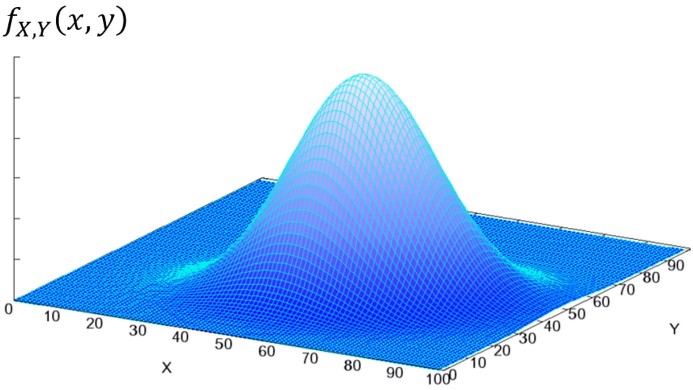
Continuous Random Variables, Joint PDF, Conditioning

Joint PDF

- What is the probability that two continuous random variables X and Y take values inside region S?
  - Calculate the volume under $f_{X,Y}(x,y)$ surface on region S.

$$\mathrm{P}((X,Y) \in S) = \int\int_S f_{X,Y}(x,y)\,dx\,dy$$



Joint PDF

- Interpretation:

$$\mathbf{P}(x \leq X \leq x+\delta,\ y \leq Y \leq y+\delta) \approx f_{X,Y}(x,y)\cdot\delta^2$$

- In other words, $f_{X,Y}(x,y)$ is approximately, the probability that X and Y take values inside the small region $[x, x+\delta] \times [y, y+\delta]$ over area of the region.

$$f_{X,Y}(x,y) \approx \frac{\text{Probability}\Big((X,Y) \in [x, x+\delta] \times [y, y+\delta]\Big)}{\text{Area of } [x, x+\delta] \times [y, y+\delta] = \delta^2}$$

Joint Expectation

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

Similar to:

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

for discrete random variables. Replace PMF with PDF and summations with Double Integral.

Example:

$$E[X^2 + Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2 + y^2)\ f_{X,Y}(x,y)\ d_x\ d_y$$

$$f_{X,Y}(x,y) = \begin{cases} 1 \ \text{ for } \ (x,y) \in [0,1] \times [0,1] \\ 0 \ \textit{otherwise}. \end{cases}$$

$$E[X^2 + Y^2] = \int_0^1\int_0^1 (x^2 + y^2)\ d_x\ d_y = \frac{2}{3}$$

From the joint PDF to the marginal PDF

- The marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

- The marginal PDF of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

- **Example:** $f_{X,Y}(x,y) = \begin{cases} 1 \quad \text{for} \quad (x,y) \in [0,1] \times [0,1] \\ 0 \; otherwise. \end{cases}$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; d_x = \int_0^1 \; d_x = 1 \quad \text{for} \quad y \in [0,1] \\ 0 \; otherwise. \end{cases}$$

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; d_y = \int_0^1 \; d_y = 1 \quad \text{for} \quad x \in [0,1] \\ 0 \; otherwise. \end{cases}$$

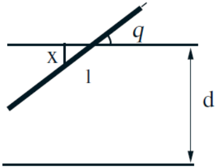
Independence

- X and Y are called independent if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y), \qquad \text{for all } x, y$$

Example – Buffon's needle

- There are several Parallel lines at distance d from each other.
- We have a needle of length l $(l < d)$
- Find Probability(needle intersects one of the lines)



- Let X be distance of needle midpoint to nearest line. Then $X \in [0, \frac{d}{2}]$.
- Also, let Θ be the angle between the needle and the nearest line, and $\Theta \in [0, \frac{\pi}{2}]$.
- Model: X and θ, both are uniform. Also, they are independent.
- Therefore, $f_{X,\Theta}(x, \theta) = f_X(x) f_\Theta(\theta)$

Example – Buffon's needle

- What is $f_X(x)$? because X is uniform, then:

$$f_X(x) = \begin{cases} c & \text{for} \quad x \in [0, \frac{d}{2}] \\ 0 & otherwise. \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f_X(x) \; d_x = \int_0^{\frac{d}{2}} c \; d_x = c\frac{d}{2} \Rightarrow c = \frac{2}{d}$$

Therefore,

$$f_X(x) = \begin{cases} \frac{2}{d} & \text{for} \quad x \in [0, \frac{d}{2}] \\ 0 & otherwise. \end{cases}$$

Similarly, Θ is uniform, then:

$$f_\Theta(\theta) = \begin{cases} c' & \text{for} \quad \theta \in [0, \frac{\pi}{2}] \\ 0 & otherwise. \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f_\Theta(\theta) \; d_\theta = \int_0^{\frac{\pi}{2}} c' \; d_\theta = c'\frac{\pi}{2} \Rightarrow c' = \frac{2}{\pi}$$

$$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi} & \text{for} \quad x \in [0, \frac{\pi}{2}] \\ 0 & otherwise. \end{cases}$$

