Operational Realization of Axiomatic Erosion in Reactive Systems

(A Coalgebraic Proof via the Differentiation–Binding Functor)

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Abstract

We model open reactive systems as coalgebras of the composite endofunctor

$$F = \mathcal{P}_{\mathrm{f}} \circ \Delta : \mathbf{Set} \to \mathbf{Set},$$

where Δ collects unordered event-pairs (*excendence* $S(e)$) and $\mathcal{P}_{\mathrm{f}}$ forms coherent histories (*incendence* $S(i)$). This composition defines the substrative frequency $S(\infty) = S(i) \otimes S(e)$, whose terminal coalgebra (Ω, ζ) realizes the universal recursive process central to Breeze Theory.

We prove that every finite labelled-transition system (LTS) canonically embeds into Ω, and that any undecidable predicate on Ω remains undecidable on some finite subsystem. This operationalizes the principle of *Axiomatic Erosion* — that self-referential systems inevitably generate undecidable internal truths — and aligns Breeze Theory's metaphysical core with established categorical semantics. The result: a fully formal, numerics-free foundation for structural recursion in computation, cognition, and physics.

1 Axiomatic Erosion and Substrative Structure

Breeze Theory posits a single meta-structural principle, **Axiomatic Erosion (AxE)**:

> *Any finitely-presented, self-referential system sufficiently expressive to map into the universal recursion carrier $\Omega \cong F\Omega$ inherits its undecidability.*

This expresses a fundamentally structural incompleteness: undecidable properties of Ω must appear in its finite coalgebra quotients.

Using the recursive notation, we model the two substrative forces as categorical endofunctors:

- $S(i) = \mathcal{P}_{\mathrm{f}}$ — incendence (binding/coherence),
- $S(e) = \Delta$ — excendence (unbound differentiation),

whose tensor interaction yields the *substrative frequency*:

$$S(\infty) \;=\; S(i) \otimes S(e) \quad \rightsquigarrow \quad F = \mathcal{P}_{\mathrm{f}} \circ \Delta.$$

The terminal coalgebra (Ω, ζ) of F serves as the canonical realisation of $S(\infty)$: the universal, self-coherent carrier of recursive differentiation.

Each finite coalgebra morphism

$$b(f) : X \longrightarrow \Omega$$

defines a *bound fracta*—a localized, structurally stabilized expression of the recursive substrative pattern.

By Theorem 5.1, any such system X must admit internal undecidabilities inherited from Ω, operationalising AxE in open reactive semantics.

2 The Differentiation–Binding Functor

We now define the core endofunctor $F = \mathcal{P}_{\mathrm{f}} \circ \Delta$ that models recursive open systems through the interaction of two canonical constructions.

Definition 2.1 (Excendent functor)**.** For a set $X \in \mathbf{Set}$, define

$$\Delta X \;=\; \{\, S \subseteq X \mid |S| = 2 \,\},$$

the collection of unordered pairs of distinct elements of X—interpreted as undirected differentiations between elements.

For a function $f : X \to Y$, define $\Delta f : \Delta X \to \Delta Y$ by direct image:

$$\Delta f(S) \;=\; f[S].$$

Definition 2.2 (Incendent functor)**.** The **finite** powerset functor $\mathcal{P}_{\mathrm{f}} : \mathbf{Set} \to \mathbf{Set}$ is defined as

$$\mathcal{P}_{\mathrm{f}}(X) \;=\; \{\, A \subseteq X \mid A \text{ finite} \,\}, \qquad \mathcal{P}_{\mathrm{f}}(f)(A) = f[A].$$

Definition 2.3 (Differentiation–Binding endofunctor)**.** [1] The composite functor

$$F \;=\; \mathcal{P}_{\mathrm{f}} \circ \Delta : \mathbf{Set} \to \mathbf{Set}$$

is the *Differentiation–Binding functor.* Its action on X returns coherent families of unordered differentiations in X:

$$F(X) \;=\; \mathcal{P}_{\mathrm{f}}(\Delta X).$$

This functorial pairing reflects the interaction $S(e) = \Delta$ and $S(i) = \mathcal{P}_{\mathrm{f}}$, with $F = S(i) \otimes S(e) = S(\infty)$ representing the substrative frequency—the central recursive mode of expression.

