Einstein field equation, recursion operators, Noether and master symmetries in conformable Poisson manifolds

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Abstract: We show that a Minkowski phase space endowed with a bracket relatively to a conformable differential realizes a Poisson algebra, confering a bi-Hamiltonian structure to the resulting manifold. We infer that the related Hamiltonian vector field is an infinitesimal Noether symmetry, and compute the corresponding deformed recursion operator. Besides, using the Hamiltonian-Jacobi separability, we construct recursion operators for Hamiltonian vector fields in conformable Poisson-Schwarzschild and Friedmann-Lemaître-Robertson-Walker (FLRW) manifolds, and derive related constants of motion, Christoffel symbols, components of Riemann and Ricci tensors, Ricci constant and components of Einstein tensor. We highlight the existence of a hierarchy of bi-Hamiltonian structures in both the manifolds, and compute a family of recursion operators and master symmetries generating the constants of motion.

Keywords: Einstein field equation, recursion operator, Noether symmetry, master symmetry, conformable differential, Poisson manifold.

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1 Introduction

Conformable fractional calculus has a long and rich history. In 1695, Gottfried Leibniz asked Guillaume l'Hôspital if the (integer) order of derivatives and integrals could be extended [41]. Was it possible if the order was some irrational, fractional or complex number? This idea motivated many mathematicians, physicists and engineers to develop the concept of fractional calculus in diverse fields of science and engineers, see e.g., [6, 30, 16, 1, 2, 3, 21, 12], and references therein. During four centuries, many famous mathematicians contributed to this development. It is still nowadays one of the most intensively developing areas of

mathematical analysis, including several definitions of fractional operators like Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz and Weyl definitions [22], [28], [40], [6]. Two of these definitions, namely Riemann-Liouville and Caputo ones are famous. Mathematicians prefer Riemann-Liouville fractional derivative while physicists and engineers use Caputo fractional one. Indeed, the Riemann-Liouville fractional derivative of a constant is not zero, and it requires fractional initial conditions which are not generally specified [6]. In contrast, Caputo derivative of a constant is zero, and a fractional differential equation expressed in terms of Caputo fractional derivative requires standard boundary conditions. Unfortunately, the Riemann-Liouville derivative and Caputo derivative do not obey the Leibniz rule and chain rule, which sometimes prevent us from applying these derivatives to ordinary physical systems with standard Newton derivative. In 2014, Khalil et al. [23] introduced the new fractional derivative called conformable fractional derivative and integral obeying the Leibniz rule and chain rule. One year later, i.e., in 2015, Chung [6] used this conformable fractional derivative and integral to discuss the fractional version of the Newtonian mechanics. In that work, he constructed the fractional Euler-Lagrange equation from the fractional version of the calculus of variations and used this equation to discuss some mechanical problems such as fractional harmonic oscillator problem, the fractional damped oscillator problem and the forced oscillator problem. In 2017, Chung et al. [7] discussed the dynamics of a particle in a viscoelastic medium using the conformable fractional derivative of order α with respect to time. Further, in 2019, the same authors [8] discussed the fractional classical mechanics and applied it to the anomalous diffusion relation from the α -deformed Langevin equation. During the same year, Kiskinov et al. [25] investigated the Cauchy problem for nonlinear systems with conformable derivatives and variable delays. Furthermore, Khalil et al. gave the geometric meaning of conformable derivative via fractional cords in 2019 [24]. In 2020, Chung et al. [9] studied the deformed special relativity based on α -deformed binary operations. In that work, they gave the α -translation invariant distance (α -distance) of infinitesimally close space-time based on the definition of α -translation invariant infinitesimal displacement and α -translation invariant infinitesimal time interval.

In addition, in the last few decades, there was a renewed interest in completely integrable Hamiltonian systems (IHS), the concept of which goes back to Liouville in 1897 [27] and Poincaré in 1899 [32]. In short, IHS are defined as nonlinear differential equations admitting a Hamiltonian description and possessing sufficiently many constants of motion so that they can be integrated by quadratures [14]. This Liouville formalism does not provide a method for obtaining the integrals of motion; therefore it has been necessary to elaborate different methods for obtaining constants of motion (Hamilton-Jacobi separability, Lax pairs formalism, Noether symmetries, Hidden symmetries, etc). A relevant progress in the analysis of the integrability was the important remark that many of these systems are Hamiltonian dynamics with respect to two compatible symplectic structures [29, 15, 42], permitting a geometrical interpretation of the so-called recursion operator [26, 18, 19]. A description of integrability working both for systems with finitely many degrees of freedom and for field theory can be given in terms of invariant, diagonalizable mixed (1,1)-tensor field, having bidimensional eigenspaces and vanishing Nijenhuis torsion. One of powerful methods of describing IHS with involutive Hamiltonian functions or constants of motion uses the recursion operator admitting a vanishing Nijenhuis torsion. In 2015, Takeuchi constructed recursion operators of Hamiltonian vector fields of geodesic flows for some Riemannian and Minkowski metrics [39], and obtained related constants of motion. In his work, Takeuchi used five particular solutions of the Einstein equation in the Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman, and FLRW metrics, and constructed recursion operators inducing the complete integrability of the Hamiltonian functions. Further, in 2019, we investigated the same problem in a noncommutative Minkowski phase space [17].

In the present work, we investigate Noether symmetry and recursion operators induced by an α -deformed Poisson algebra in a Minkowski phase space. We construct recursion operators using α -deformed Schwarzschild and Friedmann-Lemaître-Robertson-Walker (FLRW) metrics and discuss their relevant master symmetries.

The paper is organized as follows. In Section 2, we give the notion of conformable differential and related formulation of the wellknown Takeuchi Lemma [39]. In Section 3, we construct an α -deformed Poisson algebra and the Lie algebra of deformed vector fields, prove the existence of infinitesimal Noether symmetry and bi-Hamiltonian structure, and compute the corresponding recursion operator in an α -deformed Minkowski phase space. In Section 4, we construct recursion operators for Hamiltonian vector fields, related constants of motion, Christoffel symbols, components of Riemann and Ricci tensors, Ricci constant and components of Einstein tensor in the framework of α -deformed Schwarzschild and FLRW metrics. In Section 5, we derive a hierarchy of master symmetries and compute the conserved quantities. In Section 6, we end with some concluding remarks.

2 Conformable differential and formulation of Takeuchi Lemma

A Hamiltonian system is a triple (\mathcal{Q}, ω, H) , where (\mathcal{Q}, ω) is a symplectic manifold and H is a smooth function on \mathcal{Q} , called *Hamiltonian* or *Hamiltonian function* [36].

Given a general dynamical system defined on the 2n-dimensional manifold \mathcal{Q} [37, 38], its evolution can be described by the equation

$$\dot{x}(t) = X(x), \quad x \in \mathcal{Q}, \quad X \in \mathcal{T}\mathcal{Q}.$$
 (1)

If the system (1) admits two different Hamiltonian representations:

$$\dot{x}(t) = X_{H_1, H_2} = \mathcal{P}_1 dH_1 = \mathcal{P}_2 dH_2,$$

its integrability as well as many other properties are subject to Magri's approach. The bi-Hamiltonian vector field X_{H_1,H_2} is defined by two pairs of Poisson bivectors $\mathcal{P}_1, \mathcal{P}_2$ and Hamiltonian functions H_1, H_2 . Such a manifold \mathcal{Q} equipped with two Poisson bivectors is called a double Poisson manifold, and the quadruple $(\mathcal{Q}, \mathcal{P}_1, \mathcal{P}_2, X_{H_1,H_2})$ is called a bi-Hamiltonian system. \mathcal{P}_1 and \mathcal{P}_2 are two compatible Poisson bivectors with vanishing Schouten-Nijenhuis bracket [11]:

$$[\mathcal{P}_1, \mathcal{P}_2]_{NS} = 0.$$

A recursion operator $T: \mathcal{TQ} \longrightarrow \mathcal{TQ}$ is defined by

$$T:=\mathcal{P}_2\circ\mathcal{P}_1^{-1}.$$

A Noether symmetry is a diffeomorphism $\Phi: \mathcal{Q} \longrightarrow \mathcal{Q}$ such that [35]:

$$\Phi^*\omega = \omega, \quad \Phi^*H = H.$$

An infinitesimal Noether symmetry is a vector field $Y \in \mathfrak{X}(\mathcal{Q})$ (the set of all differentiable vector fields on \mathcal{Q}) such that:

$$\mathcal{L}_Y \omega = 0, \quad \mathcal{L}_Y H = 0.$$

Definition 1. Consider the map g and its inverse g^{-1} :

$$g: \mathbb{R}^{2n}_{\alpha} \longrightarrow \mathbb{R}^{2n} \qquad \qquad g^{-1}: \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}_{\alpha}$$
$$z \longmapsto g(z) = |z|^{\alpha - 1}z = \mathbf{Z} \qquad \qquad \mathbf{Z} \longmapsto g^{-1}(\mathbf{Z}) = |\mathbf{Z}|^{(1/\alpha) - 1}\mathbf{Z} = z,$$

where g(0) = 0, g(1) = 1, and $g(\pm \infty) = \pm \infty$. Then, for this map, the α -addition, α -subtraction, α -multiplication, and α -division are given by:

$$a \oplus_{\alpha} b = |a|a|^{\alpha-1} + b|b|^{\alpha-1}|^{(1/\alpha)-1}(a|a|^{\alpha-1} + b|b|^{\alpha-1}),$$

$$a \ominus_{\alpha} b = |a|a|^{\alpha-1} - b|b|^{\alpha-1}|^{(1/\alpha)-1}(a|a|^{\alpha-1} - b|b|^{\alpha-1}),$$

$$a \otimes_{\alpha} b = ab,$$

$$a \oslash_{\alpha} b = \frac{a}{b},$$

where $a, b \in \mathbb{R}^{2n}_{\alpha}$.

Definition 2. Let h be a differentiable coordinates function on \mathbb{R}^{2n}_{α} . The conformable differential, also called α -differential in sequel, with respect to the position q and its associated momentum p is defined by:

$$d_{\alpha}: \mathbb{R}^{2n}_{\alpha} \longrightarrow \mathbb{R}^{2n}$$

$$h \longmapsto d_{\alpha}h := \sum_{\mu=1}^{2n} \alpha |x_{\mu}|^{\alpha-1} \frac{\partial}{\partial x_{\mu}} h, \quad (x_{\nu} = q^{\nu}, x_{\nu+n} = p_{\nu}, \ n = 4, \ \nu = 1, 2, 3, 4)$$

satisfying the following properties:

(i)
$$d_{\alpha}(ah + bf) = ad_{\alpha}h + bd_{\alpha}f$$
 for all $a, b \in \mathbb{R}$;

(ii)
$$d_{\alpha}(h^m) = mh^{m-1}d_{\alpha}h$$
, for all $m \in \mathbb{R}$;

(iii)
$$d_{\alpha}(c) = 0$$
, for all constant functions $h(q, p) = c$;

(iv)
$$d_{\alpha}(hf) = hd_{\alpha}f + fd_{\alpha}h$$
;

(iv)
$$d_{\alpha}\left(\frac{h}{f}\right) = \frac{fd_{\alpha}h - hd_{\alpha}f}{f^2}$$
, where f is also a differentiable coordinates function on \mathbb{R}^{2n}_{α} .

The α -differential produces a new deformed phase space called α -deformed phase space. The ordinary differential is obtained for $\alpha = 1$. Using the α -addition and α -subtraction, we obtain the following infinitesimal distance between two points of coordinates $(x_i, ..., x_n)$ and $(x_i \oplus_{\alpha} d_{\alpha}x_i, ..., x_n \oplus_{\alpha} d_{\alpha}x_n)$

$$d_{\alpha}s = (d_{\alpha}^2 x_i + \dots + d_{\alpha}^2 x_n)^{\frac{1}{2}}.$$

In the \mathbb{R}^{2n}_{α} , Takeuchi Lemma [39] takes the following form:

Lemma 1. Consider α -vector fields

$$X_{\alpha_i} = -|x_i|^{(1-\alpha)}|x_{n+i}|^{(1-\alpha)}\frac{\partial}{\partial x_{n+i}}, i = 1, ..., n$$

on \mathbb{R}^{2n}_{α} and

$$T_{\alpha} = \sum_{i=1}^{n} |x_{i}|^{(\alpha-1)} |x_{i} \left(\frac{\partial}{\partial x_{i}} \otimes dx_{i} + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right),$$

a (1,1)-tensor field on \mathbb{R}^{2n}_{α} . Then, we have that the Nijenhuis torsion of T_{α} is vanishing, i.e., $\mathcal{N}_{T_{\alpha}} = 0$ and $\mathcal{L}_{X_{\alpha_i}} T_{\alpha} = 0$, that is, the (1,1)-tensor field T_{α} is an α -recursion operator of X_{α_i} , (i = 1, ..., n).

Proof.