Example – Buffon's needle

- Therefore, $f_{X,\Theta}(x,\theta) = f_X(x) f_\Theta(\theta) = \frac{4}{\pi d}$.
- Requirement for intersection with at least one of the lines is: $X \leq \frac{l}{2} sin\Theta$
- So, we should calculate $P(X \leq \frac{l}{2} sin\Theta)$.

$$\begin{aligned}
\mathbf{P}\left(X \leq \frac{\ell}{2}\sin\Theta\right) &= \int\int_{x\leq\frac{\ell}{2}\sin\theta} f_X(x) f_\Theta(\theta)\, dx\, d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{(\ell/2)\sin\theta} dx\, d\theta \\
&= \frac{4}{\pi d}\int_0^{\pi/2}\frac{\ell}{2}\sin\theta\, d\theta = \frac{2\ell}{\pi d}
\end{aligned}$$

Conditioning

- Recall

$$\mathbf{P}(x \le X \le x+\delta) \approx f_X(x) \cdot \delta$$

- By analogy, would like:

$$\mathbf{P}(x \le X \le x+\delta \mid Y \approx y) \approx f_{X|Y}(x \mid y) \cdot \delta$$

- This leads us to the **definition**:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if } f_Y(y) > 0$$

- For given y, conditional PDF is a (normalized) "section" of the joint PDF

- If independent, $f_{X,Y} = f_X f_Y$, we obtain

$$f_{X|Y}(x|y) = f_X(x)$$

Conditioning

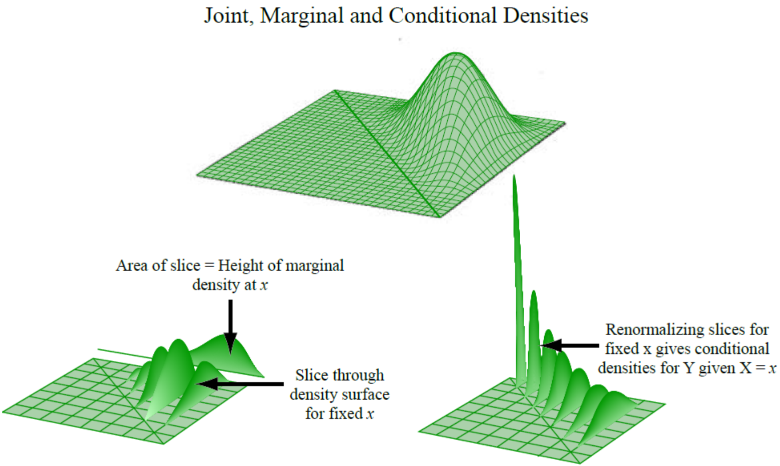
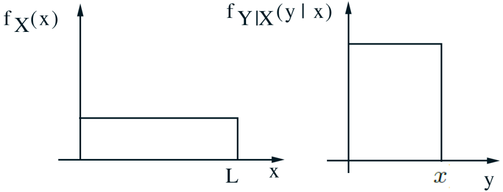


Image by MIT OpenCourseWare, adapted from *Probability*, by J. Pittman, 1999.

Stick-breaking example

- Break a stick of length l twice. First break at X, uniform in $[0, l]$. Then break again at Y, uniform in $[0, X]$.

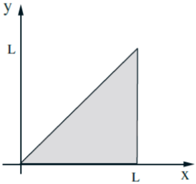


- $f_X(x) = \frac{1}{l}$
- $f_{Y|X}(y|x) = \frac{1}{x}$

$$\Rightarrow f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = \frac{1}{lx} \quad \text{for} \quad 0 \leq y \leq x \leq l$$

Stick-breaking example

- What is $f_Y(y)$ and $E[Y]$?



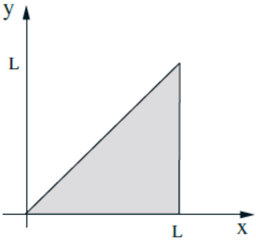
$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x,y)\,dx \\ &= \int_y^{\ell} \frac{1}{\ell x}\,dx \\ &= \frac{1}{\ell}\log\frac{\ell}{y}, \qquad 0 \le y \le \ell \end{aligned}$$

$$\mathbf{E}[Y] = \int_0^{\ell} y f_Y(y)\,dy = \int_0^{\ell} y\frac{1}{\ell}\log\frac{\ell}{y}\,dy = \frac{\ell}{4}$$

- Interpretation: in expectation, the second time you break the stick, it will be broken at the quarter of the original stick.

Stick-breaking example

- What is $E[Y|X=x]$?



- $E[Y|X=x] = \int y\ f_{Y|X}(y|x)\ d_y = \int_0^x y\ \frac{1}{x}\ d_y = \frac{1}{x}\int_0^x y\ d_y = \frac{1}{x}(\frac{x^2}{2} - 0) = \frac{x}{2}$.
- Interpretation: in expectation, the second time you break the stick, it will be broken at the middle point of the current stick, between 0 to x.