# Gaussian Multiple and Random Access in the Finite Blocklength Regime


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## Background

Wireless communication networks are often accessed through WiFi hotspots and cellular base stations. The number of users employing such access point may be unknown and time varying. We call such a communication environment a random access network.

## Motivation

- State-of-the-art methods for multiple-transmitter channels use orthogonalization (e.g., TDMA, FDMA, CDMA) or collision avoidance (e.g., CSMA, slotted ALOHA). These methods perform poorly when the number of active transmitters is unknown.
- The central aim is to improve achievable rate bounds in low-latency communication scenarios when the number of communicating devices is variable and unknown.

**Our solution**: We employ a rateless code with sporadic feedback to achieve a rate that is identical in the long run to the optimal rate for the multiple access channel in operation. We achieve this performance without either the transmitters or the receiver knowing the number of active transmitters.

## Problem Setup

- A Gaussian channel with $K$ transmitters outputs
$$Y = \sum_{i=1}^{K} X_i + Z,$$
where $Z \sim \mathcal{N}(0,1)$.
- Two problems are considered:
  1. Gaussian Multiple Access Channel (MAC): $K$ is fixed and known.
  2. Gaussian Random Access Channel (RAC): An unknown number $k$ out of $K$ transmitters are active.
- Code constraints:
  1. Average-error
  $$\mathbb{P}\left[(W_1, \dots, W_k) \neq (\hat{W}_1, \dots, \hat{W}_k)\right] \leq \epsilon_k$$
  2. Maximal-power (per-codeword)
  $$\|\mathsf{f}_i(m_i)\|^2 \leq nP_i \quad \forall i \in [K],\ m_i \in [M_i]$$

## Gaussian MAC

Capacity vector:
$$\mathbf{C}(P_1,P_2) = \begin{bmatrix} \frac{1}{2}\log(1+P_1) \\ \frac{1}{2}\log(1+P_2) \\ \frac{1}{2}\log(1+P_1+P_2) \end{bmatrix}$$

$\mathsf{V}(P_1,P_2)$ : $3 \times 3$ dispersion matrix

$Q_{\text{inv}}(\mathsf{V},\epsilon) = \left\{\mathbf{z} \in \mathbb{R}^d : \mathbb{P}\left[\mathbf{Z} \leq \mathbf{z}\right] \geq 1-\epsilon\right\}$, where $\mathbf{Z} \sim \mathcal{N}(\mathbf{0},\mathsf{V})$.

### Gaussian MAC achievability ($K = 2$)

For any $\epsilon \in (0,1)$ and $P_1, P_2 > 0$, there exists an $(n, M_1, M_2, \epsilon, P_1, P_2)$-MAC code for the two-transmitter Gaussian MAC provided that
$$\begin{bmatrix} \log M_1 \\ \log M_2 \\ \log M_1 M_2 \end{bmatrix} \in n\mathbf{C}(P_1,P_2) - \sqrt{n}Q_{\text{inv}}(\mathsf{V}(P_1,P_2),\epsilon) + \frac{1}{2}\log n\mathbf{1} + O(1)\mathbf{1}. \tag{1}$$

### A Code for the MAC

- *Encoder design*: For each transmitter $i \in \{1,2\}$, generate $M_i$ i.i.d. codewords $\mathsf{f}_i(m_i)$ on the $n$-dimensional centered sphere with radius $\sqrt{nP_i}$.
- *Decoder design*: The maximum likelihood decoder (MLD) decodes to message pair $(m_1,m_2)$ if
$$\|\mathbf{Y} - \mathsf{f}_1(m_1) - \mathsf{f}_2(m_2)\| < \|\mathbf{Y} - \mathsf{f}_1(m_1') - \mathsf{f}_2(m_2')\|$$
for all $(m_1', m_2') \neq (m_1, m_2)$.
- Our result extends to the $K$-transmitter MAC.

**Bottomline**: The third-order term $\frac{1}{2}\log n\mathbf{1} + O(1)\mathbf{1}$ improves the best achievable third-order term $O(n^{1/4})\mathbf{1}$ in the literature. Using MLD and random coding union (RCU) bound is essential to achieve $\frac{1}{2}\log n\mathbf{1} + O(1)\mathbf{1}$.

