MATH 2105– Linear Algebra
MidTerm Exam, Spring Semester 2020
Computer Science and Engineering
University of Dhaka

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Time = 1 Hour 20 Minutes **Total marks is 30**

[Answer all the following questions]

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1. (3 points) Display the following vectors using arrow on an xy-graph: $u, v, -v, 2v, u+2v, u-v$.

2. (2 points) Determine if b is a linear combination of a_1, a_2 and a_3.

$$a_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \; a_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \; a_3 = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} \; b = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

Solution: The question "Is b a linear combination of a_1, a_2 and a_3? is equivalent to the question"

- Does the vector equation $x_1a_1 + x_2a_2 + x_3a_3 = b$ have a solution?
- The equation:

$$x_1. \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + x_2. \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} + x_3. \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -3 & -3 & -1 & -1 \\ 0 & 2 & 5 & 5 \end{bmatrix}$$

- Row reduction form:

$$M \approx \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & -3 & 14 & 5 \\ 0 & 2 & 5 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -19 & -10 \\ 0 & 2 & 5 & 5 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -19 & -10 \\ 0 & 0 & 43 & 25 \end{bmatrix}$$

The linear system corresponding to M has a solution, so the equation has a solution, and therefore b is a linear combination of a_1, a_2 and a_3.

3. (3 points) Using Gauss-Jordan method, find A^-1:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 2 & 01 & \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$$

This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$

4. (3 points) Compute L and U for the symmetric matrix A.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

Solution :

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ & b-a & b-a & b-a \\ & & c-b & c-b \\ & & & d-c \end{bmatrix}$$

5. (2 points) Find the rank of the following matrix.

$$X = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution: $Rank(X) = 3$

6. (2 points) Which of the following subsets of R^3 are actually subspaces?

   1. The plane of vectors (c_1, c_2, c_3) with $c_1 = c_2$
   2. All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$

Solution: Both are subspaces.

7. (2 points) Determine if the sets of vectors are linearly independent or linearly dependent.

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ -6 \\ 4 \end{bmatrix} \right\}$$

Solution : Put theses vectors into a matrix as columns and with row-reducing we obtain,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

With $n = 3$ (3 vectors, 3 columns) and $r = 3$ (3 pivot columns) we have $n = r$ and the theorem says the vectors are linearly independent.

8. (3 points) Find the null space $N(B)$ of the matrix B.

$$X = \begin{bmatrix} -6 & 4 & -36 & 6 \\ 2 & -1 & 10 & -1 \\ -3 & 2 & -18 & 3 \end{bmatrix}$$

Solution :

$$\begin{bmatrix} -6 & 4 & -36 & 6 & 0 \\ 2 & -1 & 10 & -1 & 0 \\ -3 & 2 & -18 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 free variables: x_3 and x_4, which gives:

$$N(B) = \begin{bmatrix} -2x_3 - x_4 \\ 6x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

, where $\{x_3, x_4 \in C\}$

9. (3 points) Find the solution set of the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 8x_2 - 7x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution: The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 2 \\ 1 & -8 & -7 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For this system, we have $n = 3$ and $r = 3$. However, with a pivot column in the last column we see that the original system has no solution.

10. (2 points) Find a basis (and the dimension) for the following subspace of 3 by 3 matrices: " All diagonal matrices".
Solution : Basis are :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11. (3 points) Find the RREF R of a 3 by 4 matrix with $a_{ij} = i + j - 1$.
Solution:

$$R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

12. (2 points) Find True or False :

    1. A square matrix has no free variables. **F**
    2. An m by n matrix has no more tan n pivot variables. **T**
    3. If $C(A)$ contains only the zero vector, then A is the zero matrix. **T**
    4. The column space of $2A$ equals the column space of A. **T**