Bayes' rule—Independence

Conditional Probability—Review

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{assuming } P(B) > 0$$

- **Multiplication rule**:

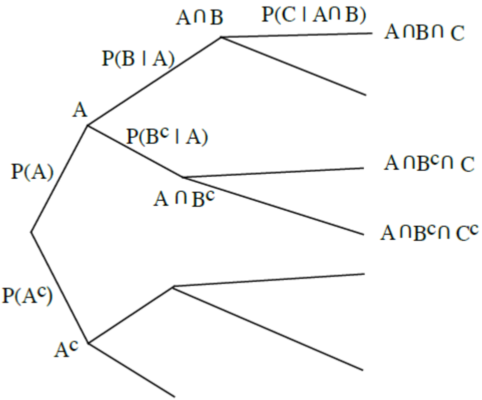
$$P(A \cap B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A)$$

- **Total probability theorem**:

$$P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$$

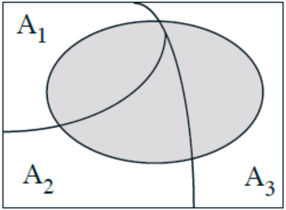
Multiplication rule—Extension

- $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$ because:
- $P((A \cap B) \cap C) = P(A \cap B)P(C|(A \cap B)) = P(A) \cdot P(B|A) \cdot P(C|(A \cap B))$



Total Probability Theorem—Extension

- Partition of sample space into A_1, A_2, A_3
- We have $P(B|A_i)$ for every i

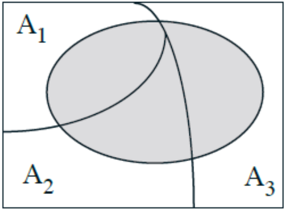


- One way of computing $P(B)$:

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

Bayes' rule

- "Prior" probabilities $P(A_i)$ (initial "beliefs")
- We know $P(B|A_i)$ for each i
- Wish to compute $P(A_i|B)$ (revise "beliefs", given that B occurred)

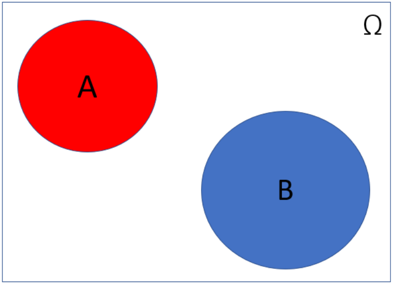


$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

Independence of two events

- Definition of independence: $P(B|A) = P(B)$
- "Occurrence of A provides no information about B's occurrence"
- Recall that $P(A \cap B) = P(A) \cdot P(B|A)$
- Definition of independence: $\boxed{P(A \cap B) = P(A) \cdot P(B)}$
- This definition applies even if $P(A) = 0$
- Independence is <u>symmetric</u> with respect to A and B $\Rightarrow P(A|B) = P(A)$

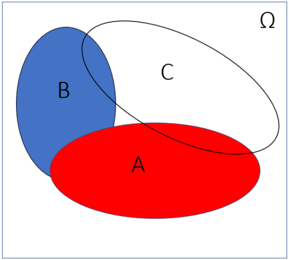
Independence— Intuition



- Are A and B independent?
- No, A and B are disjoint. So if A occurs, then B will not happen, therefore, they are dependent.
- For example, assume $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{2}$. Since they are disjoint, then $P(A \cap B) = 0$. Therefore, $P(A \cap B) \neq P(A)P(B) \Rightarrow A$ and B are not independent.

Conditioning may affect independence

- Conditional independence, given C, is defined as independence under probability law $P(.|C)$
- $P(A \cap B|C) = P(A|C)P(B|C)$
- Assume A and B are independent.



- If we are told that C occurred, are A and B independent?
- No, because in universe C, A and B are disjoint (so if A happens then B will not happen), therefore, in universe C, they are not independent.