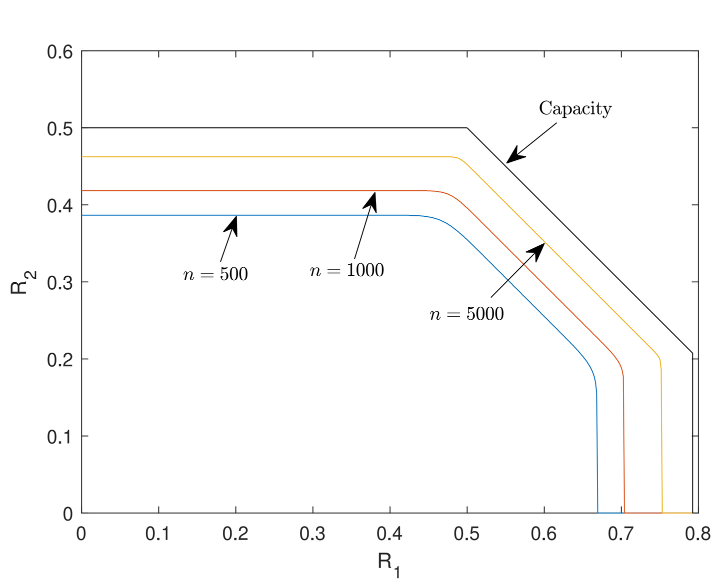


Figure 1: Achievable region for $P_1 = 2, P_2 = 1$ and $\epsilon = 10^{-3}$.

## Gaussian RAC

- Maximal-power constraint for each possible $k \in [K]$:
$$\|\mathsf{f}(m)^{n_k}\|^2 \leq n_k P \text{ for } m \in [M], k \in [K]$$

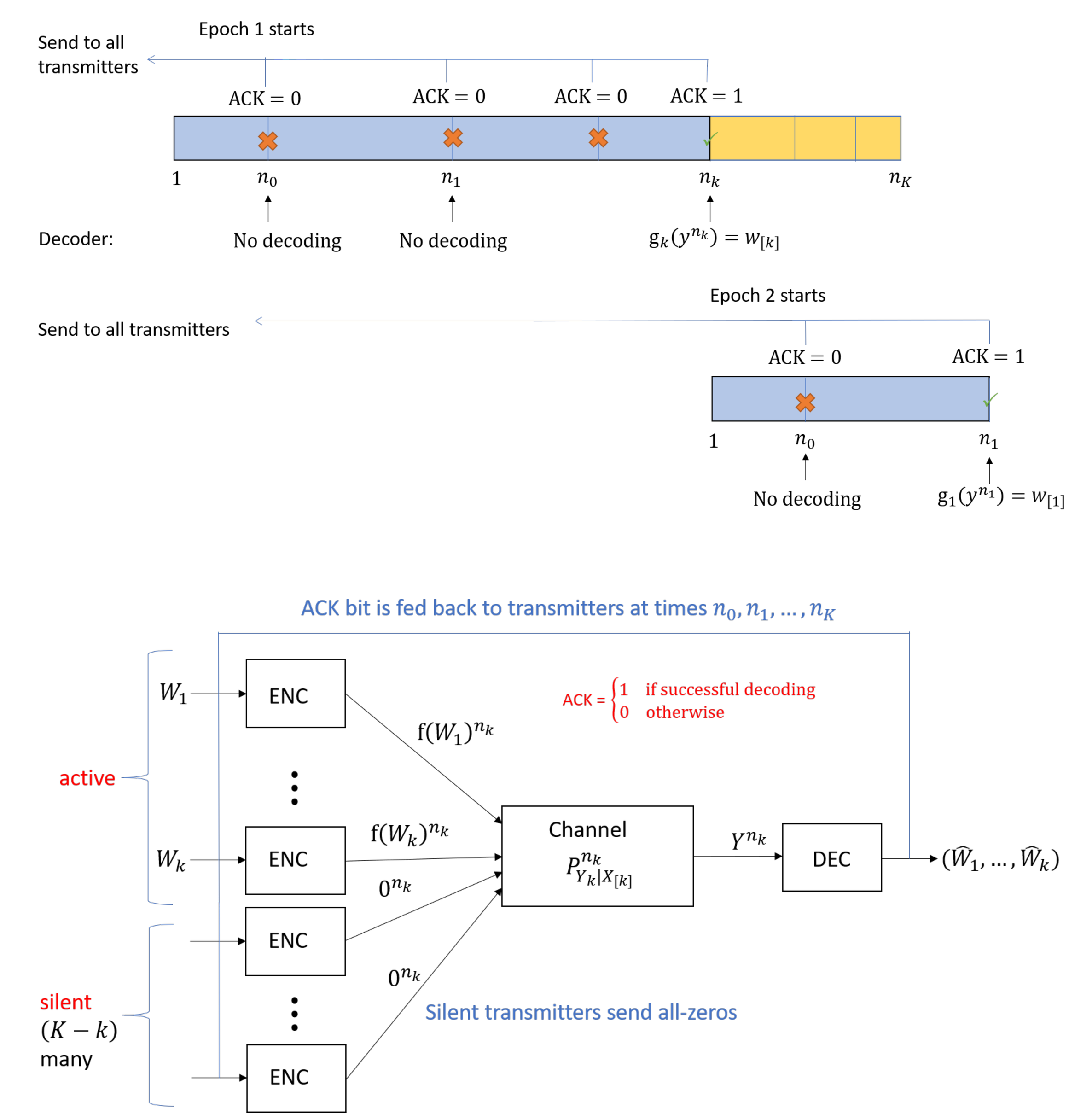


Figure 2: Epoch-based RAC scheme.

