CSc 217 – Probability and Statistics for Computer Science

Problem Set №1

by

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Problem 1:

You flip a fair coin 4 times.

All flips are mutually independent and have outcomes $\Omega = \{$ H, T $\}$.

Determine the probability of the events below.

Hint: Draw a table enumerating the all possible outcomes of 4 flips.

The full space of possible observations from four coin flips is $\Omega \times \Omega \times \Omega \times \Omega = \Omega^4$ which is enumerated below. The total number of observations in Ω^4 is $\left|\Omega^4\right| = 16$. Each observation is equally probable, hence each observation $\mathbf{C}$ in the table below has probability $P(\mathbf{C}) = {}^1/{}_{16}$.

	1st	2nd	3rd	4th
1	T	T	T	T
2	T	T	T	H
3	T	T	H	T
4	T	T	H	H
5	T	H	T	T
6	T	H	T	H
7	T	H	H	T
8	T	H	H	H
9	H	T	T	T
10	H	T	T	H
11	H	T	H	T
12	H	T	H	H
13	H	H	T	T
14	H	H	T	H
15	H	H	H	T
16	H	H	H	H

1 (a): " Four heads: (H, H, H, H) "

There is only one occurrence of four heads (H, H, H, H). Hence the answer is:

$$\frac{1}{\left|\Omega^4\right|} = \frac{1}{16}$$

1 (b): " The sequence: (H, T, H, T) "

There is only one occurrence of the sequence (H, T, H, T). Hence the answer is:

$$\frac{1}{\left|\Omega^4\right|} = \frac{1}{16}$$

1 (c): " Any sequence with 3 H and 1 T "

There are four occurrences of sequences satisfying 3 H and 1 T; specifically rows 8, 12, 14, and 15. Hence the answer is:

$$\frac{4}{|\Omega^4|} = \frac{4}{16} = \frac{1}{4}$$

1 (d): " Any sequence where the number of H is greater than or equal to T "

There are eleven occurrences of sequences satisfying the number of H is greater than or equal to T. We can arrive at this by counting the number of rolls with two H, three H, and four H, then taking thier summation (since they are mutually independent).

- Two   H: $\binom{4}{2} \times \frac{1}{16} = 6 \times \frac{1}{16} = \frac{6}{16}$
- Three H: $\binom{4}{3} \times \frac{1}{16} = 4 \times \frac{1}{16} = \frac{4}{16}$
- Four H: $\binom{4}{4} \times \frac{1}{16} = 1 \times \frac{1}{16} = \frac{1}{16}$

$$\frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16}$$

Problem 2:

You throw two fair, 6-faced dice, one green and one blue, called D_1 & D_2.
The result of each die is independent of the other.
D_1 has outcomes $\Omega_{D_1} = \left\{ \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6} \right\}$.
D_2 has outcomes $\Omega_{D_2} = \left\{ \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6} \right\}$.
Determine the probability of the events below.

Hint: Draw a 6×6 grid enumerating the all possible rolls of the dice.

D_1 \ D_2	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$
$\underline{1}$	$(\bar{1},\underline{1})$	$(\bar{2},\underline{1})$	$(\bar{3},\underline{1})$	$(\bar{4},\underline{1})$	$(\bar{5},\underline{1})$	$(\bar{6},\underline{1})$
$\underline{2}$	$(\bar{1},\underline{2})$	$(\bar{2},\underline{2})$	$(\bar{3},\underline{2})$	$(\bar{4},\underline{2})$	$(\bar{5},\underline{2})$	$(\bar{6},\underline{2})$
$\underline{3}$	$(\bar{1},\underline{3})$	$(\bar{2},\underline{3})$	$(\bar{3},\underline{3})$	$(\bar{4},\underline{3})$	$(\bar{5},\underline{3})$	$(\bar{6},\underline{3})$
$\underline{4}$	$(\bar{1},\underline{4})$	$(\bar{2},\underline{4})$	$(\bar{3},\underline{4})$	$(\bar{4},\underline{4})$	$(\bar{5},\underline{4})$	$(\bar{6},\underline{4})$
$\underline{5}$	$(\bar{1},\underline{5})$	$(\bar{2},\underline{5})$	$(\bar{3},\underline{5})$	$(\bar{4},\underline{5})$	$(\bar{5},\underline{5})$	$(\bar{6},\underline{5})$
$\underline{6}$	$(\bar{1},\underline{6})$	$(\bar{2},\underline{6})$	$(\bar{3},\underline{6})$	$(\bar{4},\underline{6})$	$(\bar{5},\underline{6})$	$(\bar{6},\underline{6})$