We have:

$$\mathcal{L}_{X_{\alpha_{i}}} T_{\alpha} = \mathcal{L}_{X_{\alpha_{i}}} \left\{ \sum_{i=1}^{n} |x_{i}|^{(\alpha-1)} |x_{i} \left(\frac{\partial}{\partial x_{i}} \otimes dx_{i} + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right) \right\}$$

$$= \sum_{i=1}^{n} \left\{ \mathcal{L}_{X_{\alpha_{i}}} (|x_{i}|^{(\alpha-1)} |x_{i}) \left(\frac{\partial}{\partial x_{i}} \otimes dx_{i} + \frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right) + |x_{i}|^{(\alpha-1)} |x_{i} \left(\mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{i}} \otimes dx_{i} \right] + \mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right] \right) \right\}$$

$$\mathcal{L}_{X_{\alpha_{i}}} T_{\alpha} = \sum_{i=1}^{n} |x_{i}|^{(\alpha-1)} |x_{i} \left(\mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{i}} \otimes dx_{i} \right] + \mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{n+i}} \otimes dx_{n+i} \right] \right)$$

because $\mathcal{L}_{X_{\alpha_i}}(|x_i|^{(\alpha-1)}|x_i) = 0.$

Then,

$$\mathcal{L}_{X_{\alpha_{i}}} T_{\alpha} = \sum_{i=1}^{n} |x_{i}|^{(\alpha-1)} |x_{i} \left(\mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{i}} \right] \otimes dx_{i} + \frac{\partial}{\partial x_{i}} \otimes \mathcal{L}_{X_{\alpha_{i}}} (dx_{i}) \right.$$
$$\left. + \mathcal{L}_{X_{\alpha_{i}}} \left[\frac{\partial}{\partial x_{n+i}} \right] \otimes dx_{n+i} + \frac{\partial}{\partial x_{n+i}} \otimes \mathcal{L}_{X_{\alpha_{i}}} (dx_{n+i}) \right)$$
$$\mathcal{L}_{X_{\alpha_{i}}} T_{\alpha} = 0.$$

The components of the Nijenhuis torsion as follows [39]:

$$(\mathcal{N}_{T_{\alpha}})_{ij}^{h} = (T_{\alpha})_{i}^{k} \frac{\partial (T_{\alpha})_{j}^{h}}{\partial x_{k}} - (T_{\alpha})_{j}^{k} \frac{\partial (T_{\alpha})_{i}^{h}}{\partial x_{k}} + (T_{\alpha})_{k}^{h} \frac{\partial (T_{\alpha})_{i}^{k}}{\partial x_{j}} - (T_{\alpha})_{k}^{h} \frac{\partial (T_{\alpha})_{j}^{k}}{\partial x_{i}}$$

$$= |x_{i}|^{(\alpha-1)} |x_{i} \frac{\partial (T_{\alpha})_{j}^{h}}{\partial x_{i}} - |x_{j}|^{(\alpha-1)} |x_{j} \frac{\partial (T_{\alpha})_{i}^{h}}{\partial x_{j}} + (T_{\alpha})_{i}^{h} \frac{\partial (|x_{i}|^{(\alpha-1)}|x_{i})}{\partial x_{j}} - (T_{\alpha})_{j}^{h} \frac{\partial (|x_{j}|^{(\alpha-1)}|x_{j})}{\partial x_{i}}$$

$$= |x_{i}|^{(\alpha-1)} |x_{i} \frac{\partial (T_{\alpha})_{j}^{h}}{\partial x_{i}} - |x_{j}|^{(\alpha-1)} |x_{j} \frac{\partial (T_{\alpha})_{i}^{h}}{\partial x_{j}} + \alpha (T_{\alpha})_{i}^{h} |x_{i}|^{(\alpha-1)} \delta_{j}^{i} - \alpha (T_{\alpha})_{j}^{h} |x_{j}|^{(\alpha-1)} \delta_{i}^{j}.$$

1. If i = j, we have $\delta_i^i = \delta_i^j = 1$ and we get

$$(\mathcal{N}_{T_{\alpha}})_{ij}^{h} = |x_{i}|^{(\alpha-1)} |x_{i} \frac{\partial (T_{\alpha})_{i}^{h}}{\partial x_{i}} - |x_{i}|^{(\alpha-1)} |x_{i} \frac{\partial (T_{\alpha})_{i}^{h}}{\partial x_{i}} + \alpha |x_{i}|^{(\alpha-1)} (T_{\alpha})_{i}^{h} - \alpha |x_{i}|^{(\alpha-1)} (T_{\alpha})_{i}^{h} = 0;$$
(2)

2. If
$$i \neq j$$
, we have $\delta_j^i = \delta_i^j = 0$ and $\frac{\partial (T_\alpha)_j^h}{\partial x_i} = \frac{\partial (T_\alpha)_i^h}{\partial x_j} = 0$. Then,
$$(\mathcal{N}_{T_\alpha})_{ij}^h = 0. \tag{3}$$

From (2) and (3), we get
$$\mathcal{N}_{T_{\alpha}} = 0$$
.

3 Recursion operator in α -deformed Minkowski phase space

In this section, we derive the recursion operator of Hamiltonian vector fields of geodesic flow for a free particle in an α -deformed Minkowski phase space and obtain the associated constants of motion.

3.1 Symplectic structure, Poisson bracket and Lie algebra

We consider our configuration space as a manifold $Q = \mathbb{R}^4_{\alpha} \setminus \{0\}$ that is, a four-dimensional real Euclidean vector space with the origin removed. The cotangent bundle $\mathcal{T}^*Q = \mathcal{Q} \times \mathbb{R}^4_{\alpha}$ has a natural α -symplectic structure

 $\omega_{\alpha}: \mathcal{TQ} \longrightarrow \mathcal{T}^*\mathcal{Q}$ which, in local coordinates (q,p), is given by

$$\omega_{\alpha} = \sum_{\mu=1}^{4} d_{\alpha} p_{\mu} \wedge d_{\alpha} q^{\mu} = \sum_{\mu=1}^{4} \alpha^{2} |p_{\mu}|^{\alpha-1} |q^{\mu}|^{\alpha-1} dp_{\mu} \wedge dq^{\mu}.$$

Since ω_{α} is non-degenerate, it induces an inverse map, called α -bivector field \mathcal{P}_{α} : $\mathcal{T}^*\mathcal{Q} \longrightarrow \mathcal{T}\mathcal{Q}$ (tangent bundle) defined by

$$\mathcal{P}_{\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{1-\alpha} |q^{\mu}|^{1-\alpha} \frac{\partial}{\partial p_{\mu}} \wedge \frac{\partial}{\partial q^{\mu}}, \quad \omega_{\alpha} \circ \mathcal{P}_{\alpha} = \mathcal{P}_{\alpha} \circ \omega_{\alpha} = 1,$$

and used to construct the α -Hamiltonian vector field X_{α_f} of an α -Hamiltonian function f by the relation

$$X_{\alpha_f} = \mathcal{P}_{\alpha} df$$
.

We consider now the next α -Minkowski metric on the manifold \mathcal{Q} :

$$d_{\alpha}s^{2} = -\alpha^{2}|q^{1}|^{2(\alpha-1)}(dq^{1})^{2} + \alpha^{2}|q^{2}|^{2(\alpha-1)}(dq^{2})^{2} + \alpha^{2}|q^{3}|^{2(\alpha-1)}(dq^{3})^{2} + \alpha^{2}|q^{4}|^{2(\alpha-1)}(dq^{4})^{2},$$

where c=1 for commodity yielding the tensor metric $(g_{\mu\nu})_{\alpha}$ and its inverse $(g^{\mu\nu})_{\alpha}$

$$(g_{\mu\nu})_{\alpha} = \alpha^2 \begin{pmatrix} -(q^1)^{2(\alpha-1)} & 0 & 0 & 0\\ 0 & (q^2)^{2(\alpha-1)} & 0 & 0\\ 0 & 0 & (q^3)^{2(\alpha-1)} & 0\\ 0 & 0 & 0 & (q^4)^{2(\alpha-1)} \end{pmatrix},$$

$$(g^{\mu\nu})_{\alpha} = \frac{1}{\alpha^2} \left(\begin{array}{cccc} -(q^1)^{2(1-\alpha)} & 0 & 0 & 0 \\ 0 & (q^2)^{2(1-\alpha)} & 0 & 0 \\ 0 & 0 & (q^3)^{2(1-\alpha)} & 0 \\ 0 & 0 & 0 & (q^4)^{2(1-\alpha)} \end{array} \right).$$

In our framework, the equation of the geodesic on the manifold Q is given by

$$\frac{d^{2}q^{\mu}}{dt^{2}} + (\Gamma^{\mu}_{\nu\lambda})_{\alpha} \frac{dq^{\nu}}{dt} \frac{dq^{\lambda}}{dt} = 0, \quad (\nu, \mu, \lambda = 1, 2, 3, 4), \tag{4}$$

where

$$(\Gamma^{\mu}_{\nu\lambda})_{\alpha} = \frac{1}{2} (g^{\mu\epsilon})_{\alpha} \left(\frac{\partial (g_{\epsilon\nu})_{\alpha}}{\partial q^{\lambda}} + \frac{\partial (g_{\epsilon\lambda})_{\alpha}}{\partial q^{\nu}} - \frac{\partial (g_{\nu\lambda})_{\alpha}}{\partial q^{\epsilon}} \right)$$
 (5)

are α -Christoffel symbols. From (5), we have

$$(\Gamma_{11}^1)_{\alpha} = \frac{\alpha - 1}{q^1}; \ (\Gamma_{22}^2)_{\alpha} = \frac{\alpha - 1}{q^2}; \ (\Gamma_{33}^3)_{\alpha} = \frac{\alpha - 1}{q^3}; \ (\Gamma_{44}^4)_{\alpha} = \frac{\alpha - 1}{q^4}; \ (\Gamma_{\nu\lambda}^{\mu})_{\alpha} = 0, \ \text{otherwise},$$

and obtain that the α -Riemann tensor components are vanished, i.e., $R_{ijkl} = 0$, (i, j, k, l = 1, 2, 3, 4). Then, the Minkowski phase space endowed with the metric $d_{\alpha}s^2$ is a flat space. Thus, we notice that this α -deformation does not change the geometric structure of the ordinary Minkowski phase space. Further, the presence of the α -Christoffel symbols $(\Gamma_{ii}^i)_{\alpha}$, (i = 1, 2, 3, 4) means that the parallel displacement of any basic vector of our considered manifold with respect to itself always remains parallel with this same basic vector. The ordinary Minkowski phase space is obtained for $\alpha = 1$.

Since the quantities $(\tilde{\Gamma}^{\mu}_{\nu\lambda})_{\alpha} = \frac{1}{\alpha+1}(\Gamma^{\mu}_{\nu\lambda})_{\alpha}$ do not change the geometric structure of the Minkowski phase space, we replace $(\Gamma^{\mu}_{\nu\lambda})_{\alpha}$ by $(\tilde{\Gamma}^{\mu}_{\nu\lambda})_{\alpha}$ in (4). Then, the equation of the geodesic becomes:

$$\frac{d^2q^{\mu}}{dt^2} + (\tilde{\Gamma}^{\mu}_{\nu\lambda})_{\alpha} \frac{dq^{\nu}}{dt} \frac{dq^{\lambda}}{dt} = 0, \quad (\nu, \mu, \lambda = 1, 2, 3, 4).$$

If we put $v^{\mu} = \frac{dq^{\mu}}{dt}$, we have a first order differential equation on the tangent bundle $\mathcal{T}(\mathcal{Q})$ of the manifold \mathcal{Q} :

$$\dot{q^{\mu}} = v^{\mu}, \quad \dot{v^{\mu}} = -\frac{1}{\alpha + 1} (\Gamma^{\mu}_{\nu\lambda})_{\alpha} v^{\nu} v^{\lambda}.$$

From the above equations, we get the α -geodesic spray

$$X_{\alpha} := v^{\mu} \frac{\partial}{\partial q^{\mu}} - \frac{1}{\alpha + 1} (\Gamma^{\mu}_{\nu \lambda})_{\alpha} v^{\nu} v^{\lambda} \frac{\partial}{\partial v^{\mu}}.$$

By setting $p_{\mu} = \varepsilon_{\mu\epsilon} v^{\epsilon}$, $\varepsilon = sgn(-, +, +, +)$, the α -vector field X_{α} is equivalently transformed to the α -vector field X_{α} on the cotangent bundle $\mathcal{T}^*(\mathcal{Q})$ such that

$$X_{\alpha} = -p_1 \frac{\partial}{\partial q^1} + \sum_{k=2}^{4} p_k \frac{\partial}{\partial q^k} + \left(\frac{\alpha - 1}{\alpha + 1}\right) \frac{p_1^2}{q^1} \frac{\partial}{\partial p_1} - \sum_{k=2}^{4} \left(\frac{\alpha - 1}{\alpha + 1}\right) \frac{p_k^2}{q^k} \frac{\partial}{\partial p_k},\tag{6}$$

The vector field X_{α} is an α -Hamiltonian vector field of a certain α -Hamiltonian function H_{α} .

Proposition 1. The set \mathfrak{F} of differentiable functions defined on $\mathcal{T}^*(\mathcal{Q})$ endowed with the bracket

$$\{f,g\}_{\alpha} := \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}} \right)$$

is an α -deformed Poisson algebra.

Proof.

To prove this Proposition, we just have to prove that the bracket $\{.,.\}_{\alpha}$ is an α -deformed Poisson bracket.

Let us consider f, g, and h be three arbitrary elements of \mathfrak{F} .

