For a variable interest rate we now have

$$\alpha \to \alpha(t)$$
$$r \to r(t)$$

and hence

$$\frac{dB}{dt} - \alpha(t) B = -r(t)$$

We will solve by the integrating factor method:

Multiply by $\mu(t)$

$$\mu(t) \frac{dB}{dt} - \mu(t)\alpha(t) B = -\mu(t) r(t)$$

Choose $\mu(t)$ s.t. $\frac{d\mu}{dt} = \mu(t)(-\alpha(t))$ (Δ)

$$\Rightarrow \int_1^{\mu} \frac{d\nu}{\nu} = \int_0^t -\alpha(v)\,dv$$

$$\Rightarrow \ln|\mu(t)| = -\int_0^t \alpha(v)\,dv$$

$$\Rightarrow \mu(t) = e^{-\int_0^t \alpha(v)\,dv}$$

$\therefore$ taking definite integrals on (Δ)

$$\int_0^t \frac{d}{ds}[\mu(s) B]\,ds = \int_0^t -\mu(s) r(s)\,ds$$

$$\Rightarrow [\mu(s) B(s)]_0^t = -\int_0^t \mu(s) r(s)\,ds$$

$$\Rightarrow \mu(t)B(t) - \mu(0)B_0 = -\int_0^t \mu(s)\, r(s)\, ds$$

$$\Rightarrow B(t)e^{-\int_0^t \alpha(v)dv} - \underbrace{e^{-\overbrace{\int_0^0 \alpha(v)dv}^{=0}}}_{=1} B_0 = -\int_0^t e^{-\int_0^s \alpha(v)dv} r(s)\, ds$$

$$\Rightarrow B(t)e^{-\int_0^t \alpha(v)dv} = B_0 - \int_0^t e^{-\int_0^s \alpha(v)dv} r(s)\, ds$$

$$\Rightarrow B(t) = e^{\int_0^t \alpha(v)dv}\left(B_0 - \int_0^t e^{-\int_0^s \alpha(v)dv} r(s)\, ds\right)$$

Since we used $B(0) = B_0$ in evaluating the definite integrals, we will find $r(t)$ using $B(T) = 0$:

$$0 = e^{\int_0^T \alpha(v)dv}\left(B_0 - \int_0^T e^{-\int_0^s \alpha(v)dv} r(s)\, ds\right)$$

$$\Rightarrow B_0 - \int_0^T e^{-\int_0^s \alpha(v)dv} r(s)\, ds = 0$$

$$\Rightarrow B_0 = \int_0^T e^{-\int_0^s \alpha(v)dv} r(s)\, ds$$

This is a constant equal to a definite integral so to find $r(t)$ we need to 'guess' at a function such that $\int_0^T (\ )\, ds = B_0$.

One such integral that satisfies this is $\int_0^T \frac{B_0}{T}\, ds = \left[\frac{B_0}{T}s\right]_0^T = B_0$.

Hence, $e^{-\int_0^s \alpha(v)dv} r(s) = \frac{B_0}{T} \Rightarrow r(s) = \frac{B_0}{T} e^{\int_0^s \alpha(v)dv}$

So we can simplify $B(t) = e^{\int_0^t \alpha(v)dv}\left(B_0 - \int_0^t \frac{B_0}{T}\, ds\right)$

and $r(t) = \frac{B_0}{T} e^{\int_0^t \alpha(v)dv}$

which satisfies both $B(0) = B_0$ and $B(T) = 0$.

- As an addendum it may be possible to find other functions of $r(t)$ that satisfy the boundary conditions, this is just the simplest value we were able to find.
Other values may not increase even as the interest rate decreases.