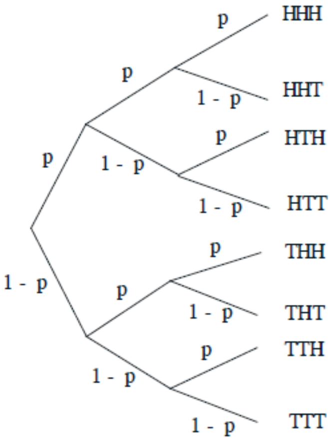
Independence

Example

- 3 tosses of a biased coin: $P(H) = p, P(T) = 1 - p$



- $P(THT) = ?$

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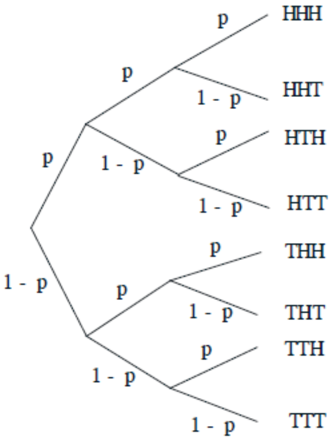

- $P(THT) = (1 - p)p(1 - p)$

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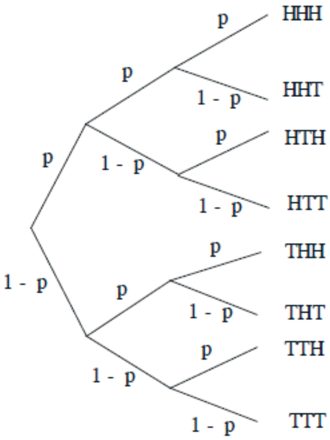

- $P(1 \text{ head}) = ?$

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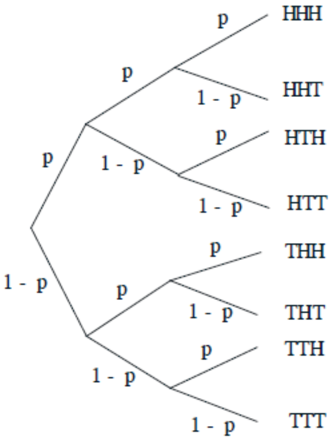

- $P(1 \text{ head}) = 3p(1-p)^2$

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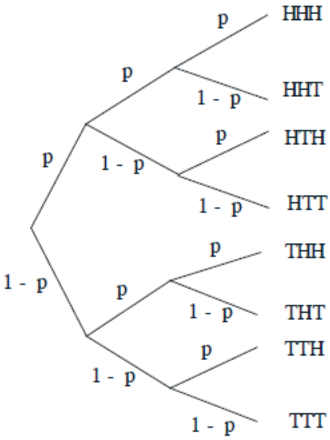

- $P($ first toss is $H \mid 1$ head$) = ?$

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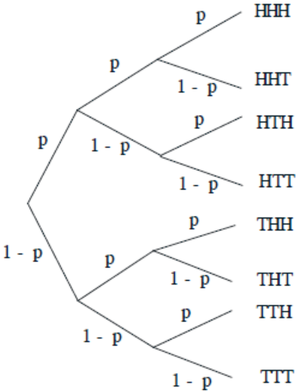
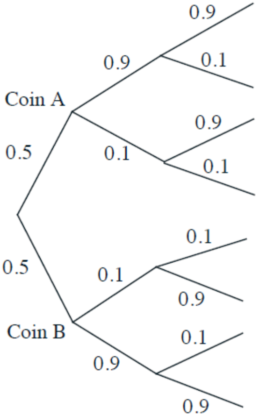


- $P($ first toss is $H \mid 1$ head$) =$

$$\frac{P(\text{ 1st toss is } H \text{ and 1 head})}{P(1 \text{ head})} = \frac{P(HTT)}{P(1 \text{ head})} = \frac{p(1-p)^2}{3p(1-p)^2} = \frac{1}{3}$$

Conditioning may affect independence

- Two unfair coins, A and B
- $P(H|\text{ coin } A) = 0.9$ and $P(H|\text{ coin } B) = 0.1$
- Choose either coin with equal probability $= \frac{1}{2}$



- Answer the following questions:

Conditioning may affect independence

- Two unfair coins, A and B
- $P(H|\text{ coin } A) = 0.9$ and $P(H|\text{ coin } B) = 0.1$
- Choose either coin with equal probability $= \frac{1}{2}$
- Question 1) Once we know it is coin A, are tosses independent?