• Antisymmetry

$$\begin{split} \{f,g\}_{\alpha} &= \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}} \right), \\ &= -\sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial g}{\partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} - \frac{\partial g}{\partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}} \right), \\ &= -\{g,f\}_{\alpha}. \end{split}$$

Jacobi identity

$$\begin{aligned}
\{f, \{g, h\}_{\alpha}\}_{\alpha} &= \left\{f, \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial g}{\partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} - \frac{\partial g}{\partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} \right) \right\}_{\alpha} \\
&= \sum_{\mu,\nu=1}^{4} \alpha^{-4} |p_{\nu}|^{(1-\alpha)} |q^{\nu}|^{(1-\alpha)} \left[\frac{\partial f}{\partial p_{\nu}} \left(\sigma_{1} |p_{\mu}|^{(1-\alpha)} (q^{\mu})^{-\alpha} \left(\frac{\partial g}{\partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} - \frac{\partial g}{\partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} \right) \right. \\
&+ |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial^{2} g}{\partial q^{\nu} \partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} + \frac{\partial g}{\partial p_{\mu}} \frac{\partial^{2} h}{\partial q^{\nu} \partial q^{\mu}} - \frac{\partial^{2} g}{\partial q^{\nu} \partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} - \frac{\partial g}{\partial q^{\nu}} \frac{\partial^{2} h}{\partial p_{\mu}} \right) \right. \\
&- \frac{\partial f}{\partial q^{\nu}} \left(\sigma_{2} |q^{\mu}|^{(1-\alpha)} (p_{\mu})^{-\alpha} \left(\frac{\partial g}{\partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} - \frac{\partial g}{\partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} \right) + |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \right. \\
&\times \left. \left(\frac{\partial^{2} g}{\partial p_{\nu} \partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} + \frac{\partial g}{\partial p_{\mu}} \frac{\partial^{2} h}{\partial p_{\nu} \partial q^{\mu}} - \frac{\partial^{2} g}{\partial p_{\nu} \partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} - \frac{\partial g}{\partial q^{\mu}} \frac{\partial^{2} h}{\partial p_{\nu} \partial p_{\mu}} \right) \right. \right], \tag{7}
\end{aligned}$$

$$\{h, \{f, g\}_{\alpha}\}_{\alpha} = \left\{h, \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}}\right)\right\}_{\alpha}$$

$$= \sum_{\mu,\nu=1}^{4} \alpha^{-4} |p_{\nu}|^{(1-\alpha)} |q^{\nu}|^{(1-\alpha)} \left[\frac{\partial h}{\partial p_{\nu}} \left(\sigma_{1} |p_{\mu}|^{(1-\alpha)} (q^{\mu})^{-\alpha} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}}\right)\right]$$

$$+ |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial^{2} f}{\partial q^{\nu} \partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} + \frac{\partial f}{\partial p_{\mu}} \frac{\partial^{2} g}{\partial q^{\nu} \partial q^{\mu}} - \frac{\partial^{2} f}{\partial q^{\nu} \partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}} - \frac{\partial f}{\partial q^{\nu}} \frac{\partial^{2} g}{\partial p_{\mu}} - \frac{\partial f}{\partial q^{\nu}} \frac{\partial^{2} g}{\partial p_{\mu}}\right)$$

$$- \frac{\partial h}{\partial q^{\nu}} \left(\sigma_{2} |q^{\mu}|^{(1-\alpha)} (p_{\mu})^{-\alpha} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}}\right) + |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)}$$

$$\times \left(\frac{\partial^{2} f}{\partial p_{\nu} \partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} + \frac{\partial f}{\partial p_{\mu}} \frac{\partial^{2} g}{\partial p_{\nu} \partial q^{\mu}} - \frac{\partial^{2} f}{\partial p_{\nu} \partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial^{2} g}{\partial p_{\nu} \partial p_{\mu}}\right)\right], \tag{8}$$

$$\{g, \{h, f\}_{\alpha}\}_{\alpha} = \left\{g, \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial h}{\partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} - \frac{\partial h}{\partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}}\right)\right\}_{\alpha}$$

$$= \sum_{\mu,\nu=1}^{4} \alpha^{-4} |p_{\nu}|^{(1-\alpha)} |q^{\nu}|^{(1-\alpha)} \left[\frac{\partial g}{\partial p_{\nu}} \left(\sigma_{1} |p_{\mu}|^{(1-\alpha)} (q^{\mu})^{-\alpha} \left(\frac{\partial h}{\partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} - \frac{\partial h}{\partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}}\right)\right]$$

$$+ |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial^{2} h}{\partial q^{\nu} \partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} + \frac{\partial h}{\partial p_{\mu}} \frac{\partial^{2} f}{\partial q^{\nu} \partial q^{\mu}} - \frac{\partial^{2} h}{\partial q^{\nu} \partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}} - \frac{\partial h}{\partial q^{\nu}} \frac{\partial^{2} f}{\partial p_{\mu}}\right)$$

$$- \frac{\partial g}{\partial q^{\nu}} \left(\sigma_{2} |q^{\mu}|^{(1-\alpha)} (p_{\mu})^{-\alpha} \left(\frac{\partial h}{\partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} - \frac{\partial h}{\partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}}\right) + |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)}\right)$$

$$\times \left(\frac{\partial^{2} h}{\partial p_{\nu} \partial p_{\mu}} \frac{\partial f}{\partial q^{\mu}} + \frac{\partial h}{\partial p_{\mu}} \frac{\partial^{2} f}{\partial p_{\nu} \partial q^{\mu}} - \frac{\partial^{2} h}{\partial p_{\nu} \partial q^{\mu}} \frac{\partial f}{\partial p_{\mu}} - \frac{\partial h}{\partial q^{\mu}} \frac{\partial^{2} f}{\partial p_{\nu} \partial p_{\mu}}\right)\right], \tag{9}$$

where $\sigma_1 = (1 - \alpha)(sgn(q^{\mu}))^{1-\alpha}$ and $\sigma_2 = (1 - \alpha)(sgn(p_{\mu}))^{1-\alpha}$. Summing (7), (8), and (9), we get

$${f, {g,h}_{\alpha}}_{\alpha} + {g, {h,f}_{\alpha}}_{\alpha} + {h, {f,g}_{\alpha}}_{\alpha} = 0,$$

which is the Jacobi identity.

• Derivation

$$\begin{split} \{f,gh\}_{\alpha} &= \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial (gh)}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial (gh)}{\partial p_{\mu}} \right) \\ &= \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left[\frac{\partial f}{\partial p_{\mu}} \left(\frac{\partial g}{\partial q^{\mu}} h + g \frac{\partial h}{\partial q^{\mu}} \right) - \frac{\partial f}{\partial q^{\mu}} \left(\frac{\partial g}{\partial p_{\mu}} h + g \frac{\partial h}{\partial p_{\mu}} \right) \right] \\ &= \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left[g \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial h}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial h}{\partial p_{\mu}} \right) + \left(\frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial q^{\mu}} - \frac{\partial f}{\partial q^{\mu}} \frac{\partial g}{\partial p_{\mu}} \right) h \right], \end{split}$$

which proves the derivative property: $\{f,gh\}_{\alpha} = g\{f,h\}_{\alpha} + \{f,g\}_{\alpha}h$.

Thus, the bracket $\{.,.\}_{\alpha}$ is antisymmetry and satisfies the Jacobi identity and the derivation property. Therefore, it is a Poisson bracket and $(\mathfrak{F}, \{.,.\}_{\alpha})$ is an α -deformed Poisson algebra. \square

Proposition 2. The set of α -Hamiltonian vector fields $\mathfrak{X}_{\alpha_{\mathfrak{F}}}$ endowed with the Lie bracket given by the commutator [.,.] is an α -deformed Lie algebra.

Proof.

Using the Jacoby identity, we have:

$$\{f, \{g, h\}_{\alpha}\}_{\alpha} + \{g, \{h, f\}_{\alpha}\}_{\alpha} + \{h, \{f, g\}_{\alpha}\}_{\alpha} = 0.$$

The left hand side of this identity can be handled as:

$$\begin{split} \{f, \{g, h\}_{\alpha}\}_{\alpha} + \{g, \{h, f\}_{\alpha}\}_{\alpha} + \{h, \{f, g\}_{\alpha}\}_{\alpha} &= \{f, \{g, h\}_{\alpha}\}_{\alpha} - \{g, \{f, h\}_{\alpha}\}_{\alpha} - \{\{f, g\}_{\alpha}, h\}_{\alpha} \\ &= X_{\alpha_{f}}\{g, h\}_{\alpha} - \{g, X_{\alpha_{f}}h\}_{\alpha} - \{X_{\alpha_{f}}g, h\}_{\alpha} \\ &= X_{\alpha_{f}}X_{\alpha_{g}}h - X_{\alpha_{g}}X_{\alpha_{f}}h - X_{\alpha_{\{f, g\}_{\alpha}}}h \\ &= [X_{\alpha_{f}}, X_{\alpha_{g}}]h - X_{\alpha_{\{f, g\}_{\alpha}}}h \end{split}$$

leading to

$$[X_{\alpha_f}, X_{\alpha_g}]h = X_{\alpha_{\{f,g\}_\alpha}}h.$$

Then, the map $f \mapsto X_{\alpha_f} = \{f,.\}_{\alpha}$, $\{f,.g\}_{\alpha} \mapsto X_{\alpha_{\{f,g\}_{\alpha}}}$ is an α -deformed Lie algebra morphism $(\mathfrak{F},\{.,.\}_{\alpha}) \to (\mathfrak{X}_{\alpha_{\mathfrak{F}}},[.,.])$. Therefore, $(\mathfrak{X}_{\alpha_{\mathfrak{F}}},[.,.])$ is an α -deformed Lie algebra.

3.2 Noether symmetry and recursion operator

By definition, we have

$$X_{\alpha} := \{H_{\alpha}, .\}_{\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial H_{\alpha}}{\partial p_{\mu}} \frac{\partial}{\partial q^{\mu}} - \frac{\partial H_{\alpha}}{\partial q^{\mu}} \frac{\partial}{\partial p_{\mu}} \right). \tag{10}$$

Using (6) and (10), we obtain the following set of equations:

$$\begin{cases} \alpha^{-2}|p_1|^{(1-\alpha)}|q^1|^{(1-\alpha)}\frac{\partial H_\alpha}{\partial p_1} &= -p_1\\ \alpha^{-2}|p_1|^{(1-\alpha)}|q^1|^{(1-\alpha)}\frac{\partial H_\alpha}{\partial q^1} &= -\left(\frac{\alpha-1}{\alpha+1}\right)\frac{p_1^2}{q^1}\\ \alpha^{-2}|p_k|^{(1-\alpha)}|q^k|^{(1-\alpha)}\frac{\partial H_\alpha}{\partial p_k} &= p_k, \qquad k=2,3,4\\ \alpha^{-2}|p_k|^{(1-\alpha)}|q^k|^{(1-\alpha)}\frac{\partial H_\alpha}{\partial q^k} &= \left(\frac{\alpha-1}{\alpha+1}\right)\frac{p_k^2}{q^k}, \qquad k=2,3,4 \end{cases}$$

leading to

$$H_{\alpha} = -\frac{\alpha^2}{\alpha + 1} |q^1|^{\alpha - 1} |p_1|^{\alpha + 1} + \sum_{k=2}^{4} \frac{\alpha^2}{\alpha + 1} |q^k|^{\alpha - 1} |p_k|^{\alpha + 1}.$$

This function is called the α -Hamiltonian function. For $\alpha = 1$, we naturally obtain the Hamiltonian function of a free particle on the ordinary Minkowski phase space.

The vector field

$$Y_{\alpha} = -\frac{1}{2\alpha^{2}}|p_{1}|^{1-\alpha}p_{1}^{-2}|q^{1}|^{1-\alpha}|q^{1}|^{\frac{1-\alpha}{1+\alpha}}\frac{\partial}{\partial p_{1}} + \frac{1}{2\alpha^{2}}\sum_{k=2}^{4}|p_{k}|^{1-\alpha}p_{k}^{-2}|q^{k}|^{1-\alpha}|q^{k}|^{\frac{1-\alpha}{1+\alpha}}\frac{\partial}{\partial p_{k}}$$

is a master symmetry, i.e.,

$$[[Y_{\alpha}, X_{\alpha}], X_{\alpha}] = 0,$$

and the following relations hold:

$$\begin{split} L_{\alpha} &:= \mathcal{L}_{Y_{\alpha}} H_{\alpha} = \frac{1}{2} \bigg(p_{1}^{-1} (q^{1})^{\frac{1-\alpha}{1+\alpha}} + \sum_{k=2}^{4} p_{k}^{-1} (q^{k})^{\frac{1-\alpha}{1+\alpha}} \bigg), \\ \omega_{\alpha_{1}} &:= \mathcal{L}_{Y_{\alpha}} \omega_{\alpha} = d\iota_{Y_{\alpha}} \omega_{\alpha} + \iota_{Y_{\alpha}} d\omega_{\alpha} \\ &= p_{1}^{-3} |q^{1}|^{\frac{1-\alpha}{1+\alpha}} dp_{1} \wedge dq^{1} - \sum_{k=2}^{4} p_{k}^{-3} |q^{k}|^{\frac{1-\alpha}{1+\alpha}} dp_{k} \wedge dq^{k}, \\ X_{\alpha_{1}} &:= [X_{\alpha}, Y_{\alpha}] \\ &= -\frac{1}{2\alpha^{2}} \bigg[\frac{1-\alpha}{(1+\alpha)} G_{1} |p_{1}|^{-\alpha} |q^{1}|^{\frac{1-2\alpha-\alpha^{2}}{1+\alpha}} \frac{\partial}{\partial p_{1}} + |p_{1}|^{1-\alpha} p_{1}^{-2} |q^{1}|^{\frac{2-\alpha-\alpha^{2}}{1+\alpha}} \frac{\partial}{\partial q^{1}} \bigg] \\ &- \frac{1}{2\alpha^{2}} \sum_{k=2}^{4} \bigg[\frac{1-\alpha}{(1+\alpha)} G_{k} p_{k}^{-\alpha} |q^{k}|^{\frac{1-2\alpha-\alpha^{2}}{1+\alpha}} \frac{\partial}{\partial p_{k}} + |p_{k}|^{1-\alpha} p_{k}^{-2} |q^{k}|^{\frac{2-\alpha-\alpha^{2}}{1+\alpha}} \frac{\partial}{\partial q^{k}} \bigg], \end{split}$$

where $G_i = sgn(p_i)sgn(q^i), i = 1, 2, 3, 4.$

We notice that X_{α_1} satisfies the relation

$$\iota_{X_{\alpha_1}}\omega_{\alpha}=-dL_{\alpha},$$

where $\iota_{X_{\alpha_1}}\omega_{\alpha}$ is the interior product of ω_{α} with respect to the α -vector fied X_{α_1} . Since X_{α_1} is a dynamical symmetry, *i.e.*, $[X_{\alpha}, X_{\alpha_1}] = 0$, L_{α} is a first integral, also called a constant of motion. Thus, we arrive at the following property:

Proposition 3. The α -vector field X_{α_1} is an infinitesimal Noether symmetry.

Proof.