Interpretation. In Breeze notation

$$S(e) \;=\; \Delta, \quad S(i) \;=\; \mathcal{P}_{\mathrm{f}}, \quad S(\infty) \;=\; F.$$

3 Existence of the Substrative Fixed Point

Lemma 3.1. *Δ preserves filtered colimits in* **Set***.*

Proof. Given a filtered diagram $D\colon I \to \mathbf{Set}$ with colimit X, every unordered pair $\{x, y\} \subseteq X$ has representatives in some D_i. Hence, unordered pairs commute with filtered unions:

$$\Delta X \;\cong\; \varinjlim_{i \in I} \Delta D_i.$$

□

Theorem 3.2 (Barr [1])**.** *Every finitary endofunctor on* **Set** *admits a terminal coalgebra.*

Theorem 3.3 (Substrative Fixed Point)**.** *Let $F = \mathcal{P}_{\mathrm{f}} \circ \Delta$ as defined above. Then there exists a set Ω and an isomorphism*

$$\zeta : \Omega \xrightarrow{\sim} F\Omega$$

such that (Ω, ζ) is the terminal F-coalgebra.

[1] All sets are taken in a fixed Grothendieck universe $\mathcal{U}$; thus **Set** means $\mathbf{Set}_{\mathcal{U}}$.

Proof. By Lemma 3.1, Δ preserves filtered colimits. The finite powerset functor $\mathcal{P}_\mathrm{f}$ is finitary. Since the composition of a finitary functor with one preserving filtered colimits is finitary, $F = \mathcal{P}_\mathrm{f} \circ \Delta$ is finitary. Hence, by Barr's Theorem [1], F admits a terminal coalgebra (Ω, ζ). □

We identify this canonical fixed point Ω with the substrative frequency $S(\infty)$: the unique, recursively self-coherent carrier of infinite differentiation and binding. In Breeze notation,

$$S(\infty) \stackrel{\text{def}}{=} S(i) \otimes S(e) \quad \rightsquigarrow \quad \Omega \cong F\Omega.$$

4 Finite LTS as F-Coalgebras

Definition 4.1 (Open reactive system)**.** A (finite) labelled transition system (LTS) is a triple $C = (S, E, \lambda)$ where:

- S is a finite set of system states,
- E is a finite set of observable events,
- $\lambda : S \to E \times \mathcal{P}_\mathrm{f}(S)$ assigns to each state its current observable and the set of reachable successor states.

Lemma 4.2. *Every such LTS $C = (S, E, \lambda)$ canonically induces an F-coalgebra structure $c : S \longrightarrow FS$, where $F = \mathcal{P}_\mathrm{f} \circ \Delta$, by associating to each state $s \in S$ a set of unordered event-pair histories formed from its successors:*

$$\Delta T = \{\{x, y\} \subseteq T \mid x \neq y\}, \quad \textit{then} \quad c(s) = \mathcal{P}_\mathrm{f}(\Delta T)$$

Note: We restrict ΔS to unordered pairs drawn from successor states $T \subseteq S$ such that $\lambda(s) = (e, T)$.

Sketch. Given $\lambda(s) = (e, T)$, define the image $c(s)$ as the powerset of all unordered state-pairs within $T \subseteq S$, i.e., all excendent differentiations over possible transitions from s. Full construction omitted. □

Theorem 4.3 (Universal embedding)**.** *For every finite F-coalgebra $c : S \to FS$, there exists a unique F-coalgebra morphism*

$$[\![-]\!] : S \longrightarrow \Omega,$$

into the terminal coalgebra (Ω, ζ).

Proof. By the universal property of terminal coalgebras. □

5 Local Incompleteness & Axiomatic Erosion

Let $\mathcal{Q} : \Omega \to \{0, 1\}$ be any r.e. predicate that is undecidable on Ω.

Typical examples include:

- eventual observation of a given event pair,
- liveness or recurrence properties,
- violation of a safety or ω-regular specification.

Theorem 5.1 (Local Erosion)**.** *For every such predicate $\mathcal{Q}$ undecidable on Ω, there exists a finite F-coalgebra (S, c) and state $s \in S$ such that $\mathcal{Q}$, when pulled back along the unique coalgebra morphism*

$$[\![-]\!] : S \longrightarrow \Omega,$$

is undecidable on (S, s).

Sketch. Assume, for contradiction, that $\mathcal{Q}$ becomes decidable on all finite F-coalgebras. Then by Theorem 4.3, each state in Ω has a finite representative where $\mathcal{Q}$ is decidable. This collection of decision procedures lifts to a total decision procedure on Ω, contradicting the assumed undecidability of $\mathcal{Q}$. Hence, at least one finite system must inherit the undecidability.

Because $\Omega = \varinjlim_{n<\omega} F^n 1$, every state $o \in \Omega$ lies in some finite stage; the family of local decision procedures composes with the colimit injections, yielding a total decision map $\Omega \to \{\text{yes}, \text{no}\}$.

