Variable-length Feedback Codes with Several Decoding Times for the Gaussian Channel

Recep Can Yavas

California Institute of Technology

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Joint work with Victoria Kostina and Michelle Effros

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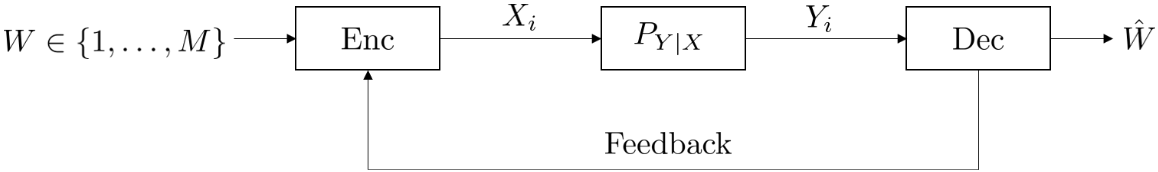
Variable-length stop-feedback (VLSF) codes



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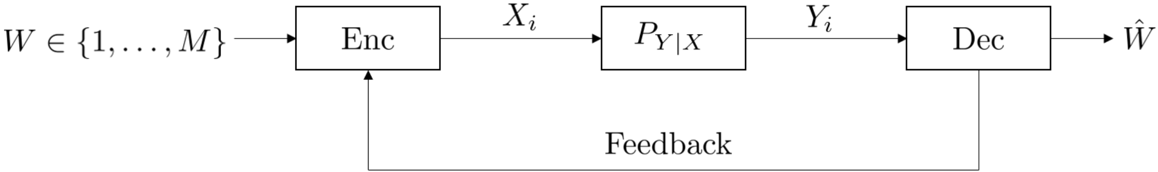

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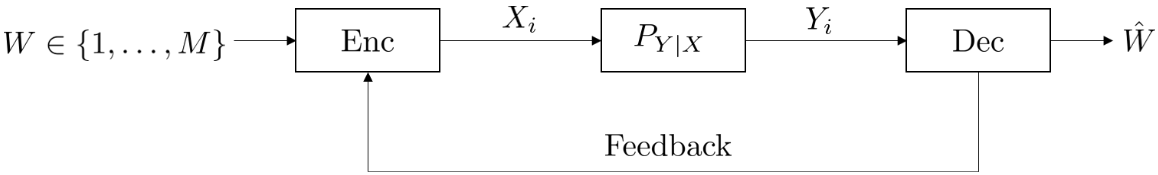

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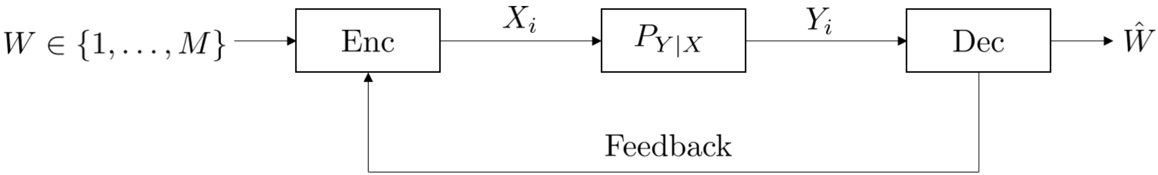

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  $\Longrightarrow$ High reliability
- Feedback at each time instant is impractical!

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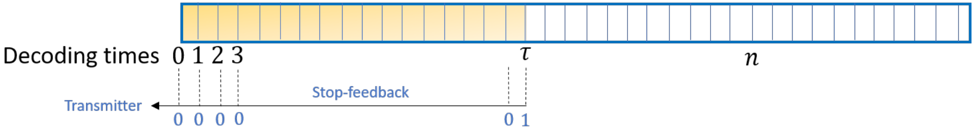

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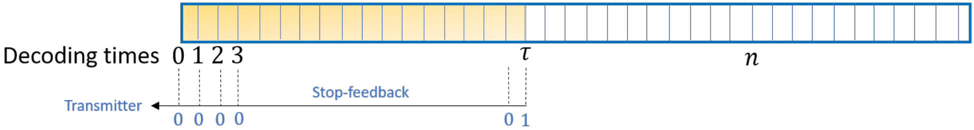

