Principles of Counting

Discrete Uniform Law

- Uniform sample space: all sample points are equally likely
- Discrete sample space: has finite or countably infinite number of possible outcomes.
- If sample space Ω is discrete and uniform, then:

$$P(A) = \frac{\text{number of elements of A}}{\text{total number of sample points}} = \frac{|A|}{|\Omega|}$$

- Therefore, you need to <u>count</u> number of elements of A, and total number of sample points.
- For example, if $|\Omega| = N$ and $|A| = M$, then each sample point will happen with probability $\frac{1}{N}$. Therefore, A will happen with probability $\frac{M}{N}$.

Basic Counting Principle

- There are r stages.
- There are n_i choices at stage i.
- Then, total number of choices will be $n_1 n_2 \cdots n_r$.

Example:

- Number of license plates with 3 letters and 4 digits:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

- Number of license plates with 3 letters and 4 digits, with no repetition:

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$$

Basic Counting Principle

- **Permutations**: all different ways of ordering n elements
- Number of permutations for n elements:

$$n \cdot (n-1) \cdots 2 \cdot 1 = n!$$

Example:

- Permutations of ABC:

$$ABC, ACB, BAC, BCA, CAB, CBA$$

- $n! = 3! = 6$

Basic Counting Principle

- Number of subsets of $\{1, 2, \cdots, n\}$:
- Each element i can be either in the subset or not, therefore we have two options for each element.
- Total number of subsets:
$$2 \cdot 2 \cdots 2 = 2^n$$
- How many subsets include n? (only one option for n: include it)
$$2 \cdot 2 \cdots 2 \cdot 1 = 2^{n-1}$$
- How many subsets do not include n? (only one option for n: exclude it)
$$2 \cdot 2 \cdots 2 \cdot 1 = 2^{n-1}$$

Example:

- All subsets of $\{1, 2, 3\}$: $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
- $2^3 = 8$ subsets.

Basic Counting Principle

- Probability that six rolls (independent rolls) of a six-sided dice (fair dice) all give different numbers?
  - Number of outcomes that make the event happen: 6!
  - Number of elements in the sample space: 6^6
  - Answer: $\frac{6!}{6^6}$
- What is the probability of seeing a specific sequence of rolls?
  - $P((2,3,3,1,6,5)) = ?$
  - $P((2,3,3,1,6,5)) = P(2)P(3)P(3)P(1)P(6)P(5) = \frac{1}{6^6}$

Combinations

- How many ways do we have to choose k elements out of n elements?
- Number of k-element subsets of a given n-element set?
- The answer is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Two ways of constructing an ordered sequence of k distinct items:
  - 1) Choose the k items one at a time:

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

  - 2) Choose k items, then order them ($k!$ possible orders):

$$\binom{n}{k} k!$$

  - Therefore,

$$\binom{n}{k} k! = \frac{n!}{(n-k)!}$$

Combinations

- $\binom{n}{k}$ is also called "binomial coefficient"
- $\sum_{k=0}^{n}\binom{n}{k}$ = Total number of subsets$= 2^n$
- $\binom{n}{0} = \binom{n}{n} = \frac{n!}{n!0!} = 1 \Rightarrow 0! = 1$
- Note that $\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}$
- Binomial formula:

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$$

- If $p = q = 1$:

$$(2)^n = \sum_{k=0}^{n}\binom{n}{k}$$

Binomial Probabilities

- We have n independent coin tosses.
- $P(\text{Head}) = P(H) = p$ and $P(\text{Tail}) = P(T) = 1 - p$.
- $P(\text{sequence}) = p^{\text{ \# of heads}}(1-p)^{\text{ \# of tails}}$

$$P(k \ \ \text{heads}) = \sum_{k-\text{head} \ \ \text{seq}} P(\text{seq}) = (\text{ \# of k-head seqs }) p^k (1-p)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

- $\#$ of k-head seqs $= \binom{n}{k}$: because out of n tosses k of them are heads.

