Properties of Expectations, Variance, Conditional PMF, Conditional Expectation, Geometric PMF

Properties of Expectations

- Let X be a random variable.
- Expectation of X:

$$E[X] = \sum_x x p_X(x)$$

- Let random variable Y be: $Y = g(X)$

$\longrightarrow$ What is the expectation of Y?

- Based on definition:

$$E[Y] = \sum_y y p_Y(y)$$

- Expectation of Y based on PMF of X:

$$E[Y] = E[g(X)] = \sum_x g(x) p_X(x)$$

- Caution: In general, $E[g(X)] \neq g(E[X])$

Properties of Expectations—Example

- Let X be a random variable and PMF of X is:

$$p_X(x) = \begin{cases} \frac{1}{3} & \text{if} \quad x = 1 \\ \frac{1}{3} & \text{if} \quad x = -1 \\ \frac{1}{3} & \text{if} \quad x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Let random variable Y be: $Y = g(X) = X^2$.
- Expectation of Y using PMF of X:

$$E[Y] = E[g(X)] = E[X^2] = \sum_x g(x) p_X(x) = \sum_x x^2 p_X(x)$$

$$\Rightarrow E[Y] = \sum_x x^2 p_X(x) = (1)^2 p_X(1) + (-1)^2 p_X(-1) + (0)^2 p_X(0) =$$

$$= (1)^2 \frac{1}{3} + (-1)^2 \frac{1}{3} + (0)^2 \frac{1}{3} = \frac{2}{3}$$

Properties of Expectations—Example

- You can also find PMF of Y from PMF of X.
- X takes values $-1, 0, 1$ with non zero probability. Because $Y = X^2$, then Y takes values 0 and 1 with non zero probability.

- $P(Y = 1) = P(X = 1) + P(X = -1) = \frac{2}{3}$
- $P(Y = 0) = P(X = 0) = \frac{1}{3}$

$$p_Y(y) = \begin{cases} \frac{2}{3} & \text{if } \ y = 1 \\ \frac{1}{3} & \text{if } \ y = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Expectation of Y using PMF of Y:

$$E[Y] = \sum_y y p_Y(y) = 1 \cdot p_Y(1) + 0 \cdot p_Y(0) = 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{2}{3}$$

Properties of Expectations

- If α and β are constants, then:
- $E[\alpha] = \alpha$
  - for example $E[2] = 2$
- $E[\alpha X] = \alpha E[X]$
  - for example $E[2X] = 2E[X]$
- $E[\alpha X + \beta] = \alpha E[X] + \beta$
  - for example $E[2X + 3] = 2E[X] + 3$ or $E[2X - 3] = 2E[X] - 3$

Proof:

$$E[\alpha X + \beta] = \sum_x (\alpha x + \beta) p_X(x) = \alpha \underbrace{\sum_x x p_X(x)}_{E[X]} + \beta \underbrace{\sum_x p_X(x)}_{1} =$$

$$= \alpha E[X] + \beta$$

Variance

- **Second moment**: $E[X^2] = \sum_x x^2 p_X(x)$
- **Variance**:

$$\begin{aligned}\mathrm{Var}(\mathrm{X}) &= \mathrm{E}[(X - E[X])^2] \\ &= \sum_x (x - E[X])^2 \ p_X(x) \\ &= \mathrm{E}[X^2] - (E[X])^2\end{aligned}$$

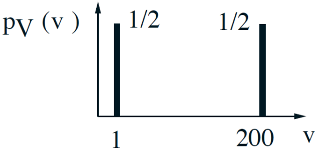
- **Standard deviation**: $\sigma_X = \sqrt{\mathrm{Var}(X)}$
- **Properties**:
- $\mathrm{Var}(X) \geq 0$
- $\mathrm{Var}(\alpha X + \beta) = \alpha^2 \mathrm{Var}(X)$

Example

- $E[X - E[X]] = ?$
- $E[X]$ is a constant. Suppose $\alpha = E[X]$.
- $E[X - E[X]] = E[X - \alpha] = E[X] - \alpha = E[X] - E[X] = 0$

Example—Random speed

- Traverse a 200 mile distance at constant but random speed V.