We have:

$$\mathcal{L}_{X_{\alpha_1}}\omega_{\alpha} = d\iota_{X_{\alpha_1}}\omega_{\alpha} + \iota_{X_{\alpha_1}}d\omega_{\alpha} = d\iota_{X_{\alpha_1}}\omega_{\alpha} = -d^2L_{\alpha} = 0.$$
(11)

Since X_{α_1} is a dynamical symmetry, then

$$\mathcal{L}_{X_{\alpha_1}} H_{\alpha} = X_{\alpha_1}(H_{\alpha}) = 0. \tag{12}$$

Equations (11) and (12) show that X_{α_1} is both an infinitesimal geometric symmetry, i.e., leaving invariant the geometric structure (the symplectic form ω_{α}), and an infinitesimal Hamiltonian symmetry leaving invariant the dynamics (the Hamiltonian function H_{α}). Hence, X_{α_1} is an infinitesimal Noether symmetry.

In the sequel, we consider the following α -Poisson bivector

$$\mathcal{P}_{\alpha_1} = p_1^3 |q^1|^{\frac{\alpha-1}{1+\alpha}} \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} - \sum_{k=2}^4 p_k^3 |q^k|^{\frac{\alpha-1}{1+\alpha}} \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial q^k}$$

and define the α -deformed Poisson bracket

$$\{f,g\}_{\alpha_1} := p_1^3 |q^1|^{\frac{\alpha-1}{1+\alpha}} \left(\frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q^1} - \frac{\partial f}{\partial q^1} \frac{\partial g}{\partial p_1} \right) - \sum_{k=2}^4 p_k^3 |q^k|^{\frac{\alpha-1}{1+\alpha}} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k} \frac{\partial g}{\partial p_k} \right),$$

with respect to the α -symplectic form ω_{α_1} .

Thus, the vector field X_{α} is a bi-Hamiltonian vector field with respect to $(\omega_{\alpha}, \omega_{\alpha_1})$, i.e.,

$$\iota_{X_{\alpha}}\omega_{\alpha}=-dH_{\alpha}$$
 and $\iota_{X_{\alpha}}\omega_{\alpha_{1}}=-d\tilde{L}_{\alpha}, \quad X_{\alpha}=\{H_{\alpha},.\}_{\alpha}=\{\tilde{L}_{\alpha},.\}_{\alpha_{1}},$

where

$$\tilde{L}_{\alpha} = \sum_{\mu=1}^{4} |q^{\mu}|^{\frac{1-\alpha}{1+\alpha}} p_{\mu}^{-1}$$

are first integrals for $X_{H_{\alpha}}$.

Therefore, the associated α -recursion operator T_{α} is given by:

$$\begin{split} T_{\alpha} &:= \mathcal{P}_{\alpha_{1}} \circ \mathcal{P}_{\alpha}^{-1} \\ &= \left(p_{1}^{3} |q^{1}|^{\frac{\alpha-1}{1+\alpha}} \frac{\partial}{\partial p_{1}} \wedge \frac{\partial}{\partial q^{1}} - \sum_{k=2}^{4} p_{k}^{3} |q^{k}|^{\frac{\alpha-1}{1+\alpha}} \frac{\partial}{\partial p_{k}} \wedge \frac{\partial}{\partial q^{k}} \right) \circ \left(\sum_{\mu=1}^{4} \alpha^{2} |p_{\mu}|^{(\alpha-1)} |q^{\mu}|^{(\alpha-1)} dp_{\mu} \wedge dq^{\mu} \right) \\ &= \alpha^{2} p_{1}^{3} |p_{1}|^{(\alpha-1)} |q^{1}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \frac{\partial}{\partial p_{1}} \otimes dp_{1} - \alpha^{2} \sum_{k=2}^{4} p_{k}^{3} |p_{k}|^{(\alpha-1)} |q^{k}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \frac{\partial}{\partial p_{k}} \otimes dp_{k} \\ &+ \alpha^{2} p_{1}^{3} |p_{1}|^{(\alpha-1)} |q^{1}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \frac{\partial}{\partial q^{1}} \otimes dq^{1} - \alpha^{2} \sum_{k=2}^{4} p_{k}^{3} |p_{k}|^{(\alpha-1)} |q^{k}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \frac{\partial}{\partial q^{k}} \otimes dq^{k}, \end{split}$$

providing the constants of motion

$$Tr(T_{\alpha}^{h}) = 2^{h} \alpha^{2h} \left\{ \left(p_{1}^{3} |p_{1}|^{(\alpha-1)} |q^{1}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \right)^{h} + (-1)^{h} \left(\sum_{k=2}^{4} p_{k}^{3} |p_{k}|^{(\alpha-1)} |q^{k}|^{\frac{-2+\alpha^{2}+\alpha}{1+\alpha}} \right)^{h} \right\}, \ h \in \mathbb{N}.$$

This work can be considered as an α -deformed case of previous investigations [39], [17]. The only difference resides in the fact that we use here the method of Noether symmetry to obtain the integrals of motion instead of the method of Hamilton-Jacobi separability developed in [39] [17], and [18].

4 α -deformed Einstein field equation

In this section, we investigate the α -deformed solutions of the Einstein field equation in the Schwarzschild and Friedmann-Lemaître-Robertson-Walker (FLRW) metrics. We consider the α -deformed Einstein field equation shortly written in the tensor form as:

$$G_{\alpha} + \Lambda g_{\alpha} = \kappa T_{\alpha},$$

where the tensor

$$G_{\alpha} = R_{\alpha} - \frac{1}{2} g_{\alpha} \mathbf{R}_{\alpha} \tag{13}$$

is the α -deformed Einstein tensor, the constant Λ is the cosmological constant, κ is a constant; T_{α} and R_{α} are the α -deformed tress-energy tensor and Ricci tensor measuring the geodesic deviation, respectively. g_{α} is the α -deformed metric tensor telling us how to measure distances and time, and \mathbf{R}_{α} , analogous to Gaussian curvature κ for surfaces, is the α -deformed scalar curvature measuring the spacetime curvature. The right hand side of the Einstein field equation is the matter content of spacetime, in here we can put electromagnetic fields, fluids, scalar fields, etc. Therefore, Einstein equations express the compromise between spacetime geometry and the existing matter, in such a way that each one influences the other, being the curvature the manifestation of the massive content of that spacetime. The main difference with respect to the old approach is that matter evolves no longer through a static spatial scenario, where all clocks in universe agree in their time measurements, but rather now spacetime is an active actor that affects matter dynamics; and in turn the matter content, through the energy-momentum tensor T_{α} , determines how the geometry is. Moreover, Einstein equations are not linear, where by linear effects we understand those that are proportional to the causes, those where small variations in initial conditions lead to small changes in the response, not much different than the former. In nonlinear processes, this is not the case.

4.1 Recursion operator of the α -deformed Schwarzschild metric

The Schwarzschild metric is the simplest one among the particular solutions of the Einstein field equation.

Here, we consider the following α -Schwarzschild metric

$$d_{\alpha}s^{2} = -\left(1 - \frac{2M}{q^{2}}\right)(q^{1})^{2(\alpha - 1)}(dq^{1})^{2} + \left(1 - \frac{2M}{q^{2}}\right)^{-1}(q^{2})^{2(\alpha - 1)}(dq^{2})^{2} + (q^{2})^{2}(q^{3})^{2(\alpha - 1)}(dq^{3})^{2} + (q^{2})^{2}(q^{4})^{2(\alpha - 1)}\sin^{2}q^{3}(dq^{4})^{2},$$

where $t=q^1,\ r=q^2,\ \theta=q^3,\ \phi=q^4,\ M$ is a positive constant representing the mass of the black hole, $t\in(-\infty,\infty),\ r\in(2M,\infty),\ \theta\in(0,\pi),$ and $\phi\in(0,2\pi).$

The metric is defined on a manifold

$$\mathcal{Q} = \{(q^1, q^2, q^3, q^4) | 0 \neq q^1 \in (-\infty, \infty), q^2 \in (2M, \infty), 0 \neq q^3 \in (0, \pi), \text{ and } 0 \neq q^4 \in (0, 2\pi)\}.$$
 (14)

For $\alpha=1$, we recover the metric found by Karl Schwarzschild in 1916 as the simplest possible solution of Einstein equations, which was of huge usefulness to derive some effects or predictions as well as to lead to the prediction of the existence of black holes, very compact massive objects. It was the only spherically symmetric, vacuum solution of Einstein equations. It describes the geometry exterior to any spherical collapsing body. And even in the case of nonspherical collapse, perturbative treatment of spherical collapse indicates that a blackhole will form and the spacetime singularity resulting from the gravitational collapse will be hidden within the black hole [4].

For our purpose, let us consider the phase space $\mathcal{T}^*\mathcal{Q} \ni (q,p), q \in \mathcal{Q}$, and the α -Hamiltonian function

$$H_{S\alpha} = -\frac{1}{2} \left(1 - \frac{2M}{q^2} \right)^{-1} (q^1)^{2(1-\alpha)} p_1^2 + \frac{1}{2} \left(1 - \frac{2M}{q^2} \right) (q^2)^{2(1-\alpha)} p_2^2 + \frac{1}{2(q^2)^2} (q^3)^{2(1-\alpha)} p_3^2 + \frac{1}{2(q^2)^2 \sin^2 q^3} (q^4)^{2(1-\alpha)} p_4^2.$$

The α -Hamiltonian vector field of $H_{S\alpha}$ in α -Schwarzschild metric with respect to the canonical α -symplectic structure $\omega_{\alpha} = \sum_{\mu=1}^{4} \alpha^2 |p_{\mu}|^{(\alpha-1)} |q^{\mu}|^{(\alpha-1)} dp_{\mu} \wedge dq^{\mu}$ is given by

$$X_{S\alpha} := \{H_{S\alpha}, .\}_{\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\frac{\partial H_{\alpha}}{\partial p_{\mu}} \frac{\partial}{\partial q^{\mu}} - \frac{\partial H_{\alpha}}{\partial q^{\mu}} \frac{\partial}{\partial p_{\mu}} \right)$$
$$= \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(V_{\mu} \frac{\partial}{\partial q^{\mu}} + U_{\mu} \frac{\partial}{\partial p_{\mu}} \right),$$

where

$$V_{1} = -\left(1 - \frac{2M}{q^{2}}\right)^{-1}\eta_{1}, \quad V_{2} = \left(1 - \frac{2M}{q^{2}}\right)\eta_{2}, \quad V_{3} = \frac{1}{(q^{2})^{2}}\eta_{3}, \quad V_{4} = \frac{1}{(q^{2})^{2}\sin^{2}q^{3}}\eta_{4},$$

$$U_{1} = (1 - \alpha)\left(1 - \frac{2M}{q^{2}}\right)^{-1}\zeta_{1}, \quad U_{3} = -\left(\frac{1 - \alpha}{(q^{2})^{2}}\zeta_{3} - \frac{\cos q^{3}}{(q^{2})^{2}\sin^{3}q^{3}}\eta_{4}p_{4}\right), \quad U_{4} = -\frac{1 - \alpha}{(q^{2})^{2}\sin^{2}q^{3}}\zeta_{4},$$

$$U_{2} = -\left\{\frac{M}{(q^{2})^{2}}\left(1 - \frac{2M}{q^{2}}\right)^{-2}\eta_{1}p_{1} + (1 - \alpha)\left(1 - \frac{2M}{q^{2}}\right)\zeta_{2} + \frac{M}{(q^{2})^{2}}\eta_{2}p_{2} - \frac{1}{(q^{2})^{3}}\eta_{3}p_{3} - \frac{1}{(q^{2})^{3}\sin^{2}q^{3}}\eta_{4}p_{4}\right\},$$

with $\eta_{\nu} = (q^{\nu})^{2(1-\alpha)} p_{\nu}$, and $\zeta_{\nu} = (q^{\nu})^{(1-2\alpha)} p_{\nu}^2$, $\nu = 1, 2, 3, 4$.

Then, we get in α -Schwarzschild metric, the α -Christoffel symbols $(\Gamma_{ij}^k)_{\alpha}$, the components of the α -Riemann $(R_{ijkl})_{\alpha}$ and α -Ricci tensors $(R_{ii})_{\alpha}$, the α -Ricci scalar \mathbf{R} , and the components of the α -Einstein tensor $(G_{ij})_{\alpha}$, i, j, k, l = 1, 2, 3, 4, see Appendix.

Note that the components of defined geometric objects are obtained in the usual undeformed Schwarzschild metric by setting $\alpha = 1$.

Now, we consider the Hamilton-Jacobi equation for the α -Hamiltonian function $H_{S\alpha}$

$$E_{S} = H_{S\alpha} \left(q, \frac{\partial W}{\partial q} \right)$$

$$= -\frac{1}{2} \left(1 - \frac{2M}{q^{2}} \right)^{-1} (q^{1})^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^{1}} \right)^{2} + \frac{1}{2} \left(1 - \frac{2M}{q^{2}} \right) (q^{2})^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^{2}} \right)^{2}$$

$$+ \frac{1}{2(q^{2})^{2}} (q^{3})^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^{3}} \right)^{2} + \frac{1}{2(q^{2})^{2} \sin^{2} q^{3}} (q^{4})^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^{4}} \right)^{2},$$

where E_S is a constant and $W = \sum_{\mu=1}^4 W_{\mu}(q_{\mu})$ is the generating function. In particular, we put

 $W_1 = \frac{a}{\alpha} |q^1|^{\alpha}$, where a is a constant. This equation is a type of separation of variables; then, the above Hamilton-Jacobi equation becomes

$$2E_S(q^2)^2 + \left(1 - \frac{2M}{q^2}\right)^{-1} (q^2)^2 a^2 - \left(1 - \frac{2M}{q^2}\right) (q^2)^{2(2-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2$$

$$= (q^3)^{2(1-\alpha)} \left(\frac{dW_3}{dq^3}\right)^2 + \frac{1}{\sin^2 q^3} (q^4)^{2(1-\alpha)} \left(\frac{dW_4}{dq^4}\right)^2, \tag{15}$$

which can be rewritten through a constant K as:

$$K = 2E_S(q^2)^2 + \left(1 - \frac{2M}{q^2}\right)^{-1} (q^2)^2 a^2 - \left(1 - \frac{2M}{q^2}\right) (q^2)^{2(2-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2 \tag{16}$$

$$K = (q^3)^{2(1-\alpha)} \left(\frac{dW_3}{dq^3}\right)^2 + \frac{1}{\sin^2 q^3} (q^4)^{2(1-\alpha)} \left(\frac{dW_4}{dq^4}\right)^2.$$
 (17)

From the above, we set:

$$\left(K - (q^3)^{2(1-\alpha)} \left(\frac{dW_3}{dq^3}\right)^2\right) \sin^2 q^3 = G$$
(18)

$$(q^4)^{2(1-\alpha)} \left(\frac{dW_4}{dq^4}\right)^2 = G. \tag{19}$$

and obtain

$$W_4 = \frac{\sqrt{G}}{\alpha} |q^4|^{\alpha - 1} q^4 + A,$$

where A is a constant.