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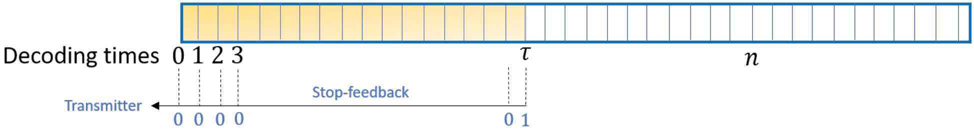

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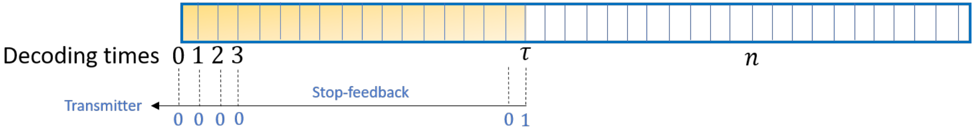

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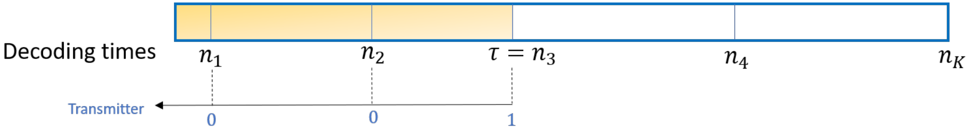

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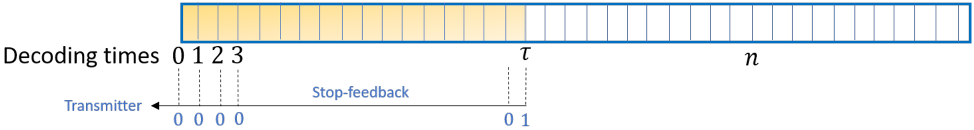

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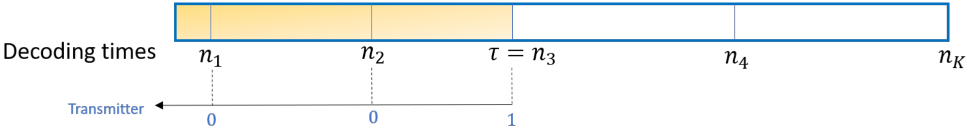

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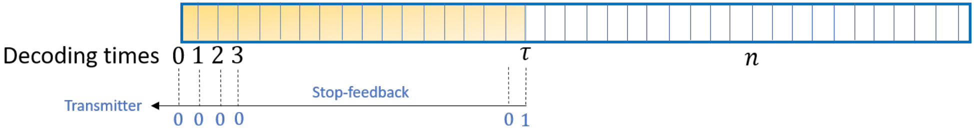

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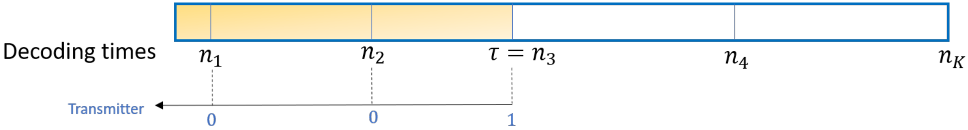

  - Sporadic feedback
  - Practical codes: Incremental redundancy hybrid automatic repeat request codes

Prior work: $K = \infty$

- [Burnashev (1976)]: error exponent $\frac{-\ln P_e}{\mathbb{E}[\tau]}$ for DMCs
- [Polyanskiy et al. (2011)]: VLSF codes for DMCs under non-vanishing error value ϵ

$$\ln M^*(N, 1, \epsilon) = NC - \sqrt{NV} Q^{-1}(\epsilon) + O(\ln N)$$
$$\frac{NC}{1-\epsilon} - \ln N + O(1) \leq \ln M^*(N, \infty, \epsilon) \leq \frac{NC}{1-\epsilon} + O(1)$$

where C = capacity, V = dispersion, $M^*(N, K, \epsilon)$ = maximum achievable message size compatible with average decoding time N, average error probability ϵ and K decoding times

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- Then they search for the optimal $n_1 \implies$ sequential differential optimization
- They do not solve the problem analytically $\implies$ no second-order analysis

Channel model

- Memoryless Gaussian channel: the channel output at time i is

$$
\begin{aligned}
Y_i &= X_i + Z_i \\
Z_i &\sim \mathcal{N}(0,1),
\end{aligned}
$$

where Z_i's are i.i.d. and X_i and Z_i are independent.