2 (a): " At least one 3 occurs; $P\left(D_1 = \bar{3} \right)$ or $P\left(D_2 = \underline{3} \right)$ "

$$P\left(D_1 = \bar{3} \right) + P\left(D_2 = \underline{3} \right) - P\left(D_1 = \bar{3} \wedge D_2 = \underline{3} \right) = \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$$

2 (b): " The larger of the two dice is greater than 4; $P\left(\max\left(D_1, D_2 \right) > 4 \right)$ "

$$P\left(D_1 > \bar{4} \right) + P\left(D_2 > \underline{4} \right) - P\left(D_1 > \bar{4} \wedge D_2 > \underline{4} \right) = \frac{12}{36} + \frac{12}{36} - \frac{4}{36} = \frac{20}{36} = \frac{5}{9}$$

2 (c): " The faces of the dice match; $P\left(D_1 = D_2 \right)$ "

$$
\begin{aligned}
P\left(D_1 = D_2 \right) &= \sum_{x \in \Omega} P\left(D_1 = D_2 = x \right) \\
&= P\left(D_1 = D_2 = 1 \right) + P\left(D_1 = D_2 = 2 \right) + P\left(D_1 = D_2 = 3 \right) \\
&\quad + P\left(D_1 = D_2 = 4 \right) + P\left(D_1 = D_2 = 5 \right) + P\left(D_1 = D_2 = 6 \right) \\
&= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} \\
&= \frac{6}{36} \\
&= \frac{1}{6}
\end{aligned}
$$

2 (d): " The sum of the dice is 7; $P\left(\mathbf{D}_1 + \mathbf{D}_2 = 7\right)$ "

$$P\left(\mathbf{D}_1 + \mathbf{D}_2 = 7\right) = P\left(\mathbf{D}_1 = \overline{1} \wedge \mathbf{D}_2 = \underline{6}\right) + P\left(\mathbf{D}_1 = \overline{2} \wedge \mathbf{D}_2 = \underline{5}\right) + P\left(\mathbf{D}_1 = \overline{3} \wedge \mathbf{D}_2 = \underline{4}\right)$$
$$+ P\left(\mathbf{D}_1 = \overline{4} \wedge \mathbf{D}_2 = \underline{3}\right) + P\left(\mathbf{D}_1 = \overline{5} \wedge \mathbf{D}_2 = \underline{2}\right) + P\left(\mathbf{D}_1 = \overline{6} \wedge \mathbf{D}_2 = \underline{1}\right)$$
$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$$
$$= \frac{6}{36}$$
$$= \frac{1}{6}$$

Problem 3:

You the same D_1 and D_2 from Problem 2. However, this time you are using D_1 and D_2 as the dice to play the game of Monopoly. In the game of Monopoly, if you "roll doubles," meaning that the faces of the dice are the same number, then you get to roll the dice again. You can roll the dice a maximum of three times, if you happen to "roll doubles" on both the first and second rolls. The number of spaces you move is the sum of all dice rolls.

What is the probability that you move *at least* 12 spaces?

Hint: Draw a 6×6 grid enumerating the all possible rolls of the dice. Then sub-divide the cells where you "roll doubles" into another, smaller 6×6 grid within that cell. Each cell, no matter how small, is one possible outcome.

