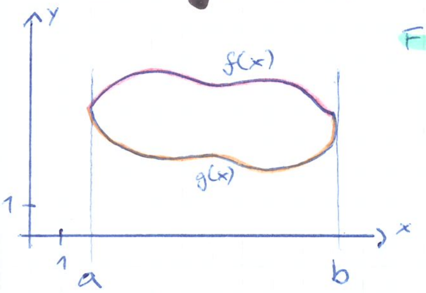
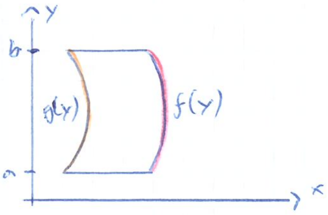
SFPM - Mathe

Mehrfachintegrale

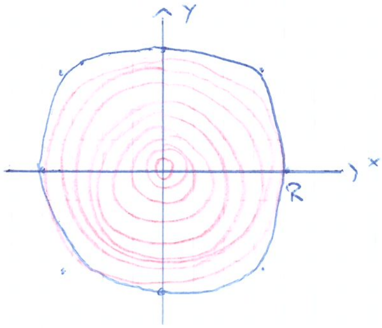


Flächen

$$\int_A dA = \int_A dx\,dy = \int_a^b \int_{g(x)}^{f(x)} dy\,dx$$

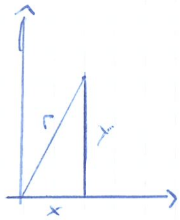


$$\int_a^b \int_{g(y)}^{f(y)} dx\;dy$$



$$\int_0^R \int_0^{2\pi} \underbrace{r}_{} \,d\varphi\,dr$$

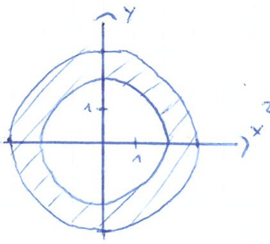
Funktionaldeterminante der Jacobi-Matrix ->



$$J = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\varphi} \\ \frac{dy}{dr} & \frac{dy}{d\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & r\cdot(-\sin\varphi) \\ \sin\varphi & r\cdot\cos\varphi \end{pmatrix}$$

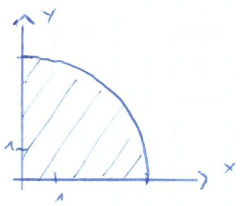
$$\det(J) = r\cdot\cos^2\varphi + r\cdot\sin^2\varphi = r$$



$$\int_2^3 \int_0^{2\pi} r\,d\varphi\,dr$$

$$= [\;\varphi]_0^{2\pi}\cdot\left[\tfrac{1}{2}r^2\right]_2^3$$

$$= 5\pi$$

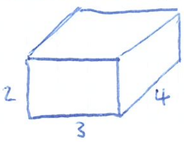


$$\int_0^4 \int_0^{\frac{\pi}{2}} r\,d\varphi\,dr$$

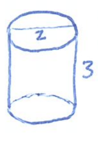
$$= [\varphi]_0^{\frac{\pi}{2}}\;\left[\tfrac{1}{2}r^2\right]_0^4 \quad = 4\pi$$

Kreisflächenformel: $\int_0^{2\pi}\int_0^{r'} r\;d\varphi\,dr = \int_0^{2\pi} \frac{1}{2}r'^2 = r'^2\cdot\pi$

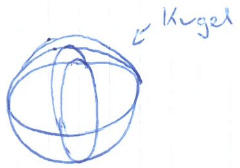
Volumen integrieren: $\int 1\;dV = \iiint dx\;dy\;dz$



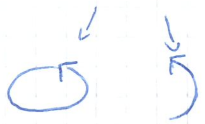
$$V = \int_0^2\int_0^4\int_0^3 dx\,dy\,dz = 2\cdot4\cdot3 = 24$$



$$V = \int_0^3\int_0^1\int_0^{2\pi} r\,d\varphi\,dr\,dz = [z]_0^3\;\left[\tfrac{1}{2}r^2\right]_0^1\;[\varphi]_0^{2\pi} = 3\pi$$



