_2(m_2); y^n)$ (Ties occur with probability $0 \to$ disregard) – ML Decoder
- ML decoding under the Gaussian noise is equivalent to minimum distance decoding.

Main Tool: Random-Coding Union (RCU) Bound

Using the maximum likelihood decoder, for a general MAC:

Theorem (RCU bound)

For arbitrary input distributions P_{X_1} and P_{X_2}, there exists a (M_1, M_2, ϵ)-MAC code such that

$$\epsilon \le \mathbb{E}\Big[\min\Big\{1, (M_1-1)\mathbb{P}\left[\imath_1(\bar{X}_1; Y_2|X_2) \ge \imath_1(X_1; Y_2|X_2) \mid X_1, X_2, Y_2\right]$$
$$+ (M_2-1)\mathbb{P}\left[\imath_2(\bar{X}_2; Y_2|X_1) \ge \imath_2(X_2; Y_2|X_1) \mid X_1, X_2, Y_2\right]$$
$$+ (M_1-1)(M_2-1)\mathbb{P}\left[\imath_{1,2}(\bar{X}_1, \bar{X}_2; Y_2) \ge \imath_{1,2}(X_1, X_2; Y_2) \mid X_1, X_2, Y_2\right]\Big\}\Big],$$

where $P_{X_1,\bar{X}_1,X_2,\bar{X}_2,Y_2}(x_1, \bar{x}_1, x_2, \bar{x}_2, y) = P_{X_1}(x_1)P_{X_1}(\bar{x}_1)P_{X_2}(x_2)P_{X_2}(\bar{x}_2)P_{Y_2|X_1X_2}(y|x_1, x_2)$.

Proof of RCU bound

Draw M_1 and M_2 i.i.d. distributed codewords with distribution P_{X_1} and P_{X_2}, respectively. Decode the message pair with the maximum $\imath_{1,2}(X_1(m_1), X_2(m_2); Y_2)$. We bound the probability of error as

$$
\begin{aligned}
\epsilon &\le \mathbb{P}\left[\bigcup_{(j,k)\neq(1,1)} \{\imath_{1,2}(X_1(j), X_2(k); Y_2) \ge \imath_{1,2}(X_1(1), X_2(1); Y_2)\}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\bigcup_{(j,k)\neq(1,1)} \{\imath_{1,2}(X_1(j), X_2(k); Y_2) \ge \imath_{1,2}(X_1(1), X_2(1); Y_2)\} \,\middle|\, X_1(1), X_2(1), Y_2\right]\right] \\
&\le \mathbb{E}\Big[\min\Big\{1, (M_1-1)\Big[\imath_{1,2}(\bar{X}_1, X_2; Y_2) \ge \imath_{1,2}(X_1, X_2; Y_2) \mid X_1, X_2, Y_2\Big] \\
&\quad +(M_2-1)\mathbb{P}\Big[\imath_{1,2}(X_1, \bar{X}_2; Y_2) \ge \imath_{1,2}(X_1, X_2; Y_2) \mid X_1, X_2, Y_2\Big] \\
&\quad +(M_1-1)(M_2-1)\mathbb{P}\Big[\imath_{1,2}(\bar{X}_1, \bar{X}_2; Y_2) \ge \imath_{1,2}(X_1, X_2; Y_2) \mid X_1, X_2, Y_2\Big]\Big\}\Big],
\end{aligned}
$$

Proof Sketch

- Show that (modified from the P2P case [Tan-Tomamichel 15'])

$$g_{1,2}(t;\mathbf{y}) = \mathbb{P}\left[\imath_{1,2}(\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2; \mathbf{Y}_2) \geq t \mid \mathbf{Y}_2 = \mathbf{y}\right] \leq \frac{G_{1,2}\exp\{-t\}}{\sqrt{n}}$$

  for typical values of $\mathbf{y}$
  $(\frac{1}{n}\|\mathbf{y}\|^2 \in [1 + P_1 + P_2 - n^{-1/3}, 1 + P_1 + P_2 + n^{-1/3}])$.
- Define the event

$$\mathcal{A} = \left\{ \boldsymbol{\imath}_2 \geq \log \begin{bmatrix} M_1 G_1 \\ M_2 G_2 \\ M_1 M_2 G_{1,2} \end{bmatrix} - \frac{1}{2}\log n\mathbf{1} \right\}$$