We put the solutions of equations (16) and (18) in the form:

$$W_2 = W_2(q^2, E_S, a, K), \quad W_3 = W_3(q^3, K, G).$$

Then, a generating function W takes the form:

$$W = \frac{a}{\alpha} |q^1|^{\alpha} + W_2(q^2, E_S, a, K) + W_3(q^3, K, G) + \frac{\sqrt{G}}{\alpha} |q^4|^{\alpha - 1} q^4 + A.$$

Now, we consider the canonical system (Q, P), where

$$Q^{1} = E_{S}, \ Q^{2} = a, \ Q^{3} = K, \ Q^{4} = \sqrt{G},$$

$$P_{1} := -\frac{\partial W}{\partial Q^{1}} = -\frac{\partial W_{2}}{\partial Q^{1}}, \quad P_{2} := -\frac{\partial W}{\partial Q^{2}} = -\frac{a}{\alpha}(q^{1})^{\alpha} - \frac{\partial W_{2}}{\partial Q^{2}},$$

$$P_{3} := -\frac{\partial W}{\partial Q^{3}} = -\frac{\partial W_{2}}{\partial Q^{3}} - \frac{\partial W_{3}}{\partial Q^{3}}, \text{ and } P_{4} := -\frac{\partial W}{\partial Q^{4}} = -\frac{\partial W_{4}}{\partial Q^{4}} - \frac{\partial W_{3}}{\partial Q^{4}} = -\frac{1}{\alpha}|q^{4}|^{\alpha-1}q^{4} - \frac{\partial W_{3}}{\partial Q^{4}}.$$

In this new canonical system, we define the following Poisson bracket

$$\{f,g\}_{\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |P_{\mu}|^{(1-\alpha)} |Q^{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial P_{\mu}} \frac{\partial g}{\partial Q^{\mu}} - \frac{\partial f}{\partial Q^{\mu}} \frac{\partial g}{\partial P_{\mu}} \right), \tag{20}$$

with respect to the α -symplectic form

$$\omega_{\alpha} = \sum_{\mu=1}^{4} \alpha^{2} |P_{\mu}|^{(\alpha-1)} |Q^{\mu}|^{(\alpha-1)} dP_{\mu} \wedge dQ^{\mu}.$$

Then, the α -Hamiltonian vector field takes the form:

$$X_{S\alpha} := \{H_{S\alpha}, .\}_{\alpha} = -\alpha^{-2} |P_1|^{(1-\alpha)} |Q^1|^{(1-\alpha)} \frac{\partial}{\partial P_1}.$$

Now, we consider a (1,1)-tensor field $T_{S\alpha}$ as

$$T_{S\alpha} = \sum_{\mu=1}^{4} |Q^{\mu}|^{\alpha-1} Q^{\mu} \left(\frac{\partial}{\partial P_{\mu}} \otimes dP_{\mu} + \frac{\partial}{\partial Q^{\mu}} \otimes dQ^{\mu} \right).$$

We can put $Q^{\mu} = x_{\mu}$ and $P_{\mu} = x_{\mu+n}$, where n=4 in this case and $\mu=1,2,3,4$. Then, by Lemma 1, $T_{S\alpha}$ satisfies $\mathcal{L}_{X_{S\alpha}}T_{S\alpha}=0$, $\mathcal{N}_{T_{S\alpha}}=0$, and $degQ^{\mu}=2$. Hence, $T_{S\alpha}$ is an α -recursion operator of $X_{S\alpha}$. The constants of motion $Tr(T_{\alpha}^{l})$ $(l \in \mathbb{N})$ of the α -Hamiltonian vector field $X_{S\alpha}$ for the α -Schwarzschild metric are finally obtained as:

$$Tr(T_{F\alpha}^l) = 2((Q^1)^l + (Q^2)^l + (Q^3)^l + (Q^4)^l), \quad l \in \mathbb{N}.$$

4.2 Recursion operator in α -FLRW metric

Now, we consider the following α -Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$d_{\alpha}s^{2} = -|q^{1}|^{2(\alpha-1)}(dq^{1})^{2} + R^{2}(q^{1}) \left\{ \frac{|q^{2}|^{2(\alpha-1)}}{1 - k(q^{2})^{2}}(dq^{2})^{2} + (q^{2})^{2} \left(|q^{3}|^{2(\alpha-1)}(dq^{3})^{2} + |q^{4}|^{2(\alpha-1)}\sin^{2}q^{3}(dq^{4})^{2} \right) \right\}$$

defined on the same manifold Q (14), where $R(q^1)$ is a scale factor and k is a constant representing the curvature of the space. Considering the α -Hamitonian function

$$H_{F\alpha} = -\frac{1}{2}(q^1)^{2(1-\alpha)}p_1^2 + \frac{1-k(q^2)^2}{2R^2(q^1)}(q^2)^{2(1-\alpha)}p_2^2 + \frac{(q^3)^{2(1-\alpha)}}{2(q^2)^2R^2(q^1)}p_3^2 + \frac{(q^4)^{2(1-\alpha)}}{2(q^2)^2R^2(q^1)\sin^2(q^3)}p_4^2$$

we obtain the following α -Hamiltonian vector field

$$X_{F\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |p_{\mu}|^{(1-\alpha)} |q^{\mu}|^{(1-\alpha)} \left(\tilde{V}_{\mu} \frac{\partial}{\partial q^{\mu}} + \tilde{U}_{\mu} \frac{\partial}{\partial p_{\mu}} \right),$$

with respect to the α -symplectic structure $\omega_{\alpha} = \sum_{\mu=1}^{4} \alpha^2 |p_{\mu}|^{(\alpha-1)} |q^{\mu}|^{(\alpha-1)} dp_{\mu} \wedge dq^{\mu}$, where

$$\begin{split} \tilde{V}_1 &= \eta_1, \quad \tilde{V}_2 = \frac{1 - k(q^2)^2}{2R^2(q^1)} \eta_2, \quad \tilde{V}_3 = \frac{1}{(q^2)^2 R^2(q^1)} \eta_3, \quad \tilde{V}_4 = \frac{1}{(q^2)^2 R^2(q^1) \sin^2(q^3)} \eta_4, \\ \tilde{U}_1 &= (1 - \alpha)\zeta_1 + \frac{1}{R^3(q^1)} \bigg((1 - k(q^2)^2) \eta_2 p_2 + \frac{1}{(q^2)^2} \eta_3 p_3 + \frac{1}{\sin^2 q^3} \eta_4 p_4 \bigg) \frac{dR(q^1)}{dq^1}, \\ \tilde{U}_2 &= -\frac{p_2^2}{R^2(q^1)} \bigg(-kq^2 \eta_2 p_2 + (1 - \alpha)(1 - k(q^2)^2) \zeta_2 \bigg) + \frac{1}{(q^2)^3 R^2(q^1)} \eta_3 p_3 + \frac{1}{(q^2)^3 R^2(q^1) \sin^2 q^3} \eta_4 p_4, \\ \tilde{U}_3 &= -\frac{(1 - \alpha)}{(q^2)^2 R^2(q^1)} \zeta_3 + \frac{\cos q^3}{(q^2)^2 R^2(q^1) \sin^3 q^3} \eta_4 p_4, \quad \tilde{U}_4 &= -\frac{(1 - \alpha)}{(q^2)^2 R^2(q^1) \sin^2 q^3} \zeta_4, \end{split}$$

with
$$\eta_{\nu} = (q^{\nu})^{2(1-\alpha)} p_{\nu}$$
, $\zeta_{\nu} = (q^{\nu})^{(1-2\alpha)} p_{\nu}^2$, and $\nu = 1, 2, 3, 4$.

Here, we perform the computation of the α -Christoffel symbols, the components of the α -Riemann and α -Ricci tensors, the α -Ricci scalar, and the components of the Einstein tensor, see Appendix.

Remark that for $\alpha = 1$, we recover the components of these geometric objects in the usual FLRW metric as expected.

The Hamiltonian-Jacobi equation takes here the form:

$$2E_F = -(q^1)^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^1}\right)^2 + \frac{1 - k(q^2)^2}{R^2(q^1)} (q^2)^{2(1-\alpha)} \left(\frac{\partial W}{\partial q^2}\right)^2 + \frac{(q^3)^{2(1-\alpha)}}{(q^2)^2 R^2(q^1)} \left(\frac{\partial W}{\partial q^3}\right)^2 + \frac{(q^4)^{2(1-\alpha)}}{(q^2)^2 R^2(q^1) \sin^2(q^3)} \left(\frac{\partial W}{\partial q^4}\right)^2,$$

where E_F is a constant and $W = \sum_{\mu=1}^{4} W_{\mu}(q_{\mu})$ is the generating function. The above equation can be rewritten as

$$2E_F R^2(q^1) + (q^1)^{2(1-\alpha)} R^2(q^1) \left(\frac{dW_1}{dq^1}\right)^2 = (1 - k(q^2)^2)(q^2)^{2(1-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2 + \frac{(q^3)^{2(1-\alpha)}}{(q^2)^2} \left(\frac{dW_3}{dq^3}\right)^2 + \frac{(q^4)^{2(1-\alpha)}}{(q^2)^2 \sin^2(q^3)} \left(\frac{dW_4}{dq^4}\right)^2,$$

which is of a type of separation of variables. Thus, we can also express them via a constant K as:

$$K = 2E_F R^2(q^1) + (q^1)^{2(1-\alpha)} R^2(q^1) \left(\frac{dW_1}{dq^1}\right)^2, \tag{21}$$

$$K = (1 - k(q^2)^2)(q^2)^{2(1-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2 + \frac{(q^3)^{2(1-\alpha)}}{(q^2)^2} \left(\frac{dW_3}{dq^3}\right)^2 + \frac{(q^4)^{2(1-\alpha)}}{(q^2)^2 \sin^2(q^3)} \left(\frac{dW_4}{dq^4}\right)^2. \tag{22}$$

Moreover, from equation (22), we get

$$(1 - k(q^2)^2)(q^2)^{2(2-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2 - (q^2)^2 K = -(q^3)^{2(1-\alpha)} \left(\frac{dW_3}{dq^3}\right)^2 - \frac{(q^4)^{2(1-\alpha)}}{\sin^2(q^3)} \left(\frac{dW_4}{dq^4}\right)^2.$$
(23)

Since equation (23) is of a type of separation of variables, we can introduce a constant L such that

$$L = (q^2)^2 K - (1 - k(q^2)^2)(q^2)^{2(2-\alpha)} \left(\frac{dW_2}{dq^2}\right)^2, \tag{24}$$

$$L = (q^3)^{2(1-\alpha)} \left(\frac{dW_3}{dq^3}\right)^2 + \frac{(q^4)^{2(1-\alpha)}}{\sin^2(q^3)} \left(\frac{dW_4}{dq^4}\right)^2,\tag{25}$$

and the equation (25) can be expressed as

$$L\sin^2(q^3) - (q^3)^{2(1-\alpha)}\sin^2(q^3) \left(\frac{dW_3}{dq^3}\right)^2 = (q^4)^{2(1-\alpha)} \left(\frac{dW_4}{dq^4}\right)^2.$$

Setting

$$G = L\sin^2(q^3) - (q^3)^{2(1-\alpha)}\sin^2(q^3) \left(\frac{dW_3}{dq^3}\right)^2,$$
(26)

$$G = (q^4)^{2(1-\alpha)} \left(\frac{dW_4}{dq^4}\right)^2, \tag{27}$$

we can formulate the solutions of the equations (21), (24), and (26) as:

$$W_1 = W_1(q^1; E_F, K), W_2 = W_2(q^2; K, L), W_3 = W_3(q^3; L, G).$$

From (27), we obtain

$$W_4 = \frac{\sqrt{G}}{\alpha} |q^4|^{\alpha - 1} q^4 + C$$

where C is a constant, and, hence,

$$W = W_1(q^1; E_F, K) + W_2(q^2; K, L) + W_3(q^3; L, G) + \frac{\sqrt{G}}{\alpha} |q^4|^{\alpha - 1} q^4 + C.$$

Considering now the canonical system (Q, P), where

$$\begin{split} Q^1 &= E_F, \ Q^2 = K, \ Q^3 = \sqrt{L}, \ Q^4 = \sqrt{G}, \\ P_1 &:= -\frac{\partial W}{\partial Q^1} = -\frac{\partial W_1}{\partial Q^1}, \quad P_2 := -\frac{\partial W}{\partial Q^2} = -\frac{\partial W_1}{\partial Q^2} - \frac{\partial W_2}{\partial Q^2}, \\ P_3 &:= -\frac{\partial W}{\partial Q^3} = -\frac{\partial W_2}{\partial Q^3} - \frac{\partial W_3}{\partial Q^3}, \text{ and } P_4 := -\frac{\partial W}{\partial Q^4} = -\frac{\partial W_3}{\partial Q^4} - \frac{\partial W_4}{\partial Q^4} = -\frac{1}{\alpha} |q^4|^{\alpha-1} q^4 - \frac{\partial W_3}{\partial Q^4}, \end{split}$$

the α -Hamiltonian vector field $X_{F\alpha}$ and the (1,1)-tensor field $T_{F\alpha}$ are given by

$$X_{F\alpha} := \{H_{F\alpha}, .\}_{\alpha} = -\alpha^{-2} |P_1|^{(1-\alpha)} |Q^1|^{(1-\alpha)} \frac{\partial}{\partial P_1}, T_{F\alpha} = \sum_{\mu=1}^4 |Q^{\mu}|^{\alpha-1} Q^{\mu} \left(\frac{\partial}{\partial P_{\mu}} \otimes dP_{\mu} + \frac{\partial}{\partial Q^{\mu}} \otimes dQ^{\mu}\right),$$

respectively.