Code definition

Definition

An $(N, \{n_i\}_{i=1}^{K}, M, \epsilon, P)$ VLSF code comprises

1. encoding functions $\mathsf{f}_n \colon [M] \to \mathbb{R}$, $n = 1, \ldots, n_K$:

$$\mathsf{f}(m)^{n_K} \triangleq (\mathsf{f}_1(m), \ldots, \mathsf{f}_{n_K}(m))$$

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such that

$$\begin{aligned} \text{Maximal power constraint:} \quad & \|\mathsf{f}(m)^{n_k}\|^2 \leq n_k P \quad \forall\, m \in [M], k \in [K] \\ \text{Average decoding time:} \quad & \mathbb{E}[\tau] \leq N \\ \text{Average error probability:} \quad & \mathbb{P}[\mathsf{g}_\tau(Y^\tau) \neq W] \leq \epsilon \end{aligned}$$

where the message W is uniformly distributed on the set $[M]$.

Main Result

Theorem (Achievability)

Fix $K \geq 2$, $P > 0$ and $\epsilon \in (0, 1)$. For the Gaussian channel

$$\ln M^*(N, K, \epsilon, P) \geq \frac{NC(P)}{1-\epsilon} - \sqrt{N \ln_{(K-1)}(N) \frac{V(P)}{1-\epsilon}} + o(\sqrt{N})$$

The decoding times satisfy $n_1 = 0$ and the equations

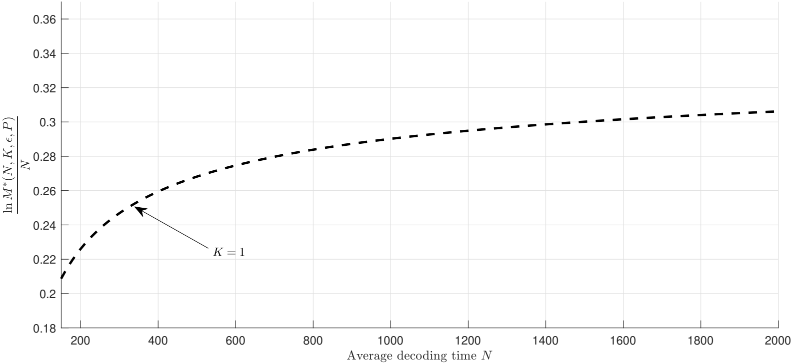
$$\ln M^*(N, K, \epsilon, P) = n_k C(P) - \sqrt{n_k \ln_{(K-k+1)}(n_k) V(P)} - \ln n_k + O(1)$$

for $k \in \{2, \ldots, K\}$.

$C(P) = \frac{1}{2}\ln(1+P) = \text{capacity}$, $V(P) = \frac{P(P+2)}{2(1+P)^2} = \text{dispersion}$

$\ln_{(K)}(\cdot) \triangleq \underbrace{\ln(\ln(\ldots(\ln(\cdot))))}_{K \text{ times}}$

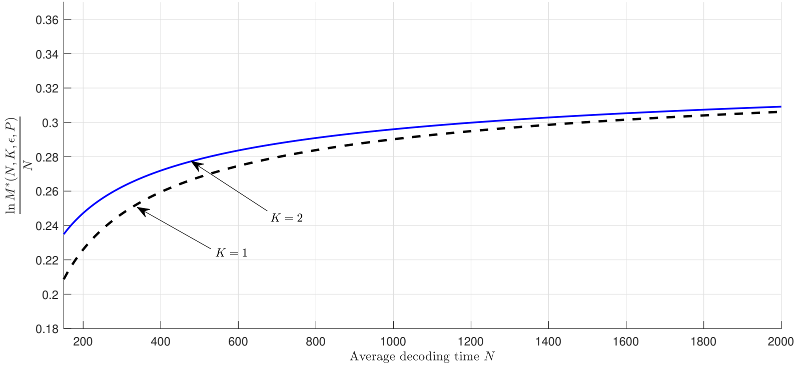
- **Bottom-line:** We derive an achievability bound and optimize the choices of the decoding times $n_1, \ldots, n_K$ to minimize average decoding time N for the given ϵ and M.