- Applying the RCU bound gives

Proof Sketch (RCU bound)

$$
\begin{aligned}
&\mathbb{E}\Big[\min\Big\{1, (M_1-1)\mathbb{P}\left[\imath_1(\bar{\mathbf{X}}_1;\mathbf{Y}_2|\mathbf{X}_2) \ge \imath_1(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2) \mid \mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}_2\right] \\
&\quad +(M_2-1)\mathbb{P}\left[\imath_2(\bar{\mathbf{X}}_2;\mathbf{Y}_2|\mathbf{X}_1) \ge \imath_2(\mathbf{X}_2;\mathbf{Y}_2|\mathbf{X}_1) \mid \mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}_2\right] \\
&\quad +(M_1-1)(M_2-1)\mathbb{P}\left[\imath_{1,2}(\bar{\mathbf{X}}_1,\bar{\mathbf{X}}_2;\mathbf{Y}_2) \ge \imath_{1,2}(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}_2) \mid \mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}_2\right]\Big\}\Big] \\
&= \mathbb{E}\Big[\min\Big\{1,(M_1-1)g_1+(M_2-1)g_2+(M_1-1)(M_2-1)g_{1,2}\Big\}1\left\{\mathcal{A}^c\cup\mathcal{E}^c\right\}\Big] \\
&\quad +\mathbb{E}\Big[\min\Big\{1,(M_1-1)g_1+(M_2-1)g_2+(M_1-1)(M_2-1)g_{1,2}\Big\}1\left\{\mathcal{A}\cap\mathcal{E}\right\}\Big] \\
&\le \mathbb{P}\left[\mathcal{A}^c\cup\mathcal{E}^c\right]+\mathbb{P}\left[\mathcal{E}(\{1\})\right]\, M_1\,\mathbb{E}\left[g_1 1\left\{\imath_1(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2)\ge\log\frac{M_1G_1}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{1\})\right] \\
&\quad +\mathbb{P}\left[\mathcal{E}(\{2\})\right]\, M_2\,\mathbb{E}\left[g_2 1\left\{\imath_2(\mathbf{X}_2;\mathbf{Y}_2|\mathbf{X}_1)\ge\log\frac{M_2G_2}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{2\})\right] \\
&\quad +\mathbb{P}\left[\mathcal{E}(\{1,2\})\right]\, M_1M_2\,\mathbb{E}\left[g_{1,2} 1\left\{\imath_{1,2}(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}_2)\ge\log\frac{M_1M_2G_{1,2}}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{1,2\})\right] \\
&\le \mathbb{P}\left[\mathcal{A}^c\cup\mathcal{E}^c\right]+\frac{M_1G_1}{\sqrt{n}}\mathbb{E}\left[\exp\{-\imath_1(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2)\}1\left\{\imath_1(\mathbf{X}_1;\mathbf{Y}_2|\mathbf{X}_2)\ge\log\frac{M_1G_1}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{1\})\right] \\
&\quad +\frac{M_2G_2}{\sqrt{n}}\mathbb{E}\left[\exp\{-\imath_2(\mathbf{X}_2;\mathbf{Y}_2|\mathbf{X}_1)\}1\left\{\imath_2(\mathbf{X}_2;\mathbf{Y}_2|\mathbf{X}_1)\ge\log\frac{M_2G_2}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{2\})\right] \\
&\quad +\frac{M_1M_2G_{1,2}}{\sqrt{n}}\mathbb{E}\left[\exp\{-\imath_{1,2}(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}_2)\}1\left\{\imath_{1,2}(\mathbf{X}_1,\mathbf{X}_2;\mathbf{Y}_2)\ge\log\frac{M_1M_2G_{1,2}}{\sqrt{n}}\right\}\,\middle|\,\mathcal{E}(\{1,2\})\right] \\
&\le \mathbb{P}\left[\mathcal{A}^c\right]+\mathbb{P}\left[\mathcal{E}^c\right]+\frac{G_1+G_2+G_{1,2}}{\sqrt{n}} \\
&\le \mathbb{P}\left[\mathcal{A}^c\right]+\exp\left\{-c_2n^{1/3}\right\}+\frac{G_1+G_2+G_{1,2}}{\sqrt{n}}
\end{aligned}
$$