The diagonal contains the *six* possibilities for another roll to occur:

D_1 \ D_2	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
$\underline{1}$	$\| \overline{1}, \underline{1} \|$	$\left(\overline{2}, \underline{1}\right)$	$\left(\overline{3}, \underline{1}\right)$	$\left(\overline{4}, \underline{1}\right)$	$\left(\overline{5}, \underline{1}\right)$	$\left(\overline{6}, \underline{1}\right)$
$\underline{2}$	$\left(\overline{1}, \underline{2}\right)$	$\| \overline{2}, \underline{2} \|$	$\left(\overline{3}, \underline{2}\right)$	$\left(\overline{4}, \underline{2}\right)$	$\left(\overline{5}, \underline{2}\right)$	$\left(\overline{6}, \underline{2}\right)$
$\underline{3}$	$\left(\overline{1}, \underline{3}\right)$	$\left(\overline{2}, \underline{3}\right)$	$\| \overline{3}, \underline{3} \|$	$\left(\overline{4}, \underline{3}\right)$	$\left(\overline{5}, \underline{3}\right)$	$\left(\overline{6}, \underline{3}\right)$
$\underline{4}$	$\left(\overline{1}, \underline{4}\right)$	$\left(\overline{2}, \underline{4}\right)$	$\left(\overline{3}, \underline{4}\right)$	$\| \overline{4}, \underline{4} \|$	$\left(\overline{5}, \underline{4}\right)$	$\left(\overline{6}, \underline{4}\right)$
$\underline{5}$	$\left(\overline{1}, \underline{5}\right)$	$\left(\overline{2}, \underline{5}\right)$	$\left(\overline{3}, \underline{5}\right)$	$\left(\overline{4}, \underline{5}\right)$	$\| \overline{5}, \underline{5} \|$	$\left(\overline{6}, \underline{5}\right)$
$\underline{6}$	$\left(\overline{1}, \underline{6}\right)$	$\left(\overline{2}, \underline{6}\right)$	$\left(\overline{3}, \underline{6}\right)$	$\left(\overline{4}, \underline{6}\right)$	$\left(\overline{5}, \underline{6}\right)$	$\| \overline{6}, \underline{6} \|$

$$30 + 6 \times (30 + 6 \times (30 + 6)) = 1506 = |\Omega|$$

- When $\text{Roll}_1 \left(D_1 \neq D_2 \right)$ then $P(\text{ spaces} \geq 12) = {}^{0}/{}_{1506}$

- When $\text{Roll}_1 \left(D_1 = D_2 = 6 \right)$ then $P(\text{ spaces} \geq 12) = {}^{30+6\times36}/{}_{1506} = {}^{246}/{}_{1506}$
  Because the first roll is $6 + 6 \geq 12$, and all subsequent rolls are monotonically increasing the number of spaces.

- When $\text{Roll}_1 \left(D_1 = D_2 = 5 \right)$ then $P(\text{ spaces} \geq 12) = {}^{30+6\times36}/{}_{1506} = {}^{246}/{}_{1506}$
  Because the first roll plus the minimum possible second roll is $5 + 5 + 1 + 1 \geq 12$, and all subsequent rolls are monotonically increasing the number of spaces.

- When $\text{Roll}_1 \left(D_1 = D_2 = 4 \right)$ then $P(\text{ spaces} \geq 12) = {}^{28+6\times36}/{}_{1506} = {}^{244}/{}_{1506}$
  Because the first roll is $4 + 4 = 8$, there are *two* possible second rolls which do not result in a another roll and also do not have a cumulative sum ≥ 12, notably $\left(\overline{1}, \underline{2}\right)$ and $\left(\overline{2}, \underline{1}\right)$. All possible doubles will yield a result ≥ 12, either immediately on the second roll or on the third and final roll.

- When $\text{Roll}_1 \left(D_1 = D_2 = 3 \right)$ then $P(\text{ spaces} \geq 12) = {}^{33}/{}_{1506} + {}^{36}/{}_{1506} + {}^{22+4\times36}/{}_{1506} = {}^{235}/{}_{1506}$
  Because the first roll is $3 + 3 = 6$, there are *eight* possible second rolls which do not result in a another roll and also do not have a cumulative sum ≥ 12, notably:
  $\left(\overline{1}, \underline{2}\right), \left(\overline{1}, \underline{3}\right), \left(\overline{1}, \underline{4}\right), \left(\overline{2}, \underline{1}\right), \left(\overline{2}, \underline{3}\right), \left(\overline{3}, \underline{1}\right), \left(\overline{3}, \underline{2}\right)$ and $\left(\overline{4}, \underline{1}\right)$.
  There are *four* possible second rolls which are doubles and do immediately yield a result ≥ 12, resulting in a partial answer of $\langle ? \rangle + {}^{22+4\times36}/{}_{1506}$.