$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \cdot \sin\theta \; dr \; d\theta \; d\varphi$$



$$J = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{d\varphi} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{d\varphi} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{d\varphi} \end{pmatrix} \qquad \det(J) = r^2 \cdot \sin\theta$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{1} r^2 \cdot \sin\theta \; dr \; d\theta \; d\varphi = \left[\tfrac{1}{3} r^3\right]_0^1 \cdot \left[-\cos\theta\right]_0^{\frac{\pi}{6}} \cdot \left[\varphi\right]_0^{2\pi}$$

$$= 0.089\,\pi$$

Masse: $m = V \cdot \rho$

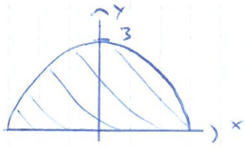
$\rho(r) = 1 - \frac{r}{R}$

$$M = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{R-r}{R} \, r^2 \cdot \sin\theta \; dr \; d\theta \; d\varphi$$

$$= \left[\tfrac{1}{3} r^3 - \tfrac{1}{4}\tfrac{r^4}{R}\right]_0^R \cdot \left[-\cos\theta\right]_0^{\pi} \cdot \left[\varphi\right]_0^{2\pi}$$

$$= \tfrac{1}{3} R^3 \pi$$

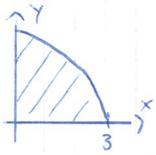
Schwerpunkt



$x_s = 0 \qquad y_s = ?$

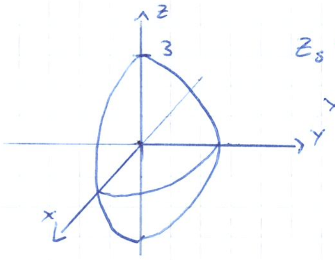
$$y_s = \underbrace{\frac{1}{4.5\pi}}_{\frac{1}{A}} \int_0^3 \int_0^{\pi} \underbrace{r \cdot \sin\varphi}_{y} \cdot r \; d\varphi \; dr$$

$$= \frac{2}{9\pi} \left[\tfrac{1}{3} r^3\right]_0^3 \left[-\cos\varphi\right]_0^{\pi} = \frac{2}{\pi}$$



$x_s = y_s$

$$x_s = \frac{4}{9\pi} \int_0^3 \int_0^{\frac{\pi}{2}} r \cdot \cos\varphi \; r \; d\varphi \; dr = \frac{4}{\pi}$$



$z_s = 0$

$x_s = y_s$

$$x_s = \frac{1}{V} \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^{3} r \cdot \cos\varphi \; \sin\theta \cdot r^2 \cdot \sin\theta \; dr \; d\theta \; d\varphi$$

$V = \frac{1}{4} \cdot \frac{4}{3} \cdot \pi \cdot R^3$

$$= \frac{3}{27\pi} \cdot \left[\tfrac{1}{4} r^4\right]_0^3 \left[\tfrac{1}{2} \cdot ((-\cos\theta) \cdot \sin\theta + \theta)\right]_0^{\pi} \left[\sin\varphi\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9}{8}$$

$x = r \cdot \cos\varphi \cdot \sin\theta$

$y = r \cdot \sin\varphi \cdot \cos\theta$

$z = r \cdot \cos\theta$

$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

Massenträgheitsmoment

$$I_{0,\text{Scheibe}} = \int r^2 \; dm = \int r^2 \frac{m}{A} \; dA = \frac{m}{R^2\pi} \int_0^R \int_0^{2\pi} r^2 \; r \; d\varphi \; dr = \frac{1}{2} m R^2$$

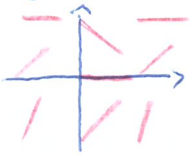
$$I_{z,\text{Zylinder}} = \frac{m}{V} \int r^2 \; dV = \frac{m}{R^2 \cdot \pi \cdot h} \int_0^h \int_0^{2\pi} \int_0^R r^2 \cdot r \cdot dr \; d\varphi \; dz = \frac{1}{2} m R^2$$

$$I_{K,\text{Kugel}} = \frac{m}{\frac{4}{3} R^3 \pi} \int_0^{2\pi} \int_0^{\pi} \int_0^R r^2 \sin^2\theta \; r^2 \sin\theta \; dr \; d\theta \; d\varphi = \frac{2}{5} m R^2$$

Differentialgleichungen

Differentialgleichungen 1. Ordnung:

$y' = x^2 - y$

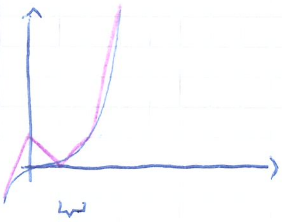


Randbedingung: $y(x') = y'$

Anfangsbedingung: $y(0) = y'$

Isokline: Linien gleicher Steigung

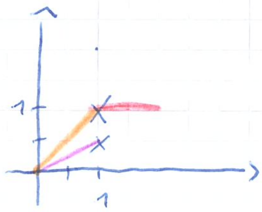
Euler-Methode



Schrittweite

$h = 1$ → h immer kleiner

Heun-Methode



— Steigung wie bei Euler

— 2. Steigung

— Mittelwert aus — und —

gewöhnliche homogene lineare Differentialgleichungen mit konstanten Koeffizienten