Proof Sketch (Tricky part)

- To bound $\mathbb{P}[\mathcal{A}^c]$, we cannot directly use Berry-Esséen theorem since
  1. $\imath_2$ is not a sum.
  2. $\mathbf{X}_1, \mathbf{X}_2$ are not i.i.d.
- Solution:
  - Define the modified mutual info. r.v. $\tilde{\imath}_2$, where the input distributions $P_{\tilde{\mathbf{X}}_1}$ and $P_{\tilde{\mathbf{X}}_2}$ are Gaussian with variance P_1 and P_2 $\Rightarrow$ $\tilde{\imath}_2$ is a sum.
  - Conditional dist. $\tilde{\imath}_2 | \langle X_1^n, X_2^n \rangle = q$ is sum of independent, but not identical r.v.s
  - Apply Berry Esséen Theorem after conditioning on the inner product $\langle X_1^n, X_2^n \rangle$
  - Use [Stam 82'] to approximate the normalized inner product by a standard Gaussian r.v.

$$Q = \frac{\langle X_1^n, X_2^n \rangle}{\sqrt{nP_1P_2}} = \sqrt{n}\cos(\angle(X_1^n, X_2^n)) \to \mathcal{N}(0,1)$$

$$\Longrightarrow X_1^n \text{ and } X_2^n \text{ almost orthogonal in high dimensions}$$

Specifically, we show

$$\mathrm{TV}(P_Q, \mathcal{N}(0,1)) \leq \frac{4}{n}.$$

Extension to K-transmitter ($P_i = P,\ M_i = M\ \forall i \in [K]$)

Corollary

For any $\epsilon \in (0,1)$, and $P > 0$, an $(n, M\mathbf{1}, \epsilon, P\mathbf{1})$-MAC code for the K-transmitter Gaussian MAC exists provided that

$$K \log M \leq nC(KP) - \sqrt{n(V(KP) + V_{\mathrm{cr}}(K,P))}Q^{-1}(\epsilon) + \frac{1}{2}\log n + O(1).$$

$C(\cdot)$ and $V(\cdot)$ are the capacity and the dispersion functions, respectively, and $V_{\mathrm{cr}}(K,P)$ is the cross dispersion term

$$V_{\mathrm{cr}}(K,P) = \frac{K(K-1)P^2}{2(1+KP)^2}.$$

Critical step in the extension

Let $Q_{ij} = \sqrt{n}\langle \mathbf{X}_i, \mathbf{X}_j \rangle$ for $1 \le i < j \le r$, and $\mathbf{Q} = (Q_{ij} \colon 1 \le i < j \le r)$. Then

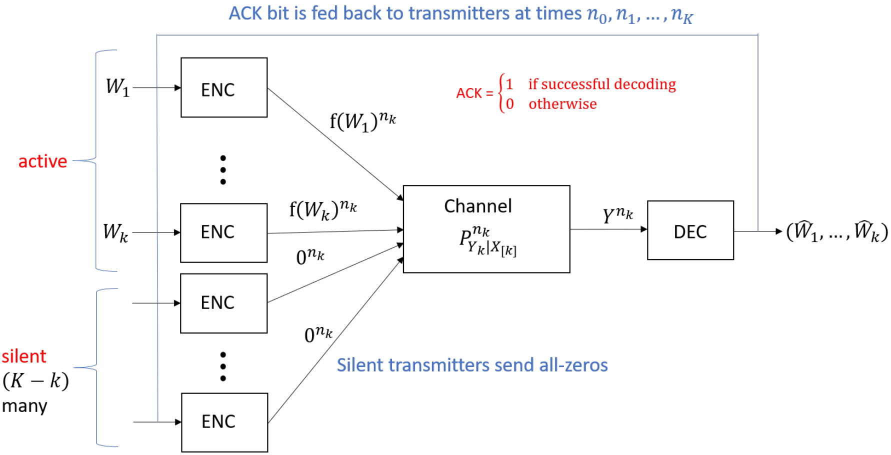
$$\mathrm{TV}\left(P_{\mathbf{Q}}, \mathcal{N}\left(\mathbf{0}, \mathsf{I}_{\binom{r}{2}}\right)\right) \le \frac{C_r}{\sqrt{n}}$$

