Matrix Equation for Simple Model

The Equation for the model is:

$$h(t) = h_0 + a\cos(\Omega t) + b\sin(\Omega t) \tag{1}$$

Three sets of arbitrary constants will be used to create a matrix equation.

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} h_0 & a\cos(\Omega t_1) & b\sin(\Omega t_1) \\ h_0 & a\cos(\Omega t_2) & b\sin(\Omega t_2) \\ h_0 & a\cos(\Omega t_3) & b\sin(\Omega t_3) \end{bmatrix} \tag{2}$$

In order to solve this set of equations to determine values of h_0, a and b, we will put this system of equations in row echelon form. Here are the steps:

1. Subtract the Row 1 from Row 2 and Row 3 to get

$$\begin{bmatrix} h_1 \\ h_2 - h_1 \\ h_3 - h_1 \end{bmatrix} = \begin{bmatrix} 1 & \cos(\Omega t_1) & \sin(\Omega t_1) \\ 0 & \cos(\Omega t_2) - \cos(\Omega t_1) & \sin(\Omega t_2) - \sin(\Omega t_1) \\ 0 & \cos(\Omega t_3) - \cos(\Omega t_1) & \sin(\Omega t_3) - \sin(\Omega t_1) \end{bmatrix} \tag{3}$$

2. Multiply Row 3 by $\frac{\cos(\Omega t_2) - \cos(\Omega t_1)}{\cos(\Omega t_3) - \cos(\Omega t_1)}$ to get:

$$\begin{bmatrix} h_1 \\ h_2 - h_1 \\ (h_3 - h_1)(\frac{\cos(\Omega t_2) - \cos(\Omega t_1)}{\cos(\Omega t_3) - \cos(\Omega t_1)}) \end{bmatrix} = \begin{bmatrix} 1 & \cos(\Omega t_1) & \sin(\Omega t_1) \\ 0 & \cos(\Omega t_2) - \cos(\Omega t_1) & \sin(\Omega t_2) - \sin(\Omega t_1) \\ 0 & \cos(\Omega t_2) - \cos(\Omega t_1) & \frac{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)]}{\cos(\Omega t_3) - \cos(\Omega t_1)} \end{bmatrix} \tag{4}$$

3. Subtracting Row 2 from Row 3, the left hand side becomes:

$$\begin{bmatrix} h_1 \\ h_2 - h_1 \\ (h_3 - h_1)(\frac{\cos(\Omega t_2) - \cos(\Omega t_1)}{\cos(\Omega t_3) - \cos(\Omega t_1)}) - h_2 + h_1 \end{bmatrix} \tag{5}$$

and the left hand side becomes:

$$\begin{bmatrix} 1 & \cos(\Omega t_1) & \sin(\Omega t_1) \\ 0 & \cos(\Omega t_2) - \cos(\Omega t_1) & \sin(\Omega t_2) - \sin(\Omega t_1) \\ 0 & 0 & \frac{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)]}{\cos(\Omega t_3) - \cos(\Omega t_1)} - \sin(\Omega t_2) + \sin(\Omega t_1) \end{bmatrix} \tag{6}$$

We will now rearrange the last line of the matrix to find the equation for b.

$$(h_3 - h_1)(\frac{\cos(\Omega t_2) - \cos(\Omega t_1)}{\cos(\Omega t_3) - \cos(\Omega t_1)}) - h_2 + h_1 = b(\frac{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)]}{\cos(\Omega t_3) - \cos(\Omega t_1)} - \sin(\Omega t_2) + \sin(\Omega t_1)) \tag{7}$$

$$b = \frac{(h_3 - h_1)(\frac{\cos(\Omega t_2) - \cos(\Omega t_1)}{\cos(\Omega t_3) - \cos(\Omega t_1)}) - h_2 + h_1}{\frac{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)]}{\cos(\Omega t_3) - \cos(\Omega t_1)} - \sin(\Omega t_2) + \sin(\Omega t_1)} \tag{8}$$

$$b = \frac{(h_3 - h_1)(\frac{\cos(\Omega t_2) - \cos(\Omega t_1) - h_2(\cos(\Omega t_3) - \cos(\Omega t_1)) + h_1(\cos(\Omega t_3) - \cos(\Omega t_1))}{\cos(\Omega t_3) - \cos(\Omega t_1)})}{\frac{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)] - \sin(\Omega t_2)[\cos(\Omega t_3) - \cos(\Omega t_1)] + \sin(\Omega t_1)[\cos(\Omega t_3) - \cos(\Omega t_1)]}{\cos(\Omega t_3) - \cos(\Omega t_1)}} \tag{9}$$

$$b = \frac{(h_3 - h_1)(\cos(\Omega t_2) - \cos(\Omega t_1) - h_2(\cos(\Omega t_3) - \cos(\Omega t_1)) + h_1(\cos(\Omega t_3) - \cos(\Omega t_1)))}{[\cos(\Omega t_2) - \cos(\Omega t_1)][\sin(\Omega t_3) - \sin(\Omega t_1)] - \sin(\Omega t_2)[\cos(\Omega t_3) - \cos(\Omega t_1)] + \sin(\Omega t_1)[\cos(\Omega t_3) - \cos(\Omega t_1)]} \tag{10}$$

Next, a can be determined by:

$$a[\cos(\Omega t_2) - \cos(\Omega t_1)] + b[\sin(\Omega t_2) - \sin(\Omega t_1)] = h_2 - h_1 \tag{11}$$

$$a = \frac{h_2 - h_1 - b[\sin(\Omega t_2) - \sin(\Omega t_1)]}{\cos(\Omega t_2) - \cos(\Omega t_1)} \tag{12}$$

Finally, h_0 can be determined by:

$$h_0 + a\cos(\Omega t_1) + b\sin(\Omega t_1) = h_1 \tag{13}$$

$$h_0 = h_1 - a\cos(\Omega t_1) - b\sin(\Omega t_1) \tag{14}$$