We are given

$$\frac{dB}{dt} - \alpha B = -r \quad (1)$$

$$B(0) = B_0 \quad (2)$$

$$B(T) = 0 \quad (3)$$

Solve (1) for $B(t)$:

$$\frac{dB}{dt} - \alpha B = -r$$

$$\Rightarrow \frac{dB}{dt} = \alpha B - r$$

$$\Rightarrow \int \frac{dB}{\alpha B - r} = \int dt$$

$$\Rightarrow \frac{1}{\alpha} \ln |\alpha B - r| = t + c, \quad c \in \mathbb{R}$$

$$\Rightarrow \ln |\alpha B - r| = \alpha t + c$$

$$\Rightarrow \alpha B - r = e^{\alpha t + c}$$

$e^c \to C$

$$\Rightarrow \alpha B = Ce^{\alpha t} + r$$

$$\Rightarrow B(t) = \frac{1}{\alpha} Ce^{\alpha t} + \frac{r}{\alpha}$$

At $t=0$: $B(0) = \frac{c}{\alpha} + \frac{r}{\alpha} \Rightarrow B_0 - \frac{r}{\alpha} = \frac{c}{\alpha}$

$\Rightarrow c = B_0 \alpha - r$

$$\therefore \quad B(t) = \left(B_0 - \frac{r}{\alpha}\right)e^{\alpha t} + \frac{r}{\alpha}$$

To find r:

$$B(t) = B_0 e^{\alpha t} - \frac{r}{\alpha} e^{\alpha t} + \frac{r}{\alpha}$$

$$\Rightarrow B(t) - B_0 e^{\alpha t} = \frac{r}{\alpha}\left(1 - e^{\alpha t}\right)$$

$$\Rightarrow r = \frac{\alpha\left(B(t) - B_0 e^{\alpha t}\right)}{1 - e^{\alpha t}}$$

and $B(T) = 0$

$$\therefore \quad r = \frac{\alpha\left(B(T) - B_0 e^{\alpha T}\right)}{1 - e^{\alpha T}}$$

$$\Rightarrow r = \frac{\alpha B_0 e^{\alpha T}}{e^{\alpha T} - 1}$$

$$\text{or} \quad r = \frac{\alpha B_0}{1 - e^{-\alpha T}}$$

$\therefore$ we can simplify $B(t)$ to

$$B(t) = B_0 e^{\alpha t} + \frac{B_0 e^{\alpha T}}{e^{\alpha T} - 1}\left(1 - e^{\alpha t}\right).$$