Gaussian RAC

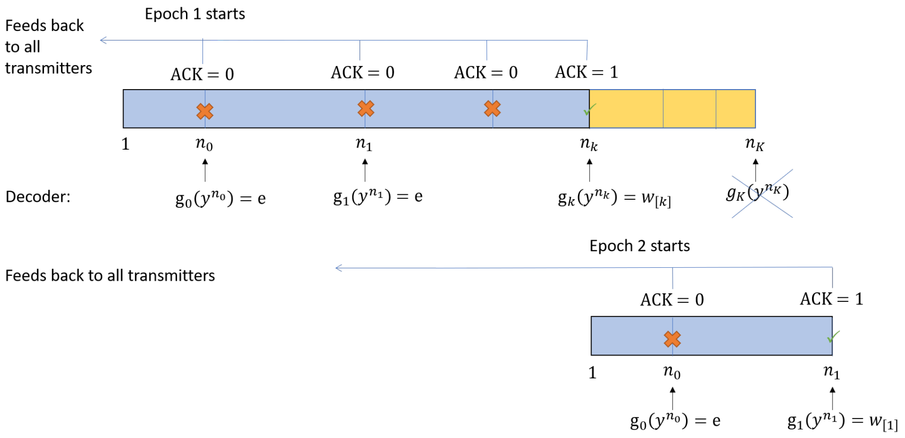
RAC communication setup

- We consider an epoch-based communication.
- There are a total of K transmitters. k of them are active. The set of active transmitters is unknown to the transmitters and the receiver.
- No probability of being active is assigned to the transmitters.
- Decoder only needs to decode the list of messages sent, not who sent which message (no transmitter identity).
- Identical encoding: all transmitters use the same codebook to encode their messages.
- Possible decoding times $n_0, n_1, \ldots, n_K$ are predetermined. At time n_t, decoder decodes t messages.
- When the decoder can decode, it broadcasts a positive ACK bit to all transmitters, thereby ending the current epoch and staring the next. If the decoder cannot decode at n_t, it broadcasts a negative ACK.
- Maximal power constraint: $\|\mathsf{f}(m)^{n_k}\|^2 \leq n_k P$ for all k and m

Problem setup: A New Random Access Strategy



Problem setup: A New Random Access Strategy



RAC Code Definition

Definition

An $\left(\{n_j, \epsilon_j\}_{j=0}^{K}, M, P\right)$-RAC code for with K transmitters consists of a single encoding function $\mathsf{f}\colon \mathcal{U} \times [M] \to \mathbb{R}^{n_K}$ and decoding functions $\mathsf{g}_k\colon \mathcal{U} \times \mathbb{R}^{n_k} \to [M]^k \cup \{\mathsf{e}\}$ for $k = 0, \ldots, K$. The codewords satisfy the maximal-power constraints

$$\left\|\mathsf{f}(u, m)^{[n_j]}\right\|^2 \leq n_j P \text{ for } m \in [M], u \in \mathcal{U}, j \in [K].$$

If k transmitters are active, then the average probability of error in decoding k messages at time n_k is bounded as

$$\frac{1}{M^k} \sum_{m_{[k]} \in [M]^k} \mathbb{P}\Bigg[\Bigg\{ \bigcup_{t<k} \left\{\mathsf{g}_t(U, \mathbf{Y}_k^{[n_t]}) \neq \mathsf{e}\right\} \Bigg\} \bigcup \left\{\mathsf{g}_k(U, \mathbf{Y}_k^{[n_k]}) \overset{\pi}{\neq} m_{[k]}\right\} \Bigg| \mathbf{X}_{[k]}^{[n_k]} = \mathsf{f}(U, m_{[k]})^{[n_k]}\Bigg] \leq \epsilon_k$$