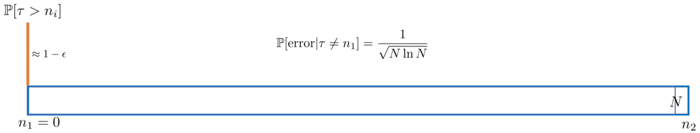


0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
200
400
600
800
1000
1200
1400
1600
1800
2000
$\frac{\ln M^*(N, K, \epsilon, P)}{N}$
Average decoding time N
$K = 1$

$n_1 = N$

Example: $\epsilon = 10^{-3}$, $P = 1$





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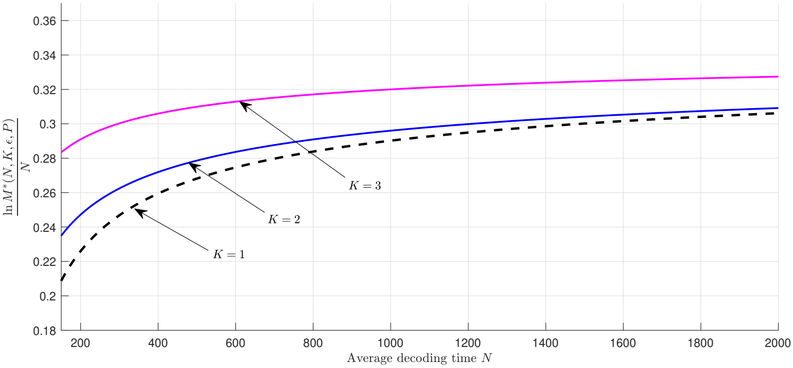

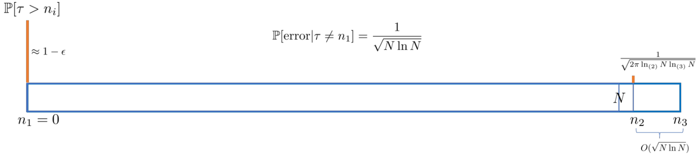

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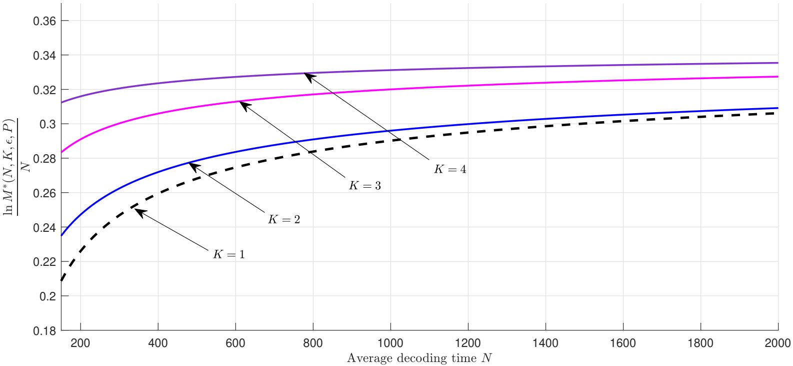

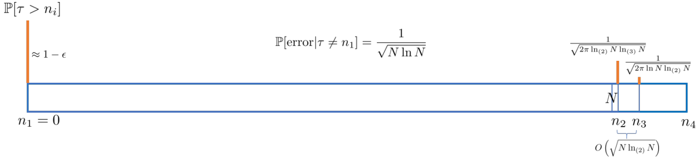

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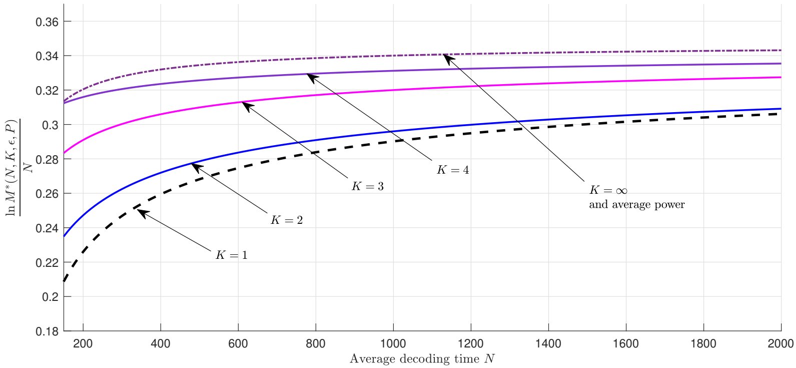