There are *two* possible second rolls which are doubles and *do not* immediately yield a result ≥ 12 which we must consider to determine that $\langle ? \rangle = {}^{33}/{}_{1506} + {}^{36}/{}_{1506}$:

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = \mathbb{1}}$) then P (**spaces** ≥ 12) $= {}^{33}/{}_{1506}$
  Because the first roll plus the second roll is $3 + 3 + 1 + 1 = 8$, there are *three* possible third rolls which do not yield a number of spaces ≥ 12, notably $\left(\overline{\mathbb{1}}, \underline{\mathbb{1}} \right)$, $\left(\overline{\mathbb{1}}, \underline{2} \right)$ and $\left(\overline{2}, \underline{\mathbb{1}} \right)$.

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 2}$) then P (**spaces** ≥ 12) $= {}^{36}/{}_{1506}$
  Because the first roll plus the second roll plus the minimum possible third roll is $3 + 3 + 2 + 2 + 1 + 1 \geq 12$

- When **Roll$_1$** ($\mathbf{D_1 = D_2 = 2}$) then P (**spaces** ≥ 12) $= {}^{26}/{}_{1506} + {}^{33}/{}_{1506} + {}^{36}/{}_{1506} + {}^{12+3\times36}/{}_{1506} = {}^{215}/{}_{1506}$
Because the first roll is $2 + 2 = 4$, there are *eighteen* possible second rolls which do not result in a another roll and also do not have a cumulative sum ≥ 12, notably:
$\left(\overline{\mathbb{1}}, \underline{2} \right)$, $\left(\overline{\mathbb{1}}, \underline{3} \right)$, $\left(\overline{\mathbb{1}}, \underline{4} \right)$, $\left(\overline{\mathbb{1}}, \underline{5} \right)$, $\left(\overline{\mathbb{1}}, \underline{6} \right)$, $\left(\overline{2}, \underline{\mathbb{1}} \right)$, $\left(\overline{2}, \underline{3} \right)$, $\left(\overline{2}, \underline{4} \right)$, $\left(\overline{2}, \underline{5} \right)$,
$\left(\overline{3}, \underline{\mathbb{1}} \right)$, $\left(\overline{3}, \underline{2} \right)$, $\left(\overline{3}, \underline{4} \right)$, $\left(\overline{4}, \underline{\mathbb{1}} \right)$, $\left(\overline{4}, \underline{2} \right)$, $\left(\overline{4}, \underline{3} \right)$, $\left(\overline{5}, \underline{\mathbb{1}} \right)$, $\left(\overline{5}, \underline{2} \right)$, and $\left(\overline{6}, \underline{\mathbb{1}} \right)$.
There are *three* possible second rolls which are doubles and do immediately yield a result ≥ 12, resulting in a partial answer of $\langle ? \rangle + {}^{12+3\times36}/{}_{1506}$.
There are *three* possible second rolls which are doubles and *do not* immediately yield a result ≥ 12 which we must consider to determine that $\langle ? \rangle = {}^{26}/{}_{1506} + {}^{33}/{}_{1506} + {}^{36}/{}_{1506}$:

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = \mathbb{1}}$) then P (**spaces** ≥ 12) $= {}^{26}/{}_{1506}$
  Because the first roll plus the second roll is $2 + 2 + 1 + 1 = 6$, there are *ten* possible third rolls which do not yield a number of spaces ≥ 12, notably:
  $\left(\overline{\mathbb{1}}, \underline{\mathbb{1}} \right)$, $\left(\overline{\mathbb{1}}, \underline{2} \right)$, $\left(\overline{\mathbb{1}}, \underline{3} \right)$, $\left(\overline{\mathbb{1}}, \underline{4} \right)$, $\left(\overline{2}, \underline{\mathbb{1}} \right)$, $\left(\overline{2}, \underline{2} \right)$, $\left(\overline{2}, \underline{3} \right)$, $\left(\overline{3}, \underline{\mathbb{1}} \right)$, $\left(\overline{3}, \underline{2} \right)$, and $\left(\overline{4}, \underline{\mathbb{1}} \right)$.