Gaussian RAC - Nonasymptotic bound

Theorem

Fix constants $K < \infty$, $\epsilon_k \in (0,1)$, $\lambda_k > 0$ for $k \in \{0, \ldots, K\}$, $n_0 < n_1 < \cdots < n_K$, $P > 0$, M, and a distribution $P_{\mathbf{X}}$ on $\mathbb{R}^{n_K}$. Then, there exists an $\left(\{n_j, \epsilon_j\}_{j=0}^K, M, P\right)$-RAC code with

$$\epsilon_0 \leq \mathbb{P}\left[\left|\left\|\mathbf{Y}_0^{[n_0]}\right\|^2 - n_0\right| > n_0\lambda_0\right]$$

$$\epsilon_k \leq \frac{k(k-1)}{2M} + \mathbb{P}\left[\bigcup_{i=1}^k \bigcup_{j=1}^k \left\{\left\|\mathbf{X}_i^{[n_j]}\right\|^2 > n_j P\right\}\right]$$

$$+ \mathbb{P}\left[\bigcup_{t=0}^{k-1}\left\{\left|\left\|\mathbf{Y}_k^{[n_t]}\right\|^2 - n_t(1+tP)\right| \leq n_t\lambda_t\right\} \bigcup \left\{\left|\left\|\mathbf{Y}_k^{[n_k]}\right\|^2 - n_k(1+kP)\right| > n_k\lambda_k\right\}\right]$$

$$+ \mathbb{E}\left[\min\left\{1, \sum_{s=1}^k \binom{k}{s}\binom{M-k}{s}\right.\right.$$

$$\left.\left.\mathbb{P}\left[\imath_{[s]}(\bar{\mathbf{X}}_{[s]}^{[n_k]}; \mathbf{Y}_k^{[n_k]} | \mathbf{X}_{[s+1:k]}^{[n_k]}) \geq \imath_{[s]}(\mathbf{X}_{[s]}^{[n_k]}; \mathbf{Y}_k^{[n_k]} | \mathbf{X}_{[s+1:k]}^{[n_k]}) \mid \mathbf{X}_{[k]}^{[n_k]}, \mathbf{Y}_k^{[n_k]}\right]\right\}\right]$$

for all $k \in [K]$, where $\mathbf{X}_{[K]}, \bar{\mathbf{X}}_{[K]}, \mathbf{Y}_k \in \mathbb{R}^{n_K}$ are distributed according to $P_{\mathbf{X}_{[K]}, \bar{\mathbf{X}}_{[K]}, \mathbf{Y}_k}(\mathbf{x}_{[K]}, \bar{\mathbf{x}}_{[K]}, \mathbf{y}_k) = \left(\prod_{j\in[K]} P_{\mathbf{X}}(\mathbf{x}_j) P_{\mathbf{X}}(\bar{\mathbf{x}}_j)\right) P_{\mathbf{Y}_k|\mathbf{X}_{[k]}}(\mathbf{y}_k|\mathbf{x}_{[k]})$.

Proof of the nonasymptotic bound

- The main idea: The number of active transmitters can be distinguished via the output power $\left\|\mathbf{Y}_k^{[n_k]}\right\|^2$ since $\frac{1}{n_k}\mathbb{E}\left[\left\|\mathbf{Y}_k^{[n_k]}\right\|^2\right] = 1 + kP$ is distinct for all $k \in [K]$.
- Decoder: Decode at time n_t only if $\frac{1}{n_t}\left|\left\|\mathbf{y}^{[n_t]}\right\|^2 - (1+tP)\right| < \lambda_t$. If the condition is satisfied, use ML decoding rule to decode t messages. Otherwise skip n_t and broadcast a negative ACK indicating that decoding did not occur.

Gaussian RAC - Main Result

Theorem

For any $\epsilon_k \in (0,1)$ and any P, an $(M, \{(n_k, \epsilon_k)\}_{k=0}^{K}, P)$-code for the AWGN-RAC exists provided that

$$k \log M \le n_k C(kP) - \sqrt{n_k(V(kP) + V_c(k,P))}\,Q^{-1}(\epsilon_k) + \frac{1}{2}\log n_k + O(1)$$