Similarly, by Lemma 1, $T_{F\alpha}$ satisfies $\mathcal{L}_{X_{F\alpha}}T_{F\alpha}=0$, $\mathcal{N}_{T_{F\alpha}}=0$, and $degQ^{\mu}=2$. Thus, $T_{F\alpha}$ is an α -recursion operator of $X_{F\alpha}$, and the constants of motion $Tr(T_{F\alpha}^l)$ $(l \in \mathbb{N})$ of the α -vector field $X_{F\alpha}$ for the α -FLRW metric are provided in the form

$$Tr(T_{F\alpha}^l) = 2((Q^1)^l + (Q^2)^l + (Q^3)^l + (Q^4)^l), \quad l \in \mathbb{N}.$$

5 Family of conserved quantities

In this section, we consider the Hamiltonian system $(\mathcal{T}^*\mathcal{Q}, \omega, Q^1)$, for which the α -Hamiltonian function H_{α} , the α -vector field X_{α} , the α -symplectic form ω_{α} , the α -bivector field \mathcal{P}_{α} , and the α -recursion operator T_{α} are given in both the α -Schwarzschild and α -FLRW metrics by: $H_{\alpha} = Q^1 > 0$,

$$X_{\alpha} = \{H_{\alpha}, .\}_{\alpha} = -\alpha^{-2} |P_{1}|^{(1-\alpha)} |Q^{1}|^{(1-\alpha)} \frac{\partial}{\partial P_{1}}, \ \omega_{\alpha} = \sum_{\mu=1}^{4} \alpha^{2} |P_{\mu}|^{(\alpha-1)} |Q^{\mu}|^{(\alpha-1)} dP_{\mu} \wedge dQ^{\mu},$$

$$\mathcal{P}_{\alpha} = \sum_{\mu=1}^{4} \alpha^{-2} |P_{\mu}|^{(1-\alpha)} |Q^{\mu}|^{(1-\alpha)} \frac{\partial}{\partial P_{\mu}} \wedge \frac{\partial}{\partial Q^{\mu}}, \text{ and } T_{\alpha} = \sum_{\mu=1}^{4} |Q^{\mu}|^{\alpha-1} Q^{\mu} \left(\frac{\partial}{\partial P_{\mu}} \otimes dP_{\mu} + \frac{\partial}{\partial Q^{\mu}} \otimes dQ^{\mu} \right).$$

In the sequel, we introduce the functions

$$\tilde{H}_{\alpha_j} = -\sum_{\mu=1}^4 \alpha |Q^{\mu}|^{\alpha(1-j)-1} Q^{\mu} |P_{\mu}|^{\alpha-1} P_{\mu}$$

and obtain the vector fields $Z_{\alpha_i} \in \mathcal{T}^*\mathcal{Q}$,

$$Z_{\alpha_j} := \{ \tilde{H}_{\alpha_j}, . \}_{\alpha} = \sum_{\mu=1}^4 |Q^{\mu}|^{-\alpha_j} \bigg((1-j) P_{\mu} \frac{\partial}{\partial P_{\mu}} - Q^{\mu} \frac{\partial}{\partial Q^{\mu}} \bigg).$$

satisfying the relation

$$\iota_{Z_{\alpha_j}}\omega_{\alpha} = -d\tilde{H}_{\alpha_j}.$$

Then, it is straightforward to notice that the α -symplectic structure ω_{α} generates a set of Hamiltonian systems on the same manifold $\mathcal{T}^*\mathcal{Q}$. The Lie bracket between the vector fields X_{α_i} and Z_{α_j} obeys the relations

$$[X_{\alpha_i}, Z_{\alpha_j}] = X_{\alpha_{i+1}}, \ [X_{\alpha_i}, X_{\alpha_{i+1}}] = 0, \ i, j \in \mathbb{N}, \ X_{\alpha_0} = X_{\alpha},$$
 (28)

with

$$X_{\alpha_{i+j}} = -\alpha^{-2}(1 - \alpha i)[1 - (i+j)\alpha]|Q^1|^{1 - \alpha(i+j+1)}|P_1|^{(1-\alpha)}\frac{\partial}{\partial P_1}.$$

These relations are well diagrammatically represented in Fig 1. In terms of differential geometry, Z_{α_j} and \tilde{H}_{α_j} are called α -master symmetries for X_{α_i} and α -master integrals, respectively. For more details on these symmetries, see [5, 10, 13, 33, 34].

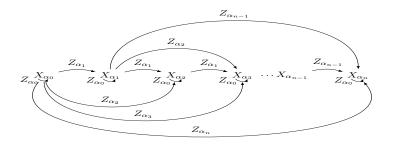


Figure 1: Diagrammatical illustration of equation (28).

Thus, we can generate a family of α -Hamiltonian functions:

$$H_{\alpha_{i+j}} := \{ H_{\alpha_i}, \tilde{H}_{\alpha_j} \} = (1 - \alpha i)(Q^1)^{1 - \alpha(i+j)}, \text{ with } H_{\alpha_0} = H_{\alpha}, i, j \in \mathbb{N}.$$

The α -recursion operator T_{α} can be written as:

$$T_{\alpha} = \mathcal{P}_{\alpha_1} \circ \mathcal{P}_{\alpha}^{-1},$$

where

$$\mathcal{P}_{\alpha_1} = \sum_{\mu=1}^{4} \alpha^{-2} Q^{\mu} |P_{\mu}|^{(1-\alpha)} \frac{\partial}{\partial P_{\mu}} \wedge \frac{\partial}{\partial Q^{\mu}}$$

and \mathcal{P}_{α} are two compatible α -Poisson bivectors with vanishing Schouten-Nijenhuis bracket $[\mathcal{P}_{\alpha}, \mathcal{P}_{\alpha_1}]_{NS} = 0$.

Furthermore, we put
$$\mathcal{P}_{\alpha_1}^{k+1} = S_{k+1}\mathcal{P}_{\alpha_1} = S_{k+1}\sum_{\mu=1}^4 \alpha^{-2}Q^{\mu}|P_{\mu}|^{(1-\alpha)}\frac{\partial}{\partial P_{\mu}}\wedge\frac{\partial}{\partial Q^{\mu}}$$
, with

 $S_{k+1} = \frac{1-k\alpha}{1-(k+1)\alpha}, k=i+j \in \mathbb{N}, \ (1-(k+1)\alpha) \neq 0, \text{ and introduce the following } \alpha\text{-Poisson}$ bracket $\{.,.\}_{\alpha}^{k}$

$$\{f,g\}_{\alpha_1}^{k+1} := \sum_{\mu=1}^{4} \alpha^{-2} S_{k+1} Q^{\mu} |P_{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial P_{\mu}} \frac{\partial g}{\partial Q^{\mu}} - \frac{\partial f}{\partial Q^{\mu}} \frac{\partial g}{\partial P_{\mu}} \right)$$
(29)

with respect to the α -symplectic form $\omega_{\alpha_1}^{k+1} = \sum_{m=1}^4 \alpha^2 S_{k+1}^{-1}(Q^\mu)^{-1} |P_\mu|^{(\alpha-1)} dP_\mu \wedge dQ^\mu$ and get

$$X_{\alpha_k} = \{H_{\alpha_k}, .\}_{\alpha} = \{H_{\alpha_{k+1}}, .\}_{\alpha_1}^{k+1}$$

proving that X_{α_k} are bi-Hamiltonian vector fields defined by the two α -Poisson bivectors \mathcal{P}_{α} and $\mathcal{P}_{\alpha_1}^{k+1}$. Then, the quadruple $(\mathcal{Q}, \mathcal{P}_{\alpha}, \mathcal{P}_{\alpha_1}^{k+1}, X_{\alpha_k})$ is a bi-Hamiltonian system for each k.

The associated α -recursion operators are given by

$$T_{(k+1)\alpha} := \mathcal{P}_{\alpha_1}^{k+1} \circ \mathcal{P}_{\alpha}^{-1} = \sum_{\mu=1}^{4} S_k |Q^{\mu}|^{\alpha-1} Q^{\mu} \left(\frac{\partial}{\partial P_{\mu}} \otimes dP_{\mu} + \frac{\partial}{\partial Q^{\mu}} \otimes dQ^{\mu} \right).$$

In addition, we have

$$\mathcal{L}_{Z_{\alpha_0}}(\mathcal{P}_{\alpha}) = 0, \ (\tilde{\alpha} = 0), \quad \mathcal{L}_{Z_{\alpha_0}}(\mathcal{P}_{\alpha_1}^{k+1}) = -\alpha \sum_{\mu=1}^4 \alpha^{-2} S_{k+1} Q^{\mu} |P_{\mu}|^{(1-\alpha)} \frac{\partial}{\partial P_{\mu}} \wedge \frac{\partial}{\partial Q^{\mu}} = -\alpha \mathcal{P}_{\alpha_1}^{k+1},$$

$$(\tilde{\beta} = -\alpha), \ \mathcal{L}_{Z_{\alpha_0}}(H_{\alpha}) = -Q^1 = -H_{\alpha}, \ (\tilde{\gamma} = -1)$$

permitting to conclude that the vector field

$$Z_{\alpha_0} = \sum_{\mu=1}^{4} \left(P_{\mu} \frac{\partial}{\partial P_{\mu}} - Q^{\mu} \frac{\partial}{\partial Q^{\mu}} \right)$$

is a conformal symmetry for \mathcal{P}_{α} , $\mathcal{P}_{\alpha_{1}}^{k+1}$ and H_{α} [13]. Defining now the families of quantities $X_{\alpha_{l}}^{k+1}$, $Z_{\alpha_{l}}^{k+1}$, $\mathcal{P}_{\alpha_{l}}^{k+1}$, $\omega_{\alpha_{l}}^{k+1}$ and $dH_{\alpha_{l}}^{k+1}$ by $X_{\alpha_{l}}^{k+1} := T_{(k+1)\alpha}^{l} X_{\alpha}$, $Z_{\alpha_{l}}^{k+1} := T_{(k+1)\alpha}^{l} Z_{\alpha_{0}}$, $\mathcal{P}_{\alpha_{l}}^{k+1} := T_{(k+1)\alpha}^{l} \mathcal{P}_{\alpha}$, $\omega_{\alpha_{l}}^{k+1} := ((T_{(k+1)\alpha}^{l})^{*})\omega_{\alpha}$, $dH_{\alpha_{l}}^{k+1} := (T_{(k+1)\alpha}^{l})^{*}dH_{\alpha}$, where $l \in \mathbb{N}$, and $T_{(k+1)\alpha}^{*} := \mathcal{P}_{\alpha}^{-1} \circ \mathcal{P}_{\alpha_{1}}^{k+1}$ denoting the adjoint of $T_{(k+1)\alpha} := \mathcal{P}_{\alpha_{1}}^{k+1} \circ \mathcal{P}_{\alpha}^{-1}$, we obtain

$$\begin{split} X_{\alpha_{l}}^{k+1} &= -\alpha^{-2} (S_{k+1})^{l} (Q^{1})^{1+\alpha(l-1)} |P_{1}|^{(1-\alpha)} \frac{\partial}{\partial P_{1}}; \\ Z_{\alpha_{l}}^{k+1} &= \sum_{\mu=1}^{4} (S_{k+1})^{l} |Q^{\mu}|^{l(\alpha-1)} (Q^{\mu})^{l} \bigg(P_{\mu} \frac{\partial}{\partial P_{\mu}} - Q^{\mu} \frac{\partial}{\partial Q^{\mu}} \bigg); \\ \mathcal{P}_{\alpha_{l}}^{k+1} &= \sum_{\mu=1}^{4} \alpha^{-2} (S_{k+1})^{l} (Q^{\mu})^{l} |P_{\mu}|^{(1-\alpha)} |Q^{\mu}|^{(1-\alpha)(1-l)} \frac{\partial}{\partial P_{\mu}} \wedge \frac{\partial}{\partial Q^{\mu}}; \end{split}$$

$$\omega_{\alpha_l}^{k+1} = \sum_{\mu=1}^4 \alpha^2 (S_{k+1})^l (Q^\mu)^l |P_\mu|^{(\alpha-1)} |Q^\mu|^{(\alpha-1)(l+1)} dP_\mu \wedge dQ^\mu;$$

$$dH_{\alpha_l}^{k+1} = (S_{k+1})^l (Q^1)^{\alpha l} dQ^1; \text{ and } H_{\alpha_l}^{k+1} = \frac{1}{l\alpha+1} (S_{k+1})^l (Q^1)^{\alpha l+1}$$

and for each $l \in \mathbb{N}$ we derive the following plethora of conserved quantities:

$$\mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(Z_{\alpha_{h}}^{k+1}) = \alpha(l-h)(S_{k+1})^{l+h} \sum_{\mu=1}^{4} |Q^{\mu}|^{(l+h)(\alpha-1)} (Q^{\mu})^{l+h} \left(P_{\mu} \frac{\partial}{\partial P_{\mu}} - Q^{\mu} \frac{\partial}{\partial Q^{\mu}} \right)$$

$$= \alpha(l-h)Z_{\alpha_{l+h}}^{k+1};$$

$$\mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(X_{\alpha_{h}}^{k+1}) = \alpha^{-2}(S_{k+1})^{l+h} (h\alpha+1)(Q^{1})^{1+\alpha((l+h)-1)} |P_{1}|^{(1-\alpha)} \frac{\partial}{\partial P_{1}}$$