Binomial Probabilities—Example 1

- $P(HTT) = P(H)P(T)P(T) = p(1-p)(1-p) = p(1-p)^2$
- $P(THT) = P(T)P(H)P(T) = (1-p)p(1-p) = p(1-p)^2$
- $P(TTH) = P(T)P(T)P(H) = (1-p)(1-p)p = p(1-p)^2$

$$P(1 \;\; \text{heads}) = \binom{3}{1} p(1-p)^2 = 3p(1-p)^2$$

$$P(1 \;\; \text{heads}) = P(HTT) + P(THT) + P(TTH) = 3p(1-p)^2$$

Binomial Probabilities—Example 2

- $P(HHTT) = P(H)P(H)P(T)P(T) = pp(1-p)(1-p) = p^2(1-p)^2$
- $P(HTHT) = P(H)P(T)P(H)P(T) = p(1-p)p(1-p) = p^2(1-p)^2$
- $P(HTTH) = P(H)P(T)P(T)P(H) = p(1-p)(1-p)p = p^2(1-p)^2$
- $P(THHT) = P(T)P(H)P(H)P(T) = (1-p)pp(1-p) = p^2(1-p)^2$
- $P(THTH) = P(T)P(H)P(T)P(H) = (1-p)p(1-p)p = p^2(1-p)^2$
- $P(TTHH) = P(T)P(T)P(H)P(H) = (1-p)(1-p)pp = p^2(1-p)^2$

$$P(\text{ 2 heads out of 4 tosses}) = \binom{4}{2} p^2(1-p)^2 = 6p^2(1-p)^2$$

$$= P(HHTT) + P(HTHT) + P(HTTH) + P(THHT) + P(THTH) + P(TTHH)$$

$$= 6p^2(1-p)^2$$

Binomial Probabilities

- Claim:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

- Proof:
- $\binom{n}{k} p^k (1-p)^{n-k}$ is P(seeing k heads out of n tosses)
- $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} =$

$$P(0 \text{ heads }) + P(\text{ 1 heads}) + P(2 \text{ heads}) + \cdots + P(\text{n heads}) = 1$$

- Second proof:
- Use Binomial formula for p and $q = 1 - p$:

$$1 = (p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Coin Tossing Problem

- We have 10 independent coin tosses.
- $P(\text{Head}) = P(H) = p$ and $P(\text{Tail}) = P(T) = 1 - p$
- Event $B = 3$ out of 10 tosses are "heads"
- Event $A =$ first 2 tosses are "heads", what is $P(A|B)$?
- Number of outcomes in B: $\binom{n}{k} = \binom{10}{3}$
- Number of outcomes in $A \cap B$:
  - first 2 tosses are "heads" and 3 out of 10 tosses are "heads"
  - HH _ _ _ _ _ _ _ _
  - There are 8 positions for the third "head"
  - Therefore, number of outcomes in $A \cap B$ is 8
- All outcomes in set B are equally likely with probability $p^3(1-p)^7$
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{8\ p^3(1-p)^7}{\binom{10}{3}\ p^3(1-p)^7} = \frac{8}{\binom{10}{3}}$

Conditional Probability—Discrete Uniform Law

- B is a subset of the sample space Ω and number of outcomes in B is N
- All outcomes in set B are equally likely with probability q
- A is also a subset of the sample space Ω, number of outcomes in A is M and number of outcomes in $A \cap B$ is K
- Then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{K \cdot q}{N \cdot q} = \frac{K}{N}$
- **Another Interpretation**:
    - In universe B, there are N outcomes that are equally likely
    - Therefore, probability of seeing one single element in B is $\frac{1}{N}$
    - $A \cap B$ is an event in universe B and has K outcomes.
    - In discrete uniform universe B:

$$P(\text{seeing K elements}) = K \cdot P(\text{seeing one element}) = K \cdot \frac{1}{N}$$