$$n_0 \ge C \log n_1 + o(\log n_1)$$

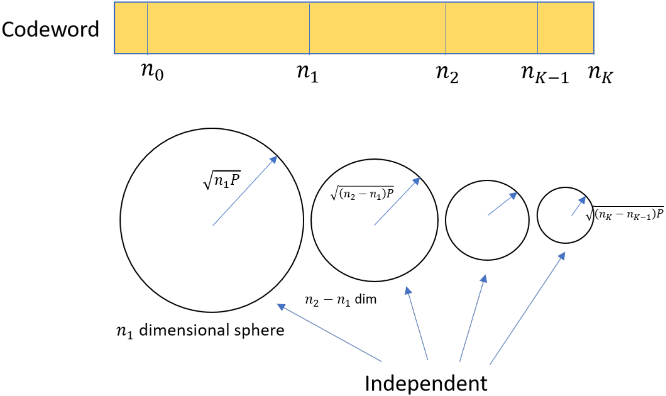
for some constant C where $V_c(k,P) = \frac{k(k-1)P^2}{2(1+kP)^2}$ is the cross dispersion term.

Proof:

- Use concatenation of independent spherical codewords with lengths $n_1,\ n_2 - n_1,\ \ldots,\ n_K - n_{K-1}$.
- Show that the modified input dist. only affects the $O(1)$ term in the bound

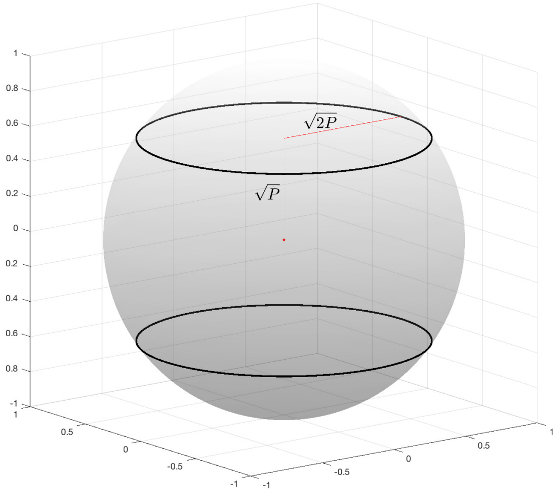
Gaussian RAC - Input distribution

- To satisfy the maximal power constraints for all decoding times, we modify the input distribution as the concatenation of K independent spherical distribution.



Feasible codeword set for Gaussian RAC

- $n_1 = 2, n_2 = 3, P = \frac{1}{3}$



Implications of the main theorem

- We show that for the Gaussian RAC, our proposed rateless code **performs as well in the first-, second-, and third-order terms as the best known communication scheme** when the set of active transmitters is known (Corollary).
- The constant term $O(1)$ is independent of K, but grows with k as $O(2^k\, k^{3/4})$. This term becomes $O(2^{K+k}\, k^{3/4})$ when transmitter identities are decoded.
- Another strategy is to estimate the number of active transmitters in one shot from the received power at time n_0, and to inform the transmitters about the estimate t of the number of active transmitters via a $\lceil \log(K+1) \rceil$-bit feedback at time n_0, so that they can modify their encoding function based on t. Employing this modified coding strategy only affects the $O(1)$ term in the expansion.
- The blocklengths can be written as

$$n_k = n_1 \frac{kC_1}{C_k} + \sqrt{n_1}\left(\frac{1}{C_k}\sqrt{\frac{kC_1 V_k}{C_k}} Q^{-1}(\epsilon_k) - \frac{1}{C_k}\sqrt{kV_1} Q^{-1}(\epsilon_1) \right) + \frac{k-1}{2} \log n_1 + O(1)$$

where $C_k = C(kP)$ and $V_k = V(kP) + V_{\mathrm{cr}}(k, P)$.

Discussions

- The best converse result on the Gaussian MAC has the second order term $O(\sqrt{n \log n})$, not matching the dispersion term $O(\sqrt{n})$ in the achievability direction [Fong-Tan 15'],. This is mostly due to the wringing technique used in the converse. It is difficult to enforce the codebooks from transmitter 1 and 2 to be independent.
- In [Polyanskiy 17'], a different regime is considered, where $K = O(n)$ (no feedback, decoding only occurs at n, the number of transmitters is too large). Unlike our code definition, in that paper, per user probability of error is considered

$$\frac{1}{K}\sum_{i=1}^{K} \mathbb{P}\left[W_j \notin g(\mathbf{Y})\right]$$

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Thanks

Thank you for joining the meeting.