### Gaussian RAC Achievability

Fix $K < \infty$, $\epsilon_k \in (0,1)$ for $k \in \{0,1,\dots,K\}$, and $M$. An $(\{n_j,\epsilon_j\}_{j=0}^K, M, P)$-RAC code exists for the Gaussian RAC with a total of $K$ transmitters provided that
$$k\log M \leq n_k C(kP) - \sqrt{n_k(V(kP) + V_{\text{cr}}(k,P))} \cdot Q^{-1}(\epsilon_k) + \frac{1}{2}\log n_k + O(1) \tag{2}$$
for $k \in [K]$, and
$$n_0 \geq c\log n_1 + o(\log n_1)$$
for some constant $c > 0$, where $C(P) = \frac{1}{2}\log(1+P)$, $V(P) = \frac{P(P+2)}{2(1+P)^2}$, and $V_{\text{cr}}(k,P) = \frac{k(k-1)P^2}{2(1+kP)^2}$ are the capacity, dispersion and cross-dispersion terms.

**Bottomline**: Gaussian RAC achieves the same first three order terms as the Gaussian MAC in operation even though the number of transmitters $k$ is unknown to the transmitters and receiver!

### A Rateless Code for the RAC

- *Encoder design*: Use a concatenation of $K$ codewords distributed uniformly on $\{n_1, n_2 - n_1, \dots, n_K - n_{K-1}\}$-dimensional spheres, independent of each other.
- *Decoder design*: At time $n_k$, if the received output power $\frac{1}{n_k}\|Y^{n_k}\|^2$ is in its typical range, decode $k$ messages using MLD; otherwise do not decode and broadcast a negative ACK bit to all transmitters.

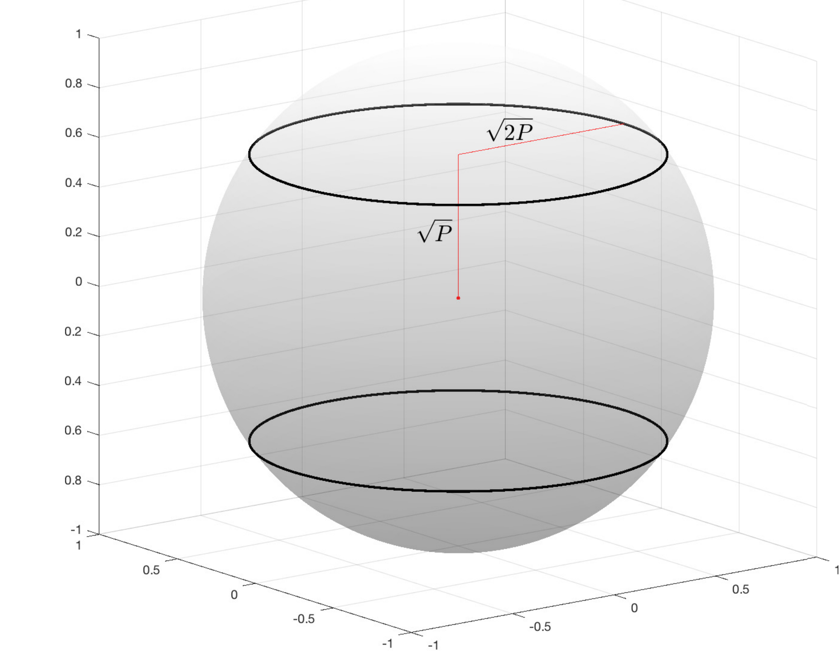


Figure 3: $K = 2$, $n_1 = 2$, $n_2 = 3$ and $P = \frac{1}{3}$. The codewords for the Gaussian RAC are drawn uniformly from the Cartesian product of $\mathbb{S}^{n_1-1}(\sqrt{n_1 P})$ (here a circle with radius $\sqrt{2P}$) and $\mathbb{S}^{n_2-n_1-1}(\sqrt{(n_2-n_1)P})$ (here the set $\{-\sqrt{P}, \sqrt{P}\}$.)

## Summary

- We improve the Gaussian MAC achievability bound.
- We propose a new Gaussian RAC code and bound its performance.
- Our proposed RAC code design with limited feedback performs as well in its first, second, and third-order terms as the best known code for the $k$-transmitter MAC for every $k$.

## Acknowledgments


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## References


[1] R. C. Yavas, V. Kostina, and M. Effros, "Gaussian multiple and random access in the finite blocklength regime," submitted to ISIT 2020.

[2] M. Effros, V. Kostina, and R. C. Yavas, "Random access channel coding in the finite blocklength regime," in *2018 IEEE International Symposium on Information Theory (ISIT)*, June 2018, pp. 1261–1265.