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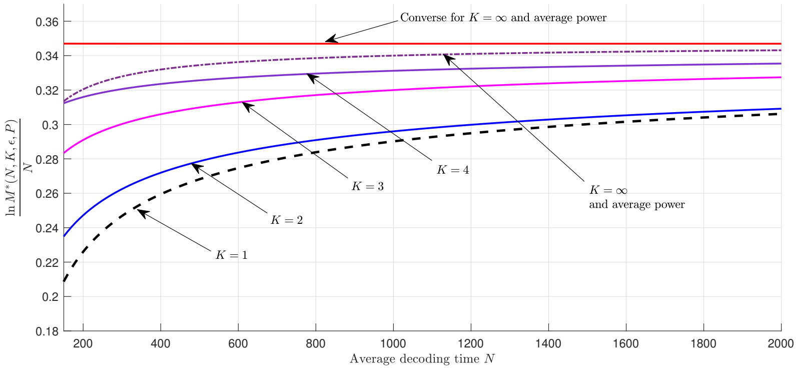

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Comparison with prior work in extreme scenarios

- $2 \leq K < \infty$, maximal power:

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  [Polyanskiy et al. (2010) and Tan-Tomamichel (2015)]

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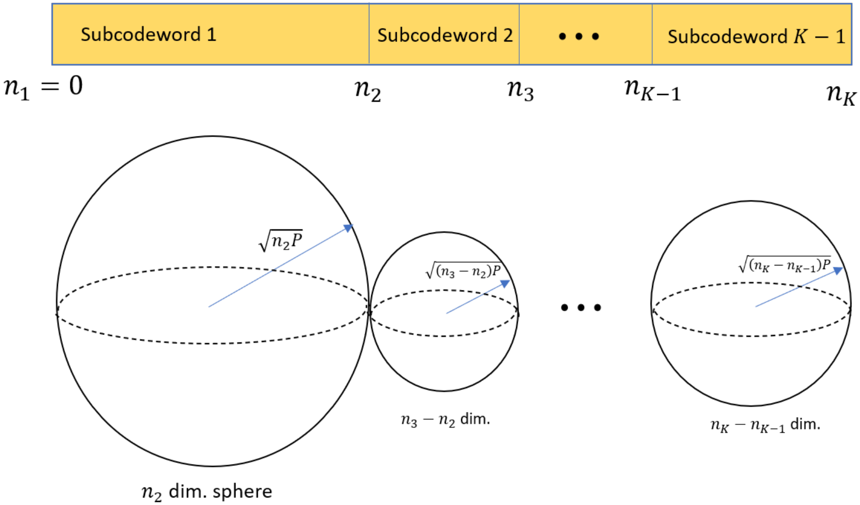
- $K = \infty$, average power:
  [Truong-Tan (2018)]

$$\ln M^*(N, \infty, \epsilon, P)_{\mathrm{ave}} \geq \frac{NC(P)}{1-\epsilon} - \ln N + O(1)$$
$$\ln M^*(N, \infty, \epsilon, P)_{\mathrm{ave}} \leq \frac{NC(P)}{1-\epsilon} + \frac{h_b(\epsilon)}{1-\epsilon}$$

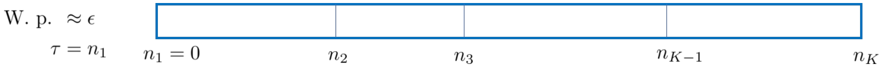
Random encoder design

- We generate M i.i.d. codewords of length-n_K so that maximal power constraint is satisfied with equality for each n_k, and the subcodewords are drawn independent of each other.