  Yes. Denote the result of k-th toss by T_k, then:

  $$P(T_{k+1} = H|T_1 \cap T_2 \cap \cdots \cap T_k \cap A) = P(T_{k+1} = H|A) = P(H|A) = 0.9$$

  Similarly,

  $$P(T_{k+1} = T|T_1 \cap T_2 \cap \cdots \cap T_k \cap A) = P(T_{k+1} = T|A) = P(T|A) = 0.1$$

- $\Rightarrow$ Once we know we are tossing Coin A, then the "history", which is the results of the first k coins, will not affect the result of the next coin, which is $(k+1)$-th coin.

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- Choose either coin with equal probability $= \frac{1}{2}$

- Question 2) If we do not know which coin it is, are tosses independent?

  No! Compare:

$$\rightarrow P\Big(T_{11} = H \mid (T_1 = H) \cap (T_2 = H) \cap \cdots \cap (T_{10} = H)\Big) =?$$

with

$$\rightarrow P(T_{11} = H) = ?$$

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  No! Compare:

$$\rightarrow P\Big(T_{11} = H \mid (T_1 = H) \cap (T_2 = H) \cap \cdots \cap (T_{10} = H)\Big) \approx 0.9$$

with

$$\rightarrow P(T_{11} = H) = P\big((T_{11} = H) \ \cap A\big) + P\big((T_{11} = H) \ \cap B\big) =$$

$$= P(A)P(T_{11} = H|A) + P(B)P(T_{11} = H|B) = \frac{1}{2}0.1 + \frac{1}{2}0.9 = \frac{1}{2}$$

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  No! Compare:

$$\rightarrow P\Big(T_{11} = H \mid (T_1 = H) \cap (T_2 = H) \cap \cdots \cap (T_{10} = H)\Big) \approx 0.9$$

  with

$$\rightarrow P(T_{11} = H) = \frac{1}{2}$$

- $\Rightarrow$ If we don't know which coin we are tossing, then the "history", which is the results of the first k tosses ($k = 10$ here), will change our perspective about the result of $k+1$-th (here 11-th) toss.

Independence— Collection of Events

- Intuitive definition:

  Information on some of the events tells us nothing about probabilities related to the remaining events.

- Mathematical definition:

  Events $A_1, A_2, \cdots, A_n$ are called **independent** if:

  $$P(A_i \cap A_j \cap \cdots \cap A_q) = P(A_i)P(A_j)\cdots P(A_q)$$

  for any distinct indices $i, j, \cdots, q$ chosen from $\{1, 2, \cdots, n\}$

- $\Rightarrow$ Probability of intersection of any subset of events is multiplication of probability of each event.

Independence vs. Pairwise Independence

- Two independent fair coin tosses. Sample space: $\{HH, HT, TT, TH\}$
- Event A: First toss is H
- Event B: Second toss is H
- $P(A) = P(B) = \frac{1}{2} \Rightarrow A, B$ are independent
- Event C: First and second toss give same result
- Calculate the followings:
  - $P(C) = \frac{1}{2}$
  - $P(C \cap A) = \frac{1}{4} \Rightarrow P(C \cap A) = P(C)P(A) \Rightarrow A, C$ are independent
  - $P(C \cap B) = \frac{1}{4} \Rightarrow P(C \cap B) = P(C)P(B) \Rightarrow B, C$ are independent
  - $P(A \cap B \cap C) = \frac{1}{4}$
  - $P(C | A \cap B) = 1$
  - Therefore, $P(C|A \cap B) \neq P(C) \Rightarrow C$ and $A \cap B$ are not independent
- **Conclusion:** Pairwise independence does not imply independence.

Example

- The king comes from a family of two children. What is the probability that his sibling is a girl?
- Suppose $P(B) = P(G) = \frac{1}{2}$
- Sample space: $\{BB, GG, BG, GB\}$
- We know there is a king $\Rightarrow$ at least one of the children is a boy $\Rightarrow$ set of possible outcomes: $\{BB, BG, GB\}$
- $\Rightarrow P(\text{his sibling is a female}) = \frac{2}{3}$.
- Is this correct? depends what is your model. There are hidden assumptions here, and there could be two different models:
- Model 1) The family "decide to have exactly two children", one is a boy, what is the probability that the other one is a girl? $\frac{2}{3}$
- Model 2) The family "decide to have children until they get a boy", then they will stop. Here set of possible outcomes is $\{GB\}$, therefore, the probability that king's sibling is a female is 1.