Sample Space

We define Ω as the set of *all* possible outcomes of a random event. $\Omega = \{\cdots\}$

Collectively Exhaustive

We define the probability of *no element* in Ω occurring as 0.

$$P\,(\,\emptyset\,) = 0$$

Nonnegativity

For all **A** in Ω the probability of an event **A** is not negative:

$$\forall\,\mathbf{A}\,\in\,\Omega \quad P\,(\,\mathbf{A}\,) \geq 0$$

Normalization

We define the sum of the probability of all the elements in Ω as being equal to 1.

$$1 = P\,(\,\Omega\,) = \sum_{\mathbf{A}\,\in\,\Omega} P\,(\,\mathbf{A}\,)$$

Mutually Exclusive

We say **A** and **B** are mutually independent *if and only if*, **A** and **B** have no overlap.

$$\mathbf{A} \cap \mathbf{B} = \emptyset \iff \text{mutually exclusive random variables}$$

Complement

The probability of **A** not occurring is equal to 1 minus the probability **A** occurring:

$$P\left(\,\overline{\mathbf{A}}\,\right) = 1 - P\,(\,\mathbf{A}\,) = P\,(\,\Omega\,) - P\,(\,\mathbf{A}\,) = P\,(\,\Omega \setminus \mathbf{A}\,)$$

Independence

Define **A** as independent of **B** *if and only if*, the probabilities of **A** given **B** and **A** are equal:

$$P\,(\,\mathbf{A}\,|\,\mathbf{B}\,) = P\,(\,\mathbf{A}\,)$$

Additivity

The probability of **A** or **B** occurring is:

$$P\,(\,\mathbf{A} \cup \mathbf{B}\,) = P\,(\,\mathbf{A}\,) + P\,(\,\mathbf{B}\,) - P\,(\,\mathbf{A} \cap \mathbf{B}\,)$$
$$P\,(\,\mathbf{A} \cup \mathbf{B}\,) = P\,(\,\mathbf{A}\,) + P\,(\,\mathbf{B}\,) - 0 \iff \mathbf{A}\ \&\ \mathbf{B} \text{ are mutually exclusive}$$

Multiplicativity / Joint Probability

The probability of **A** and **B** occurring is:

$$P\,(\,\mathbf{A} \cap \mathbf{B}\,) = P\,(\,\mathbf{A}\,) * P\,(\,\mathbf{B}\,|\,\mathbf{A}\,) = P\,(\,\mathbf{B}\,) * P\,(\,\mathbf{A}\,|\,\mathbf{B}\,)$$

Derivations :

$$\begin{aligned}
&P\,(\,\mathbf{A} \cap \mathbf{B}\,) = P\,(\,\mathbf{A}\,) * P\,(\,\mathbf{B}\,|\,\mathbf{A}\,) && \Longleftrightarrow \; \mathbf{A}\ \&\ \mathbf{B} \text{ dependent} \\
&P\,(\,\mathbf{A} \cap \mathbf{B}\,) = P\,(\,\mathbf{A}\,) * P\,(\,\mathbf{B}\,) && \Longleftrightarrow \; \mathbf{A}\ \&\ \mathbf{B} \text{ independent} \\
&P\,(\,\mathbf{A} \cap \mathbf{B}\,) = 0 && \Longleftrightarrow \; \mathbf{A}\ \&\ \mathbf{B} \text{ mutually exclusive}
\end{aligned}$$

Conditional Probability

The probability of **A** occurring *given* that **B** has occurred is:

$$P\,(\,\mathbf{A}\,|\,\mathbf{B}\,) = \frac{P\,(\,\mathbf{A} \cap \mathbf{B}\,)}{P\,(\,\mathbf{B}\,)}$$

Bayes' Theorem

The probability of **A** occurring *given* that **B** has occurred is:

$$P\,(\,\mathbf{A}\,|\,\mathbf{B}\,) = \frac{P\,(\,\mathbf{B}\,|\,\mathbf{A}\,)}{P\,(\,\mathbf{B}\,)}$$

Total Probability Theorem

$$P\,(\,\mathbf{A}\,) = P\,(\,\mathbf{B}\,) * P\,(\,\mathbf{A}\,|\,\mathbf{B}\,) + P\left(\,\overline{\mathbf{B}}\,\right) * P\left(\,\mathbf{A}\,|\,\overline{\mathbf{B}}\,\right)$$