Decoder design: Polyanskiy et al. (2011)

- With probability $\approx \epsilon$, do not transmit any symbols and decode to an arbitrary message at time $n_1 = 0$.



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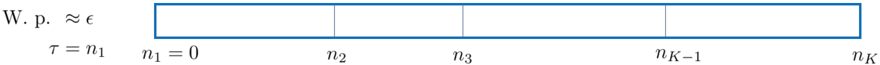
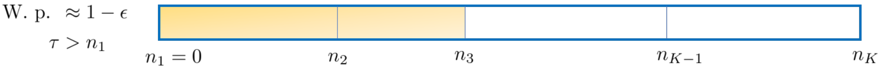


- With probability $\approx 1 - \epsilon$, transmit symbols and use
  Threshold decoder: Decode at the first time $n_k \in \{n_2, \ldots, n_K\}$ s.t. $\imath(\mathsf{f}(m)^{n_k}; Y^{n_k}) \geq \gamma$ for some m.

$$\overbrace{\imath(x^n; y^n)}^{\text{information density}} \triangleq \ln \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{Y^n}(y^n)}$$



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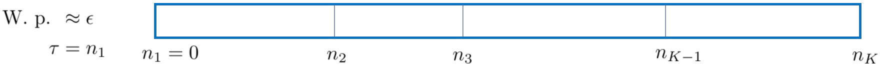

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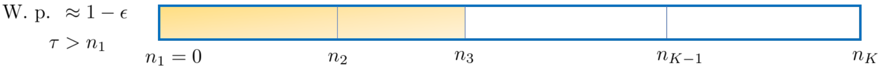

- Goal: optimize $n_2, \ldots, n_K$.

Optimizing decoding times $n_2, \ldots, n_K$ to minimize N

- (N', ϵ_N): average decoding time and error probability given $\tau > n_1$

$$
\begin{aligned}
\min \quad & N(n_2, \ldots, n_K, \gamma) = \frac{N'(1-\epsilon)}{1-\epsilon_N} \\
\text{s.t.} \quad & N' = n_2 + \sum_{i=2}^{K-1} (n_{i+1} - n_i) \mathbb{P}\left[\tau > n_i\right] \\
& \epsilon_N = \mathbb{P}\left[\imath(X^{n_K}; Y^{n_K}) < \gamma\right] + M \exp\{-\gamma\}
\end{aligned}
$$

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$$\epsilon_N = Q\left(\frac{n_K C(P) - \gamma}{\sqrt{n_K V(P)}}\right)(1 + o(1)) + M \exp\{-\gamma\}$$

Using moderate deviations theorem

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- Find the optimal $(n_2^*, \ldots, n_K^*, \gamma^*)$ by solving $\nabla N = 0$.

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- (N', ϵ_N): average decoding time and error probability given $\tau > n_1$

$$\begin{aligned} \min \quad & N(n_2, \ldots, n_K, \gamma) = \frac{N'(1-\epsilon)}{1-\epsilon_N} \\ \text{s.t.} \quad & N' = n_2 + \sum_{i=2}^{K-1} (n_{i+1} - n_i) Q\left(\frac{n_i C(P) - \gamma}{\sqrt{n_i V(P)}}\right)(1 + o(1)) \\ & \epsilon_N = Q\left(\frac{n_K C(P) - \gamma}{\sqrt{n_K V(P)}}\right)(1 + o(1)) + M \exp\{-\gamma\} \end{aligned}$$

- Find the optimal $(n_2^*, \ldots, n_K^*, \gamma^*)$ by solving $\nabla N = 0$.
- The optimal $\epsilon_N^* = \frac{1}{\sqrt{N \ln N}}$.

Conclusion

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Future work

- Improve the converse result for $K < \infty$ and maximal power constraint.

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- Investigate maximal power constraint vs. average power constraint for VLSF codes with $K = \infty$.

$$\frac{NC(P)}{1-\epsilon} - \ln N + O(1) \leq \ln M^*_{\mathrm{ave}}(N, \infty, \epsilon, P) \leq \frac{NC(P)}{1-\epsilon} + O(1)$$

We show for the maximal power constraint:

$$\ln M^*(N, \infty, \epsilon, P) \geq \frac{NC(P)}{1-\epsilon} - O(\sqrt{N})$$

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