$$= -(h\alpha+1)X_{\alpha_{l+h}}^{k+1};$$

$$\mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(\mathcal{P}_{\alpha_{h}}^{k+1}) = \alpha^{-1}(S_{k+1})^{l+h}(l-h)(Q^{\mu})^{l+h}|P_{\mu}|^{(1-\alpha)}|Q^{\mu}|^{(1-\alpha)(1-(l+h))}\frac{\partial}{\partial P_{\mu}} \wedge \frac{\partial}{\partial Q^{\mu}}$$

$$= \alpha(l-h)\mathcal{P}_{\alpha_{l+h}}^{k+1};$$

$$\mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(\omega_{\alpha_{h}}^{k+1}) = -\alpha^{3}(S_{k+1})^{l+h}(l+h)\sum_{\mu=1}^{4}(Q^{\mu})^{l+h}|P_{\mu}|^{(\alpha-1)}|Q^{\mu}|^{(\alpha-1)((l+h)+1)}dP_{\mu} \wedge dQ^{\mu}$$

$$= -\alpha(l+h)\omega_{\alpha_{l+h}}^{k+1};$$

$$< dH_{\alpha_{l}}^{k+1}, Z_{\alpha_{h}}^{k+1} > = -(S_{k+1})^{l+h}\frac{\alpha(l+h)+1}{\alpha(l+h)+1}(Q^{1})^{1+\alpha(l+h)}$$

$$= -(\alpha(l+h)+1)H_{\alpha_{l+h}}^{k+1};$$

$$\mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(T_{(k+1)\alpha}) = -\alpha\sum_{\mu=1}^{4}(S_{k+1})^{l+1}|Q^{\mu}|^{(\alpha-1)(l+1)}(Q^{\mu})^{l+1}\left(\frac{\partial}{\partial P_{\mu}} \otimes dP_{\mu} + \frac{\partial}{\partial Q^{\mu}} \otimes dQ^{\mu}\right)$$

$$= -\alpha T_{(k+1)\alpha}^{l+1},$$

satisfying the following relations linking the master symmetries Z_{α_j} to the conformal symmetry Z_{α_0} for \mathcal{P}_{α} , $\mathcal{P}_{\alpha_1}^{k+1}$ and H_{α} , and to a set of conformal symmetries generated by successive applications of the recursion operator $T_{(k+1)\alpha}$ on Z_{α_0} :

$$\begin{split} \mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(Z_{\alpha_{h}}^{k+1}) &= (\tilde{\beta} - \tilde{\alpha})(h - l)Z_{\alpha_{l+h}}^{k+1}, \ \mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(X_{\alpha_{h}}^{k+1}) = (\tilde{\beta} + \tilde{\gamma} + (h - 1)(\tilde{\beta} - \tilde{\alpha}))X_{\alpha_{l+h}}^{k+1}, \\ \mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(\mathcal{P}_{\alpha_{h}}^{k+1}) &= (\tilde{\beta} + (h - l - 1)(\tilde{\beta} - \tilde{\alpha}))\mathcal{P}_{\alpha_{l+h}}^{k+1}, \ \mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(\omega_{\alpha_{h}}^{k+1}) = (\tilde{\beta} + (l + h - 1)(\tilde{\beta} - \tilde{\alpha}))\omega_{\alpha_{l+h}}^{k+1}, \\ \mathcal{L}_{Z_{\alpha_{l}}^{k+1}}(T_{(k+1)\alpha}) &= (\tilde{\beta} - \tilde{\alpha})T_{(k+1)\alpha}^{1+l}, \ \langle dH_{\alpha_{h}}^{k+1}, Z_{\alpha_{l}}^{k+1} \rangle = (\tilde{\gamma} + (l + h)(\tilde{\beta} - \tilde{\alpha}))H_{\alpha_{l+h}}^{k+1}. \end{split}$$

This is reminiscent to the well known Oevel formulas (see [31, 13, 37, 38, 19, 20]). Finally, it is worth mentioning a generalization of the α -Poisson brackets (29) as follows:

$$\{f,g\}_{\alpha_t}^{k+t} := \sum_{\mu=1}^4 \alpha^{-2} S_{k+t} |Q^{\mu}|^{1+\alpha(t-1)} |P_{\mu}|^{(1-\alpha)} \left(\frac{\partial f}{\partial P_{\mu}} \frac{\partial g}{\partial Q^{\mu}} - \frac{\partial f}{\partial Q^{\mu}} \frac{\partial g}{\partial P_{\mu}} \right),$$

where $S_{k+t} = \frac{1-k\alpha}{1-(k+t)\alpha}, k,t \in \mathbb{N}, \ (1-(k+t)\alpha) \neq 0, \ \text{and} \ \{f,g\}_{\alpha_0}^0 = \{f,g\}_{\alpha}, \ \text{with} \ S_0 = 1,$ leading to a set of α - generalized bi-Hamiltonian vector fields

$$X_{\alpha_k} = \{H_{\alpha_k}, .\}_{\alpha} = \{H_{\alpha_{k+t}}, .\}_{\alpha_t}^{k+t}$$

main ingredients governing the Hamiltonian dynamics and pertaining symmetries.

6 Concluding remarks

In this work, we have proved that a Minkowski phase space endowed with a bracket relatively to a conformable differential realizes a conformable Poisson algebra, confering a bi-Hamiltonian structure to the resulting manifold. We have deduced that the related α -Hamiltonian vector field for a free particle is an infinitesimal Noether symmetry. We have computed the corresponding α -deformed recursion operator. Using the Hamiltonian-Jacobi separability, we have constructed recursion operators in the framework of α -Schwarzschild and Friedmann-Lemaître-Robertson-Walker (FLRW) metrics, and obtained related constants of motion. We have highlighted the existence of a hierarchy of bi-Hamiltonian structures in both the metrics, and derived a family of α -recursion operators and master symmetries generating the constants of motion. This study has also shown that Hamiltonian dynamics hints at a connection between the geometry of our physical system, (α -deformes symplectic manifolds and related Hamiltonian vector fields), and conservation laws. In this connection, the free particle positions on the α -deformed manifolds are viewed as states and vector fields as laws governing how those states evolve.

Further, we have calculated the α -deformed Christoffel symbols, Ricci scalar, the components of the α -deformed Riemann, Ricci, and Einstein tensors. This study revealed that the α -Christoffel symbols $((\Gamma^1_{11})_{\alpha}, (\Gamma^2_{22})_{\alpha}, (\Gamma^3_{33})_{\alpha})_{\alpha}$, and $(\Gamma^4_{44})_{\alpha})$ in α -deformed Minkowski metric are no longer null, contrary to the undeformed case corresponding to $\alpha=1$. Similarly, the α -Christoffel symbols $((\Gamma^1_{11})_{\alpha}, (\Gamma^3_{33})_{\alpha})_{\alpha}$, and $(\Gamma^4_{44})_{\alpha})$ are not equal zero in α -deformed Schwarzschild and FLRW metrics. The existence of these symbols $(\Gamma^i_{ii})_{\alpha}$, (i=1,2,3,4) informs us about the way in which the parallel displacement of any basic vector on the α -deformed manifolds with respect to itself always remains parallel to the same basic vector.

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A Appendix

A.1 α -Christoffel symbols $(\Gamma^k_{ij})_{\alpha}$ in α -Schwarzschild metric

	$(\Gamma_{24}^4)_\alpha = \frac{1}{q^2}$	$(\Gamma^4_{34})_\alpha = \cot(q^3)$	$ \cos(q^3) = (\Gamma_{44}^4)_{\alpha} = \frac{\alpha - 1}{q^4}$
	$\left (\Gamma^3_{23})_\alpha = \frac{1}{q^2} \right $	$\left (\Gamma^3_{33})_\alpha = \frac{\alpha - 1}{q^3} \right $	$\left \begin{array}{c} (\Gamma_{44}^3)_{\alpha} = -\frac{(q^4)^{2(\alpha-1)}\sin(q^3)\cos(q^3)}{(q^3)^{2(\alpha-1)}} \end{array} \right $
$(\Gamma_{11}^2)_{\alpha} = -\frac{M(2M - q^2)(q^1)^{2(\alpha - 1)}}{(q^2)^{2\alpha + 1}}$	$(\Gamma_{22}^2)_{\alpha} = \frac{(1-\alpha)q^2 + (2\alpha - 1)M}{q^2(2M-q^2)}$	$(\Gamma_{33}^2)_{\alpha} = -\frac{(2M - q^2)(q^3)^{2(\alpha - 1)}}{(q^2)^{2(\alpha - 1)}}$	$(\Gamma_{44}^2)_{\alpha} = \frac{(2M - q^2)(q^4)^{2(\alpha - 1)}\sin^2(q^3)}{(q^2)^{2(\alpha - 1)}}$
$(\Gamma_{11}^1)_\alpha = \frac{\alpha - 1}{q^1}$	$(\Gamma_{12}^1)_{\alpha} = -\frac{M}{q^2(2M-q^2)}$		

Components of the α -Riemann tensor α - $(R_{ijkl})_{\alpha}$ in α -Schwarzschild metric A.2

$(R_{1414})_{\alpha} = -\frac{\alpha^2 M(q^1)^{2(\alpha-1)} (q^4)^{2(\alpha-1)} \sin^2(q^3) (2M - q^2)}{(q^2)^{2\alpha}}$	$(R_{3434})_{\alpha} = -\frac{\alpha^2 q^2(q^4)^{2(\alpha - 1)}}{(q^2)^{2(\alpha - 1)}q^3} \left[\sin^2(q^3)(q^2q^3)(q^3)^{2(\alpha - 1)} - (q^2)^{2(\alpha - 1)}) - 2M(q^3)^{2\alpha - 1} \right] $ $+ (1 - \alpha)(q^2)^{2(\alpha - 1)} \sin(q^3) \cos(q^3) \right]$
$(R_{1313})_{\alpha} = -\frac{\alpha^2 M(q^1)^{2(\alpha-1)}(q^3)^{2(\alpha-1)}(2M - q^2)}{(q^2)^{2\alpha}}$	$(R_{2424})_{\alpha} = \frac{\alpha^2 (M(2\alpha - 1))}{2M - q^2} + \frac{(1 - \alpha)q^2)(q^4)^{2(\alpha - 1)}\sin^2(q^3)}{2M - q^2}$
$(R_{1212})_{\alpha} = -\frac{(1+\alpha)\alpha^2 M(q^1)^{2(\alpha-1)}}{(q^2)^3}$	$(R_{2323})_{\alpha} = rac{lpha^2 (M(2\alpha - 1))}{2M - q^2} + rac{(1 - lpha)q^2)(q^3)^{2(\alpha - 1)}}{2M - q^2}$

Components of the α -Ricci tensor $(R_{ij})_{\alpha}$ in α -Schwarzschild metric A.3

$(R_{22})_{\alpha} = -\frac{(1-\alpha)(3M - 2q^2)}{(q^2)^2(-2M + q^2)}$	$(R_{44})_{\alpha} = -\frac{1}{(q^2)^{2\alpha - 1}(q^3)^{2\alpha - 1}} \left[(q^4)^{2(\alpha - 1)} [(\alpha - 2)q^2(q^3)^{2\alpha - 1} + (q^2)^{2\alpha - 1}q^3 + 2(1 - \alpha)M(q^3)^{2\alpha - 1}] \sin^2(q^3) + (\alpha - 1)(q^2)^{2\alpha - 1} \sin(q^3) \cos(q^3) \right]$
$(R_{11})_{\alpha} = -\frac{(\alpha - 1)M(2M - q^2)(q^1)^{2(\alpha - 1)}}{(q^2)^{2(\alpha - 1)}}$	$(R_{33})_{\alpha} = \frac{1}{(q^2)^{2(\alpha-1)}q^3 \sin(q^3)} \left[(1-\alpha)(q^2)^{2(\alpha-1)} \cos(q^3) + q^2 q^3 \left[(2-\alpha)(q^3)^{2(\alpha-1)} - (q^2)^{2(\alpha-1)} \right] \sin(q^3) + 2(\alpha-1)M(q^2)^{2\alpha-1} \sin(q^3) \right]$

A.4 α -Ricci scalar R in α -Schwarzschild metric

$$\mathbf{R} = \frac{2[(1-\alpha)(q^2)^{2\alpha-1}\cos(q^3) + (3(\alpha-1)M(q^3)^{2\alpha-1} + (3-2\alpha)q^2(q^3)^{2\alpha-1} - q^3(q^2)^{2\alpha-1})\sin(q^3)]}{\alpha^2(q^2)^{2\alpha+1}(q^3)^{2\alpha-1}}$$

Components of the α -Einstein tensor G_{ij} in α -Schwarzschild metric A.5

$(G_{22})_{\alpha} = -\frac{q^3 \left((q^2)^{2(\alpha - 1)} - (q^3)^{2(\alpha - 1)} \right)}{q^2 (2M - q^2) (q^3)^{2\alpha - 1}} - \frac{(\alpha - 1)(q^2)^{2(\alpha - 1)} \cot(q^3)}{q^2 (2M - q^2)(q^3)^{2\alpha - 1}}$	$(G_{44})_{\alpha} = \frac{(\alpha - 1)(q^2 - M)(q^4)^{2(\alpha - 1)}\sin^2(q^3)}{(q^2)^{2\alpha - 1}}$
$(G_{11})_{\alpha} = -\frac{1}{(q^1)^{2(1-\alpha)}(q^2)^{2(\alpha+1)}(q^3)^{2\alpha-1}} \left[(2M-q^2)[(1-\alpha)(q^2)^{2\alpha-1}\cos(q^3) + (4(\alpha-1)M(q^3)^{2\alpha-1} + (3-2\alpha)q^2(q^3)^{2\alpha-1} - q^3(q^2)^{2\alpha-1})\sin(q^3)] \right]$	$(G_{33})_{\alpha} = \frac{(\alpha - 1)(q^2 - M)(q^3)^{2(\alpha - 1)}}{(q^2)^{2\alpha - 1}}$