- $d = 200$, distance is fixed.
- Time is a function of speed: $T = t(V) = \frac{200}{V}$

$$E[V] = \sum_v v\ p_V(v) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 200 = 100.5$$

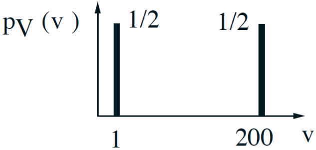
- $\text{Var}(V) =$

$$\sum_v (v - E[V])^2 p_V(v) = \frac{1}{2}(1 - 100.5)^2 + \frac{1}{2}(200 - 100.5)^2 \approx 100^2$$

- $\sigma_V = \sqrt{\text{Var}(V)} \approx 100$

Example—Average Time

- Traverse a 200 mile distance at constant but random speed V.



$$E[T] = \sum_v t(v) p_V(v) = \sum_v \frac{200}{v} p_V(v) = \frac{1}{2} \cdot 200 + \frac{1}{2} \cdot 1 = 100.5$$

- $E[TV] = E[200] \neq E[T]E[V]$

- $E[\frac{200}{V}] = E[T] \neq \frac{200}{E[V]} \approx 2$

Example

Question:

- Two fair three-sided dice are rolled simultaneously. Let X be the difference of the two rolls.
- **Part a**: Calculate the PMF, the expected value, and the variance of X.
- **Part b**: Calculate and plot the PMF of X^2.

Example

Answer Part a:

- For each value of X, we count the number of outcomes which have a difference that equals that value:

$$p_X(x) = \begin{cases} \frac{1}{9} & \text{if } \ x = 2, -2 \\ \frac{2}{9} & \text{if } \ x = -1, 1 \\ \frac{3}{9} & \text{if } \ x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=-2}^{2} x p_X(x) = -2\frac{1}{9} + -1\frac{2}{9} + 0\frac{3}{9} + 1\frac{2}{9} + 2\frac{1}{9} = 0$$

- To find the variance of X:

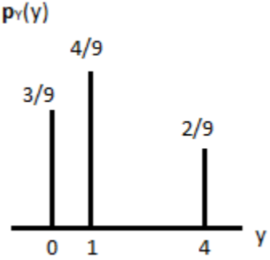
$$E[X^2] = \sum_{x=-2}^{2} x^2 p_X(x) = 4\frac{1}{9} + 1\frac{2}{9} + 0\frac{3}{9} + 1\frac{2}{9} + 4\frac{1}{9} = \frac{4}{3}$$

- $\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{4}{3} - 0 = \frac{4}{3}$

Answer Part b

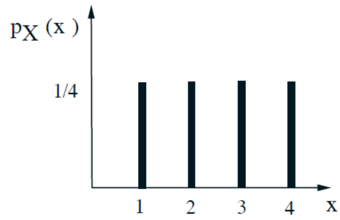
- Let $Z = X^2$. By matching the possible values of X and their probabilities to the possible values of Z, we obtain

$$p_Z(z) = \begin{cases} \frac{2}{9} & \text{if} \quad z = 4 \\ \frac{4}{9} & \text{if} \quad z = 1 \\ \frac{3}{9} & \text{if} \quad z = 0 \\ 0 & \text{otherwise} \end{cases}$$

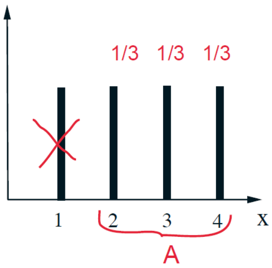


Conditional PMF

- The following PMF is given:

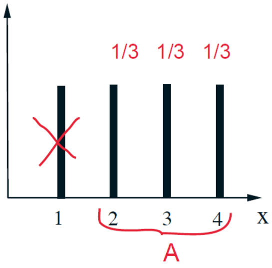


- Event $A = \{X \geq 2\}$
- $p_{X|A}(x) = P(X = x|A)$ $\Rightarrow p_{X|A}(x) = \frac{1}{3}$ for $x = 2, 3, 4$



Conditional Expectation

- $\boxed{E(X|A) = \sum_x x \, p_{X|A}(x)}$
- $p_{X|A}(x) = \frac{1}{3}$ for $x = 2, 3, 4$
- $E(X|A) = 2\frac{1}{3} + 3\frac{1}{3} + 4\frac{1}{3} = 3$



More generally:

- $E(g(X)|A) = \sum_x g(x) \, p_{X|A}(x)$

Geometric PMF

- X: number of independent coin tosses until first head.

$$p_X(k) = (1-p)^{k-1}p \quad \text{for } k = 1, 2, \cdots$$

$$E[X] = \sum_{k=1}^{\infty} k\ p_X(k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

- **Memory less property**: Given that $X > 2$, the random variable $X - 2$ has the same geometric PMF as X: