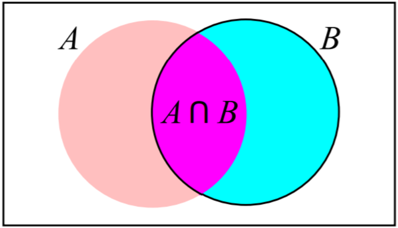
Conditional Probability

Conditional Probability—Definition

- $P(A|B)$ = probability of A, given that B occurred.
- B is our new universe:
- Meaning that $P(A|B)$ measures $A \cap B$ as a fraction of B
- Assuming that $P(B) \neq 0$,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A|B)$ is undefined if $P(B) = 0$.
- Sample space, Ω is the rectangle.



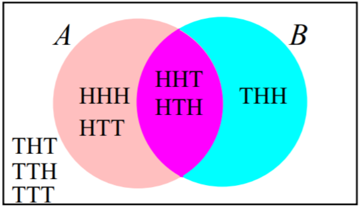
Conditional Probability—Example

- Flip a coin 3 times. The sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

- Events: $A = \{\text{"First flip is heads"}\} = \{HHH, HHT, HTH, HTT\}$

$$B = \{\text{"Two flips are heads"}\} = \{HHT, HTH, THH\}$$



- First solution: $P(A \cap B) = \frac{2}{8}$, $P(B) = \frac{3}{8} \Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{3}$
- Second solution: B is the new universe, therefore:

$$P(A|B) = \frac{|\{HHT, HTH\}|}{|\{HHT, HTH, THH\}|} = \frac{2}{3}$$

Conditional Probability

- $P(B|B) = \frac{P(B \cap B)}{P(B)} = 1$, immediate response:

  if B is given, then B is the new universe, so $P(B|B) = 1$.

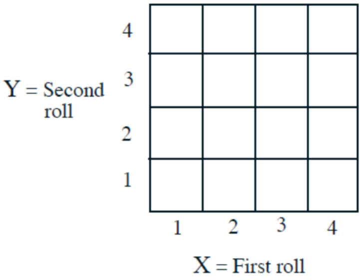
- $P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$

- $P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A)$

$$\boxed{P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)}$$

Conditional Probability—Example

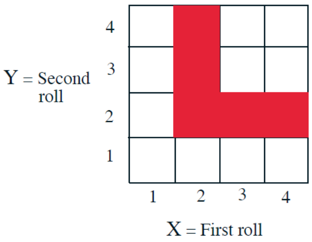
- Rolling a four-sided dice:



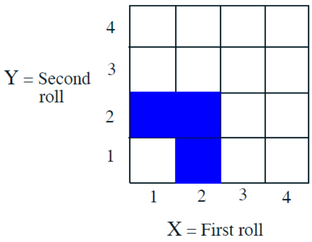
- Let B be the event: $\min(X, Y) = 2$
- Let A be the event: $\max(X, Y) = 2$
- What is $P(A|B)$?

Conditional Probability—Example

- Event $B : \min(X, Y) = 2$

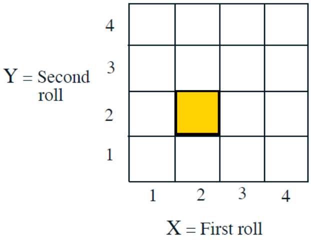


- Event $A : \max(X, Y) = 2$



Conditional Probability—Example

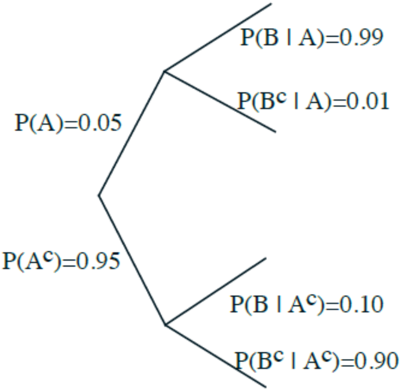
- Event $A \cap B$:



- $P(A \cap B) = \frac{1}{16}, P(B) = \frac{5}{16} \Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{5}$

Conditional Probability—Example

- Event A: Airplane is flying above
- Event B: Something registers on radar screen

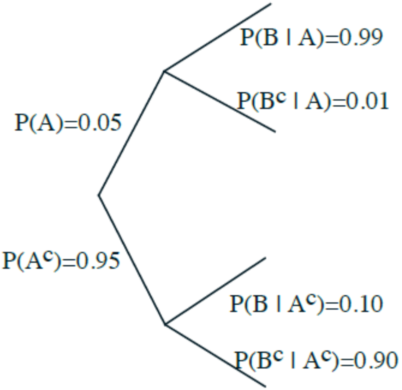


- **a)** $P(A \cap B)$?

$$P(A \cap B) = P(A)P(B|A) = 0.05 \times 0.99 = 0.0495$$

Conditional Probability—Example

- Event A: Airplane is flying above
- Event B: Something registers on radar screen

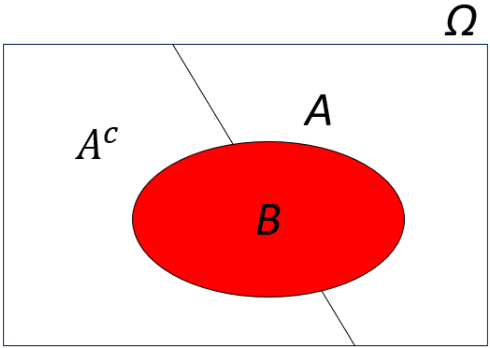


- **b)** $P(B) = ?$

Conditional Probability—Example

- **Claim**: $P(B) = P(B \cap A) + P(B \cap A^c)$

  Picturing:



Conditional Probability—Example

- **Claim**: $P(B) = P(B \cap A) + P(B \cap A^c)$

Mathematical Proof:

- Denote the "Sample Space" by Ω.
- Remind: A^c is complement of set A, meaning that:

$$A^c \cup A = \Omega$$

$$A^c \cap A = \emptyset$$

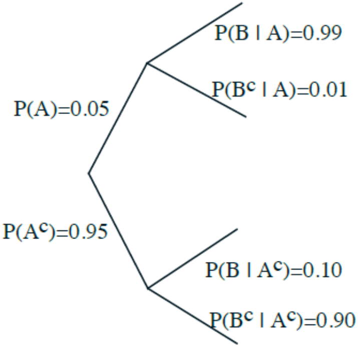
- $B \cap \Omega = B \Rightarrow B \cap (A^c \cup A) = B \Rightarrow (B \cap A) \cup (B \cap A^c) = B$
- $(B \cap A)$ and $(B \cap A^c)$ are disjoint. Therefore, based on "Addivity" rule:

$$P\Big((B \cap A) \cup (B \cap A^c)\Big) = P(B \cap A) + P(B \cap A^c)$$

- $\Rightarrow P(B \cap A) + P(B \cap A^c) = P(B)$

Conditional Probability—Example

- Back to the exmaple:



- $P(B) = P(B \cap A) + P(B \cap A^c)$

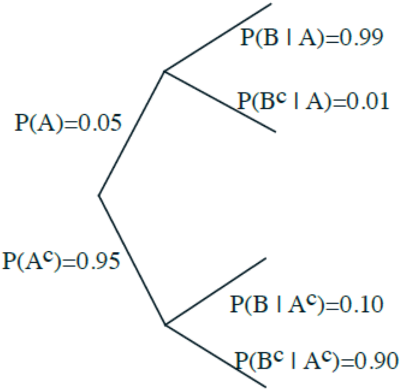
  We know: $P(B \cap A) = P(A)P(B|A)$ and $P(B \cap A^c) = P(A^c)P(B|A^c)$

  Therefore, $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c)$

  $\Rightarrow P(B) = 0.05 \times 0.99 + 0.95 \times 0.1 = 0.1445$

Conditional Probability—Example

- Event A: Airplane is flying above
- Event B: Something registers on radar screen



- **c)** $P(A|B) = ?$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0495}{0.1445} \approx 0.34$$

Conditional Probability—Example

- What does this number, 0.34, represent?
- If the radar detects something, the probability that there has been an actual airplane is 0.34! With probability 0.66, there is no airplane $\Rightarrow$ Not a reliable radar!