↑ eine Variable ↑ $\neq 0$

Bsp $3y^{(3)} - 2y'' + 4y' - 5y = 0$

Ansatz Lösung: $y = ce^{\lambda x}$ → $y' = \lambda ce^{\lambda x}$

$y^{(n)} = \lambda^n ce^{\lambda x}$

charakteristisches Polynom → $3\lambda^3 - 2\lambda^2 + 4\lambda - 5 = 0$

Anleitung:

bei mehreren reellen NST: $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots$

mehrfache (z.B. doppelte): $c_1 e^{\lambda x} + c_2 \cdot x e^{\lambda x} + \dots$

konjugiert komplexe NST: $c_1 e^{ax} \cos(bx) + \dots = c_1 e^{(a+bi)x} + \dots$

Inhomogene gewöhnliche lineare Diff'gleichungen

Form : $ay'' + by' + cy = f(x)$ $\leftarrow$ $y = y_{homogen} + y_{partikulär}$

Bsp. : $8y'' - 2y' - 15y = 3x^2 - 2x + 1$

Ansätze:
$y = ax^2 + bx + c$
$y = a \cdot \sin x + b \cdot \cos x$
$y = a e^x$

$8\lambda^2 - 2\lambda - 15 = 0$

$\lambda_1 = 1.5 \quad \lambda_2 = -1.25$

$y_{hom} = c_1 e^{1.5x} + c_2 e^{-1.25x}$

$y_{part} = ax^2 + bx + c$

$y'_{part} = 2ax + b$

$y''_{part} = 2a$

$8y'' - 2y' - 15y = 16a - 4ax - 2b - 15ax^2 - 15bx - 15c \overset{!}{=} 3x^2 - 2x + 1$

Koeffizientenvergleich: x^2 : $-15a = 3$ $\rightarrow a = -\frac{1}{5}$

x : $-4a - 15b = -2$ $\rightarrow b = \frac{14}{75}$

$=$: $16a - 2b - 15c = 1$ $\rightarrow c \approx -0.3$

$\rightarrow y = \underbrace{c_1 e^{1.5x} + c_2 e^{-1.25x}}_{y_{hom}} + \underbrace{\left(-\frac{1}{5}x^2\right) + \frac{14}{75}x - 0.3}_{y_{part}}$

Bedingte W'keit

Vierfelder-Tafel Bsp. :

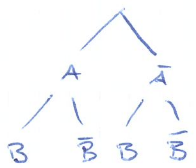
	I	$\bar{I}$	
+	15'968	31'872	47'840
−	32	...	3'952'160
	16'000	...	4'000'000

Debakel $\rightarrow P_+(I) = \frac{15'968}{47'840} = 33.38\%$

$$P(I \mid +) = P_+(I) = \frac{P(+ \cap I)}{P(+)} = \underbrace{\frac{P(I) \cdot P_I(+)}{P(+)}}_{\text{Satz von Bayes}} = \underbrace{\frac{P(I) \cdot P_I(+)}{P(I) \cdot P_I(+) + P(\bar{I}) \cdot P_{\bar{I}}(+)}}_{\text{Satz der totalen W'keit}}$$

Unabhängigkeit von Ereignissen:

$P(A) \cdot P_A(B) = P(A \cap B)$



$P_A(B) = \frac{P(A \cap B)}{P(A)}$

$\downarrow$ wenn unabhängig

$P(B) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A) \cdot P(B)$

Matrizen

Determinante:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + cdh + bfg - ceg - bdi - afh$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \cdot \det\begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \cdot \det\begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} + c \cdot \ldots$$

Vektor-Matrix-Multiplikation: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 6 \\ 3\cdot 5 + 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$

Matrix-Matrix-Multiplikation: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

Inverse: $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{\det(M)} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$

Gleichung: $\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{M} X = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$

$$M^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & \frac{3}{2} \end{pmatrix}$$

Eigenwerte /vektoren:

$$A \cdot \vec{v} = \lambda \vec{v}$$

$\det(A - \lambda E_n) \stackrel{!}{=} 0$ ← nicht invertierbar, sonst gibt es $\vec{0}$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc \stackrel{!}{=} 0$$

charakteristisches Polynom

Abbildungen:

Achsenspiegelungen: x-Achse: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ y-Achse: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Drehung: $\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$

zentrische Streckung: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

Drehstreckung am Ursprung: $\begin{pmatrix} k\cos & -k\sin \\ k\sin & k\cos \end{pmatrix}$