A.6 α -Christoffel symbols $(\Gamma^k_{ij})_{\alpha}$ in α -FLRW metric

$(\Gamma_{14}^4)_{\alpha} = \frac{\frac{dR(q^1)}{dq^1}}{R(q^1)}$	$(\Gamma_{24}^4)_\alpha = \frac{1}{q^2}$	$(\Gamma^4_{34})_\alpha = \cot(q^3)$	$(\Gamma_{44}^4)_{\alpha} = \frac{\alpha - 1}{q^4}$
$(\Gamma_{13}^3)_{\alpha} = \frac{dR(q^1)}{2}$	$(\Gamma_{23}^3)_\alpha = \frac{1}{q^2}$	$(\Gamma_{33}^3)_{\alpha} = \frac{\alpha - 1}{q^3}$	$(\Gamma_{44}^3)_{\alpha} = -\frac{(q^4)^{(2\alpha - 2)}}{(q^3)^{(2\alpha - 2)}} \times \sin(q^3) \cos(q^3)$
$(\Gamma_{12}^2)_{\alpha} = rac{dR(q^1)}{2}$	$(\Gamma_{22}^2)_{\alpha} = -\frac{\alpha(-1+k(q^2)^2)}{q^2(-1+k(q^2)^2)} - \frac{1-2k(q^2)^2}{q^2(-1+k(q^2)^2)}$	$(\Gamma_{33}^2)_{\alpha} = \frac{q^2 (q^3)^{(2\alpha - 2)}(-1 + k(q^2)^2)}{(q^2)^{(2\alpha - 2)}}$	$(\Gamma_{44}^2)_{\alpha} = \frac{q^2 (q^4)^{(2\alpha - 2)} \sin^2(q^3)}{(q^2)^{(2\alpha - 2)}} \times (-1 + k (q^2)^2)$
$(\Gamma^1_{11})_{\alpha} = \frac{\alpha - 1}{q^1}$	$(\Gamma_{22}^1)_{\alpha} = -\frac{1}{2} \frac{\frac{dR(q^1)}{dq^1} (q^2)^{(2\alpha - 2)}}{(-1 + k(q^2)^2)(q^1)^{(2\alpha - 2)}}$	$(\Gamma_{33}^1)_{\alpha} = \frac{1}{2(q^1)^{(2\alpha - 2)}} \frac{dR(q^1)}{dq^1} \times (q^2)^2 (q^3)^{(2\alpha - 2)}$	$(\Gamma_{44}^1)_{\alpha} = \frac{1}{2(q^1)^{(2\alpha - 2)}} \frac{dR(q^1)}{dq^1} \times (q^2)^2 (q^4)^{(2\alpha - 2)} \sin^2(q^3)$

A.7 Components of the α -Riemann tensor $(R_{ijkl})_{\alpha}$ in α -FLRW metric

$R_{1414} = \frac{\alpha^2 (q^2)^2 (q^4)^{(2\alpha - 2)} \sin(q^3)^2}{4q^1 R(q^1)} \times \left[-2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 2 \frac{dR(q^1)}{dq^1} R(q^1)(\alpha - 1) + \left(\frac{dR(q^1)}{dq^1} \right)^2 q^1 \right]$	$R_{3434} = \frac{\alpha^2 (q^2)^2 (q^4)^{(2\alpha - 2)} \sin^2(q^3)}{4(q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3} \times \left[4 R(q^1) (q^1)^{(2\alpha - 2)} \left((q^2)^{(2\alpha - 2)} - (q^3)^{(2\alpha - 2)} \right) q^3 + \left(\frac{dR(q^1)}{dq^1} \right)^2 (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 + 4 R(q^1)(q^1)^{(2\alpha - 2)} (q^3)^{(2\alpha - 2)} q^3 k (q^2)^2 + 4 (\alpha - 1) R(q^1) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 k (q^2)^2 \right]$
$R_{1313} = \frac{\alpha^2 (q^2)^2 (q^3)^{(2\alpha - 2)}}{4q^1 R(q^1)} \times \left[-2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 2 \frac{d R(q^1)}{dq^1} R(q^1)(\alpha - 1) + \left(\frac{d R(q^1)}{dq^1} \right)^2 q^1 \right]$	$R_{2424} = \frac{\alpha^2 (q^4)^{(2\alpha - 2)} \sin(q^3)^2}{4((q^1)^{(2\alpha - 2)} (-1 + k (q^2)^2))} \times \left[-\left(\frac{d}{dq^1} R(q^1)\right)^2 (q^2)^2 (q^2)^{(2\alpha - 2)} + 4\alpha R(q^1) (q^1)^{(2\alpha - 2)} (-1 + k(q^2)^2) \right] + 4R(q^1) (q^1)^{(2\alpha - 2)} (1 - 2k (q^2)^2) \right]$
$R_{1212} = -\frac{\alpha^2 (q^2)^{(2\alpha - 2)}}{4(-1 + k(q^2)^2) q^1 R(q^1)} \times \left[-2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 2 \frac{d R(q^1)}{dq^1} R(q^1)(\alpha - 1) + \left(\frac{d R(q^1)}{dq^1} \right)^2 q^1 \right]$	$R_{2323} = \frac{\alpha^2 (q^3)^{(2\alpha - 2)}}{4\left((q^1)^{(2\alpha - 2)}(-1 + k (q^2)^2)\right)} \times \left[-\left(\frac{dR(q^1)}{dq^1}\right)^2 (q^2)^2 (q^2)^{(2\alpha - 2)} + 4\alpha R(q^1) (q^1)^{(2\alpha (-2))}(-1 + k(q^2)^2) + 4R(q^1) (q^1)^{(2\alpha - 2)}(1 - 2k (q^2)^2) \right]$
$R_{121:}$	R_{232}

Components of the α -Ricci tensor $(R_{ii})_{\alpha}$ in α -FLRW metric A.8

$R_{22} = -\frac{1}{4(q^1)^{(2\alpha-2)}(-1+k(q^2)^2)} \frac{1}{q^1 R(q^1)(q^2)^2} \times \left[-2 (q^2)^{(2\alpha-2)} (q^2)^2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 2 (q^2)^{(2\alpha-2)} (q^2)^2 \frac{dR(q^1)}{dq^1} R(q^1) (\alpha - 1) - (q^2)^{(2\alpha-2)} (q^2)^2 \left(\frac{dR(q^1)}{dq^1} \right)^2 q^1 + 8 q^1 R(q^1) (q^1)^{(2\alpha-2)} \alpha (-1+k(q^2)^2) + 8 q^1 R(q^1) (q^1)^{(2\alpha-2)} (1-2k(q^2)^2) \right]$	$R_{44} = -\frac{(q^4)^{(2\alpha - 2)}\sin^2(q^3)}{4q^1 R(q^1) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} (q^3)^{(2\alpha - 2)} q^3} \times \left[2 (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 R(q^1) \times \left(\frac{d^2 (R(q^1))}{d(q^1)^2} q^1 - (\alpha - 1) \frac{d(R(q^1))}{dq^1} \right) \right] + 4 (q^3)^{(2\alpha - 2)} q^1 q^3 R(q^1) (q^1)^{(2\alpha - 2)} \left(q^3 + (\alpha - 1) \cot(q^3) \right) + 4 q^1 R(q^1) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} \left(q^3 + (\alpha - 1) \cot(q^3) \right) + q^1 \left(\frac{dR(q^1)}{dq^1} \right)^2 (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 \right]$
$R_{11} = -\frac{3}{4R(q^1)^2 q^1} \left[-2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 2 \frac{dR(q^1)}{dq^1} R(q^1) (\alpha - 1) + \left(\frac{dR(q^1)}{dq^1} \right)^2 q^1 \right]$	$R_{33} = \frac{1}{4 q^1 R(q^1) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3} \times \left[4 q^1 R(q^1) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} \left(\cot(q^3) (1 - \alpha) - q^3 \right) + 4 q^1 R(q^1) (q^1)^{(2\alpha - 2)} q^3 (q^3)^{(2\alpha - 2)} \times \left(2 - 3k (q^2)^2 + \alpha (-1 + k (q^2)^2) \right) - 2 (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + \frac{dR(q^1)}{dq^1} (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 \times \left(- q^1 \frac{dR(q^1)}{dq^1} + 2(\alpha - 1)R(q^1) \right) \right]$

A.9 α -Ricci scalar R in α -FLRW metric

$$\mathbf{R} = \frac{1}{(R(q^1) (q^2)^2 \alpha^2 q^1 (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} (q^3)^{(2\alpha - 2)} q^3)} \left[2 (1 - \alpha) q^1 \cot(q^3) (q^1)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} + 3(\alpha - 1) (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 \frac{dR(q^1)}{dq^1} + 4\alpha (q^3)^{(2\alpha - 2)} q^1 q^3 (q^1)^{(2\alpha - 2)} (-1 + k (q^2)^2) + 2 q^1 (q^1)^{(2\alpha - 2)} q^3 \left(3(q^3)^{(2\alpha - 2)} - (q^2)^{(2\alpha - 2)} \right) - 3 (q^2)^2 (q^3)^{(2\alpha - 2)} (q^2)^{(2\alpha - 2)} q^3 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 - 10 q^1 (q^3)^{(2\alpha - 2)} (q^1)^{(2\alpha - 2)} q^3 k (q^2)^2, \right]$$

A.10 Components of the α -Eintein tensor $(G_{ij})_{\alpha}$ in α -FLRW metric

$$G_{11} = -\frac{1}{4(R(q^1)^2 (q^2)^2 (q^2)^{(2\alpha-2)} (q^3)^{(2\alpha-2)} q^3)} \times$$

$$\left[3 \left(\frac{dR(q^1)}{dq^1} \right)^2 (q^2)^2 (q^3)^{(2\alpha-2)} (q^2)^{(2\alpha-2)} q^3 + 4(\alpha - 1) R(q^1) \cot(q^3) (q^1)^{(2\alpha-2)} (q^2)^{(2\alpha-2)} q^3 + 4 R(q^1) (q^1)^{(2\alpha-2)} (q^2)^{(2\alpha-2)} q^3 + 4 R(q^1) (q^3)^{(2\alpha-2)} (q^1)^{(2\alpha-2)} q^3 \left(-3 + 5 k (q^2)^2 \right) + 4 R(q^1) (q^3)^{(2\alpha-2)} (q^1)^{(2\alpha-2)} q^3 \left(-3 + 5 k (q^2)^2 \right) + 4 R(q^1) (q^3)^{(2\alpha-2)} (q^1)^{(2\alpha-2)} q^3 \left(-3 + 5 k (q^2)^2 \right) + 4 R(q^1) (q^2)^{(2\alpha-2)} \left(-3 + 5 k (q^2)^2 \right) \right]$$

$$G_{22} = \frac{4((q^{1})(^{2}\alpha - ^{2})(-1 + k (q^{2})^{2})q^{1}R(q^{1})(q^{2})^{2}(q^{3})(^{2}\alpha - ^{2})}{q^{2}R(q^{1})}q^{1}R(q^{1})}$$

$$= \left[- 4 (q^{2})^{2} (q^{3})(^{2}\alpha - ^{2})(q^{2})(^{2}\alpha - ^{2})q^{3}\frac{d^{2}R(q^{1})}{(dq^{1})^{2}}q^{1}R(q^{1})$$

$$+ (q^{2})^{2} (q^{3})(^{2}\alpha - ^{2})(q^{2})(^{2}\alpha - ^{2})q^{3}\frac{dR(q^{1})}{dq^{1}} \times$$

$$\left(4(\alpha - 1)R(q^{1}) + q^{1}\frac{dR(q^{1})}{dq^{1}} \right)$$

$$+ 4 q^{1}R(q^{1})(q^{1})(^{2}\alpha - ^{2})\left(- (q^{3})(^{2}\alpha - ^{2})q^{3}(-1 + k (q^{2})^{2}) \right)$$

$$+ (q^{2})(^{2}\alpha - ^{2})(-q^{3} + (1 - \alpha)\cot(q^{3})) \right] \right]$$

$$+ (q^{2})(^{2}\alpha - ^{2})(-q^{3} + (1 - \alpha)\cot(q^{3})) \right]$$

$$G_{44} = \frac{(q^{4})^{2}\alpha - ^{2}}{4q^{1}R(q^{1})(q^{1})(^{2}\alpha - ^{2})(q^{2})^{2}\frac{d^{2}R(q^{1})}{(qq^{1})^{2}}q^{1}R(q^{1})$$

$$+ 4(q^{2})(^{2}\alpha - ^{2})(q^{2})^{2}\frac{d^{2}R(q^{1})}{(qq^{1})^{2}}q^{1}R(q^{1})$$

$$- 4(\alpha - 1)(q^{2})(^{2}\alpha - ^{2})(q^{2})^{2}\frac{d^{2}R(q^{1})}{(qq^{1})^{2}}R(q^{1})$$

$$+ 4q^{1}R(q^{1})(q^{1})(^{2}\alpha - ^{2})(q^{2})^{2}\frac{d^{2}R(q^{1})}{(qq^{1})^{2}}$$

$$+ q^{1}\left(\frac{dR(q^{1})}{dq^{1}}\right)^{2}(q^{2})^{2}(q^{2})(^{2}\alpha - ^{2})\right]$$

$$G_{33} = -\frac{(q^3)^{(2\alpha-2)}}{4q^1 R(q^1) (q^1)^{(2\alpha-2)} (q^2)^{(2\alpha-2)}} \times \left[4(1-\alpha) q^1 R(q^1) (q^1)^{(2\alpha-2)} (q^2)^2 \frac{dR(q^1)}{dq^1} R(q^1) + 4(\alpha-1) (q^2)^{(2\alpha-2)} (q^2)^2 \frac{d^2 R(q^1)}{dq^1} R(q^1) - 4(q^2)^{(2\alpha-2)} (q^2)^2 \frac{d^2 R(q^1)}{(dq^1)^2} q^1 R(q^1) + 4(q^2)^{(2\alpha-2)} (q^2)^2 \left(\frac{dR(q^1)}{dq^1} \right)^2 q^1 \right] + (q^2)^{(2\alpha-2)} (q^2)^2 \left(\frac{dR(q^1)}{dq^1} \right)^2 q^1 \right]$$

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