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 2}$) then P (**spaces** ≥ 12) $= {}^{33}/{}_{1506}$
  Because the first roll plus the second roll is $2 + 2 + 2 + 2 = 8$, there are *three* possible third rolls which do not yield a number of spaces ≥ 12, notably $\left(\overline{\mathbb{1}}, \underline{\mathbb{1}} \right)$, $\left(\overline{\mathbb{1}}, \underline{2} \right)$ and $\left(\overline{2}, \underline{2} \right)$

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 3}$) then P (**spaces** ≥ 12) $= {}^{36}/{}_{1506}$
  Because the first roll plus the second roll plus the minimum possible third roll is $2 + 2 + 3 + 3 + 1 + 1 \geq 12$

- When **Roll$_1$** ($\mathbf{D_1 = D_1 = \mathbb{1}}$) then P (**spaces** ≥ 12) $= {}^{15}/{}_{1506} + {}^{26}/{}_{1506} + {}^{33}/{}_{1506} + {}^{36}/{}_{1506} + {}^{4+2\times36}/{}_{1506} = {}^{186}/{}_{1506}$
Because the first roll is $1 + 1 = 2$, there are *twenty-six* possible second rolls which do not result in a another roll and also do not have a cumulative sum ≥ 12, leaving only *four* which do, notably:
$\left(\overline{4}, \underline{6} \right)$, $\left(\overline{5}, \underline{6} \right)$, $\left(\overline{6}, \underline{4} \right)$, and $\left(\overline{6}, \underline{5} \right)$.
There are *two* possible second rolls which are doubles and do immediately yield a result ≥ 12, resulting in a partial answer of $\langle ? \rangle + {}^{4+2\times36}/{}_{1506}$.
There are *four* possible second rolls which are doubles and *do not* immediately yield a result ≥ 12 which we must consider to determine that $\langle ? \rangle = {}^{15}/{}_{1506} + {}^{26}/{}_{1506} + {}^{33}/{}_{1506} + {}^{36}/{}_{1506}$:

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = \mathbb{1}}$) then P (**spaces** ≥ 12) $= {}^{15}/{}_{1506}$
  Because the first roll plus the second roll is $1 + 1 + 1 + 1 = 4$, there are *twenty-one* possible third rolls which do not yield a number of spaces ≥ 12, leaving only *fifteen* which do, notably:
  $\left(\overline{2}, \underline{6} \right)$, $\left(\overline{3}, \underline{5} \right)$, $\left(\overline{3}, \underline{6} \right)$, $\left(\overline{4}, \underline{4} \right)$, $\left(\overline{4}, \underline{5} \right)$, $\left(\overline{4}, \underline{6} \right)$, $\left(\overline{5}, \underline{3} \right)$, $\left(\overline{5}, \underline{4} \right)$,
  $\left(\overline{5}, \underline{5} \right)$, $\left(\overline{5}, \underline{6} \right)$, $\left(\overline{6}, \underline{2} \right)$, $\left(\overline{6}, \underline{3} \right)$, $\left(\overline{6}, \underline{4} \right)$, $\left(\overline{6}, \underline{5} \right)$, and $\left(\overline{6}, \underline{6} \right)$.