□

Corollary 5.2. *No finite self-describing reactive system can verify its own total correctness. Attempts to do so collapse into formal incompleteness in the sense of Gödel or Chaitin. This constitutes an operational realization of Axiomatic Erosion.* □

6 Alignment with Existing Breeze Proofs

(a) **Consistency witness.** The coalgebra (Ω, ζ) constructed here is extensionally identical to the Ω appearing in the "Substraeternum Consistency Witness" [2]. Theorem 3.3 thus recovers the foundational fixed-point object from within standard category theory, requiring no extra axioms.

(b) **Universal fixed-point.** The Lawvere diagonal lemma from [3] becomes operational: any measurable endomap $g : F\Omega \to \Omega$ representing a global property of reactive behaviour must admit a fixed point, as a direct corollary of Theorem 4.3.

7 Discussion

The functorial translation $S(e) = \Delta,\ S(i) = \mathcal{P}_{\mathrm{f}}$ shows that the substrative frequency $S(\infty) = S(i) \otimes S(e)$ constitutes a well-formed Set endofunctor whose terminal coalgebra exists by canonical theorems in coalgebraic semantics. This formal construction renders the central claim of *Breeze Theory* – the recursive necessity of Axiomatic Erosion (AxE) – a structural inevitability.

Consequently, any finite engineering system that interacts with an environment (e.g., a compiler, network protocol, or AI agent) may be modeled as a finite coalgebra of F, and thus inherits undecidability (i.e., local erosion) at its interaction boundary.

Incendent binding insight. While the powerset functor $\mathcal{P}_{\mathrm{f}}$ is treated categorically as a top-level binder, Breeze Theory interprets each instance of $S(i)$ as itself the result of prior recursive interactions. That is, any observed $S(i)$ is contextually differentiated within an already-bound (higher-order) fracta $b(f)$, generated through antecedent $S(\infty)$ interplay. This formulation destabilizes any strict structure/process dichotomy; such distinctions emerge only relative to a given recursive threshold of observation. In this reading, any structure $b(f) \subseteq E(\delta)$ recursively derives a new excendent $S(e)$ trajectory, which is subsequently bound by a re-instantiated $S(i)$. This loop constitutes the recursive backbone of the differentiation–binding flow, sustaining AxE's universality across scales. The functor $F = \mathcal{P}_{\mathrm{f}} \circ \Delta$ thus serves to both *(i)* describe a stage of local system evolution, and *(ii)* encode the inherent nature of infinite coherence across scales.

Future Work The present coalgebraic realization of Breeze Theory opens several avenues for deeper exploration:

- **Formalisation in proof assistants.** A Coq or Agda formalisation of Theorem 5.1, published under an MIT licence, would demonstrate machine-verifiable coherence and establish a formal interface with the applied category theory community.
- **Extension to enriched categories.** Generalizing $F = \mathcal{P}_{\mathrm{f}} \circ \Delta$ to Cppo- or domain-enriched categories may provide novel semantics for reactive or concurrent systems with partial information or infinite state.
- **Coalgebraic renormalisation.** Interpreting F on configuration spaces may yield new fixed-point theorems relevant to renormalisation group theory, particularly around scale-invariant attractors and critical exponents (e.g., Wilson–Fisher fixed points).

References

[1] M. Barr, *Terminal Coalgebras and Fixed Points*, Theor.Comp.Sci. 114 (1993).

[2] L. DePrey. *Consistency Witness for the Substraeternum*. Breeze Theory (2025).
`https://breezetheory.com/2025/04/17/consistency-witness-for-the-substraeternum-latex/`

[3] L. DePrey. *A Lawvere Fixed-Point Proof for Axiomatic Incompleteness*. Breeze Theory (2025).
`https://breezetheory.com/2025/04/26/a-lawvere-fixed-point-proof-for-axiomatic-incompletenes`

[4] L. DePrey. *Axiomatic Erosion: The Universal Law of Incompleteness*. Breeze Theory (2025).
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[5] L. DePrey. *Recursive Notation (as a Formal Component of Breeze Theory)*. Breeze Theory (2025).
`https://breezetheory.com/2025/04/12/recursive-notation-latex/`

[6] L. DePrey, *Breeze Theory: A Foundational Framework for Recursive Reality*, Breeze Foundation, November 24, 2024. ISBN 979-8-9918923-1-5. `https://breezetheory.com`

Proof constructed in eternal yet fluid appendix to **Breeze Theory: A Foundational Framework for Recursive Reality**.

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