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 2}$) then P (**spaces** ≥ 12) $= {}^{26}/{}_{1506}$
  Because the first roll plus the second roll is $1 + 1 + 2 + 2 = 6$, there are *ten* possible third rolls which do not yield a number of spaces ≥ 12, notably:
  $\left(\overline{\mathbb{1}}, \underline{\mathbb{1}} \right)$, $\left(\overline{\mathbb{1}}, \underline{2} \right)$, $\left(\overline{\mathbb{1}}, \underline{3} \right)$, $\left(\overline{\mathbb{1}}, \underline{4} \right)$, $\left(\overline{2}, \underline{\mathbb{1}} \right)$, $\left(\overline{2}, \underline{2} \right)$, $\left(\overline{2}, \underline{3} \right)$, $\left(\overline{3}, \underline{\mathbb{1}} \right)$, $\left(\overline{3}, \underline{2} \right)$, and $\left(\overline{4}, \underline{\mathbb{1}} \right)$.

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 3}$) then P (**spaces** ≥ 12) $= {}^{33}/{}_{1506}$
  Because the first roll plus the second roll is $1 + 1 + 3 + 3 = 8$, there are *three* possible third rolls which do not yield a number of spaces ≥ 12, notably $\left(\overline{\mathbb{1}}, \underline{\mathbb{1}} \right)$, $\left(\overline{\mathbb{1}}, \underline{2} \right)$ and $\left(\overline{2}, \underline{2} \right)$

- – When **Roll$_2$** ($\mathbf{D_1 = D_2 = 4}$) then P (**spaces** ≥ 12) $= {}^{36}/{}_{1506}$
  Because the first roll plus the second roll plus the minimum possible third roll is $1 + 1 + 4 + 4 + 1 + 1 \geq 12$

The problem statements asks to compute $P(\textbf{spaces} \geq 12)$. The hint suggests that we use the following logic:

$$P(\textbf{spaces} \geq 12) = P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 \neq \textbf{D}_2\right)\right) + \sum_{x \in \Omega} P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = x\right)\right)$$

Since $P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 \neq \textbf{D}_2\right)\right) = 0$ we have:

$$
\begin{aligned}
P(\textbf{spaces} \geq 12) &= \sum_{x \in \Omega} P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = x\right)\right) \\
&= P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 1\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 2\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 3\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 4\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 5\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 6\right)\right) \\
&= \frac{246}{1506} + \frac{246}{1506} + \frac{244}{1506} + \frac{235}{1506} + \frac{215}{1506} + \frac{186}{1506} \\
&= \frac{1372}{1506}
\end{aligned}
$$

We have determined that out of the 1506 possible outcomes, 1374 outcome will result in moving *at least* 12 spaces!

However, each outcome does not occur with equal probability, so we cannot say $P(\textbf{spaces} \geq 12) = {}^{1374}/_{1506}$. We will need to use conditional probabilities to determine the probability of each outcome.

$$
\begin{aligned}
P(\textbf{spaces} \geq 12) &= \sum_{x \in \Omega} P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = x\right)\right) \\
&= P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 1\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 2\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 3\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 4\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 5\right)\right) \\
&\quad + P\left(\textbf{spaces} \geq 12 \mid \textbf{Roll}_1\left(\textbf{D}_1 = \textbf{D}_2 = 6\right)\right) \\
&= \frac{1}{36} \times \frac{36}{36} + \frac{1}{36} \times \frac{36}{36} + \frac{1}{36} \times \frac{34}{36} \\
&\quad + \frac{1}{36} \times \frac{(22 + 5 + {}^{33}/_{36})}{36} \\
&\quad + \frac{1}{36} \times \frac{(12 + 4 + {}^{33}/_{36} + {}^{26}/_{36})}{36} \\
&\quad + \frac{1}{36} \times \frac{(4 + 3 + {}^{33}/_{36} + {}^{26}/_{36} + {}^{15}/_{36})}{36} \\
&= \frac{36^2}{36^3} + \frac{36^2}{36^3} + \frac{34 * 36}{36^3} + \frac{27 * 36 + 33}{36^3} + \frac{16 * 36 + 59}{36^3} + \frac{7 * 36 + 74}{36^3} \\
&= \frac{5782}{36^3} \\
&= \frac{2891}{23328} \\
&\approx 0.124
\end{aligned}
$$

Extra Credit 4:

Which is the most preferable from these mutually exclusive options?

♡ **Pursue your passion**

$ **Take the money**